

Computational Geometry

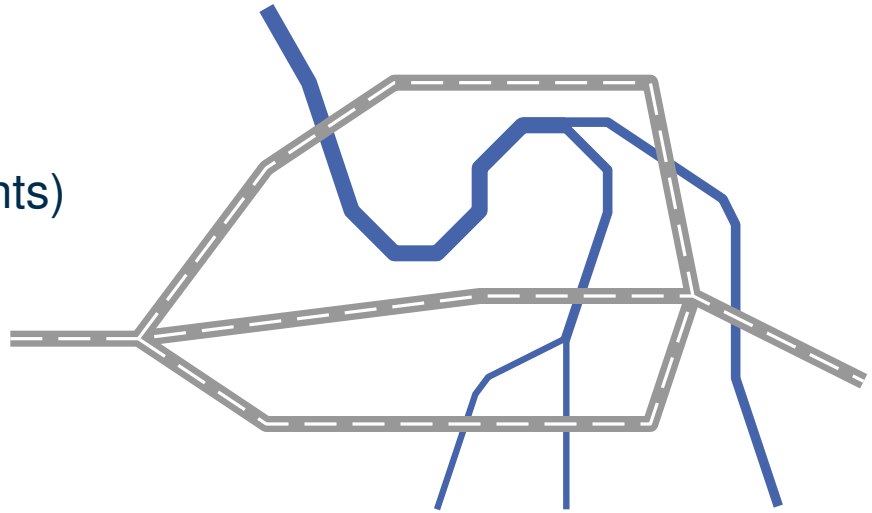
Line Segment Intersection

Thomas Bläsius

Line Segment Intersection: Motivation

Where Are Bridges?

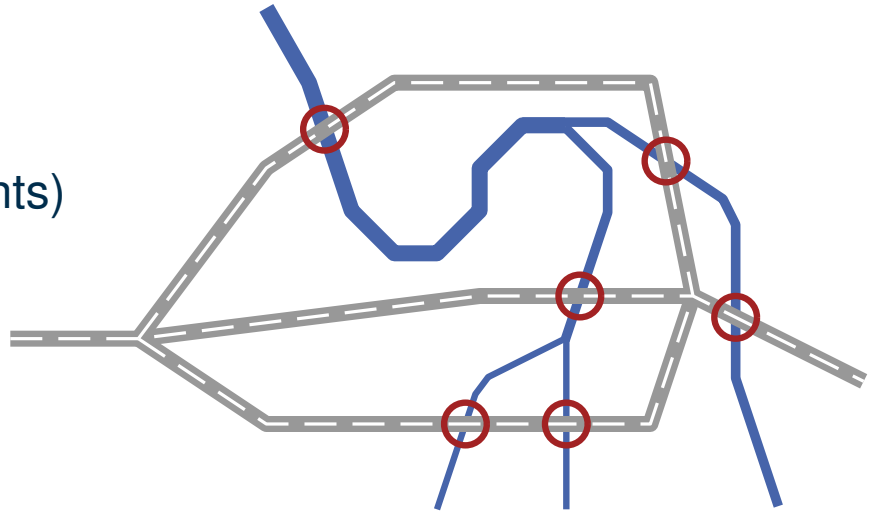
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- **goal:** find all bridges



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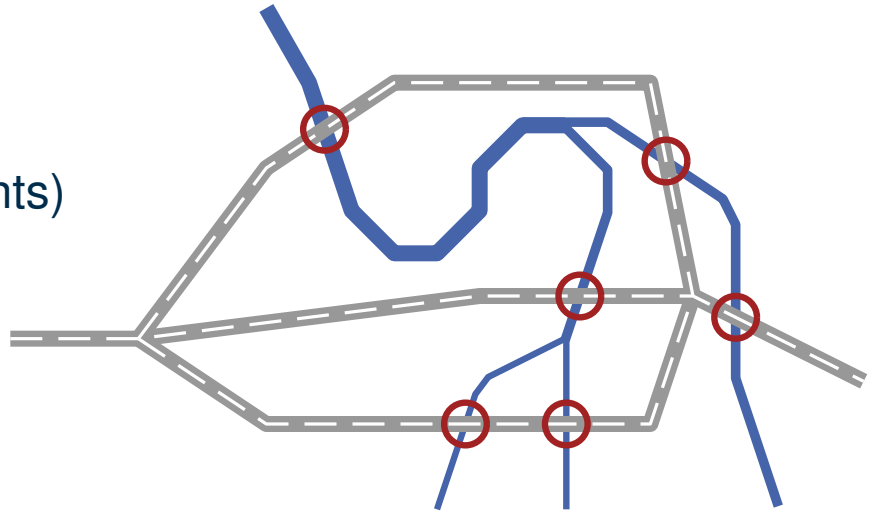
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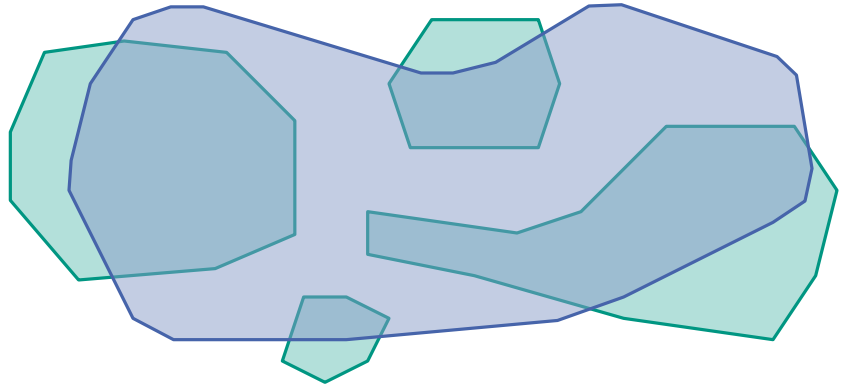
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Forests With A Lot Of Rainfall

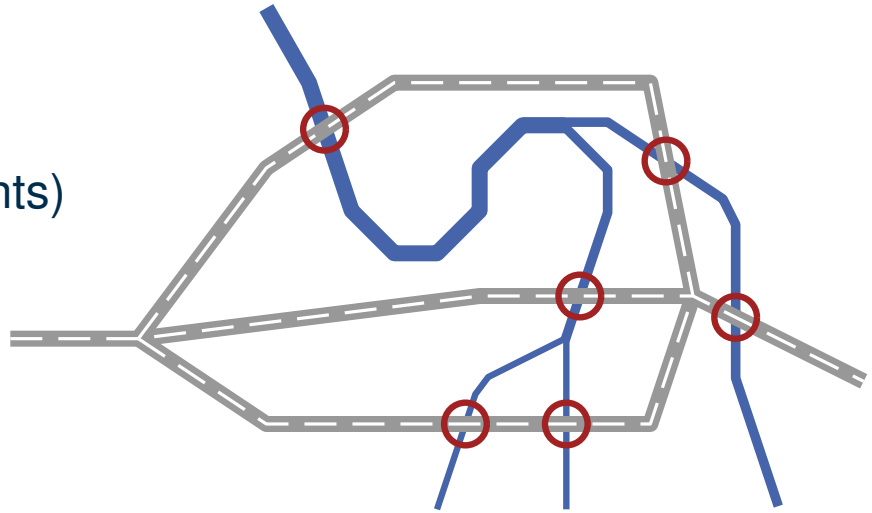
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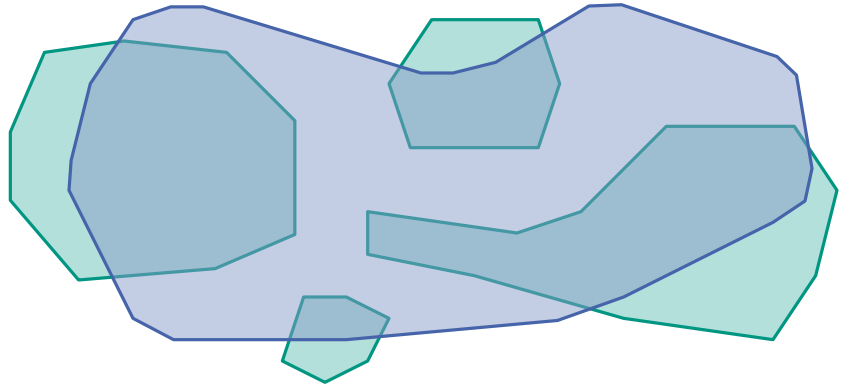
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Given n line segments, compute all pairwise intersections.

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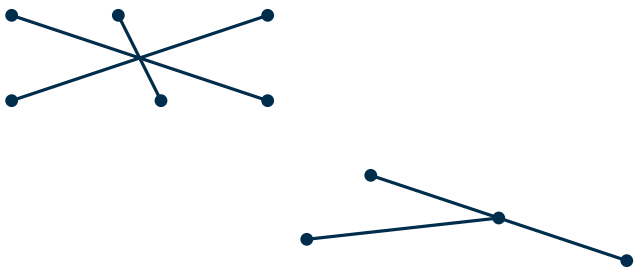


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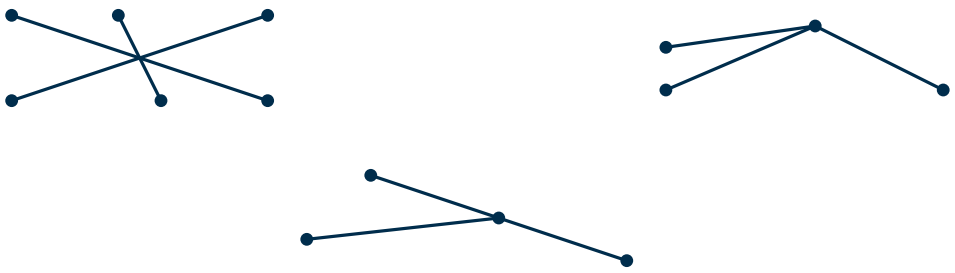


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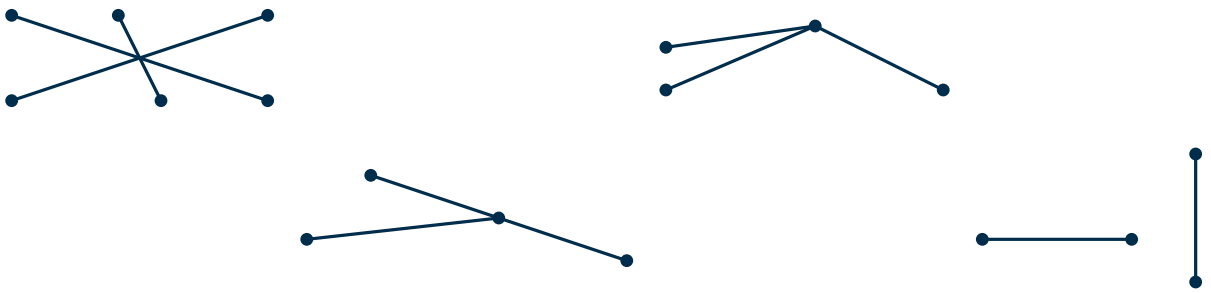


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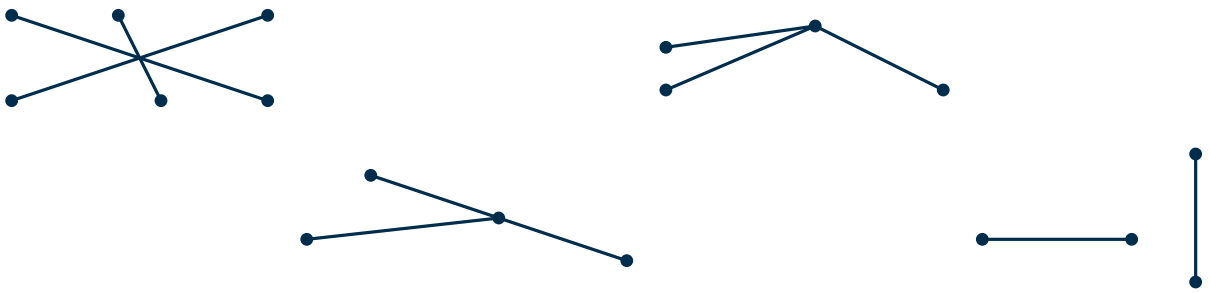


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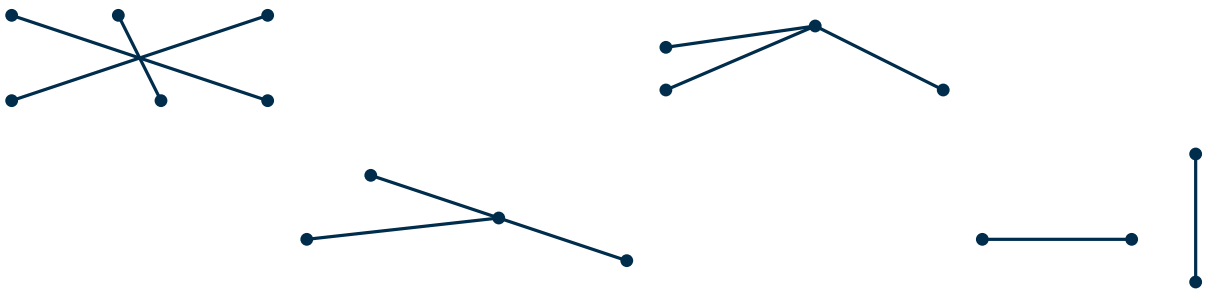
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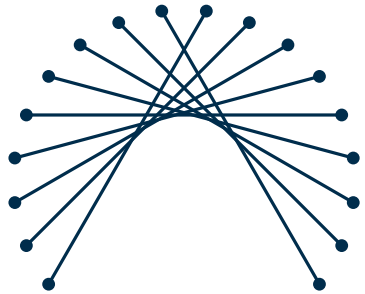


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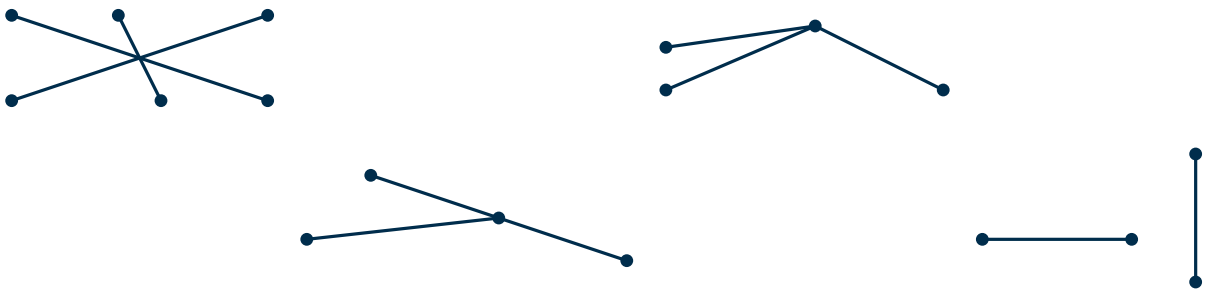
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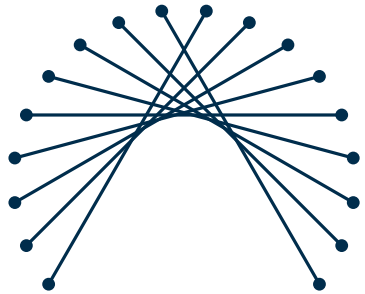


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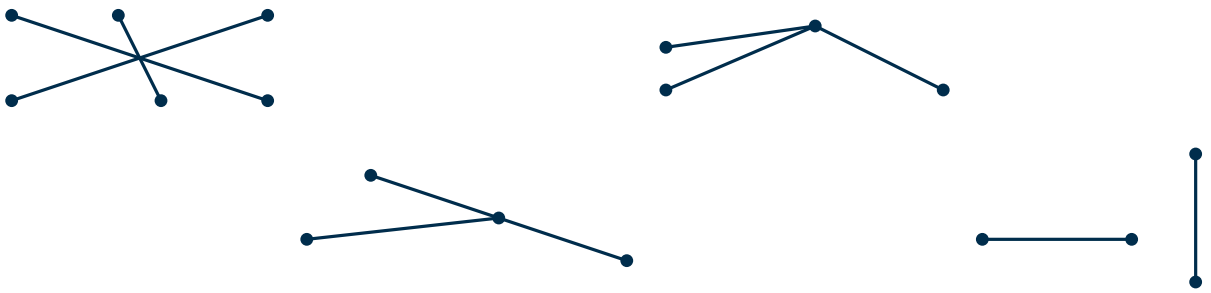
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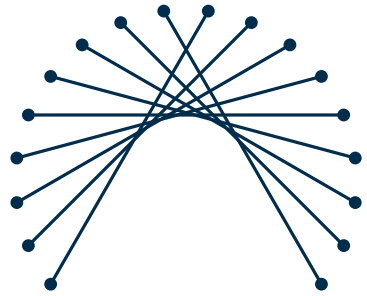
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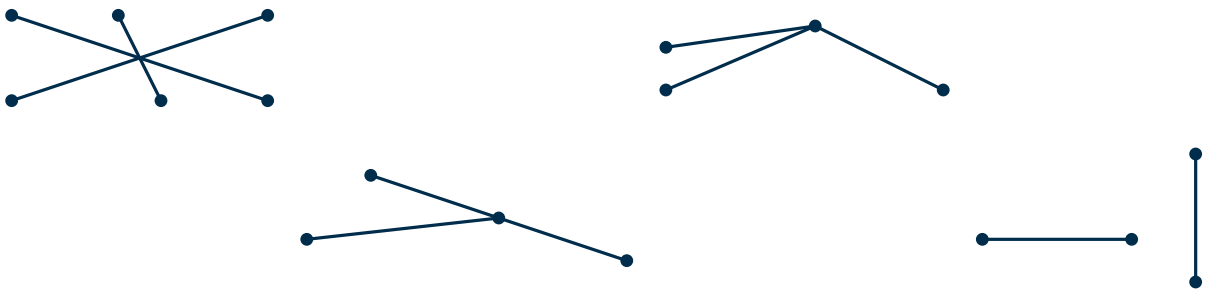
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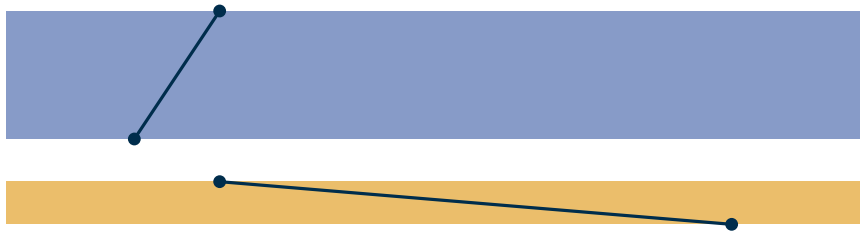
Potential Improvement

- few intersections $\Rightarrow$ better running time
- running time depends on output size
  $\rightarrow$ output sensitive algorithm

A Simple Sweep-Line Algorithm

Idea

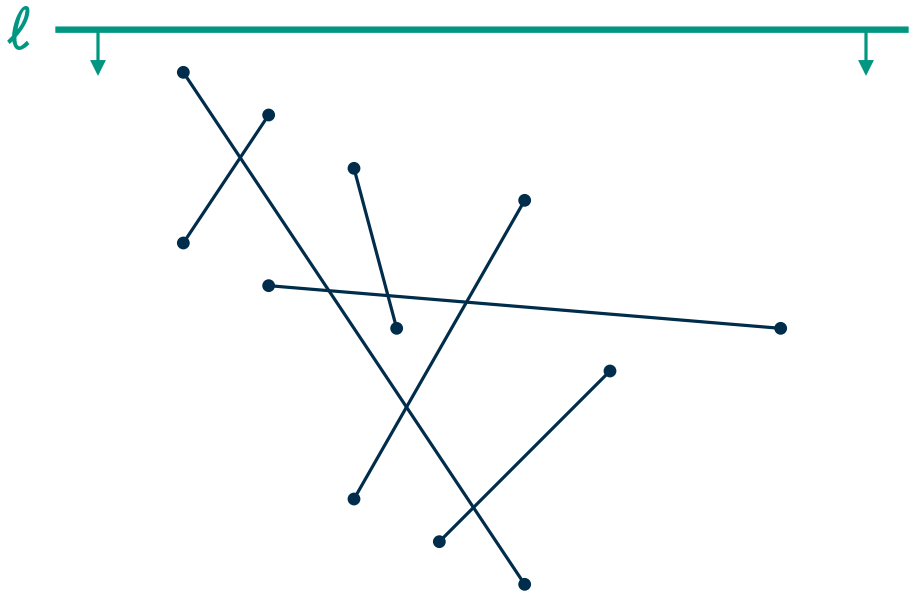
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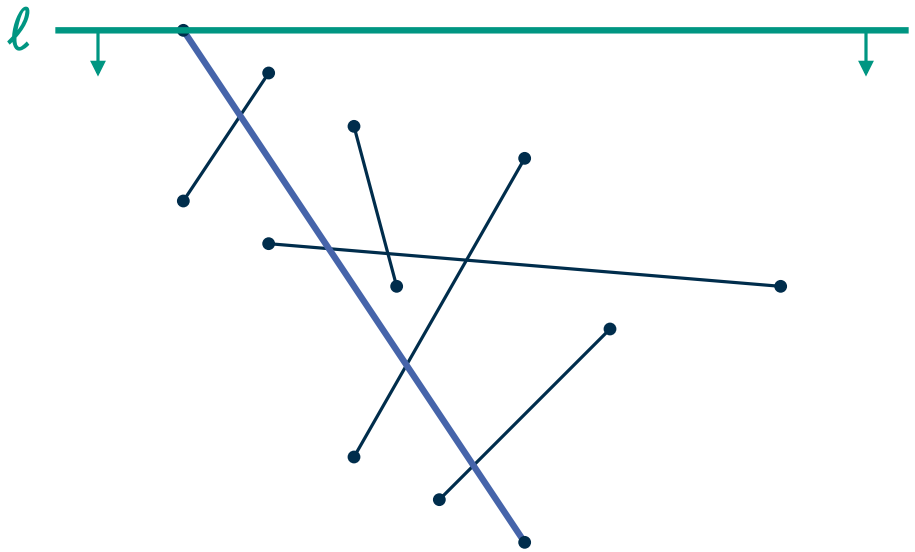
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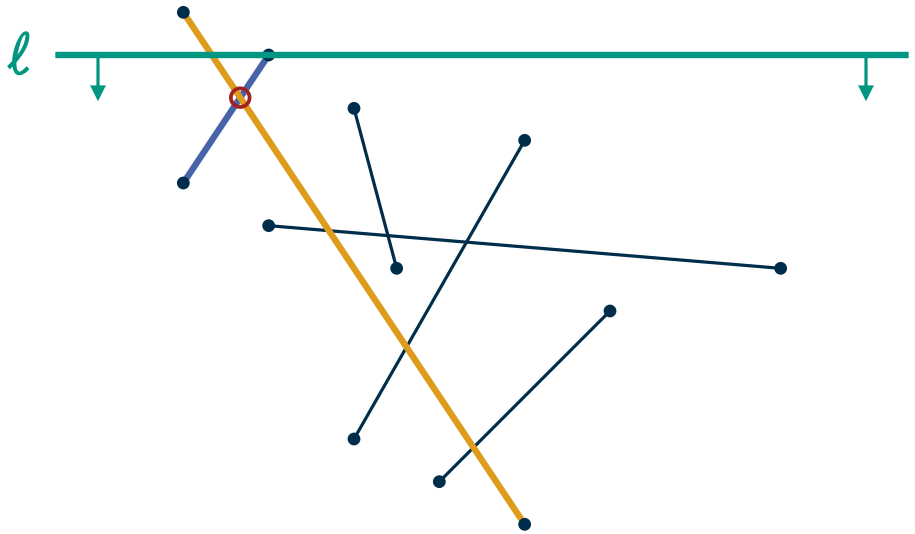
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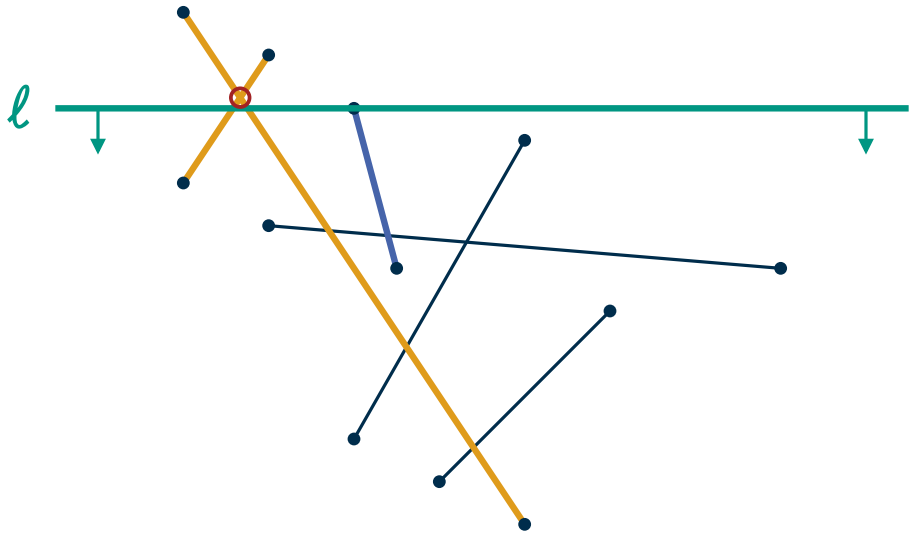
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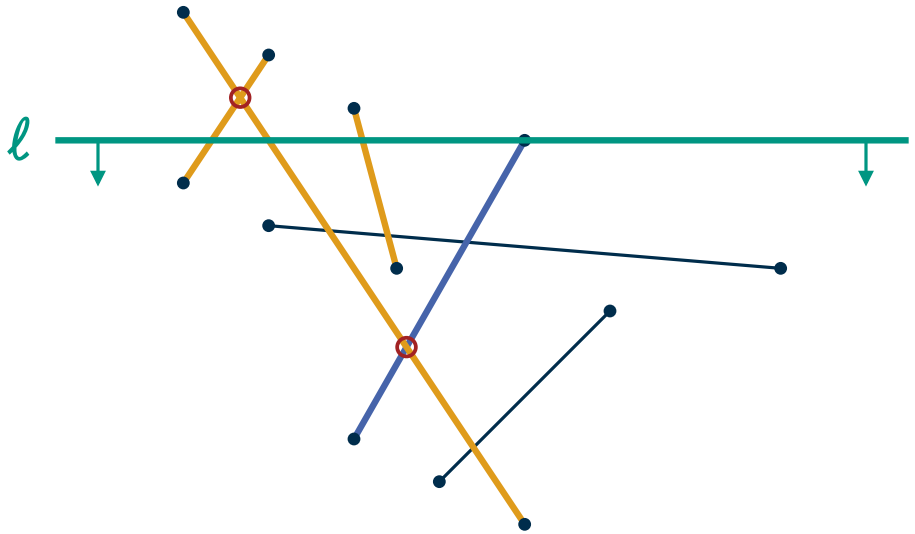
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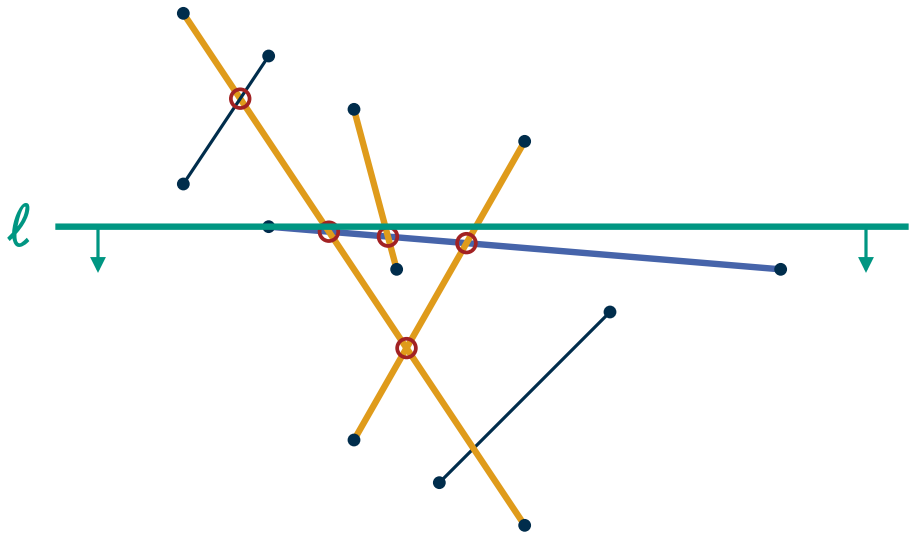
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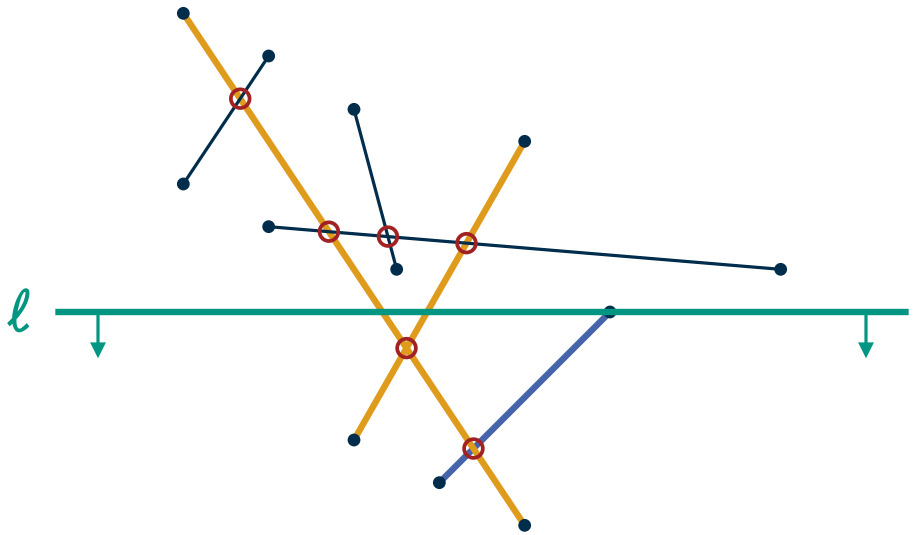
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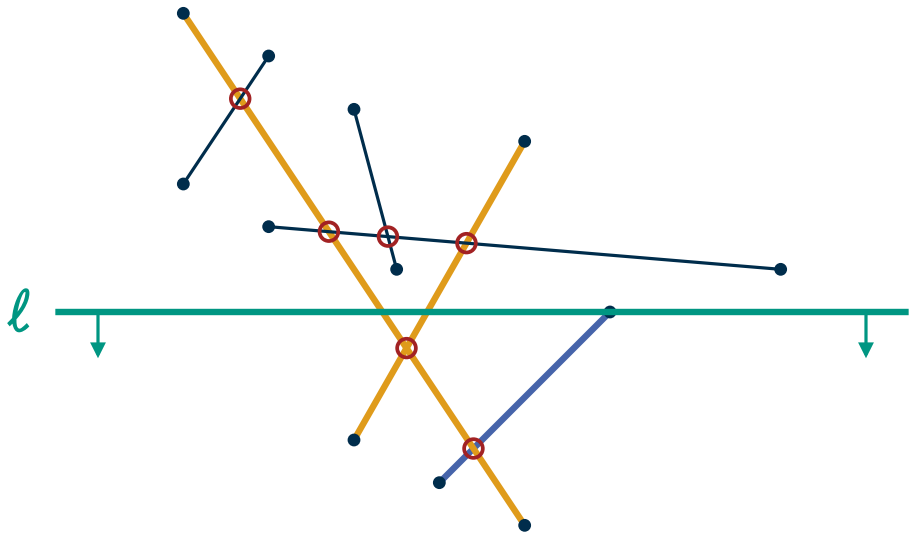
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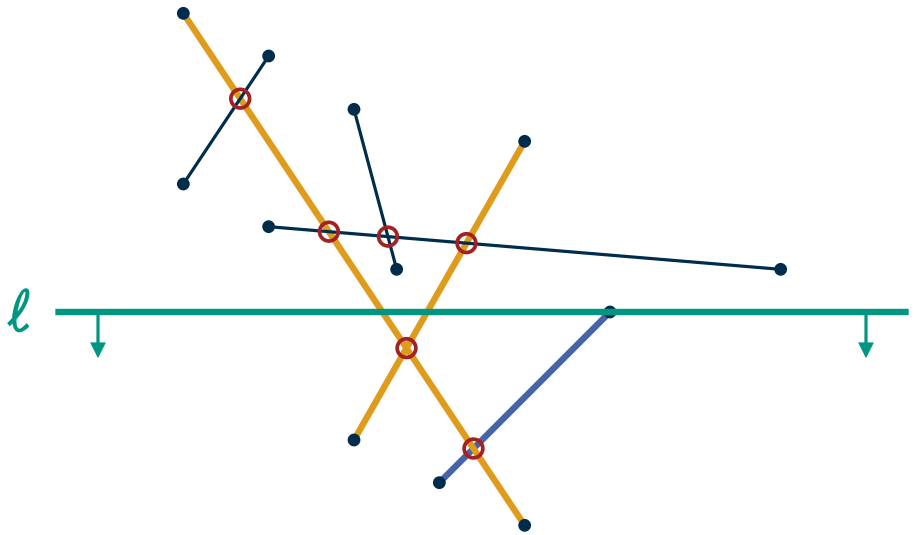
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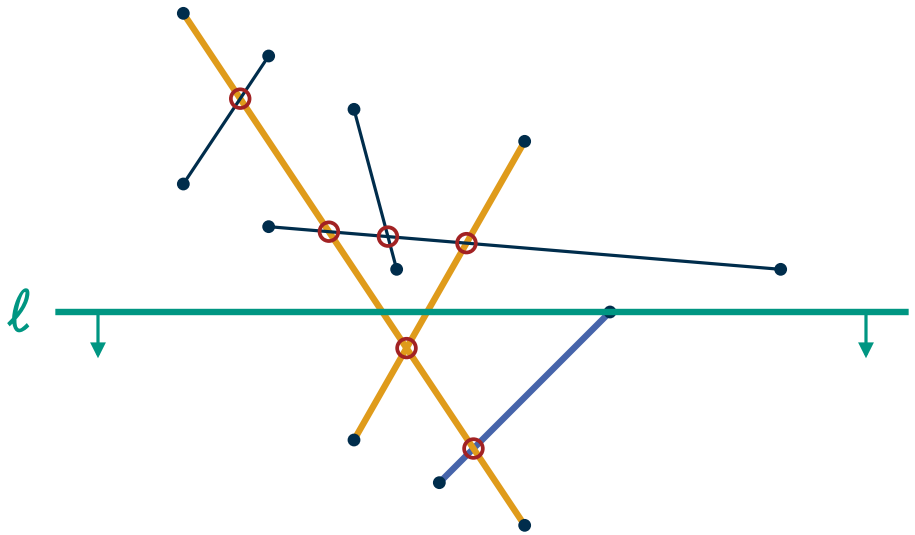
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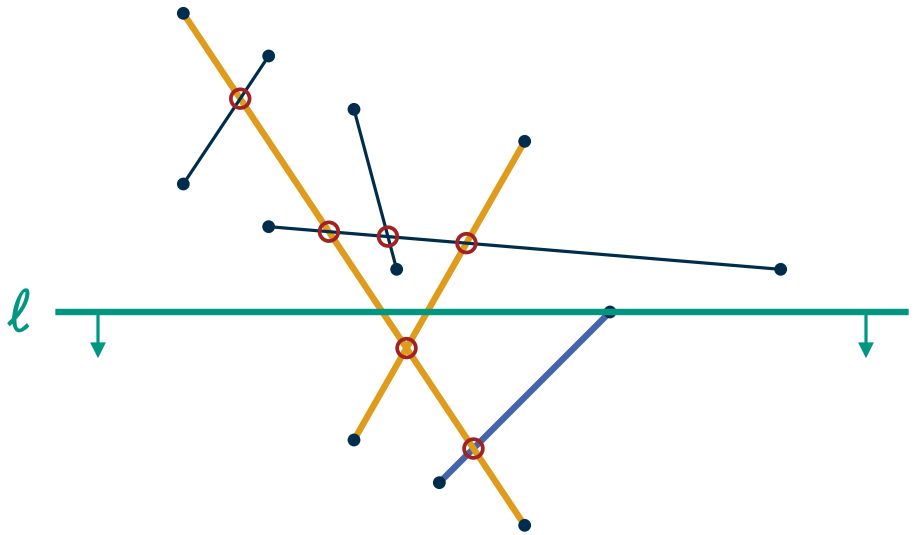
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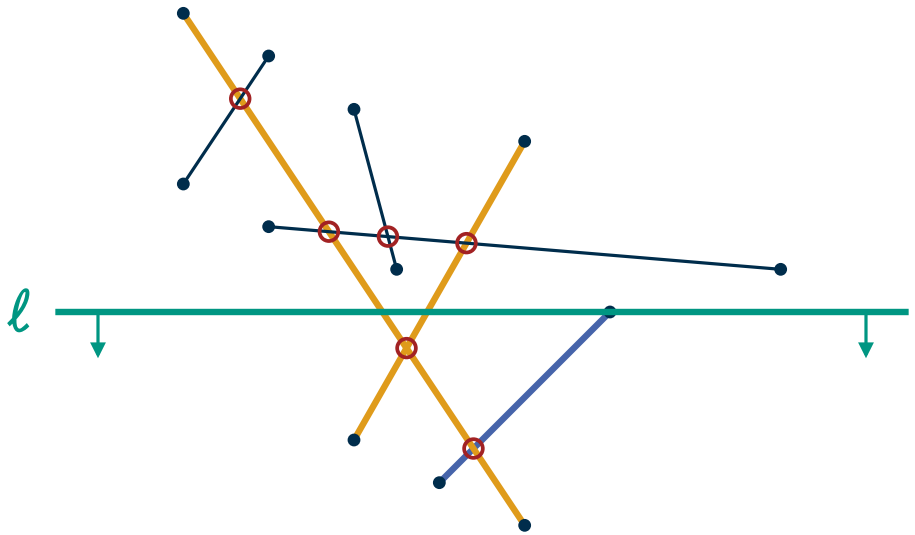
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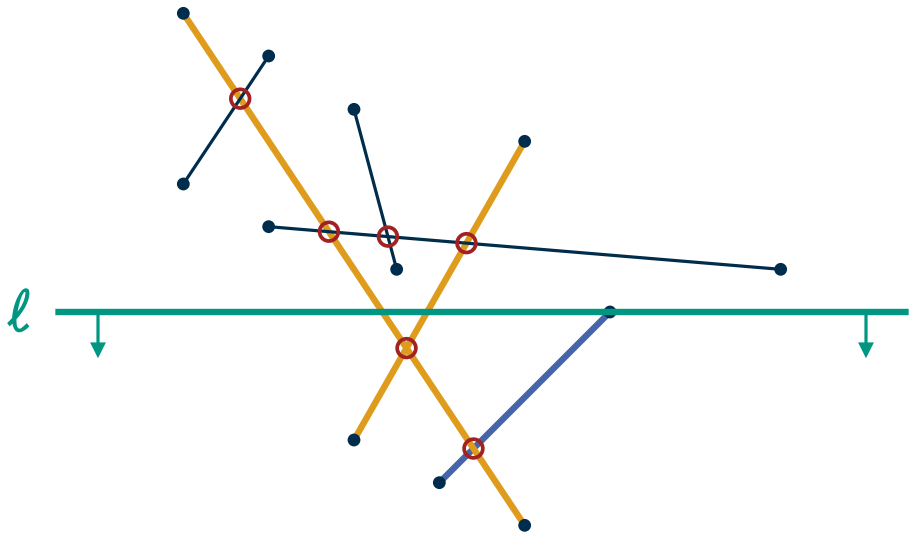
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  - endpoint of segment $s \rightarrow$ set $S_\ell = S_\ell - s$
  - start point of $s \rightarrow$ check if s intersects segments in S_ℓ and set $S_\ell = S_\ell + s$

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Solution: only compare adjacent segments (**adjacent:** next to each other on the current sweep line)

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  - segment starts: check only with adjacent segments

Only Checking Adjacent Segments

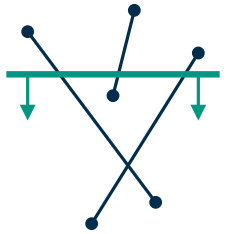
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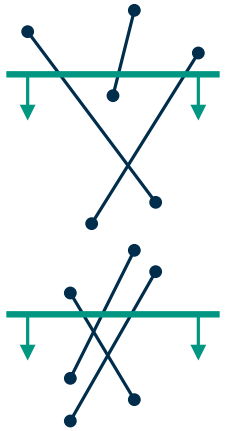
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Only Checking Adjacent Segments

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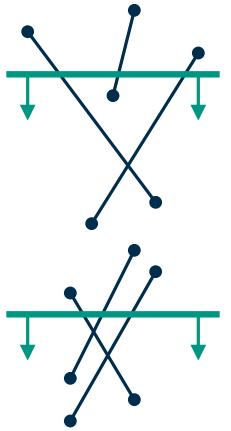
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Only Checking Adjacent Segments

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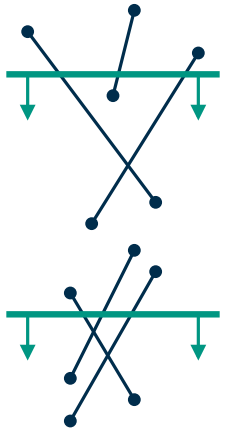
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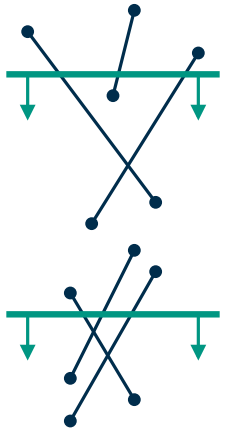


Which Data Structure?

Only Checking Adjacent Segments

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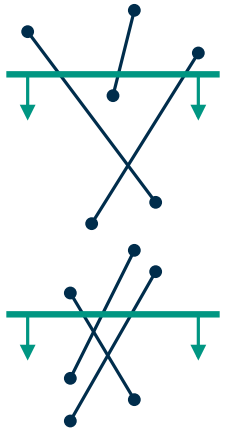
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- sweep-line status: insert, delete, find successor / predecessor
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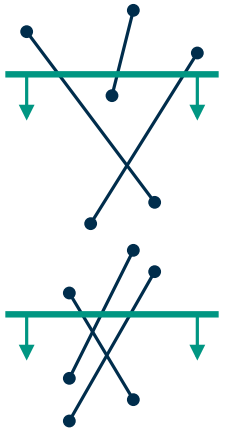
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} search tree: $O(\log n)$ (e.g., (a, b)-tree, red-black tree)

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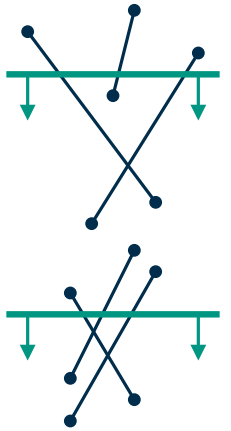
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Why not a heap for the event queue?

Only Checking Adjacent Segments

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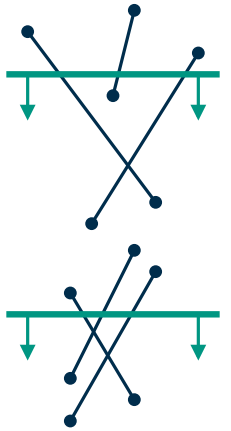
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Why do we need “search” for the event queue?

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one way to handle finding the same intersection multiple times

Improved Sweep-Line Algorithm

function FINDINTERSECTIONS(S)

Input: set of line segments S

Output: all intersections (with segment pairs)

```
Q = empty queue                 // event queue
for pq ∈ S
    Q = Q + p + q
T = search tree                 // status
while Q ≠ ∅
    p = min{Q} and Q = Q − p
    HANDLEEVENT(p)              // event handler
```

Improved Sweep-Line Algorithm

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function HANDLEEVENT(p)

```
if p is vertex of a segment
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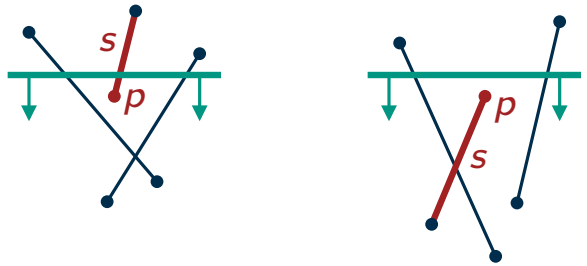
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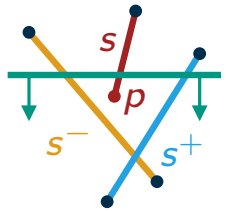
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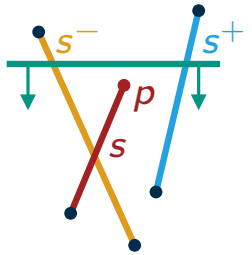
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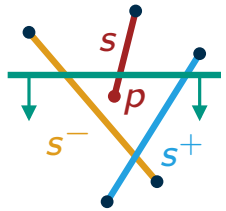
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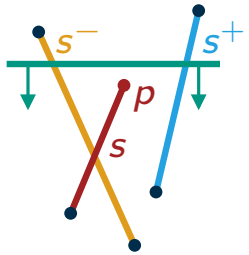
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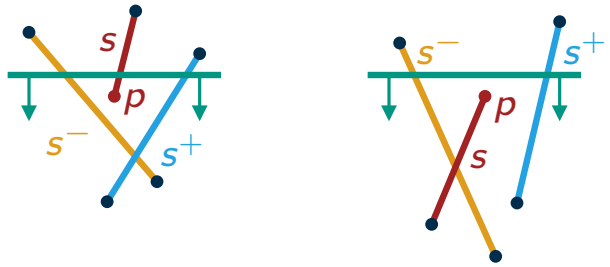
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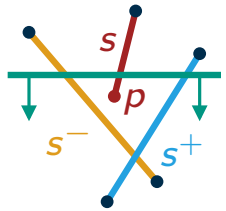
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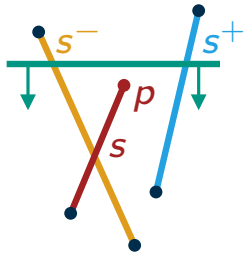
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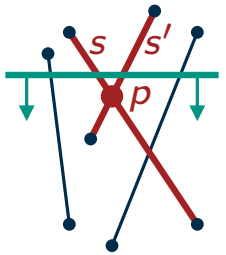
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if s ∈ T                           // segment ends
    CHECKINTERSECTION(s⁻, s⁺, p)
    T = T − s
else                               // segment starts
    T = T + s
    CHECKINTERSECTION(s⁻, s, p)
    CHECKINTERSECTION(s⁺, s, p)
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function HANDLEINTERSECTION(p)

```
s, s′ = segments intersecting in p
output (p, s, s′)                  (s < s′ in T)
```

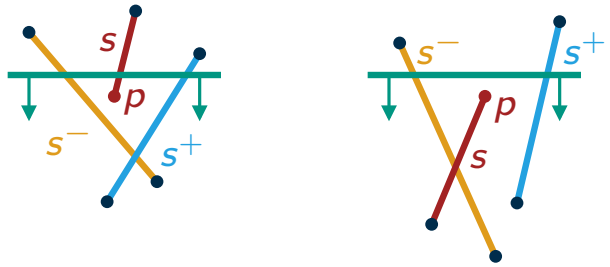
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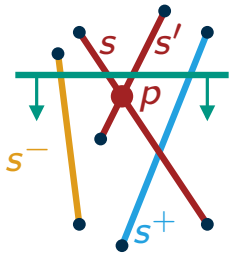
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function HANDLEINTERSECTION(p)

- $s, s' =$ segments intersecting in p
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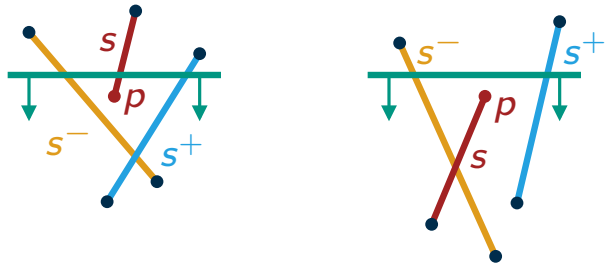
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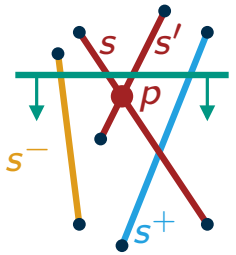
function HANDLEEVENT(p)

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function HANDLEVERTEX(p)

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function CHECKINTERSECTION(s^-, s^+, p)

- $q = s^- \cap s^+$
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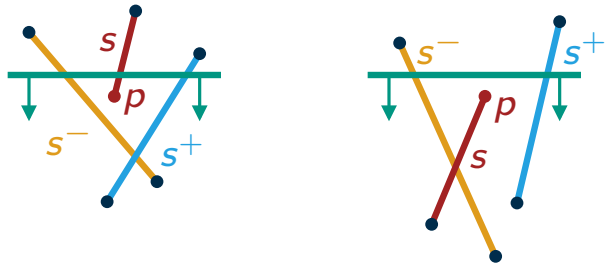
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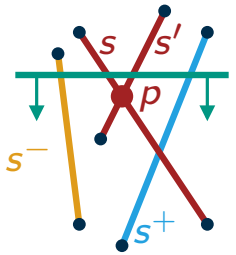
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Running time?



function HANDLEINTERSECTION(p)

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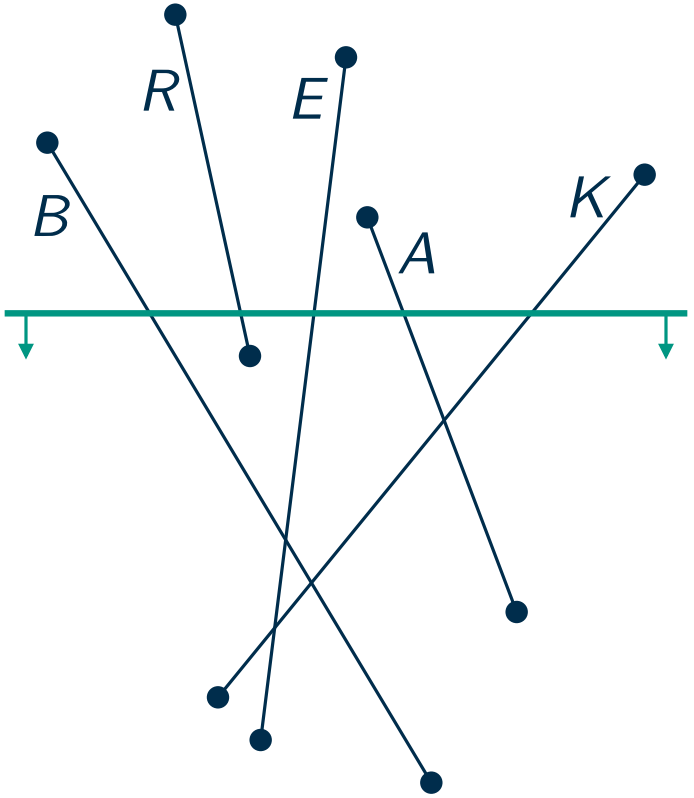
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Improved Sweep-Line Algorithm

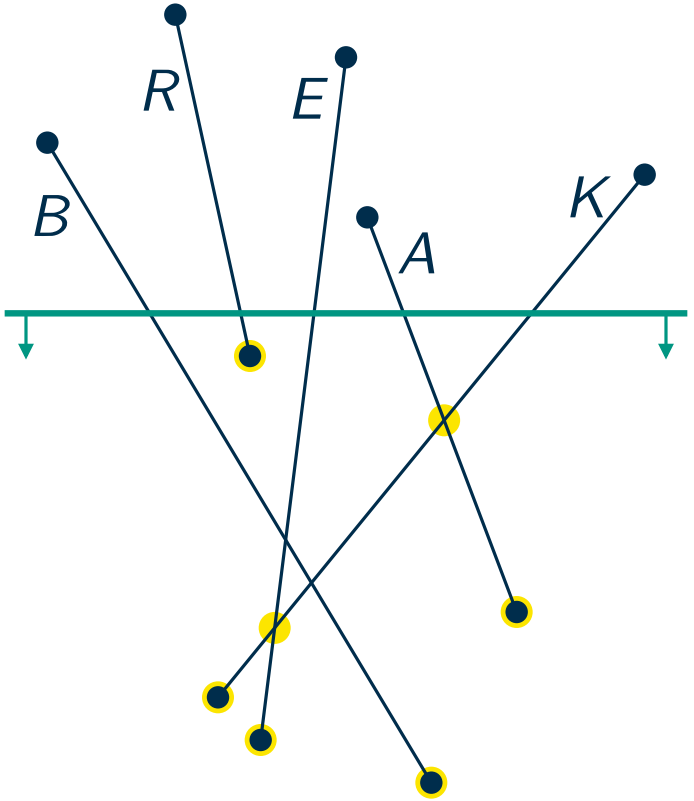
Theorem **assumption:** general position

The k intersections of n line segments can be computed in $O((n+k)\log n)$ time.

How Many Events Are In The Event Queue?



How Many Events Are In The Event Queue?



7

Do We Need General Position?

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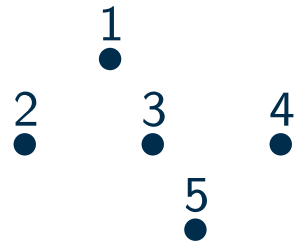
Do We Need General Position?

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Problem: Event Points With Equal y-Coordinate

- use lexicographical order with respect to (y, x)
- equivalent to slight clockwise rotation



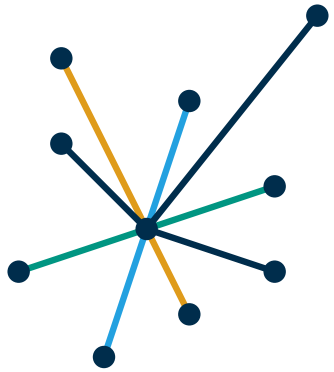
Do We Need General Position?

Theorem **assumption:** general position

The k intersections of n line segments can be computed in $O((n+k)\log n)$ time.

Problem: Multiple Events At The Same Point

- three things can happen at an event point:
  - one or multiple segments start
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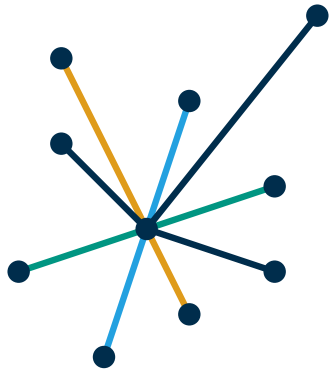
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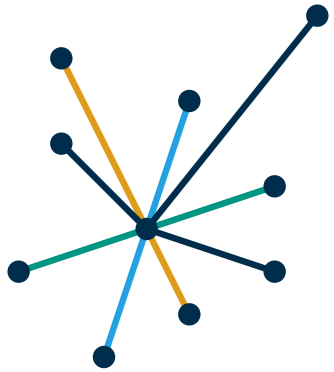
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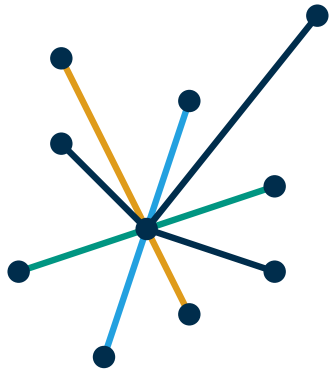
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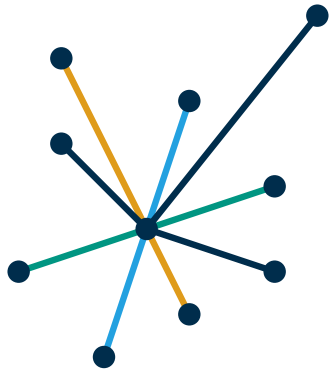
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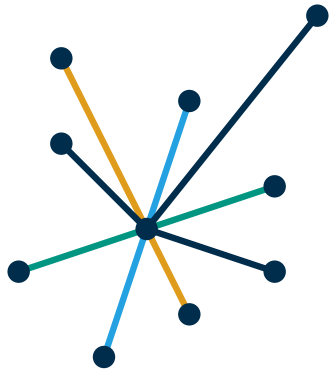
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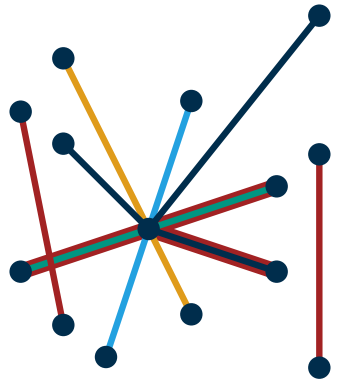
What happens to horizontal edges?

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- check newly adjacent segments afterwards

What happens to horizontal edges?

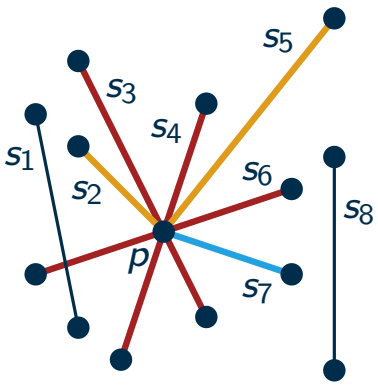
Sweep-Line Algo – No General Position Assumption

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output p (with $\text{start}(p) \cup \text{end}(p) \cup \text{int}(p)$)



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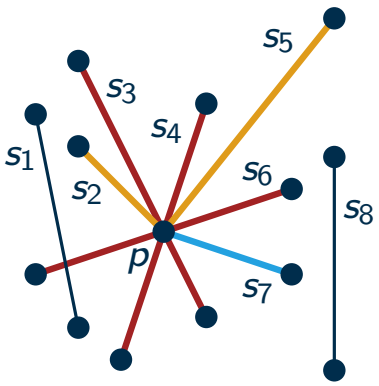
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T: s_1 s_2 s_3 s_4 s_5 s_6 s_8 ⟶ s_1 s_8

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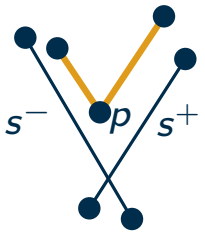
  s^- = predecessor of p in T

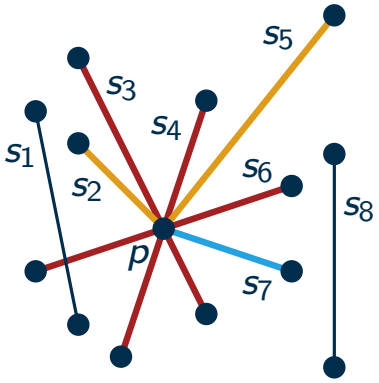
  s^+ = successor of p in T

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T: s_1 s_2 s_3 s_4 s_5 s_6 s_8 → s_1 s_8

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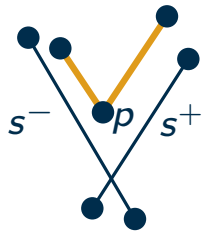
  $\hat{s}^+$ = successor of s^+ in T

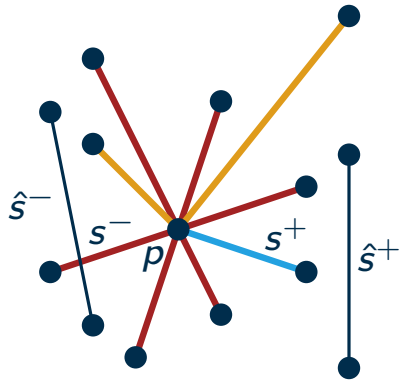
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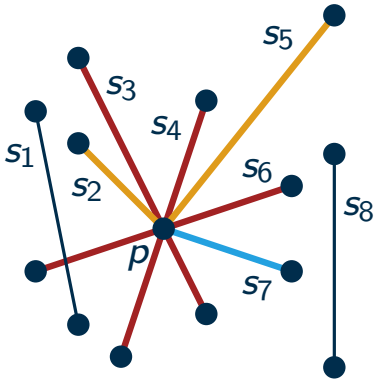
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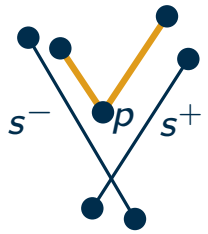
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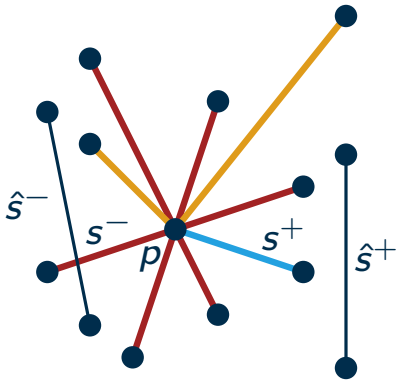
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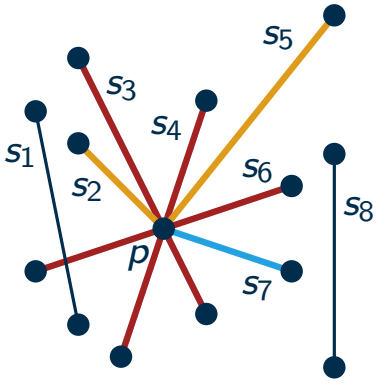
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How can this be done using a search tree?

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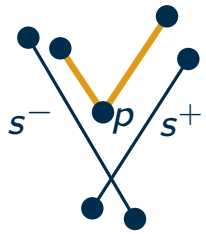
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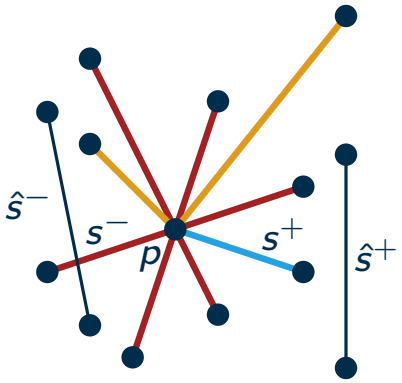
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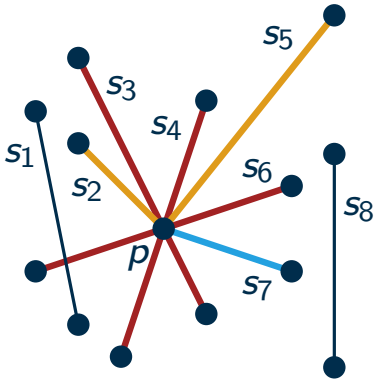
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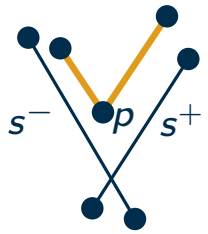
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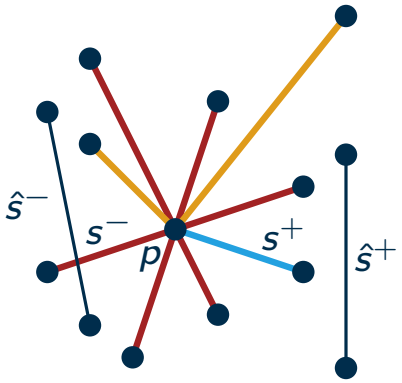
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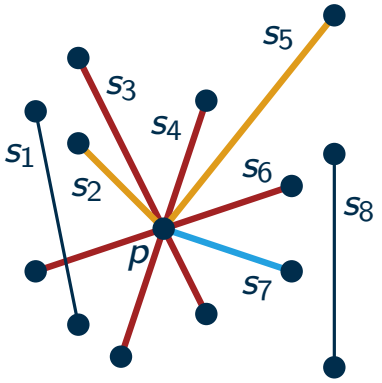
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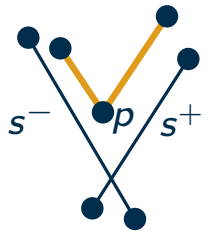
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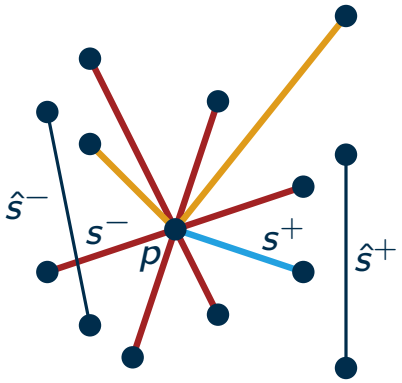
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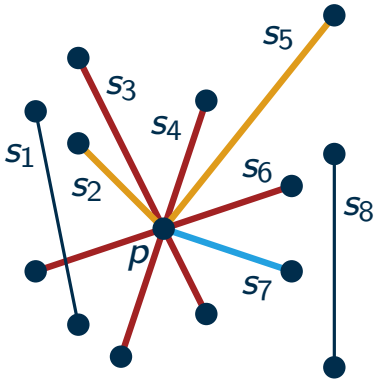
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What is left to show?

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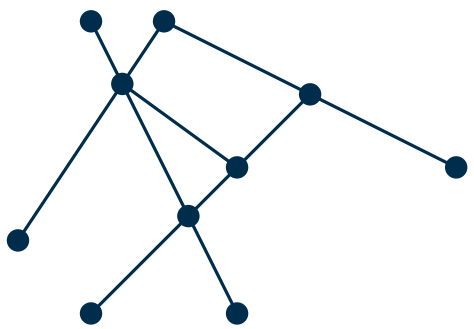
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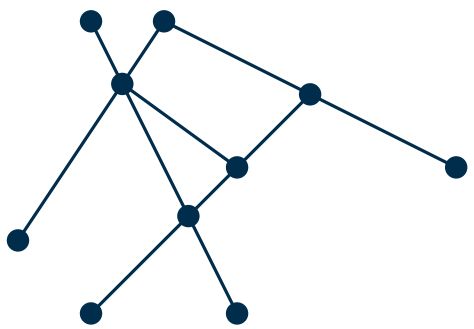
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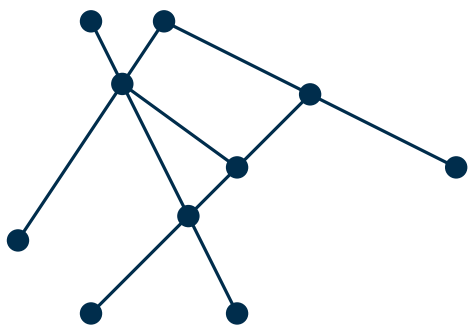
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- for planar graphs: $|E| \leq 3|V| - 6$
  $\Rightarrow m \in O(n+k)$



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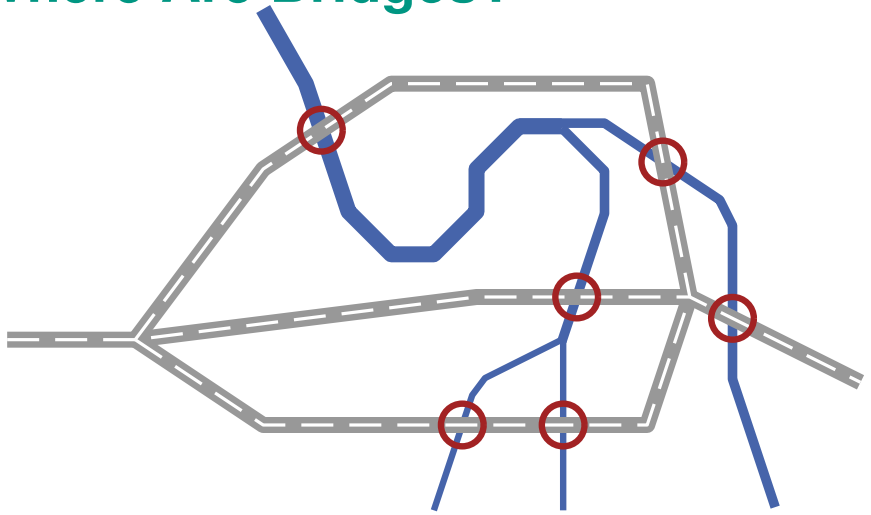
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- option: only keep intersections in the queue corresponding to adjacent segments $\rightarrow O(n)$

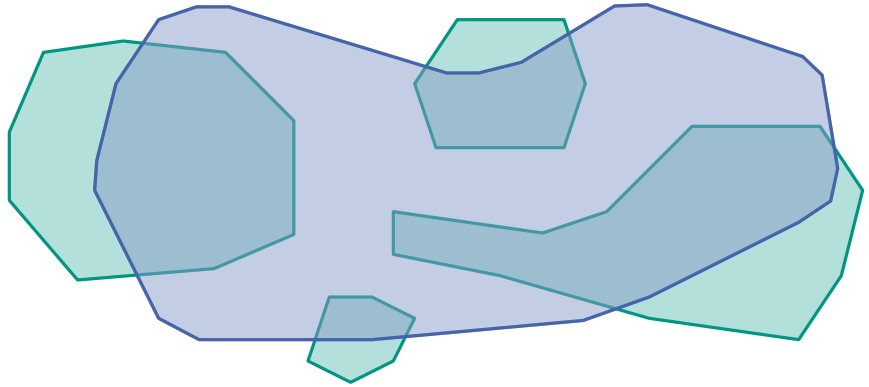
Back To Our Motivation

Did We Reach Our Goal?

Where Are Bridges?



Forests With A Lot Of Rainfall

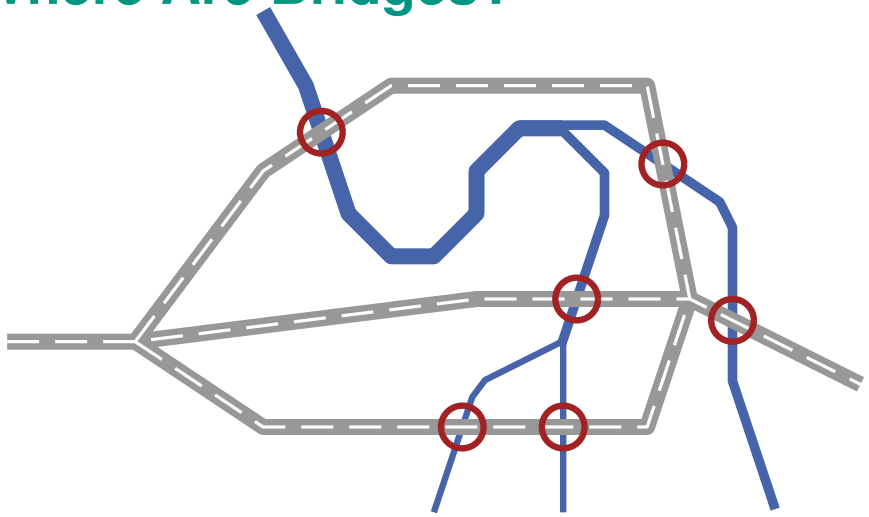


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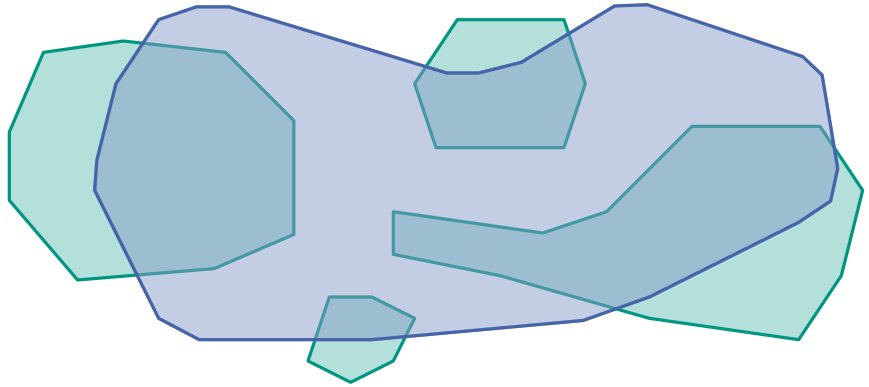
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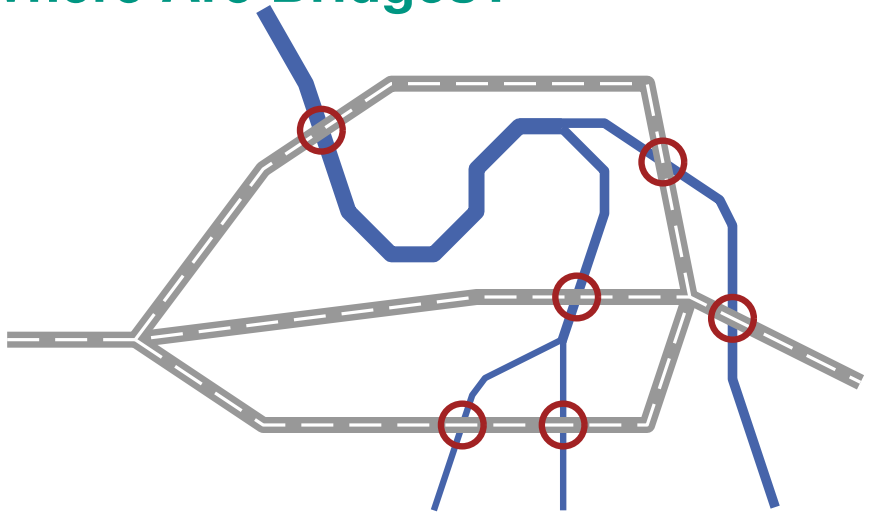


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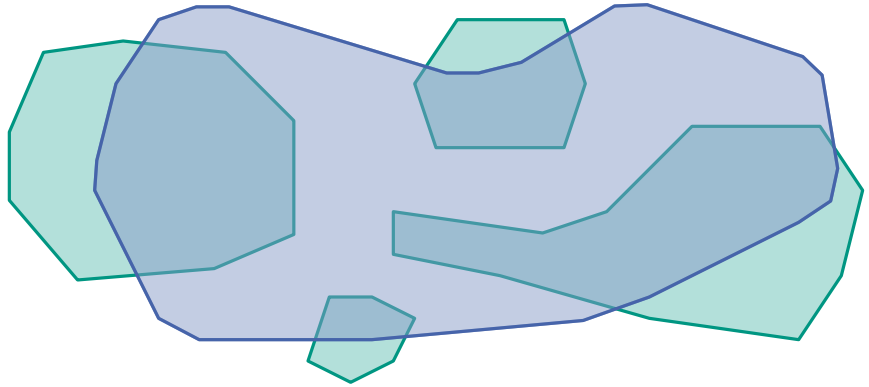
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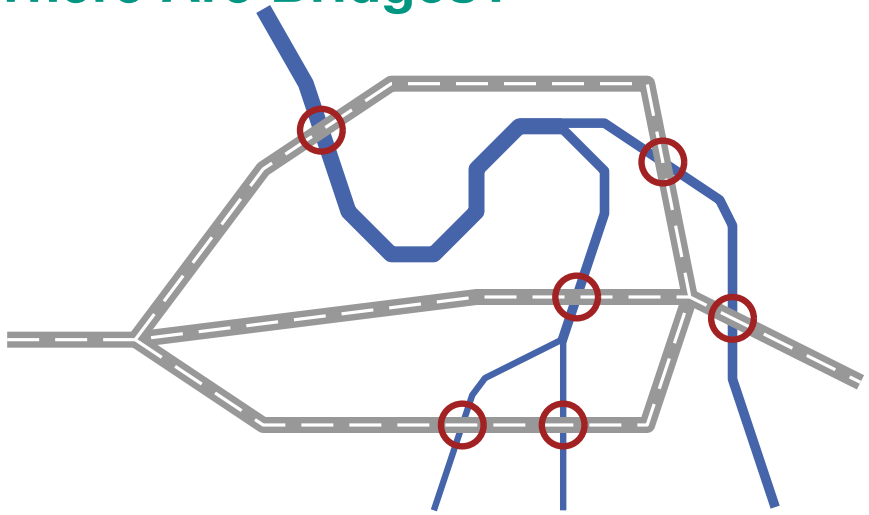
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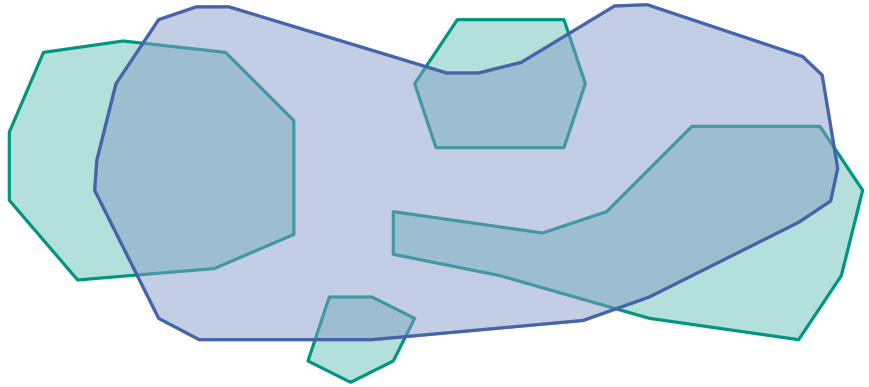
In The Following

- data structure that helps with computing the intersection
- actually using it: exercise sheet

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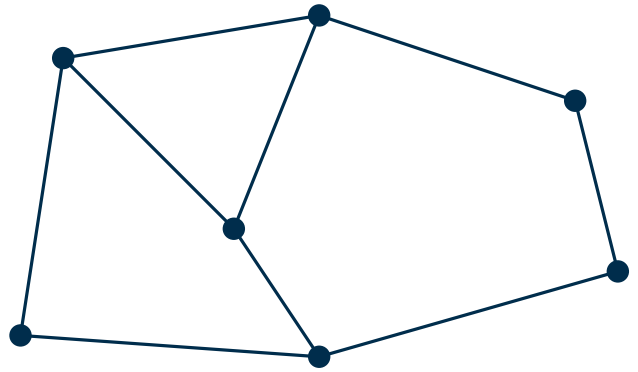


Forests With A Lot Of Rainfall



Doubly-Connected Edge List

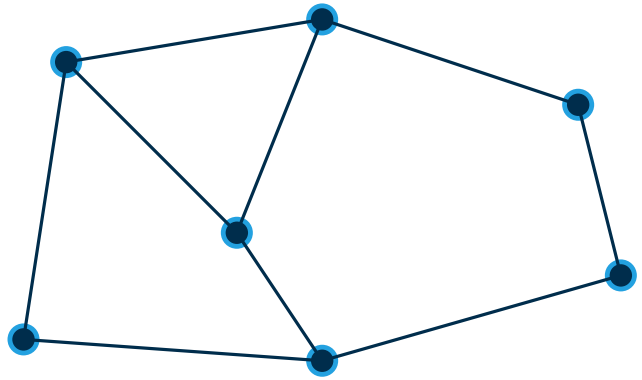
Geometric Graphs



Doubly-Connected Edge List

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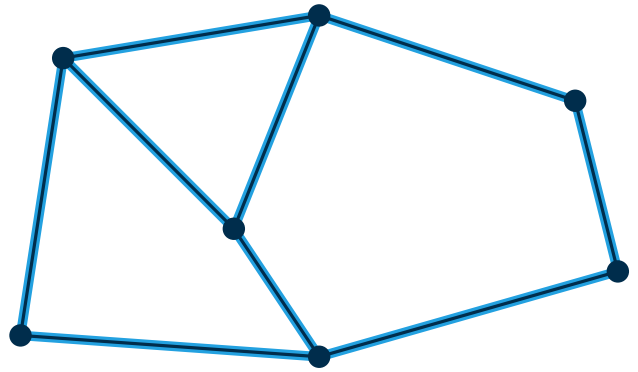
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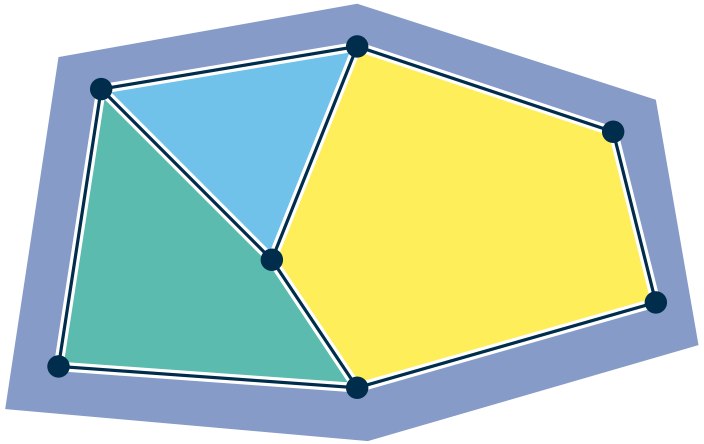
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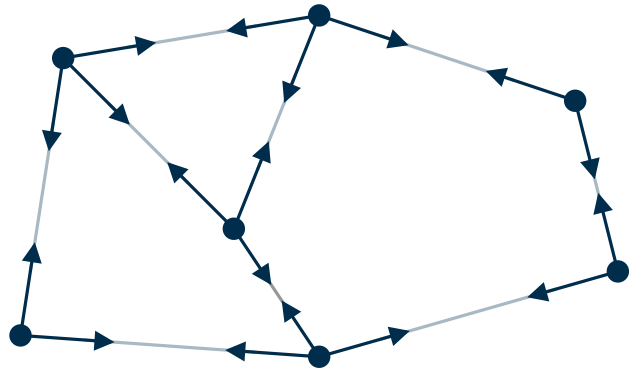
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- each edge has two incident vertices $\rightarrow$ store each edge twice



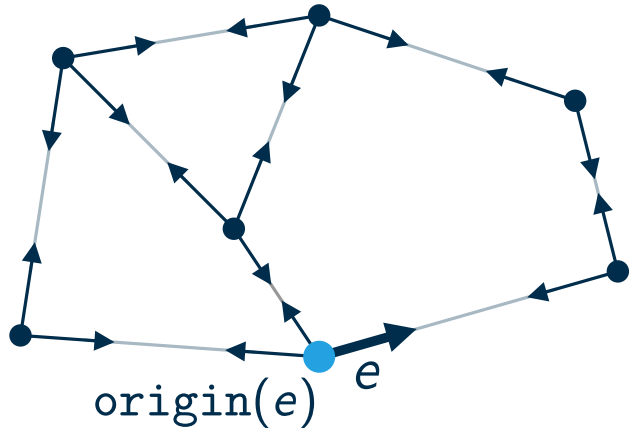
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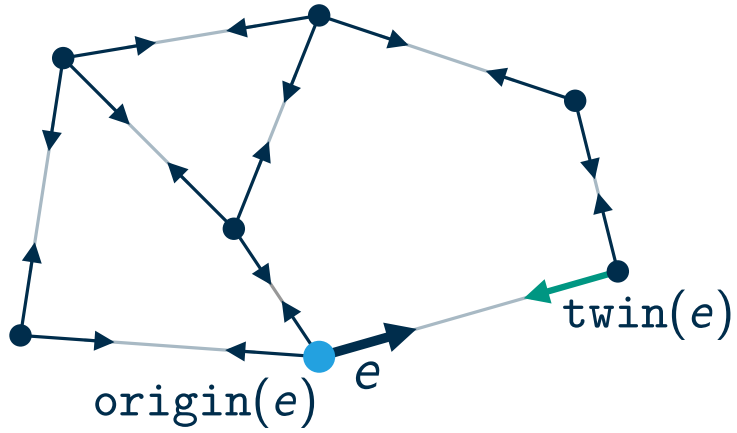
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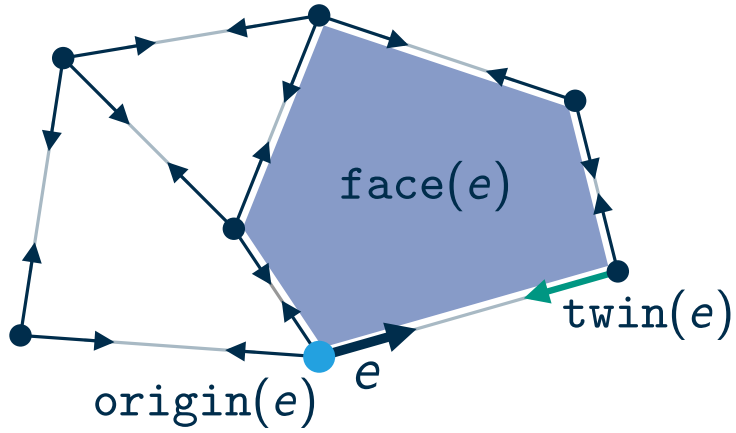
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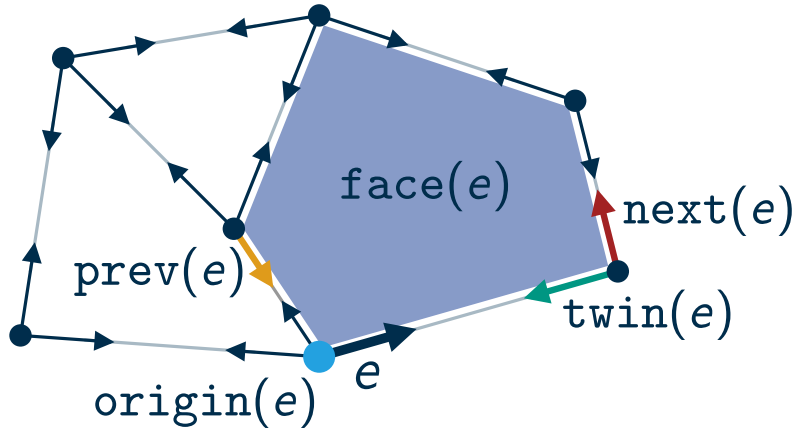
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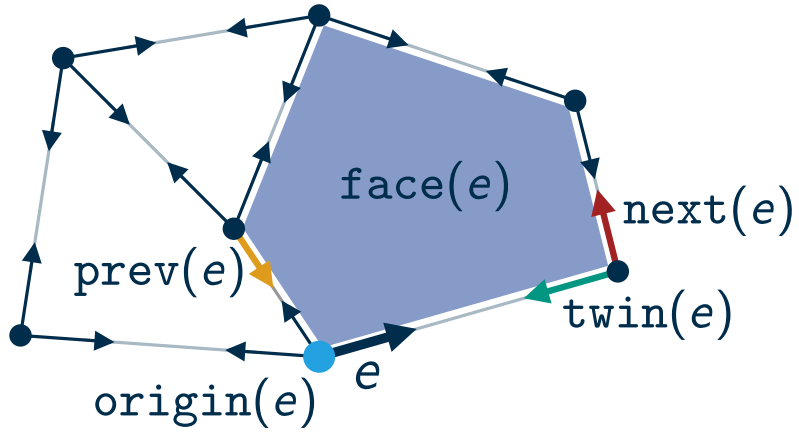
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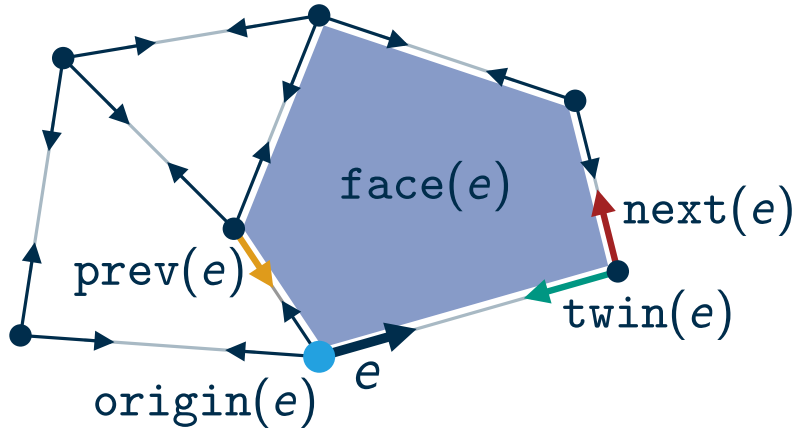
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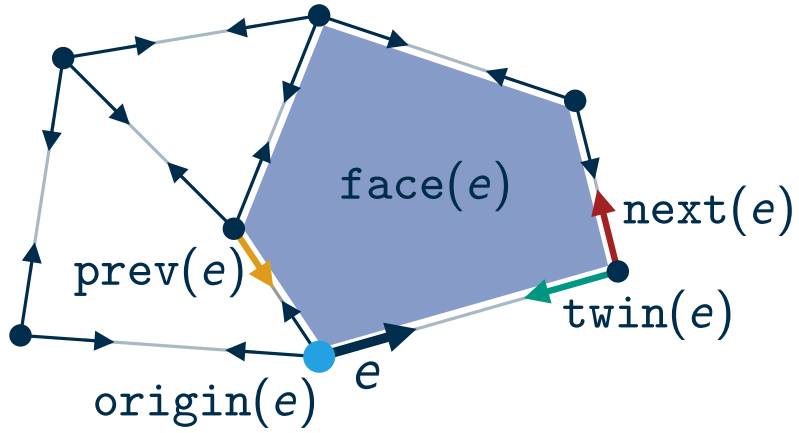
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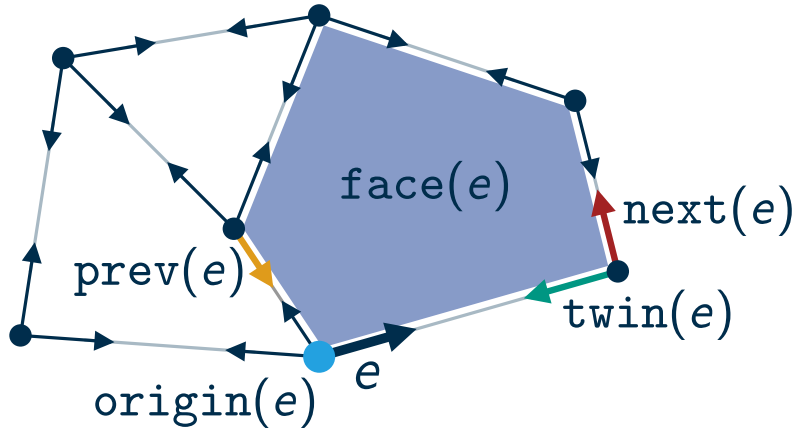
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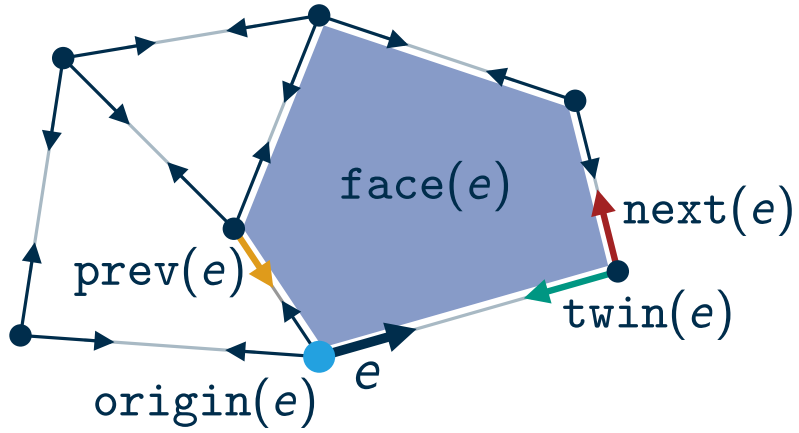
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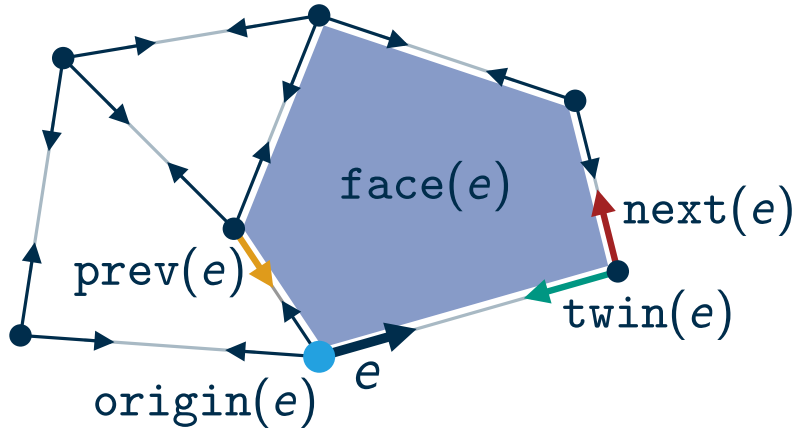
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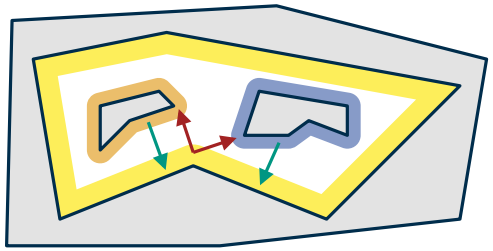


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adaptation for multiple connected components

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  - the sweep line might move differently (e.g., rotate)
  - the sweep line does not need to be a line