# Parameterized Algorithms

## Treewidth: Courcelle’s Theorem and Chordal Graphs

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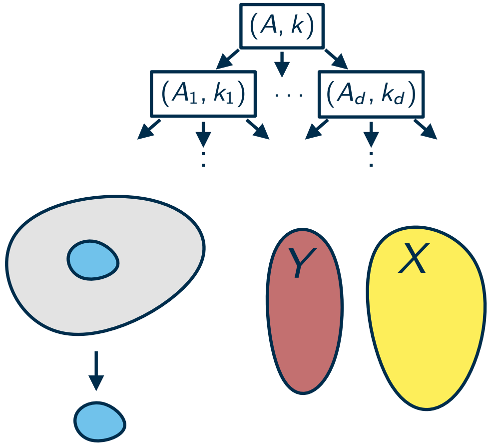
# Exam dates

| | | | | | |
|---|---|---|---|---|---|
| August | 3 | | | | 7 |
| | 10 | | | | 14 |
| | 17 | | | | 21 |
| | 24 | | | | 28 |
| | 31 | | | | 4 |
| September | 7 | | | | 11 |
| | 14 | | | | 18 |
| | 21 | | | | 25 |
| | 28 | | | | 2 |

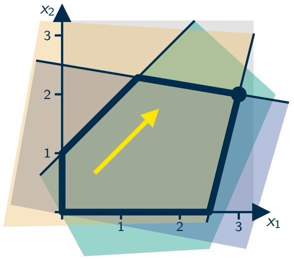
Which weeks work for you?

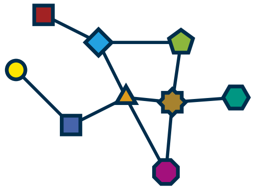
# Content

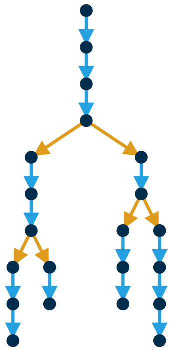
## Basic toolbox
- bounded search trees
- kernelization
- iterative compression




## Extended toolbox
- linear programs
- branch-and-reduce
- color coding

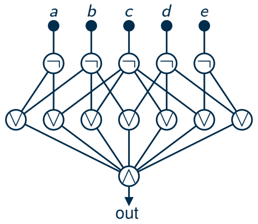


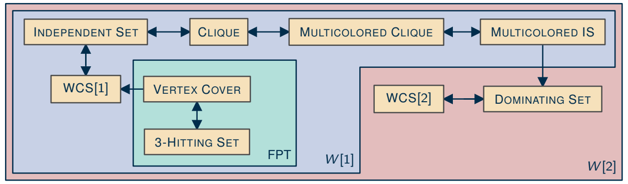





## Tree width
- dynamic programming
- chordal and planar graphs
- Courcelle's theorem

## Lower bonds
- kernel lower bounds
- parameterized reductions
- circuits and the W-hierarchy
- ETH and SETH



# Courcelle’s theorem

## Previously seen

- dynamic program on tree decomposition
- yields FPT algorithm for many problems (parameterized by treewidth):

VERTEX COVER MAXCUT STEINER TREE HAMILTON CYCLE LONGEST CYCLE CYCLE PACKING
DOMINATING SET ODD CYCLE TRANSVERSAL LONGEST PATH CONNECTED DOMINATING SET
INDEPENDENT SET FEEDBACK VERTEX SET HAMILTON PATH CONNECTED VERTEX COVER
CHROMATIC NUMBER CONNECTED FEEDBACK VERTEX SET

## Today

- meta theorem of the form:
  if a problem has property $XY$, then it is in FPT when parameterized by treewidth
- we don’t prove the theorem here
- but I’ll tell you, what $XY$ is

# $MSO_2$ on graphs

(MSO = monadic second order logic)

## Example: What is the meaning of the following formula?

- $G = (V, E)$ is a graph and $X \subseteq V$
- for $v \in V$ and $e \in E$, $\text{inc}(v, e)$ is true $\Leftrightarrow$ $v$ is a vertex of $e$

$$A(X) = \forall_{Y \subseteq V} \Big[ (\exists_{u,v \in X}\, u \in Y \land v \notin Y) \Rightarrow (\exists_{e \in E} \exists_{u,v \in X}\, \text{inc}(u, e) \land \text{inc}(v, e) \land u \in Y \land v \notin Y) \Big]$$

- solution: $A(X) = \text{true} \Leftrightarrow X \subseteq V$ induces a connected graph

## What is allowed in $MSO_2$?

**Second Order:** you can quantify over elements (first order) and over relations (second order)

**Monadic:** not all relations, only sets

**2:** you can quantify over $V$ and over $E$
(MSO$_1$: only over $V$)

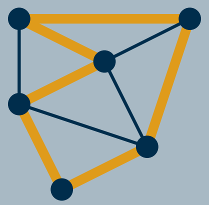
first order: $\forall_{v \in V}$

second order: $\forall_{Y \subseteq V}$ ~~$\forall_{Y \subseteq V \times V}$~~ ~~$\forall_{Y \subseteq V \times V \times V}$~~

$\forall_{v \in V}$ $\forall_{e \in E}$ $\forall_{V' \subseteq V}$ $\forall_{E' \subseteq E}$

# Formulating HAMILTONIAN CYCLE in $MSO_2$

**Problem: HAMILTONIAN CYCLE**
Given a graph $G$. Does $G$ have a cycle that visits all vertices?



**Express existence of a Hamiltonian cycle in $MSO_2$**

- note: $C \subseteq E$ is solution $\Leftrightarrow$ $G[C]$ is connected and 2-regular
- high-level formula: $\text{hamiltonian} = \exists_{C \subseteq E}\, \text{conn}(C) \wedge \forall_{v \in V}\, \text{deg2}(v, C)$
- connectedness: $\text{conn}(C) = \forall_{Y \subseteq V} \Big[ (\exists_{u,v \in V}\, u \in Y \wedge v \notin Y) \Rightarrow (\exists_{e \in C} \exists_{u \in Y} \exists_{v \notin Y}\, \text{inc}(u, e) \wedge \text{inc}(v, e) \Big]$
- 2-regularity: $\text{deg2}(v, C) = \exists_{e_1, e_2 \in C} \Big[ e_1 \neq e_2 \wedge \text{inc}(v, e_1) \wedge \text{inc}(v, e_2) \wedge (\forall_{e_3 \in C}\, \text{inc}(v, e_3) \Rightarrow (e_3 = e_1 \vee e_3 = e_2)) \Big]$

# Courcelle's theorem

**Theorem (without proof)**
Let $\varphi$ be a $MSO_2$ formula and let $G$ be a graph together with a tree decomposition of width $t$ and with an evaluation of all free variables in $\varphi$. Then there is an algorithm, that checks in $f(|\varphi|, t) \cdot n$ time whether $G$ satisfies $\varphi$ (for a computable function $f$).

## Example

- property to have a Hamiltonian cycle: can be expressed with a $MSO_2$ formula of constant size
- thus: HAMILTONIAN CYCLE parameterized with treewidth is FPT

## What are the free variables for?

- useful, if the input is not just the graph but, e.g., consists of $G = (V, E)$ and $X \subseteq V$
- example: for STEINER TREE, the input consists of a graph and a set of terminal vertices

# Courcelle's theorem – another example

**Theorem (without proof)**
Let $\varphi$ be a $MSO_2$ formula and let $G$ be a graph together with a tree decomposition of width $t$ and with an evaluation of all free variables in $\varphi$. Then there is an algorithm, that checks in $f(|\varphi|, t) \cdot n$ time whether $G$ satisfies $\varphi$ (for a computable function $f$).

**Another example: *k*-VERTEX COVER**

- high-level formula: $\text{k-vc} = \exists_{X \subseteq V} (\text{vc}(X) \wedge \text{size-k}(X))$
- covering all edges: $\text{vc}(X) = \forall_{e \in E} \exists_{v \in X} \text{inc}(v, e)$
- at most $k$ vertices: $\text{size-k}(X) = \forall_{v_0, \ldots, v_k \in V} v_0 \notin X \vee \cdots \vee v_k \notin X \vee v_0 = v_1 \vee v_0 = v_2 \vee \cdots \vee v_{k-1} = v_k$
  (for every set of $k+1$ vertices, one is not in $X$ or two are the same vertex)

**Note:** $|\varphi| = |\text{k-vc}|$ depends on $k$
$\Rightarrow$ Courcelle's theorem only gives an FPT-algorithm for parameter $k + t$

# Courcelle's theorem – optimization

**Theorem (without proof)**
Let $\varphi$ be a $MSO_2$ formula with $p$ free monadic variables $X_1, \ldots, X_p$ and let $\alpha(x_1, \ldots, x_p)$ be an affine function. Let $G$ be a graph together with a tree decomposition of width $t$ and with an evaluation of all free variables of $\varphi$ except $X_1, \ldots, X_p$. Then there is an algorithm that finds in $f(|\varphi|, t) \cdot n$ time values for $X_1, \ldots X_p$, such that $\varphi(X_1, \ldots, X_p) = \texttt{true}$ and $\alpha(|X_1|, \ldots, |X_p|)$ is minimum or maximum.

affine function: $\alpha(x_1, \ldots, x_p) = a_0 + \sum_{i=1}^{p} a_i x_i$

## Example: Vertex Cover

- one free variable $X \subseteq V$
- goal 1: $X$ is a vertex cover: $\varphi(X) = \mathrm{vc}(X) = \forall_{e \in E} \exists_{v \in X} \mathrm{inc}(v, e)$
- goal 2: $X$ is minimum among all vertex covers: $\alpha(x) = x$
- $|\varphi|$ is constant $\Rightarrow$ Vertex Cover parameterized by treewidth is FPT

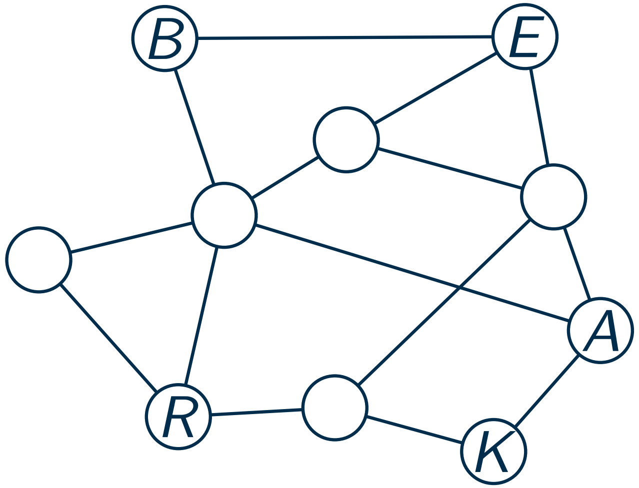
# What is the problem?

## Another example

- formula $\varphi(X)$ with free variables $X$ and $T$

$$\varphi(X) = \forall_{Y \subseteq V} \Big[ (\exists_{u,v \in T}\, u \in Y \land v \notin Y) \\ \Rightarrow (\exists_{u,v \in V} \exists_{e \in X}\, u \in Y \land v \notin Y \land \mathrm{inc}(u,e) \land \mathrm{inc}(v,e)) \Big]$$

- $\alpha(x) = x$
- instance: $G = (V, E)$ is the shown graph and $T = \{B, R, E, A, K\} \subseteq V$




**Your task:** find $X \subseteq E$, such that $\alpha(|X|)$ is minimized

# What is the problem?

**Another example**

- formula $\varphi(X)$ with free variables $X$ and $T$

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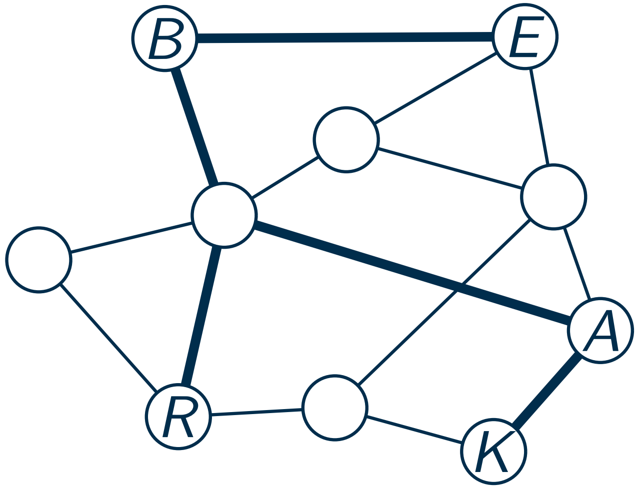



**Your task:** find $X \subseteq E$, such that $\alpha(|X|)$ is minimized

**Solution:** connected subgraph with minimum number of edges that contains all vertices of $T \Rightarrow$ STEINER TREE

# Lecture evaluation

(the exercise sessions are evaluated separately)

**What was good and should be kept?**

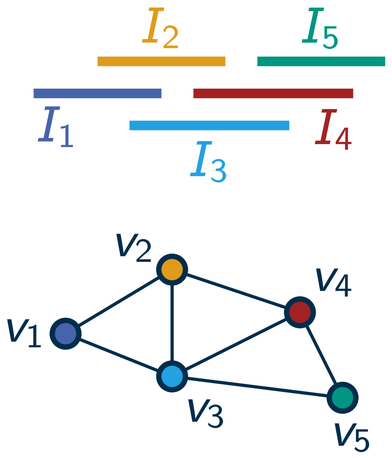
1.19 ~~What I liked most:~~

**How can the lecture be improved?**

1.20 ~~What I did not like at all:~~

https://onlineumfrage.kit.edu/evasys/online.php?p=K4TXX

# Interval graphs




## Definition

- set of $n$ intervals $\{I_1, \ldots, I_n\}$ in $\mathbb{R}$
- graph $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$, $E = \{\{v_i, v_j\} \mid I_i \cap I_j \neq \emptyset\}$
- $I_1, \ldots, I_n$ is the **interval representation** of $G$
- a graph is an **interval graph** $\Leftrightarrow$ it has an interval representation

## What is the pathwidth of an interval graph?

- let $\omega(G)$ be the clique number and $\mathrm{pw}(G)$ be the pathwidth of $G$
- then: $\mathrm{pw}(G) \leq \omega(G) - 1$ (interval representation yields path decomposition)
- and: $\mathrm{pw}(G) \geq \omega(G) - 1$ (vertices of a clique must share a bag)

$\Rightarrow \mathrm{pw}(G) = \omega(G) - 1$

## Interval width

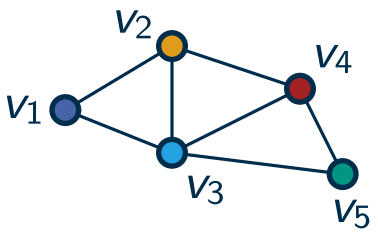
- $\text{interval-width}(G) = \min\{\omega(G') \mid G \subseteq G' \text{ and } G' \text{ is an interval graph}\}$
- $\Rightarrow \mathrm{pw}(G) = \text{interval-width}(G) - 1$

# Chordal graphs

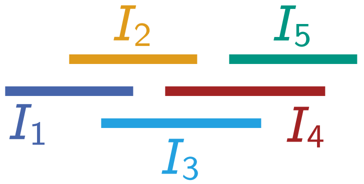
**Definition** (instead of subpaths of a path as for interval graphs)

A graph is **chordal** if it has an intersection representation with subtrees of a tree.

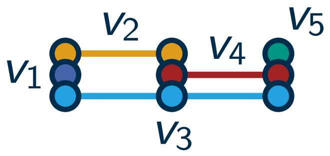
interval graph



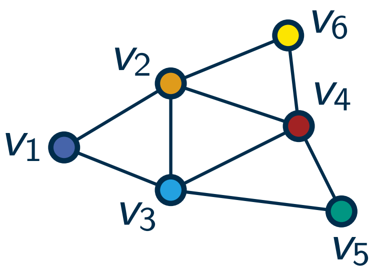

interval representation



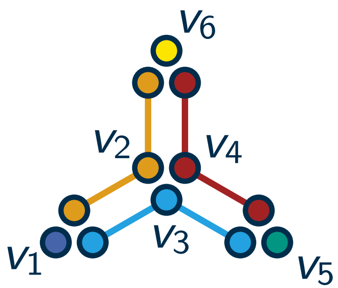

subpaths of a path




chordal graph




subtrees of a tree




## Equivalent definition

- $G$ chordal $\Leftrightarrow$ $G$ has no induced cycle of length at least 4

## Chordal width

- $\text{chordal-width}(G) = \min\{\omega(G') \mid G \subseteq G' \text{ and } G' \text{ is chordal}\}$
- $\Rightarrow \text{tw}(G) = \text{chordal-width}(G) - 1$

# Wrap-Up

## Courcelle's theorem

- problem can be formulated in $MSO_2$ $\Rightarrow$ FPT when parameterized by treewidth
- powerful tool, as many problems can be formulated in $MSO_2$
- **Warning:** Courcelle yields really bad running times:
  - $f(k)$ is multi-exponential; think of: $2^{2^{\cdot^{\cdot^{\cdot^{2^k}}}}}$
  - height of the power tower is not bounded by a constant

    (linear in the number of alternations between $\forall$ and $\exists$)

## Chordal graphs

- treewidth $=$ smallest clique number of chordal supergraph
- clique number and optimal tree decomposition of chordal graphs can be computed efficiently

# Literature

**The monadic second-order logic of graphs. I. Recognizable sets of finite graphs**

- Bruno Courcelle [1990]
- Courcelle's Theorem `https://doi.org/10.1016/0890-5401(90)90043-H`

**Automatic generation of linear-time algorithms from predicate calculus descriptions of problems on recursively constructed graph families**

- Richard B. Borie, R. Gary Parker, Craig A. Tovey [1992]
- independent rediscovery `https://doi.org/10.1007/BF01758777`

**Courcelle's theorem—A game-theoretic approach**

- Joachim Kneis, Alexander Langer, Peter Rossmanith [2011]
- Courcelle's theorem in "practice" `https://doi.org/10.1016/j.disopt.2011.06.001`