Parameterized Algorithms

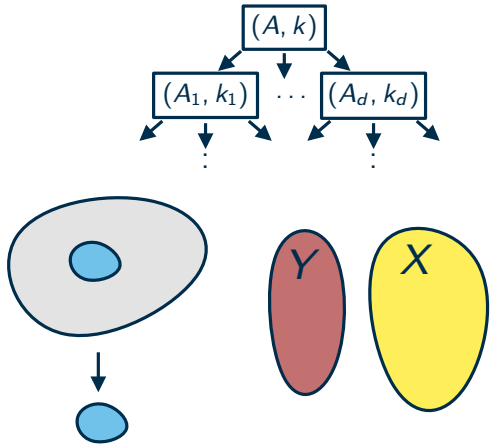
Color Coding

Thomas Bläsius

Content

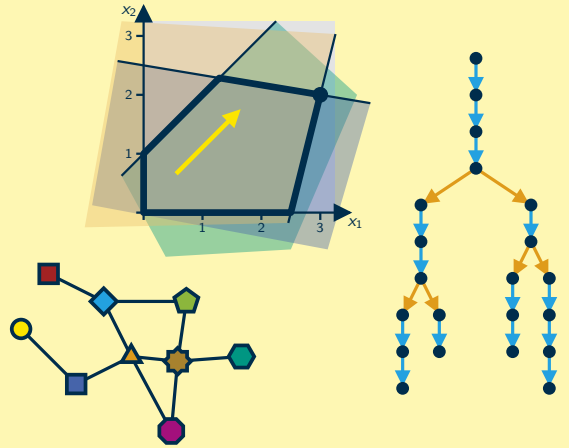
Basic toolbox

- bounded search trees
- kernelization
- iterative compression



Extended toolbox

- linear programs
- branch-and-reduce
- color coding

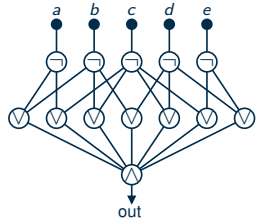


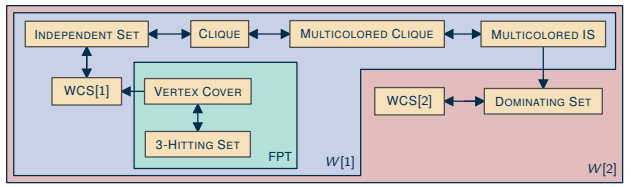
Tree width

- dynamic programming
- chordal and planar graphs
- Courcelle's theorem

Lower bonds

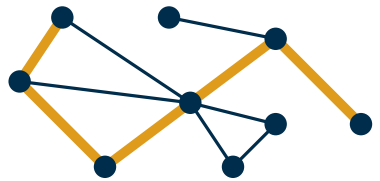
- kernel lower bounds
- parameterized reductions
- circuits and the W-hierarchy
- ETH and SETH





Best detours

Problem: LONGEST PATH
Given a graph G and a parameter k. Does G have a k-vertex path?



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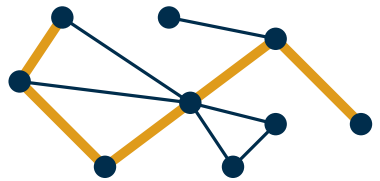

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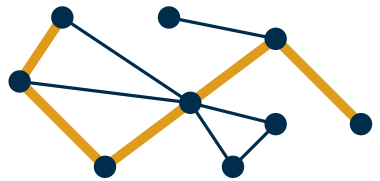

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For directed acyclic graphs (DAGs)

Can we do this in polynomial time?

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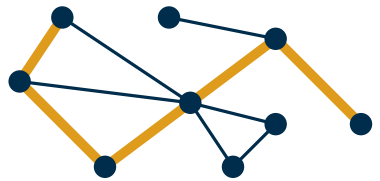

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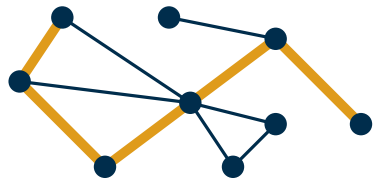

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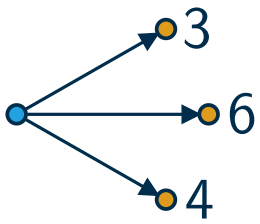

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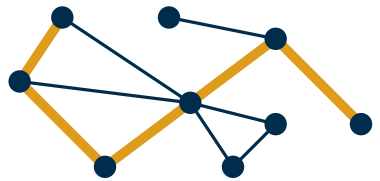

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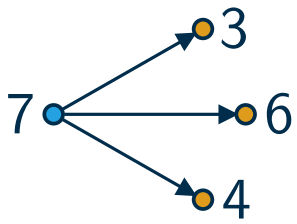

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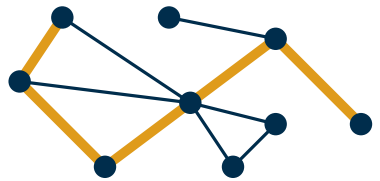

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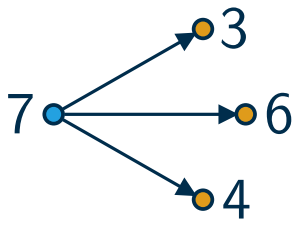

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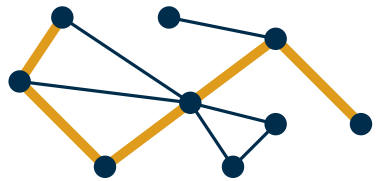

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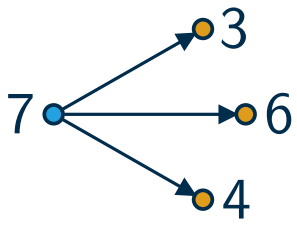
  - let A be the adjacency matrix and consider A^k
  - entry (u, v) equals the number of paths of length k from u to v



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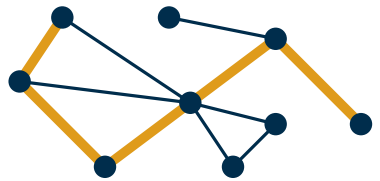

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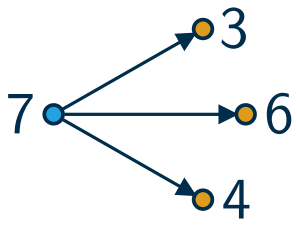
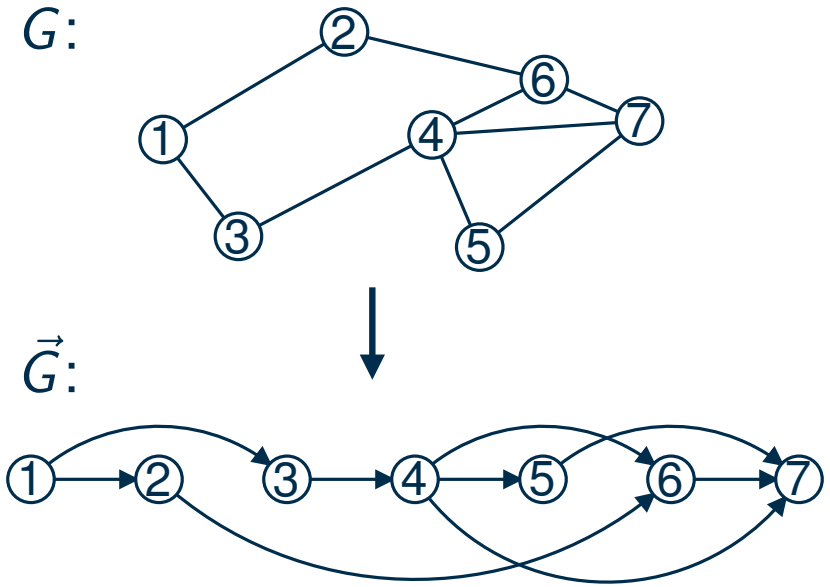


Why does this not work for undirected graphs?

Preventing cycles

Idea for undirected graphs

- transform the undirected graph G into a DAG $\vec{G}$
- find a longest path in the DAG $\vec{G}$



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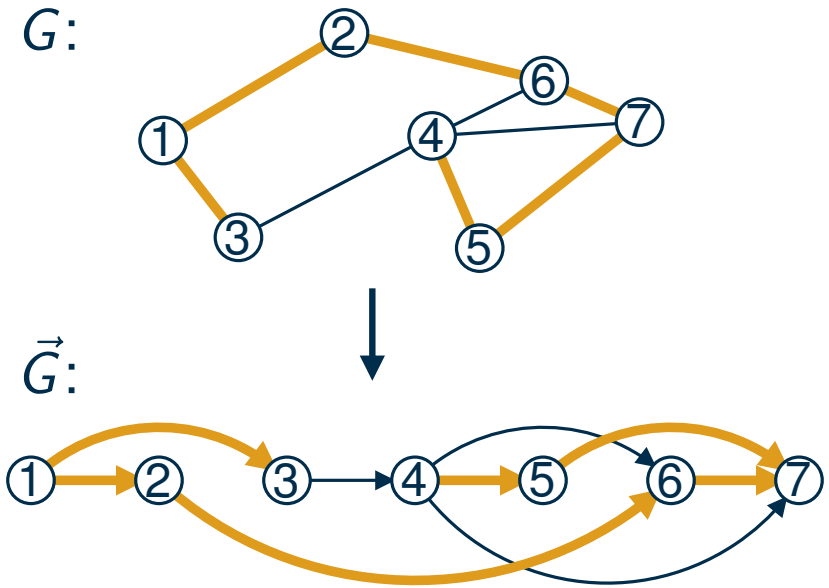

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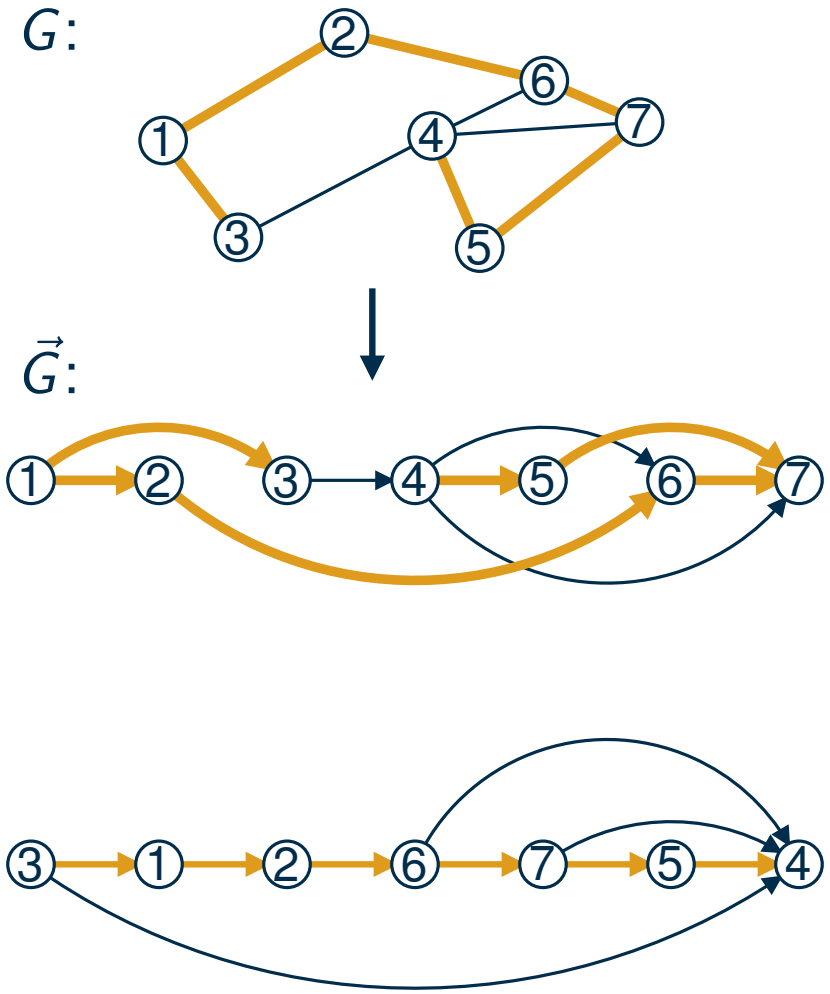

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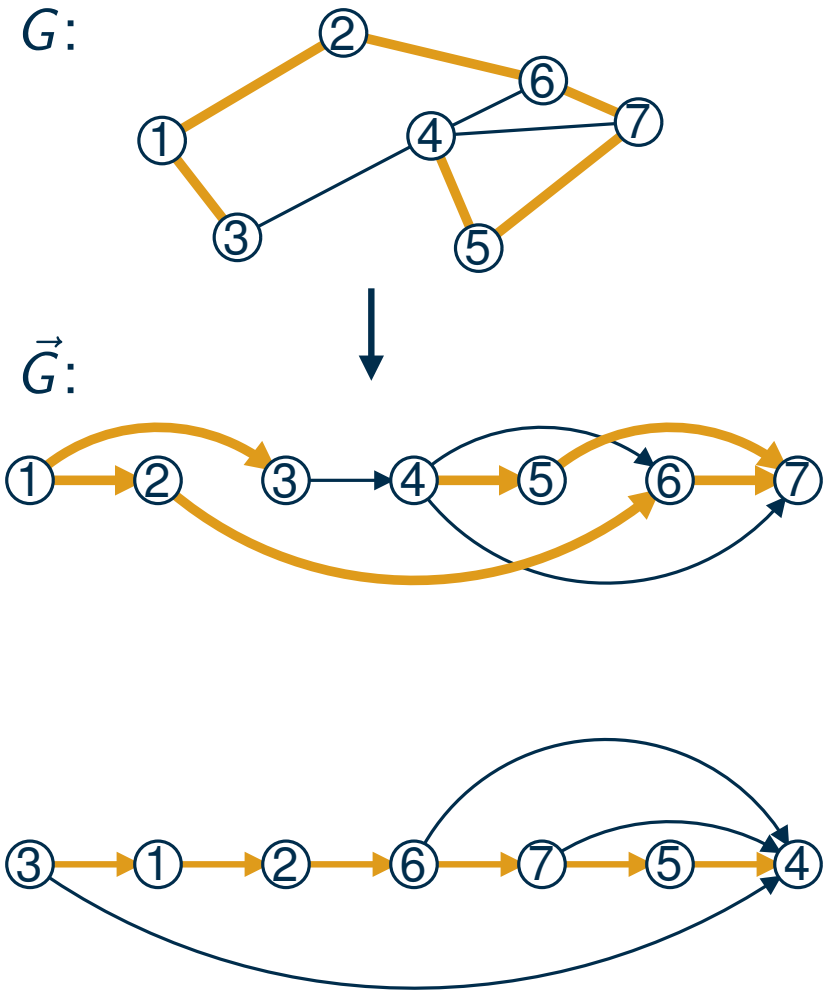

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How do we find a good orientation?

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Construction of the DAG $\vec{G}$

- choose random order for the vertices of G
- direct all edges in G from front to back

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Comments

- success probability only depends on k
- probability can be boosted by repeating the process
- to get constant probability: number of repetitions only depends on k

Randomized FPT algorithm

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With $k!$ independent trials of orienting $\vec{G}$, the probability to correctly solve LONGEST PATH for (G, k) is larger than $\frac{1}{2}$.

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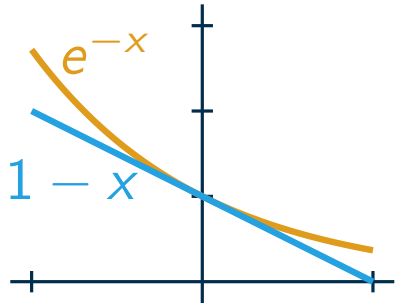

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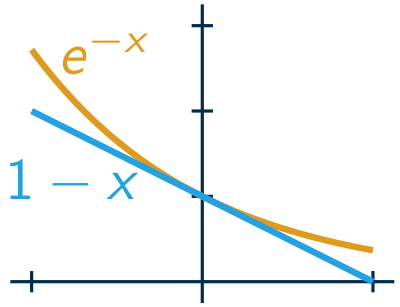

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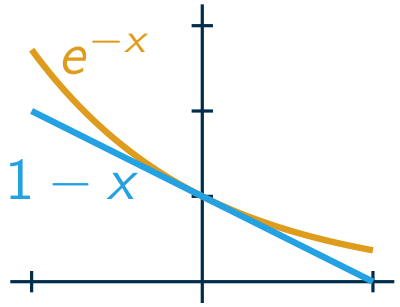

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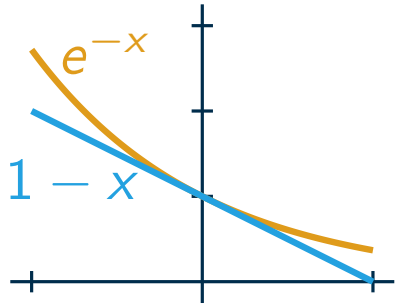

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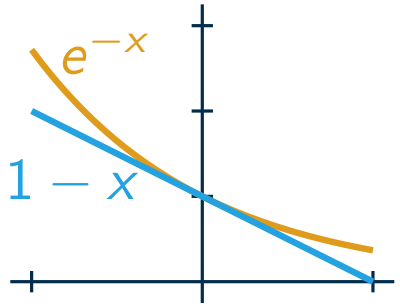

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("with high probability" for $\ell = \log n$)

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What did we just do?

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- for each solution: number of choices for the structure on the solution only depends on k

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(a path is colorful if it contains at most one vertex from each V_i)

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- show that the colors help: FPT-algo for colorful problem
- random colors $\rightarrow$ error probability only depends on k

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- wish for some additional structure (vertex order) that makes the problem easier
- for each solution: number of choices for the structure on the solution only depends on k
- probability for bad choice only dependent on k
- probability-boosting: number of trials only depends on k

Less order, more colors

- color the vertices
- only allow colorful solutions
- show that the colors help: FPT-algo for colorful problem
- random colors $\rightarrow$ error probability only depends on k
- afterwards: derandomization

Problem: Colorful Longest Path
Given a graph $G = (V, E)$, a parameter k, and a partition (coloring) $V_1, \ldots, V_k$ of V, is there a colorful k-vertex path?
(a path is colorful if it contains at most one vertex from each V_i)

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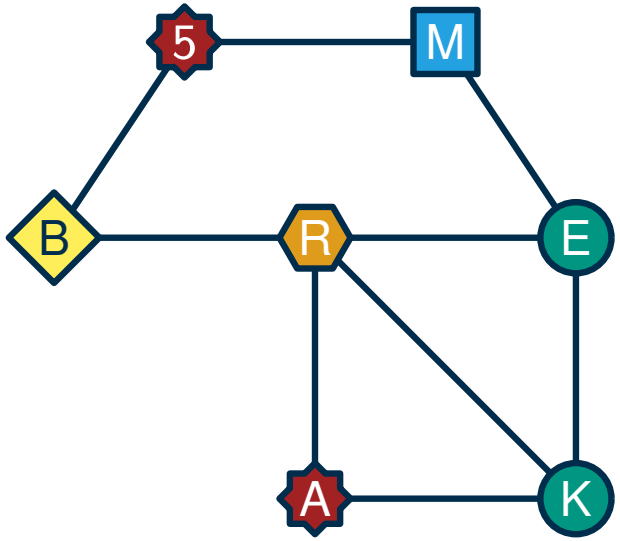
How long is the longest colorful path?



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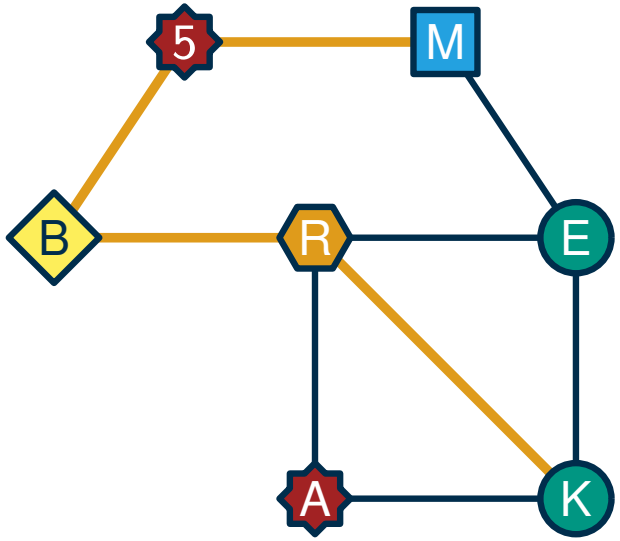

How do colors help?

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Solution for DAGs

- path of length ℓ from v and $uv \in E \Rightarrow$ path of length $\ell + 1$ from u

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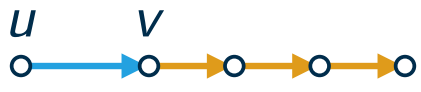

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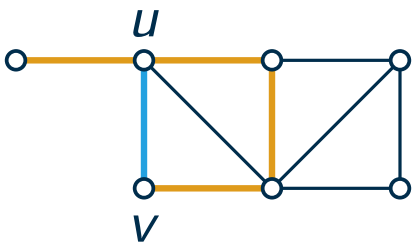

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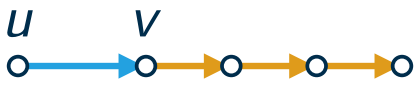

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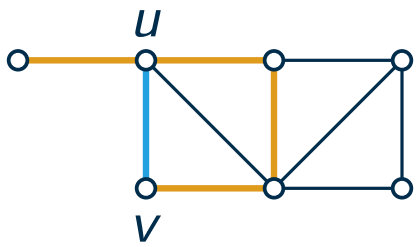

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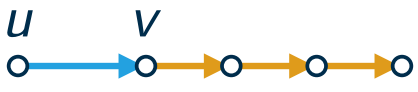

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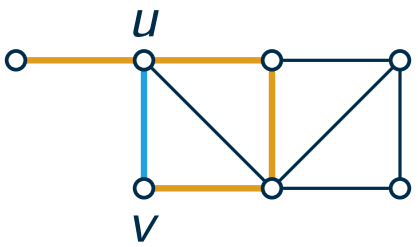

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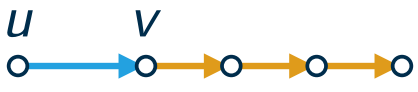

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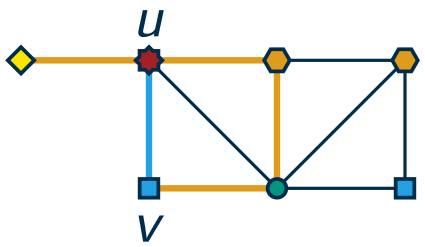

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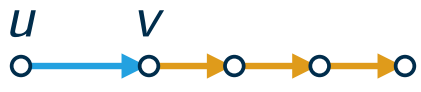

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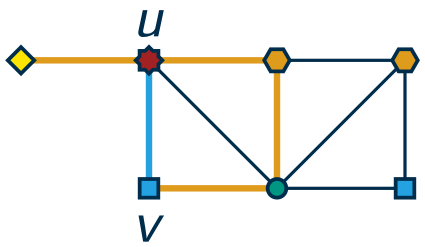

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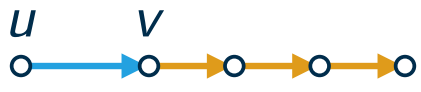

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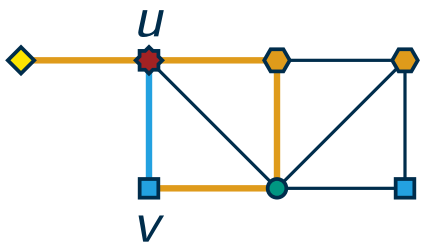

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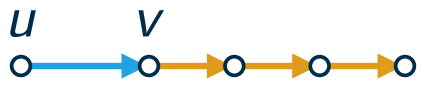

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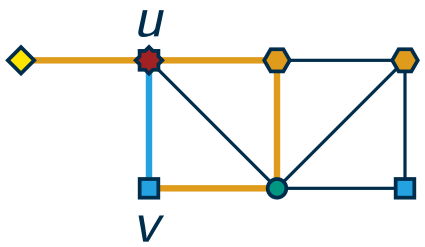

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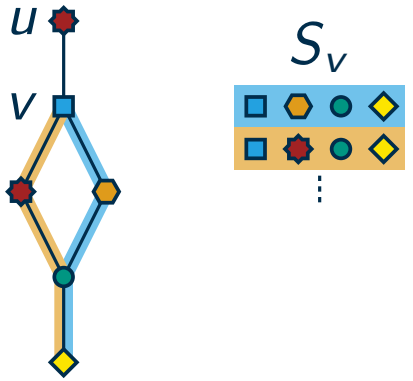

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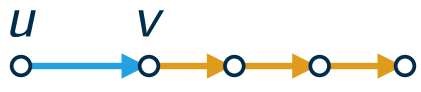

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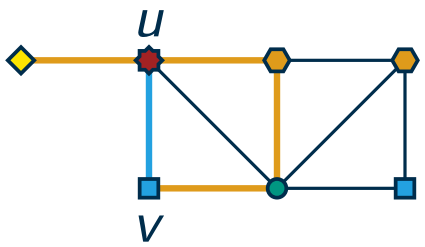

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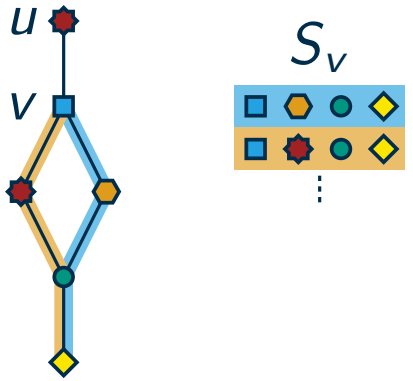

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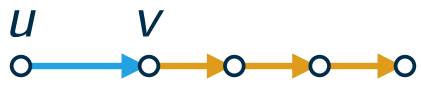

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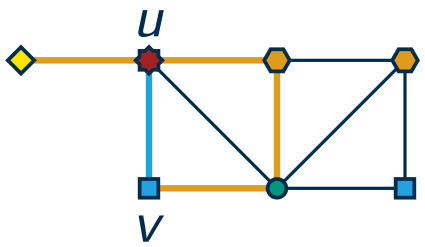

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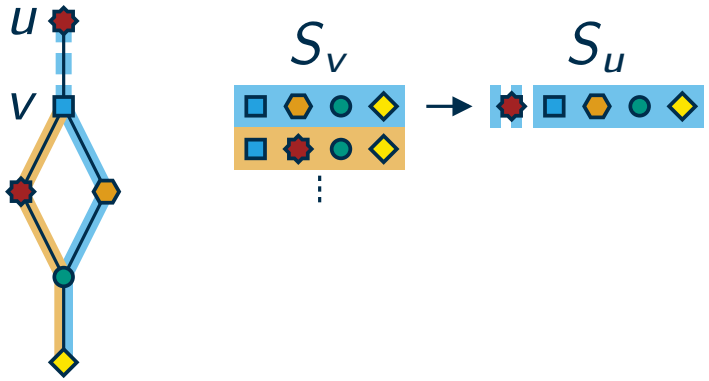
- color sets of size $\ell + 1$ can be computed from color sets of size ℓ of the neighbors



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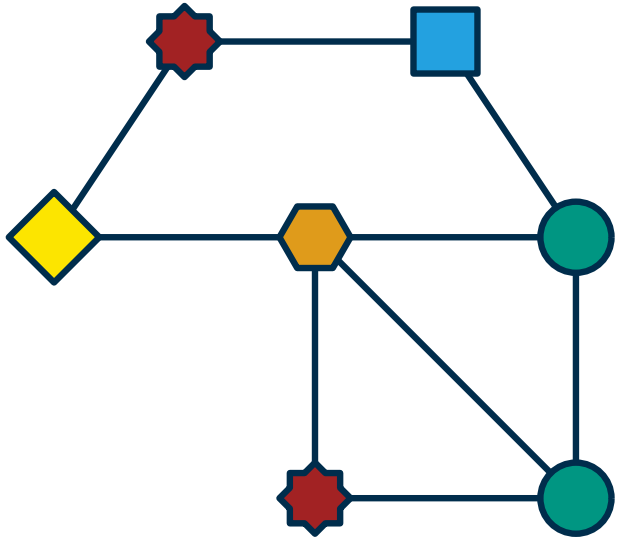

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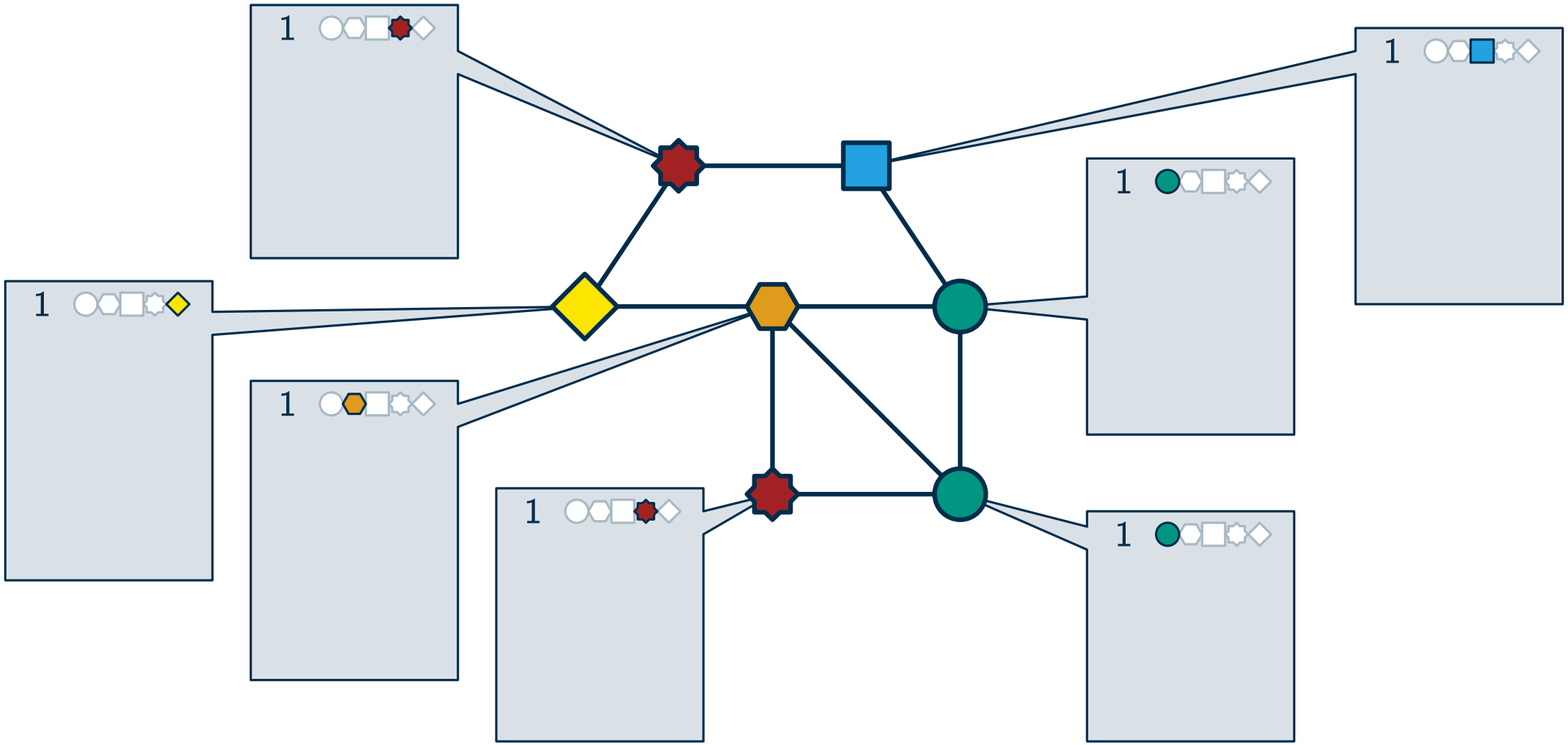

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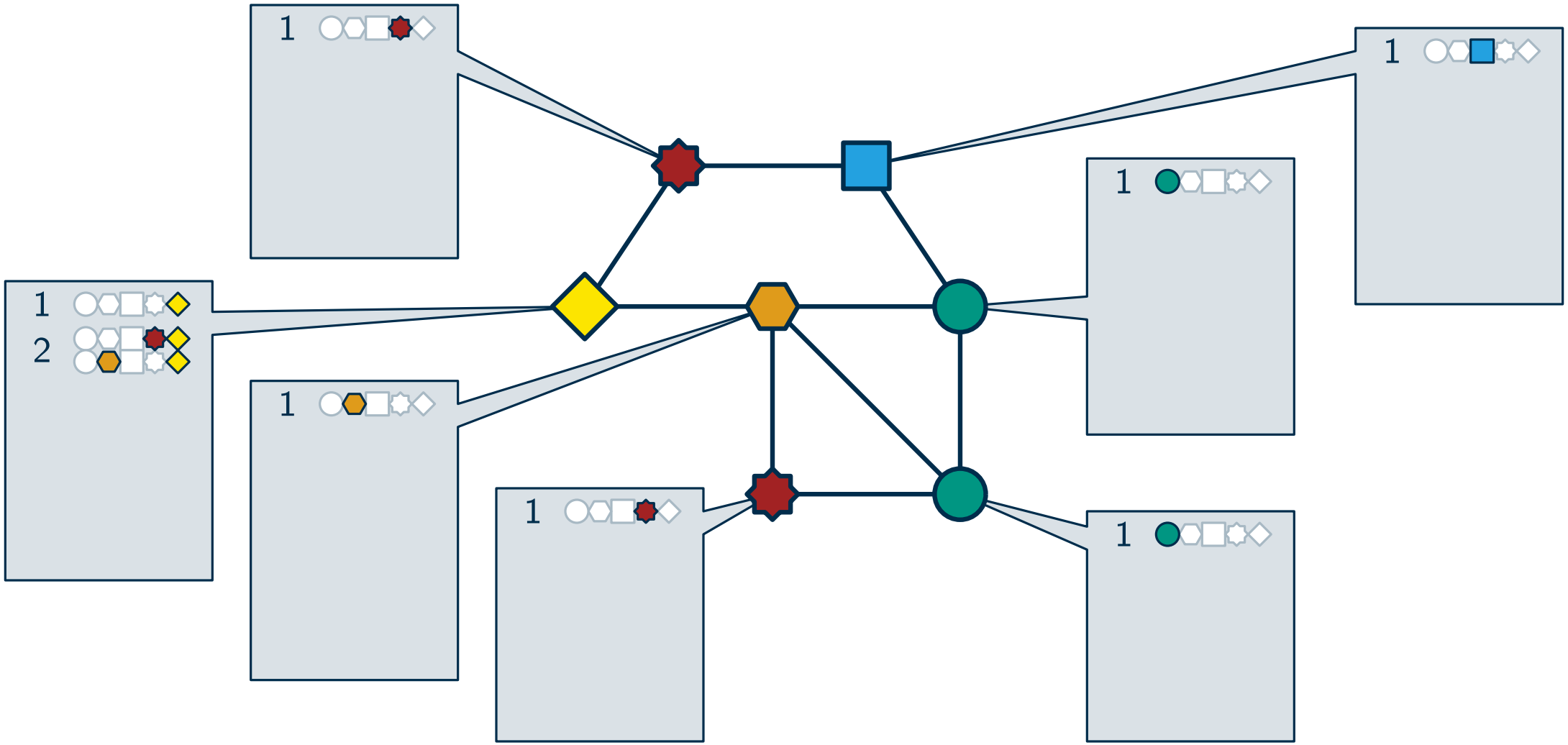

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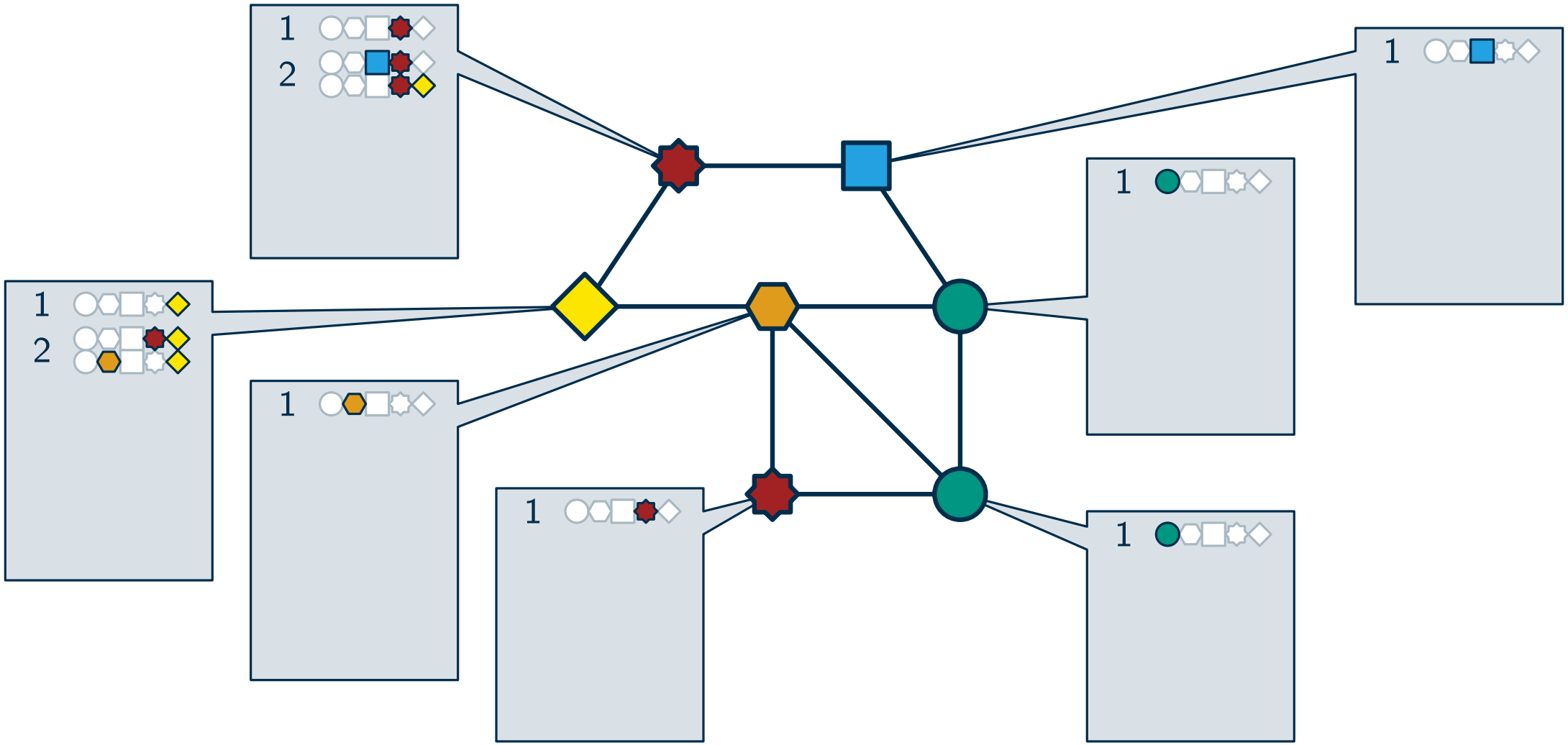

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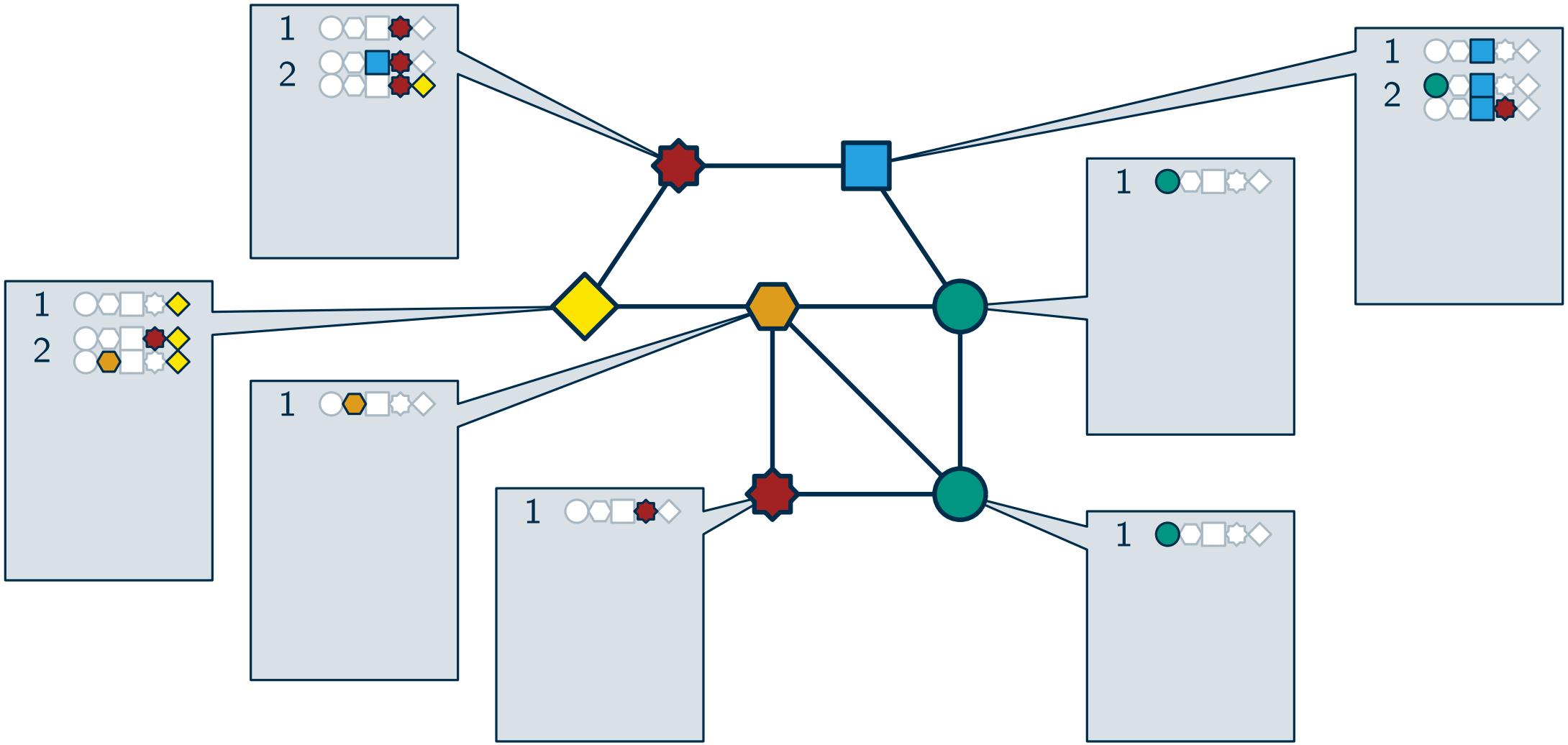

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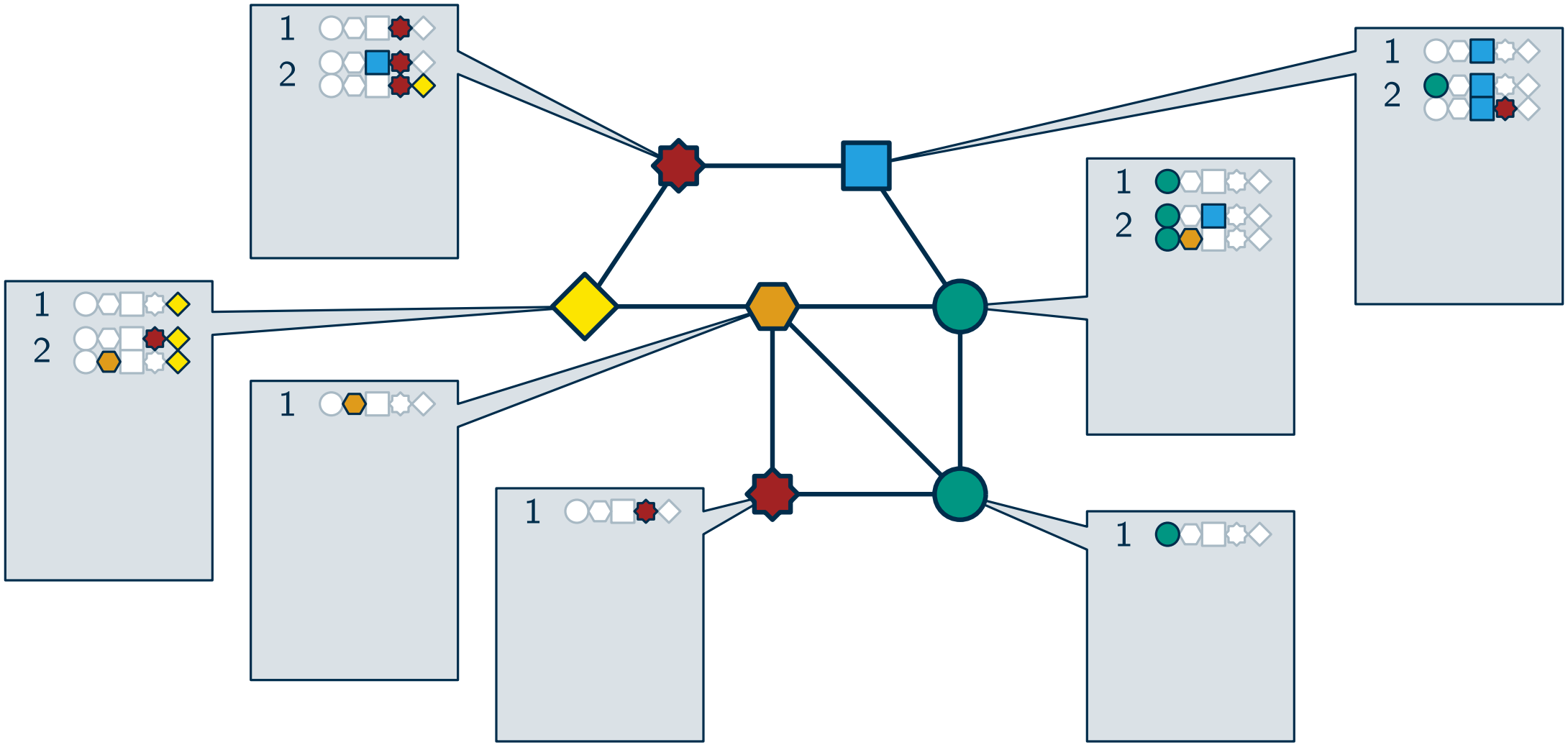

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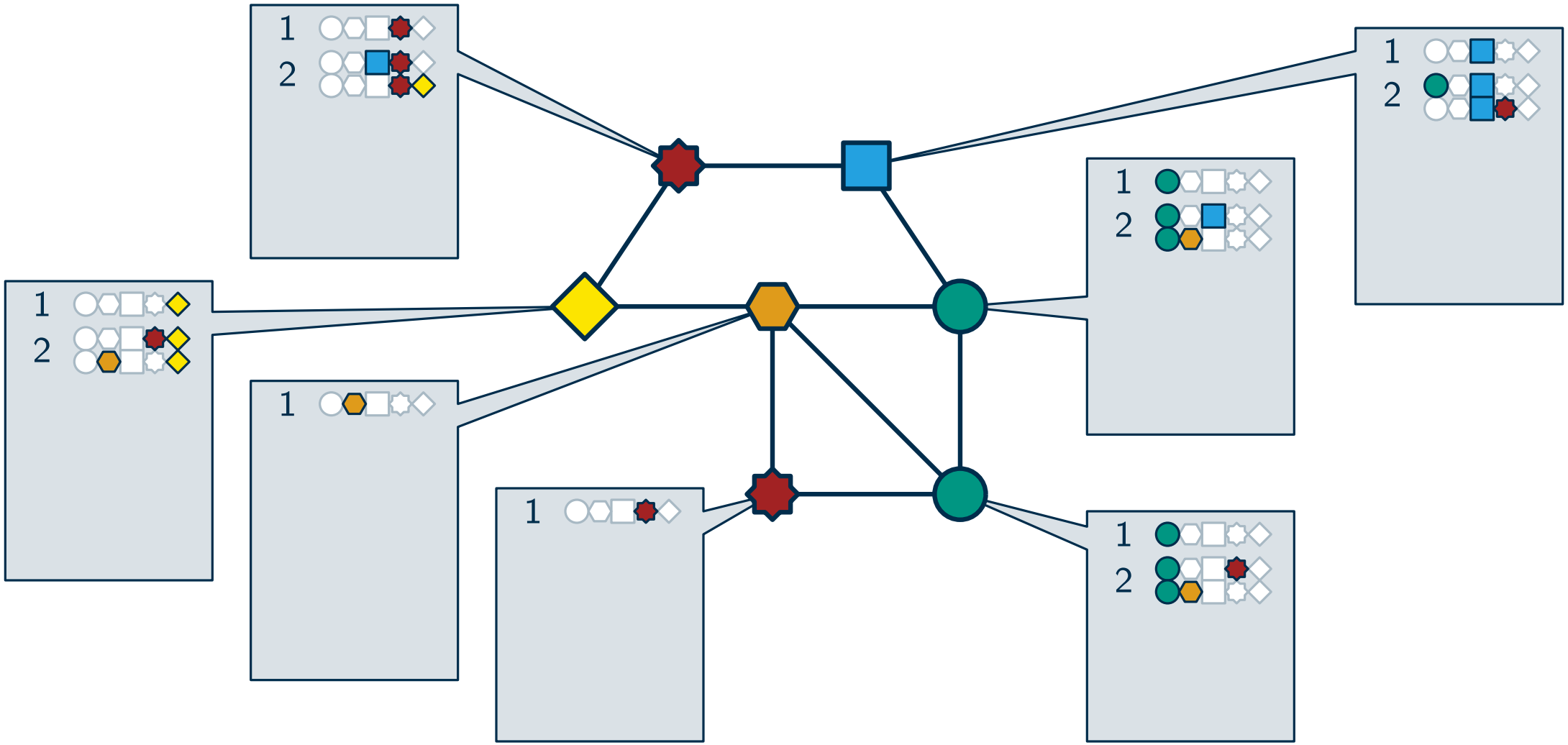

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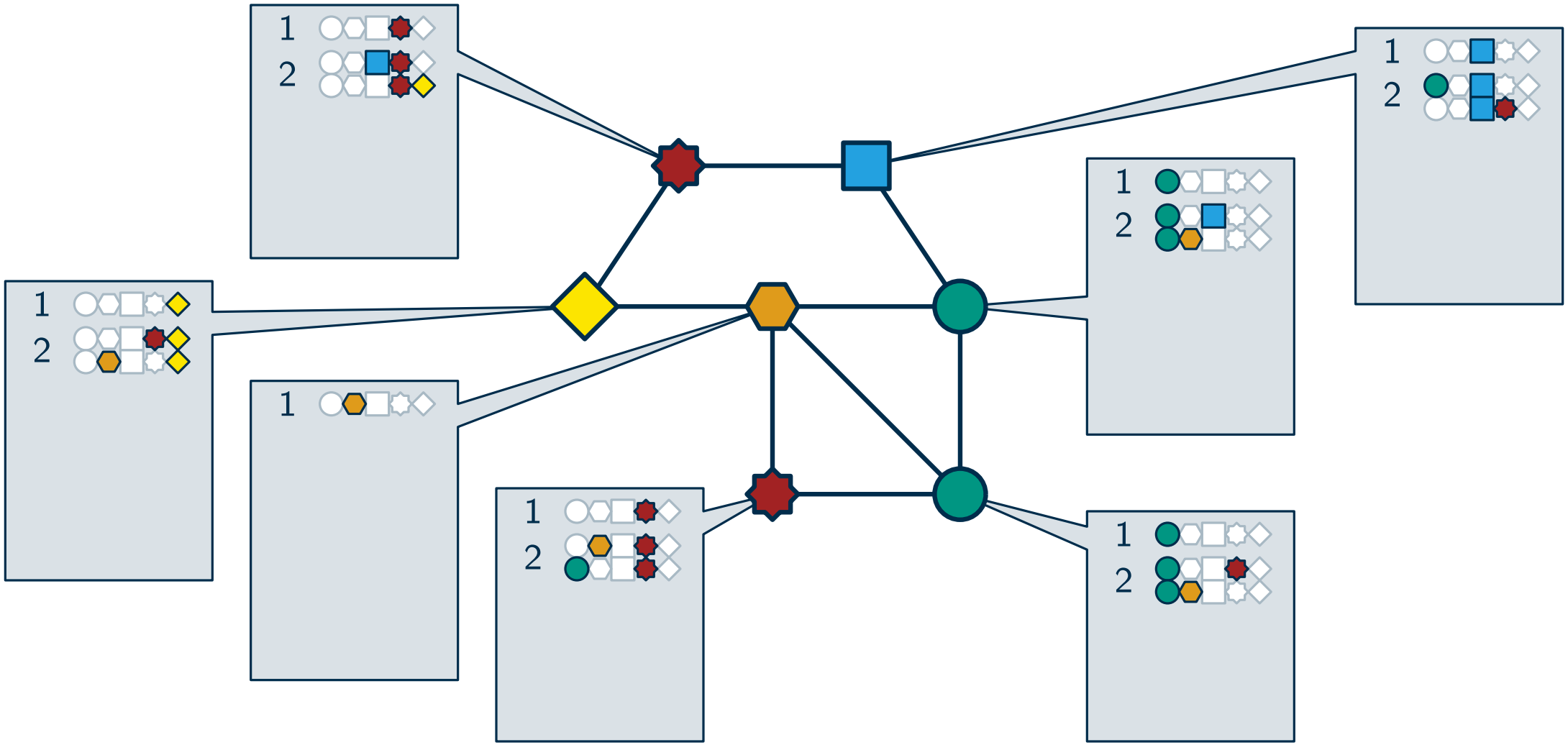

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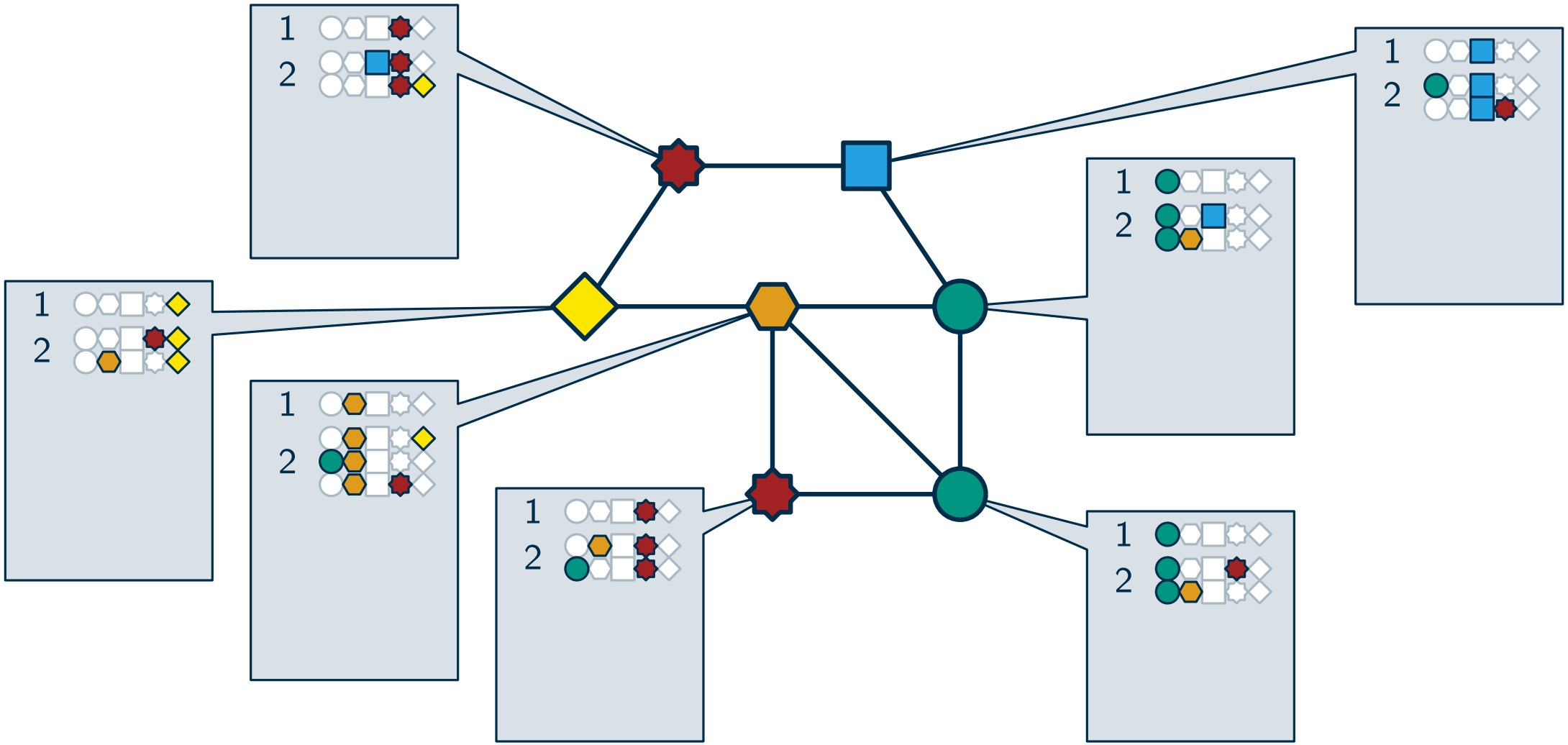

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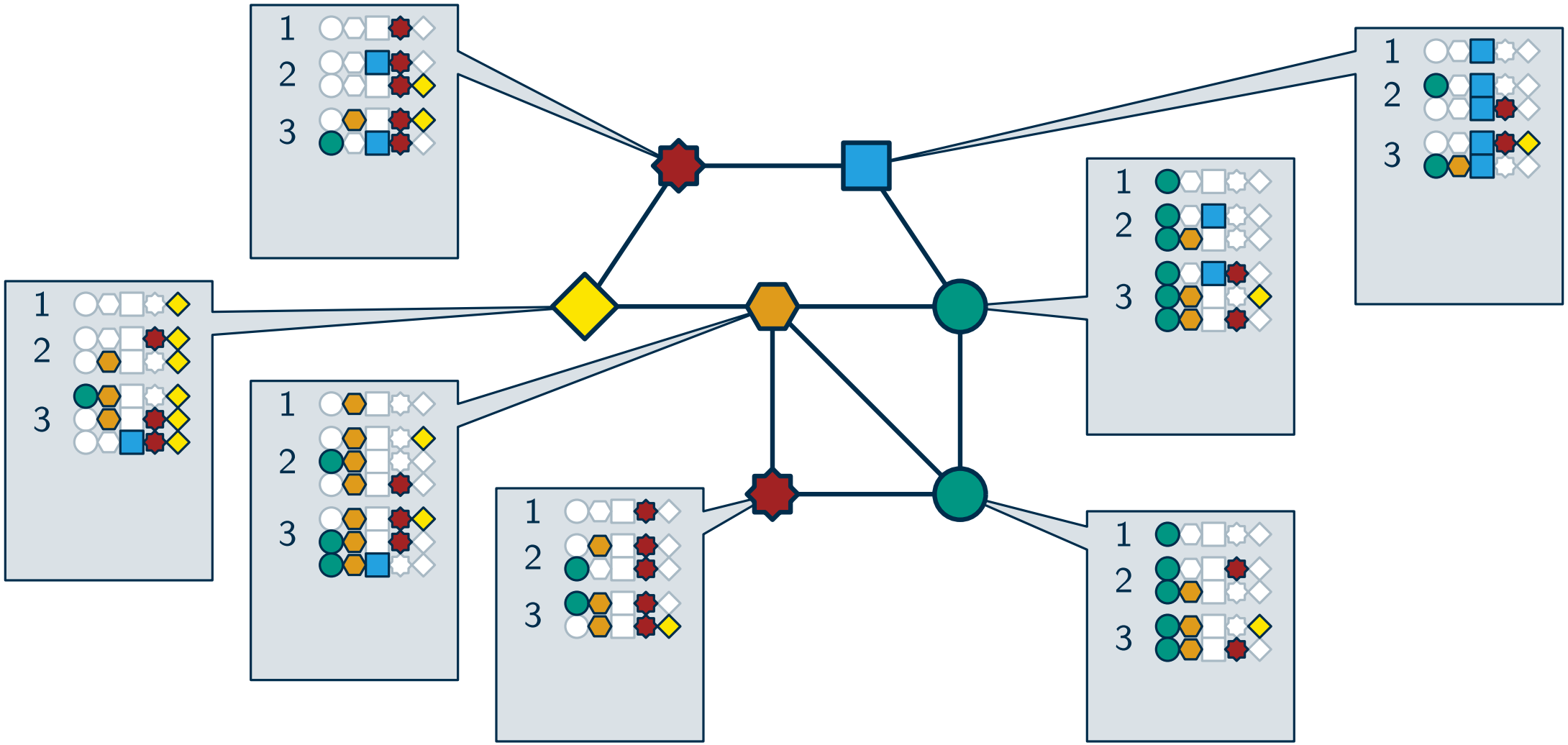

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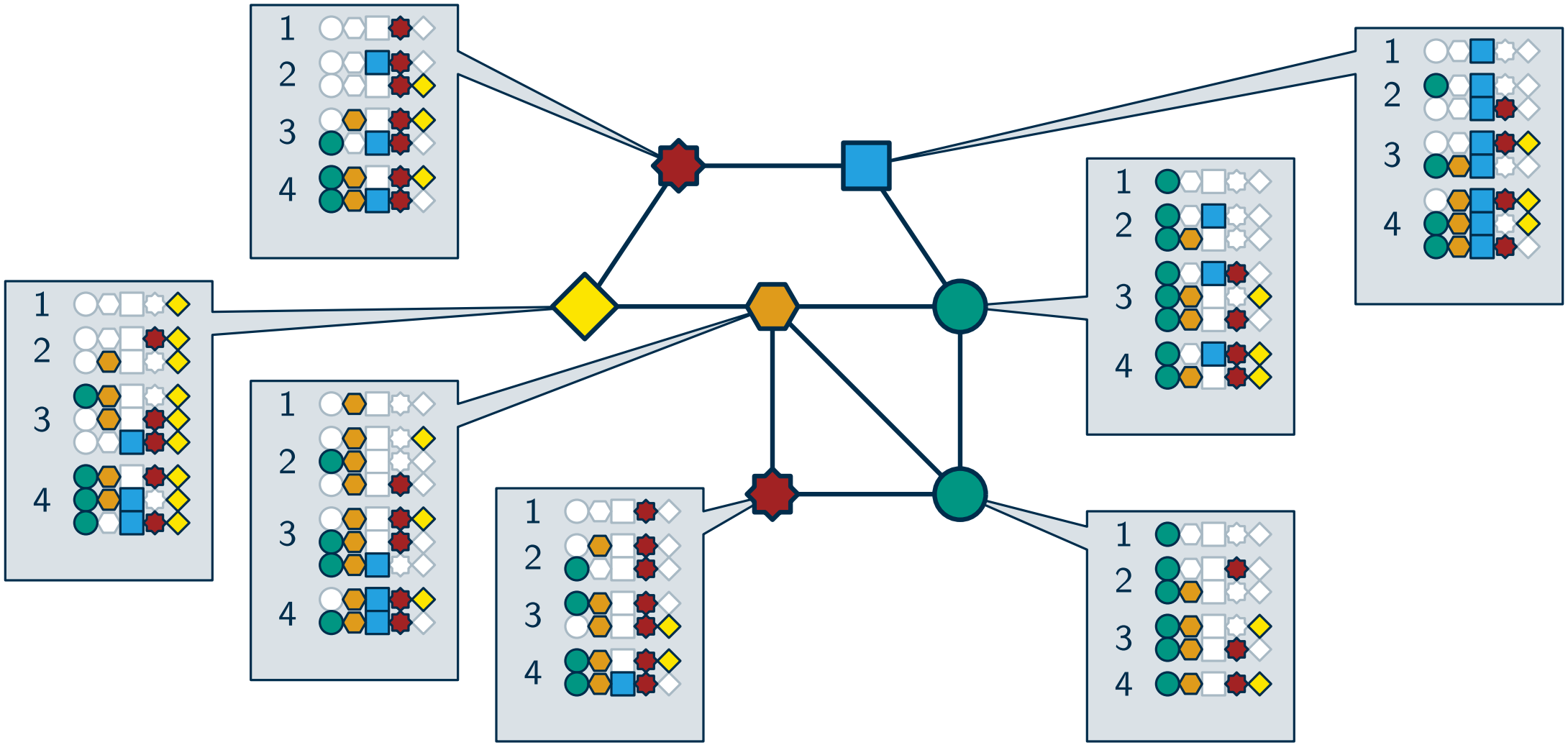

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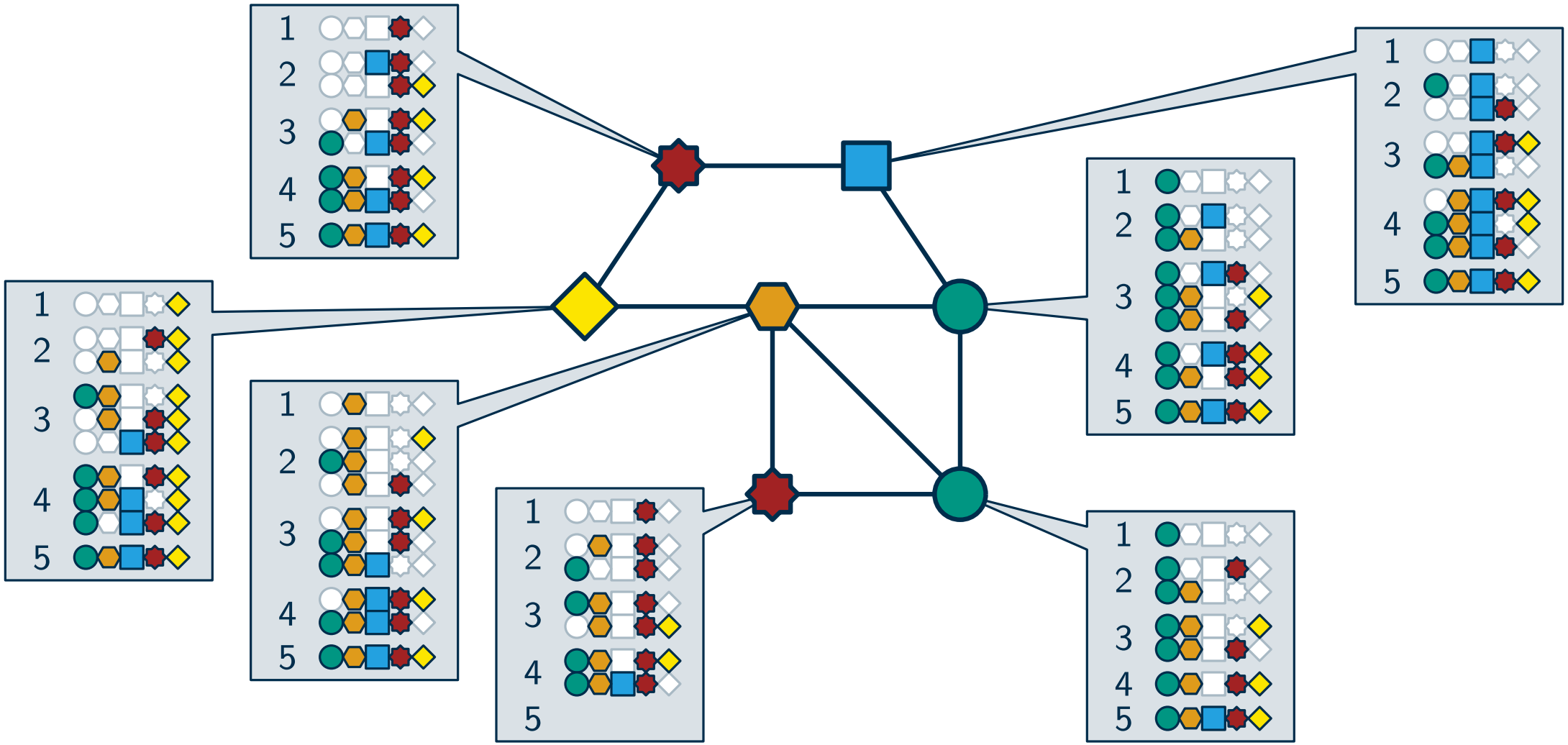
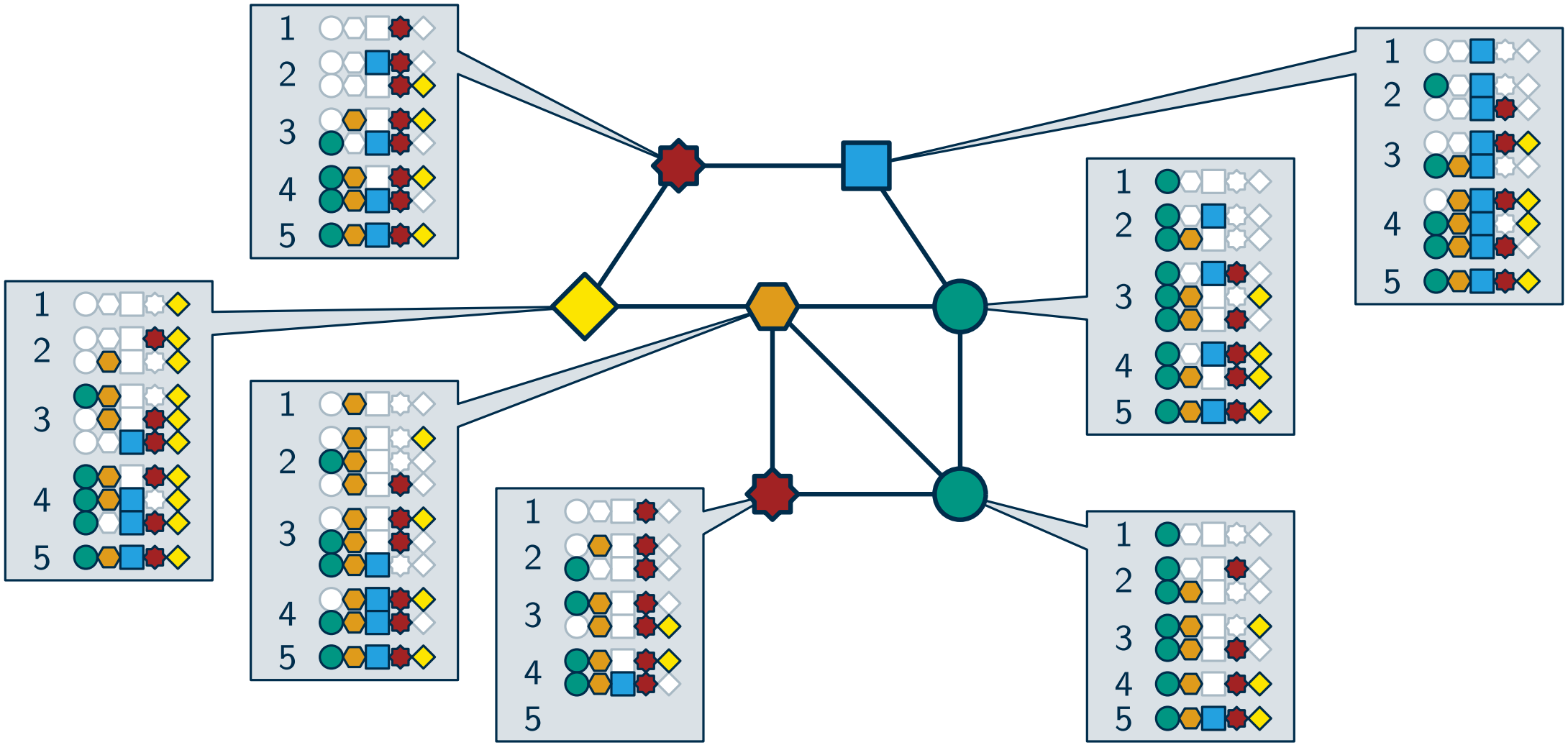


Dynamic program

Theorem
COLORFUL LONGEST PATH can be solved in $O(2^k km)$ time.



Random colors

Wassily Kandinsky: Color Study
Squares with Concentric Circles

Plan

- color vertices and solve COLORFUL LONGEST PATH
- problem: a path on k vertices is maybe not colorful
- solution: for every vertex, randomly choose one of k colors (uniformly, independent)

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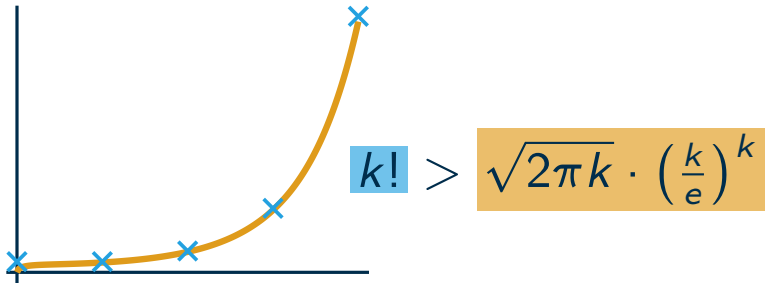
Stirling's approximation



$$k! > \sqrt{2\pi k} \cdot \left(\frac{k}{e}\right)^k$$

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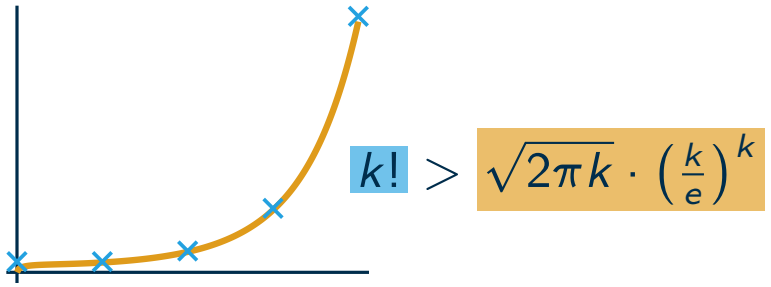

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- thus: $\frac{k!}{k^k} > \frac{1}{e^k}$

Probability boosting

- $\Theta(e^k)$ repetitions $\rightarrow$ constant success probability

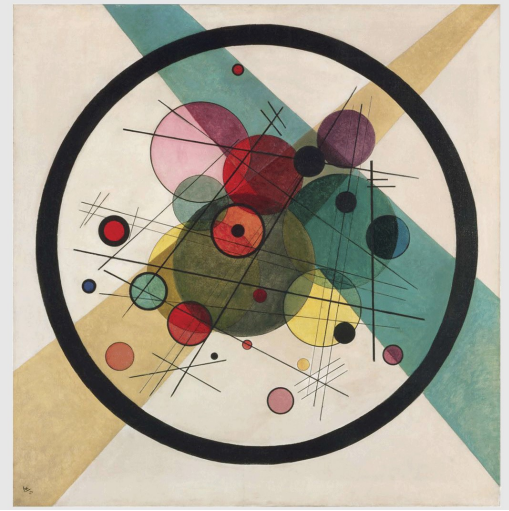
Derandomization

Random colorings

- enough random colorings
  $\rightarrow$ one is likely good

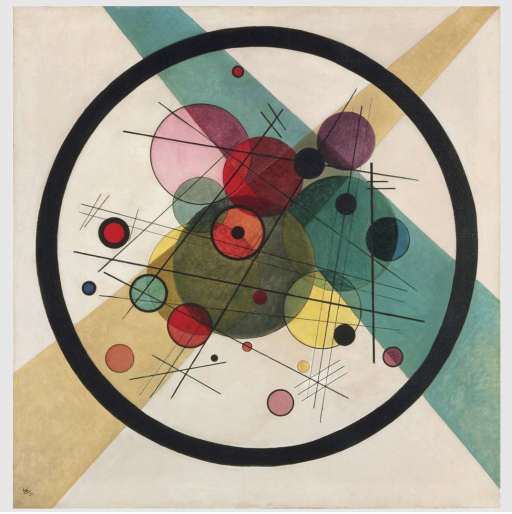
Deterministic colorings

- cleverly choose multiple colorings such that at least one is good



Wassily Kandinsky
Circles in a Circle

Derandomization



Wassily Kandinsky
Circles in a Circle

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- enough random colorings
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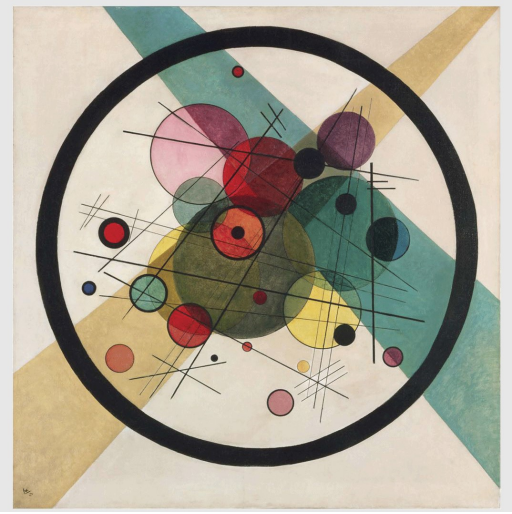

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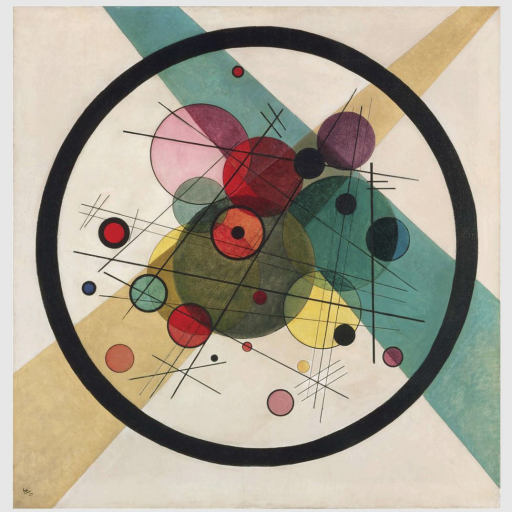

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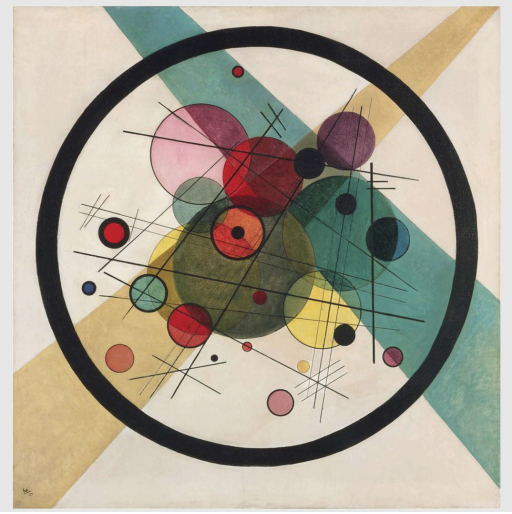

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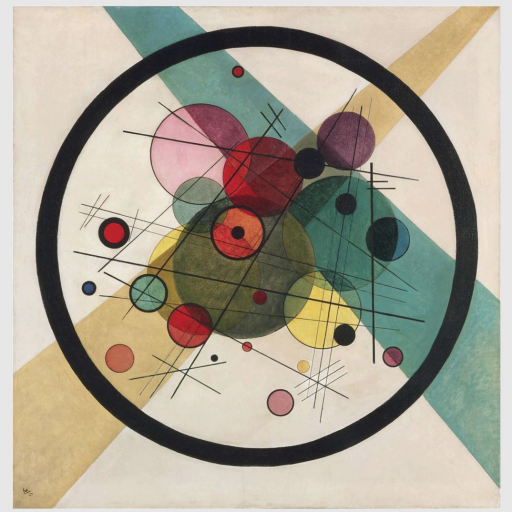

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COLORFUL LONGEST PATH can be solved in time $O(2^k km)$.

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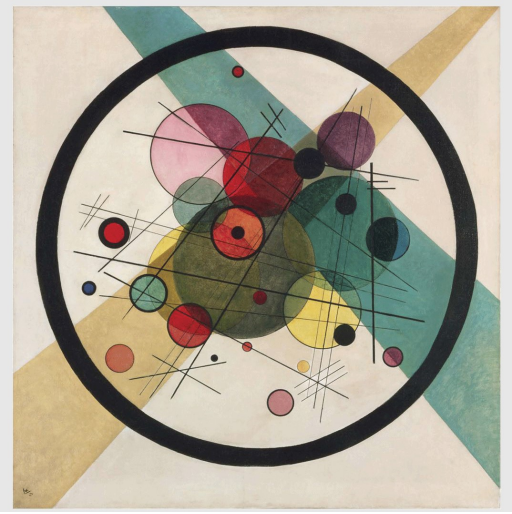

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Theorem (directly follows)
Longest Path can be solved in time $O(12.8^k km \log^2 n)$.

Color Coding

General Approach

- defined colored problem variant (with prefix COLORFUL or MULTICOLORED)
- design FPT-algorithm for the colored problem
- show:
  - `no`-instance becomes colored `no`-instance
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Tips for color coding

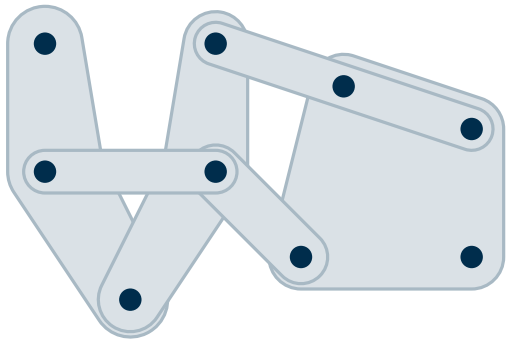
- look at a solution of an instance
- identify a witness of size only depending on k that certifies that it is a solution
- require that this witness is colorful

Set Splitting

Problem: Set Splitting (essentially Max Cut for hypergraphs)
Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with vertices V and hyperedges $\mathcal{E} \subseteq 2^V$, and a parameter s.
Is there a set $X \subseteq V$ such that X splits at least s edges of $\mathcal{H}$?

X splits $E \in \mathcal{E}$: $E \cap X \neq \emptyset$ and $E \not\subseteq X$

Example: Is there a set X that splits all hyperedges?



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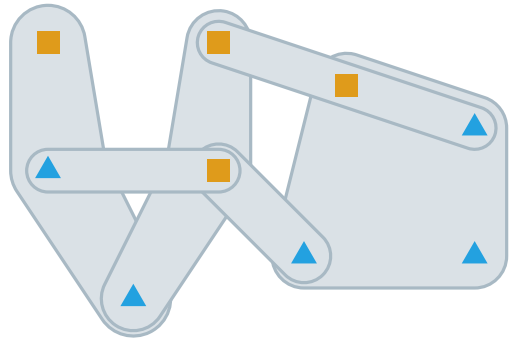

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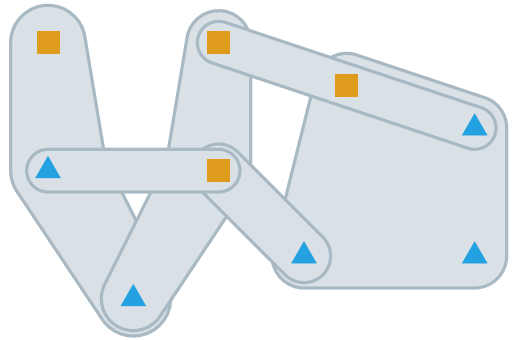

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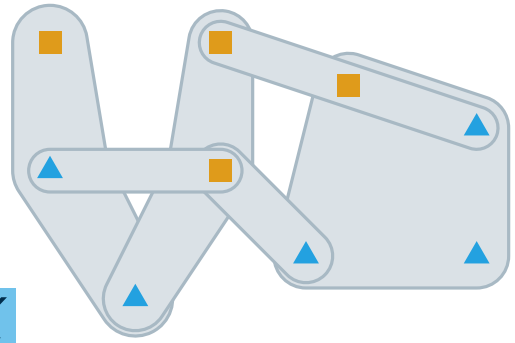

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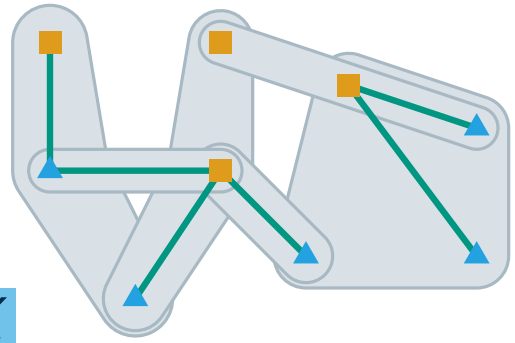

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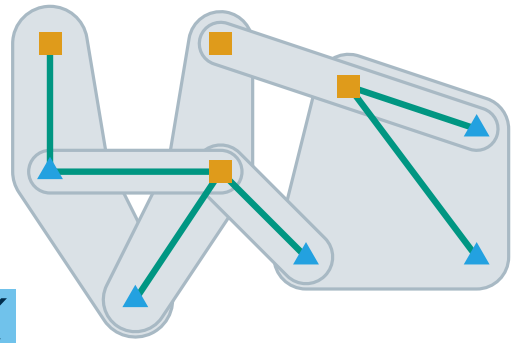

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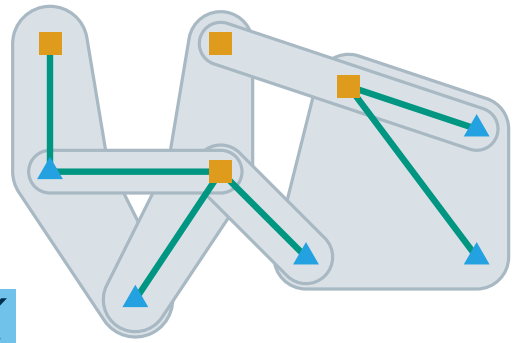

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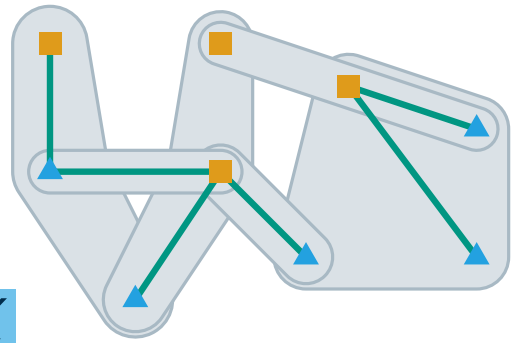

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Theorem
SET SPLITTING can be solved in $4^{s+o(s)} \cdot n^{O(1)}$ time.

Note: witness has size $2s$
$\rightarrow$ choose $k = 2s$

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- central tools: perfect hash families and universal sets
- formally, color coding is some kind of Turing reduction