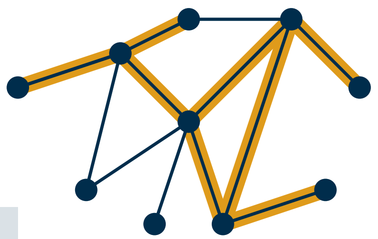
Parameterized Algorithms

Exercise 4 – Sheet 3/4, ILPs, and Coloring

Elly, Jean-Pierre, Wendy

Sheet 3 – Odd Subgraph



Find a subgraph with exactly k edges where every vertex has odd degree.

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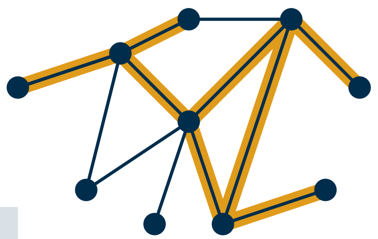

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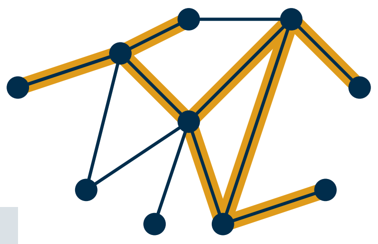

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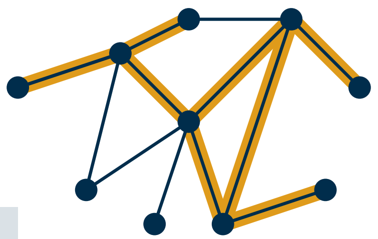

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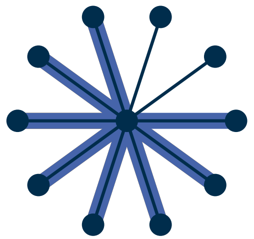
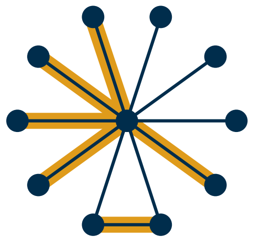
- max deg $> k$

k odd: pick star

k even and G is star: no solution



k even and G is **no** star: pick non incident edge



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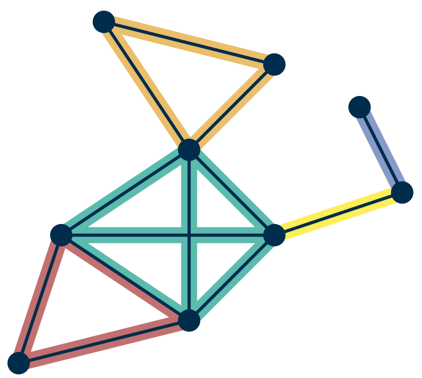

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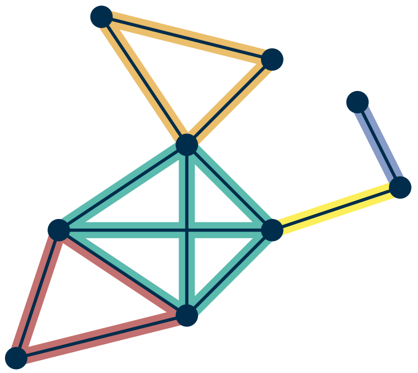

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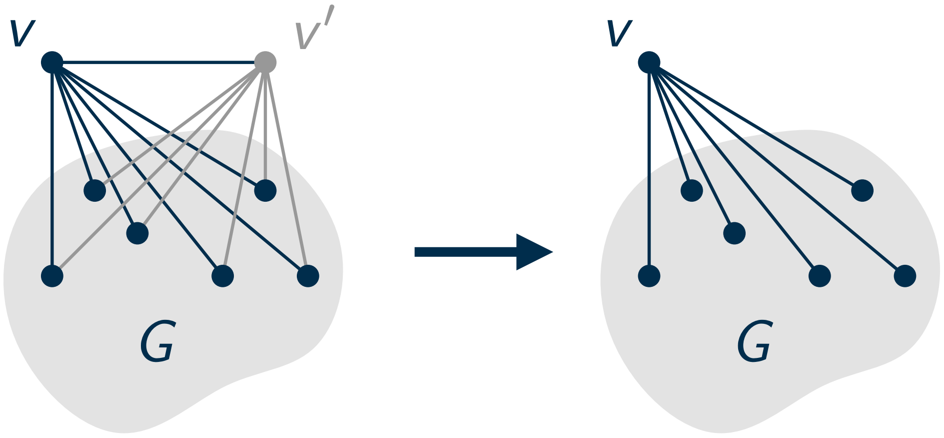
Reduction Rule

- merge **true twins**



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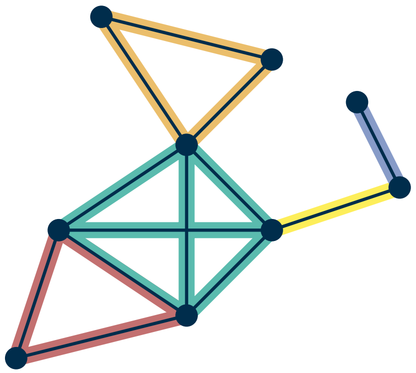

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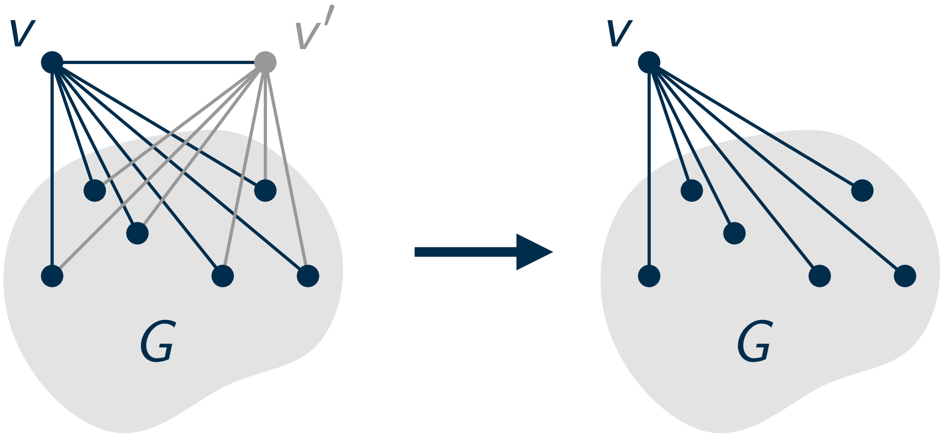

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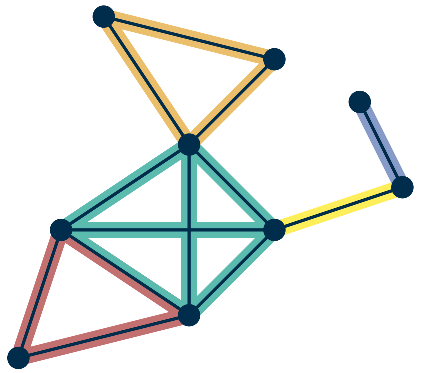

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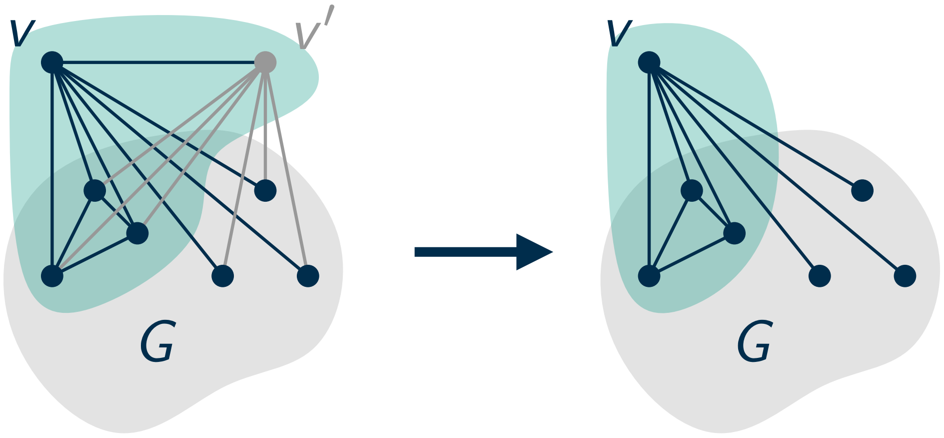

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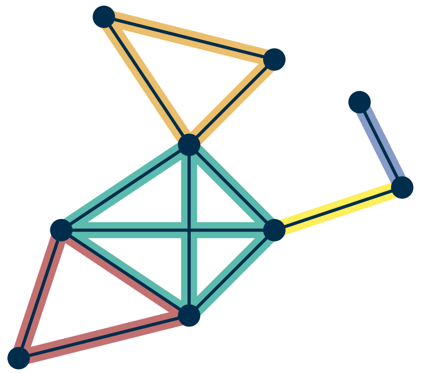

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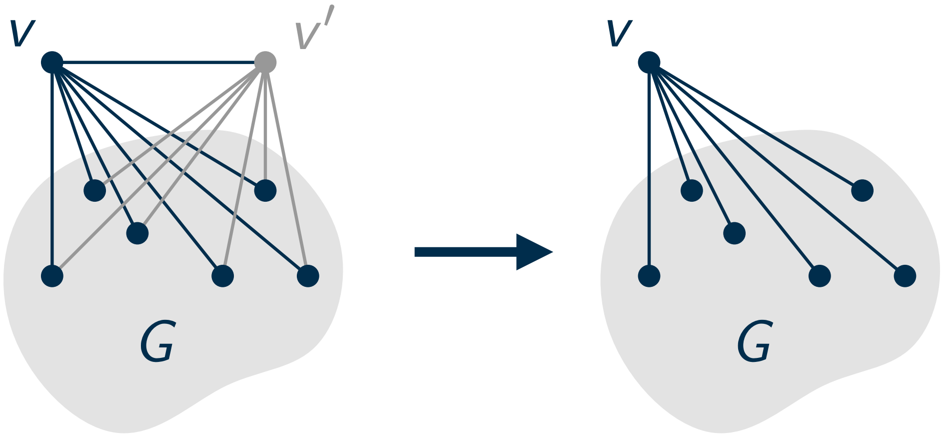

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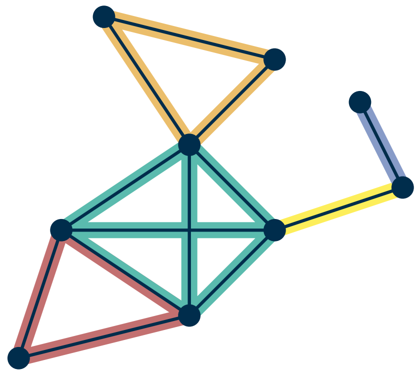

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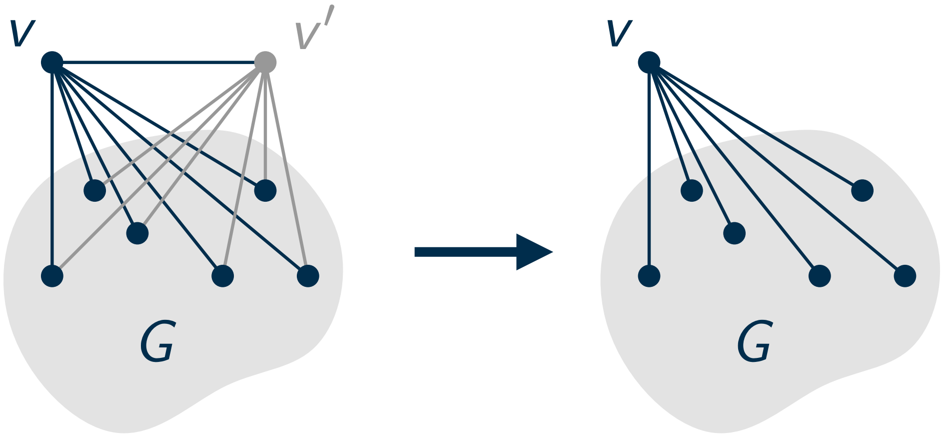

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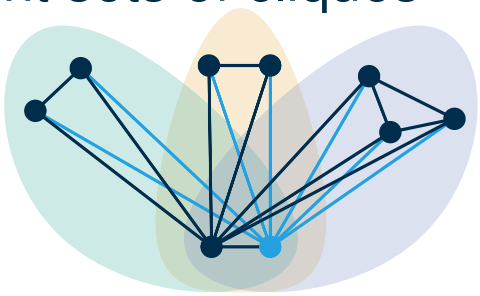

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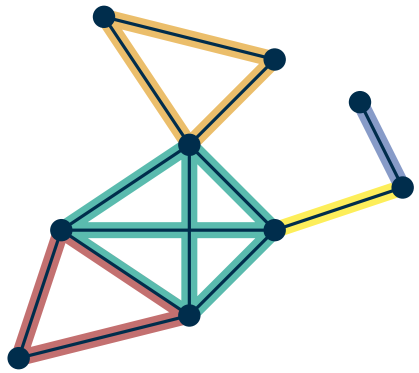

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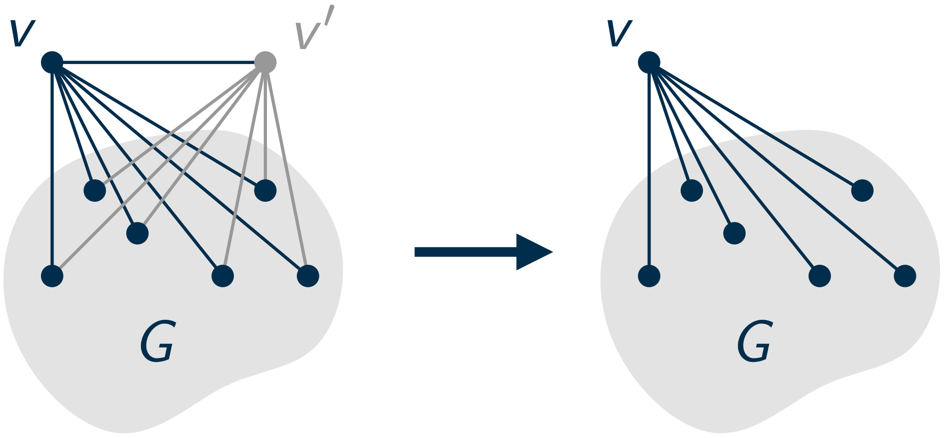

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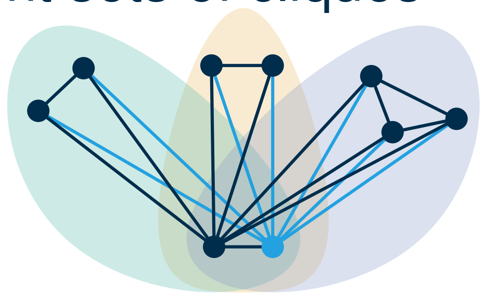
$\Rightarrow$ a solution with at most k cliques has at most 2^k vertices



Sheet 3 – VC on Bipartite Graphs

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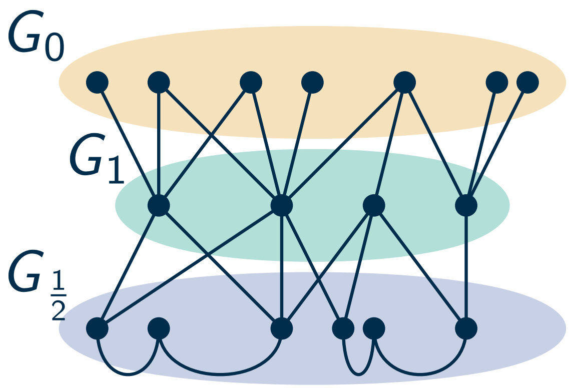

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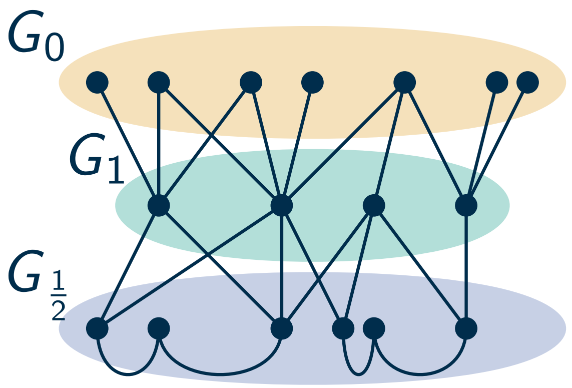

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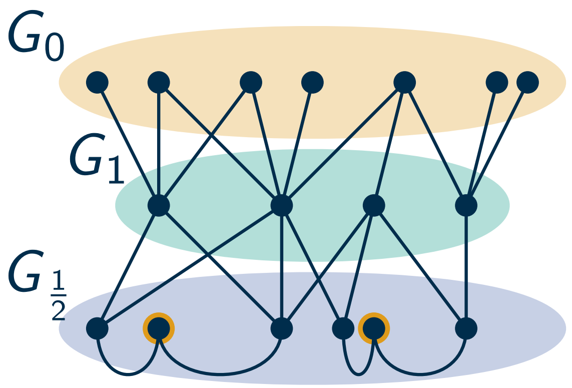

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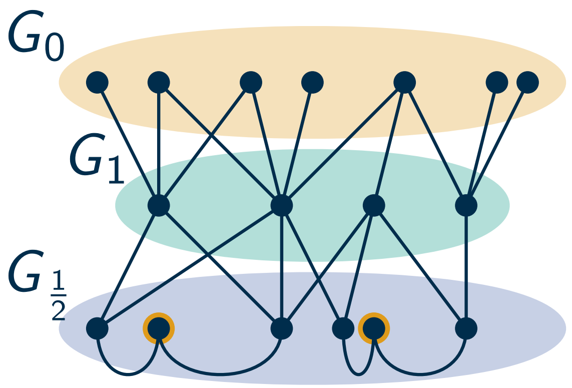

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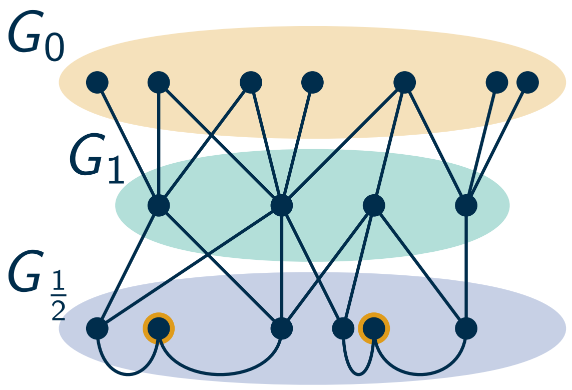

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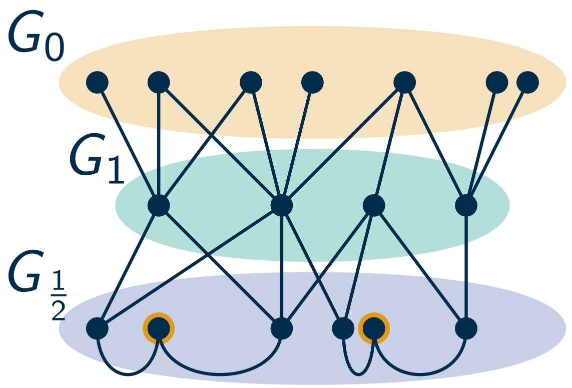

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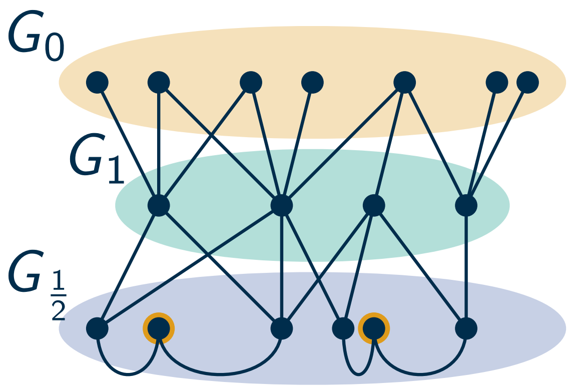

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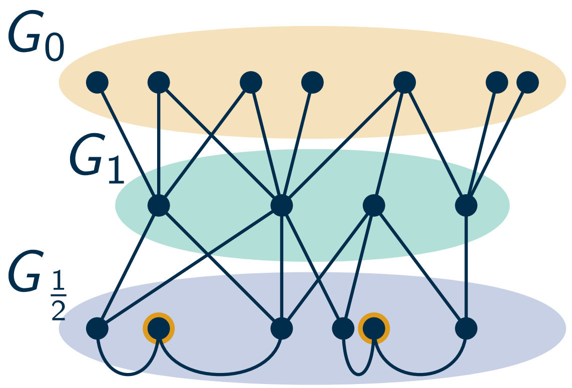

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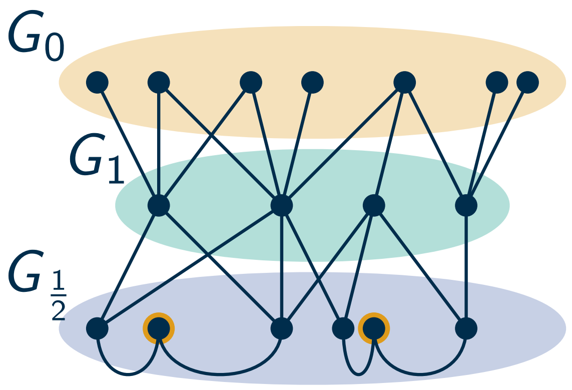

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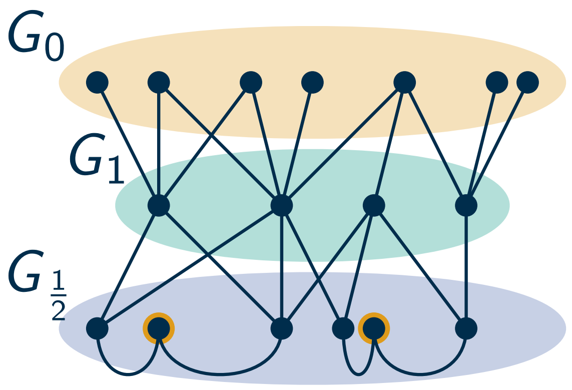

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$$\max \sum_{j=1}^{n} x_j \quad \text{with } Ax \leq \mathbf{1}$$

$$\min \sum_{i=1}^{n} y_i \quad \text{with } A^T y \geq \mathbf{1}$$

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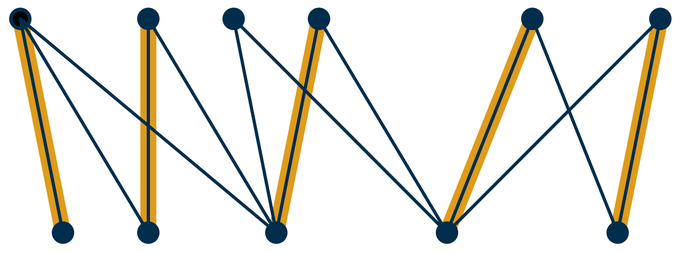

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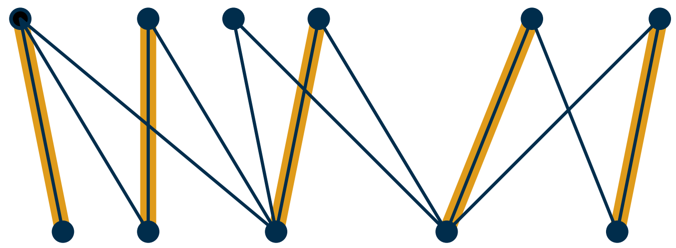

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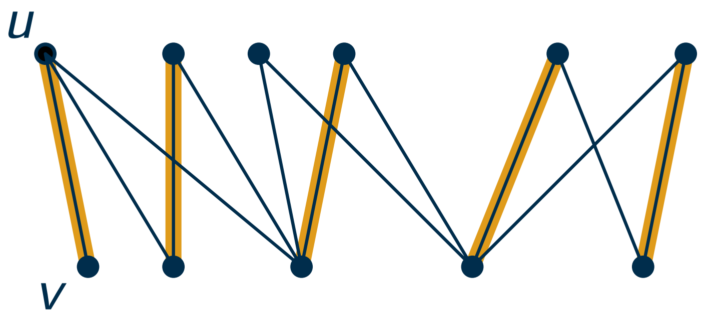

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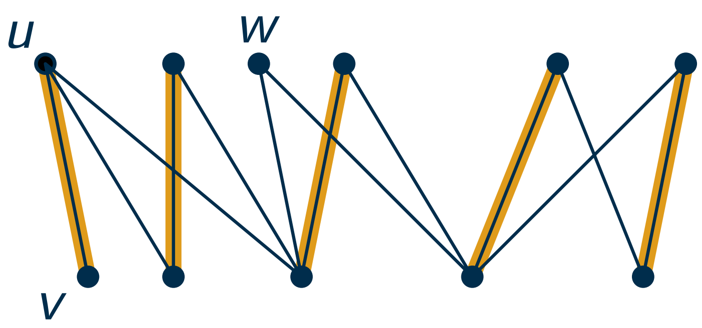

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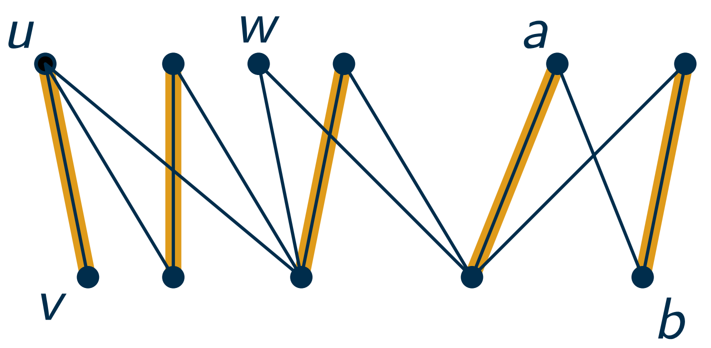
  - all edges ab are covered: $(a \vee b)$



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Given: A list L of ℓ strings. Find the minimum k and string s, such that s has Hamming distance at most k to each string.

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Which string do we branch over?

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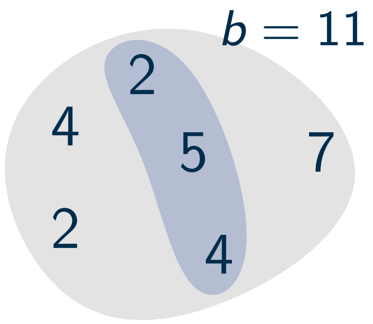

Formulate ILP

- SUBSET SUM: find sub-multiset of size b

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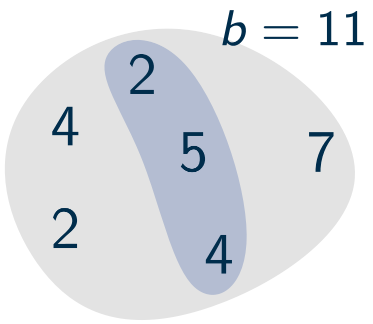

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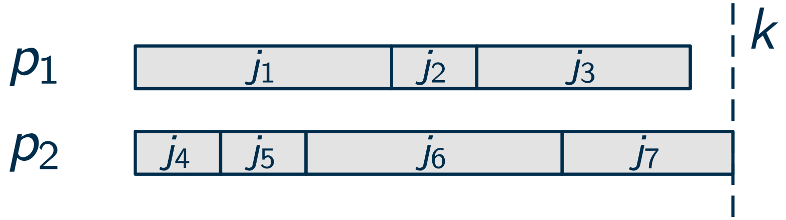

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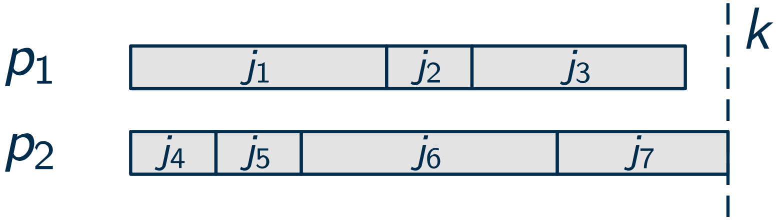

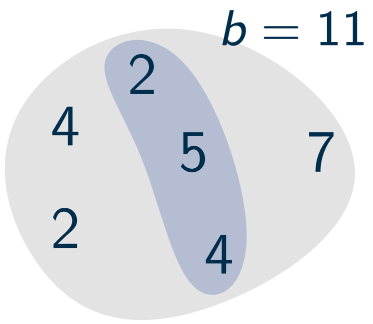

MAX SAT

given CNF, satisfy at least p clauses

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_4 \vee \neg x_5)$$

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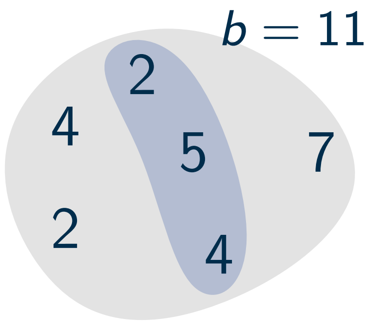

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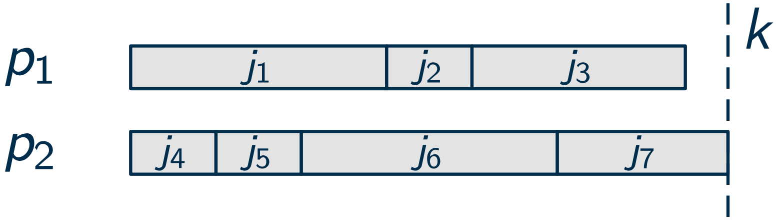
- SCHEDULING: find job assignment with makespan k



MAX SAT

given CNF, satisfy at least p clauses

- find kernel of size $O(p)$

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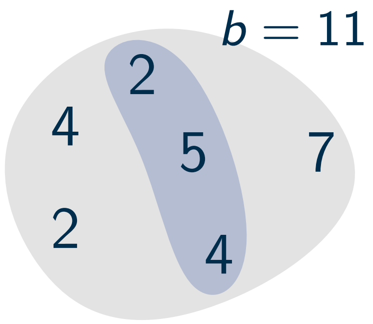
Sheet 4



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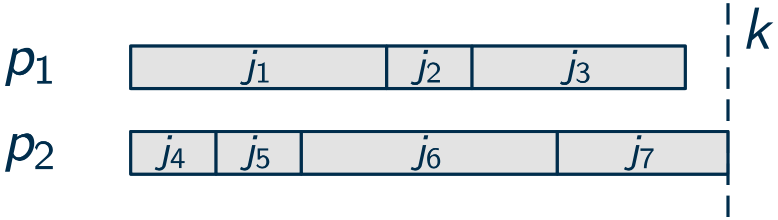

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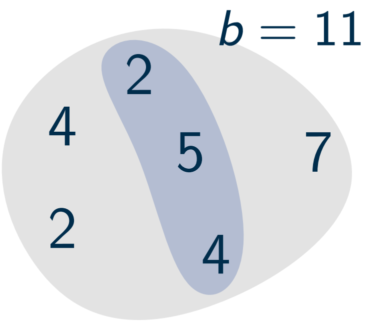

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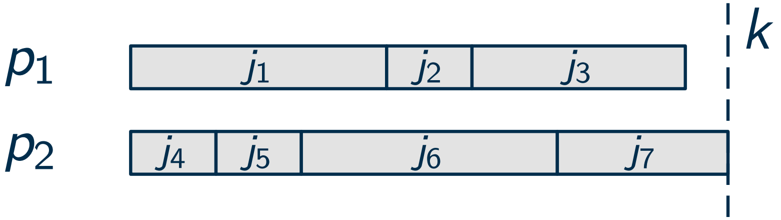

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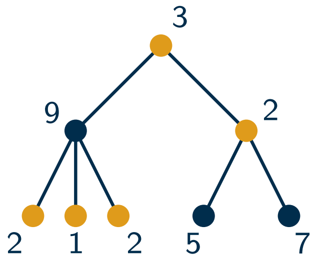
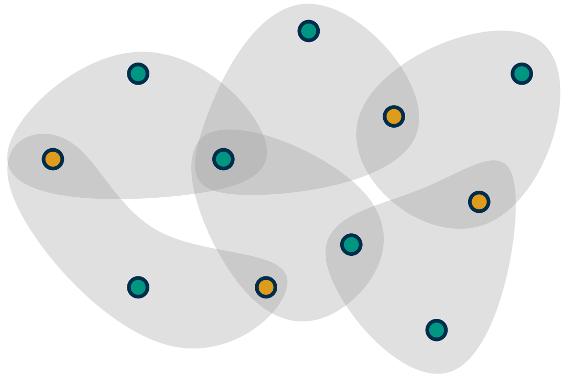
WEIGHTED VERTEX COVER on Trees

- write down DP for weighted VC on trees



Probabilistic Method

HYPERGRAPH 2-COLORING
Given hypergraph where each hyperedge has size exactly k. Is there an assignment of 2 colors such that each edge is colorful?



- **Idea:** assign every vertex a random color
- consider the probability that and edge e is single-colored

$$\mathbb{P}[\exists e \text{ that is single-colored}] \leq \sum_{e \in E} \mathbb{P}[e \text{ is single-colored}] = |E| \cdot 2^{1-k}$$

- $|E| < 2^{k-1} \Rightarrow \mathbb{P}[\exists e \text{ that is single-colored}] < 1 \Rightarrow$ there exists a colorful assignment

$\mathbb{P}[X] > 0 \Rightarrow$ there exists an event such that X holds

ILPs and Coloring

Formulate the following problems as an ILP:

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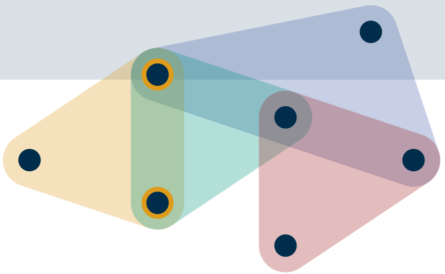

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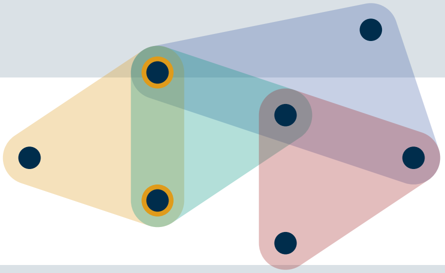

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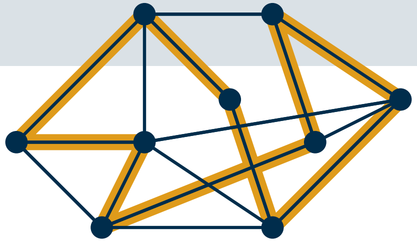
Given: Graph G. Is there a cycle in G that containes every vertex?



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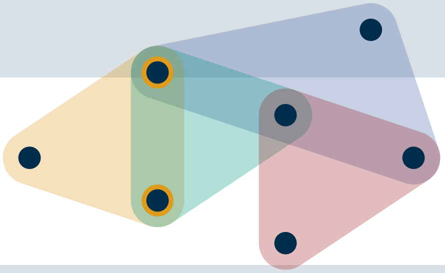

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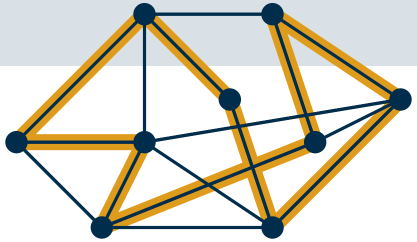

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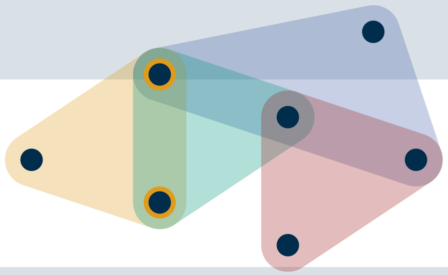

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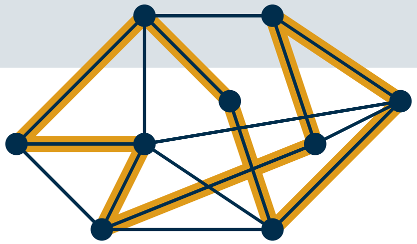

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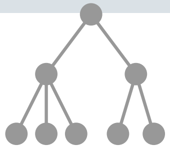
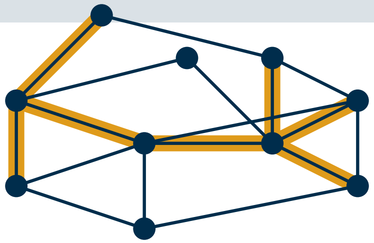

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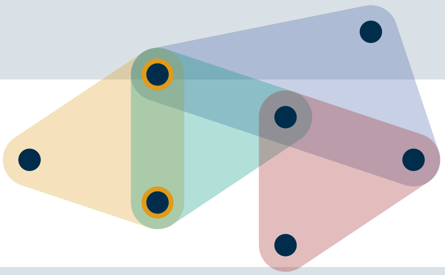

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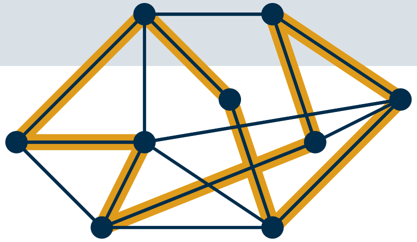

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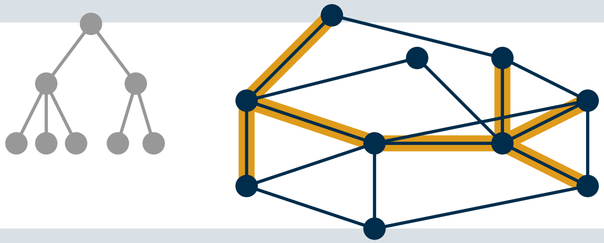

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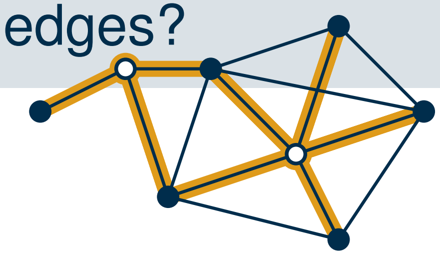
Given: Graph G, parameter k and upper bound s. Is there a subset of at most s vertices that hits at least k edges?



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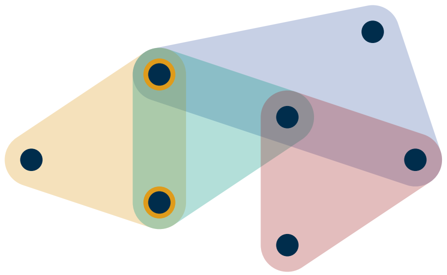

- one variable for every element $v = 1 \Leftrightarrow v$ is in hitting set
- **every set is hit:** for all $s \in S : \sum_{w \in s} w \geq 1$.
- minimize the sum of all variables

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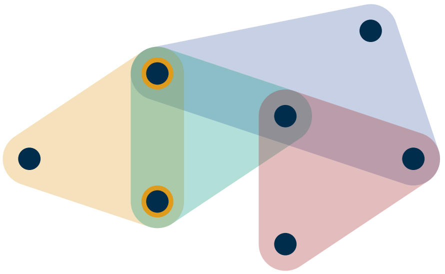

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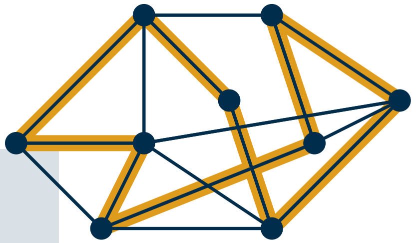

Hamilton Cycle
Given: Graph G. Is there a cycle in G that containes every vertex?

- encode permutation, $x_{v,i} = 1 \Leftrightarrow v$ is the ith vertex on the cycle
- **valid permutation:** for all vertices $w : \sum_j x_{w,j} = 1$ (same for timesteps)
- **valid cycle:** $x_{v,i} + x_{w,i+1} - e_{vw} \leq 1$ (e_{vw} is 1 when vw is an edge)

Solution Coloring

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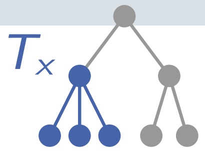

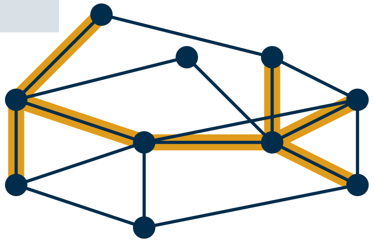

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- **DP:** for every vertex v: compute for which color sets T_x is a subgraph with v as root

Solution Coloring

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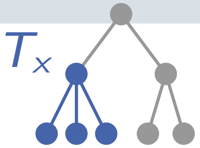

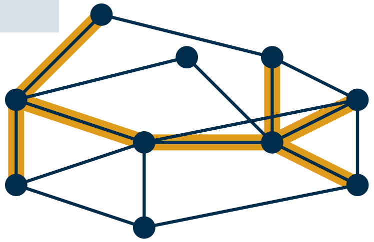

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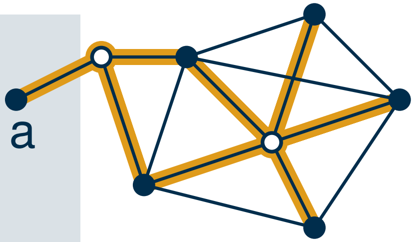

- **Color edges:** Can we select a set of s vertices that hits each color calss?

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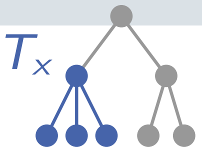

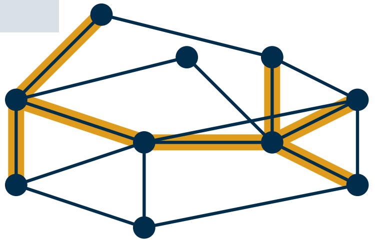

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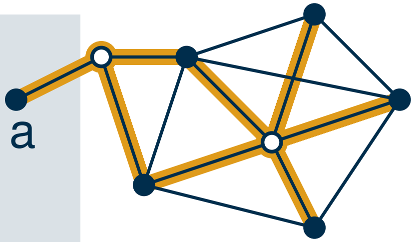

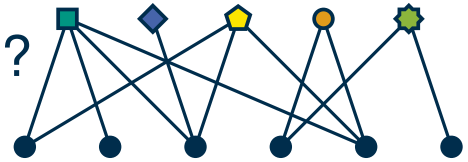

- **Color edges:** Can we select a set of s vertices that hits each color calss?
- Interpret a hyper/bipartite graph with at most 2^k vertices
- **DP:** For each combination of colors. Minimum number of vertices from $\{v_1, \ldots, v_i\}$ to cover this combination