Parameterized Algorithms

Lower Bounds: ETH and SETH

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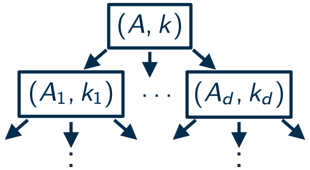
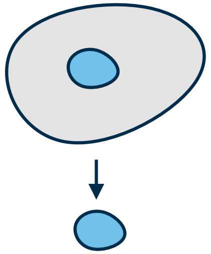
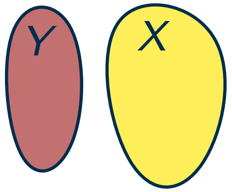
Content

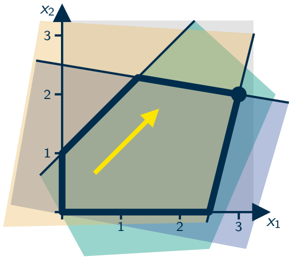
Basic toolbox
- bounded search trees
- kernelization
- iterative compression

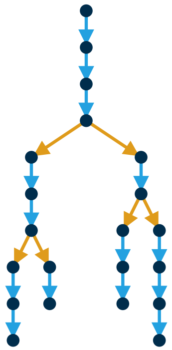


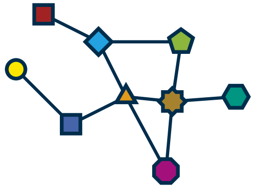




Extended toolbox
- linear programs
- branch-and-reduce
- color coding

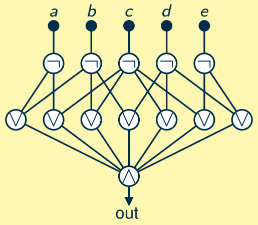


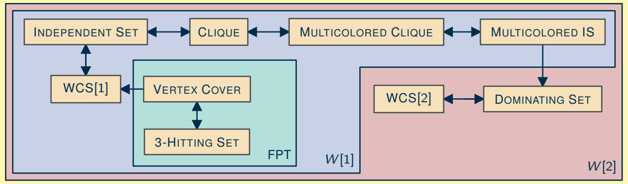




Tree width
- dynamic programming
- chordal and planar graphs
- Courcelle's theorem

Lower bonds
- kernel lower bounds
- parameterized reductions
- circuits and the W-hierarchy
- ETH and SETH





(Strong) exponential-time hypothesis

Goal

- $n^{\log \log k}$ is not FPT but still much better than n^k
- show: $f(k) \cdot n^{o(k)}$ impossible for some $W[1]$-problems ⟵ requires stronger assumption than FPT $\neq W[1]$

Definition: For $q \geq 2$, let δ_q be the infimum of all $c \in \mathbb{R}$, for which there is an $2^{cn+o(n)}$ algorithm for q-SAT with n variables.

What do we know?

- $q = 2$: $\delta_q = 0$, as 2-SAT $\in$ P and $\text{poly}(n) \in 2^{cn+o(n)}$ for every $c > 0$
- $q > 2$: $\delta_q \leq 1$, as there is an $2^{n+o(n)}$-algorithm for every q

ETH: $\delta_3 > 0$

meaning: no sub-exponential algo for 3-SAT
believe: the hypothesis is true

SETH: $\lim_{q \to \infty} \delta_q = 1$

meaning: impossible to beat brute-force for SAT
believe: unclear, but a better algorithm would be a major breakthrough

Some implications

ETH: no sub-exponential algo for 3-SAT
SETH: 2^n is the best one can do for SAT

Theorem
SETH implies ETH.

Proof idea
- reduce SAT to 3-SAT
- formula does not grow so much that something sub-exponential for 3-SAT is exponential for SAT

Theorem
ETH implies FPT $\neq W[1]$.

Proof idea
- later: ETH implies lower bound of $n^{\Omega(k)}$ for a $W[1]$-problem
- directly implies FPT $\neq W[1]$

Theorem
ETH implies, that there is no $2^{o(n+m)}$-algorithm for VERTEX COVER, DOMINATING SET, FEEDBACK VERTEX SET, 3-COLORING, or HAMILTONIAN CYCLE.

Proof idea
- linear reduction from 3-SAT to these problems

ETH and parameterization

What do we have?

- lower bound of the form $2^{\Omega(n+m)}$

What do we want?

- lower bound of the form $f(k) \cdot n^{\Omega(k)}$ for a parameterized problem

Rough plan

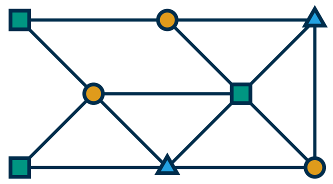
- reduce a "normal" problem to a parameterized problem
- let the reduction list possible partial solution and let the target problem "choose" k of them
- hides part of the difficulty in the input size and another part in the selection of k partial solutions

Problem we reduce from: 3-Coloring

- has no output size $\rightarrow$ no temptation to view it as parameterized problem
- local partial solutions can be combined nicely (we'll see this in a moment)

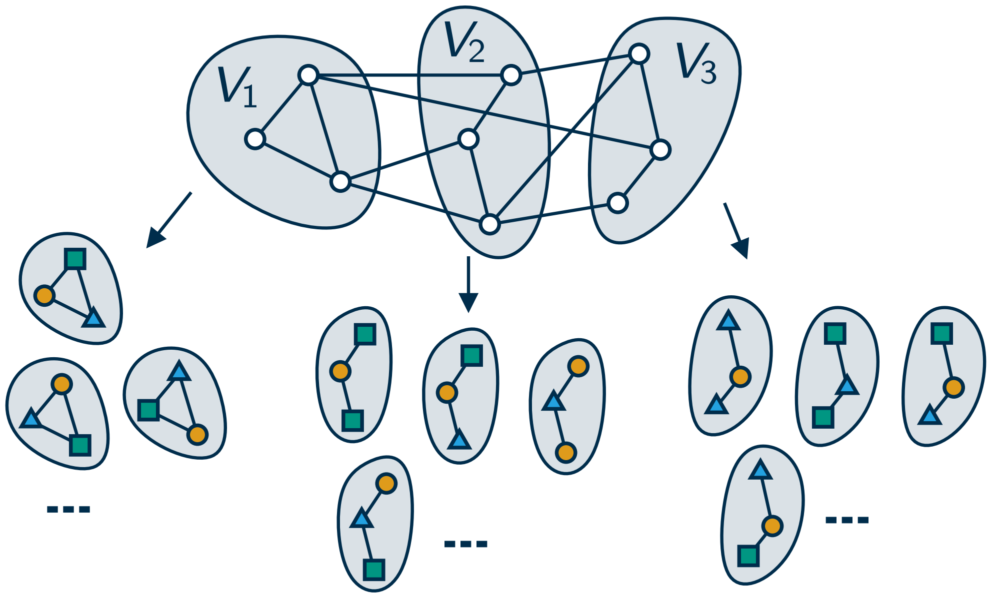
Choosing compatible partial solutions

Problem: 3-Coloring
Color vertices of G with three colors such that neighbors have different colors.



Partial solutions

- partition $V = V_1 \dot\cup \cdots \dot\cup V_k$
- C_i: set of all 3-colorings of $G[V_i]$
- choose a coloring $c_i \in C_i$ for every i such that choices are compatible

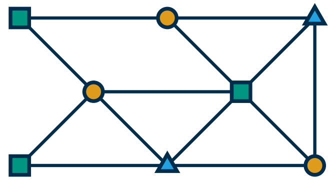


Interpretation

- part of the difficulty is hidden in the problem size (we list many colorings)
- another part lies in the choice for k fitting partial solutions

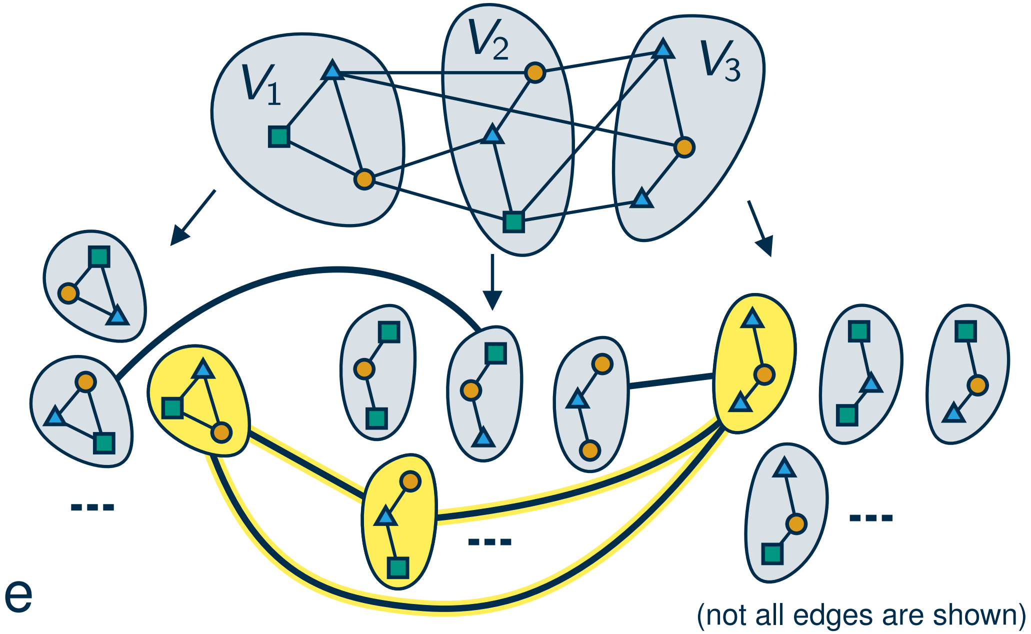
Choosing compatible partial solutions

Problem: 3-Coloring
Color vertices of G with three colors such that neighbors have different colors.



Partial solutions

- partition $V = V_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} V_k$
- C_i: set of all 3-colorings of $G[V_i]$
- choose a coloring $c_i \in C_i$ for every i such that choices are compatible



Changing perspective

- view $C_1 \cup \cdots \cup C_k$ as node set
- add edge $\{c_i, c_j\}$ if $c_i \in C_i$ and $c_j \in C_j$ are compatible

Lemma: The resulting graph with $3^{\frac{n}{k}} \cdot k$ vertices has a k-clique if and only if G is 3-colorable.

A lower bound for CLIQUE

Theorem
ETH $\Rightarrow$ no $2^{o(n+m)}$-algorithm for 3-COLORING.

Lemma: The resulting graph with $3^{\frac{n}{k}} \cdot k$ vertices has a k-clique if and only if G is 3-colorable.

Assumption: there is an $f(k) \cdot n_{\mathrm{c}}^{o(k)}$ algorithm for CLIQUE ($n_c = 3^{\frac{n}{k}}$ = #vertices in the clique CLIQUE-instance)

Resulting running time for 3-COLORING: $f(k) \cdot 3^{\frac{n}{k} \cdot o(k)}$

- goal: choose k such that this is in $2^{o(n)}$

first try: $k = n$

$$f(n) \cdot 3^{\frac{n}{n} \cdot o(n)}$$

may be too large ok

second try: $k \in O(1)$

careful:

$o(k) \neq o(1)$ for $k \in O(1)$

($o(k)$ grows sublinear with growing k; if k does not grow, then $o(k)$ is constant and not falling in n)

third try: $k = g(n) = f^{-1}(n)$

(we can assume that $g(n)$ is strictly growing)

$$n \cdot 3^{\frac{n}{g(n)} \cdot o(g(n))} = 2^{o(n)}$$

Theorem: ETH implies that there is no $f(k) \cdot n^{o(k)}$-algorithm for CLIQUE.

Implications

Theorem: ETH implies that there is no $f(k) \cdot n^{o(k)}$-algorithm for CLIQUE.

More lower bounds

- parameterized reduction from CLIQUE yield lower bounds for other problems
- simple example: ETH $\Rightarrow$ INDEPENDENT SET has no $f(k)n^{o(k)}$-algorithm
- same for MULTICOLORED CLIQUE, MULTICOLORED INDEPENDENT SET, DOMINATING SET, CONNECTED DOMINATING SET, ...
- **note:** you have to keep track of how the parameter changes in the reduction

Theorem: ETH impliziert FPT $\neq W[1]$.

Next

- additional somewhat different implications
- useful intermediate problem: GRID TILING

Grid Tiling

Problem: Grid Tiling $([n] = [1, n])$

Given a $k \times k$ matrix of sets $S_{i,j}$, with $S_{i,j} \subseteq [n] \times [n]$. Choose for each cell one $s_{i,j} \in S_{i,j}$, such that $s_{i,j}$ and $s_{i+1,j}$ coincide in the first, and $s_{i,j}$ and $s_{i,j+1}$ in the second coordinate.

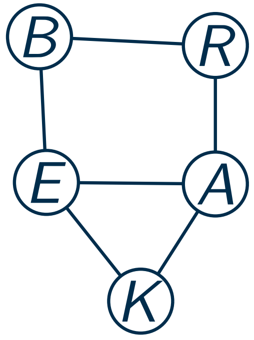
Example

- $k = 3$ and $n = 5$

$S_{1,1}$: $(1,1)$, $(3,1)$, $\boxed{(2,4)}$	$S_{1,2}$: $(5,1)$, $\boxed{(1,4)}$, $(5,3)$	$S_{1,3}$: $(1,1)$, $\boxed{(2,4)}$, $(3,3)$
$S_{2,1}$: $\boxed{(2,2)}$, $(1,4)$	$S_{2,2}$: $(3,1)$, $\boxed{(1,2)}$	$S_{2,3}$: $\boxed{(2,2)}$, $(2,3)$
$S_{3,1}$: $(1,3)$, $\boxed{(2,3)}$, $(3,3)$	$S_{3,2}$: $(1,1)$, $\boxed{(1,3)}$	$S_{3,3}$: $\boxed{(2,3)}$, $(5,3)$

What is going on?

Is this instance of GRID TILING solvable? What does it mean for the graph?



$S_{1,1}$ (1, 1) (3, 1) (2, 4)	$S_{1,2}$ (5, 1) (1, 4) (5, 3)	$S_{1,3}$ (1, 1) (2, 4) (3, 3)
$S_{2,1}$ (2, 2) (1, 4)	$S_{2,2}$ (3, 1) (1, 2)	$S_{2,3}$ (2, 2) (2, 3)
$S_{3,1}$ (1, 3) (2, 3) (3, 3)	$S_{3,2}$ (1, 1) (1, 3)	$S_{3,3}$ (2, 3) (5, 3)

(B,B) (R,R) (E,E) (A,A) (K,K)	(R,B) (E,B) (A,R) (A,E) (K,E) (K,A)	(R,B) (E,B) (A,R) (A,E) (K,E) (K,A)
(B,R) (B,E) (R,A) (E,A) (E,K) (A,K)	(B,B) (R,R) (E,E) (A,A) (K,K)	(R,B) (E,B) (A,R) (A,E) (K,E) (K,A)
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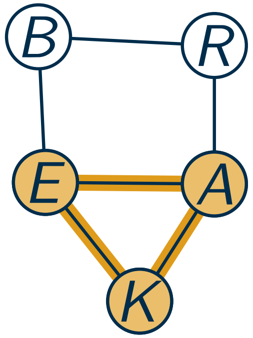
Problem: GRID TILING
Given a $k \times k$ matrix of sets $S_{i,j}$, with $S_{i,j} \subseteq [n] \times [n]$. Choose for each cell one $s_{i,j} \in S_{i,j}$, such that $s_{i,j}$ and $s_{i+1,j}$ coincide in the first, and $s_{i,j}$ and $s_{i,j+1}$ in the second coordinate.

What is going on?

Is this instance of GRID TILING solvable?
What does it mean for the graph?

- a 3-clique in G yields a solution for GRID TILING
- inverse is also true



(B,B) (R,R) (E,E) (A,A) (K,K)	(R,B) (E,B) (A,R) (A,E) (K,E) (K,A)	(R,B) (E,B) (A,R) (A,E) (K,E) (K,A)
(B,R) (B,E) (R,A) (E,A) (E,K) (A,K)	(B,B) (R,R) (E,E) (A,A) (K,K)	(R,B) (E,B) (A,R) (A,E) (K,E) (K,A)
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Note

- $n = \#$vertices, $k =$ clique size
- this is a parameterized reduction from CLIQUE to GRID TILING
- parameter k translates linearly

Theorem
ETH implies that there is no $f(k) \cdot n^{o(k)}$-algorithm for GRID TILING (and it is $W[1]$-hard).

Comments on GRID TILING

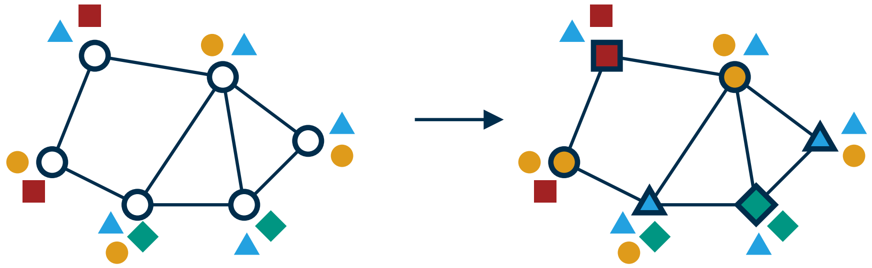
Reducing from GRID TILING

- the rough structure of GRID TILING is a grid
- all conditions are local: only between neighboring cells
- simple conditions: equality of a coordinate

well suited to prove hardness on planar graphs

Problem: PLANAR LIST COLORING
Given a planar graph and a list of colors for each vertex. For every vertex, choose a color from its list such that neighbors have different colors.

Example



Goal: reduce GRID TILING to PLANAR LIST COLORING

Grid Tiling $\rightarrow$ List Coloring

$(1,1)$ $(3,1)$ $(2,4)$	$(5,1)$ $(1,4)$ $(5,3)$	$(1,1)$ $(2,4)$ $(3,3)$
$(2,2)$ $(1,4)$	$(3,1)$ $(1,2)$	$(2,2)$ $(2,3)$
$(1,3)$ $(2,3)$ $(3,3)$	$(1,1)$ $(1,3)$	$(2,3)$ $(5,3)$

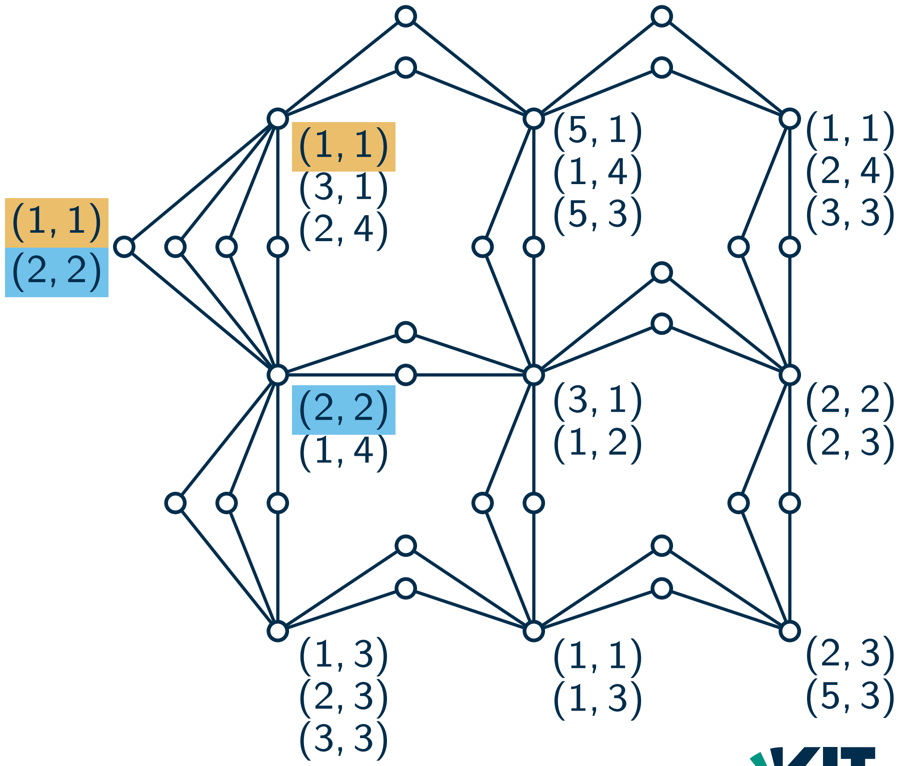
Choice of vertices and colors

- obvious attempt: one vertex per cell, one color per pair

Forbid certain color choices

- forbid choice: $(1,1)$ for $S_{1,1}$ and $(2,2)$ for $S_{2,1}$
- connect both with new vertex with colors $(1,1)$ and $(2,2)$
- same for all undesired pairs

Note: graph is planar and instances are equivalent

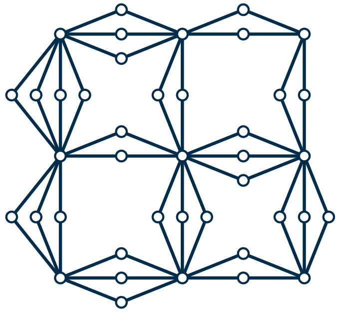


What did we show now?

Theorem
ETH implies that there is no $f(k) \cdot n^{o(k)}$-algorithm for GRID TILING (and it is $W[1]$-hard).

Theorem
ETH implies that there is no $f(k) \cdot n^{o(k)}$-algorithm for PLANAR LIST COLORING (and it is $W[1]$-hard).

Reminder
- k is the grid size



For which parameter k?
- observation: the resulting graph has treewidth k
- thus: theorem holds when parameterized by treewidth

Why?

Can we not solve LIST COLORING via DP on a tree decomposition?

we can, but it is too slow

Relaxed GRID TILING

Problem: GRID TILING WITH $\leq$ ($[n] = [1, n]$)
Given a $k \times k$ matrix of sets $S_{i,j}$, with $S_{i,j} \subseteq [n] \times [n]$. For each cell, choose one $s_{i,j} \in S_{i,j}$, such that the first and second coordinate is non-decreasing in every column and row, respectively.

Example: $k = 3$, $n = 5$

$S_{1,1}$: $(1,1)$, $(3,1)$, $(2,4)$	$S_{1,2}$: $(5,1)$, $(1,4)$, $(5,3)$	$S_{1,3}$: $(1,1)$, $(2,4)$, $(3,3)$
$S_{2,1}$: $(2,2)$, $(1,4)$	$S_{2,2}$: $(3,1)$, $(1,2)$	$S_{2,3}$: $(2,2)$, $(2,3)$
$S_{3,1}$: $(1,3)$, $(2,3)$, $(3,3)$	$S_{3,2}$: $(1,1)$, $(1,3)$	$S_{3,3}$: $(2,3)$, $(5,3)$

Note

- GRID TILING and GRID TILING WITH $\leq$ have the same instances
- a solution for GRID TILING also solves GRID TILING WITH $\leq$ (the inverse is not true)
- that does not mean that GRID TILING WITH $\leq$ is an easier problem

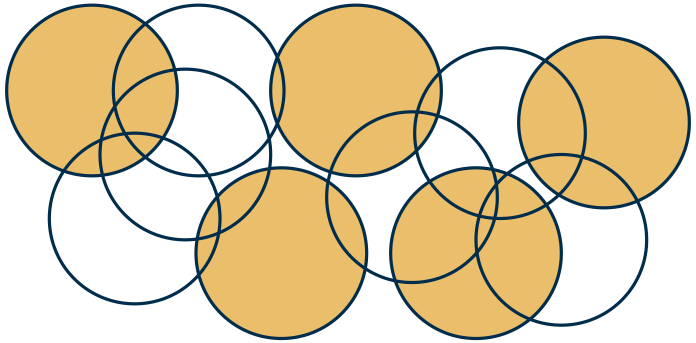
Theorem (without proof)
ETH implies that there is no $f(k) \cdot n^{o(k)}$-algorithm for GRID TILING WITH $\leq$ (and it is $W[1]$-hard).

Independent Unit Disks

Problem: Unit Disk Independent Set
Given a set of disks with radius 1 and a parameter ℓ, find a subset of ℓ disjoint disks.

Example

- are there $\ell = 6$ disjoint disks?
- no, 5 is the maximum



Note

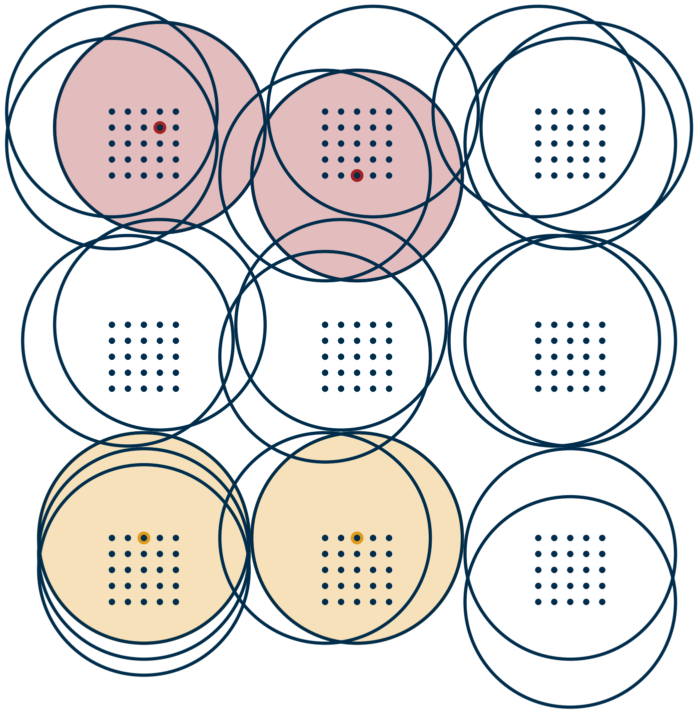
- equivalent to Independent Set on unit disk graphs
- potentially easier than Independent Set
- ETH lower bound of Independent Set: $n^{\Omega(\ell)}$
- Unit Disk Independent Set can be solved in $n^{O(\sqrt{\ell})}$

Next slide

- this is tight: lower bound of $n^{\Omega(\sqrt{\ell})}$ Unit Disk Independent Set
- reduction from Grid Tiling with $\leq$

Lower bound for UNIT DISK IS

$(1,1)$ $(3,1)$ $(2,4)$	$(5,1)$ $(1,4)$ $(5,3)$	$(1,1)$ $(2,4)$ $(3,3)$
$(2,2)$ $(1,4)$	$(3,1)$ $(1,2)$	$(2,2)$ $(2,3)$
$(1,3)$ $(2,3)$ $(3,3)$	$(1,1)$ $(1,3)$	$(2,3)$ $(5,3)$



Core idea of the reduction

- one disk for every pair of numbers
- disks of pairs from the same cell always intersect
- disks from pairs of neighboring cells intersect, if the relevant coordinate gets smaller
- grid tiling yes instance $\Leftrightarrow$ k^2 independent disks

How does the parameter change?

- grid tiling: $k \times k$ grid $\rightarrow$ parameter k
- unit disks: parameter $\ell = k^2$

$n^{o(\sqrt{\ell})}$ would imply $n^{o(k)}$

Theorem
ETH implies that there is no $f(\ell) \cdot n^{o(\sqrt{\ell})}$-algorithm for UNIT DISK INDEPENDENT SET (and it is $W[1]$-hard).

Wrap-Up

Lower bounds

- stronger assumptions (FPT $\neq W[1] \to$ ETH) yield stronger lower bounds $(f(k)n^{\omega(1)} \to f(k)n^{\Omega(k)})$
- translate bound $2^{\Omega(n)}$ (3-Coloring) to parameterized bound $f(k)n^{\Omega(k)}$ (Clique)
- yields finer subdivision of $W[1]$

Grid Tiling

- nice problem to start reductions for geometric problems

Finer subdivision of FPT

- reminder: problem in FPT $\Leftrightarrow$ there is a kernelization algo
- reminder: lower bonds for kernel sizes

Finer subdivision of P

- example: SETH $\Rightarrow$ no subquadratic algorithm for Orthogonal Vectors
- see: Fine-Grained Complexity Theory