Parameterized Algorithms

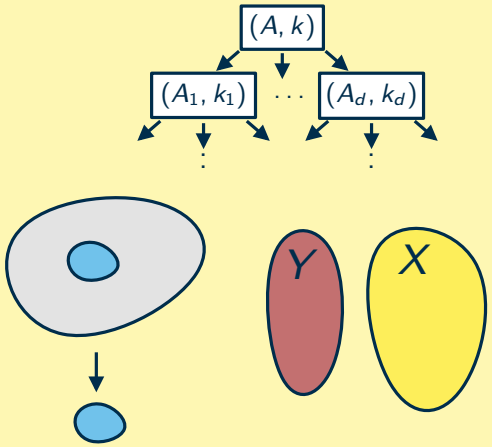
Kernelization: Similar Trees

Thomas Bläsius

Content

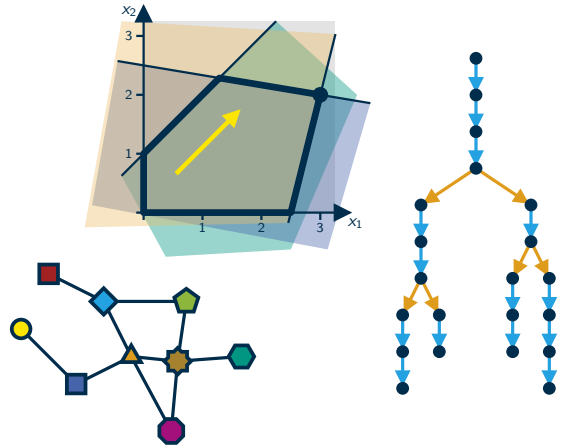
Basic toolbox

- bounded search trees
- kernelization
- iterative compression



Extended toolbox

- linear programs
- branch-and-reduce
- color coding

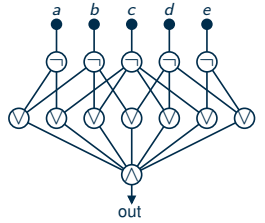


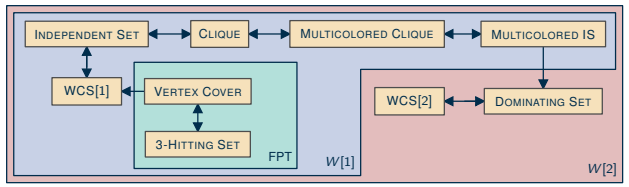
Tree width

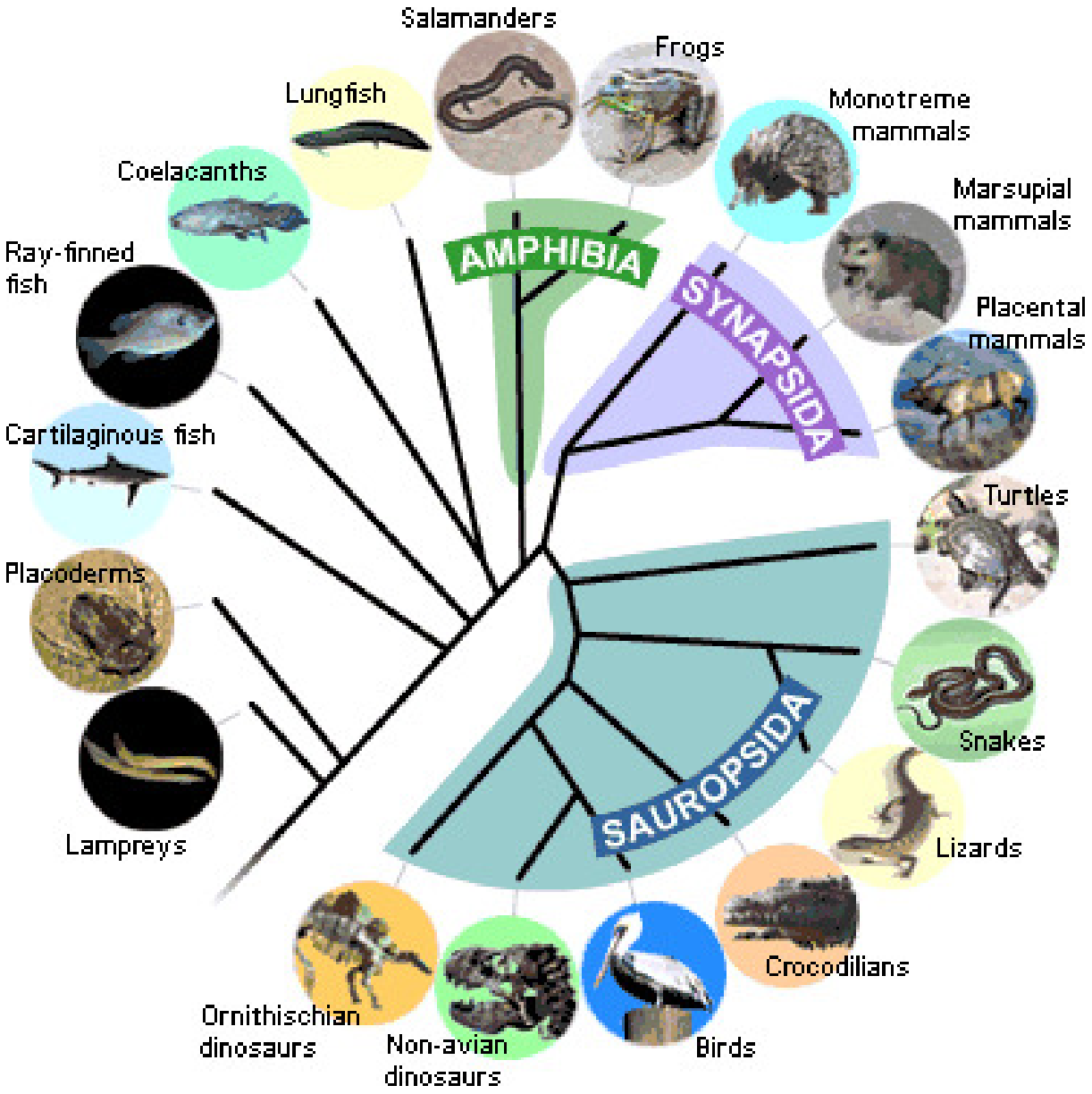
- dynamic programming
- chordal and planar graphs
- Courcelle's theorem

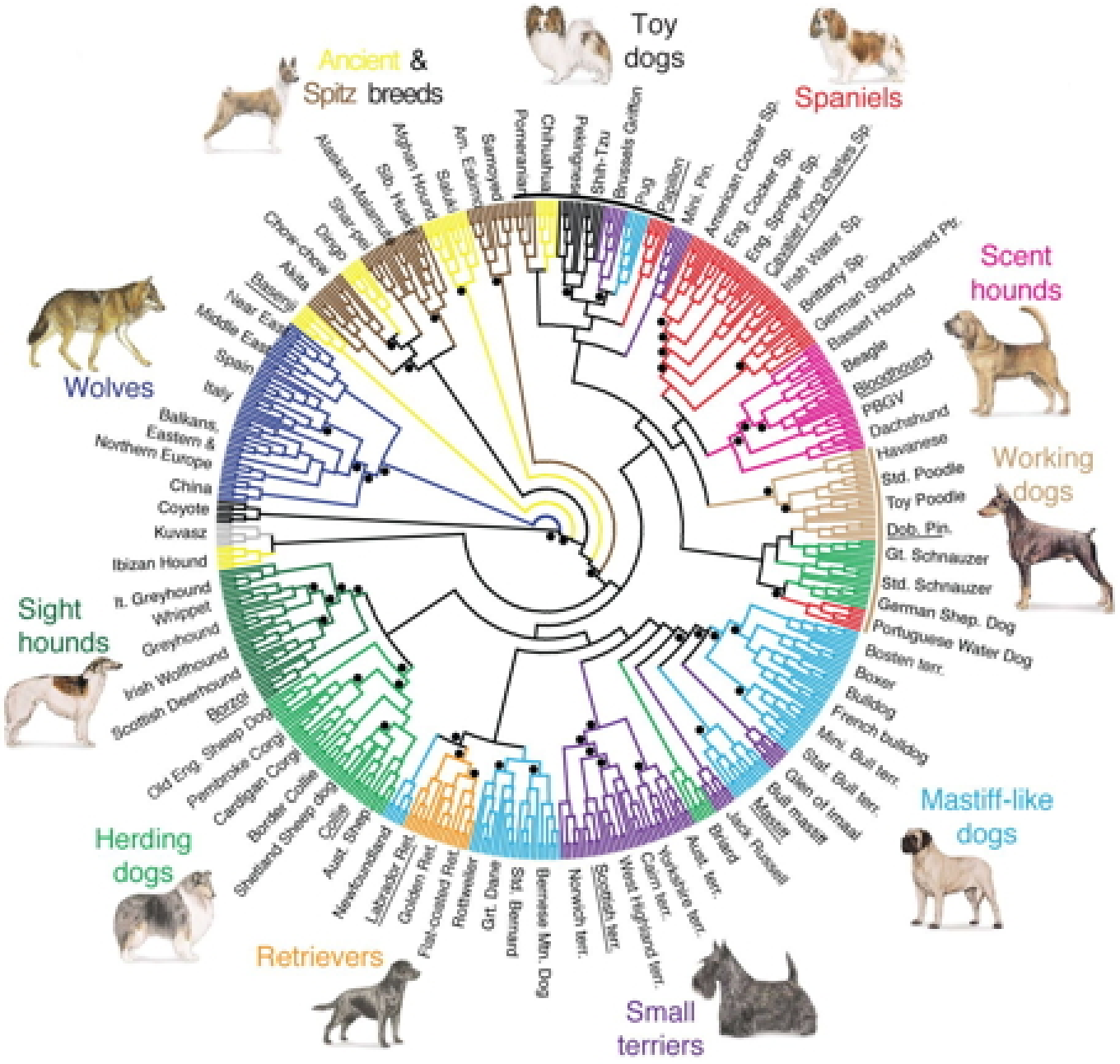
Lower bonds

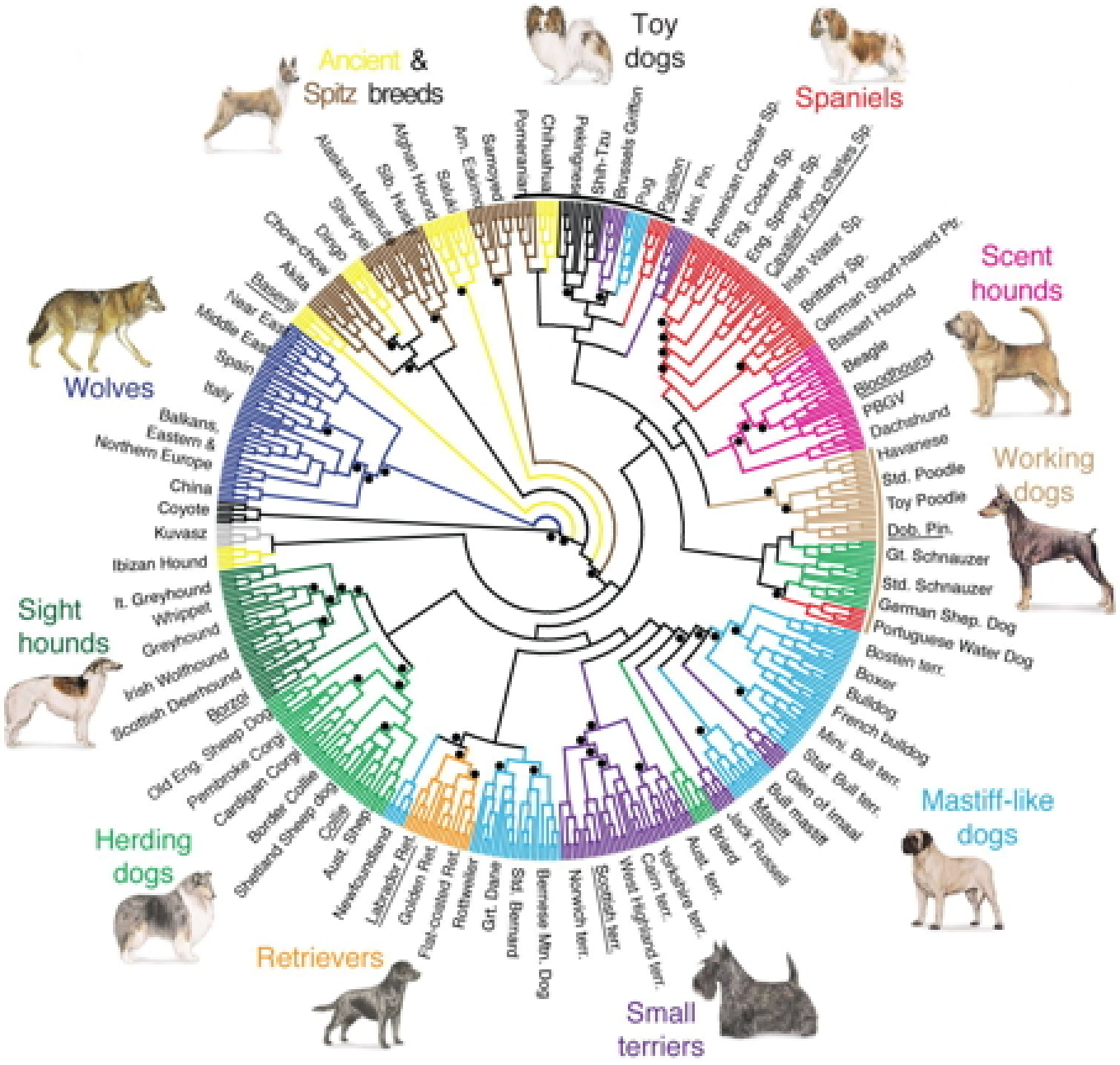
- kernel lower bounds
- parameterized reductions
- circuits and the W-hierarchy
- ETH and SETH






Salamanders
Frogs
Lungfish
Monotreme mammals
Coelacanths
Marsupial mammals
AMPHIBIA
SYNAPSIDA
Ray-finned fish
Placental mammals
Cartilaginous fish
Turtles
Placoderms
SAUROPSIDA
Snakes
Lizards
Lampreys
Crocodilians
Ornithischian dinosaurs
Non-avian dinosaurs
Birds


Toy dogs
Ancient & Spitz breeds
Spaniels
Scent hounds
Working dogs
Mastiff-like dogs
Small terriers
Retrievers
Herding dogs
Sight hounds
Wolves
Pomeranian
Chihuahua
Pekingese
Shih-Tzu
Brussels Griffon
Pug
Papillon
Min. Pin.
American Cocker Sp.
Eng. Cocker Sp.
Eng. Springer Sp.
Cavalier King Charles Sp.
Irish Water Sp.
Brittany Sp.
German Short-haired Pg.
Basset Hound
Beagle
Bloodhound
PBGV
Dachshund
Havanese
Std. Poodle
Toy Poodle
Dob. Pin.
Gt. Schnauzer
Std. Schnauzer
German Shep. Dog
Portuguese Water Dog
Boston terr.
Boxer
Bulldog
French bulldog
Mins. Bull terr.
Stds. Bull terr.
Glen of Imaal
Bull mastiff
Mastiff
Jack Russell
Brard
Aust. terr.
Yorkshire terr.
Cairn terr.
West Highland terr.
Scottish terr.
Norwich terr.
Bernese Mtn. Dog
Std. Bernard
Grt. Dane
Rottweiler
Flat-coated Ret.
Golden Ret.
Labrador Ret.
Newfoundland
Aust. Shep.
Collie
Shetland Sheep dog
Border Collie
Cardigan Corgi
Pembroke Corgi
Old Eng. Sheep Dog
Bergi
Scottish Deerhound
Irish Wolfhound
Greyhound
Whippet
It. Greyhound
Ibizan Hound
Kuvasz
Coyote
China
Balkans, Eastern & Northern Europe
Italy
Spain
Middle East
Near East
Basenji
Akita
Chow-chow
Dingo
Shar-pei
Alaskan Malamute
Sib. Husky
Afghan Hound
Saluki
Am. Eskimo
Samoyed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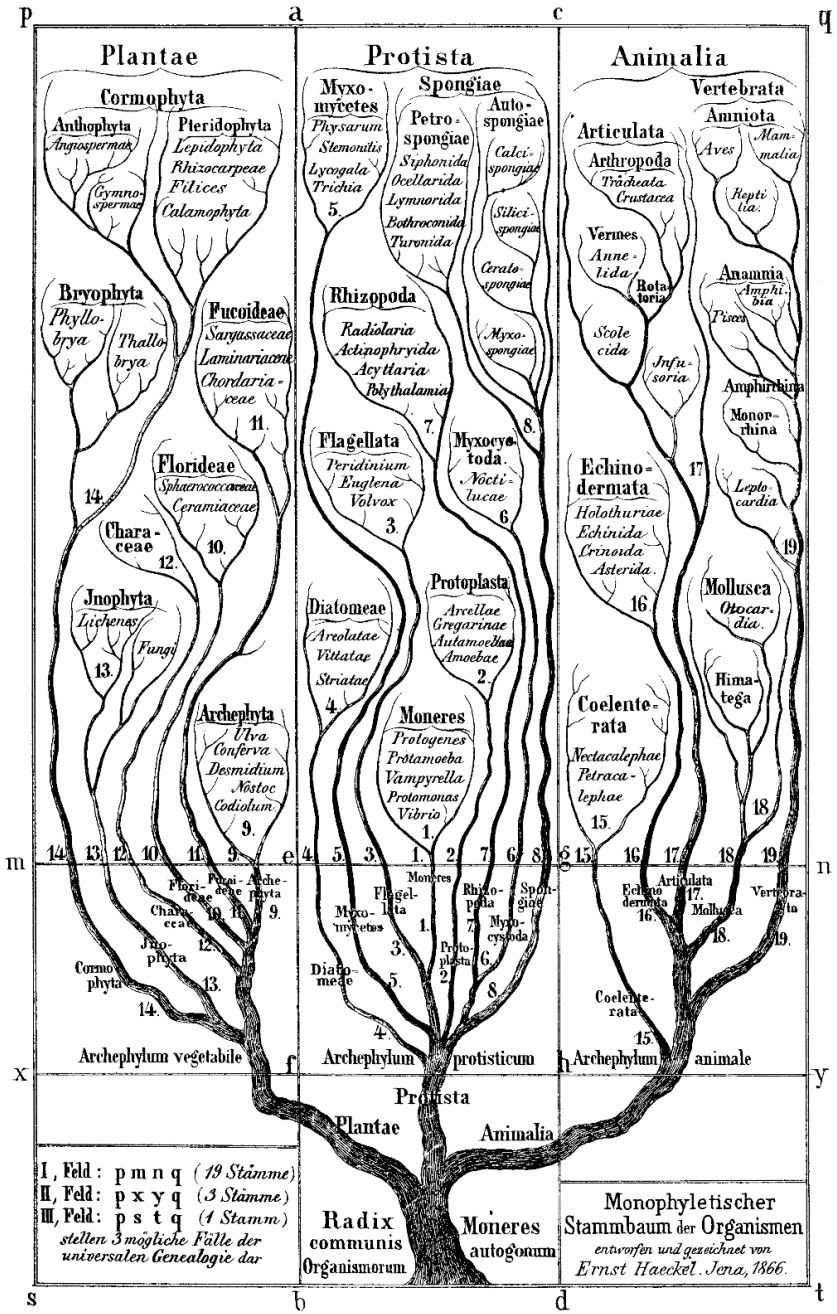
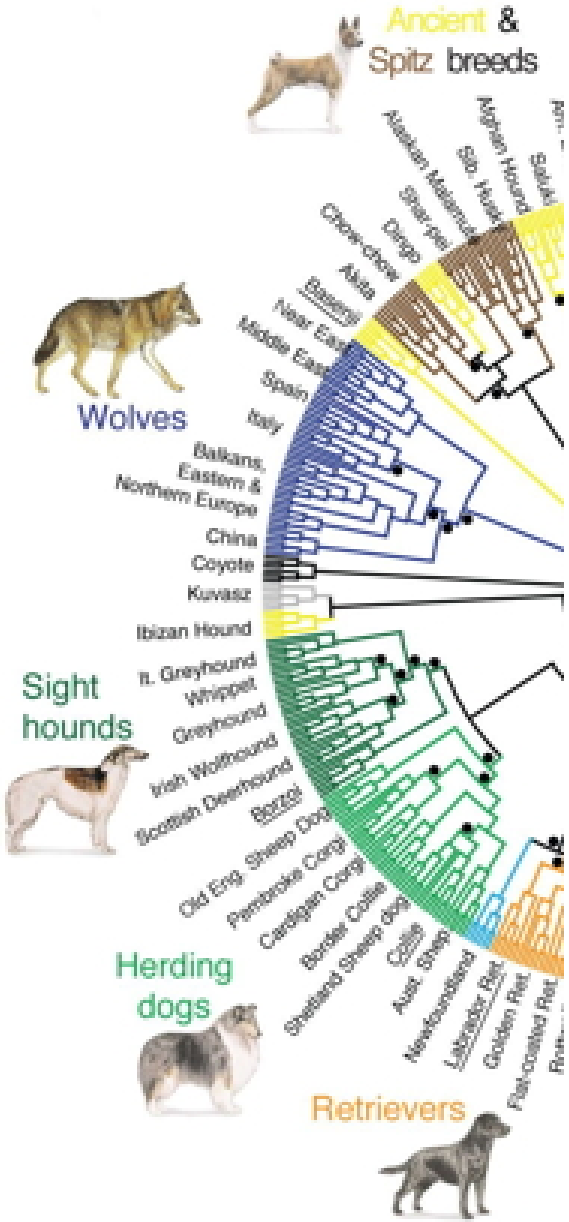

Plantae
Protista
Animalia
Cormophyta
Anthophyta
Angiospermae
Gymnospermae
Pteridophyta
Lepidophyta
Rhizocarpeae
Filices
Calamophyta
Bryophyta
Phyllobrya
Thallobrya
Fucoideae
Sargassaceae
Laminariaceae
Chordariaceae
Florideae
Sphaerococcaceae
Ceramiaceae
Characeae
Jnophyta
Lichenes
Fungi
Archephyta
Ulva
Conferva
Desmidium
Nostoc
Codiolum
Myxomycetes
Physarum
Stemonitis
Lycogala
Trichia
Spongiae
Petrospongiae
Siphonida
Ocellarida
Lymnorida
Bothrocnida
Turonida
Autospongiae
Calcispongiae
Silicispongiae
Ceratospongiae
Myxospongiae
Rhizopoda
Radiolaria
Actinophryida
Acyttaria
Polythalamia
Flagellata
Peridinium
Euglena
Volvox
Myxocytoda
Noctiluca
Diatomeae
Areolatae
Vittatae
Striatae
Protoplasta
Arcellae
Gregarinae
Autamoebae
Amoebae
Moneres
Protogenes
Protamoeba
Vampyrella
Protomonas
Vibrio
Vertebrata
Amniota
Aves
Mammalia
Reptilia
Anamnia
Amphibia
Pisces
Amphibina
Monorrhina
Leptocardia
Articulata
Arthropoda
Tracheata
Crustacea
Vermes
Annelida
Rotatoria
Scolecida
Infusoria
Echinodermata
Holothuriae
Echinida
Crinoida
Asterida
Mollusca
Otocardia
Himatega
Coelenterata
Nectacalephae
Petracalephae
Archephylum vegetabile
Archephylum protisticum
Archephylum animale
I, Feld: p m n q (19 Stämme)
II, Feld: p x y q (3 Stämme)
III, Feld: p s t q (1 Stamm)
stellen 3 mögliche Fälle der universalen Genealogie dar
Radix communis Organismorum
Moneres autogonum
Monophyletischer Stammbaum der Organismen
entworfen und gezeichnet von
Ernst Haeckel. Jena, 1866.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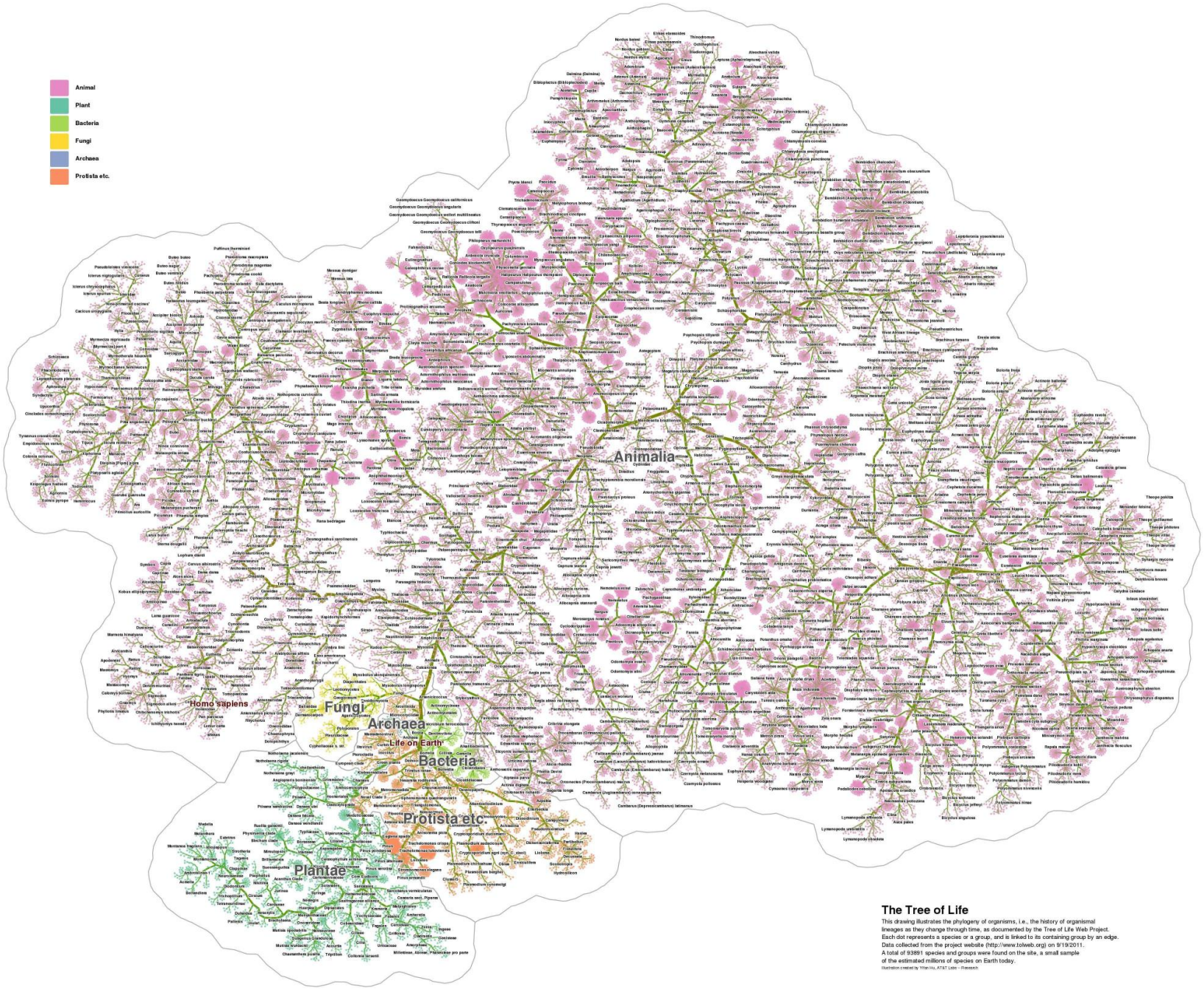
Animal
Plant
Bacteria
Fungi
Archaea
Protista etc.
Animalia
Homo sapiens
Fungi
Archaea
Life on Earth
Bacteria
Protista etc.
Plantae
The Tree of Life
This drawing illustrates the phylogeny of organisms, i.e., the history of organismal lineages as they change through time, as documented by the Tree of Life Web Project. Each dot represents a species or a group, and is linked to its containing group by an edge. Data collected from the project website (http://www.tolweb.org) on 9/19/2011. A total of 93891 species and groups were found on the site, a small sample of the estimated millions of species on Earth today.

Ancient &

Wolv

Sight hounds

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Illustration created by Yifan Hu, AT&T Labs – Research

Phylogenetic trees

Definition
A **phylogenetic tree** is an unrooted full binary tree, such that each leaf has a unique label.

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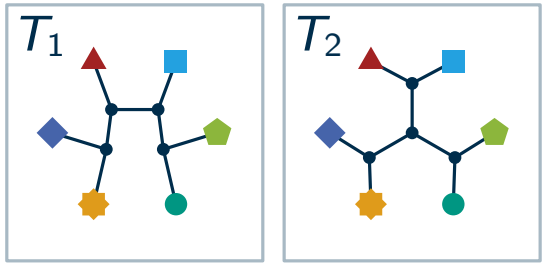
- How to compare different trees T_1 and T_2 on the same set of leaves L?



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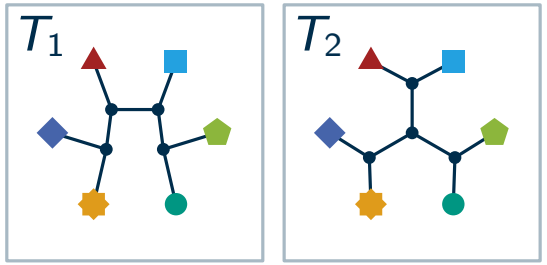

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- forest F of full binary trees with leaves L
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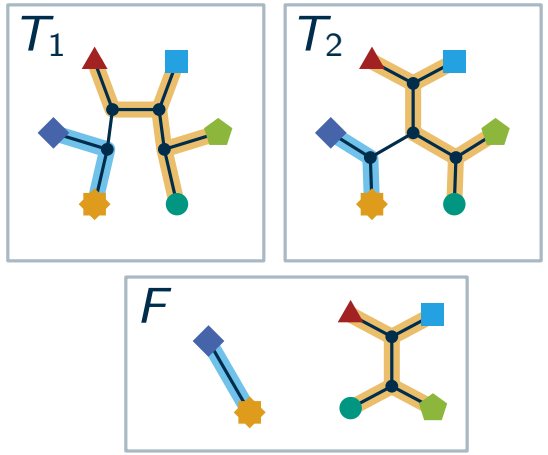

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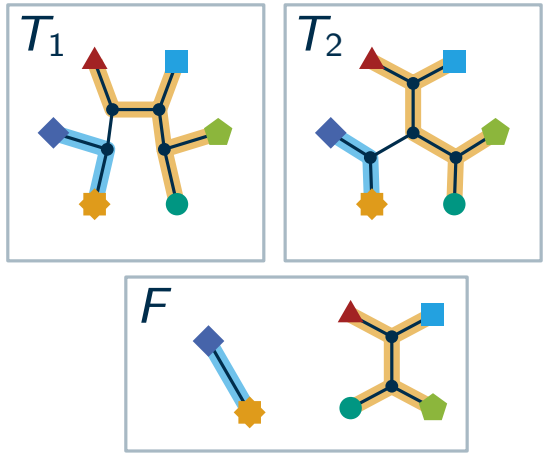


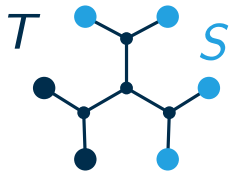
Maximum agreement forest

- forest F of full binary trees with leaves L
- trees in F are contained in T_1 and T_2 (as minors)
- minimize the number of trees in F

What does this mean exactly?

Leaf-induced subtrees

- let T be a tree with leaves $L(T)$ and let $S \subseteq L(T)$



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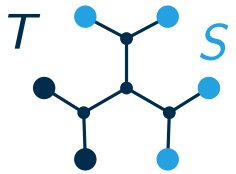

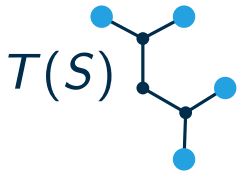

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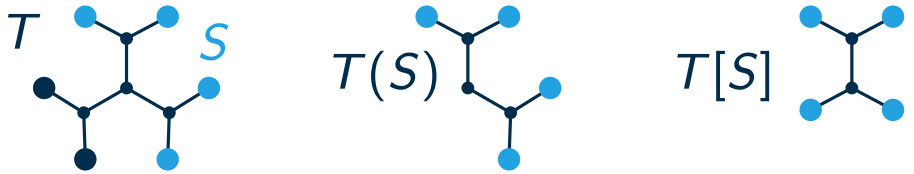
- contract all vertices of degree 2 in $T(S) \to T[S]$



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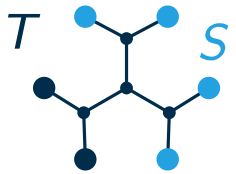
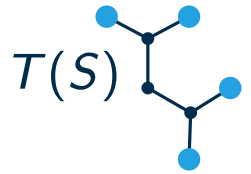
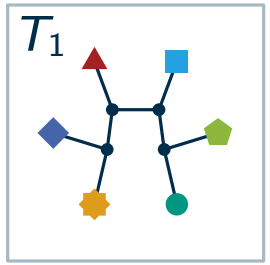
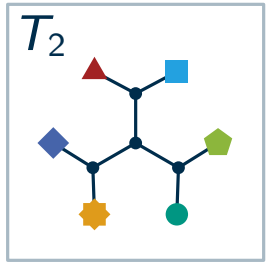


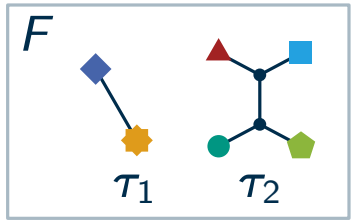


Agreement forest

- let F be a forest consisting of the trees $\tau_1, \ldots, \tau_k$
- T_1 and T_2 are trees (same leaves $L(T_1) = L(T_2) = L(F)$)







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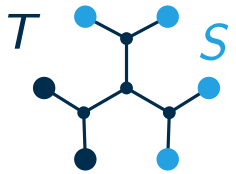

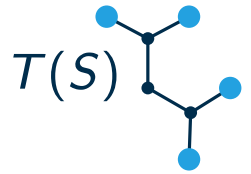

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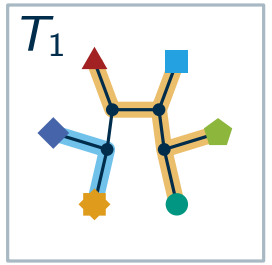

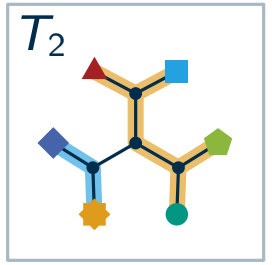

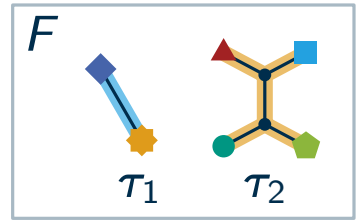

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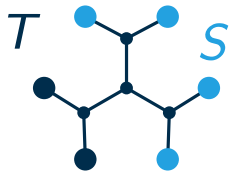

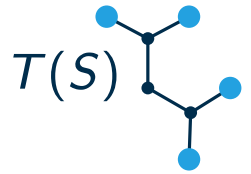

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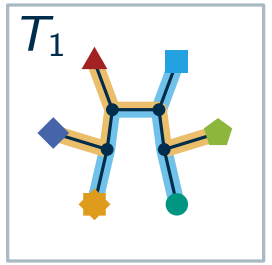

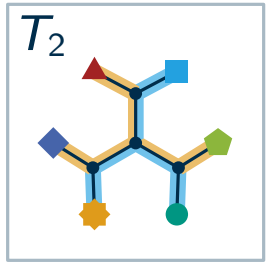

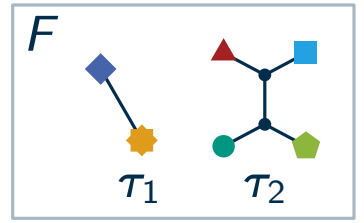

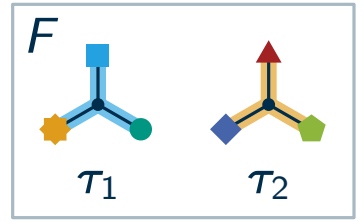

What does this mean exactly?

Leaf-induced subtrees

- let T be a tree with leaves $L(T)$ and let $S \subseteq L(T)$
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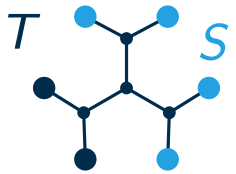

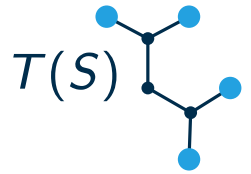

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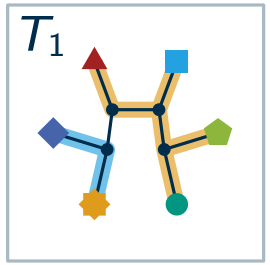

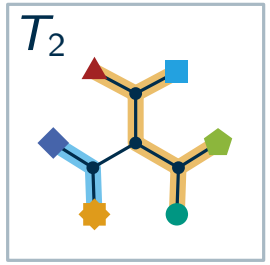

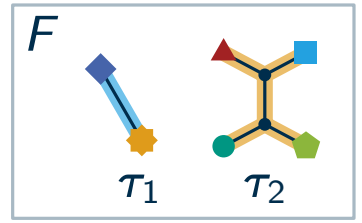

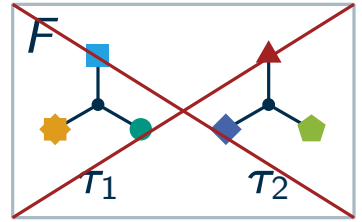

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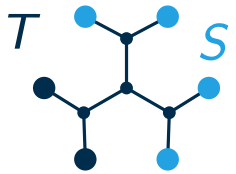

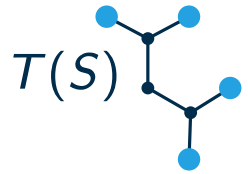

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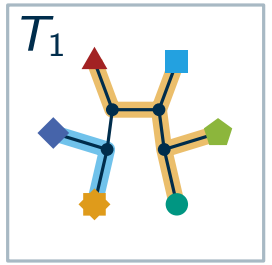

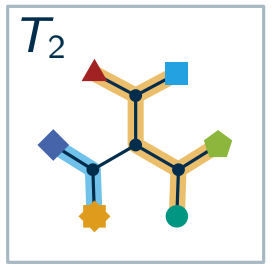

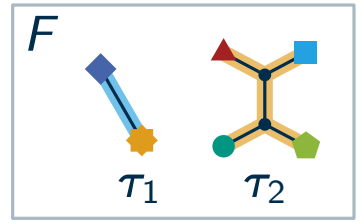

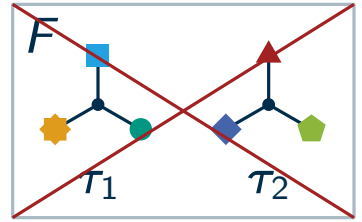

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Problem: MAXIMUM AGREEMENT FOREST
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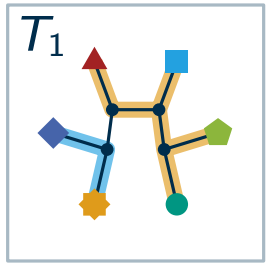

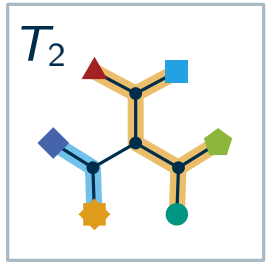

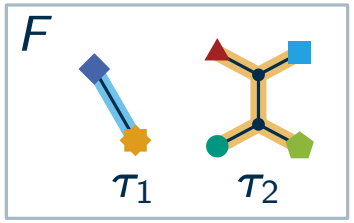

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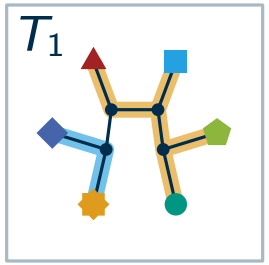

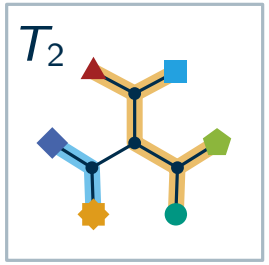

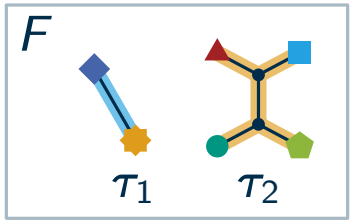

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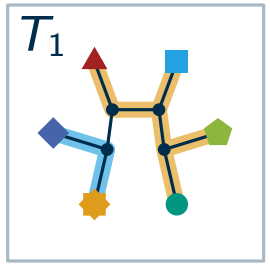

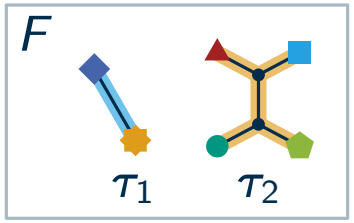

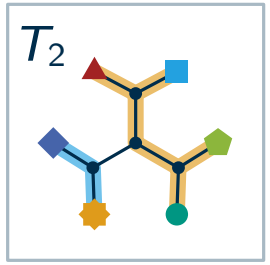

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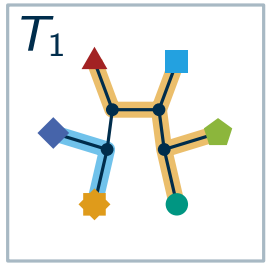

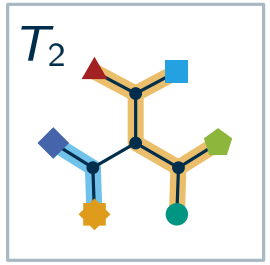

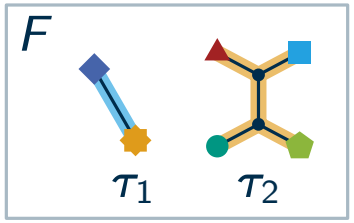

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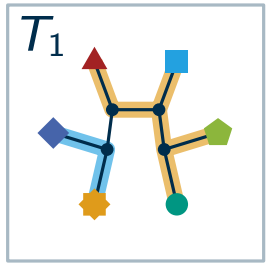

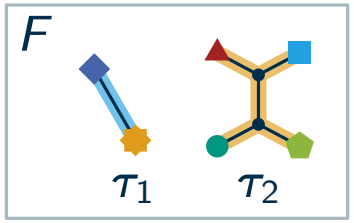

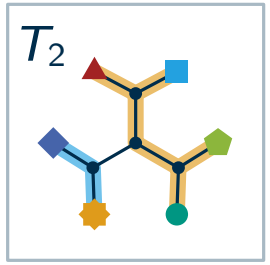

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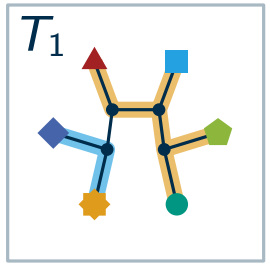

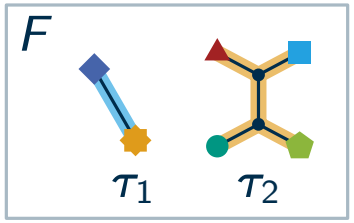

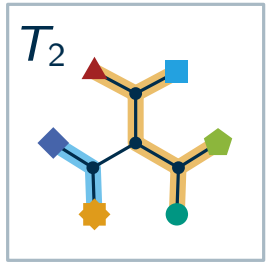

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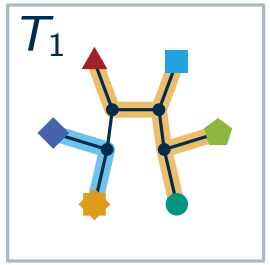

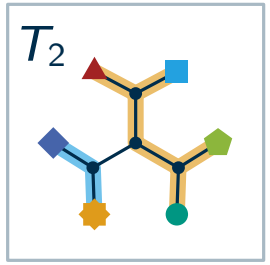

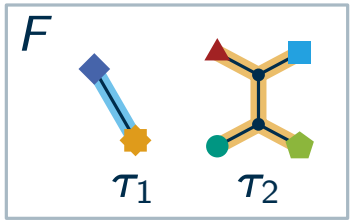

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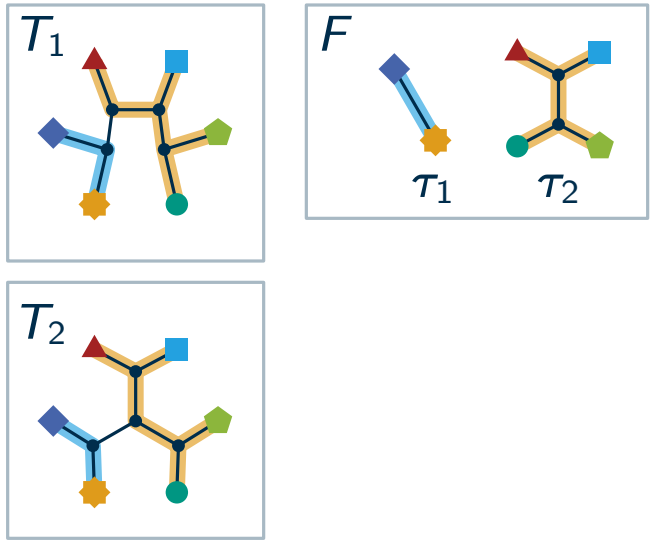

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- size of the resulting trees only depends on k
- slightly stronger but more specific claim:
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  - implies: few leaves overall $\Rightarrow$ small instance
- Which tree structures contradict this claim? Can we remove them using reduction rules?

Undesirable tree structure (1)

Claim: Every tree of the agreement forest has only few leaves.

Counterexample

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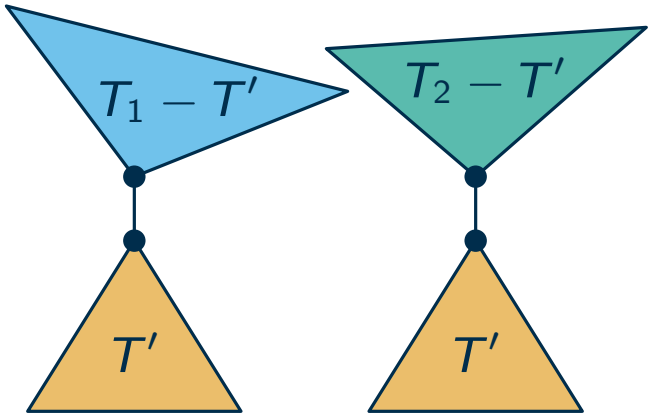

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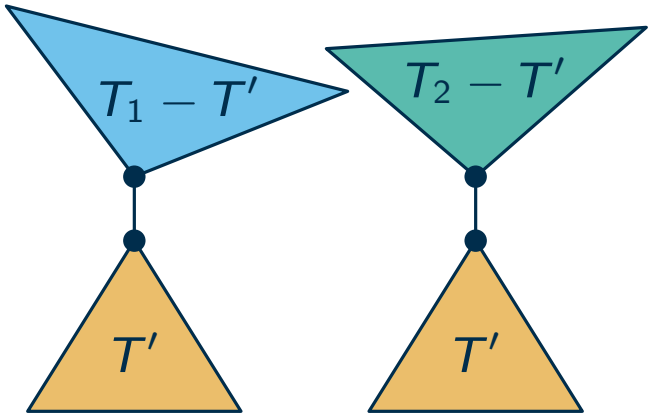

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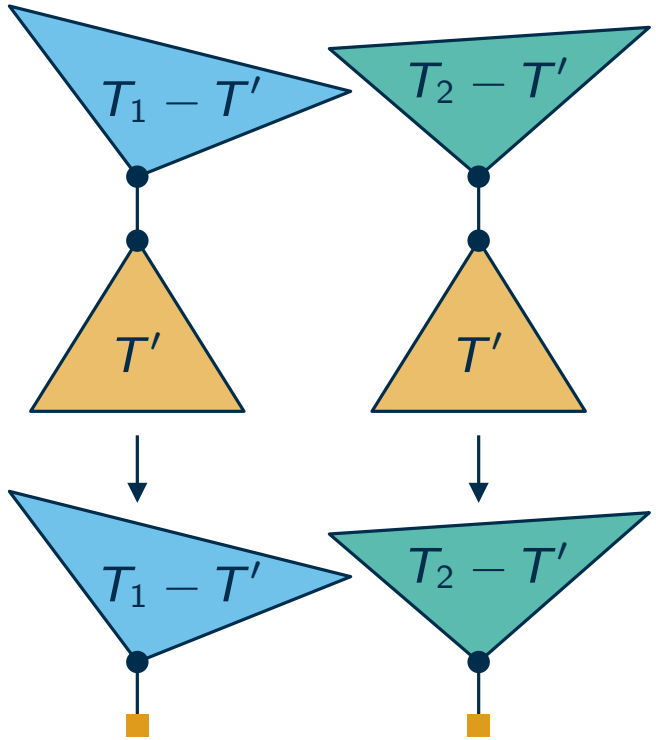

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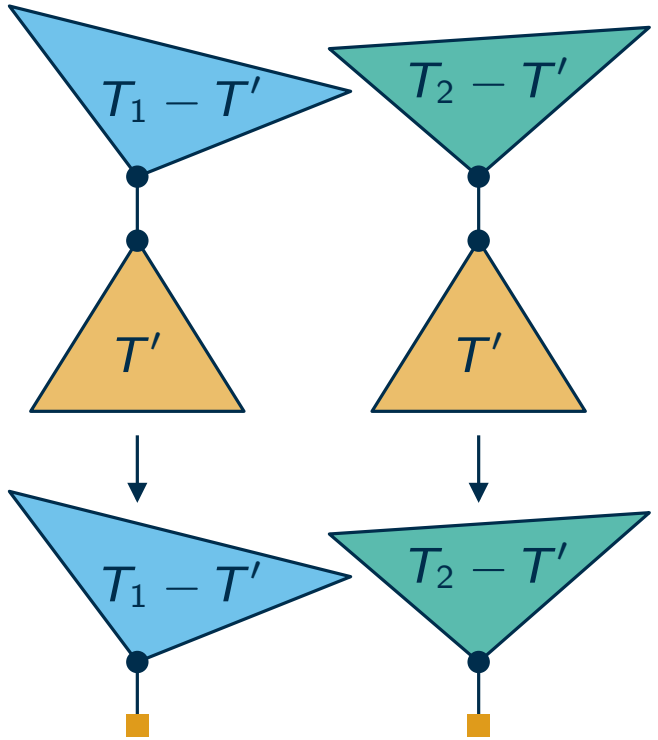
Reduction rule 1 is safe and can be applied in polynomial time.

Proof: next slide



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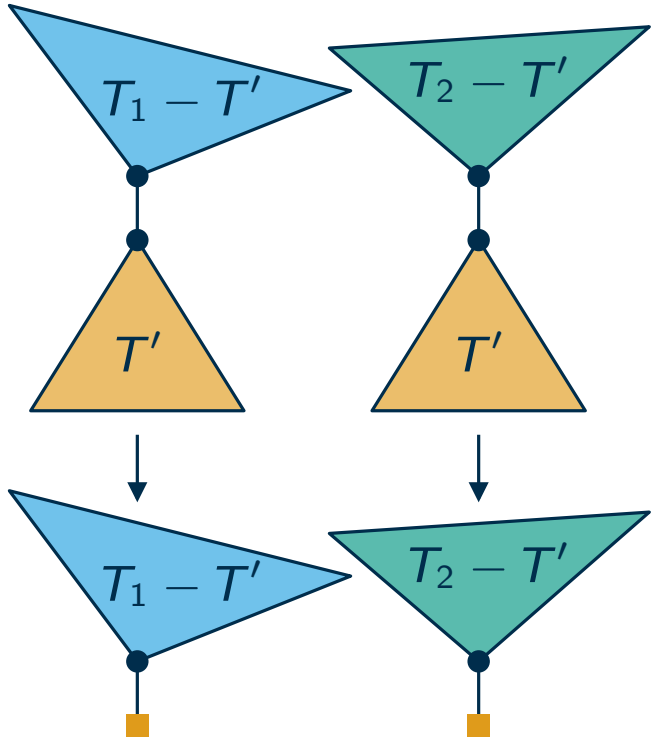

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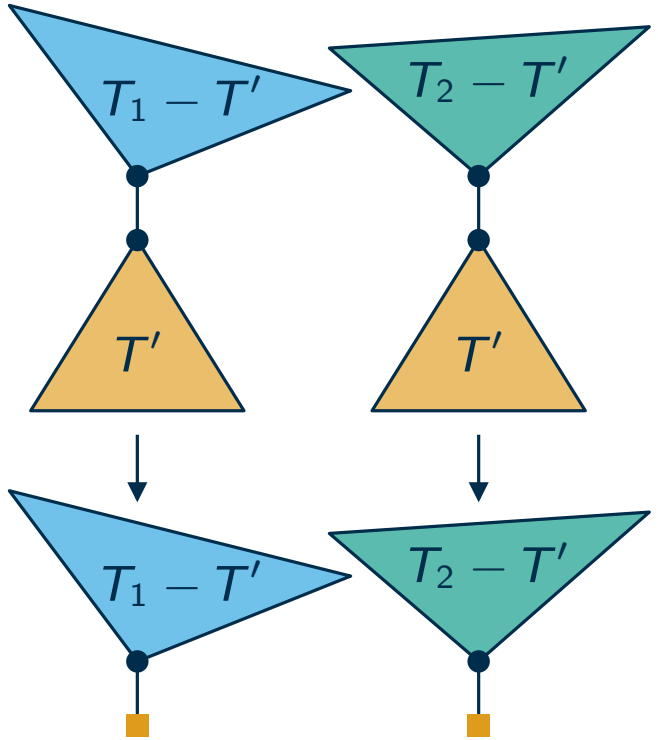

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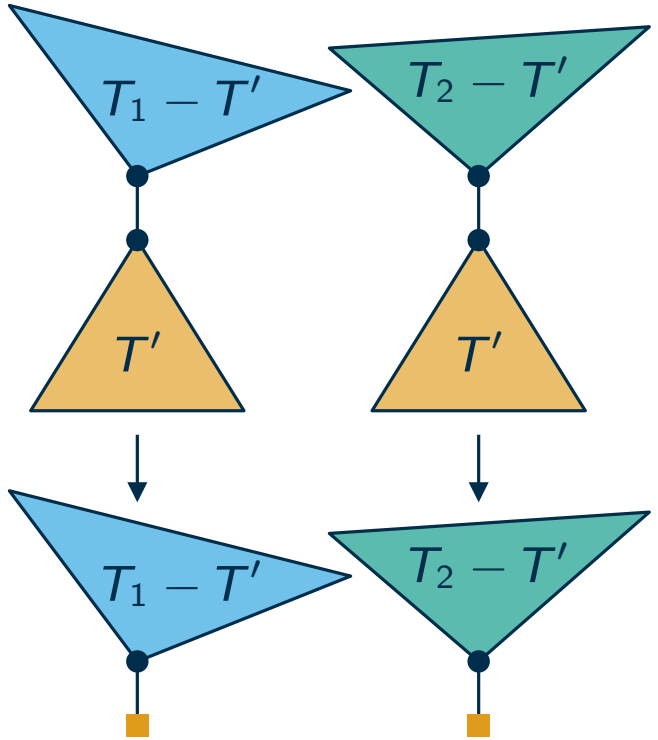

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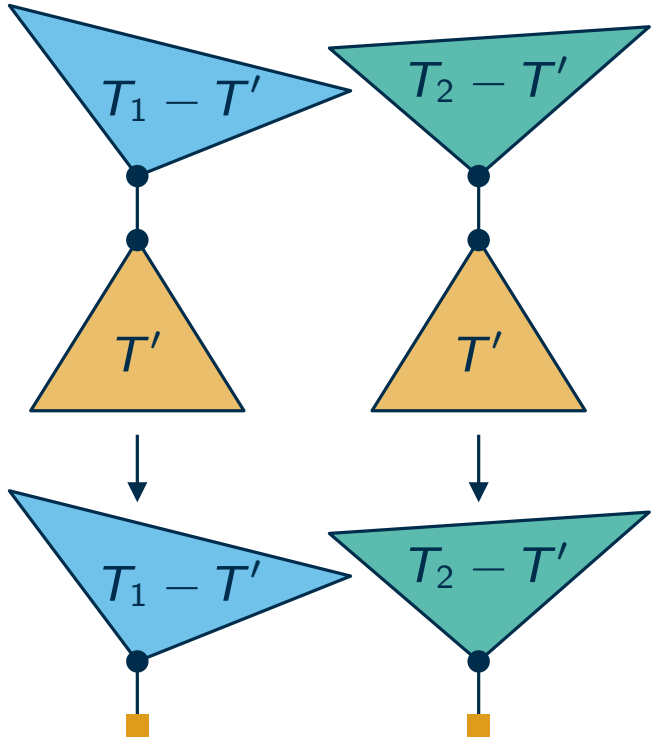
- replace this leaf in τ_1' by $T' \rightarrow$ agreement forest of the same size for T_1 and T_2
- other direction: similar

Reduction rule 1



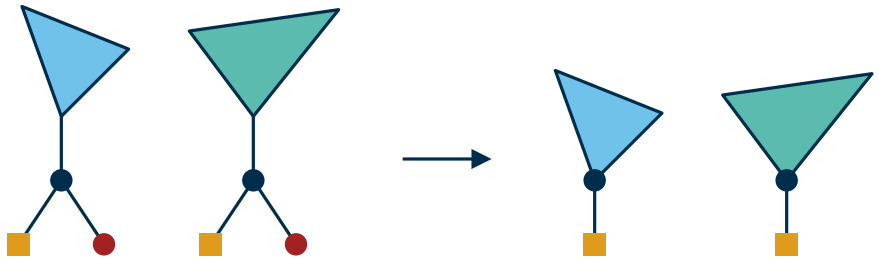
Reduction rule 1

- find edges in T_1 and T_2 that cut off the same subtree T'
- replace T' by a leaf with a new label (same label in T_1 and T_2)

Lemma
Reduction rule 1 is safe and can be applied in polynomial time.

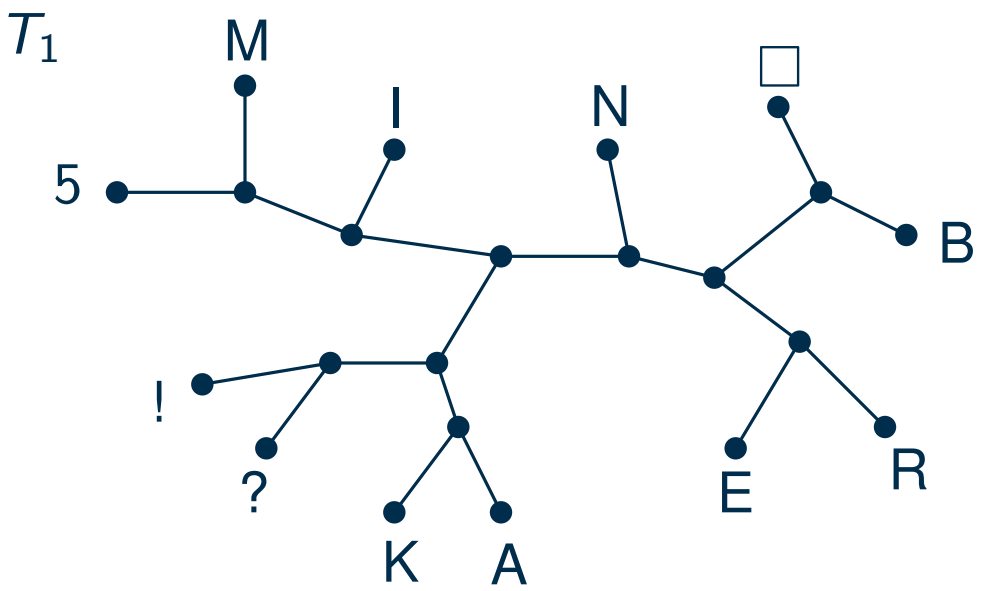
Proof

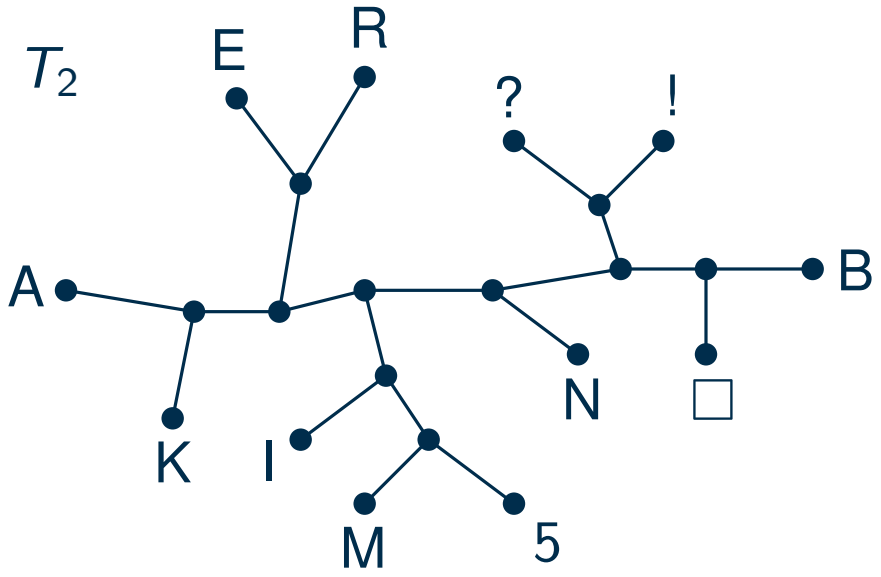
- let F' be an agreement forest of the new instance and let τ_1' be the tree with the new leaf in F'
- replace this leaf in τ_1' by $T' \rightarrow$ agreement forest of the same size for T_1 and T_2
- other direction: similar
- polynomial running time: for example by iteratively eating “cherries”



Example

Solve this instance of MAXIMUM AGREEMENT FOREST





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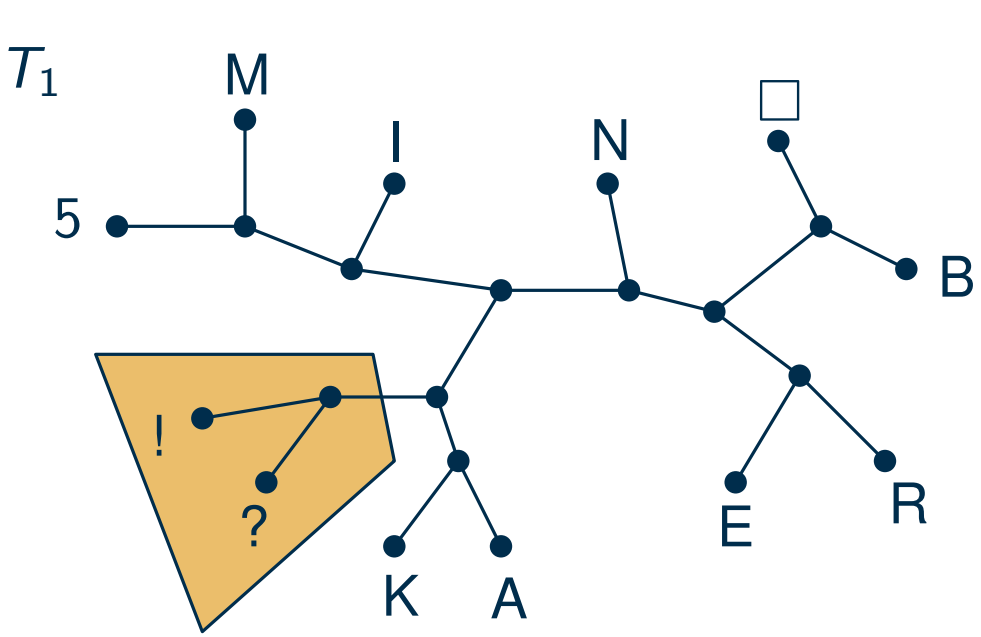

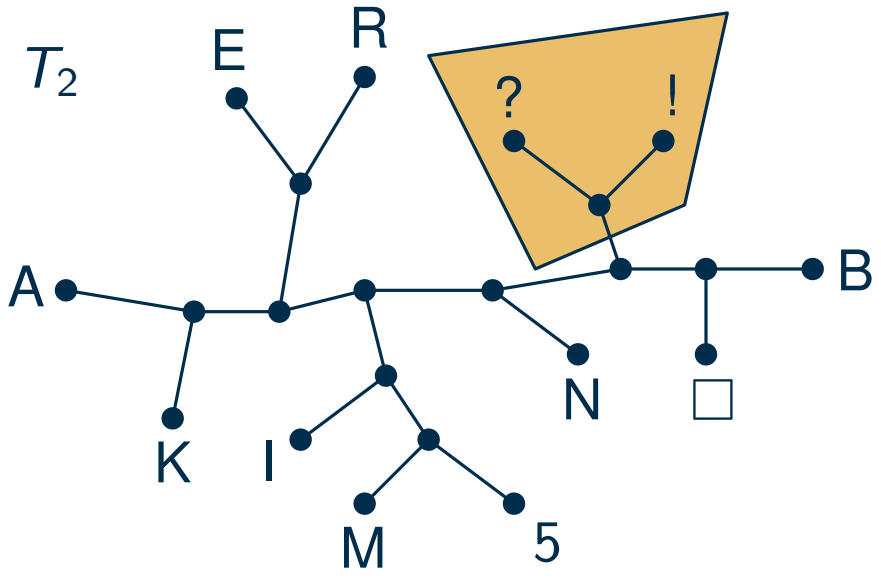

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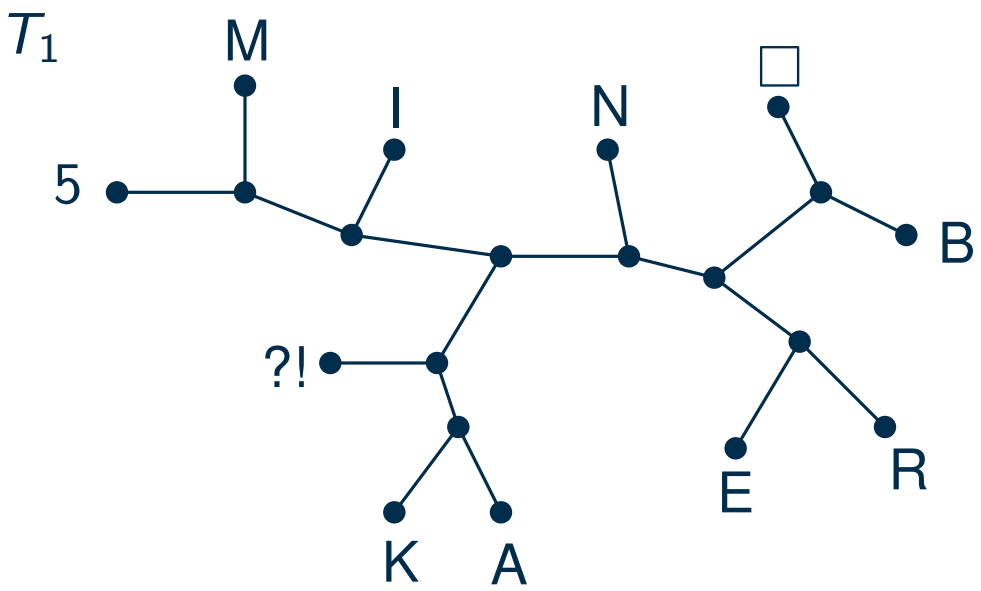

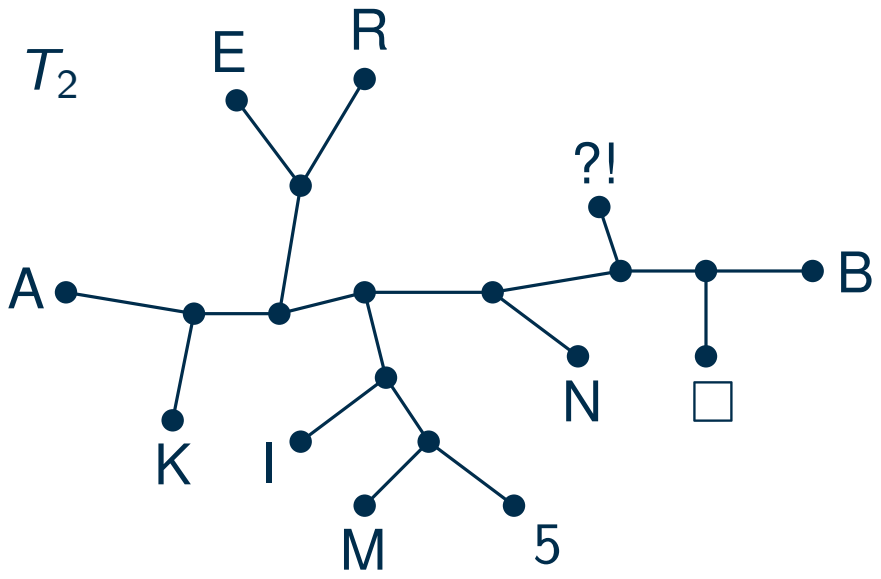

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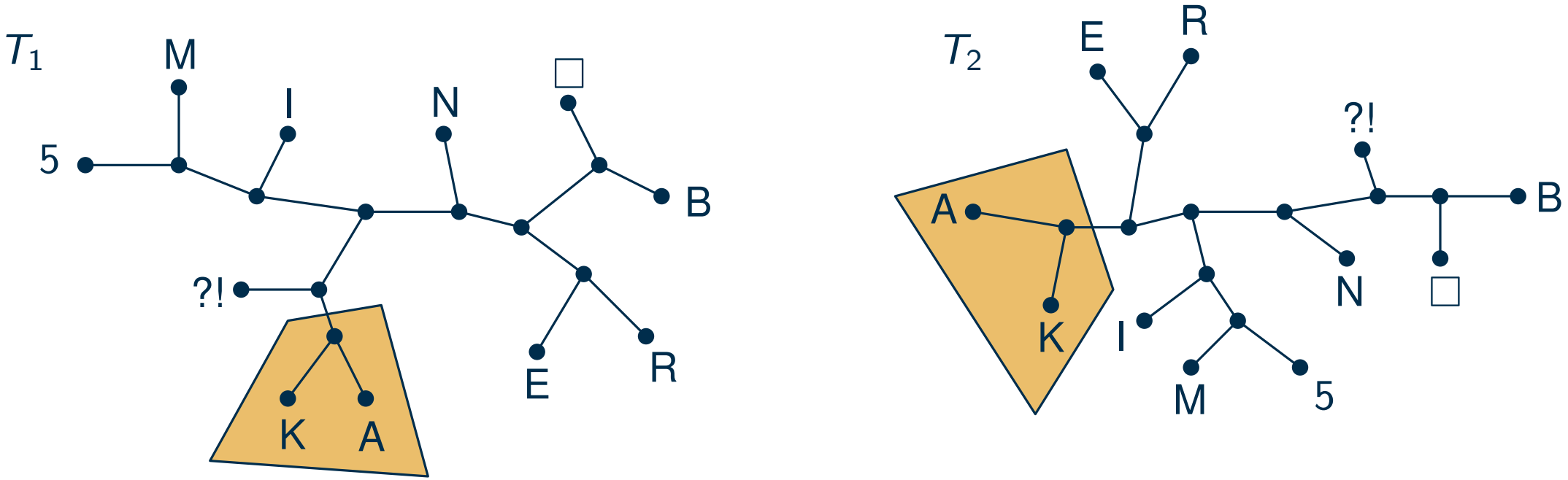

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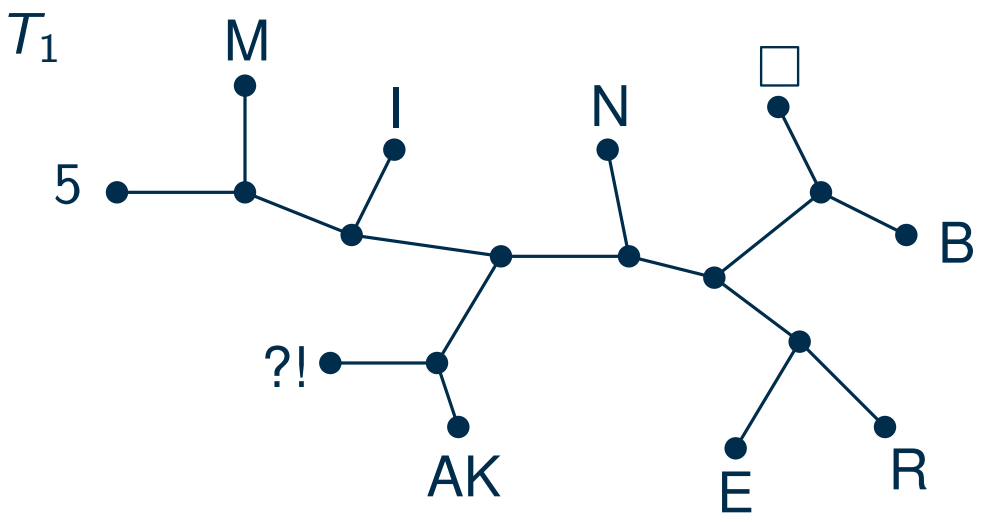

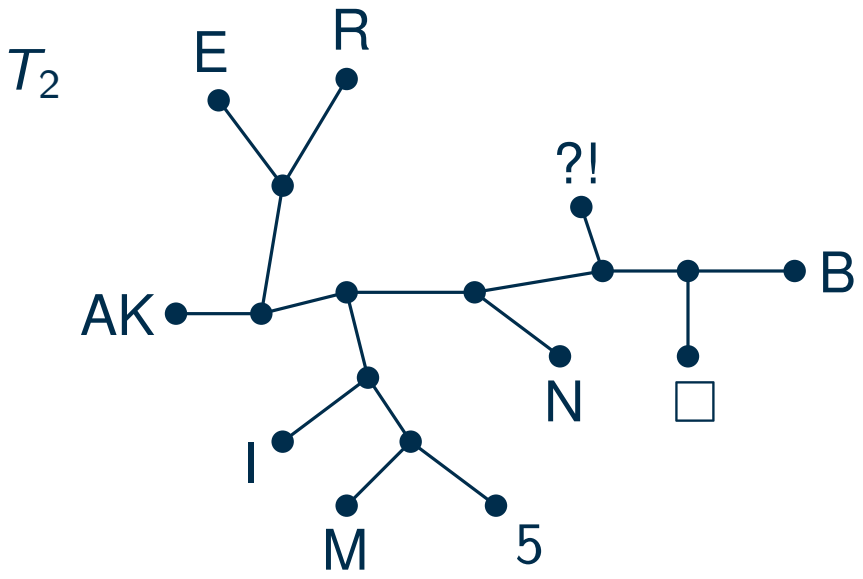

Example

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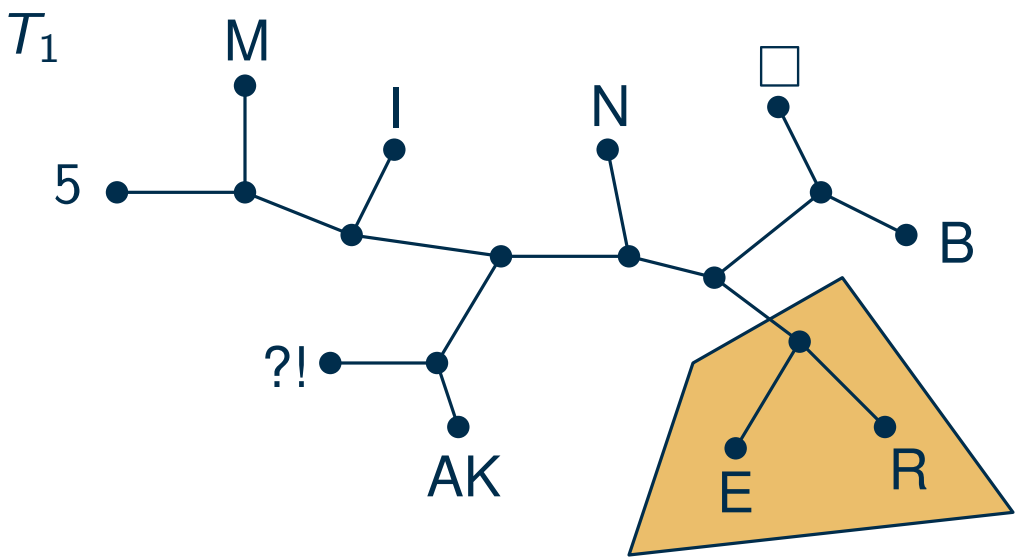

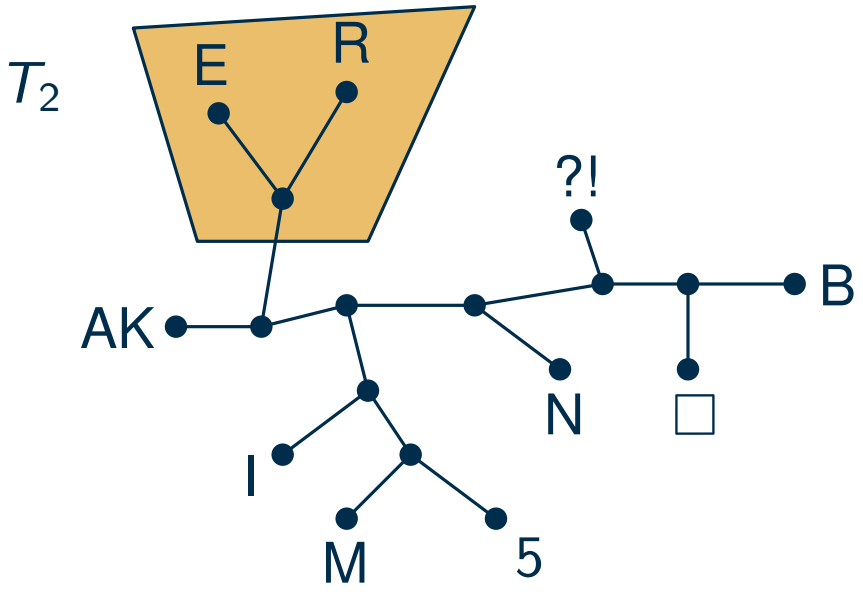

Example

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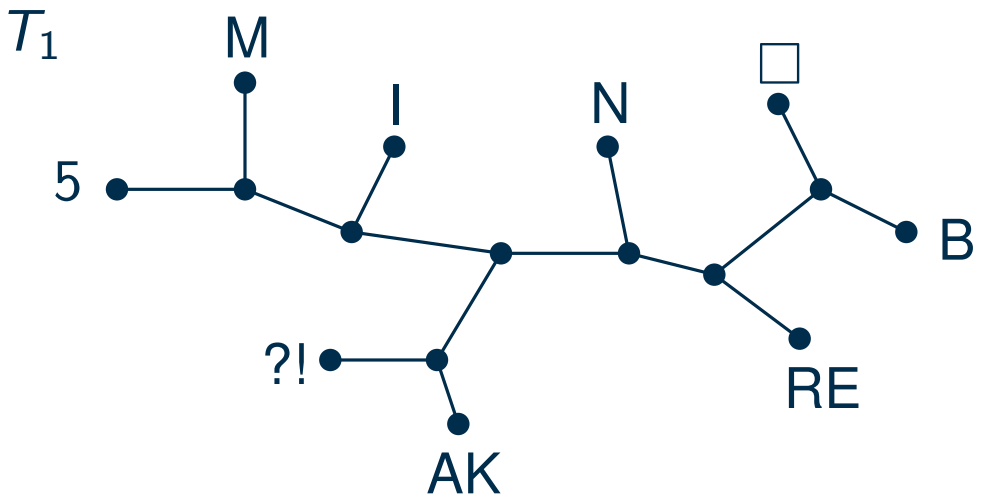
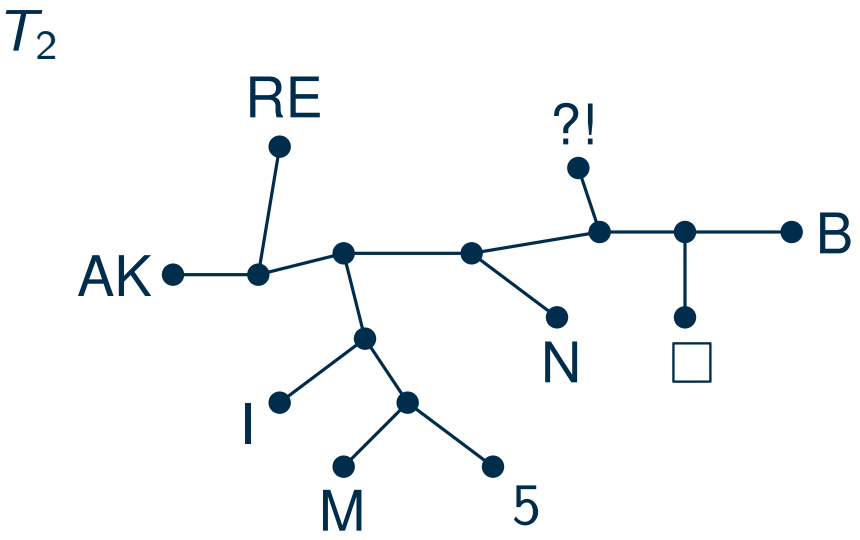
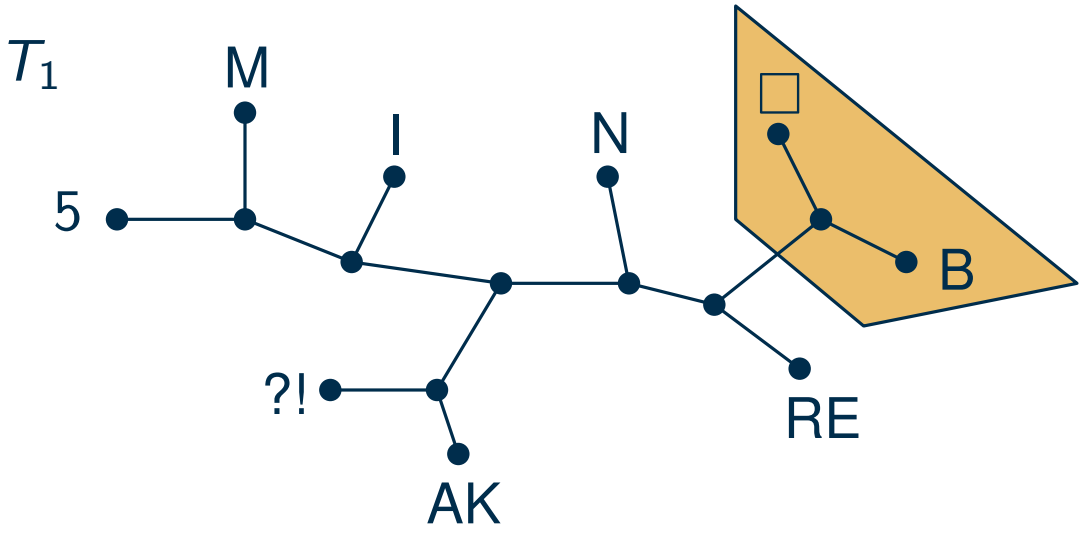
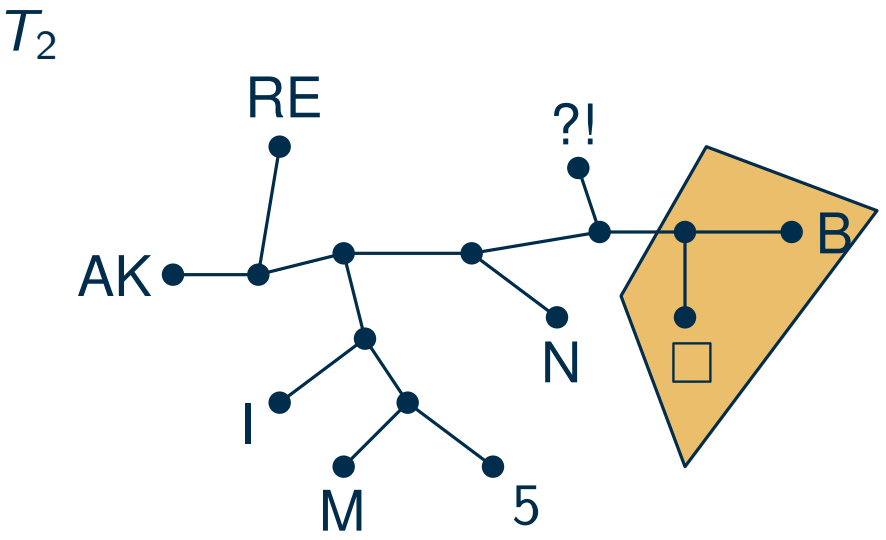




Example

Solve this instance of MAXIMUM AGREEMENT FOREST





Example

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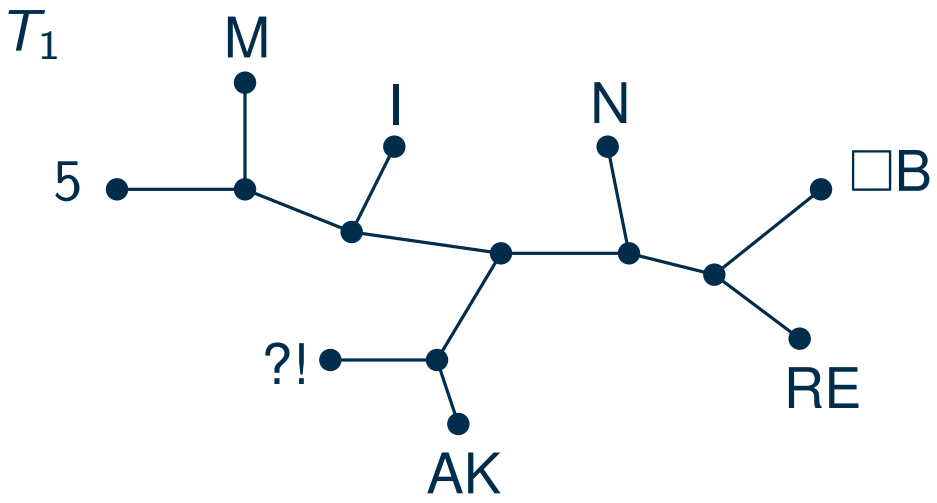

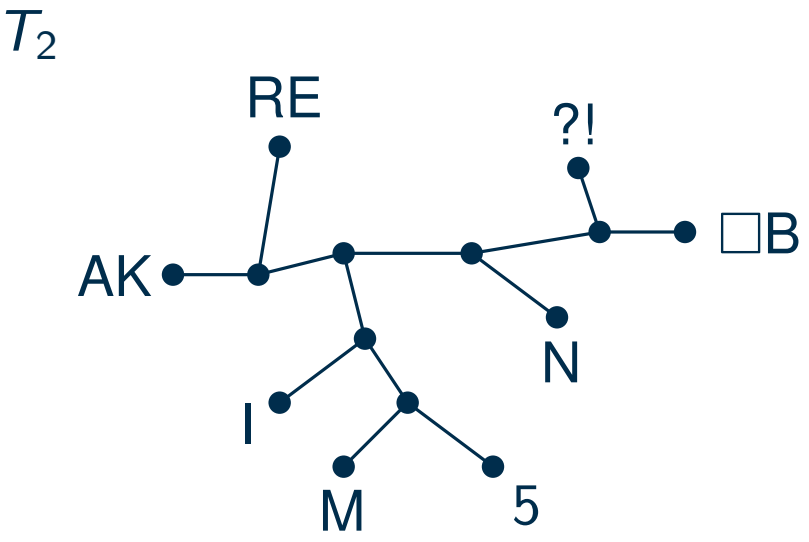

Example

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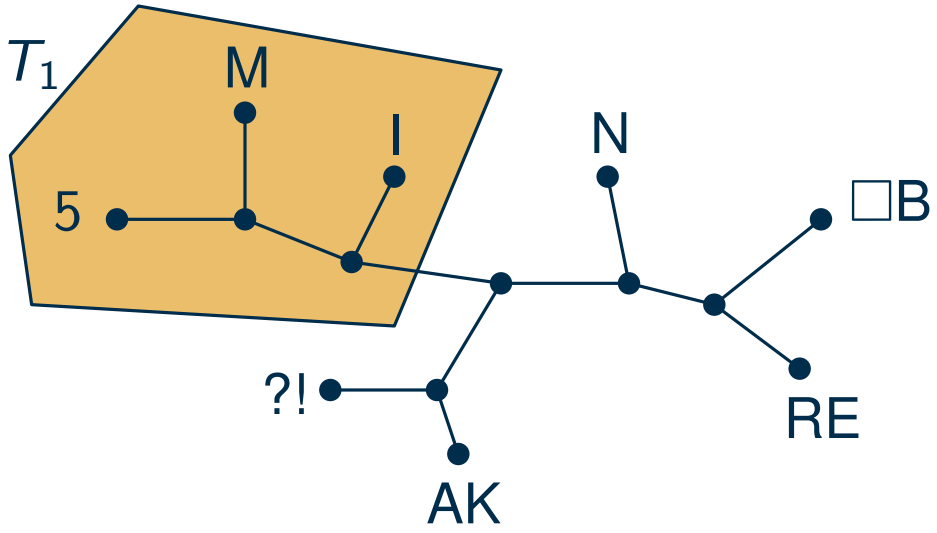
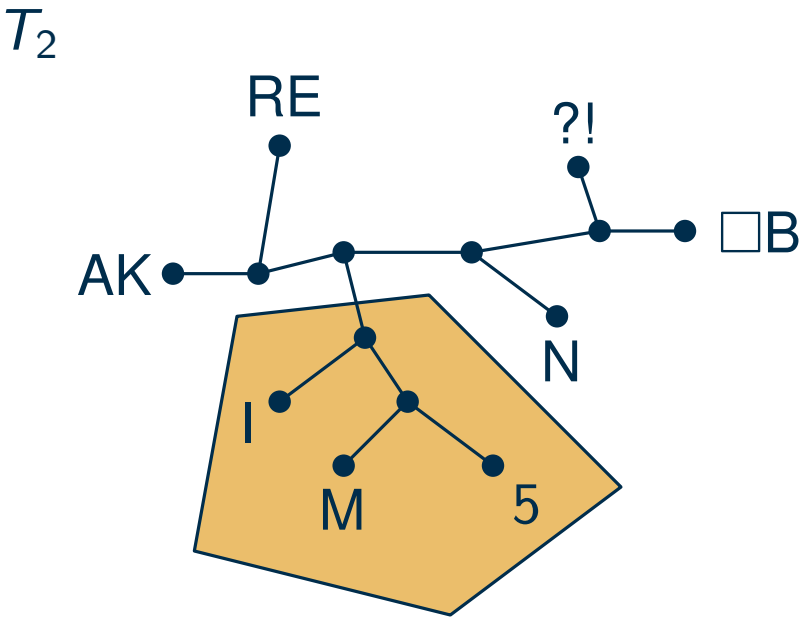
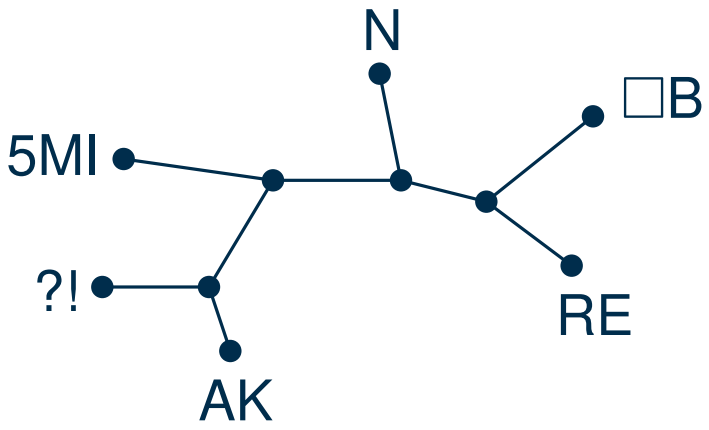




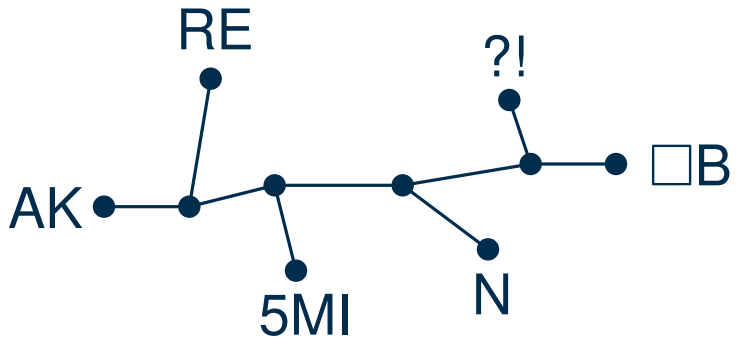
Example

Solve this instance of MAXIMUM AGREEMENT FOREST

T_1



T_2



Example

Solve this instance of MAXIMUM AGREEMENT FOREST

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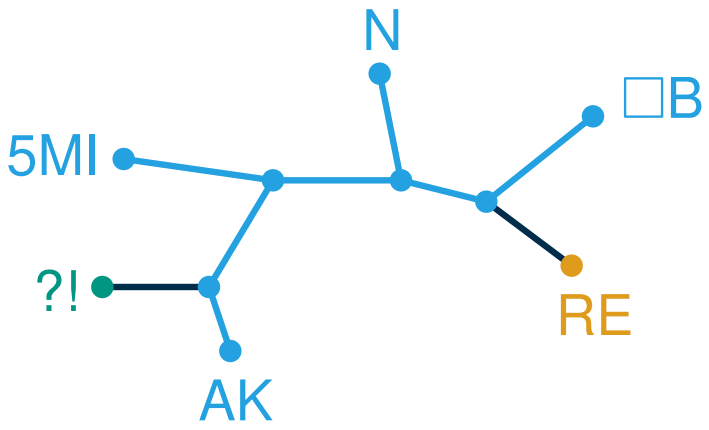
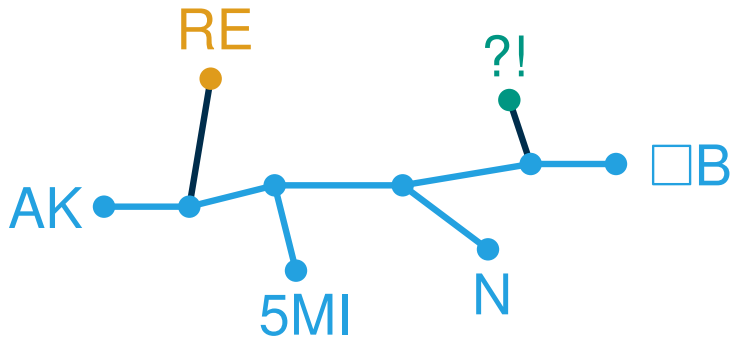
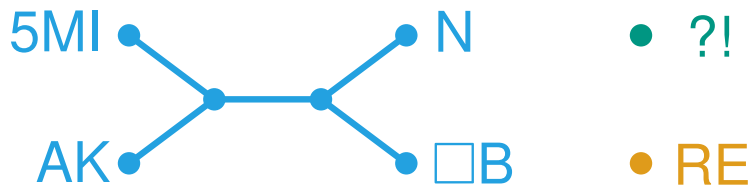


T_2



F



$\Rightarrow$ solution has size 3

Undesirable tree structures 2

Claim: Every tree of the agreement forest has only few leaves. (after applying reduction rule 1)

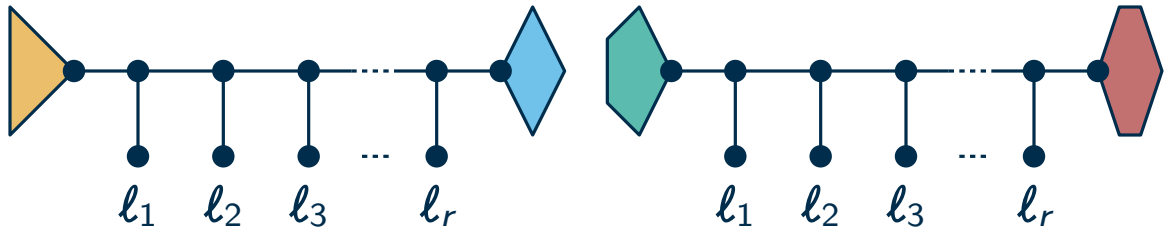
Counterexample

Undesirable tree structures 2

Claim: Every tree of the agreement forest has only few leaves. (after applying reduction rule 1)

Counterexample

- many leaves in a shared subtree



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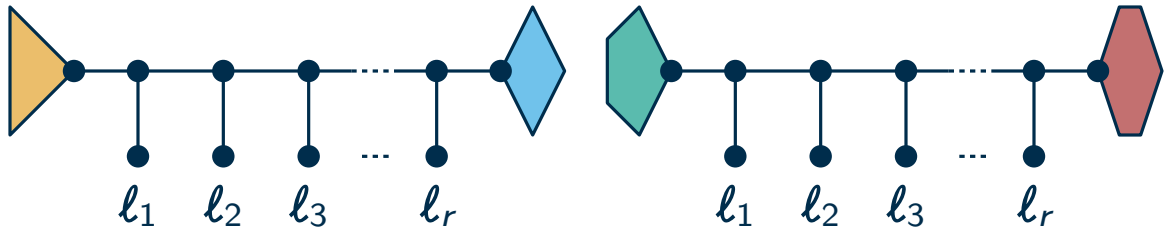

Reduction rule 2

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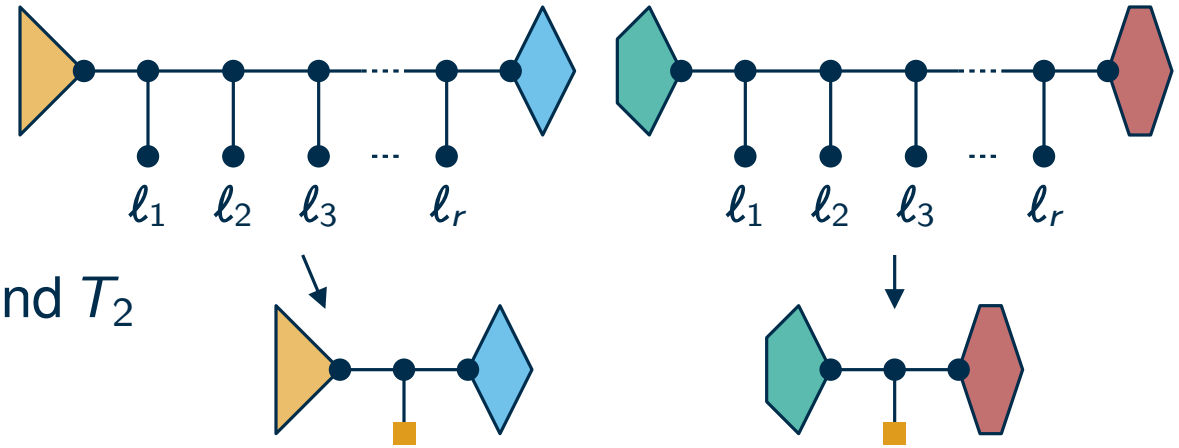

Reduction rule 2

- find path with attached leaves $\ell_1, \ldots, \ell_r$ in T_1 and T_2
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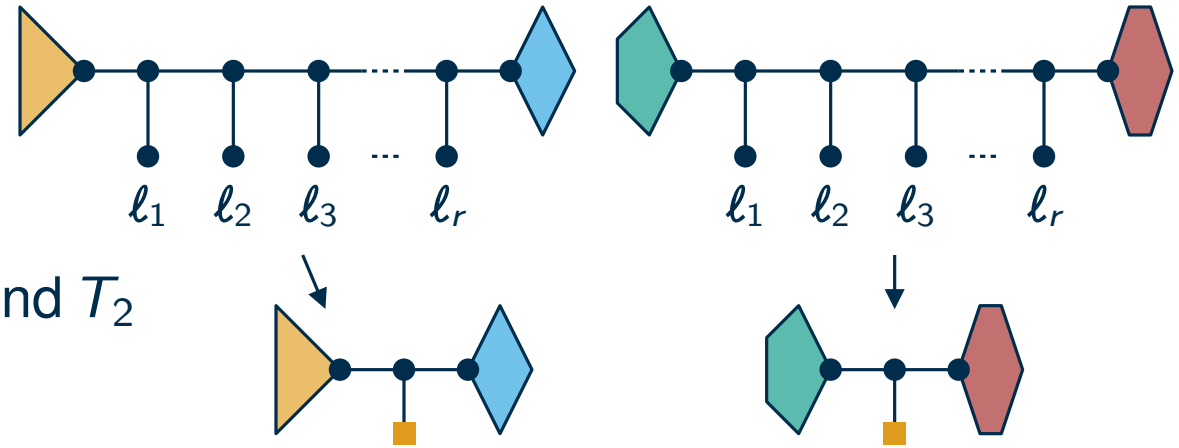

Is this rule safe?

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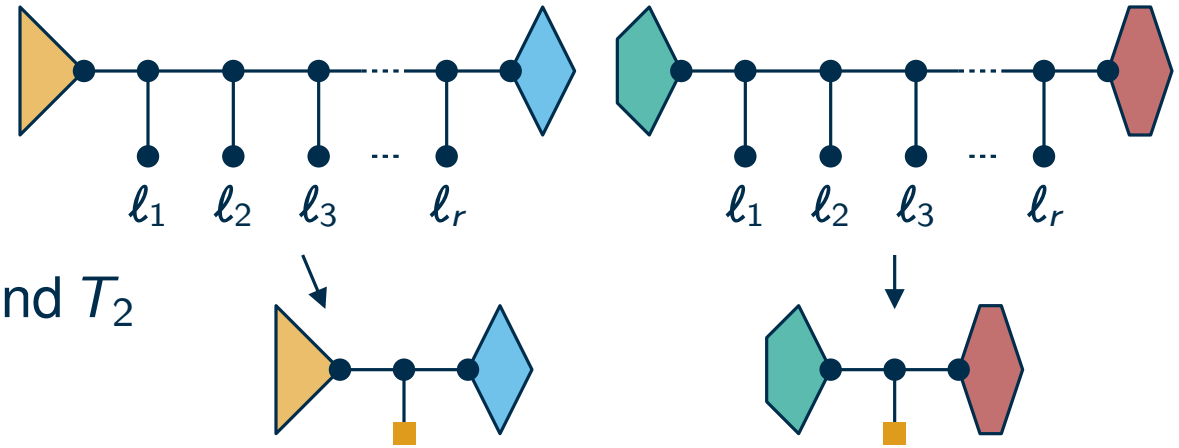

Reduction rule 2

- find path with attached leaves $\ell_1, \ldots, \ell_r$ in T_1 and T_2
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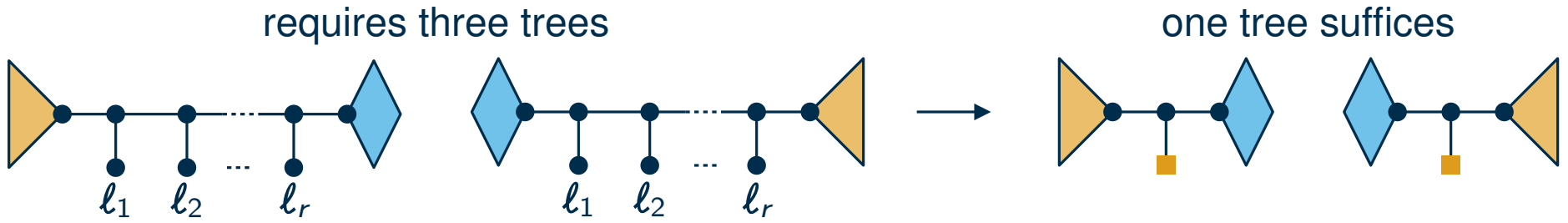
Problem

- number of required trees may be reduced

Is this rule safe?



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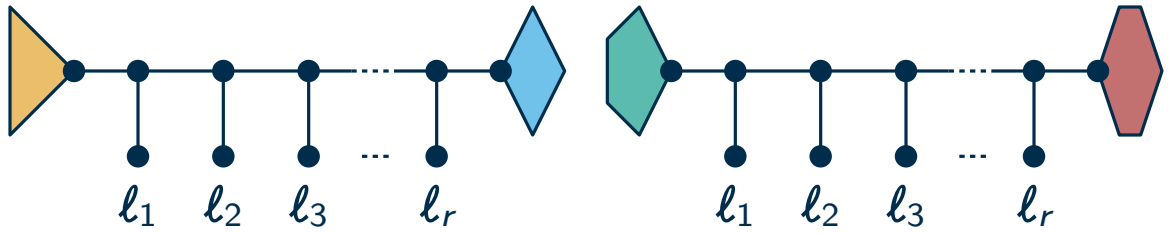

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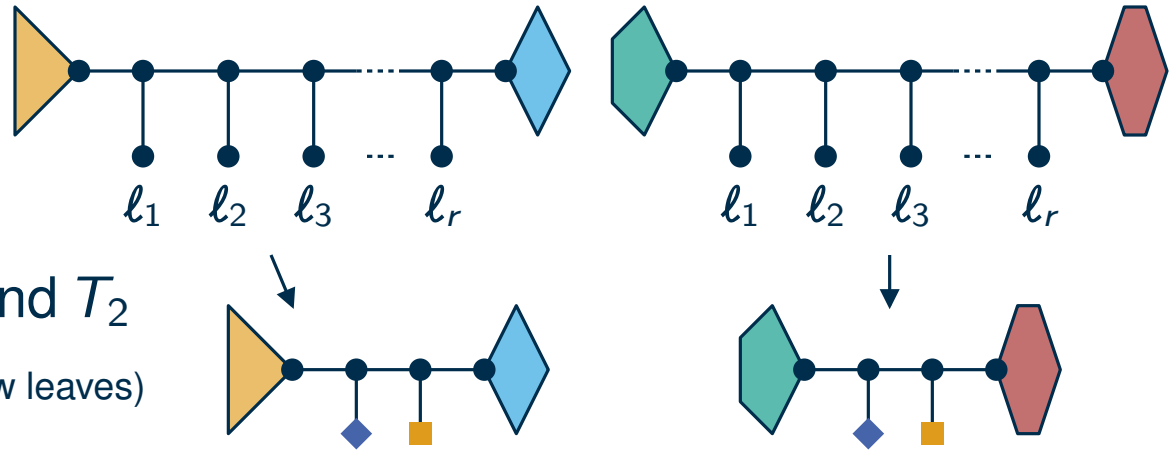

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- find path with attached leaves $\ell_1, \ldots, \ell_r$ in T_1 and T_2
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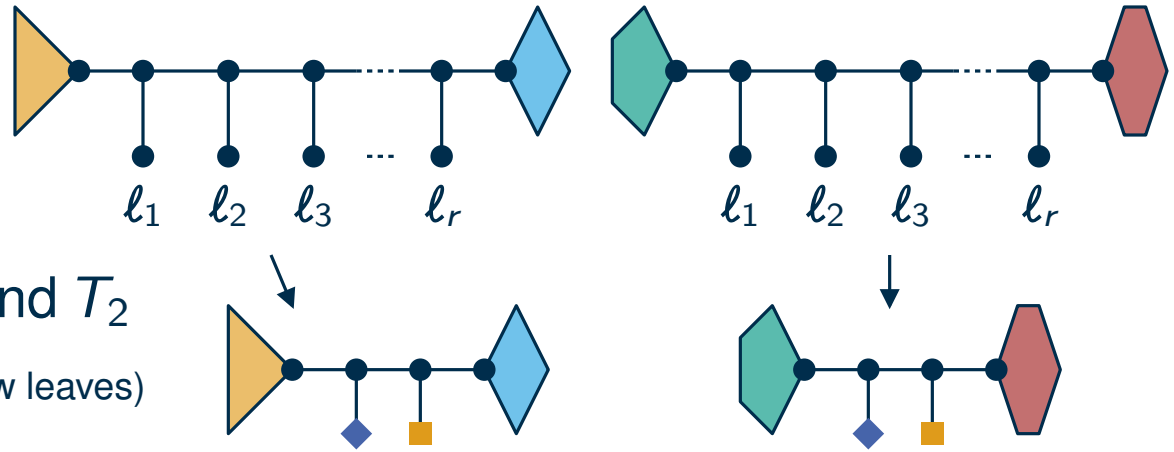

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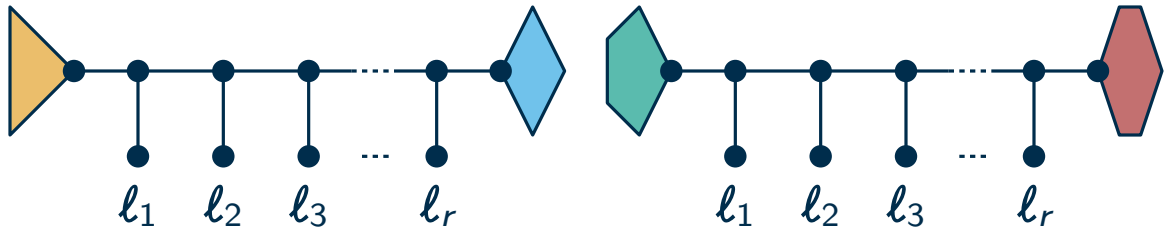

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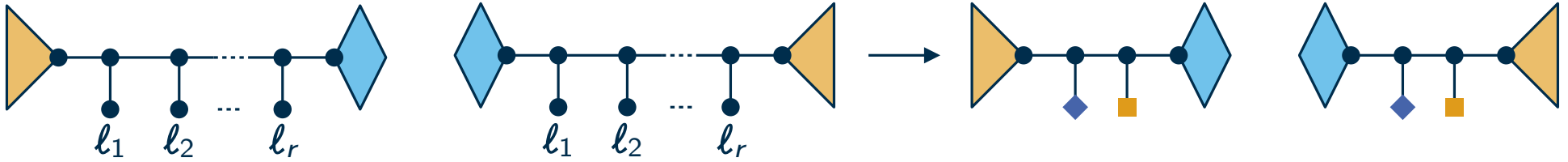
Problem

- number of required trees may be reduced

Is this rule safe?

requires three trees



two trees suffice

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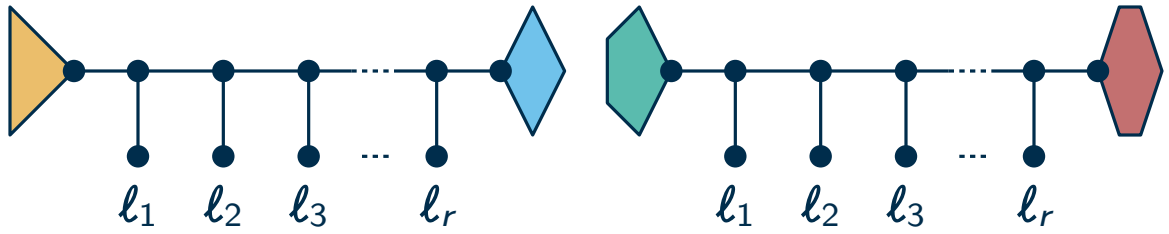

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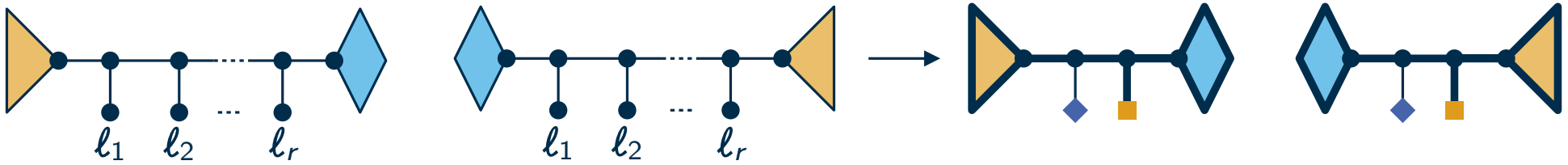

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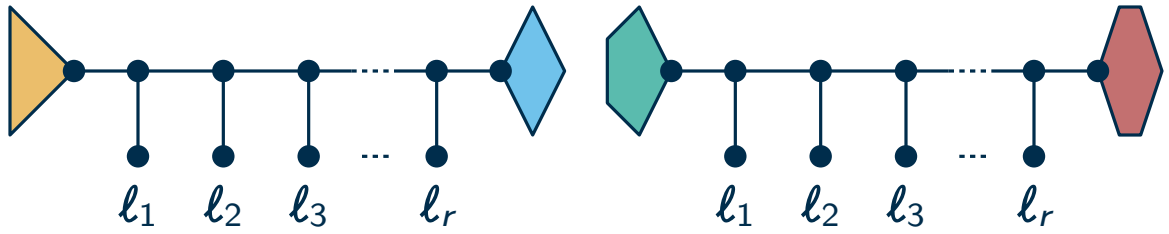

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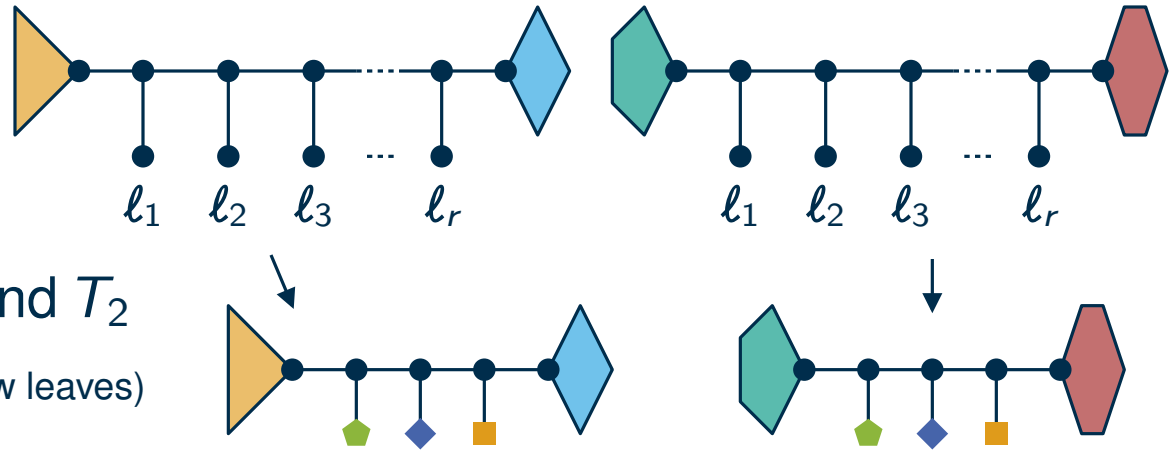

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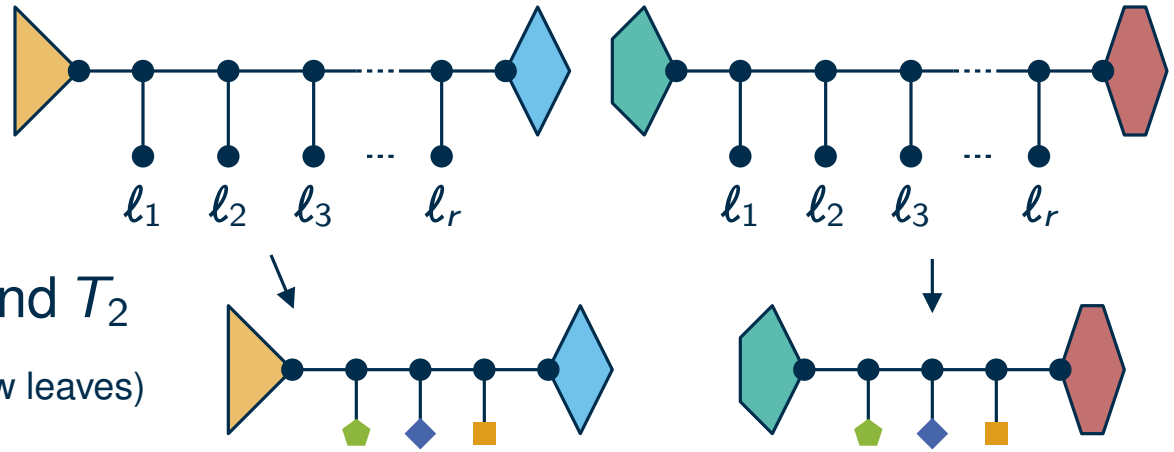


Reduction rule 2

- find path with attached leaves $\ell_1, \ldots, \ell_r$ in T_1 and T_2
- replace it with a path of length three (three new leaves)

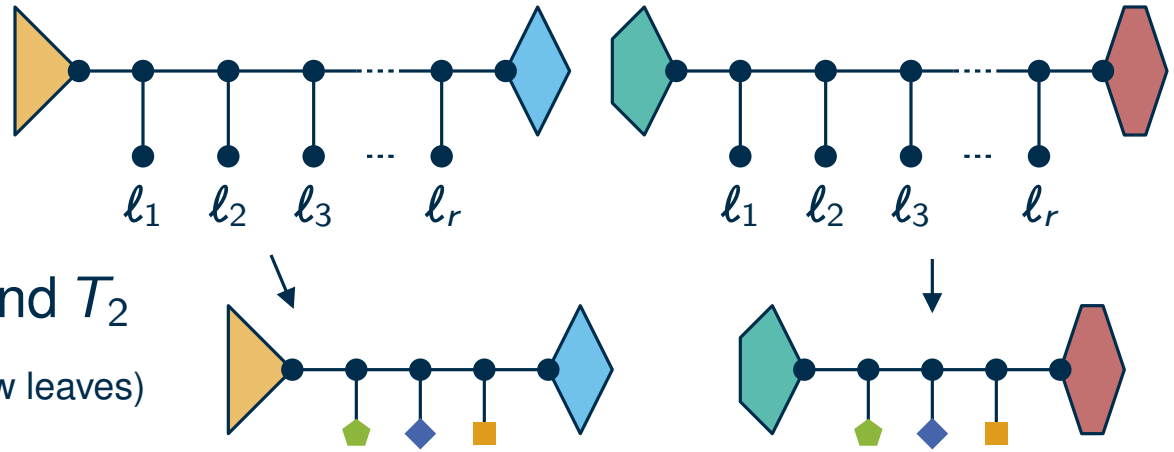
Lemma: Reduction rule 2 is safe and can be applied in polynomial time.

Undesirable tree structures 2

Claim: Every tree of the agreement forest has only few leaves. (after applying reduction rule 1)

Counterexample

- many leaves in a shared subtree



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Lemma: Reduction rule 2 is safe and can be applied in polynomial time.

Proof

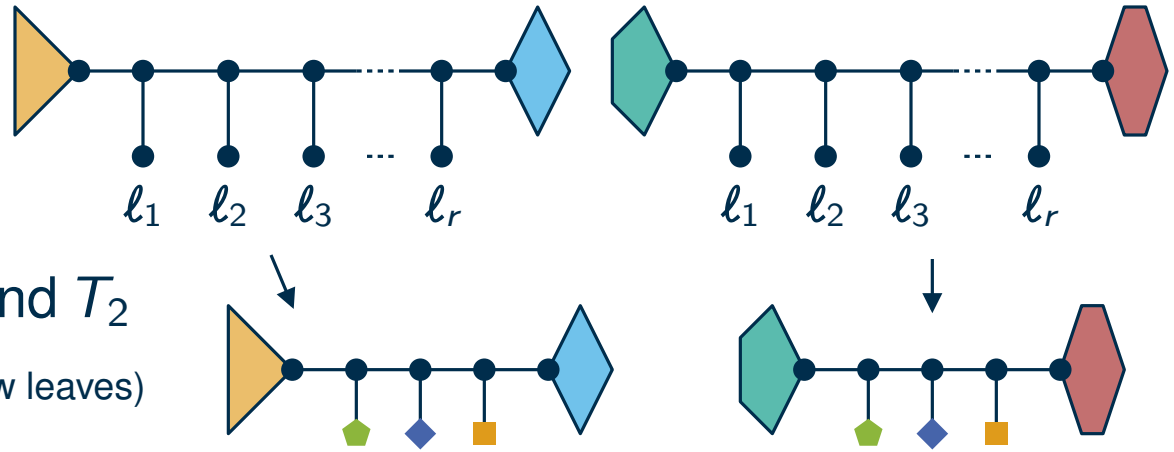
- case distinction (not here; but not very difficult)

Undesirable tree structures 2

Claim: Every tree of the agreement forest has only few leaves. (after applying reduction rule 1)

Counterexample

- many leaves in a shared subtree



Reduction rule 2

- find path with attached leaves $\ell_1, \ldots, \ell_r$ in T_1 and T_2
- replace it with a path of length three (three new leaves)

Lemma: Reduction rule 2 is safe and can be applied in polynomial time.

Proof

- case distinction (not here; but not very difficult)
- polynomial running time: for example by iteratively shortening paths of length 4

Undesirable tree structures 3

Claim: Every tree of the agreement forest has only few leaves. (after applying rules 1 and 2)

Counterexample

Undesirable tree structures 3

Claim: Every tree of the agreement forest has only few leaves. (after applying rules 1 and 2)

Counterexample

- many leaves in a shared subtree

Undesirable tree structures 3

Claim: Every tree of the agreement forest has only few leaves. (after applying rules 1 and 2)

Counterexample

- many leaves in a shared subtree

Is this really a problem?

Undesirable tree structures 3

amortized

Claim: Every tree of the agreement forest has only few leaves. (after applying rules 1 and 2)

Counterexample

- many leaves in a shared subtree

Amortized perspective

- after every three leaves, we see different subtrees

Is this really a problem?

Undesirable tree structures 3

amortized

Claim: Every tree of the agreement forest has only few leaves. (after applying rules 1 and 2)

Counterexample

- many leaves in a shared subtree

Is this really a problem?

Amortized perspective

- after every three leaves, we see different subtrees
- each of them yields at least one tree in the agreement forest

(if a, b, c, d, ... are all in the same tree)

Undesirable tree structures 3

amortized

Claim: Every tree of the agreement forest has only few leaves. (after applying rules 1 and 2)

Counterexample

- many leaves in a shared subtree

Is this really a problem?

Amortized perspective

- after every three leaves, we see different subtrees
- each of them yields at least one tree in the agreement forest (if a, b, c, d, . . . are all in the same tree)
- few trees in the agreement forest $\Rightarrow$ few leaves

Undesirable tree structures 3

amortized

Claim: Every tree of the agreement forest has only few leaves. (after applying rules 1 and 2)

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- many leaves in a shared subtree

Is this really a problem?

Amortized perspective

- after every three leaves, we see different subtrees
- each of them yields at least one tree in the agreement forest (if $a, b, c, d, \ldots$ are all in the same tree)
- few trees in the agreement forest $\Rightarrow$ few leaves

Are we sure, there are no other undesirable structures?

Undesirable tree structures 3

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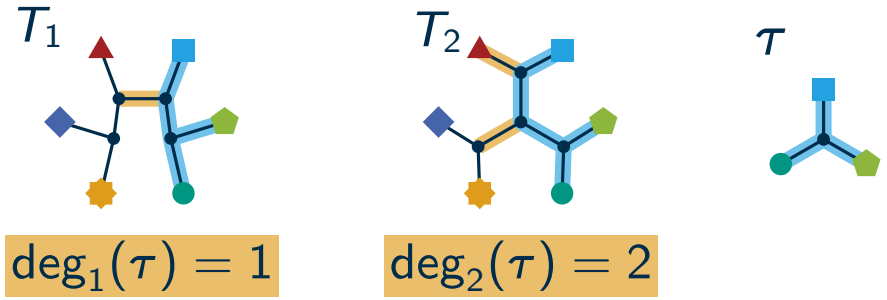
Lemma
Exhaustively applying reduction rules 1 and 2 to a yes-instance yields an equivalent instance with at most ck leaves (for a small constant c).

Rough plan of action

The degree of a subtree

- contract a tree τ of F in T_i to a single node ($i \in \{1, 2\}$)
- let $\deg_i(\tau)$ be the degree of the resulting node



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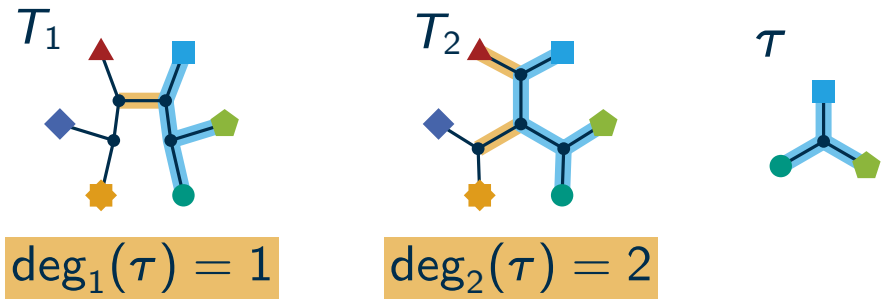

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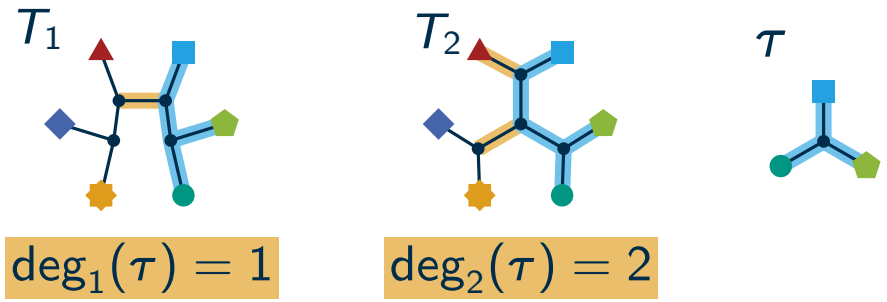

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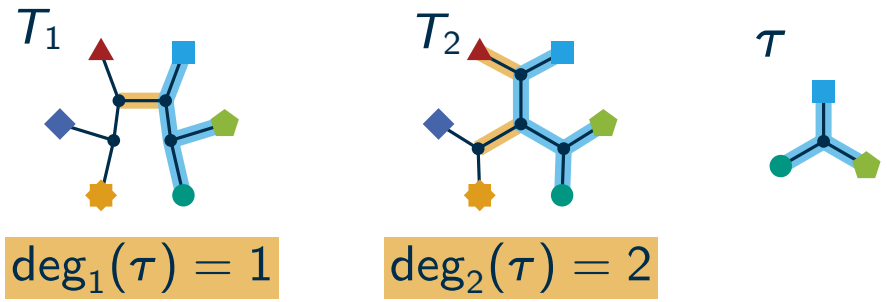

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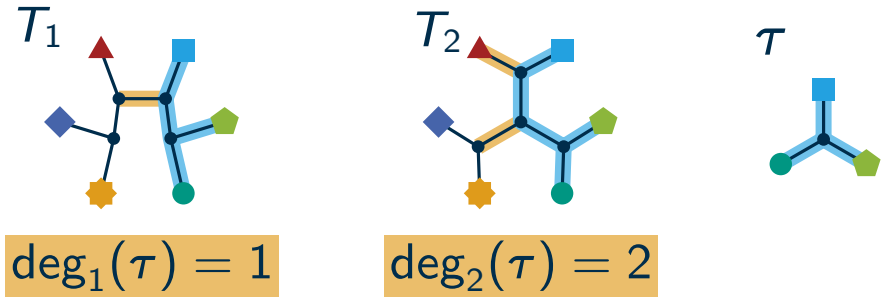

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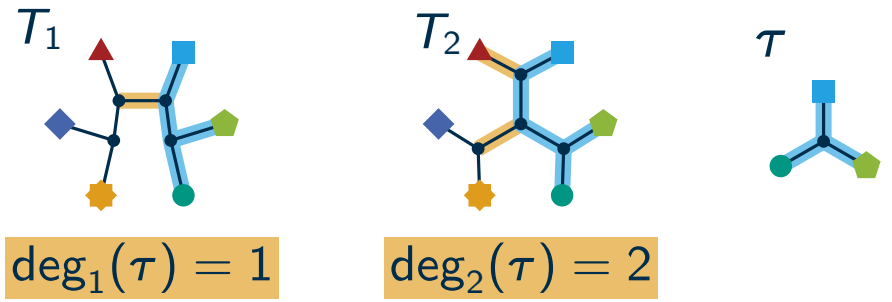

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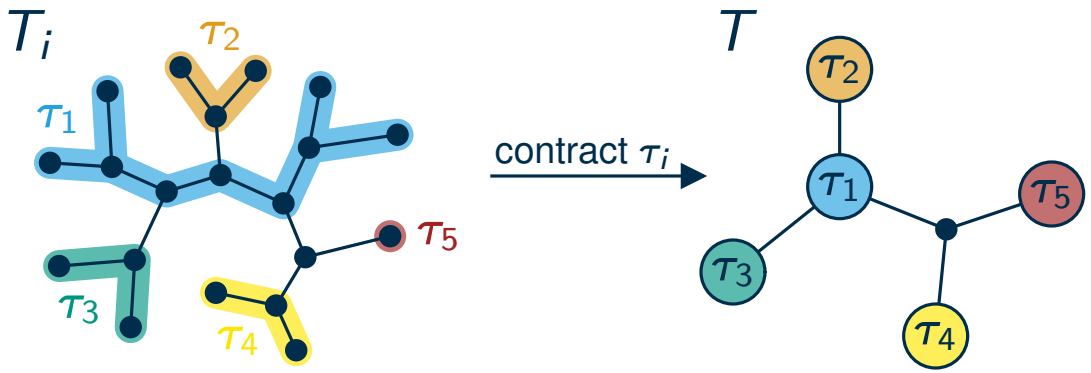

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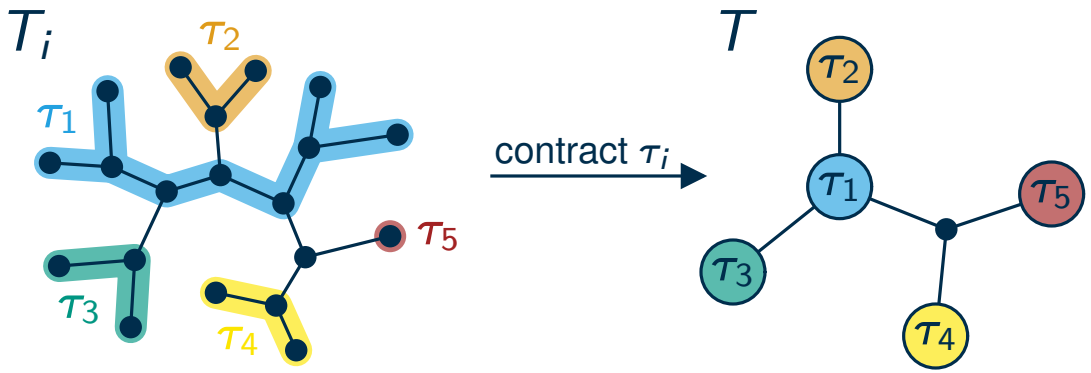

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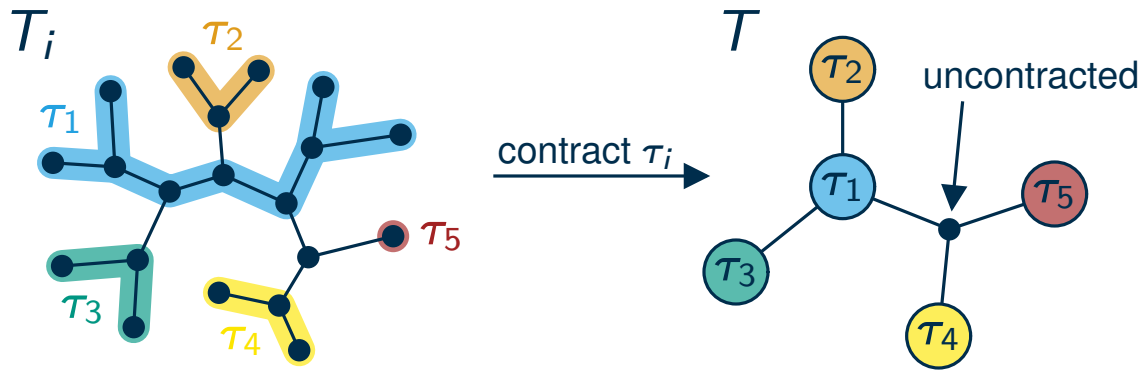

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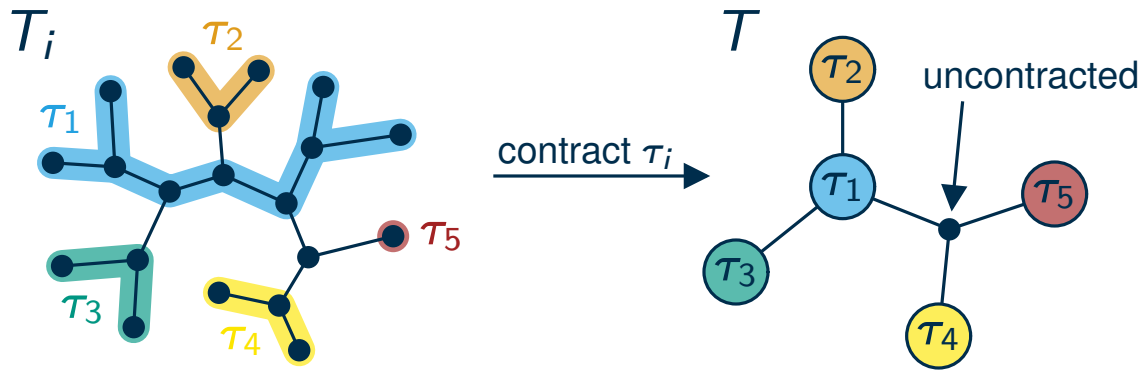

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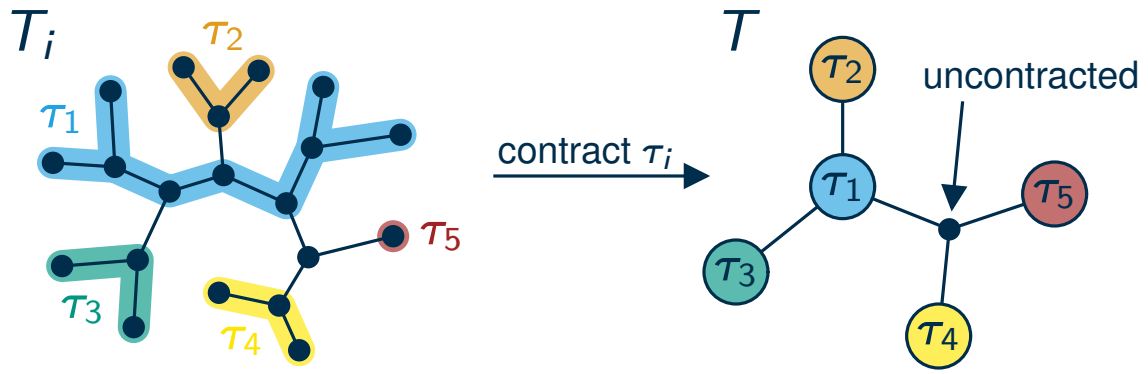

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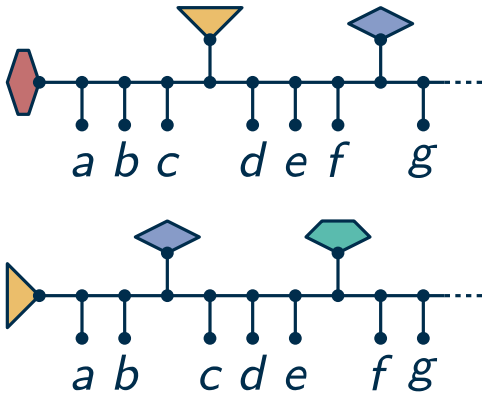

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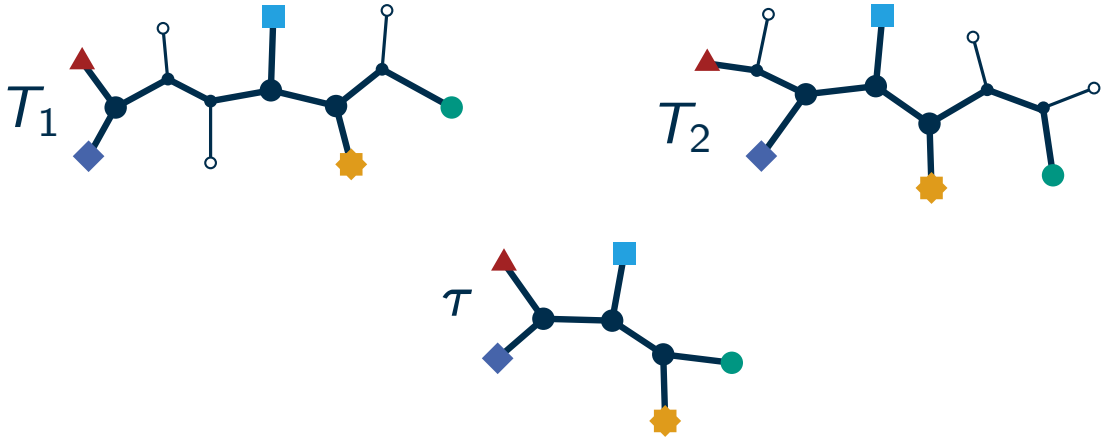

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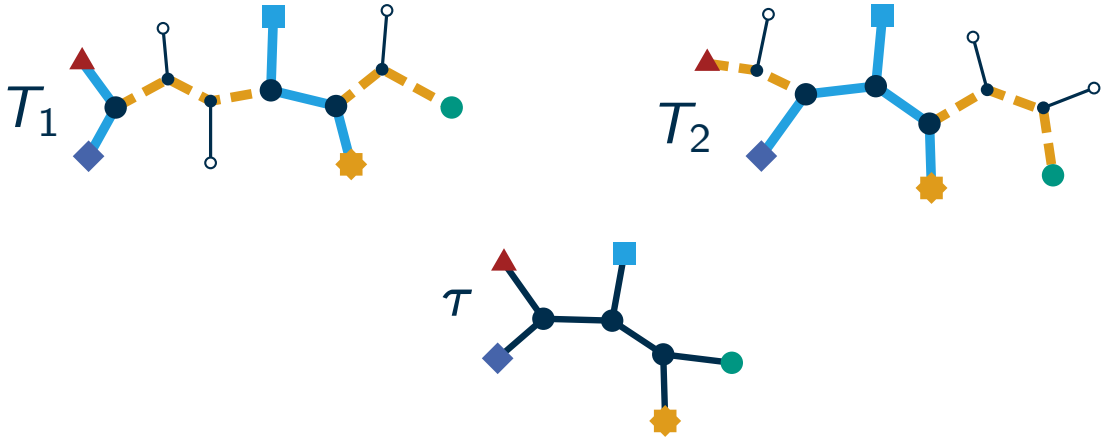

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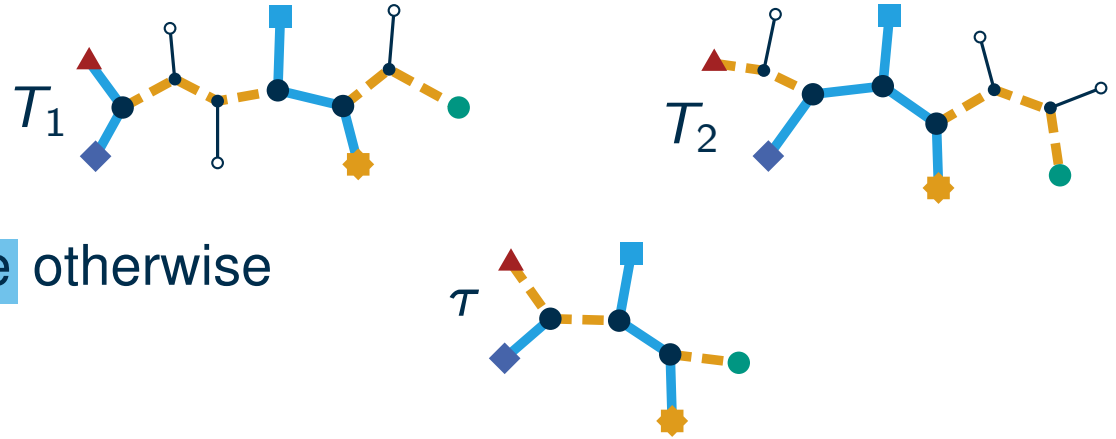

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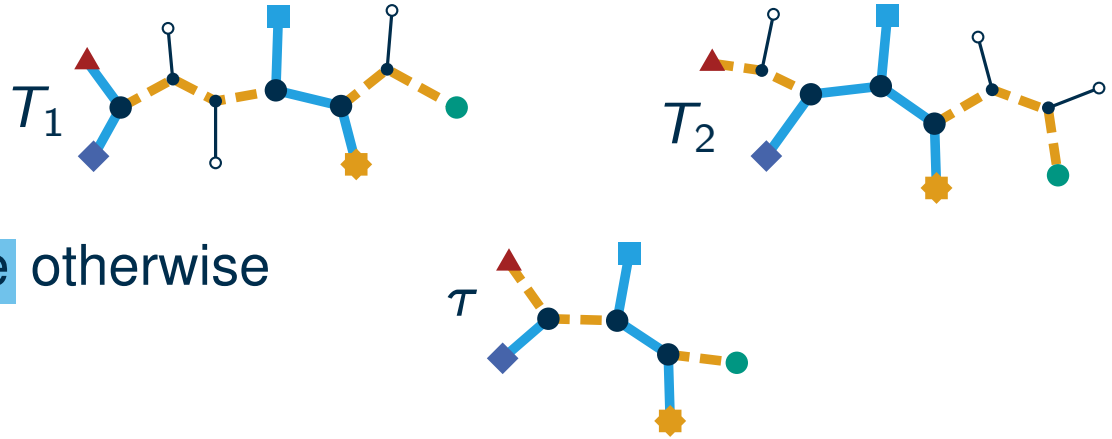

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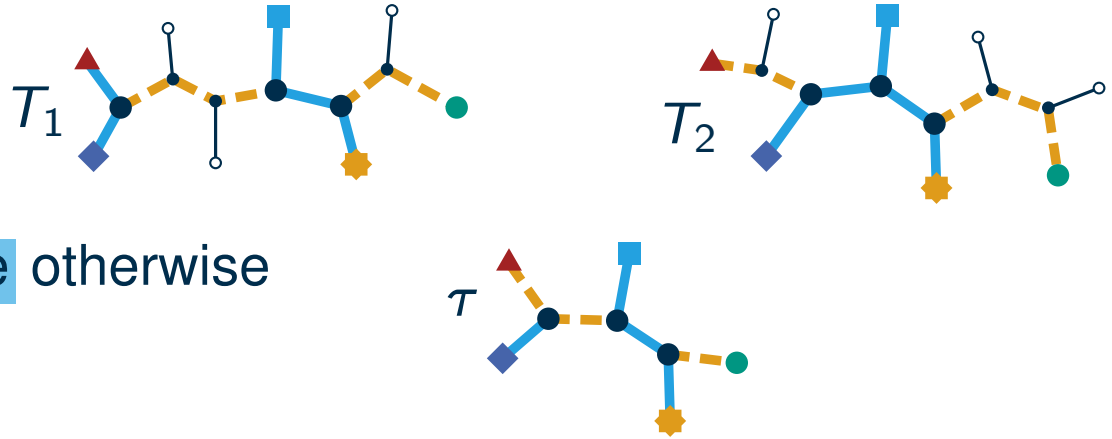

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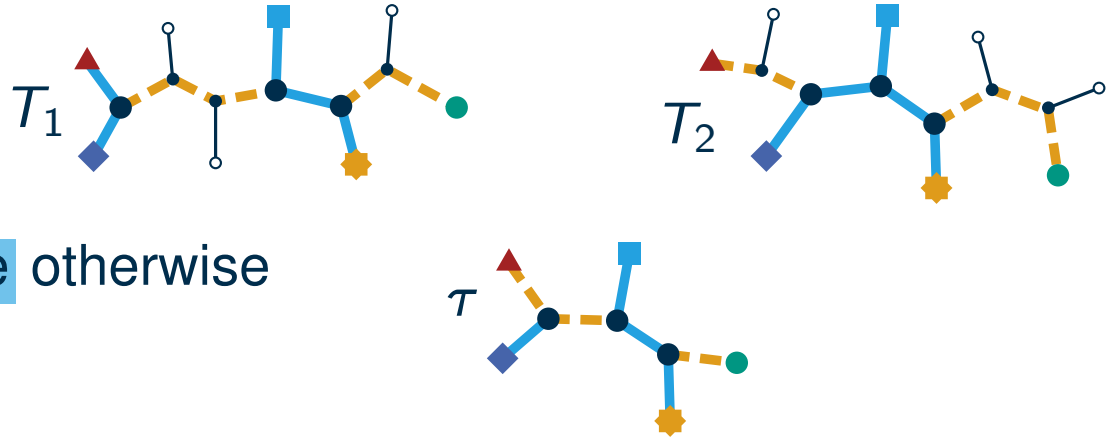

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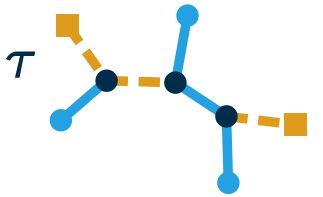
- classify each leaf by its incident edge: path leaf or edge leaf



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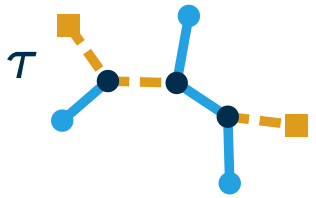

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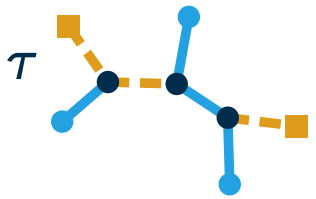

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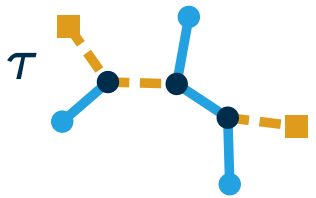

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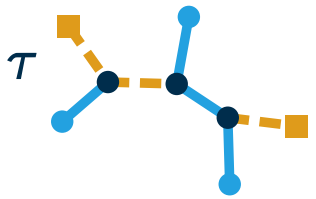

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- rule 1: no cherry with two edge leaves
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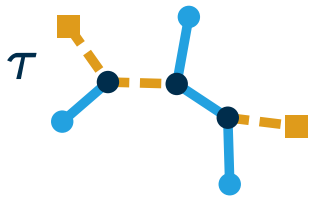

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Lemma (implies the above lemma)
Let τ be a full binary tree with m_r orange edges (non-orange edges are blue). If τ does not contain one of the two forbidden structures, then it has at most $10m_r$ leaves.

Here comes the proof

forbidden structures

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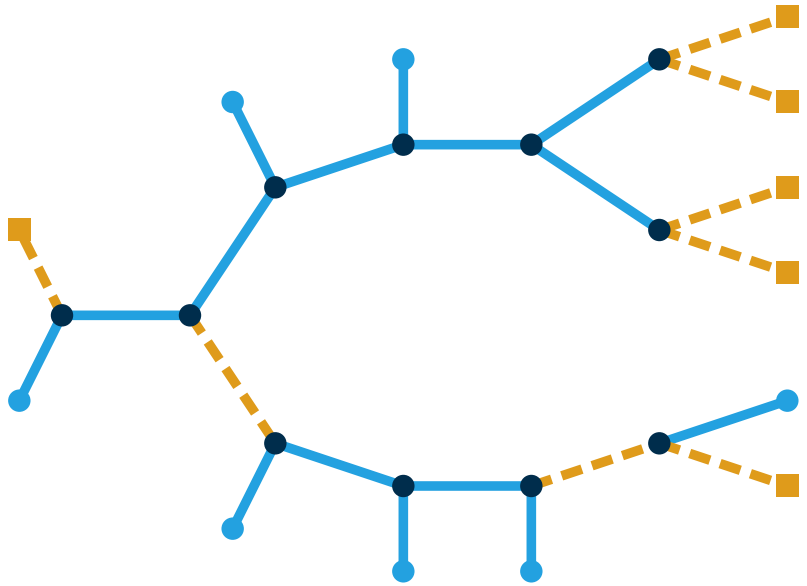

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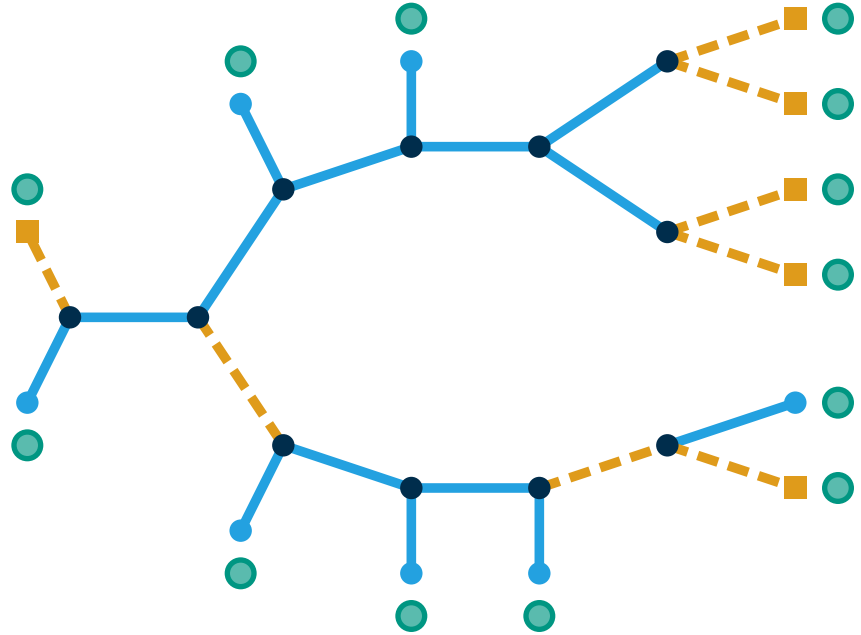

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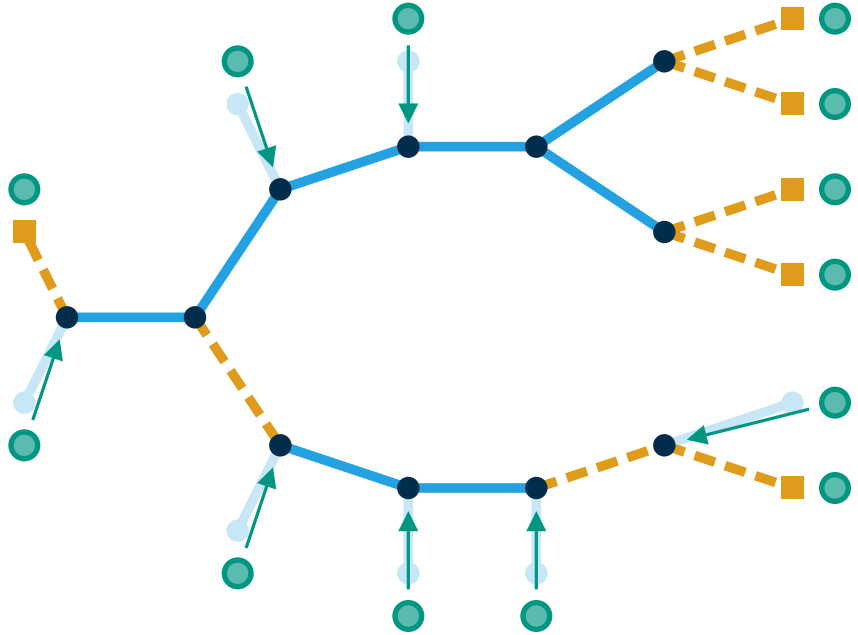

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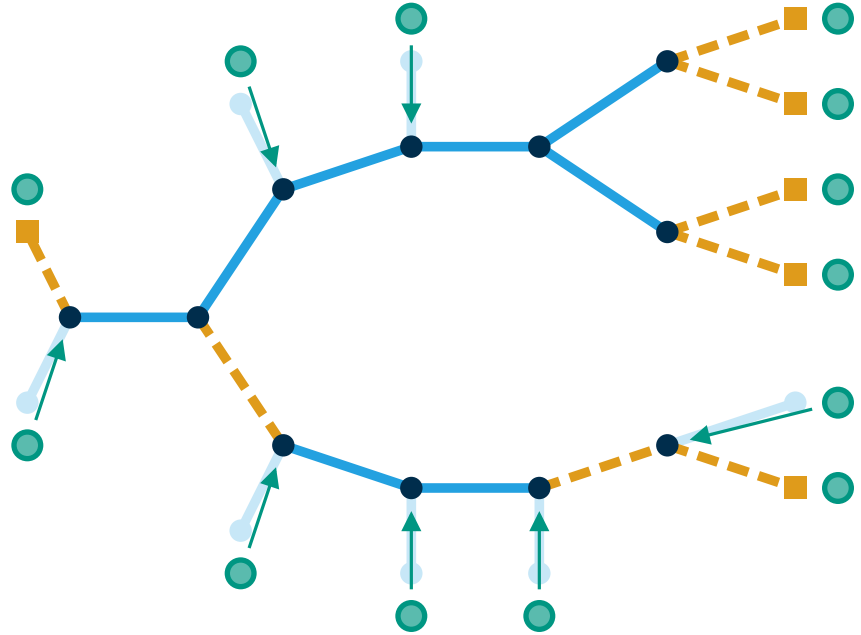
  - only orange leaves and 1● per leaf and deg-2 vertex

Why?



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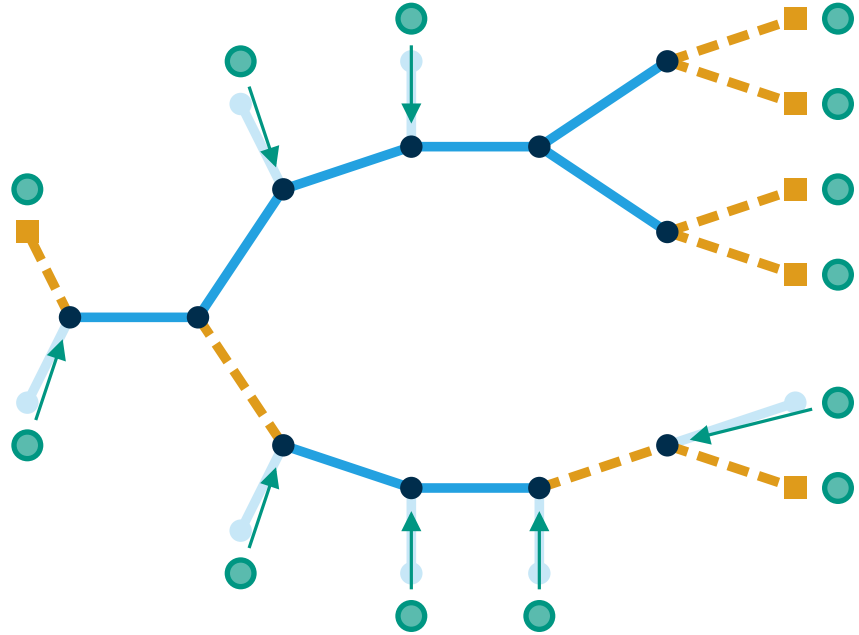
this is ok: each orange leaf is attached to an orange edge

we need to deal with this



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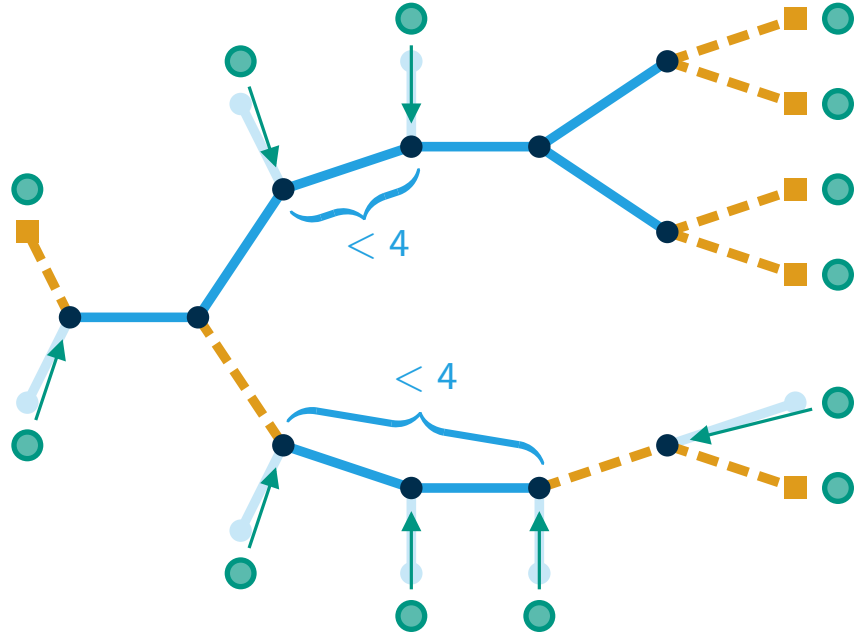

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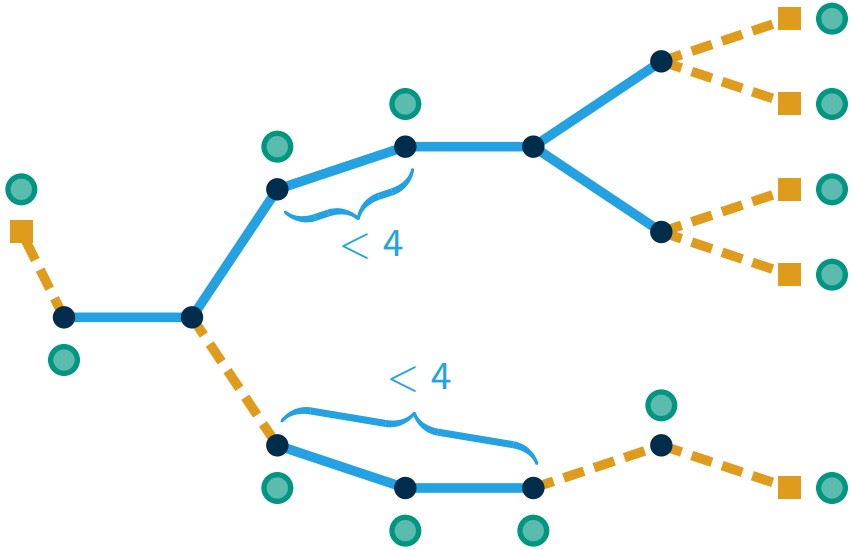

Here comes the proof

forbidden structures

Lemma (implies: small degrees ⇒ few leaves)
Let τ be a full binary tree with m_r orange edges (non-orange edges are blue). If τ does not contain one of the two forbidden structures, then it has at most $10m_r$ leaves.

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- start with one token ● per leaf → count ●
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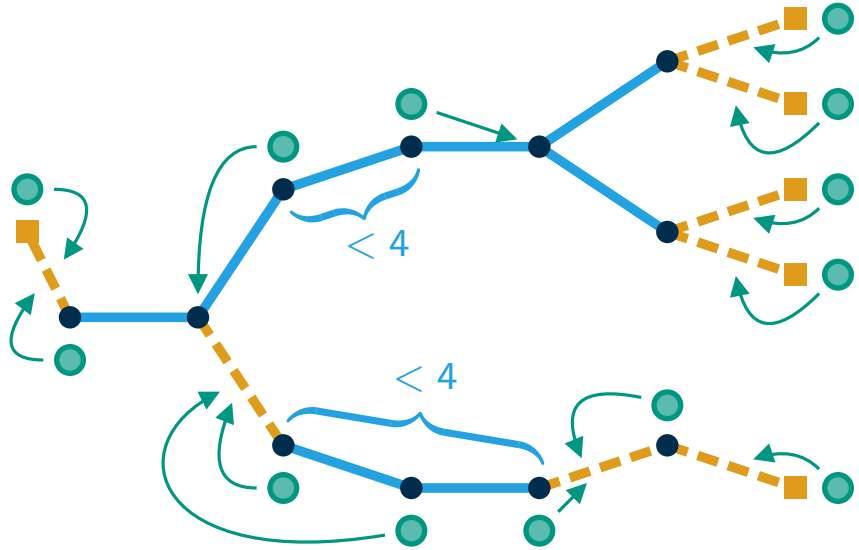

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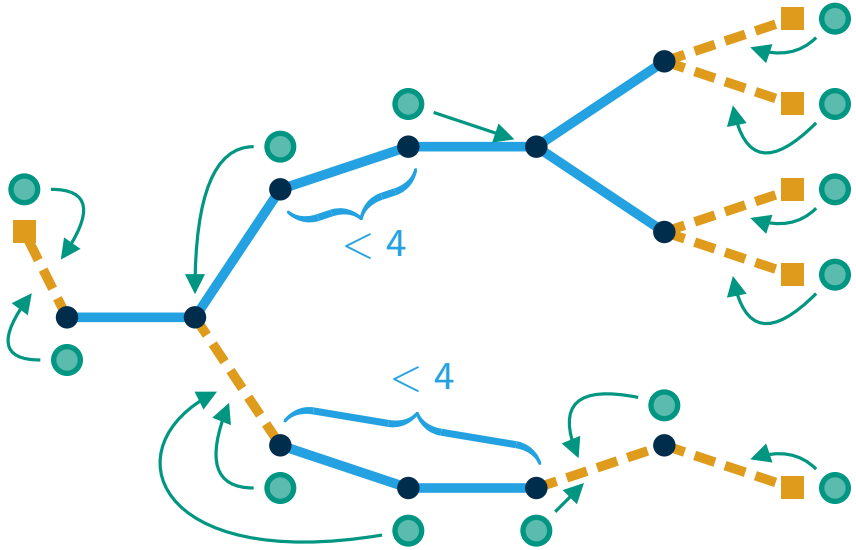
Why?



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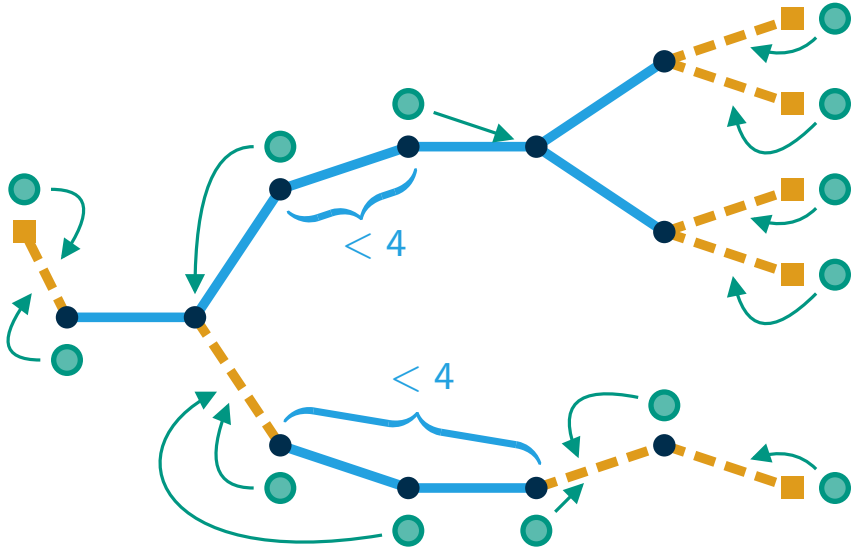
(exercise sheet: $n_3 = n_1 - 2$)



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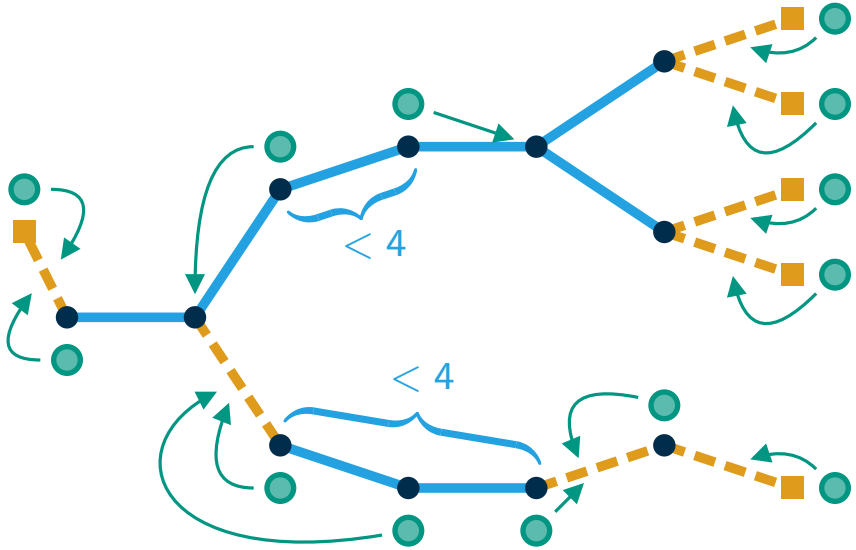
(all leaves are orange)



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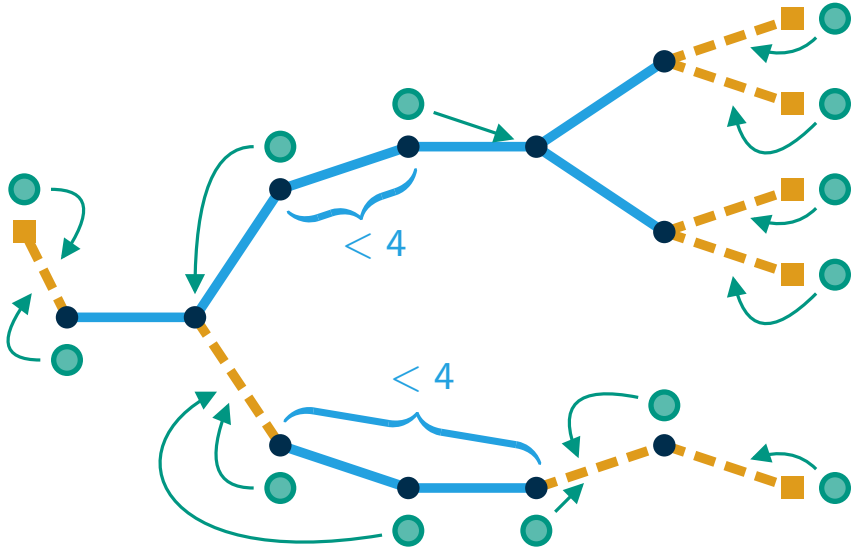

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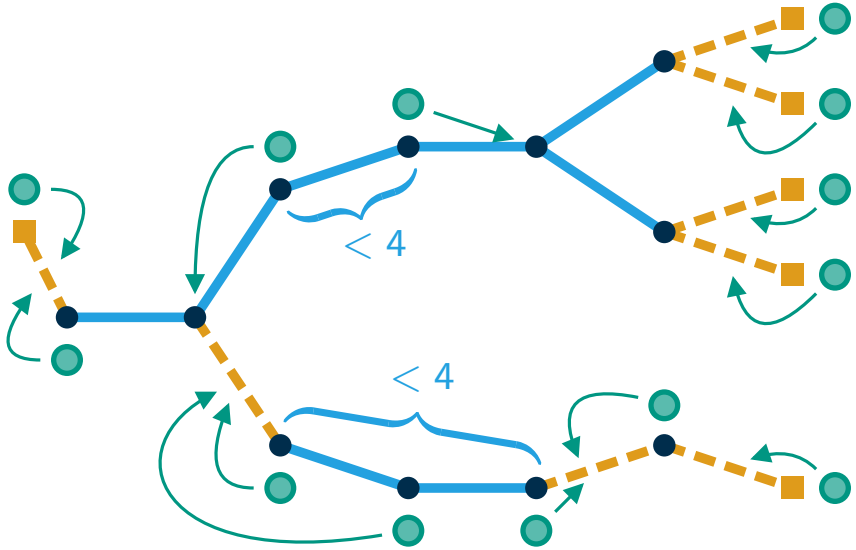
- #(leaves of τ) $=$ #● $\leq 6m_r + 4m_r$

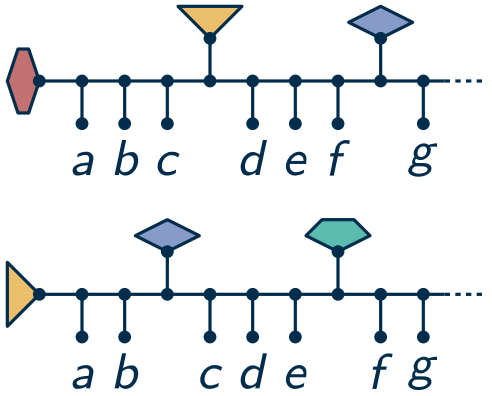


What have we done?

Lemma
Exhaustively applying reduction rules 1 and 2 to a yes-instance yields an equivalent instance with at most ck leaves (for a small constant c).

Proof

- just seen: number of leaves of $\tau \in F$ is at most $10(\deg_1(\tau) + \deg_2(\tau))$
  - τ has many leaves $\Rightarrow$ it splits T_1 or T_2 into many subtrees



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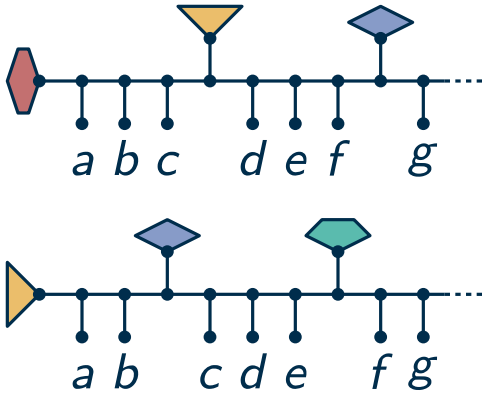

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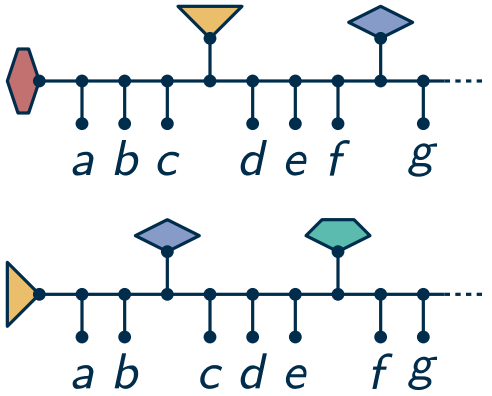
- it follows: F has at most $40k$ leaves
  - many leaves $\Rightarrow$ many trees



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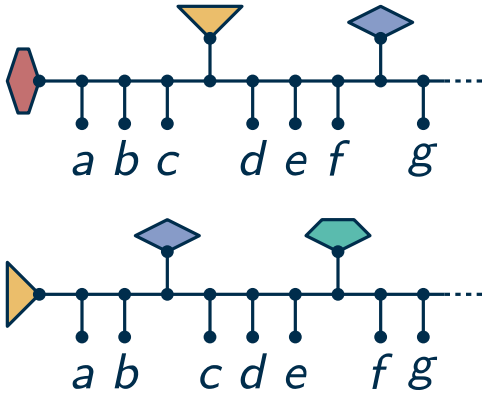

Reduction rule 3: more than $40k$ leaves after applying rule 1 and 2 $\rightarrow$ no-instance of size $O(1)$

Wrap-Up

Problem: MAXIMUM AGREEMENT FOREST
Given T_1, T_2, and a parameter k. Is there an agreement forest of at most k trees?

Theorem
MAXIMUM AGREEMENT FOREST has a kernel of size $O(k)$ that can be computed in poly time.

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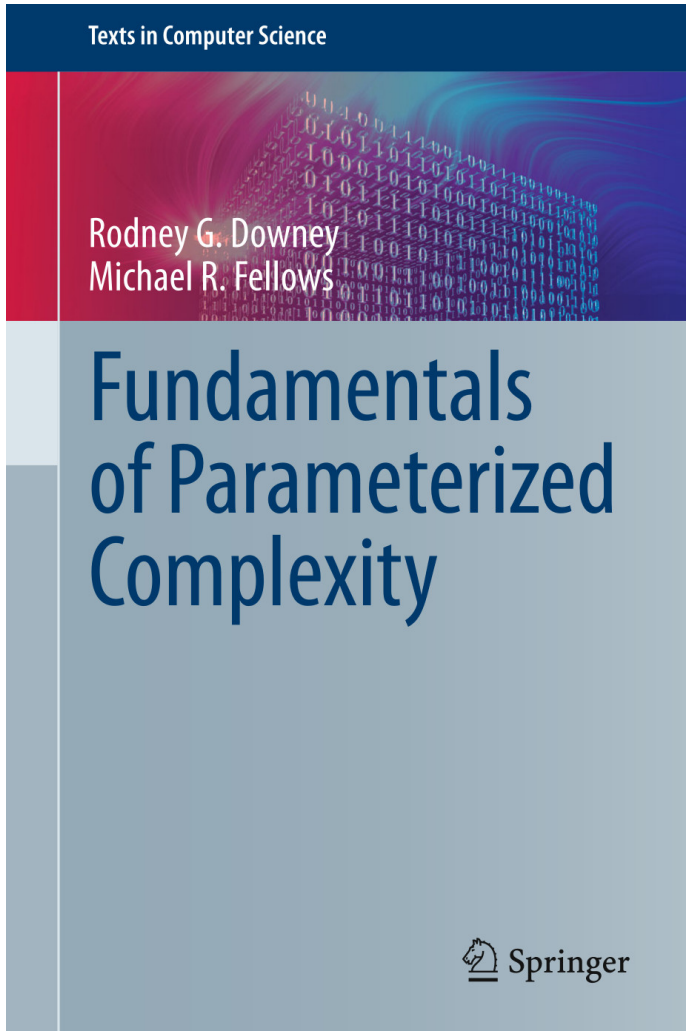
Not seen today

- specific running time for kernelization
- specific running time for brute force on the kernel $O(4^k \cdot k^5)$ is possible

Literature



Comments

- chapter 4.10 is about the topic from today
- also contains links to the original publication
- freely available for KIT members

`link.springer.com/book/10.1007/978-1-4471-5559-1`