Parameterized Algorithms

Treewidth: Courcelle's Theorem and Chordal Graphs

Thomas Bläsius

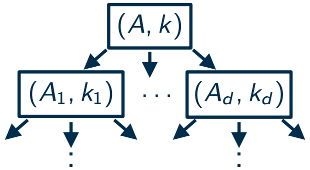
Exam dates

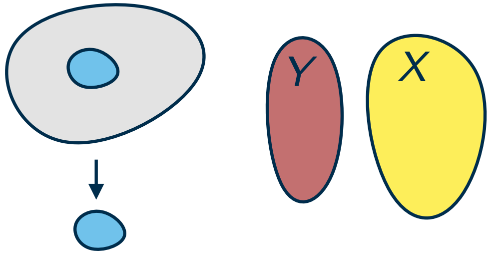
August	3				7
August	10				14
August	17				21
August	24				28
August/September	31				4
September	7				11
September	14				18
September	21				25
September	28				2

Which weeks work for you?

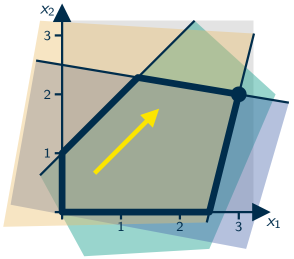
Content

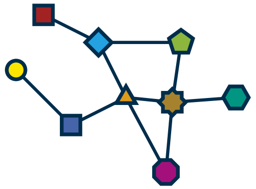
Basic toolbox
- bounded search trees
- kernelization
- iterative compression

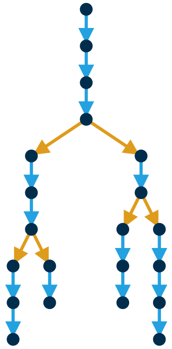




Extended toolbox
- linear programs
- branch-and-reduce
- color coding

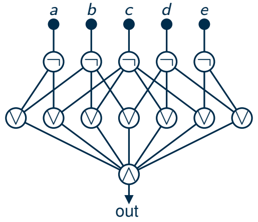


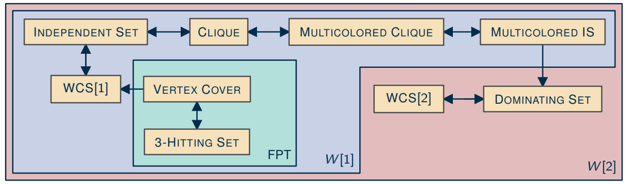




Tree width
- dynamic programming
- chordal and planar graphs
- Courcelle's theorem

Lower bonds
- kernel lower bounds
- parameterized reductions
- circuits and the W-hierarchy
- ETH and SETH





Courcelle's theorem

Previously seen

- dynamic program on tree decomposition

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- yields FPT algorithm for many problems (parameterized by treewidth):

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- meta theorem of the form:
  if a problem has property XY, then it is in FPT when parameterized by treewidth

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- we don't prove the theorem here
- but I'll tell you, what XY is

MSO_2 on graphs

(MSO = monadic second order logic)

Example: What is the meaning of the following formula?

- $G = (V, E)$ is a graph and $X \subseteq V$
- for $v \in V$ and $e \in E$, $\text{inc}(v, e)$ is true $\quad\Leftrightarrow\quad$ v is a vertex of e

$$A(X) = \forall_{Y \subseteq V} \Big[(\exists_{u,v \in X}\, u \in Y \wedge v \notin Y) \\ \Rightarrow (\exists_{e \in E} \exists_{u,v \in X}\, \text{inc}(u, e) \wedge \text{inc}(v, e) \wedge u \in Y \wedge v \notin Y) \Big]$$

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Second Order: you can quantify over elements (first order) and over relations (second order)

first order: $\forall_{v \in V}$

second order: $\forall_{Y \subseteq V}$ $\forall_{Y \subseteq V \times V}$ $\forall_{Y \subseteq V \times V \times V}$

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(MSO₁: only over V)

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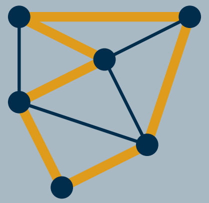
$\forall_{v \in V}$ $\forall_{e \in E}$ $\forall_{V' \subseteq V}$ $\forall_{E' \subseteq E}$

Formulating Hamiltonian Cycle in MSO_2

Problem: Hamiltonian Cycle
Given a graph G. Does G have a cycle that visits all vertices?



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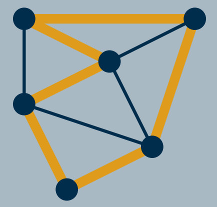

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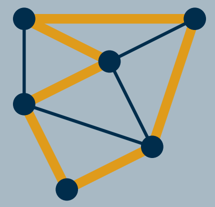

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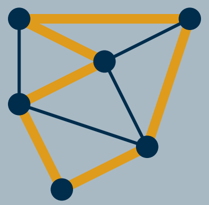

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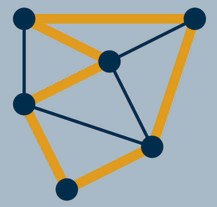

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- 2-regularity: $$\text{deg2}(v, C) = \exists_{e_1, e_2 \in C} \Big[e_1 \neq e_2 \wedge \text{inc}(v, e_1) \wedge \text{inc}(v, e_2) \wedge (\forall_{e_3 \in C}\, \text{inc}(v, e_3) \Rightarrow (e_3 = e_1 \vee e_3 = e_2)) \Big]$$

Courcelle's theorem

Theorem (without proof)
Let φ be a MSO_2 formula and let G be a graph together with a tree decomposition of width t and with an evaluation of all free variables in φ. Then there is an algorithm, that checks in $f(|\varphi|, t) \cdot n$ time whether G satisfies φ (for a computable function f).

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- example: for STEINER TREE, the input consists of a graph and a set of terminal vertices

Courcelle's theorem – another example

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- at most k vertices: $\text{size-k}(X) = \forall_{v_0, \ldots, v_k \in V} \, v_0 \notin X \vee \cdots \vee v_k \notin X \vee v_0 = v_1 \vee v_0 = v_2 \vee \cdots \vee v_{k-1} = v_k$

  (for every set of $k + 1$ vertices, one is not in X or two are the same vertex)

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Note: $|\varphi| = |\text{k-vc}|$ depends on k
$\Rightarrow$ Courcelle's theorem only gives an FPT-algorithm for parameter $k + t$

Courcelle's theorem – optimization

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Let φ be a MSO_2 formula with p free monadic variables $X_1, \dots, X_p$ and let $\alpha(x_1, \dots, x_p)$ be an affine function. Let G be a graph together with a tree decomposition of width t and with an evaluation of all free variables of φ except $X_1, \dots, X_p$. Then there is an algorithm that finds in $f(|\varphi|, t) \cdot n$ time values for $X_1, \dots X_p$, such that $\varphi(X_1, \dots, X_p) = \texttt{true}$ and $\alpha(|X_1|, \dots, |X_p|)$ is minimum or maximum.

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- $|\varphi|$ is constant $\Rightarrow$ Vertex Cover parameterized by treewidth is FPT

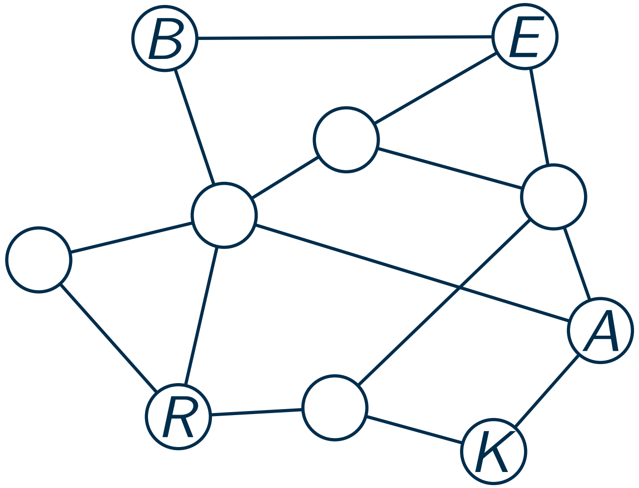
What is the problem?

Another example

- formula $\varphi(X)$ with free variables X and T

$$\varphi(X) = \forall_{Y \subseteq V} \Big[(\exists_{u,v \in T}\, u \in Y \land v \notin Y) \Rightarrow (\exists_{u,v \in V} \exists_{e \in X}\, u \in Y \land v \notin Y \land \mathrm{inc}(u,e) \land \mathrm{inc}(v,e)) \Big]$$

- $\alpha(x) = x$
- instance: $G = (V, E)$ is the shown graph and $T = \{B, R, E, A, K\} \subseteq V$



Your task: find $X \subseteq E$, such that $\alpha(|X|)$ is minimized

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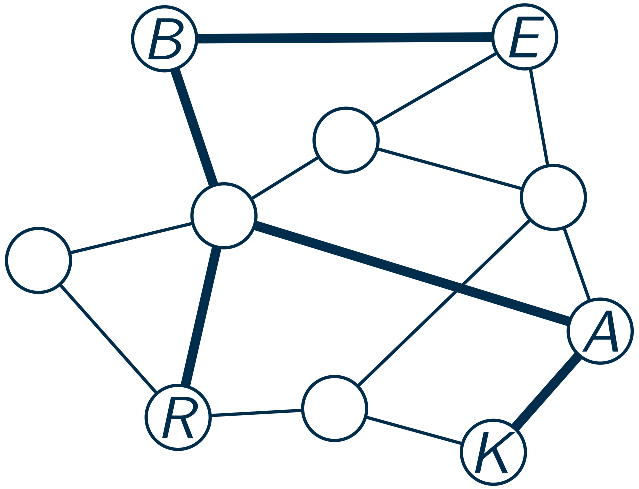
- $\alpha(x) = x$
- instance: $G = (V, E)$ is the shown graph and $T = \{B, R, E, A, K\} \subseteq V$



Your task: find $X \subseteq E$, such that $\alpha(|X|)$ is minimized

Solution: connected subgraph with minimum number of edges that contains all vertices of $T \Rightarrow$ Steiner Tree

Lecture evaluation
(the exercise sessions are evaluated separately)

1.19 What I liked most:

1.20 What I did not like at all:

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What was good and should be kept?

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https://onlineumfrage.kit.edu/evasys/online.php?p=K4TXX

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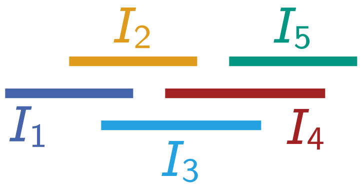
- set of n intervals $\{I_1, \ldots, I_n\}$ in $\mathbb{R}$



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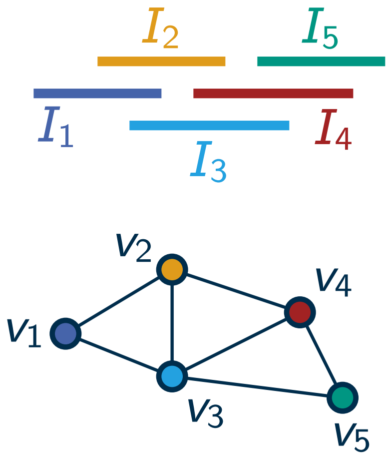

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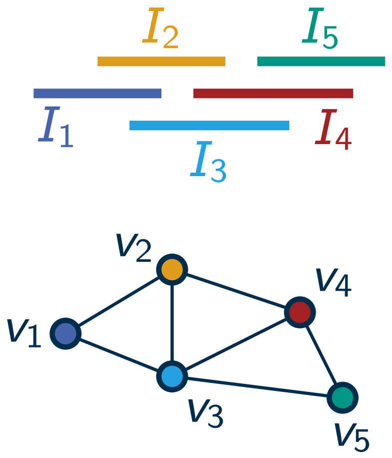

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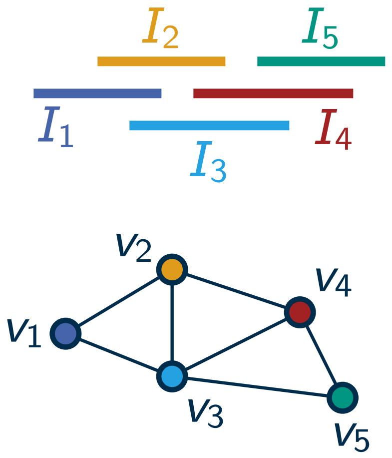

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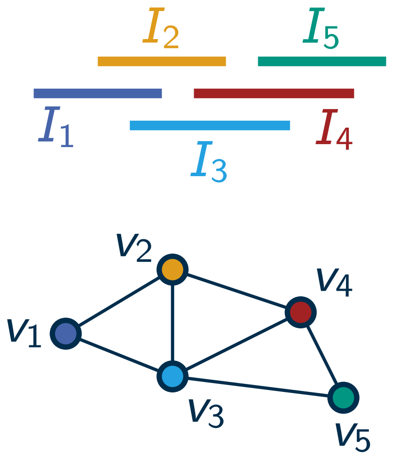

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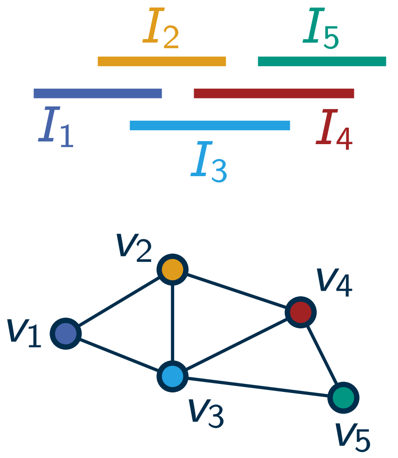

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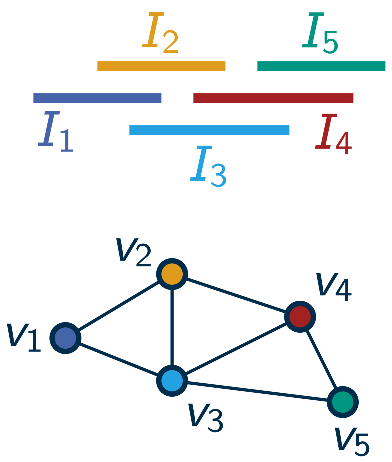

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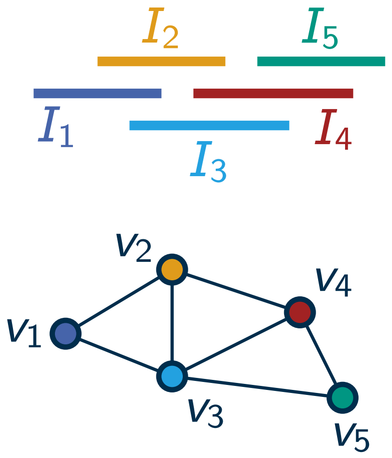

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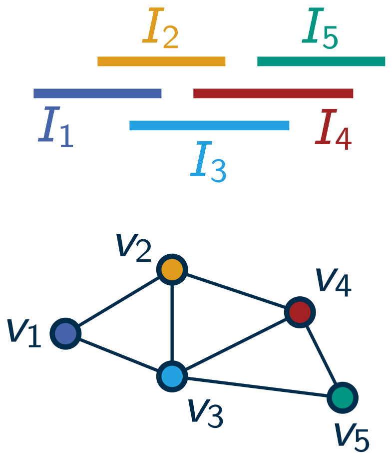

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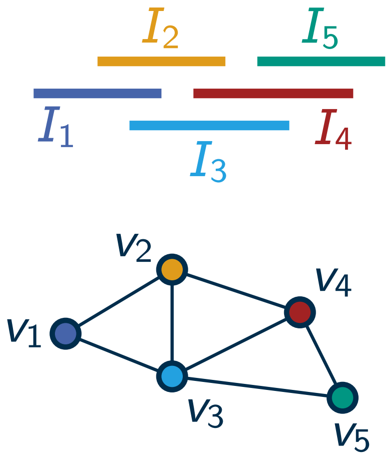

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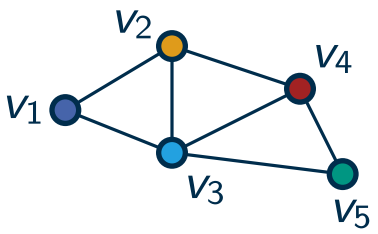
Chordal graphs

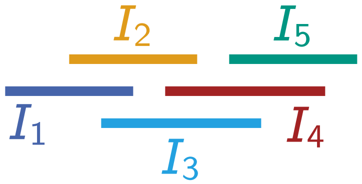
Definition (instead of subpaths of a path as for interval graphs)

A graph is **chordal** if it has an intersection representation with subtrees of a tree.

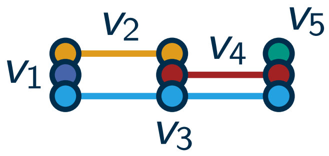
interval graph



interval representation



subpaths of a path



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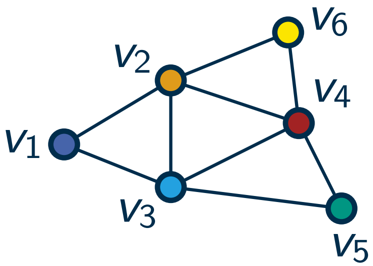

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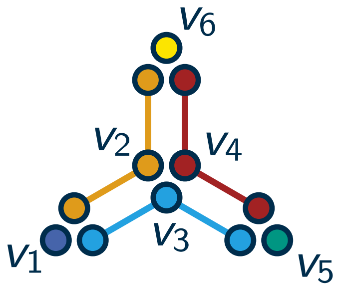

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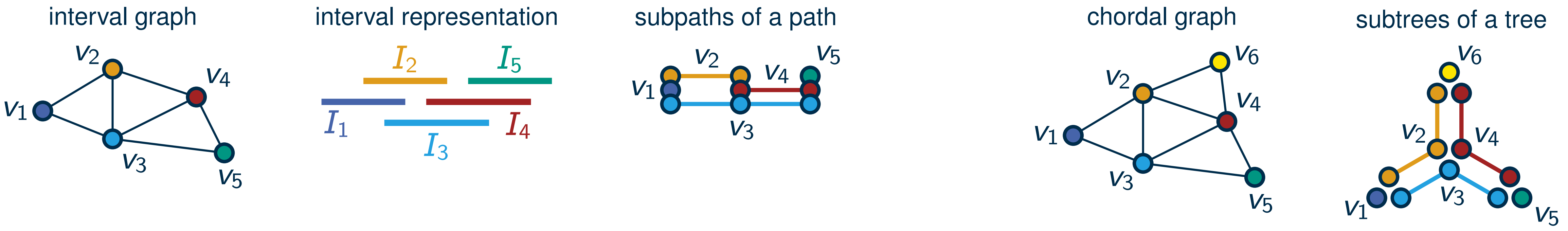

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- G chordal $\Leftrightarrow$ G has no induced cycle of length at least 4

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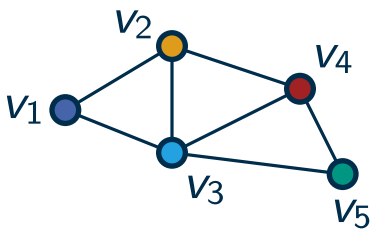

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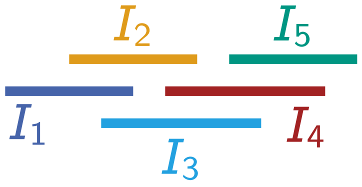

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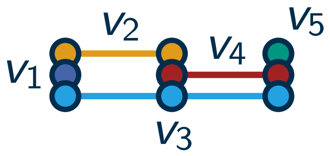

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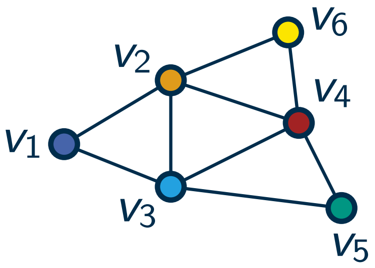

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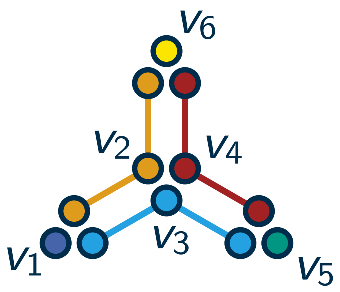

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Chordal graphs

- treewidth $=$ smallest clique number of chordal supergraph
- clique number and optimal tree decomposition of chordal graphs can be computed efficiently

Literature

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