

The background of the slide is a complex network graph. It features a dense web of white nodes connected by thin, dark lines. The nodes are distributed across the frame, with a higher concentration in the center and left side. The background color transitions from a dark teal on the left to a deep blue on the right.

Parameterized Algorithms

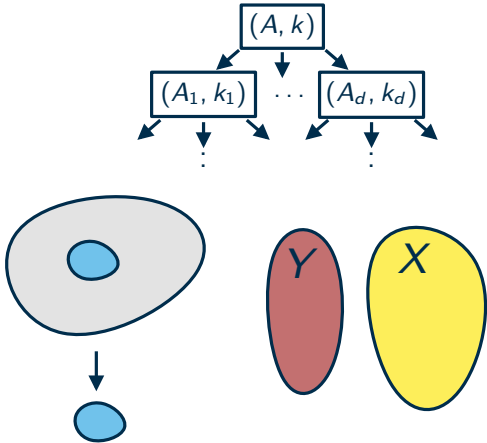
Above Lower Bound, Duality, and Lenstra

Thomas Bläsius

Content

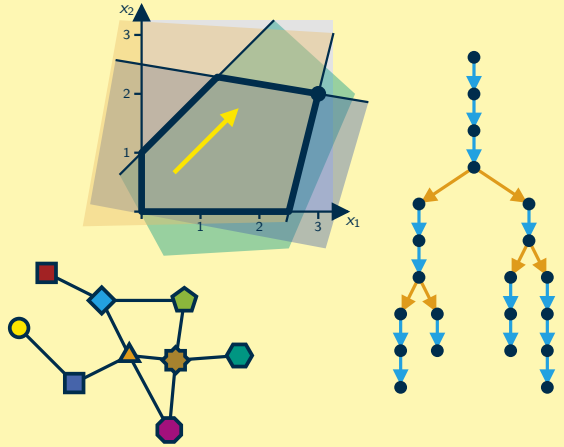
Basic toolbox

- bounded search trees
- kernelization
- iterative compression



Extended toolbox

- linear programs
- branch-and-reduce
- color coding



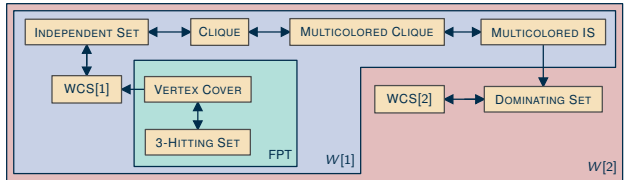
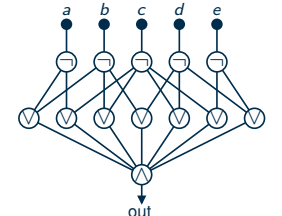
Tree width

- dynamic programming
- chordal and planar graphs
- Courcelle's theorem



Lower bounds

- kernel lower bounds
- parameterized reductions
- circuits and the W-hierarchy
- ETH and SETH



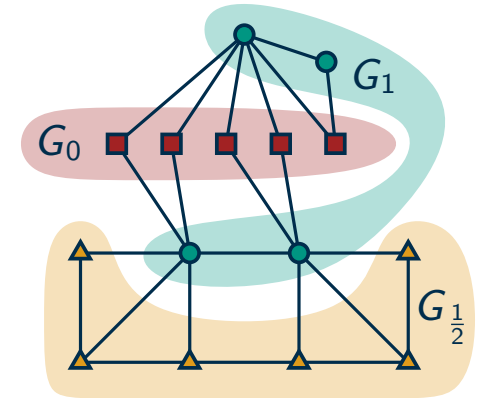
Recap: Kernelization via LP relaxation

$$\text{minimize: } \sum_{v \in V} x_v$$

$$\text{such that: } x_u + x_v \geq 1 \text{ for } uv \in E$$
$$0 \leq x_v \leq 1 \text{ for } v \in V$$

Reduction rule

- solve the LP relaxation $\rightarrow (x_v)_{v \in V}$
- if $\sum_{v \in V} x_v > k$: constant size no-instance
- otherwise: delete $V_0 \cup V_1$ and reduce k by $|V_1| \rightarrow G_{\frac{1}{2}}$ is the kernel

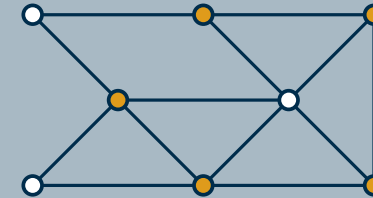


Lemma: The reduction rule is safe.

Recap: Bounded search tree

Problem: VERTEX COVER

Given a graph $G = (V, E)$ and a parameter k . Does G have a vertex cover of size k ?

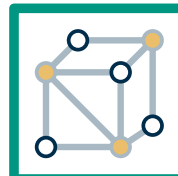
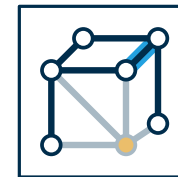


(vertex set $V' \subseteq V$ with $e \cap V' \neq \emptyset$ for all $e \in E$)

Is there a vertex cover of size at most $k = 3$?



edge $\{u, v\}$ is uncovered
 $\Rightarrow u$ or v must be selected
 \rightarrow binary decision



solution of size 3

Simple branching algorithm

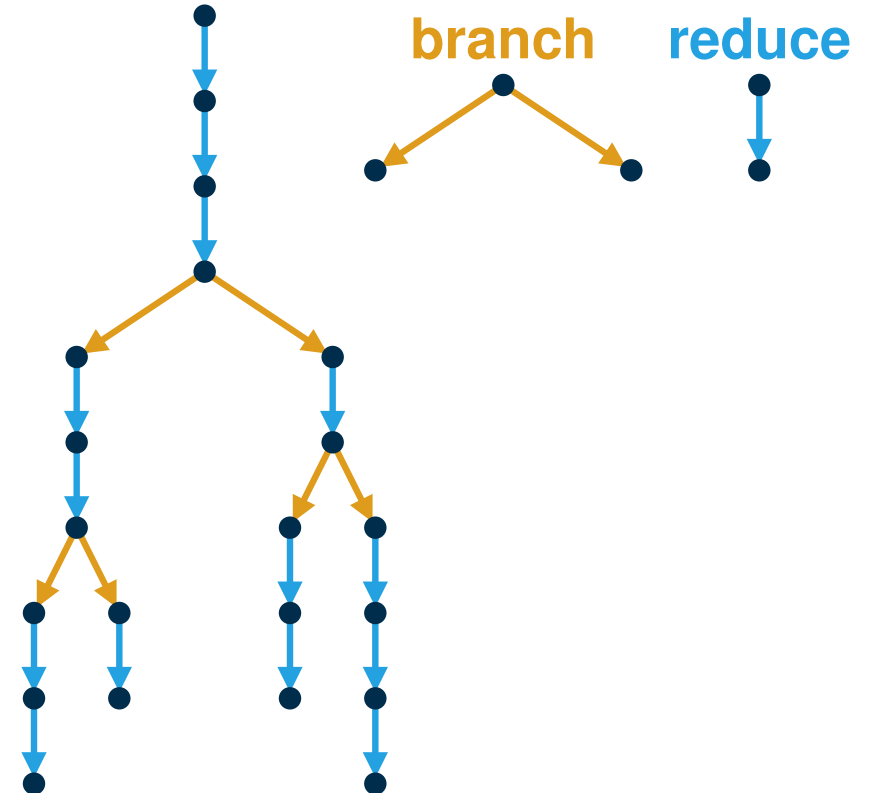
- start: $V' = \emptyset \rightarrow$ no edge covered
- step: branch on uncovered edge
- stop if: all edges covered or already k vertices selected

running time: $O(2^k m)$

Kernelization and search trees

Branch-and-reduce

- apply reduction rules whenever possible
- no rule applicable: branch once



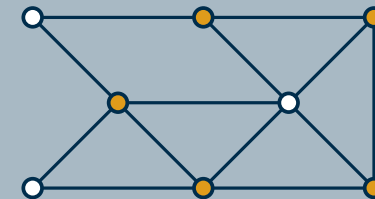
Parameterization above lower bound

Solution size and lower bounds

- denote the size of a minimum vertex cover of G with $vc(G)$
- let $\ell(G)$ be a lower bound on $vc(G)$, i.e., we know $vc(G) \geq \ell(G)$
- so far: parameter $vc(G)$; now: parameter $vc(G) - \ell(G)$
- $\ell_{LP}(G) =$ optimal solution of the LP-relaxation

Problem: VERTEX COVER ABOVE LP

Given a graph $G = (V, E)$ and a parameter k . Does G have a vertex cover of size $\ell_{LP}(G) + k$?



(vertex set $V' \subseteq V$ with $e \cap V' \neq \emptyset$ for all $e \in E$)

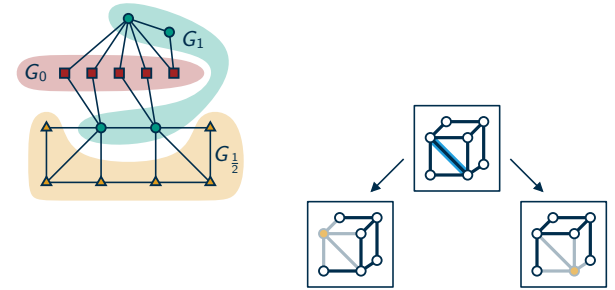
Plan for today

- FPT algorithm for VERTEX COVER ABOVE LP
- note: $vc(G)$ (previous parameter) can be much bigger than $vc(G) - \ell_{LP}(G)$ (new parameter)

Branch-and-reduce: How does k change?

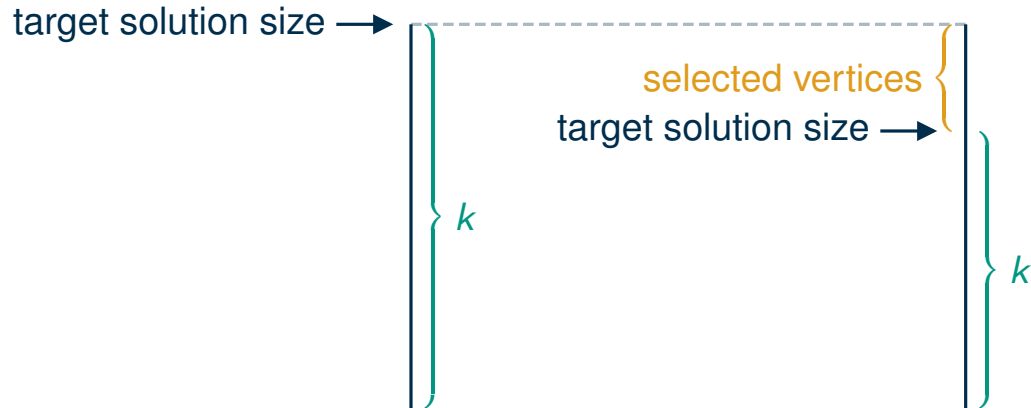
Reduce: if the LP relaxation has a solution with $G_{\frac{1}{2}} \subsetneq G$, reduce to $G_{\frac{1}{2}}$

Branch: for an edge uv , consider the child instances $G - u$ and $G - v$



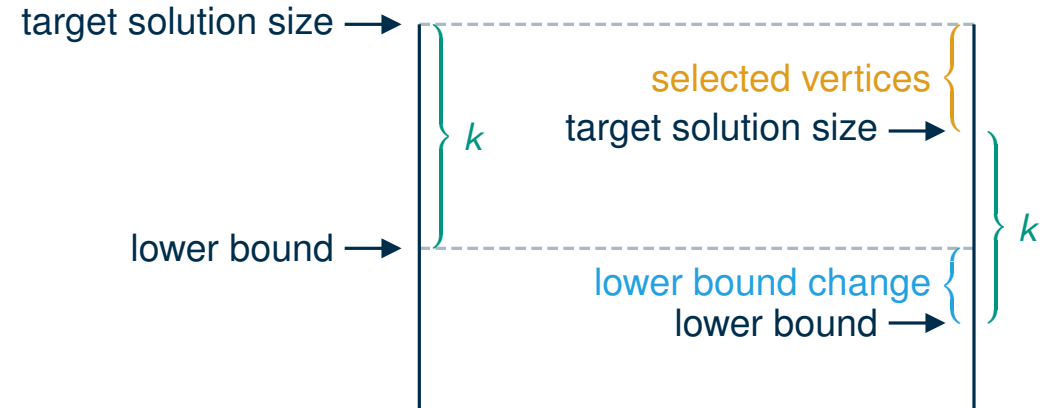
Problem: VERTEX COVER

Given a graph $G = (V, E)$ and a parameter k .
Does G have a vertex cover of size k ?



Problem: VERTEX COVER ABOVE LP

Given a graph $G = (V, E)$ and a parameter k .
Does G have a vertex cover of size $\ell_{LP}(G) + k$?

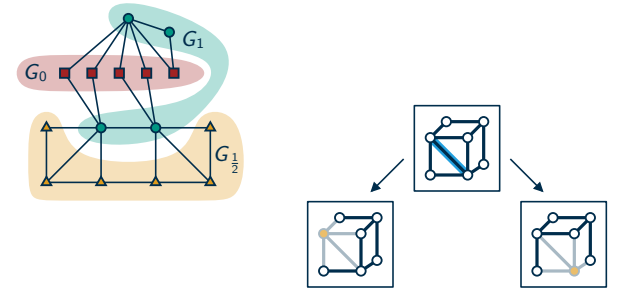


New parameter: depends on **number of selected vertices** and on how the **lower bound changes**

Branch-and-reduce: How does ℓ_{LP} change?

Reduce: if the LP relaxation has a solution with $G_{\frac{1}{2}} \subsetneq G$, reduce to $G_{\frac{1}{2}}$

Branch: for an edge uv , consider the child instances $G - u$ and $G - v$

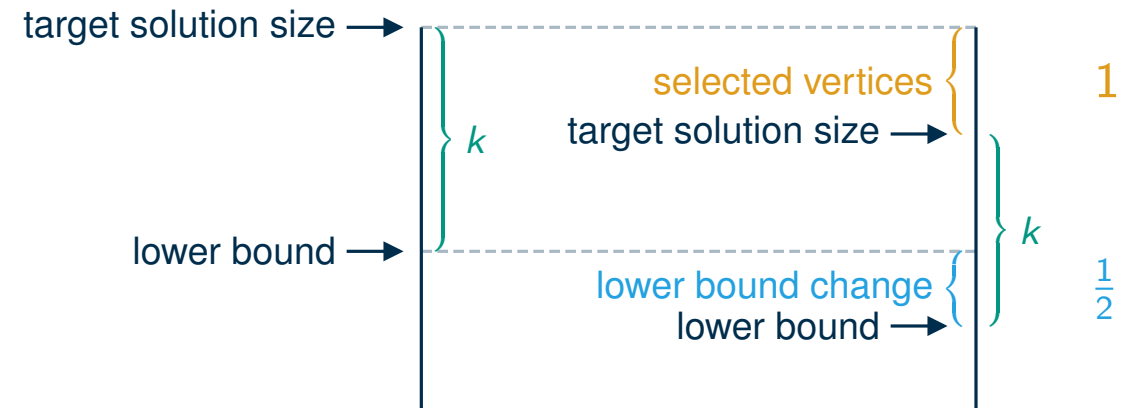


Reduction rule

- we add $|V_1|$ vertices to the vertex cover
- $\ell_{LP}(G_{\frac{1}{2}}) = \ell_{LP}(G) - |V_1|$
- the parameter k remains unchanged

Branching rule

- 1 vertex is added to the vertex cover
- next slide: ℓ_{LP} shrinks by $\frac{1}{2}$
- this implies: the parameter k shrinks by $\frac{1}{2}$



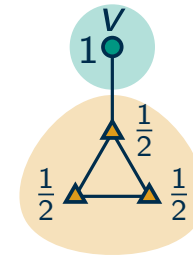
How does $\ell_{\text{LP}}(G - v)$ compare to $\ell_{\text{LP}}(G)$?

Lemma

We cannot apply the reduction rule (i.e., setting $x_v = \frac{1}{2}$ for all $v \in V$ is the only LP-solution) if and only if $\ell_{\text{LP}}(G - v) = \ell_{\text{LP}}(G) - \frac{1}{2}$ for all $v \in V$.

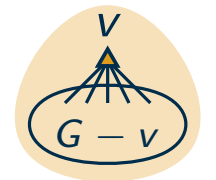
Proof for “ \Leftarrow ”: $x_v = \frac{1}{2}$ is only LP-solution $\Leftarrow \ell_{\text{LP}}(G - v) = \ell_{\text{LP}}(G) - \frac{1}{2}$ (for all $v \in V$)

- assume there is a LP-solution with $x_v = 1$ for a vertex v
- the same values (without x_v) yield LP-solution for $G - v$
- so: $G - v$ has an LP-solution of size $\ell_{\text{LP}}(G) - 1$



Proof for “ \Rightarrow ”: $x_v = \frac{1}{2}$ is only LP-solution $\Rightarrow \ell_{\text{LP}}(G - v) = \ell_{\text{LP}}(G) - \frac{1}{2}$ (for all $v \in V$)

- obvious: $\ell_{\text{LP}}(G - v) \leq \ell_{\text{LP}}(G) - \frac{1}{2}$ (all $\frac{1}{2}$ with one less vertex)
- if $\ell_{\text{LP}}(G - v) < \ell_{\text{LP}}(G) - \frac{1}{2}$: consider LP-solution for $G - v$ with value $\leq \ell_{\text{LP}}(G) - 1$
- add v with $x_v = 1$ to it \rightarrow optimal LP-solution for G



Running time

Lemma

We cannot apply the reduction rule (i.e., setting $x_v = \frac{1}{2}$ for all $v \in V$ is the only LP-solution) if and only if $\ell_{\text{LP}}(G - v) = \ell_{\text{LP}}(G) - \frac{1}{2}$ for all $v \in V$.

Branching rule

- 1 vertex is added to the vertex cover
 - previous slide: ℓ_{LP} shrinks by $\frac{1}{2}$
 - this implies: the parameter k shrinks by $\frac{1}{2}$
- } search tree height $\leq 2k \Rightarrow 2^{2k} = 4^k$ leaves

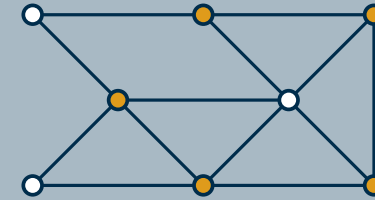
Reduction rule

- solve the LP for $G - v$ for every $v \in V$
- lemma: reduction applicable \Rightarrow we find $v \in V$ with $\ell_{\text{LP}}(G - v) < \ell_{\text{LP}}(G) - \frac{1}{2}$
- note: this means that there is an optimal LP-solution with $x_v = 1$

Wrap-Up: VERTEX COVER ABOVE LP

Problem: VERTEX COVER ABOVE LP

Given a graph $G = (V, E)$ and a parameter k . Does G have a vertex cover of size $\ell_{LP}(G) + k$?



(vertex set $V' \subseteq V$ with $e \cap V' \neq \emptyset$ for all $e \in E$)

Theorem

There is an FPT algorithm for VERTEX COVER ABOVE LP with running time $4^k \cdot n^{O(1)}$.

Branch-and-reduce

- reduce if possible using LP-relaxation
- if reduction not possible: branching makes progress (even above lower bound)

How long is the break?

$$\begin{aligned} \text{min.: } & B + r + e + a + k \\ \text{such that: } & B + k \geq 2 \\ & r - e \geq 2 \\ & 2e + a \geq 1 \end{aligned}$$

How long is the break?

$$\begin{aligned} \text{min.: } & B + r + e + a + k \\ \text{such that: } & B + k \geq 2 \\ & r - e \geq 2 \\ & 2e + a \geq 1 \end{aligned}$$

Optimal solution (value 5):

$$B = 1, r = 2.5, e = 0.5, a = 0, k = 1$$

Why is there no better solution?

Finding upper bounds

Goal: Find an upper bound for the optimal solution

- simple bound: 12
- better: 6
- combine constraints: 5

maximize: $2x_1 + 3x_2$

such that:

$$4x_1 + 8x_2 \leq 12$$

$$2x_1 + 1x_2 \leq 3$$

$$3x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

$$2x_1 + 3x_2 \leq 4x_1 + 8x_2 \leq 12$$

$$2x_1 + 3x_2 \leq \frac{4x_1 + 8x_2}{2} \leq \frac{12}{2}$$

$$2x_1 + 3x_2 \leq \frac{4x_1 + 8x_2}{3} + \frac{2x_1 + 1x_2}{3} \leq \frac{12}{3} + \frac{3}{3}$$

Systematically combine constraints with factors y_1 , y_2 , and y_3

- resulting inequality: $y_1(4x_1 + 8x_2) + y_2(2x_1 + 1x_2) + y_3(3x_1 + 2x_2) \leq 12y_1 + 3y_2 + 4y_3$
- rearranging yields: $2x_1 + 3x_2 \leq \underbrace{(4y_1 + 2y_2 + 3y_3)}_{\geq 2 \text{ (required for the first inequality)}} x_1 + \underbrace{(8y_1 + 1y_2 + 2y_3)}_{\geq 3} x_2 \leq \underbrace{12y_1 + 3y_2 + 4y_3}_{\text{minimize (for the best upper bound)}}$

Dual program

- additionally require $y_1, y_2, y_3 \geq 0$
- finds the smallest upper bound that can be achieved this way

Why do we need this?

minimize: $12y_1 + 3y_2 + 4y_3$

such that:

$$4y_1 + 2y_2 + 3y_3 \geq 2$$

$$8y_1 + 1y_2 + 2y_3 \geq 3$$



Matrix form

Primal linear program

$$\begin{aligned} \text{maximize: } & 2x_1 + 3x_2 \\ \text{such that: } & 4x_1 + 8x_2 \leq 12 \\ & 2x_1 + 1x_2 \leq 3 \\ & 3x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{max } & (2 \ 3) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \text{with } & \begin{pmatrix} 4 & 8 \\ 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 12 \\ 3 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\text{maximize } c^T x \text{ with } Ax \leq b \text{ and } x \geq 0$$

Dual linear program

$$\begin{aligned} \text{minimize: } & 12y_1 + 3y_2 + 4y_3 \\ \text{such that: } & 4y_1 + 2y_2 + 3y_3 \geq 2 \\ & 8y_1 + 1y_2 + 2y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{min } & (12 \ 3 \ 4) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ \text{with } & \begin{pmatrix} 4 & 2 & 3 \\ 8 & 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\text{minimize } b^T y \text{ with } A^T y \geq c \text{ and } y \geq 0$$

Note: the dual of the dual is the primal

Duality theorem

Theorem (without proof)

For the linear programs

$$\text{maximize } c^T x \text{ with } Ax \leq b \text{ and } x \geq 0 \text{ and} \quad (\text{P})$$

$$\text{minimize } b^T y \text{ with } A^T y \geq c \text{ and } y \geq 0, \quad (\text{D})$$

exactly one of the following statements holds:

- Neither (P) nor (D) has a feasible solution.
- (P) is unbounded and (D) has no feasible solution or vice versa.
- Both have feasible solutions. Then the maximum of (P) equals the minimum of (D).

Notes

- the dual yields a perfect upper bound for the primal
- works for arbitrary LPs; not just ones in the above form
- relation to vertex cover: the dual can be a nice tool to get a different perspective → exercise

Back to FPT: Lenstra's theorem

Theorem (without proof)

For $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, one can answer the question whether there exists an $x \in \mathbb{N}^n$ with $Ax \leq b$ in $O(n^{2,5n}|A, b|)$ time, where $|A, b|$ is the length of the binary encoding of the instance.

It follows

(n is also called the *dimension* of the ILP)

- ILP (as decision problem) with parameter $n =$ (number of variables) is in FPT
- number of inequalities contributes only polynomially to the running time
- size of the numbers contributes only logarithmically to the running time

Metatheorem

A parameterized problem with parameter k that can be formulated as an ILP with $f(k)$ many variables is in FPT.

Example: CLOSEST STRING

Problem: CLOSEST STRING

(k is our parameter)

Given k strings $s_1, \dots, s_k \in \Sigma^n$ and $d \in \mathbb{N}$. Is there a string $s \in \Sigma^n$, that has Hamming distance at most d for every s_i ?

Observation

- only equality in each column relevant \rightarrow equivalent input with $\Sigma = [k]$
- set of column types $\tau \rightarrow |\tau|$ only depends on k

(bell number $B_k \leq k!$)

Formulation as ILP

- solution boils down to: How often is char $a \in \Sigma$ selected for column-type $t \in \tau$? \rightarrow variable $x_{t,a}$
- choose one character for each column
- distance $\leq d$ for every s_i
- yields ILP with “only” $k! \cdot k$ variables

$$\tau = \left\{ \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 3 & 1 & 1 \end{matrix} \right\}$$

#chars selected for columns of type t

#columns of type t

$$\forall t \in \tau : \sum_{a \in \Sigma} x_{t,a} = \#t$$

$$\forall i \in [k] : \sum_{t \in \tau} \sum_{\substack{a \in \Sigma \\ a \neq t[i]}} x_{t,a} \leq d$$

s_1 A B C F G D C G E B A
 s_2 B B D A G D B G B E C
 s_3 B A E F C A C G B B F

↓

s_1	1	1	1	1	1	1	1	1	1	1
s_2	2	1	2	2	1	1	2	1	2	2
s_3	2	2	3	1	2	2	1	1	2	1
s	2	1	3	1	2	1	2	1	2	1

0 × 1	2 × 2	0 × 3
2 × 1	1 × 2	0 × 3
0 × 1	1 × 2	1 × 3
2 × 1	1 × 2	0 × 3
1 × 1	0 × 2	0 × 3

#columns where selection disagrees with s_i

Wrap-Up

Theorem

There is an FPT algorithm for VERTEX COVER ABOVE LP with running time $4^k \cdot n^{O(1)}$.

Other lower bounds

- the size of a maximum matching $\ell_M(G)$ is also a lower bound
- from duality it follows: $\ell_M(G) \leq \ell_{LP}(G) \Rightarrow$ today's algo is also FPT for VC ABOVE MATCHING
(see exercise sheet)
- one can show: $2\ell_{LP}(G) - \ell_M(G)$ is also a lower bound (and it is stronger)
- VERTEX COVER ABOVE $2\ell_{LP}(G) - \ell_M(G)$ is also FPT

Duality

- systematic calculation of upper bounds
- strong duality theorem: very useful tool in many areas

Literature

Raising The Bar For Vertex Cover: Fixed-parameter Tractability Above A Higher Guarantee

■ Shivam Garg, Geevarghese Philip

[2016]

■ contains the mentioned result for VERTEX COVER ABOVE $2\ell_{LP}(G) - \ell_M(G)$

■ many additional references on the topic

doi.org/10.1137/1.9781611974331.ch80

Branch-and-Reduce Exponential/FPT Algorithms in Practice: A Case Study of Vertex Cover

■ Takuya Akiba, Yoichi Iwata

[2016]

■ Branch-and-reduce for VERTEX COVER in practice

doi.org/10.1016/j.tcs.2015.09.023

WeGotYouCovered: The Winning Solver from the PACE 2019 Challenge, Vertex Cover Track

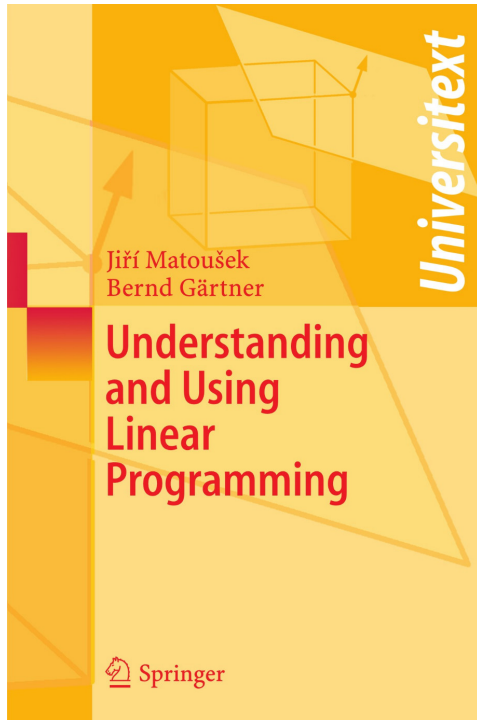
■ Demian Hesse, Sebastian Lamm, Christian Schulz, Darren Strash

[2020]

■ fast algorithm for VERTEX COVER in practice

doi.org/10.1137/1.9781611976229.1

More literature



Understanding and Using Linear Programming

- exceptionally well written and compact book
- ~~free for KIT members~~

link.springer.com/book/10.1007/978-3-540-30717-4

Integer Programming in Parameterized Complexity: Three Miniatures

- Tomás Gavenciak, Dusan Knop, Martin Koutecký [2019]
- good overview about ILPs for parameterized problems; has many additional references
drops.dagstuhl.de/opus/volltexte/2019/10222/