Parameterized Algorithms

Lower Bounds for Kernelization

Thomas Bläsius

Kernelization – formal definition

Definition
For a parameterized problem L, a **kernelization** is a polynomial-time algorithm that transforms any instance (x, k) into an equivalent instance (x', k') (the **kernel**) such that $|x'| + k' \leq g(k)$.

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  - solve (x, k) with $\mathcal{A}$ in polynomial time
  - kernelization: return trivial kernel
- **case 2:** $f(k) > |x|$
  - (x, k) itself is already a suitable kernel
  - kernelization: just return (x, k)

The size of a kernel

VERTEX COVER kernelization using the theorem

- FPT-algorithm for VERTEX COVER: $f(k) \in O(1.2529^k)$

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- today: probably no polynomial kernel for longest path

Cross-composition: Hamiltonian $\rightarrow$ longest path

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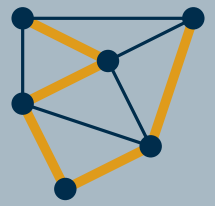
Given a graph G. Does G have a path that visits all vertices?



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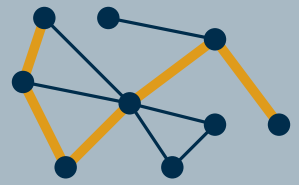

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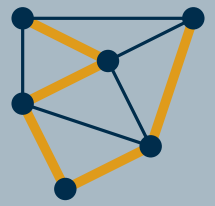

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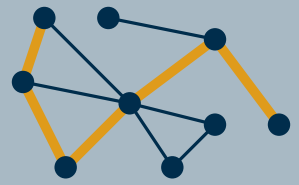

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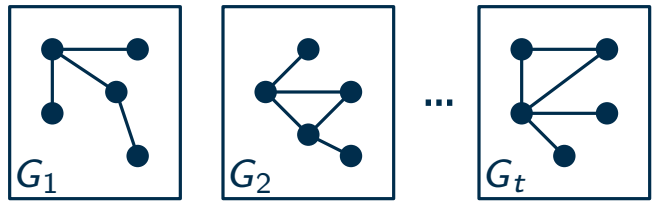

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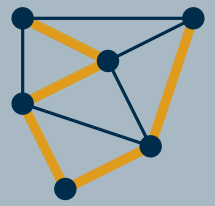

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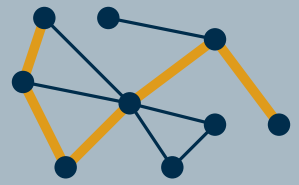

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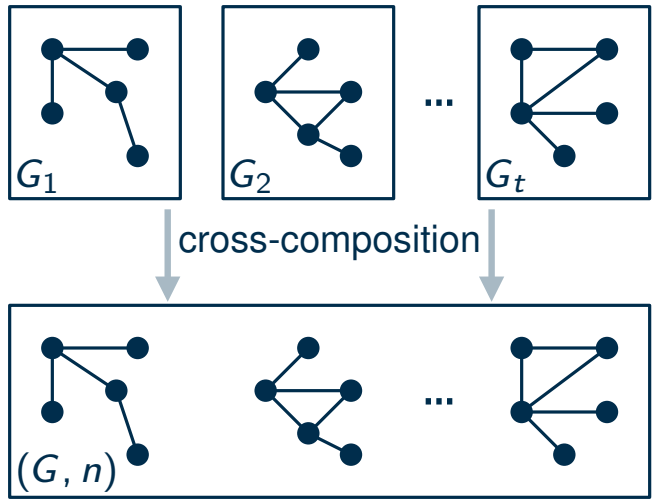

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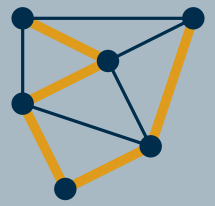

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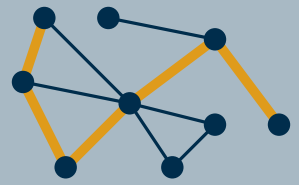

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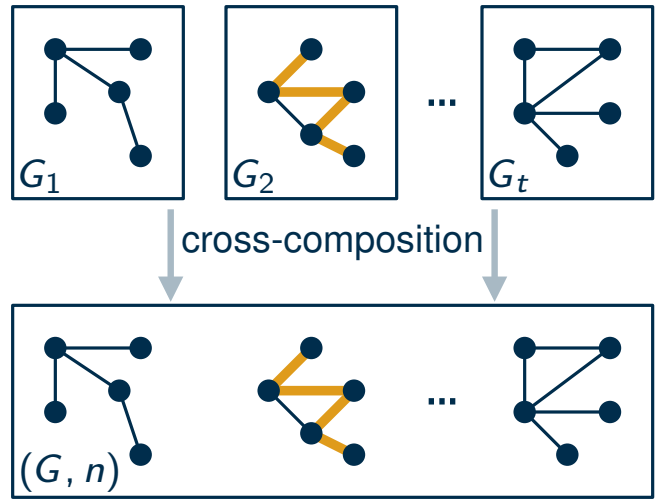

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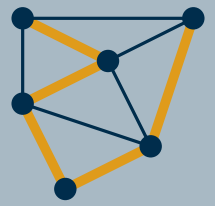

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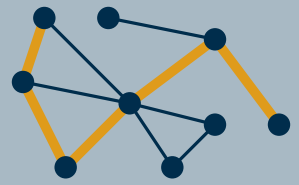

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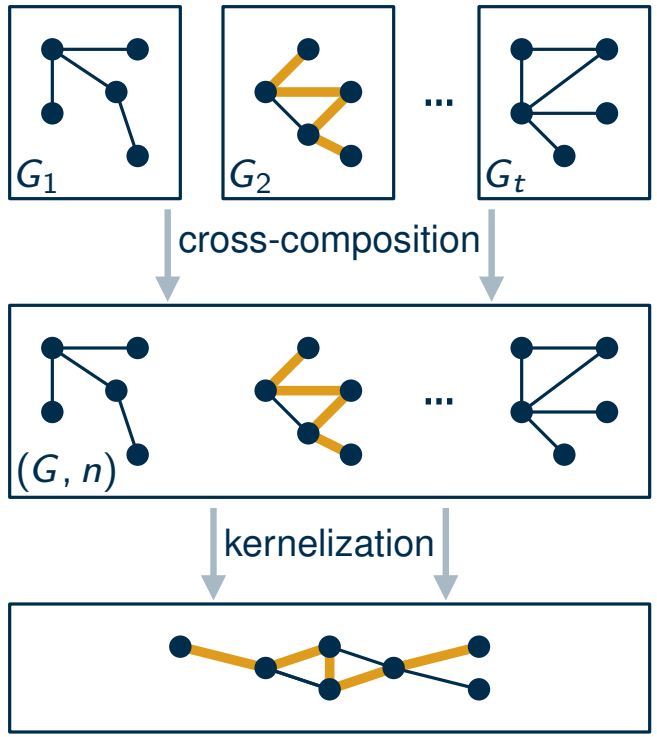

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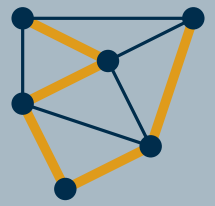

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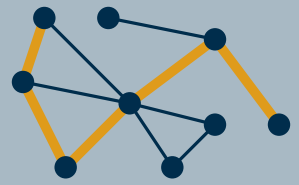

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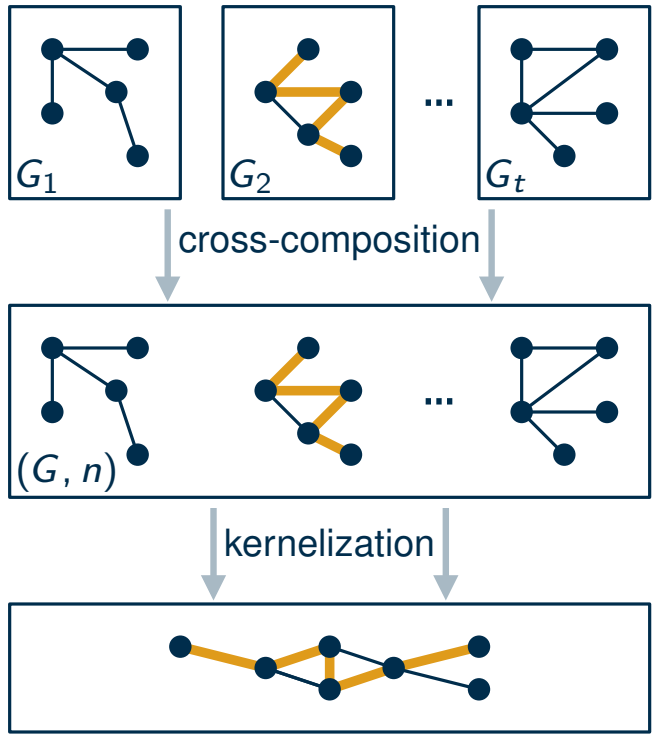

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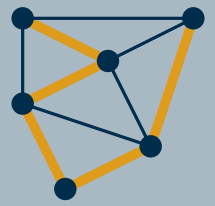

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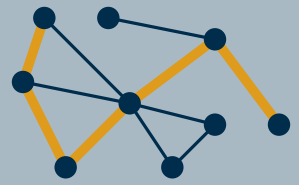

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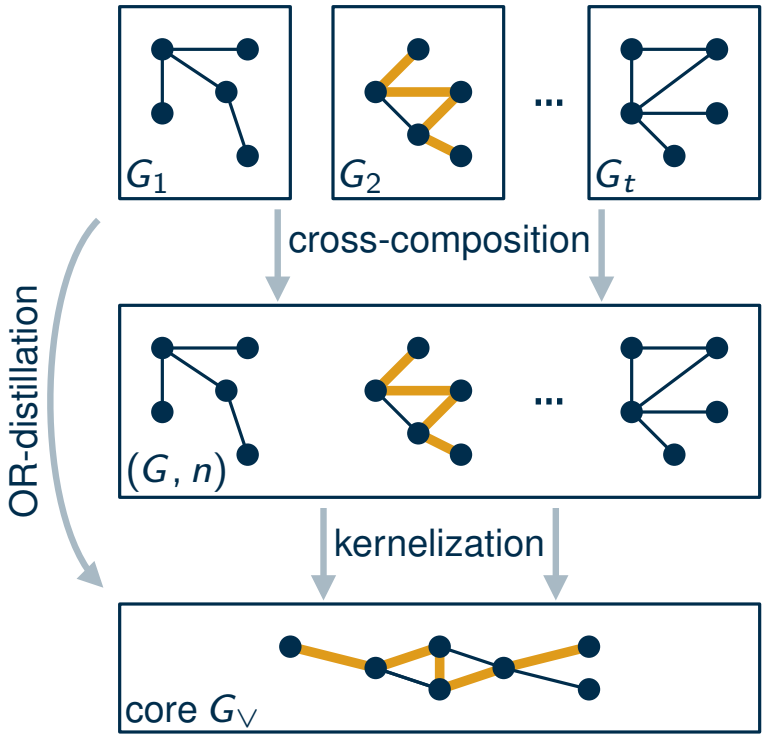

OR-Distillation

OR-Distillation algorithm $\mathcal{D}$ from HAMP into LONGP

- turns a t-tuple of n-vertex graphs $(G_1, \ldots, G_t)$ into an $n^{O(1)}$-vertex graph $G_\vee$ in time $(tn)^{O(1)}$
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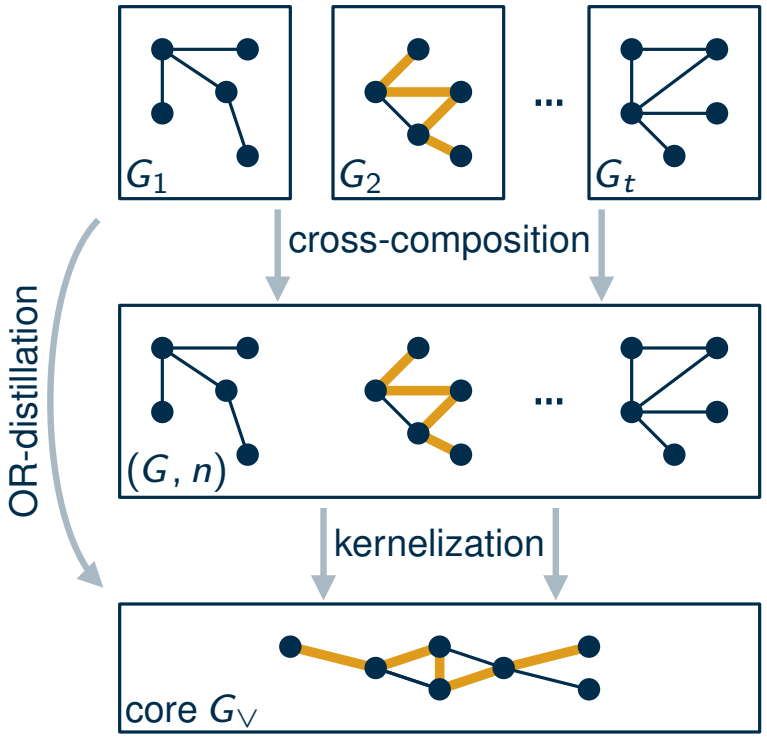

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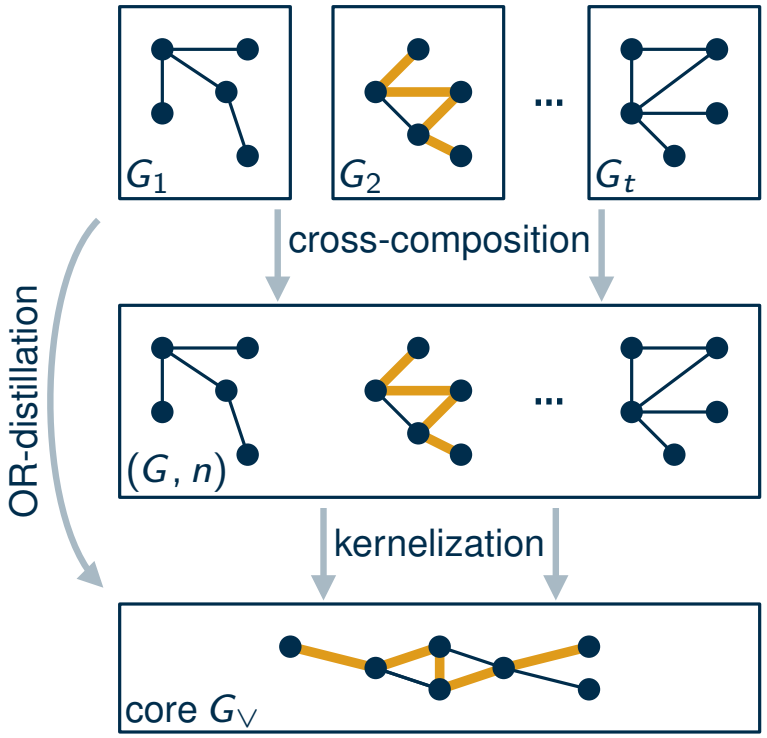

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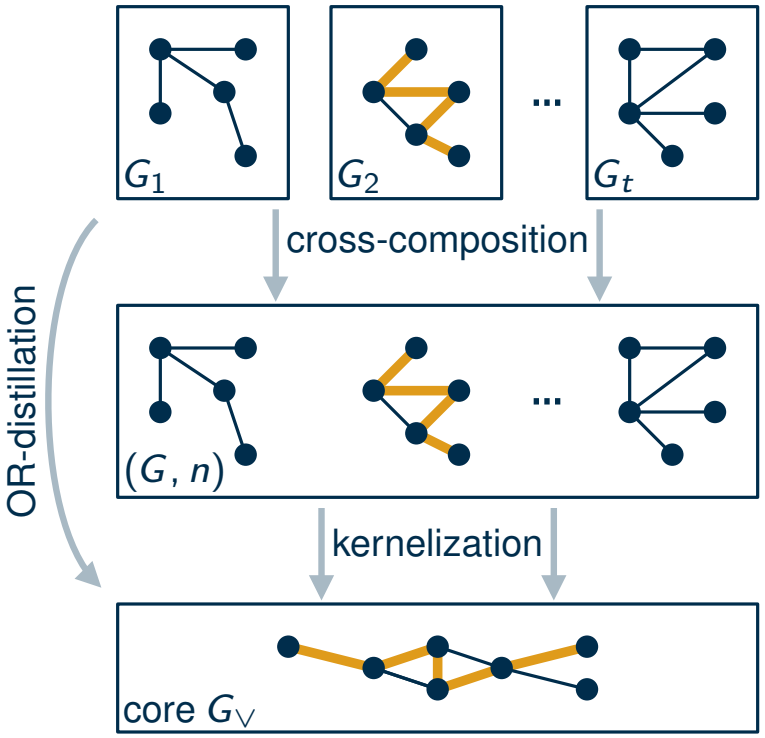

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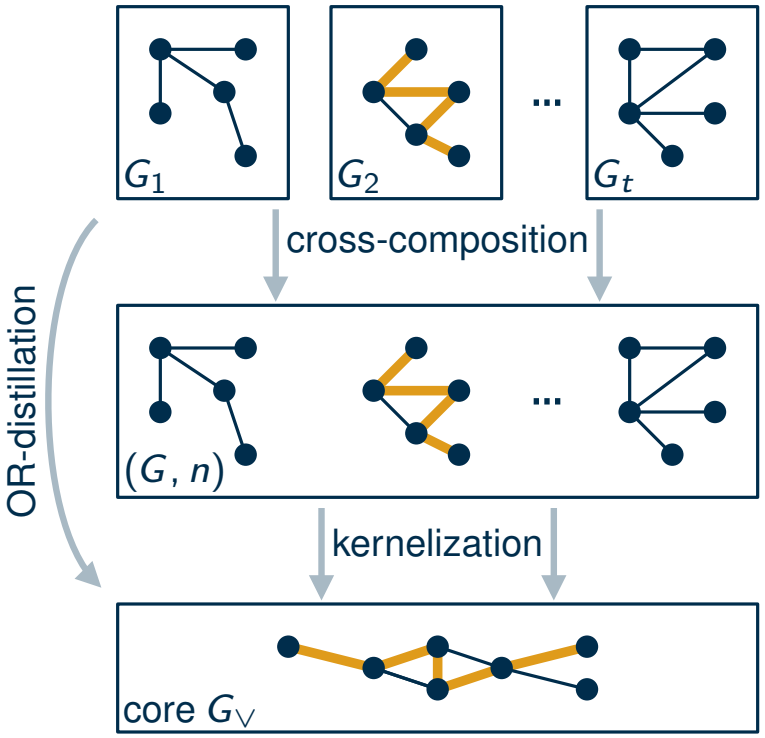

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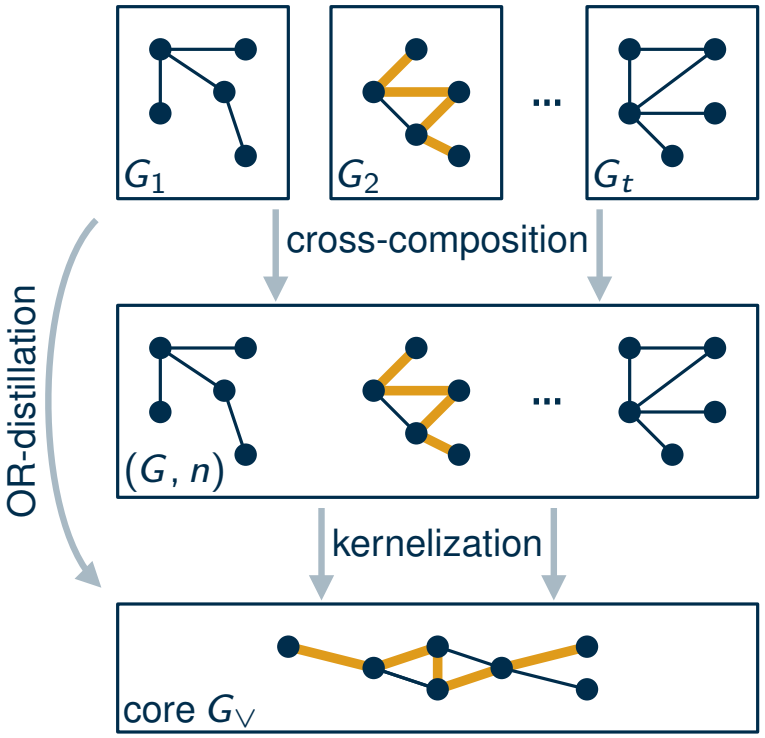

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- as we believe NP $\not\subseteq$ coNP/poly, we should believe that LONGP does not admit a polynomial kernel

coNP/poly

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Nondeterministic polynomial

- guess certificate for input being a yes-instance
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- guess certificate for input being a yes-instance
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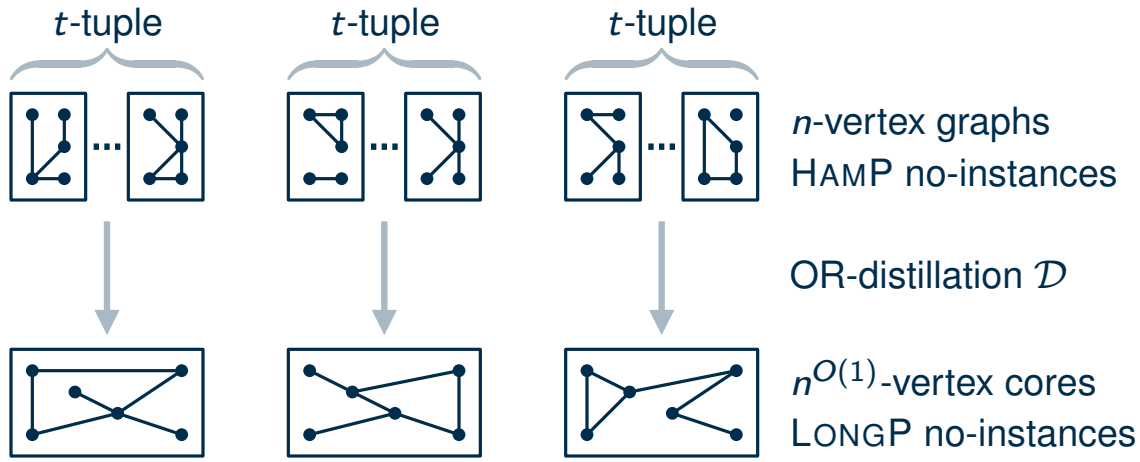

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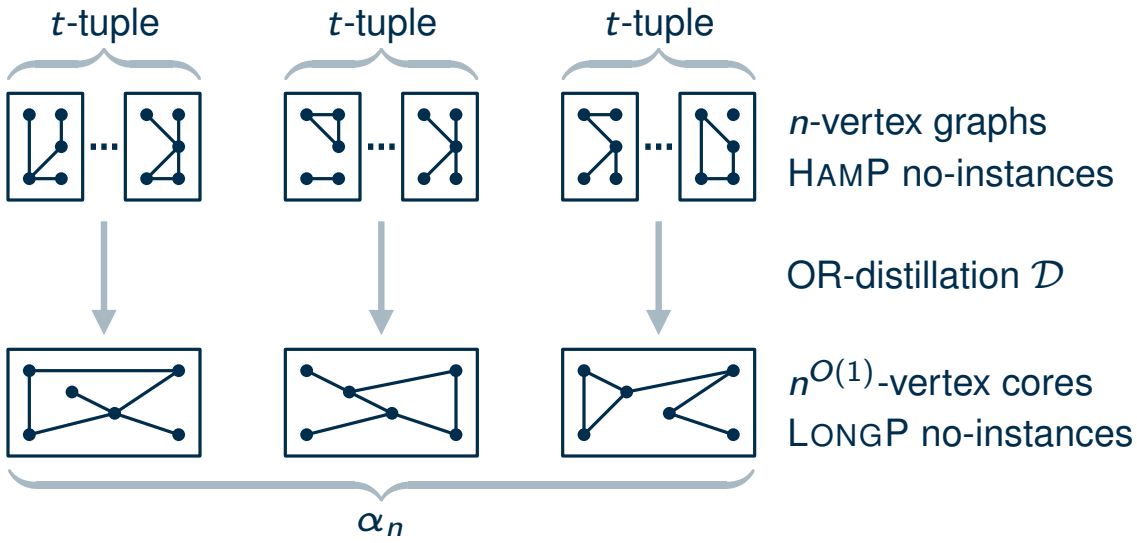

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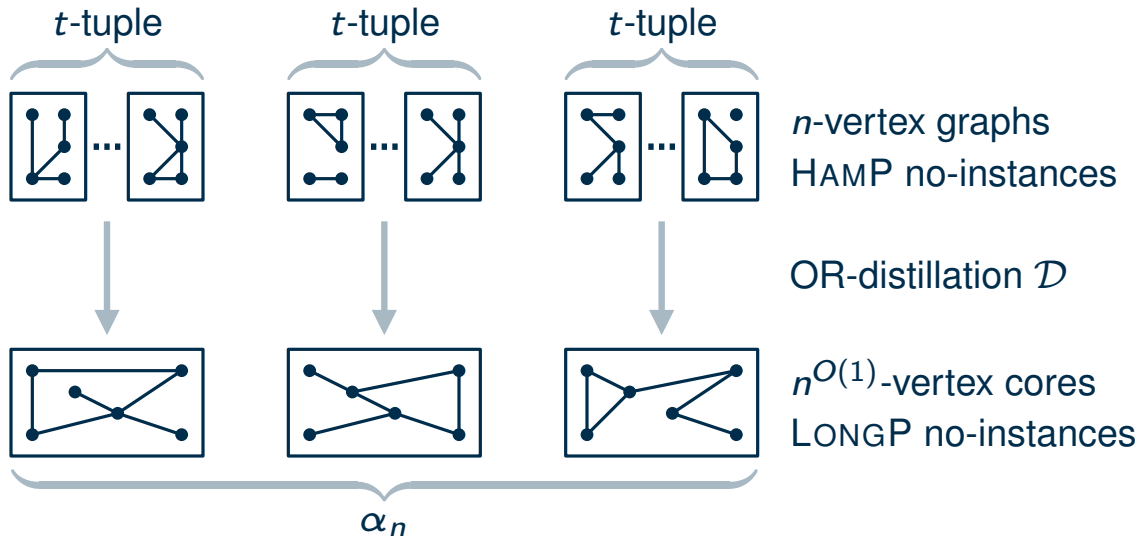

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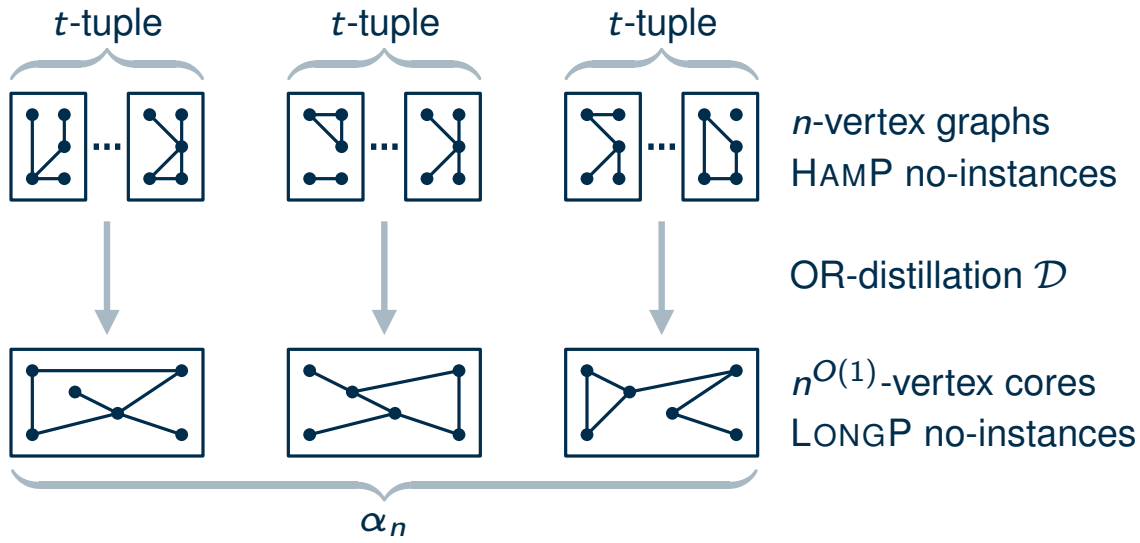

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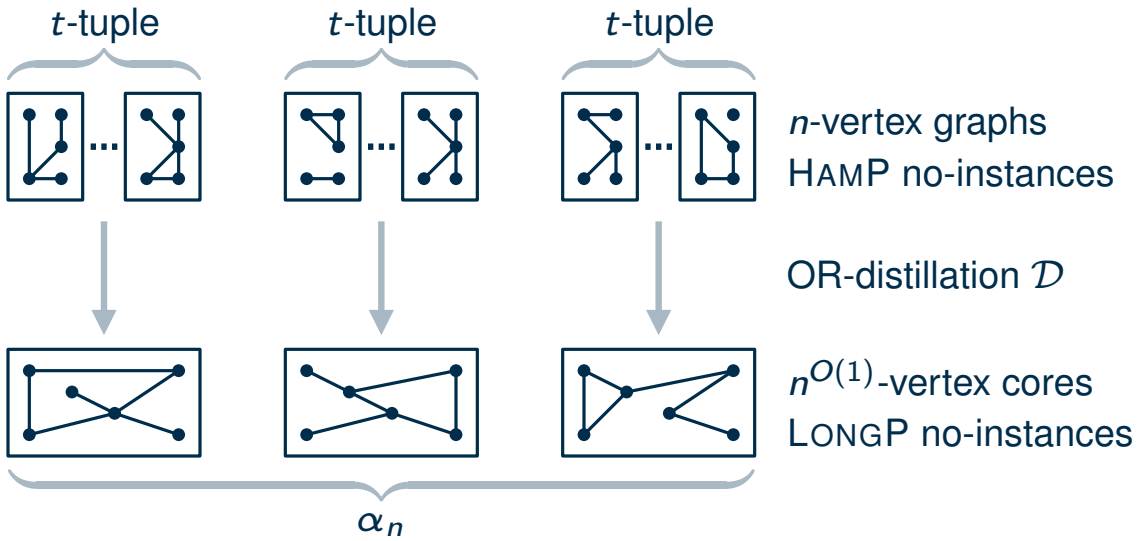

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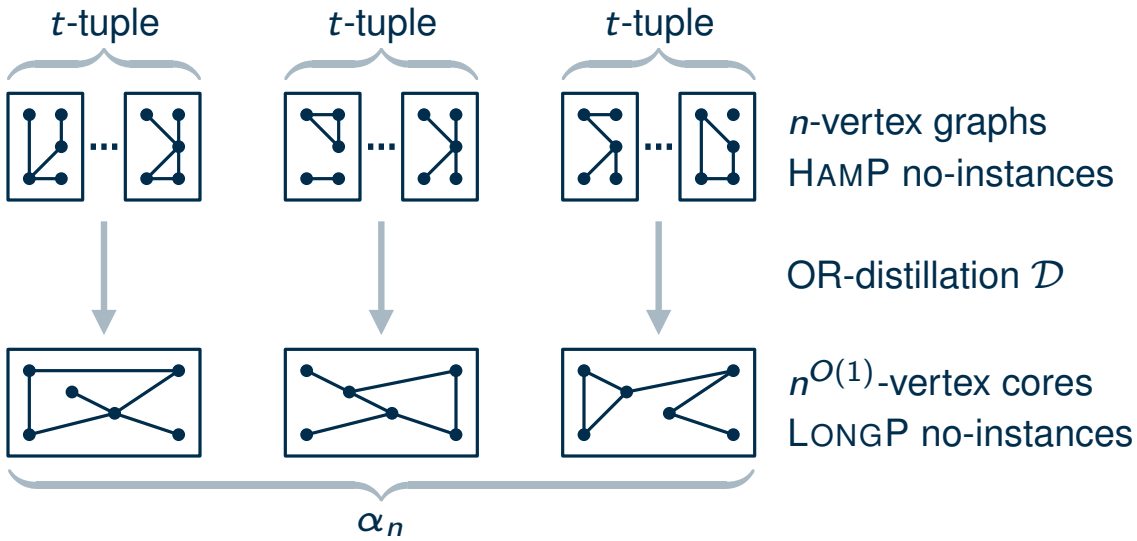

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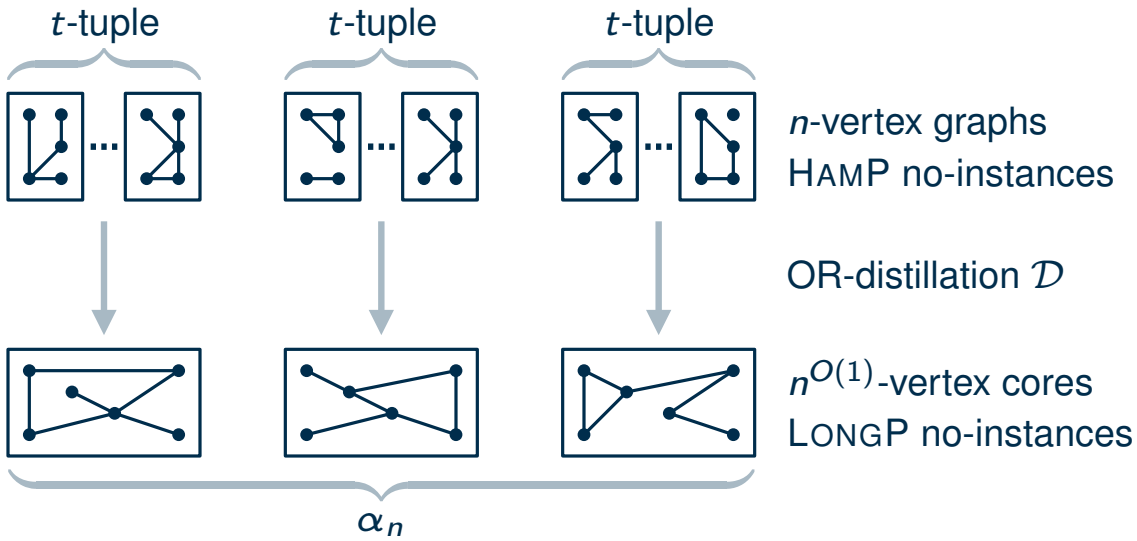

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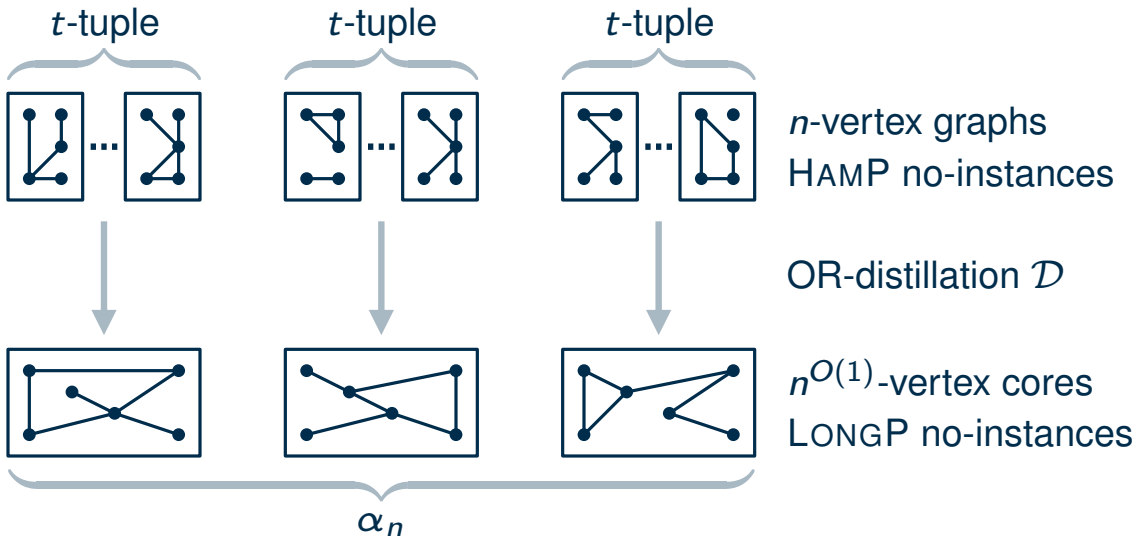

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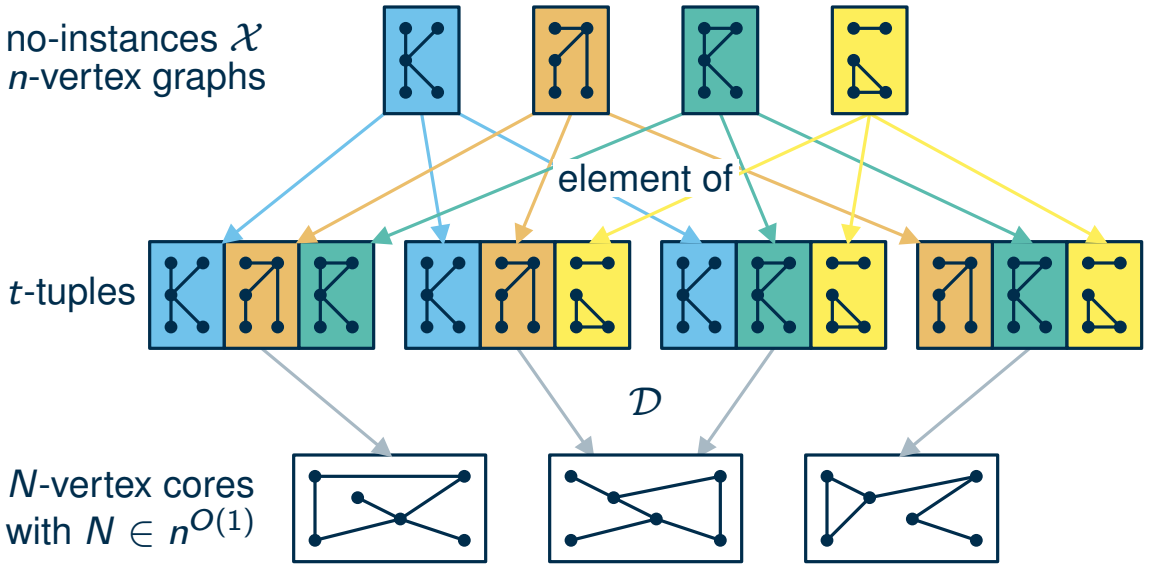

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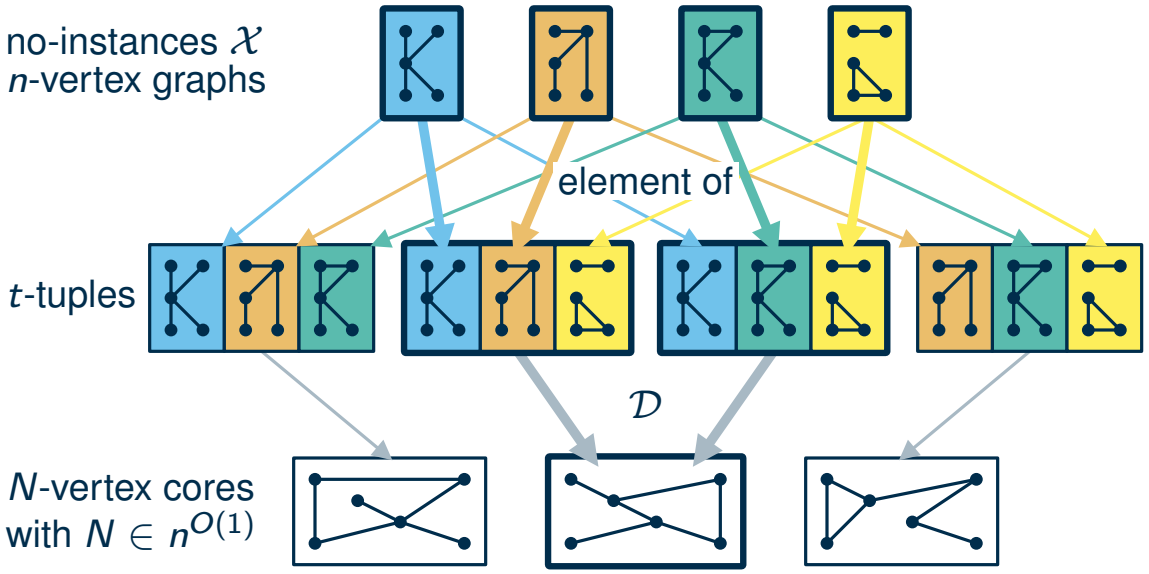

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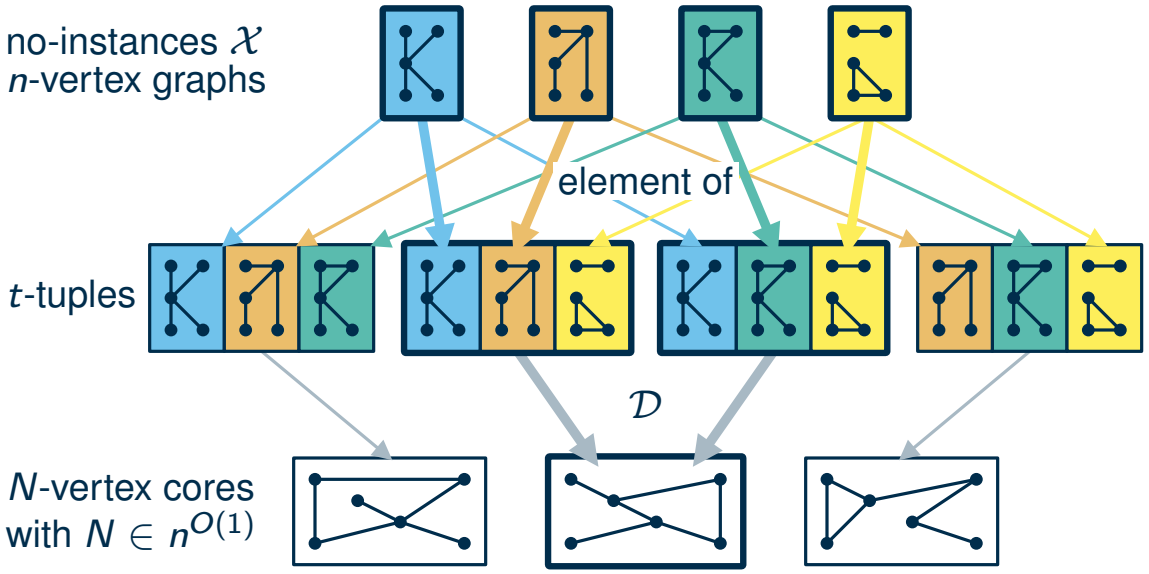

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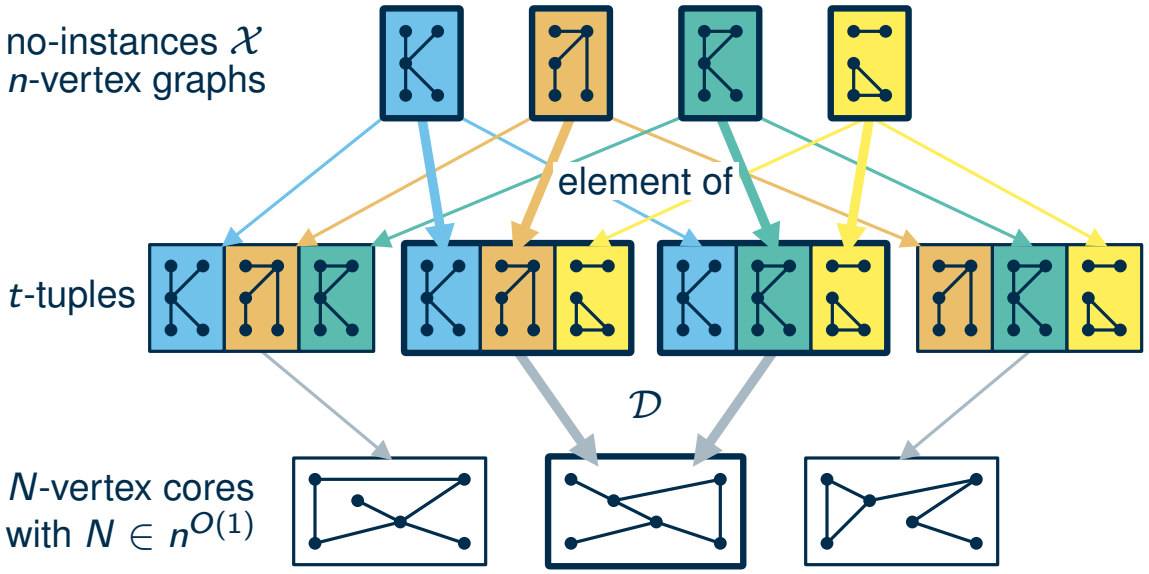

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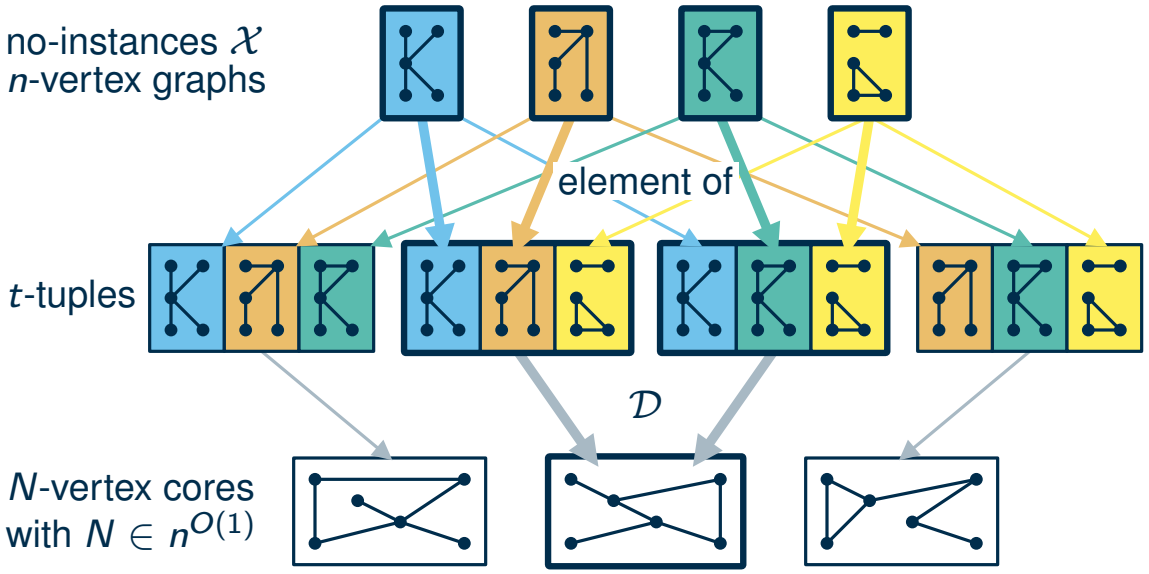

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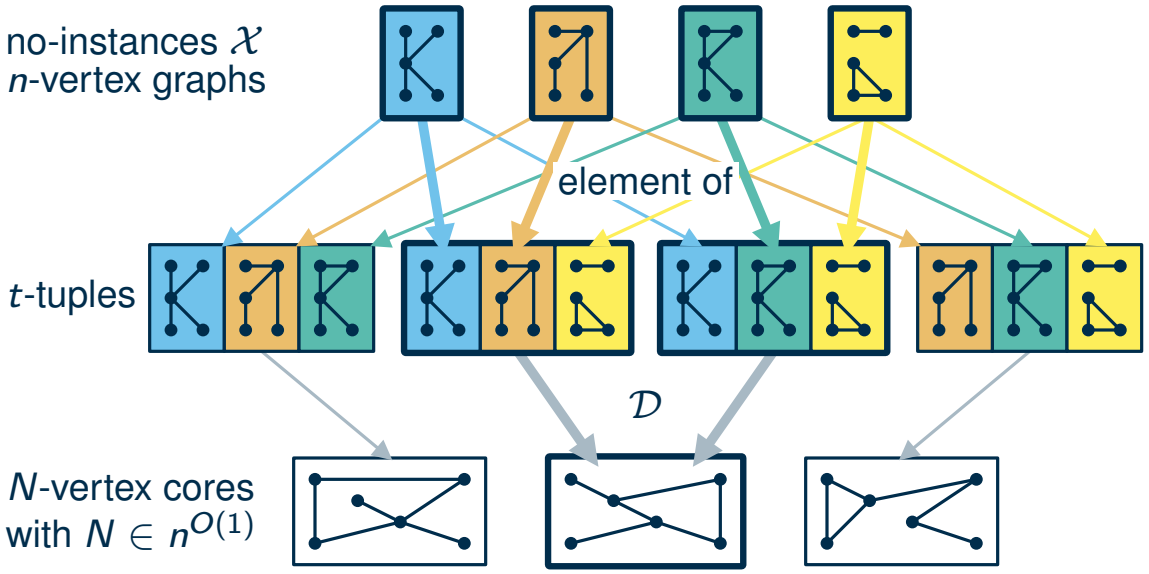

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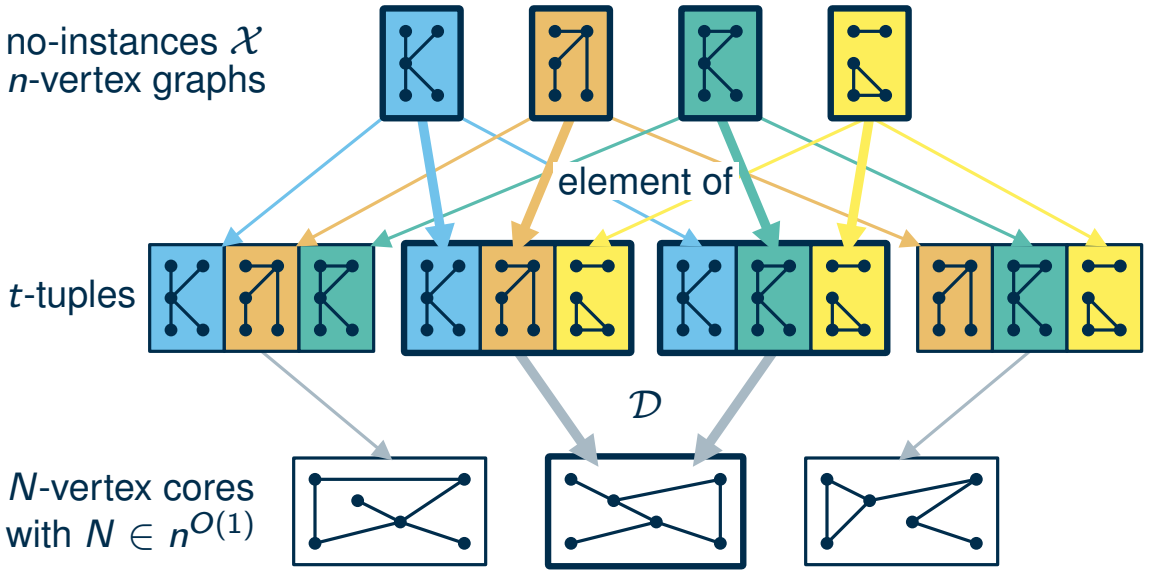

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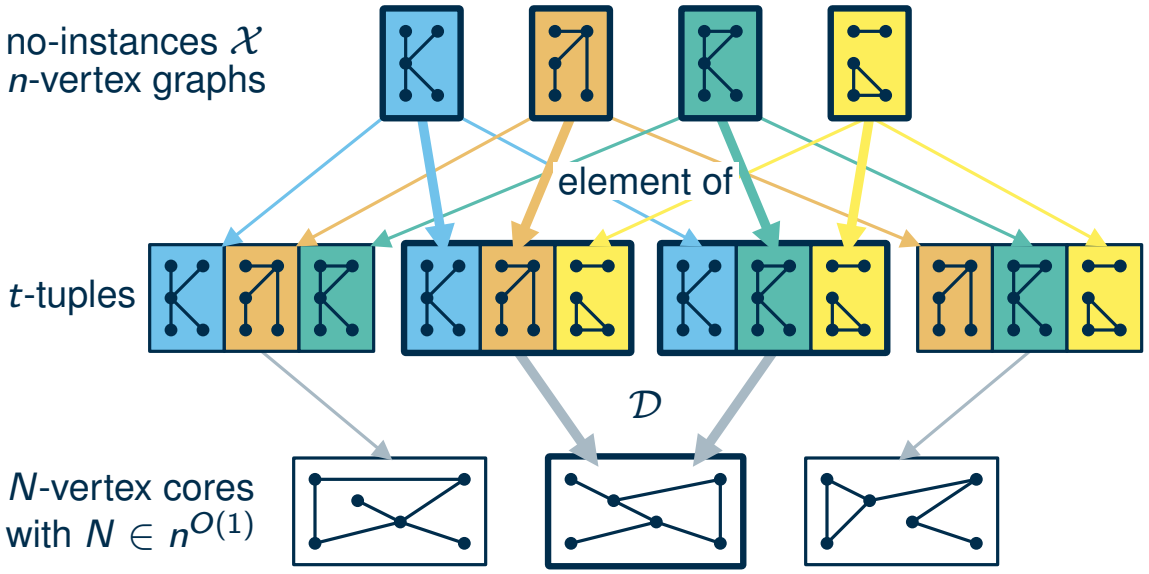

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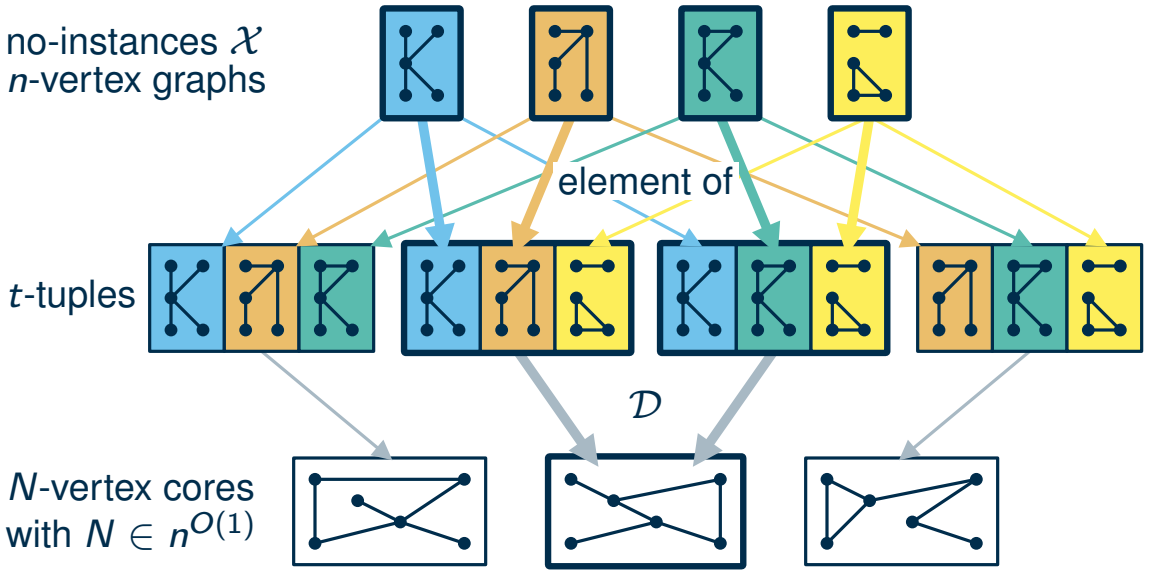

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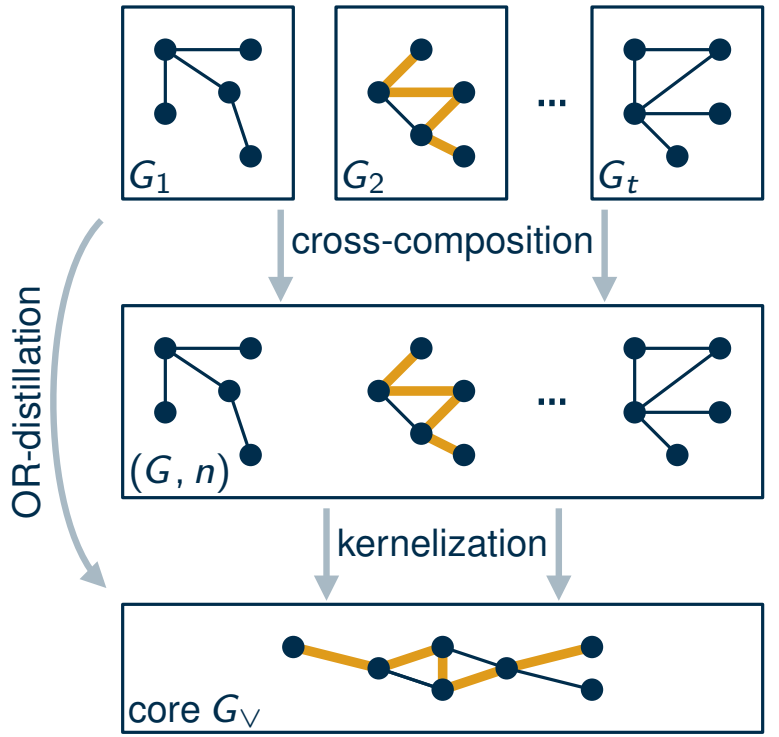

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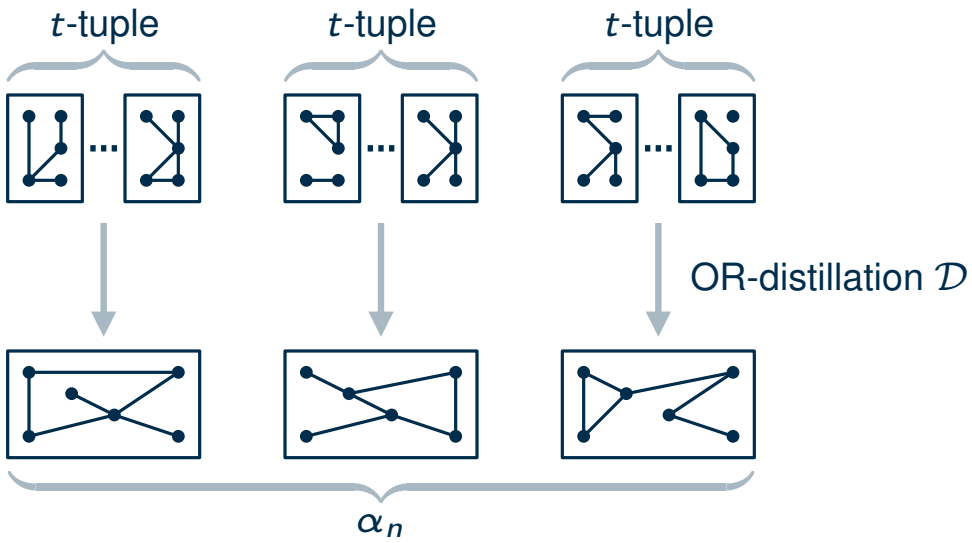

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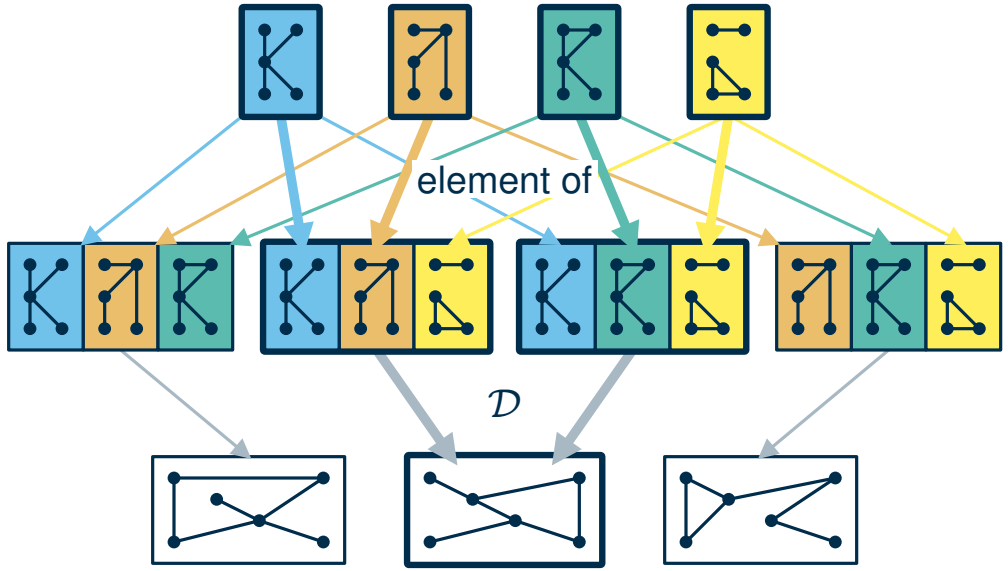

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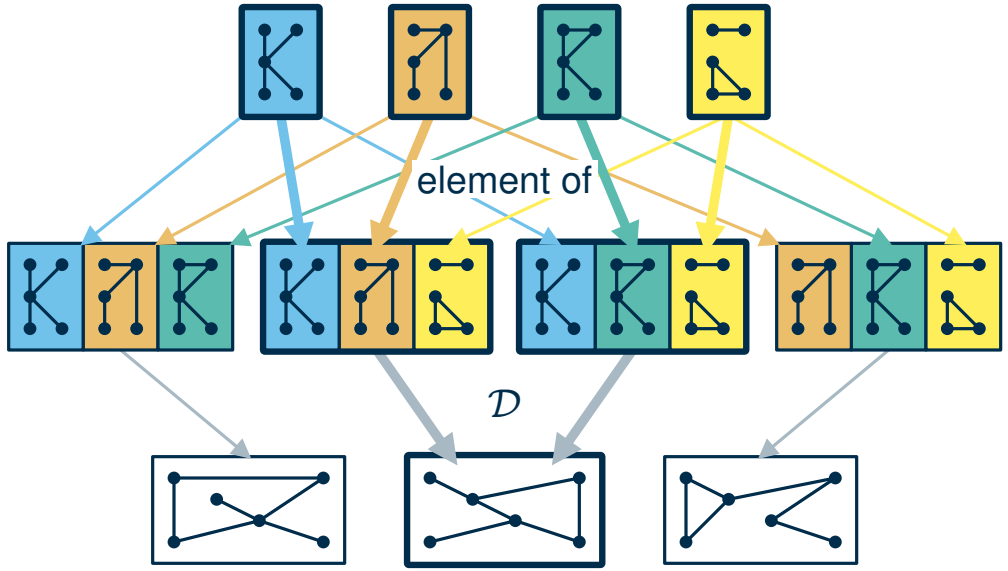

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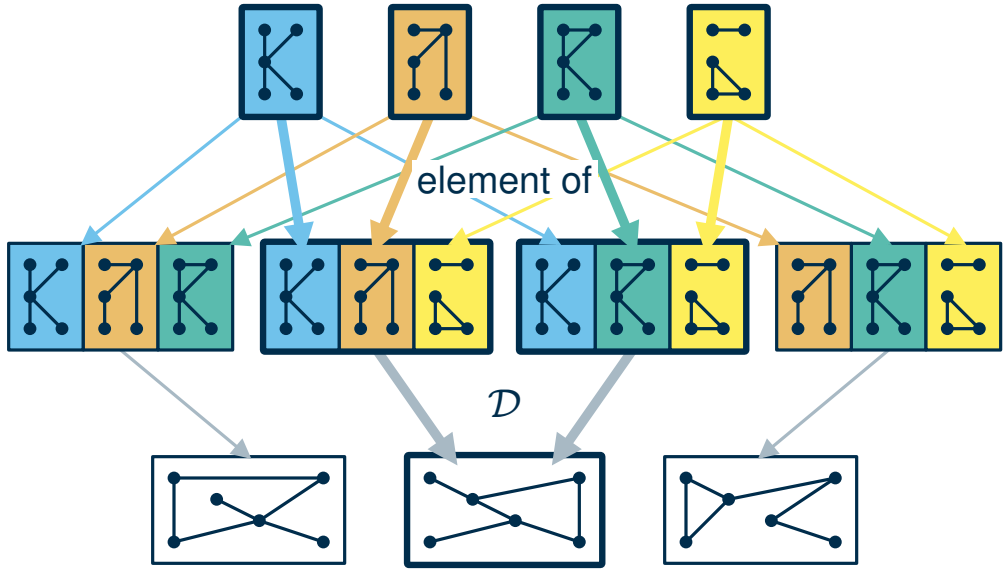

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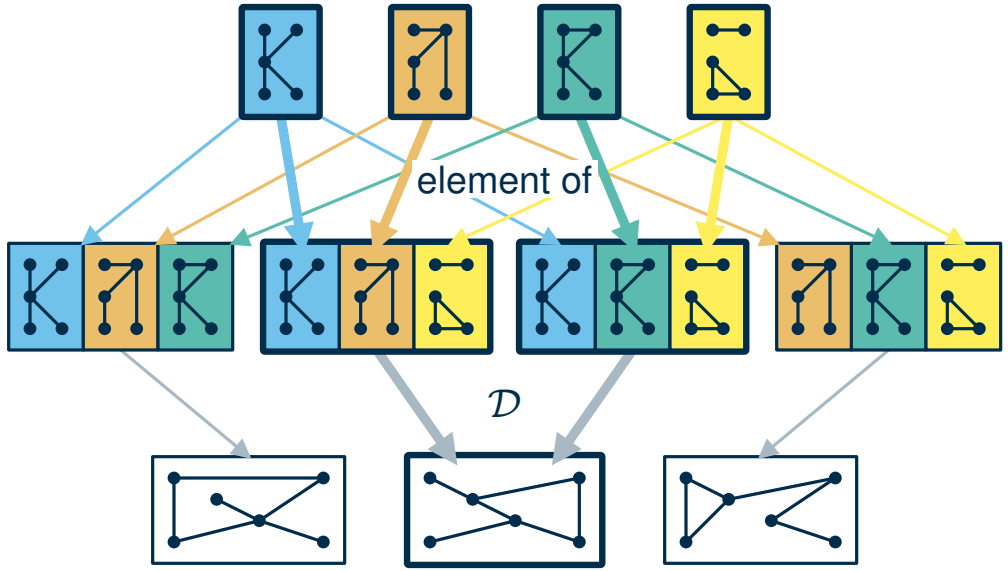

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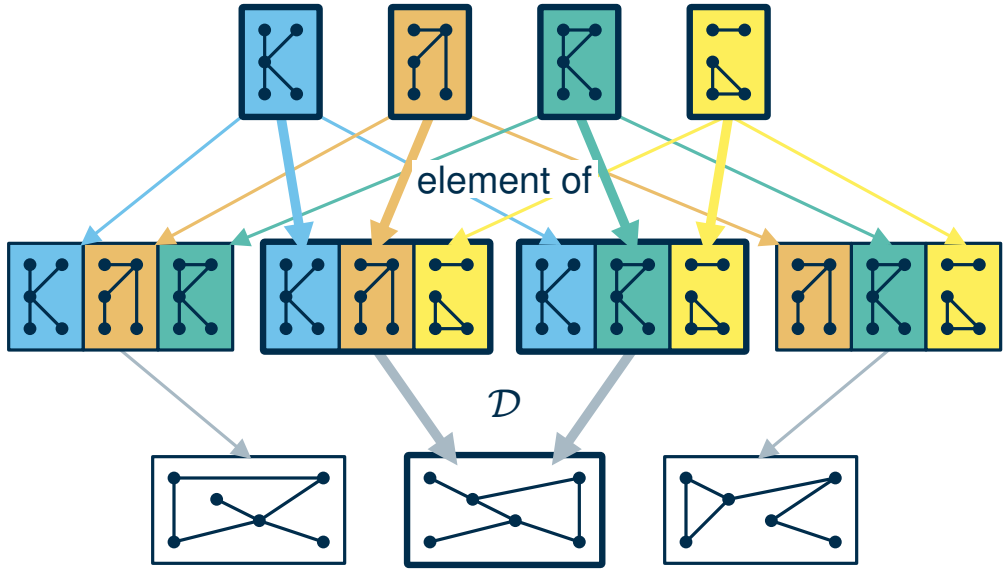

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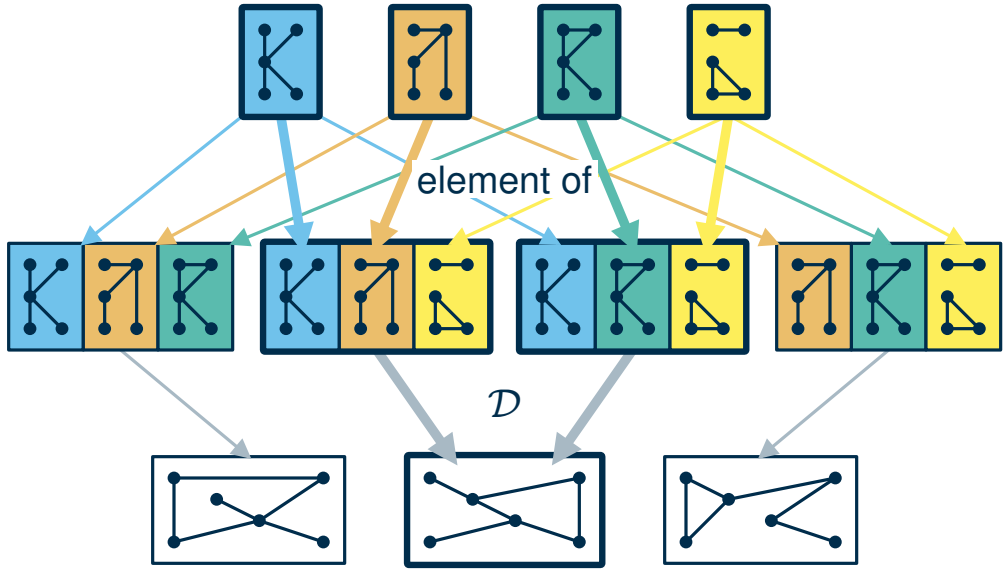
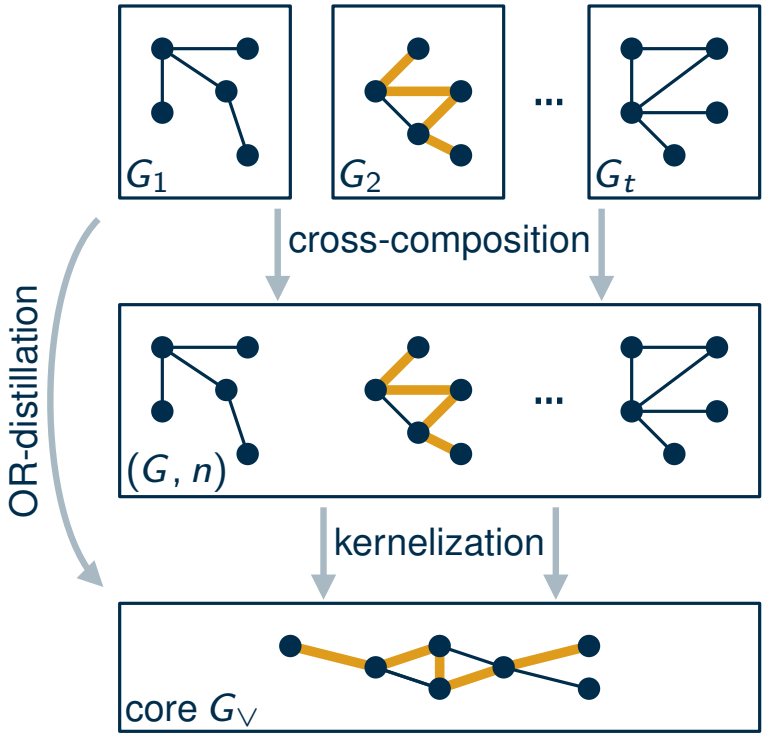


- polynomial kernel for LONGP $\Rightarrow$ OR-distillation $\mathcal{D}$
- lemma applies $\rightarrow$ defines t and advice α_n (poly size)
- coNP/poly-algo for HAMP on n-vertex graph G:
  - guess a t-tuple $(G_1, \ldots, G_t)$ containing G (poly size)
  - run $\mathcal{D}$ on $(G_1, \ldots, G_t) \rightarrow$ core C (runs in poly time)
  - return **no** if $C \in \alpha_n$, and **yes** otherwise
  - if G is a yes-instance, we return **yes**
  - if G is a no-instance, we can guess a tuple that maps to a core in α_n and return **no**
- from HAMILTONIAN PATH being NP-hard and HAMP $\in$ coNP/poly it follows NP $\subseteq$ coNP/poly

Framework

Our proof: polynomial kernel for LongP $\Rightarrow$ OR-distillation for HamP $\Rightarrow$ HamP $\in$ coNP/poly



Framework

specifically done for HAMP and LONGP — nothing special about HAMP here

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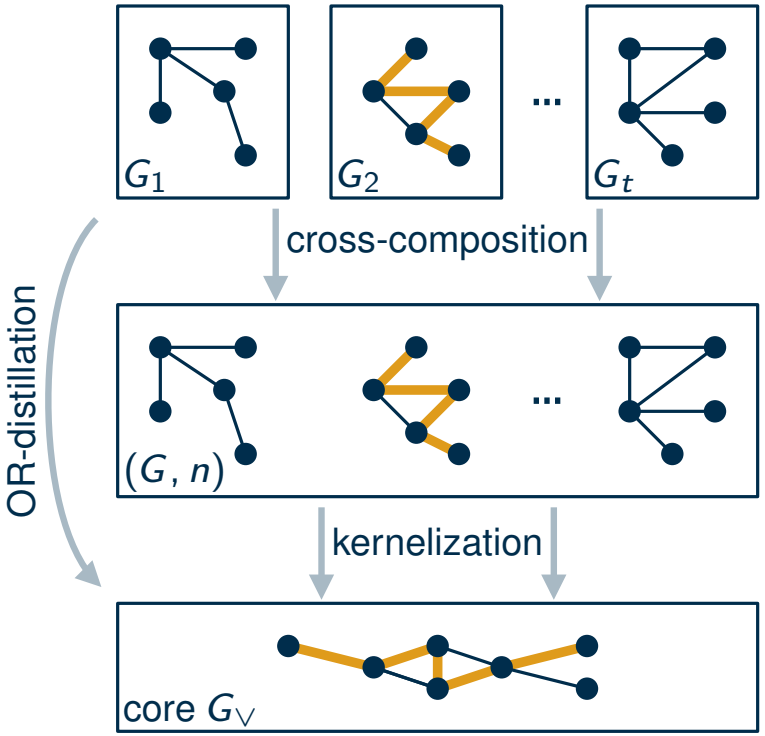

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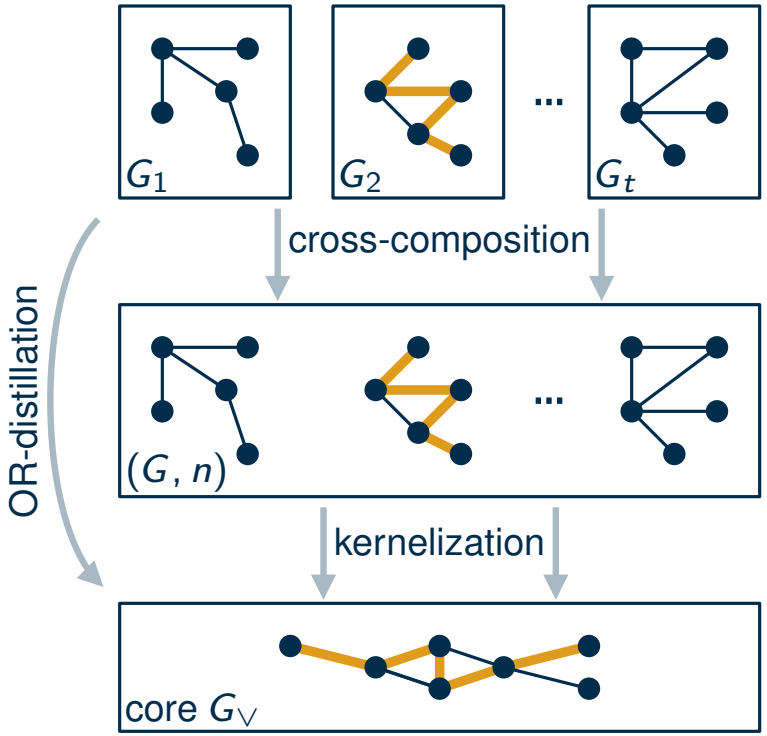
Let L and R be languages. An **OR-distillation** of L into R is an algorithm that takes t similar inputs $x_1, \ldots, x_t$, runs in time $(\sum_{i\in[t]} |x_i|)^{O(1)}$, and produces an output y such that: $|y| \in \max_{i\in[t]} |x_i|^{O(1)}$ and $y \in R \Leftrightarrow \bigvee_{i\in[t]} x_i \in L$.

Theorem
If there is an OR-distillation of L into R then $L \in$ coNP/poly.



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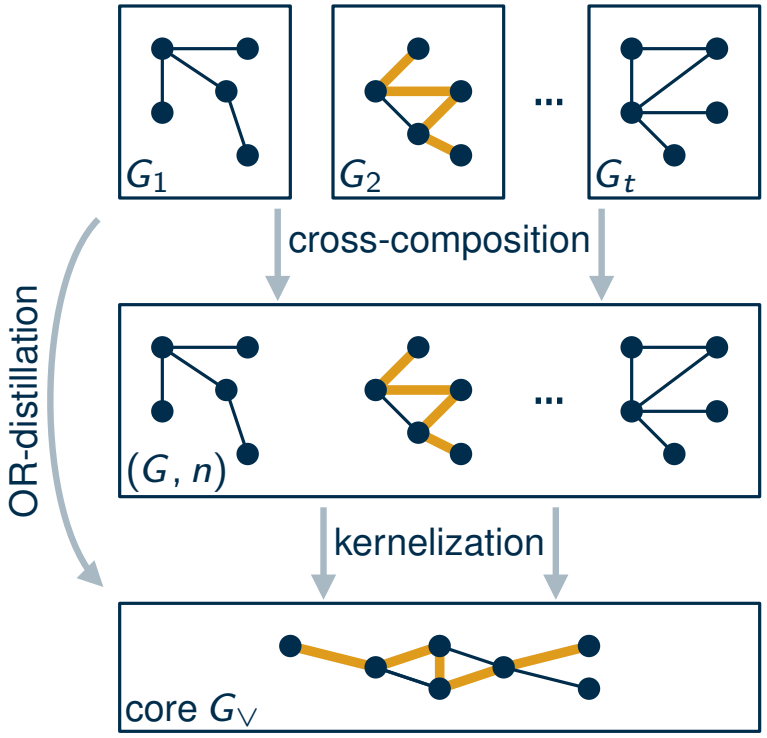

Notes

- "similar" can mean different things (e.g., n vertices)

(formalizable via equivalence relation with polynomially many classes)

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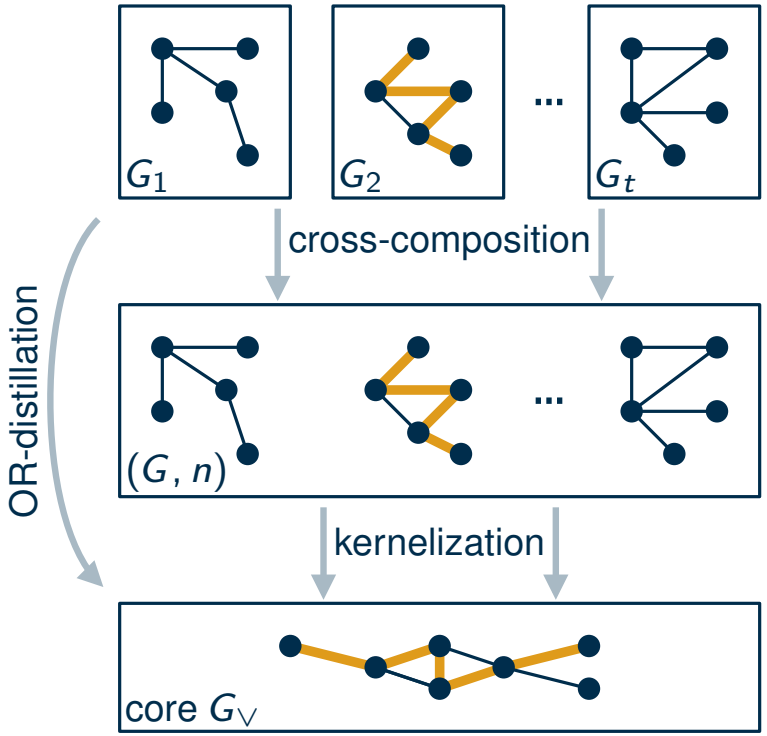


Notes

- "similar" can mean different things (e.g., n vertices) (formalizable via equivalence relation with polynomially many classes)
- R does not even need to be decidable $\rightarrow$ essentially an information-theoretic result

Taking a break and splitting sets

Input: Hypergraph

- vertex set V consists of:
  - $v_0, \ldots, v_3$
  - x_0, x_1, y_0, y_1, z_0, z_1
- 11 hyperedges: see picture

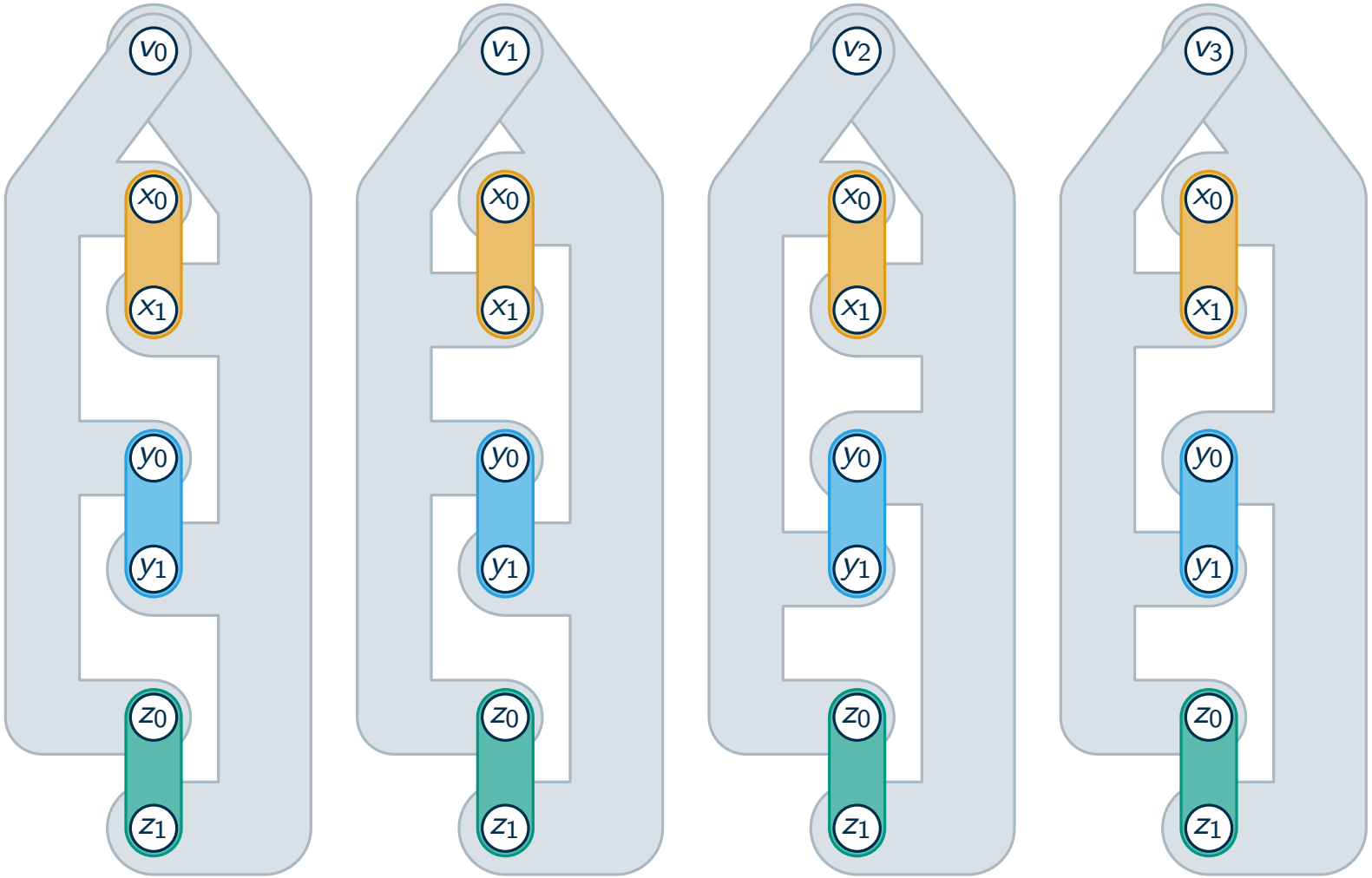
(x_i, y_i, z_i and their edges are shown multiple times)

SET SPLITTING

- find vertex set $X \subseteq V$ that splits **every** hyperedge

(every edge has a vertex from X and from $V \setminus X$)

Is this a yes-instance?



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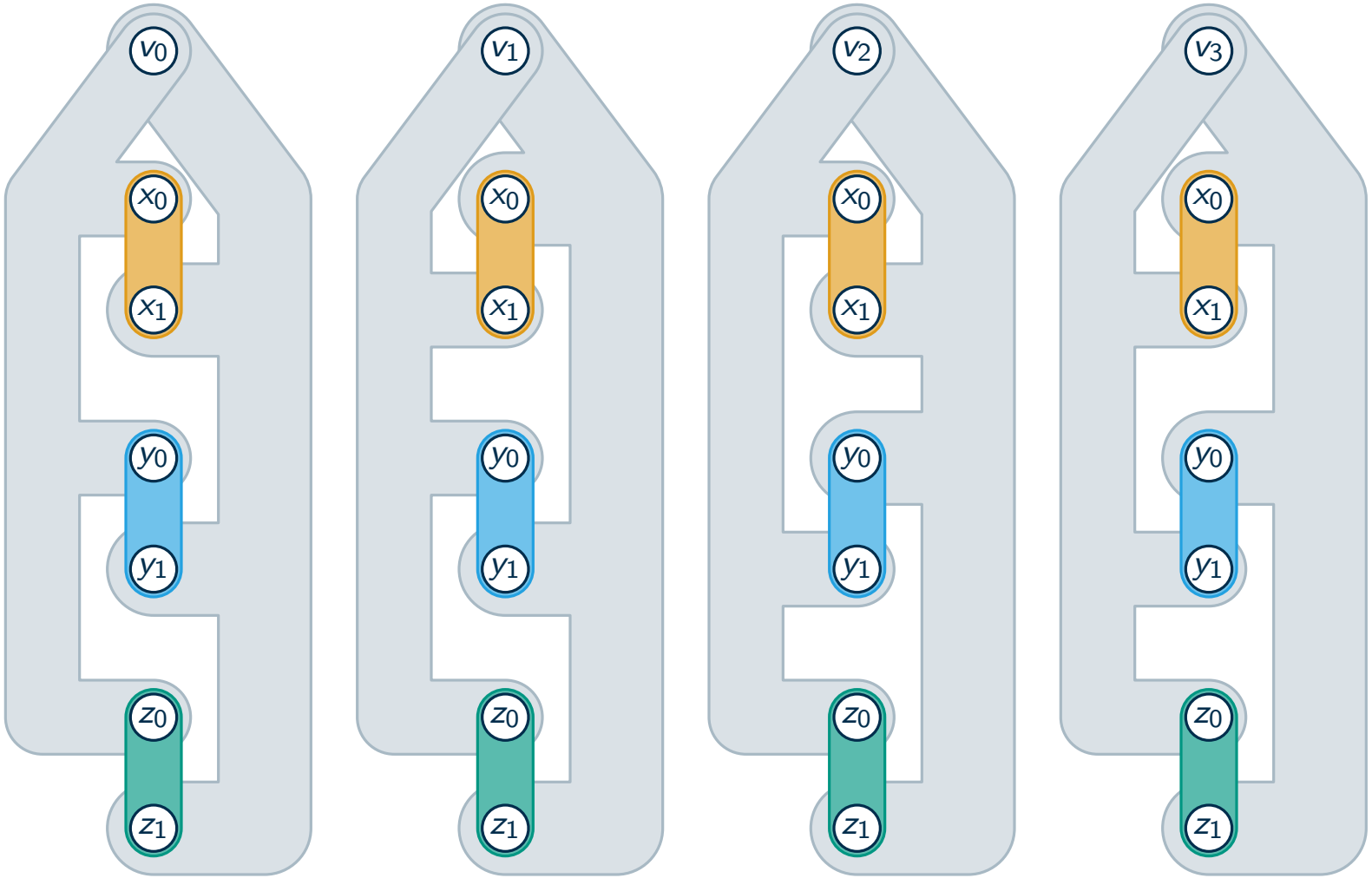

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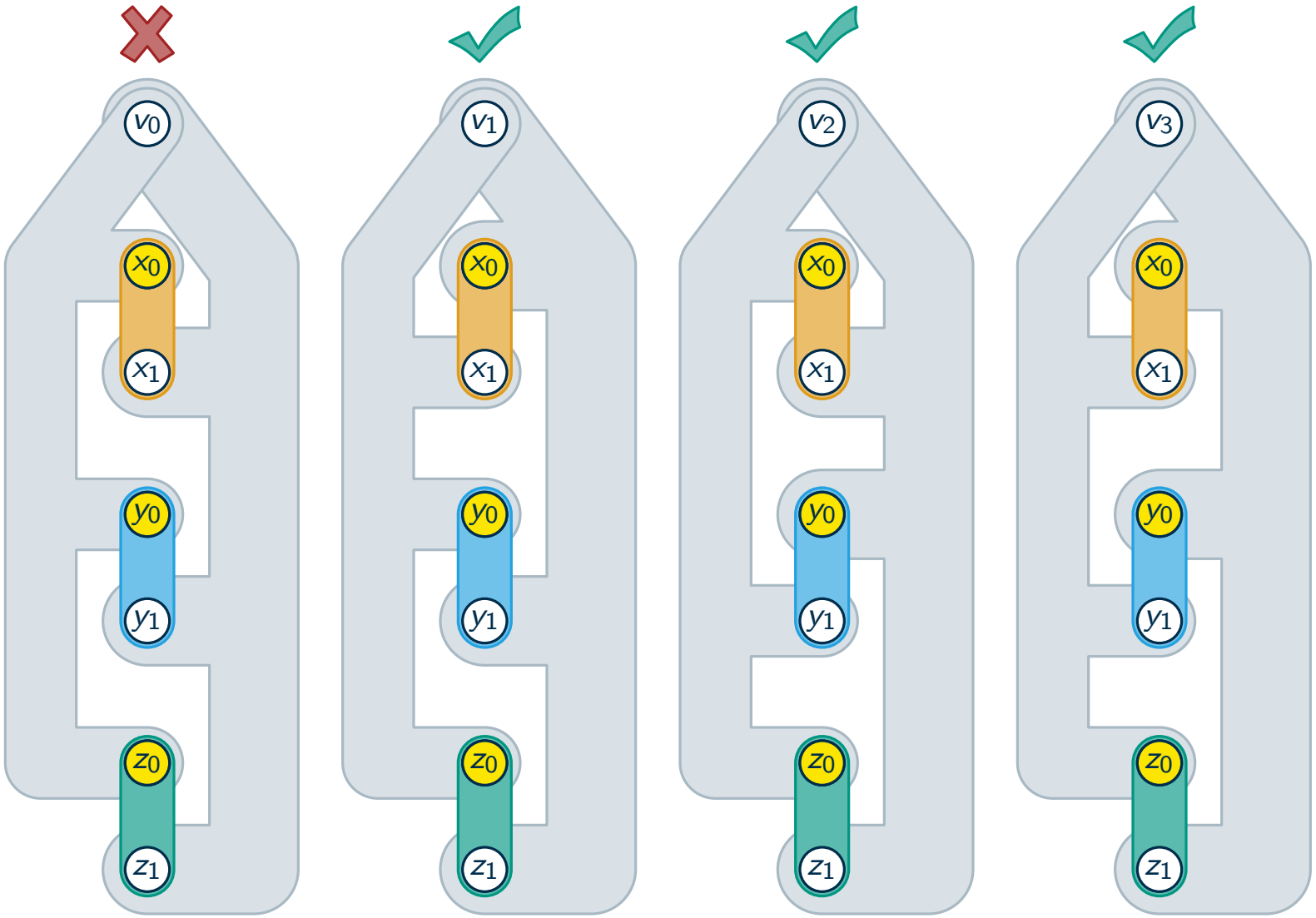

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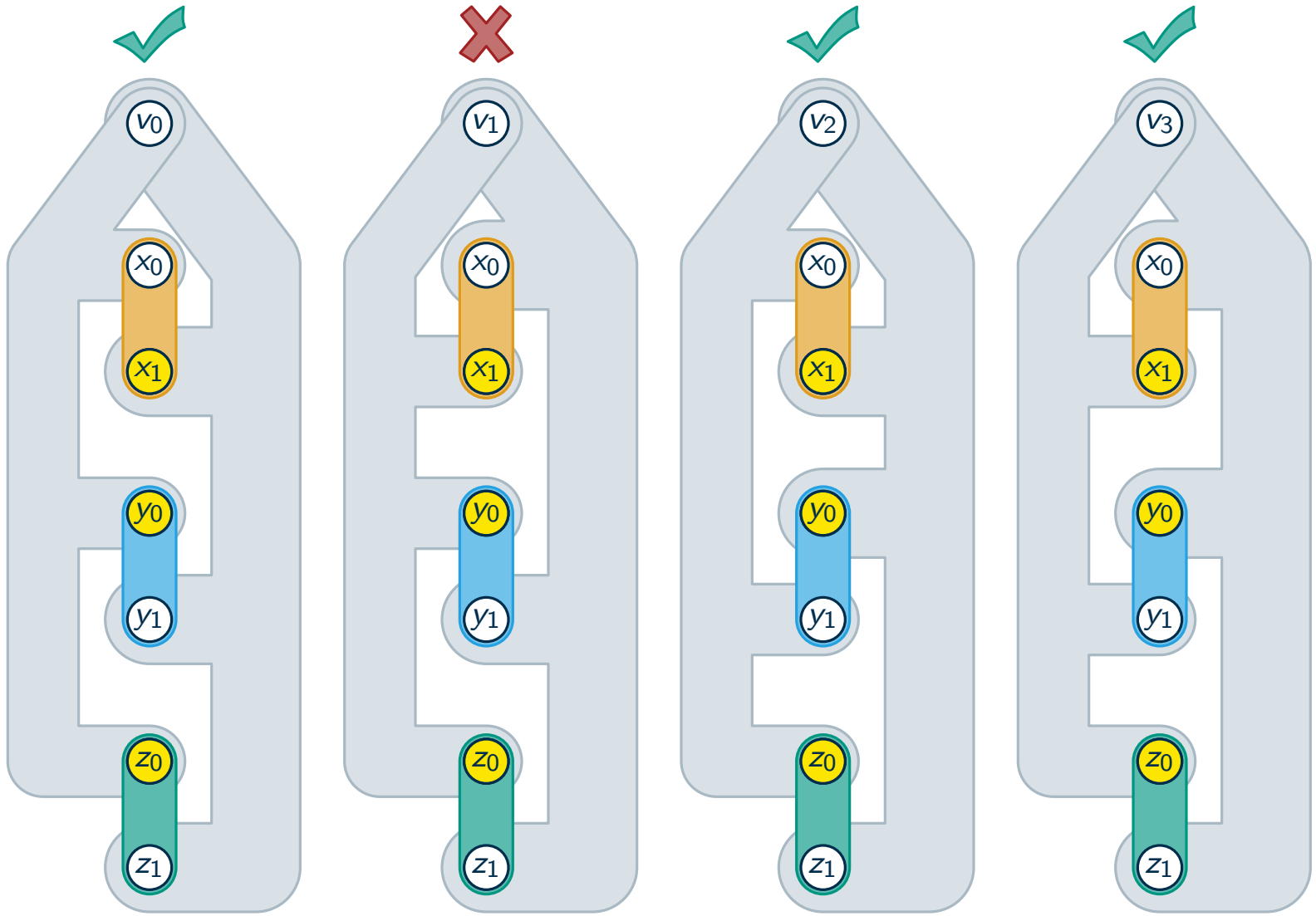

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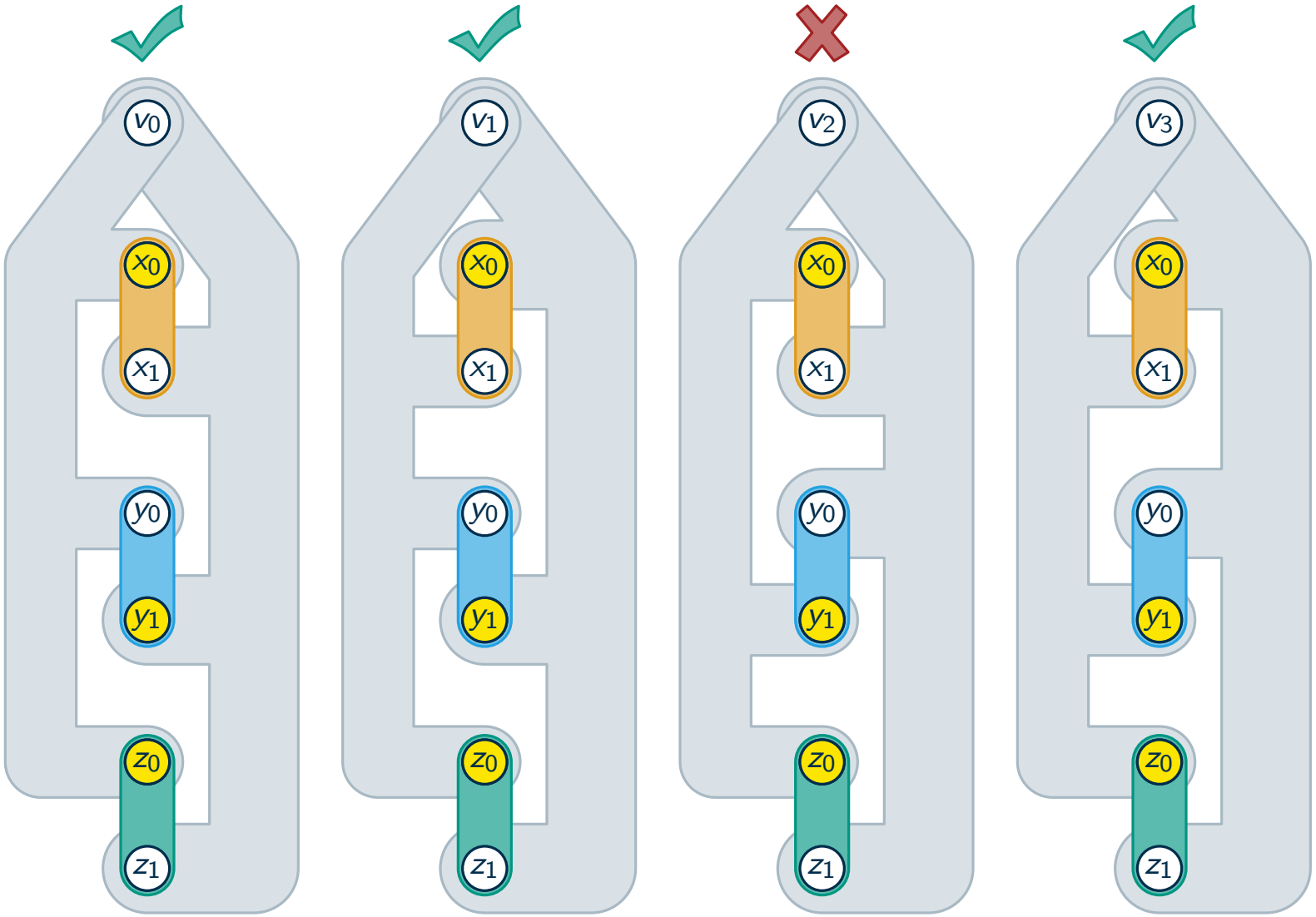

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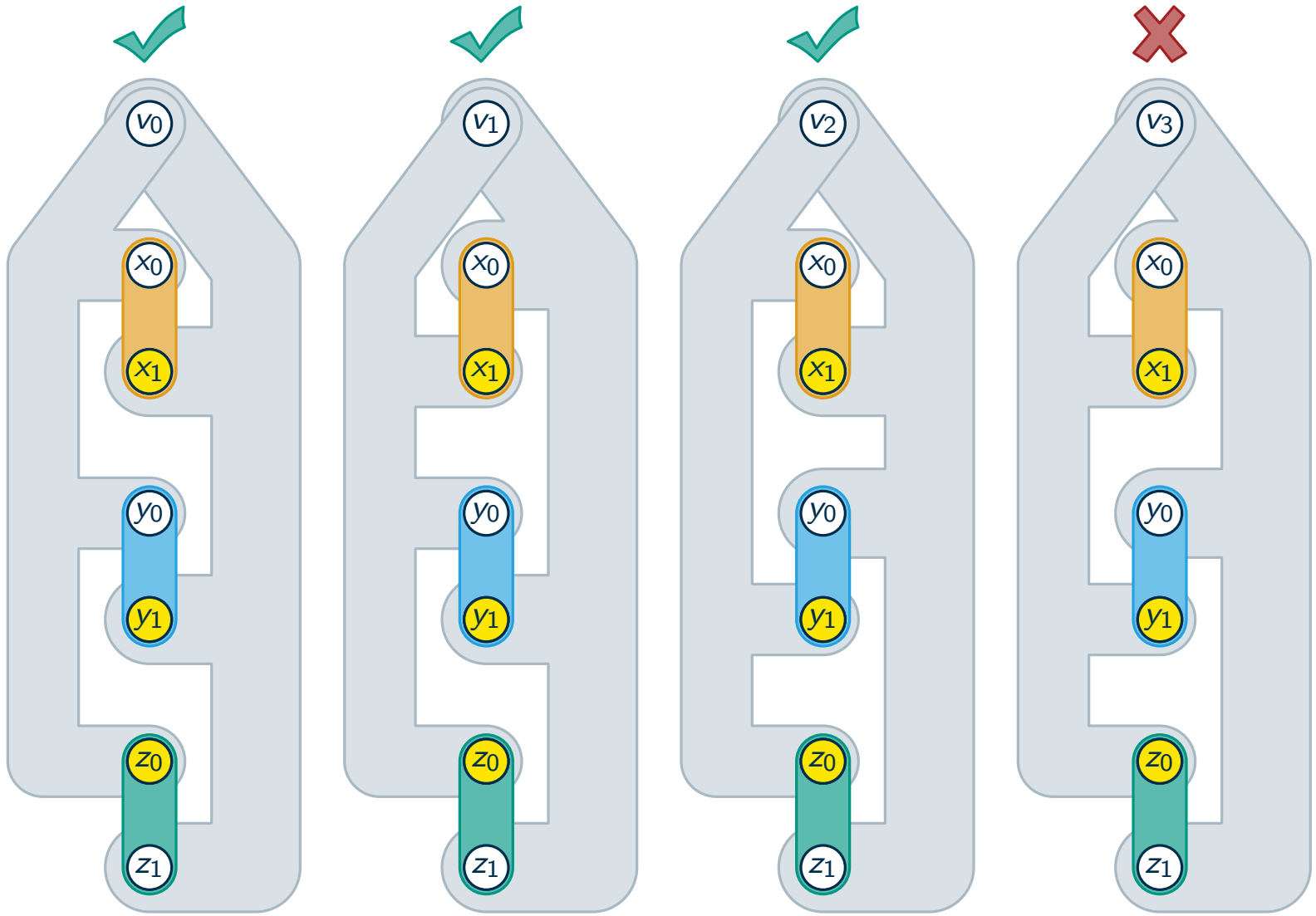

SET SPLITTING has no polynomial kernel

Problem: SET SPLITTING
Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$. Is there a set $X \subseteq V$ that splits all hyperedges of $\mathcal{H}$?

Goal: show that SET SPLITTING parameterized by $n = |V|$ has no polynomial kernel (unless...)

Set Splitting has no polynomial kernel

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Plan

- assume there is a kernelization algorithm producing a polynomial kernel
- show that this implies an OR-distillation from Set Splitting into itself
- as Set Splitting is NP-hard, this would imply NP $\subseteq$ coNP/poly

OR-Distillation of Set Splitting into itself

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Given

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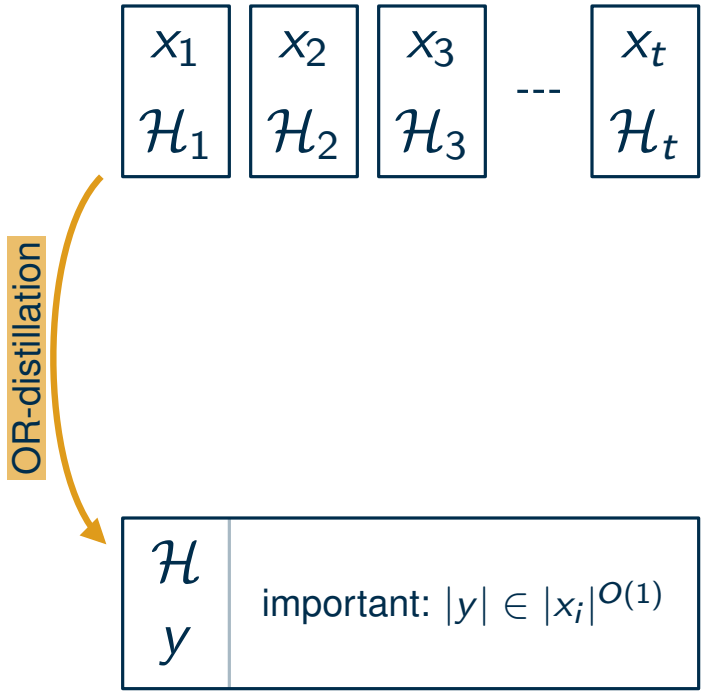

Todo

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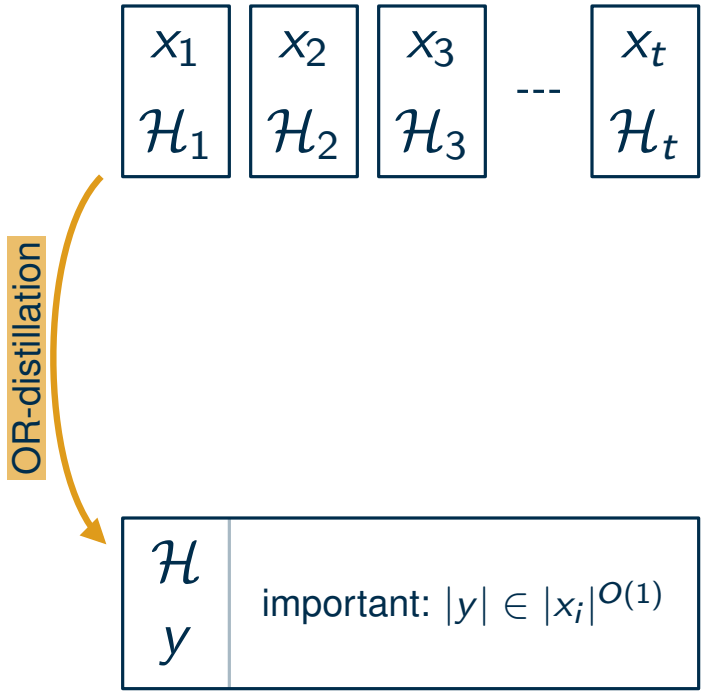

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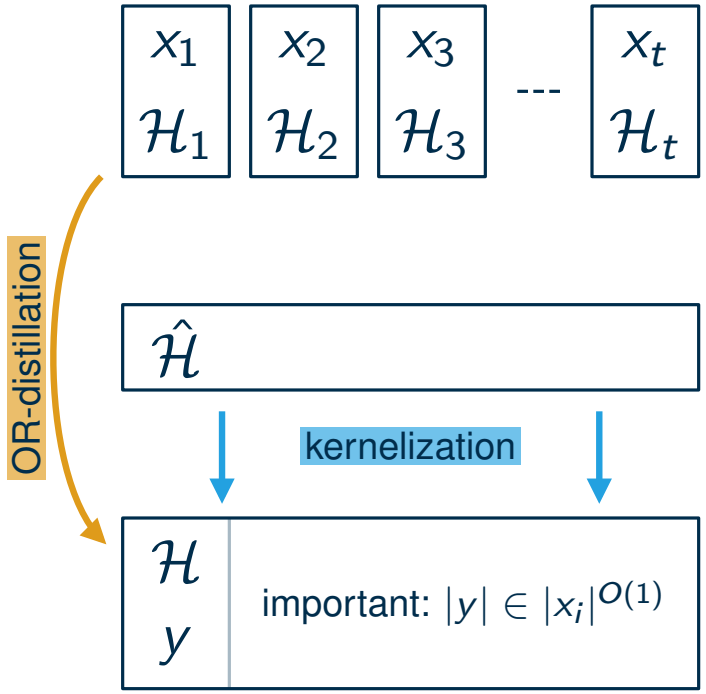

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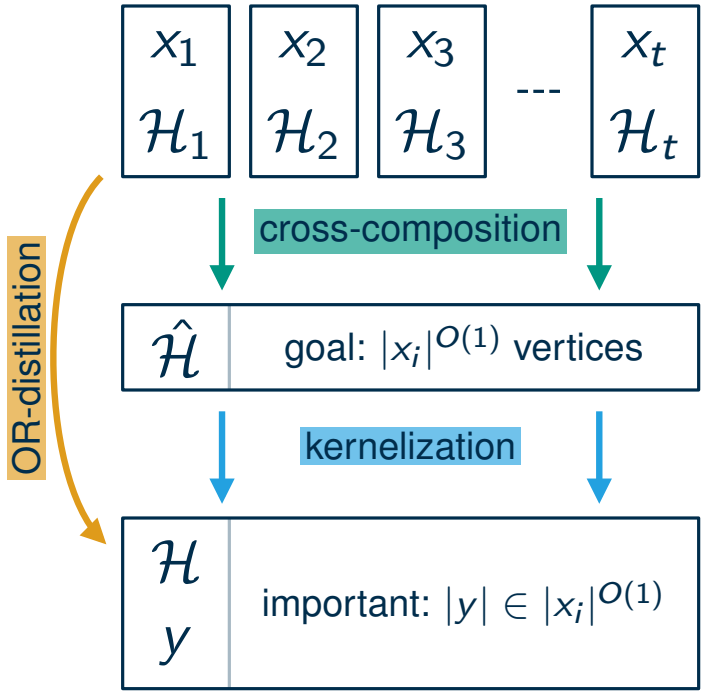

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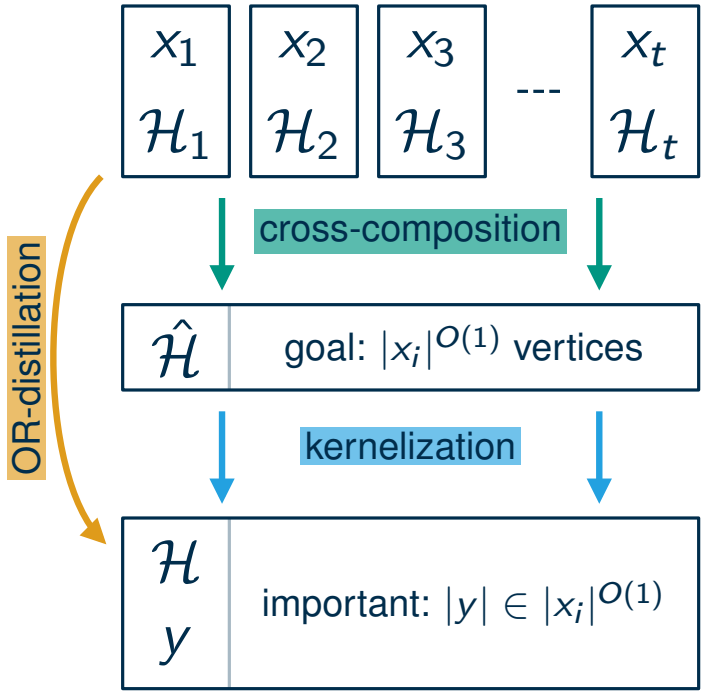
- important: number of vertices of $\hat{\mathcal{H}}$ must not depend on t



Cross-Composition

Goal (reminder):

- given $\{\mathcal{H}_1, \ldots, \mathcal{H}_t\}$ (n vertices each)
- build $\hat{\mathcal{H}}$ with $n^{O(1)}$ vertices
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Construction with selector gadget for $t = 4$

- let $V = \{v_1, \ldots, v_n\}$ for all $\mathcal{H}_i \to \hat{V} = V \cup \{x_0, x_1, y_0, y_1, z_0, z_1\}$
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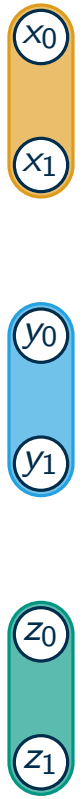

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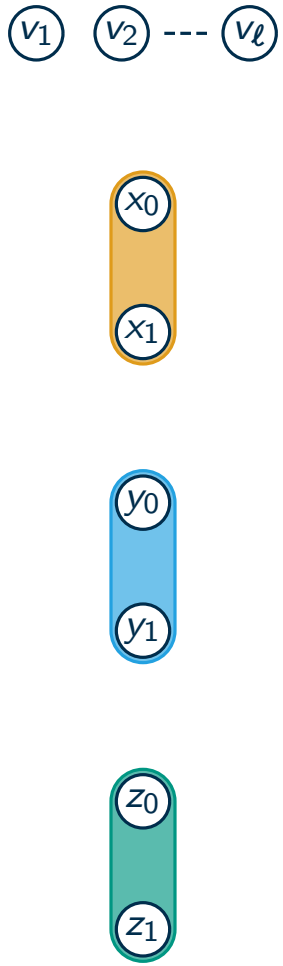

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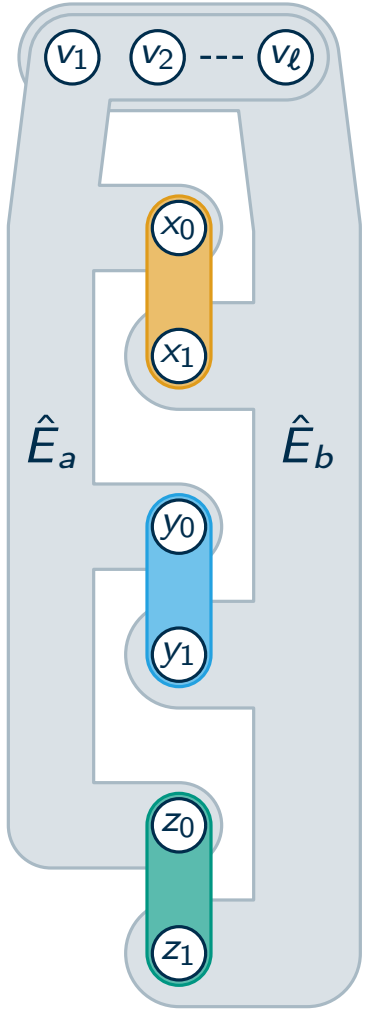

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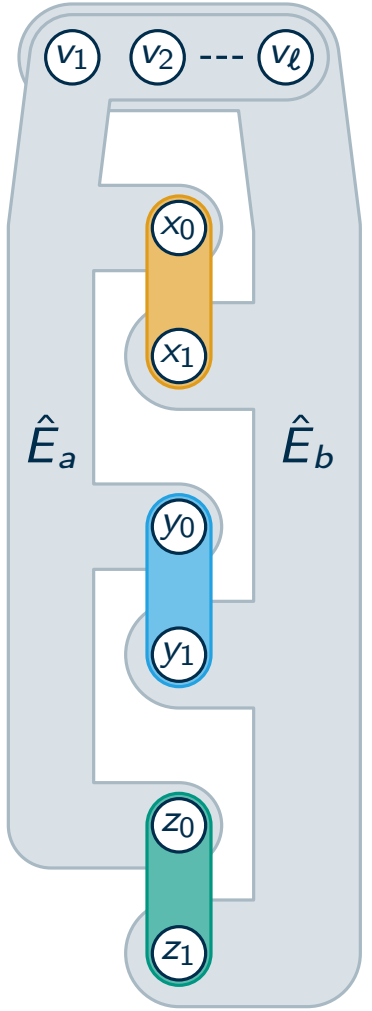

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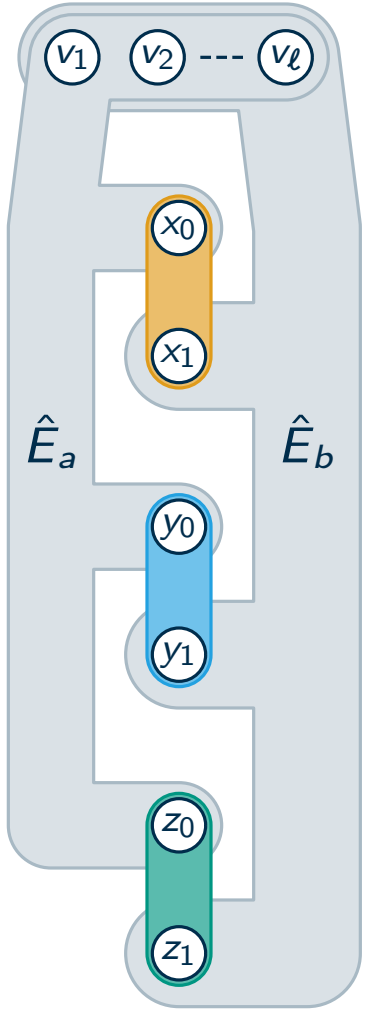

Observation for a solution candidate $X \subseteq \hat{V}$

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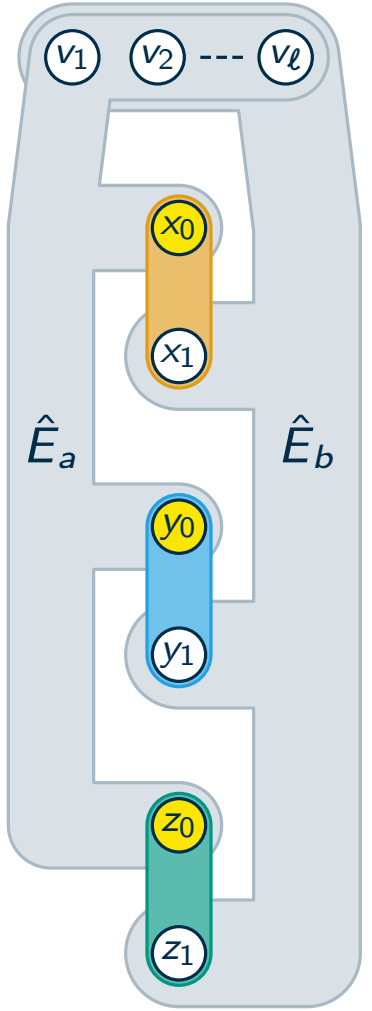

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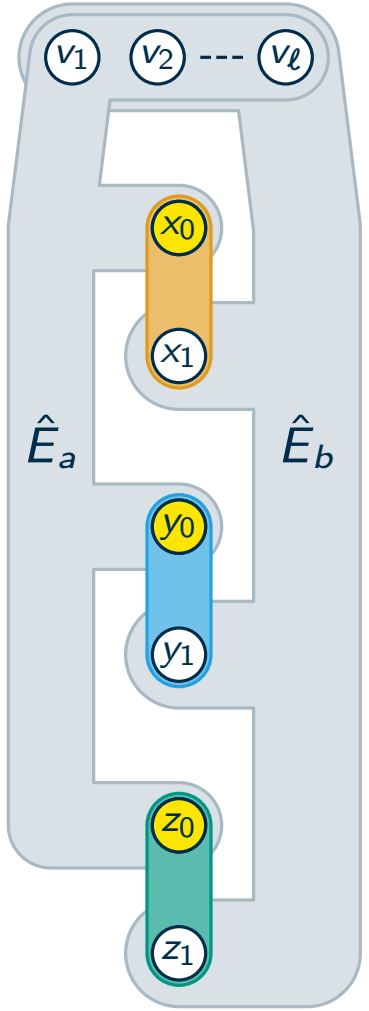

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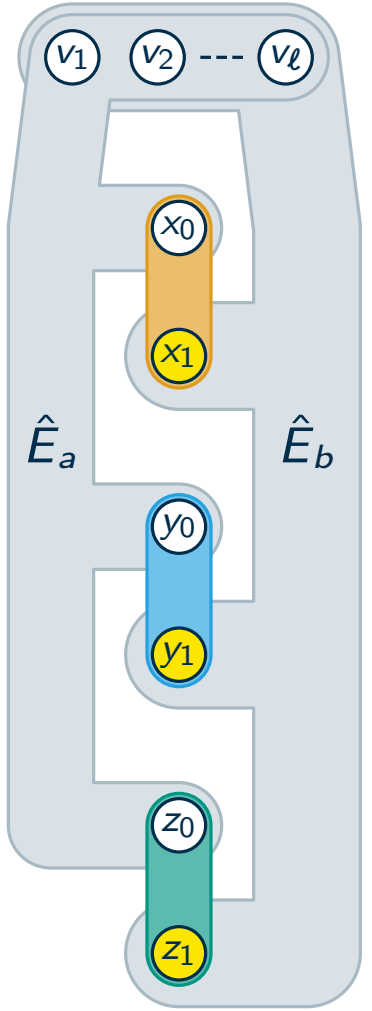

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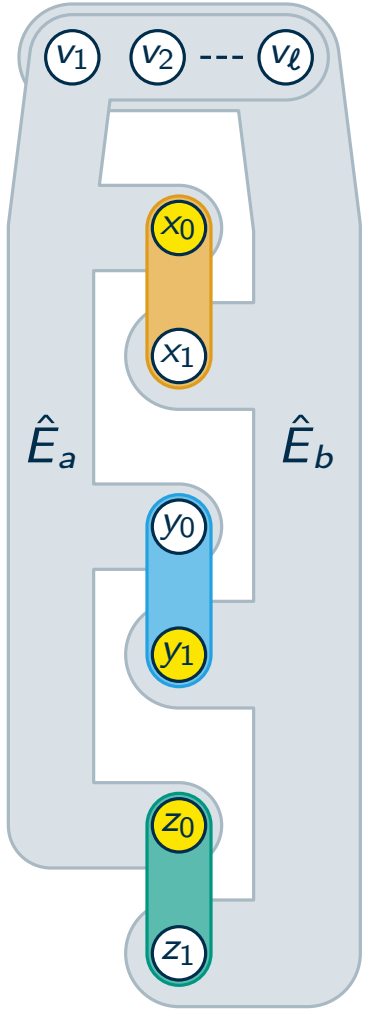

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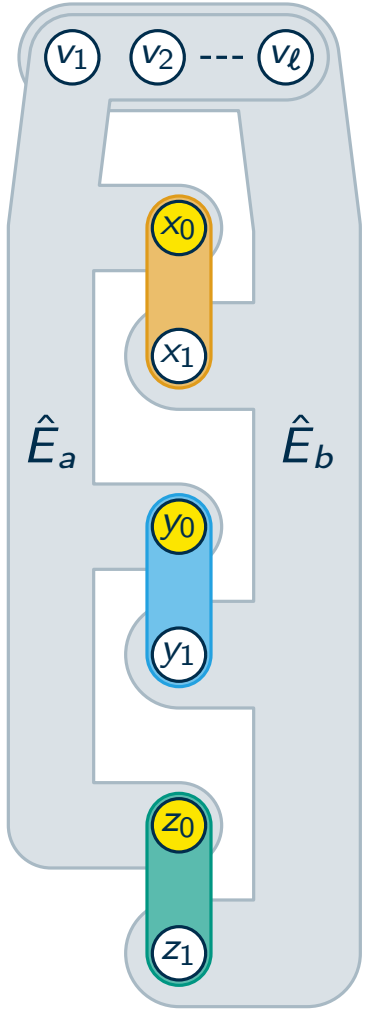

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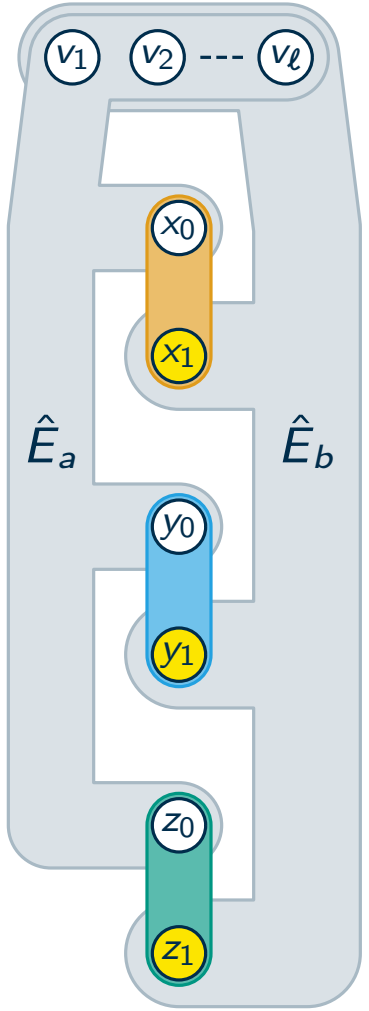

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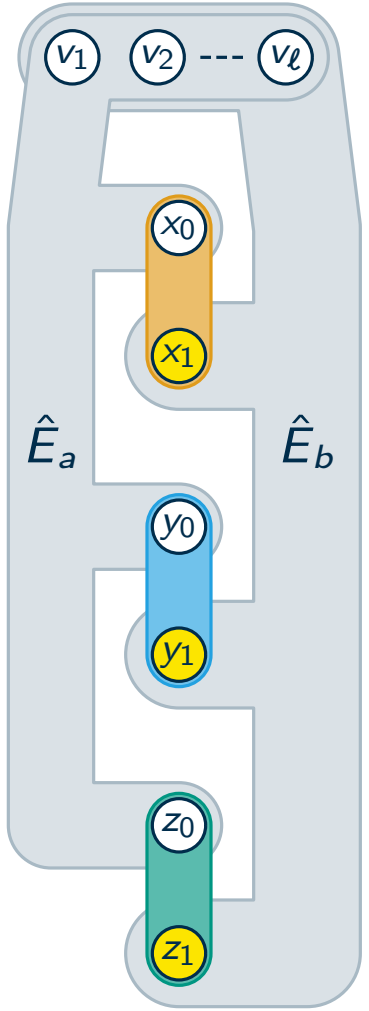

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Concluding remarks on Set Splitting

Definition
An **OR-distillation** of L into R takes t similar inputs $x_1, \ldots, x_t$, runs in time $(\sum |x_i|)^{O(1)}$, and produces an output y such that: $|y| \in \max |x_i|^{O(1)}$ and $y \in R \Leftrightarrow \bigvee_{i \in [t]} x_i \in L$.

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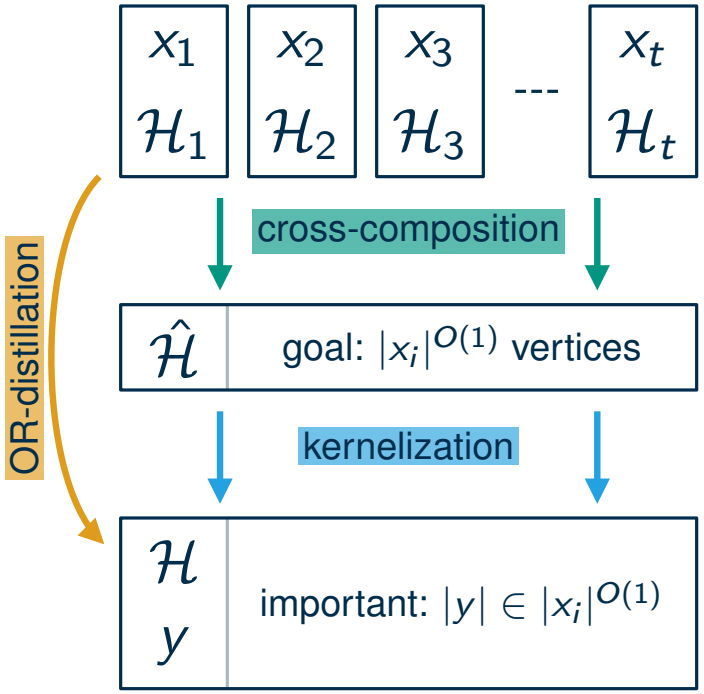

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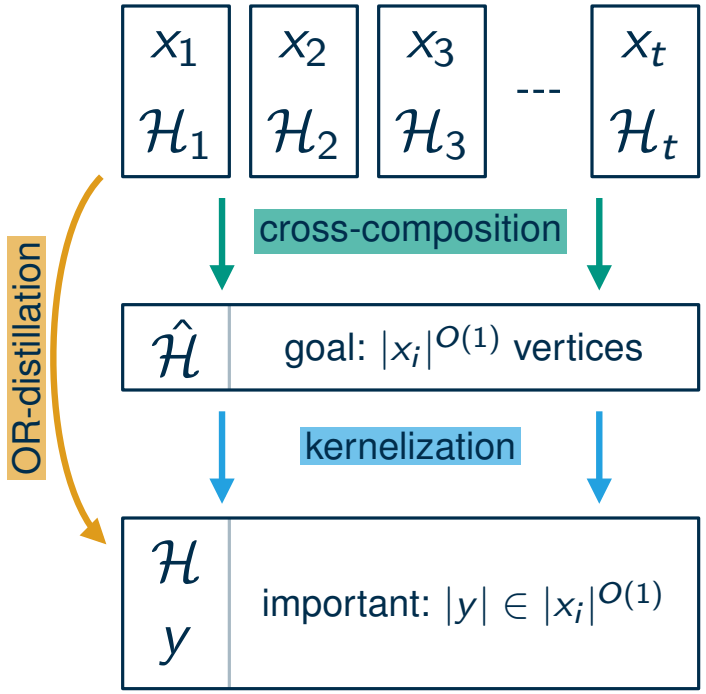

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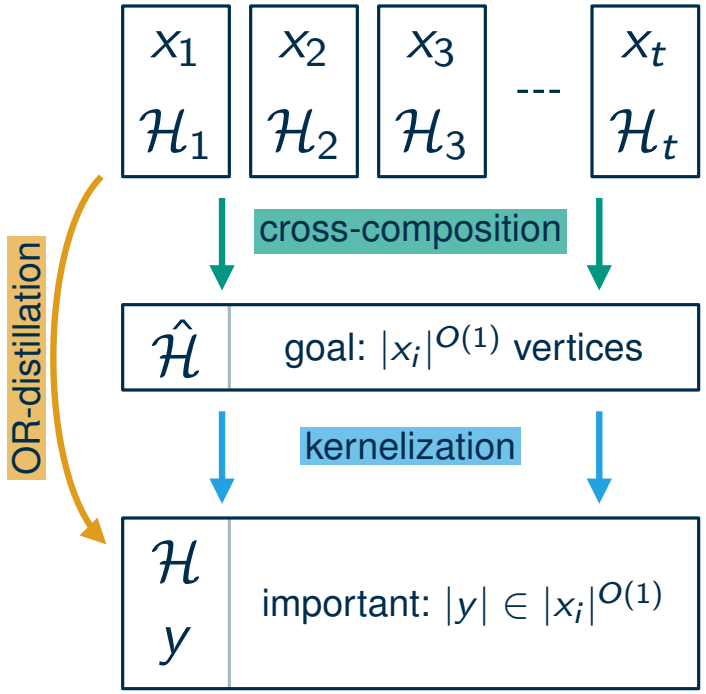

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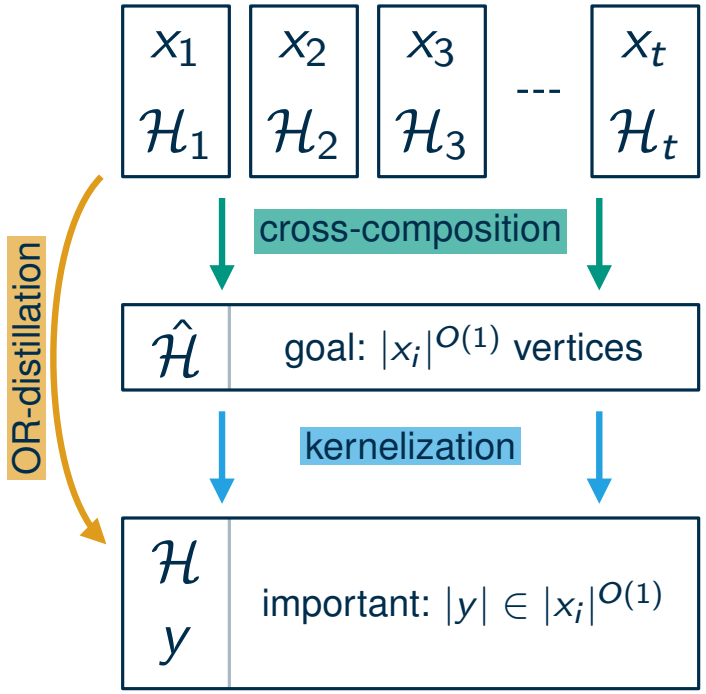

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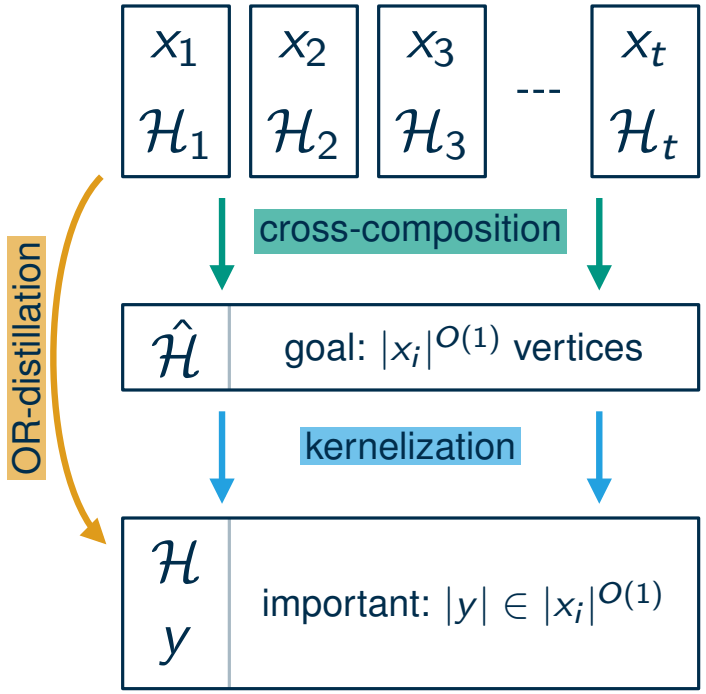

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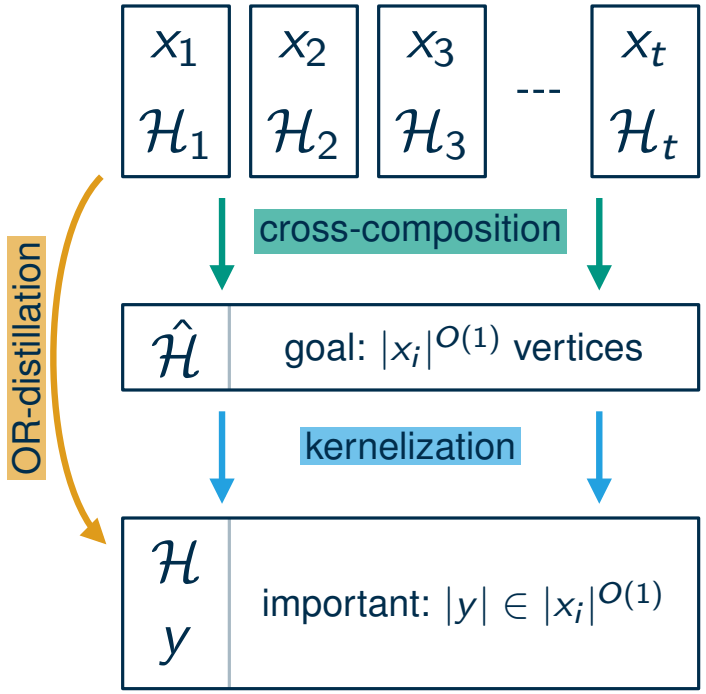

Theorem
If Set Splitting has a polynomial kernel then NP $\subseteq$ coNP/poly.

Wrap-Up

Kernelization

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Proving kernel lower bounds

- cross-composition + kernelization gives OR-distillation
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