Parameterized Algorithms

Color Coding

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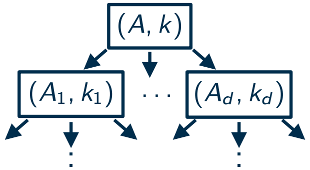
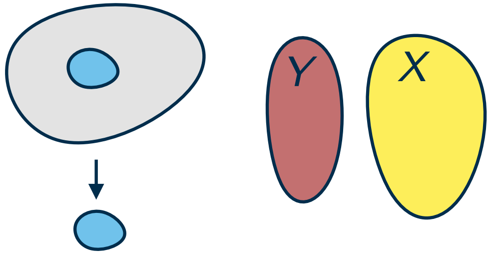
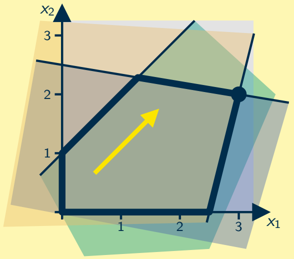
Content

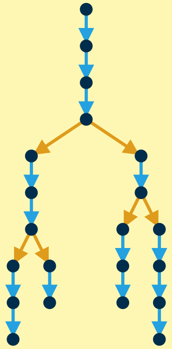
Basic toolbox
- bounded search trees
- kernelization
- iterative compression

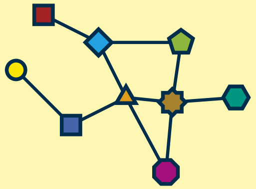




Extended toolbox
- linear programs
- branch-and-reduce
- color coding

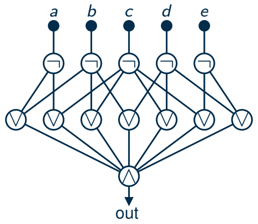


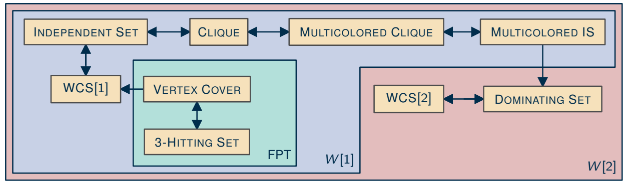




Tree width
- dynamic programming
- chordal and planar graphs
- Courcelle’s theorem

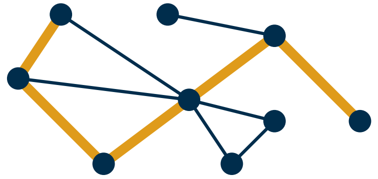
Lower bonds
- kernel lower bounds
- parameterized reductions
- circuits and the W-hierarchy
- ETH and SETH





Best detours

Problem: LONGEST PATH
Given a graph G and a parameter k. Does G have a k-vertex path?

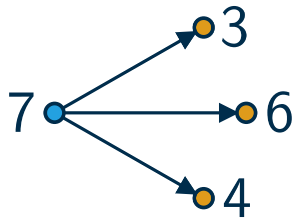


Complexity: NP-hard: reduction from HAMILTONIAN PATH $\rightarrow$ today: FPT

Can we do this in polynomial time?

For directed acyclic graphs (DAGs)

- dynamic program
  - for every vertex v: longest path starting at v
  - iterate vertices according to topological ordering
- alternative: matrix multiplication
  - let A be the adjacency matrix and consider A^k
  - entry (u, v) equals the number of paths of length k from u to v

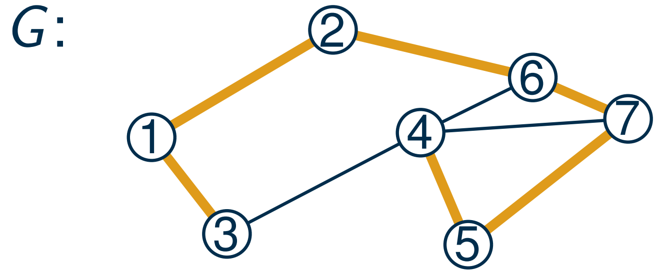


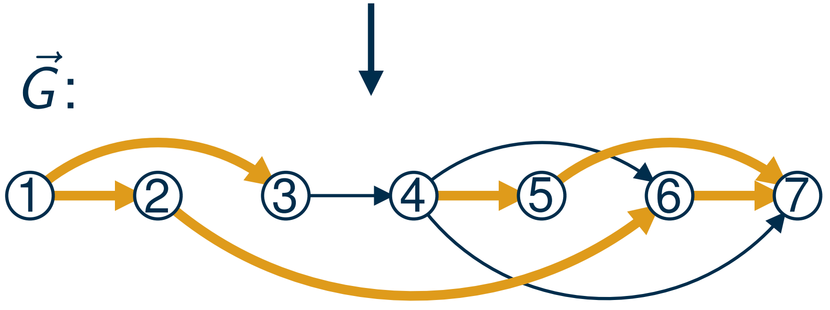
Why does this not work for undirected graphs?

Preventing cycles

Idea for undirected graphs

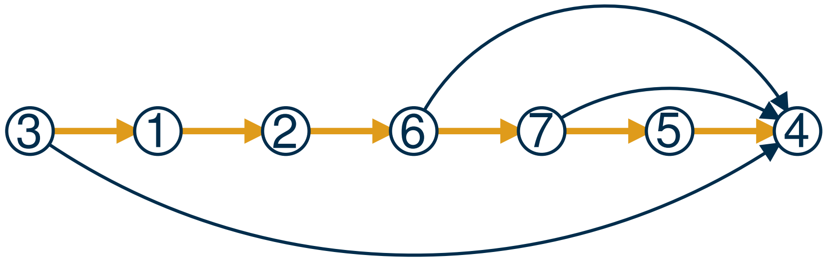
- transform the undirected graph G into a DAG $\vec{G}$
- find a longest path in the DAG $\vec{G}$





Problem: not all paths in G are directed paths in $\vec{G}$

But: there is always a good orientation



How do we find a good orientation?

Randomization

Construction of the DAG $\vec{G}$

- choose random order for the vertices of G
- direct all edges in G from front to back

Theorem
Let P be a k-vertex path k in G. Then P is a directed path in $\vec{G}$ with probability $\frac{2}{k!}$.

Proof

- let $P = v_1, \dots, v_k$
- every order on $\{v_1, \dots, v_k\}$ has the same probability
- $k!$ events, each happening with the same probability $\frac{1}{k!}$
- two of these events are good: $v_1, \dots, v_k$ and $v_k, \dots, v_1$

Comments

- success probability only depends on k
- probability can be boosted by repeating the process
- to get constant probability: number of repetitions only depends on k

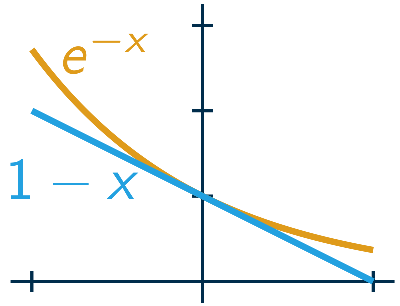
Randomized FPT algorithm

Theorem
With $k!$ independent trials of orienting $\vec{G}$, the probability to correctly solve LONGEST PATH for (G, k) is larger than $\frac{1}{2}$.

Theorem
Let P be a k-vertex path k in G. Then P is a directed path in $\vec{G}$ with probability $\frac{2}{k!}$.

Proof

- no-instances are always correctly recognized
- the algorithm fails on yes-instances with probability:
  - $1 - \frac{2}{k!}$ for an individual trial
  - $\left(1 - \frac{2}{k!}\right)^{k!}$ for $k!$ independent trials



$$\left(1 - \frac{2}{k!}\right)^{k!} \leq \left(e^{-\frac{2}{k!}}\right)^{k!} = e^{-2} < \frac{1}{2}$$

So what do we have?

- Monte Carlo algorithm with one-sided error

Taking a step back

What did we just do?

- wish for some additional structure (vertex order) that makes the problem easier
- for each solution: number of choices for the structure on the solution only depends on k
- probability for bad choice only dependent on k
- probability-boosting: number of trials only depends on k

Less order, more colors

- color the vertices
- only allow colorful solutions
- show that the colors help: FPT-algo for colorful problem
- random colors $\rightarrow$ error probability only depends on k
- afterwards: derandomization

Problem: Colorful LONGEST PATH
Given a graph $G = (V, E)$, a parameter k, and a partition (coloring) $V_1, \dots, V_k$ of V, is there a colorful k-vertex path?
(a path is colorful if it contains at most one vertex from each V_i)

Colorful paths

Problem: Colorful LONGEST PATH
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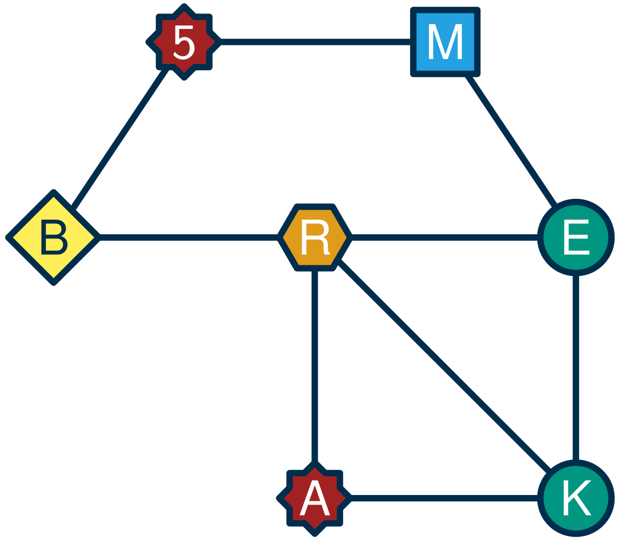
How long is the longest colorful path?



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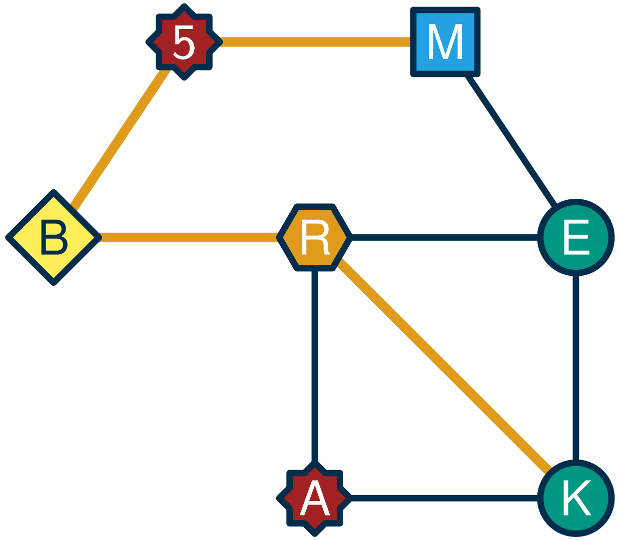
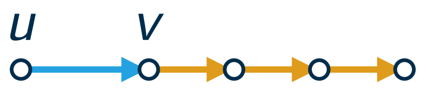


How do colors help?

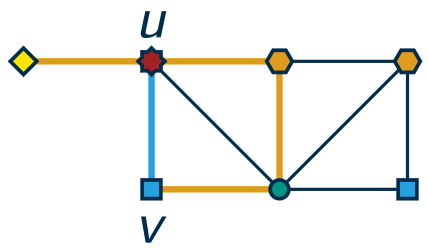
Solution for DAGs

- path of length ℓ from v and $uv \in E$ $\Rightarrow$ path of length $\ell + 1$ from u



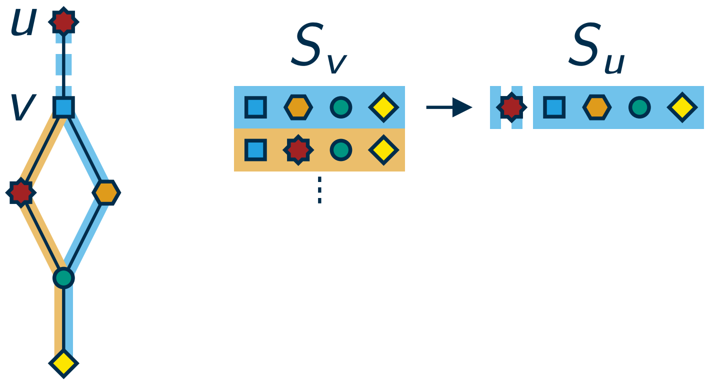
Undirected graphs

- problem: path of length ℓ from v might already contain u
- idea: remember vertices visited so far $\rightarrow n^\ell$ possibilities
- better: remember colors visited so far $\rightarrow 2^k$ possibilities



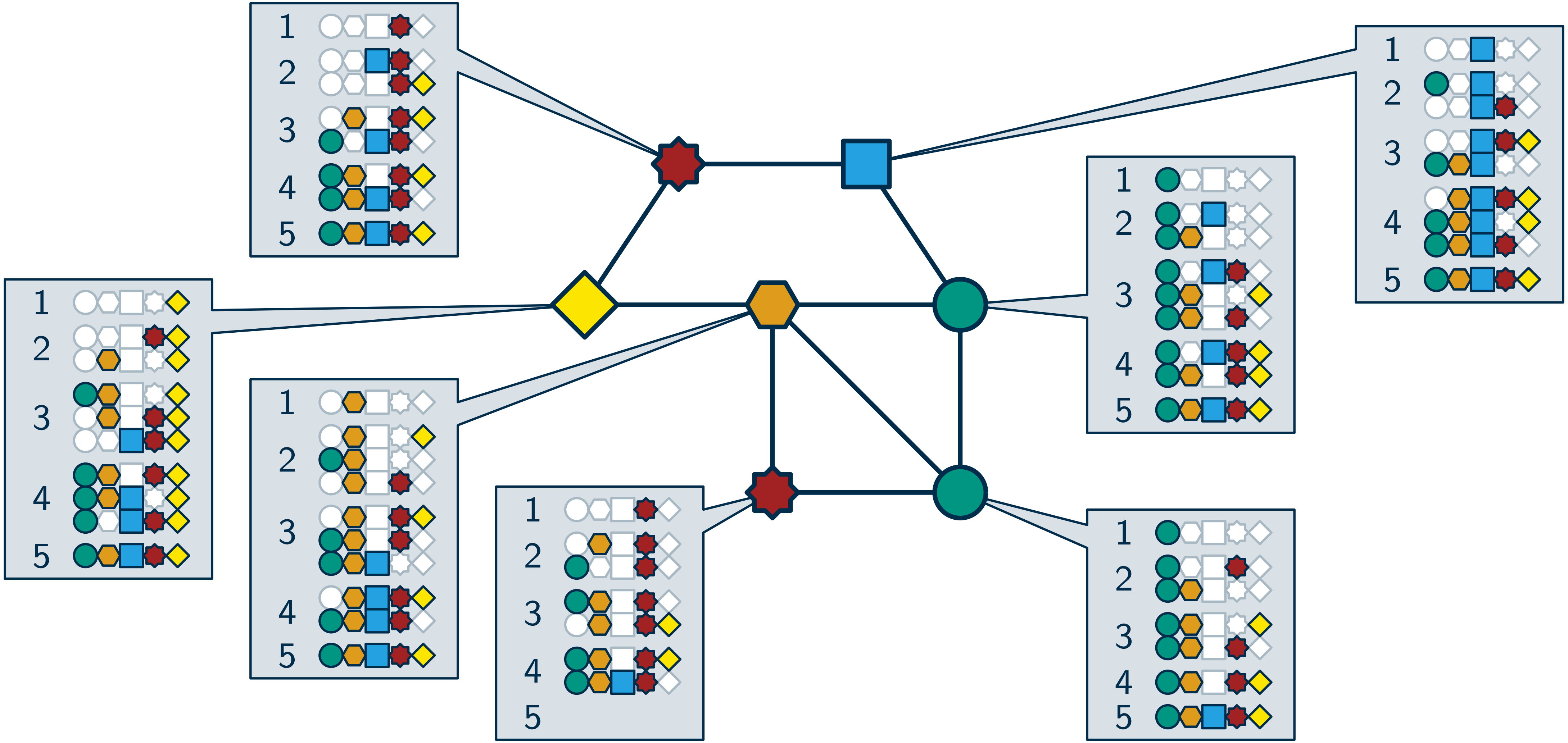
Dynamic program

- for every vertex v: compute for which color sets $S_v \subseteq [k]$ a path with these colors starting at v exists
- consider the color sets in increasing size
- color sets of size $\ell + 1$ can be computed from color sets of size ℓ of the neighbors

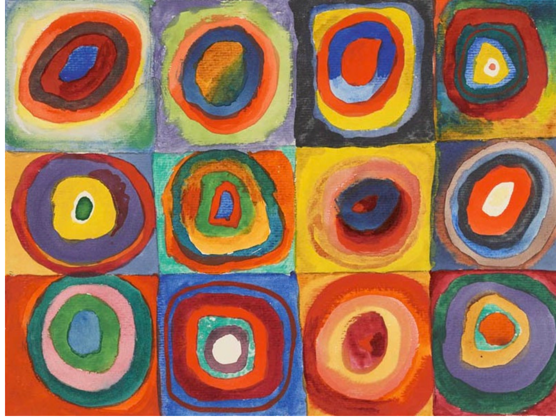


Dynamic program

Theorem
COLORFUL LONGEST PATH can be solved in $O(2^k km)$ time.



Random colors



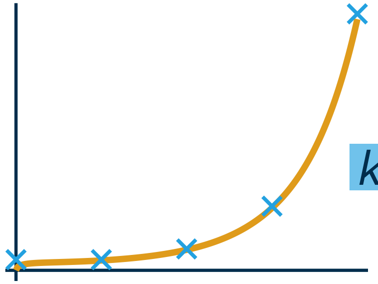
Wassily Kandinsky: Color Study Squares with Concentric Circles

Plan

- color vertices and solve COLORFUL LONGEST PATH
- problem: a path on k vertices is maybe not colorful
- solution: for every vertex, randomly choose one of k colors (uniformly, independent)

Theorem
Let P be a k-vertex path. Then P is a colorful path with probability $\geq \frac{1}{e^k}$.

Stirling's approximation



$k! > \sqrt{2\pi k} \cdot \left(\frac{k}{e}\right)^k$

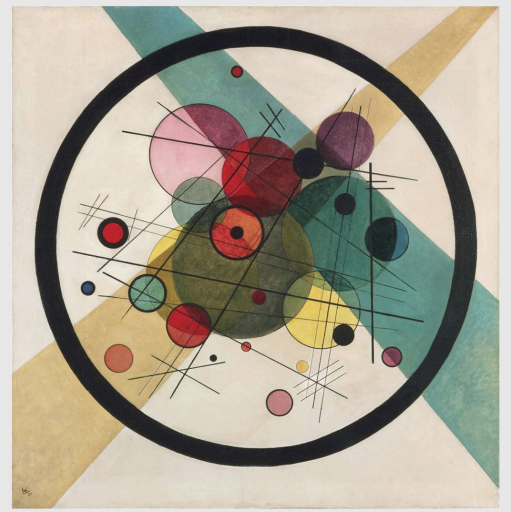
Proof

- let $P = v_1, \ldots, v_k$
- every color combination of $v_1, \ldots, v_k$ is equally likely
- k^k events, each happening with probability $\frac{1}{k^k}$
- $k!$ of these events are good
- thus: $\frac{k!}{k^k} > \frac{1}{e^k}$

Probability boosting

- $\Theta(e^k)$ repetitions $\rightarrow$ constant success probability

Derandomization



Wassily Kandinsky
Circles in a Circle

Random colorings

- enough random colorings
  $\to$ one is likely good

Deterministic colorings

- cleverly choose multiple colorings such that at least one is good

Make a wish

- set of colorings of n elements (vertices) with k colors
- for every k-element subset there is a coloring making it colorful

} (n, k)-perfect hash family

Theorem (without proof)
An (n, k)-perfect hash family of size $O(6.4^k \log^2 n)$ can be constructed in $O(6.4^k n \log^2 n)$ time.

Theorem (DP from earlier)
COLORFUL LONGEST PATH can be solved in time $O(2^k km)$.

Theorem (directly follows)
LONGEST PATH can be solved in time $O(12.8^k km \log^2 n)$.

Color Coding

General Approach

- defined colored problem variant (with prefix COLORFUL or MULTICOLORED)
- design FPT-algorithm for the colored problem
- show:
  - `no`-instance becomes colored `no`-instance
  - `yes`-instance becomes colored `yes`-instance for ≥ 1 coloring from a perfect hash family

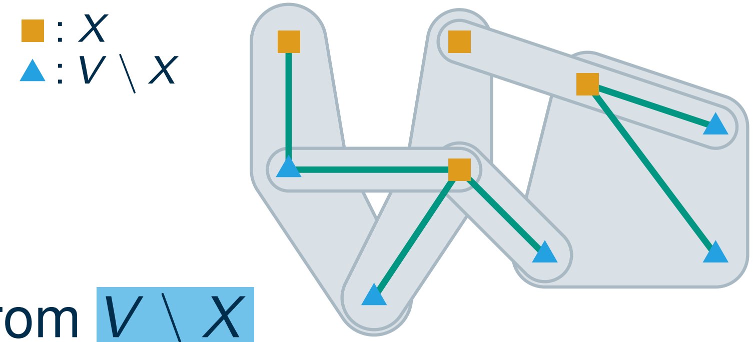
Tips for color coding

- look at a solution of an instance
- identify a witness of size only depending on k that certifies that it is a solution
- require that this witness is colorful

Set Splitting

Problem: Set Splitting (essentially Max Cut for hypergraphs)
Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with vertices V and hyperedges $\mathcal{E} \subseteq 2^V$, and a parameter s.
Is there a set $X \subseteq V$ such that X splits at least s edges of $\mathcal{H}$?
X splits $E \in \mathcal{E}$: $E \cap X \neq \emptyset$ and $E \not\subseteq X$

Example: Is there a set X that splits all hyperedges?



First step of color coding

- identify a solution witness of size only depending on s
- here: X splits $E \in \mathcal{E}$ $\Rightarrow$ E contains a vertex from X and one from $V \setminus X$
- there is a bi-colored pair for each split hyperedge $\rightarrow$ witness of size $\leq 2s$

Same but different

- randomly assign one of two colors to each vertex
- probability of bi-colored witness pairs: $\approx \frac{1}{2^s}$

} $\Theta(2^s)$ trials $\rightarrow$ constant success probability

Derandomization: Universal sets

Random 2-colorings

- enough random 2-colorings
  $\rightarrow$ one is likely good

Deterministic 2-colorings

- cleverly choose multiple 2-colorings
  such that at least one is good

Make a wish

- set of colorings of n elements (vertices) with two colors
- every k-element subset is colored in every possible way

} (n, k)-univeral set

Theorem (without proof)
An (n, k)-universal set can be computed on $O(2^{k+o(k)} \cdot n)$ time.

Theorem
SET SPLITTING can be solved in $4^{s+o(s)} \cdot n^{O(1)}$ time.

Note: witness has size $2s$
$\rightarrow$ choose $k = 2s$

Wrap-Up

Randomization

- guessing additional structure enables new solution approaches
- combines well with FPT if it suffices to be lucky on a substructure of size $f(k)$
- randomization is only an intermediate step $\rightarrow$ we can often derandomize

Color coding

- coloring or partitioning yields useful additional structure
- central tools: perfect hash families and universal sets
- formally, color coding is some kind of Turing reduction