Parameterized Algorithms

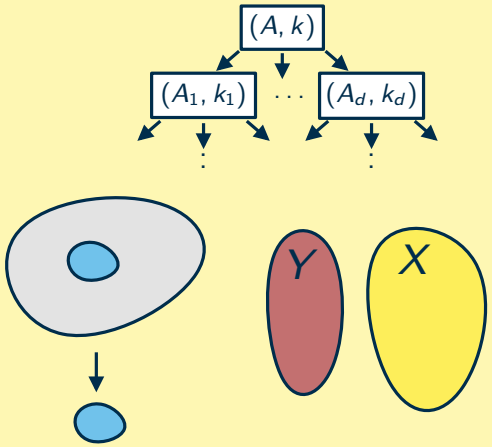
Bounded Search Trees: Improved Branching

Thomas Bläsius

Content

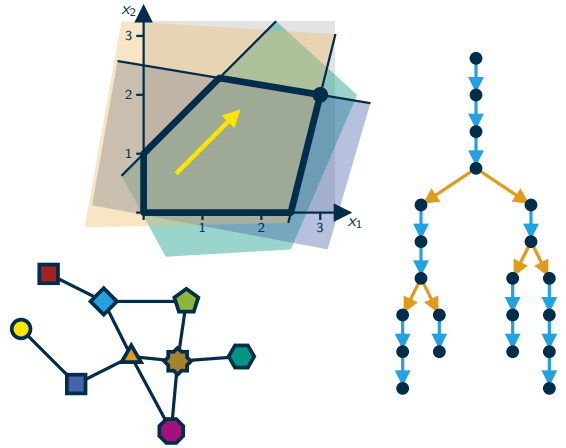
Basic toolbox

- bounded search trees
- kernelization
- iterative compression



Extended toolbox

- linear programs
- branch-and-reduce
- color coding

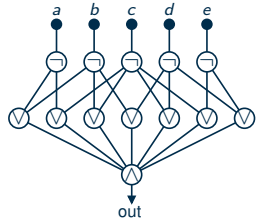


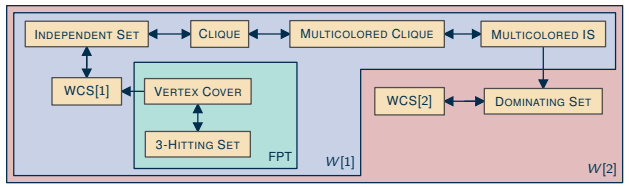
Tree width

- dynamic programming
- chordal and planar graphs
- Courcelle's theorem

Lower bonds

- kernel lower bounds
- parameterized reductions
- circuits and the W-hierarchy
- ETH and SETH





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Problem: VERTEX COVER
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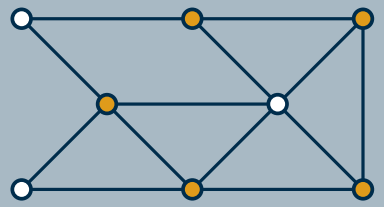

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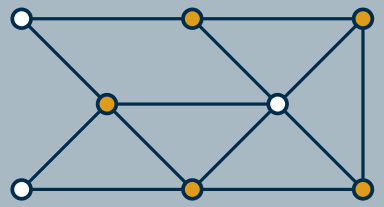

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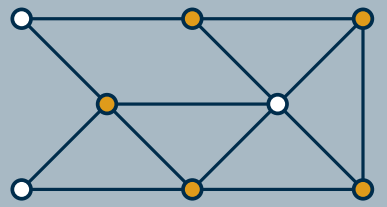

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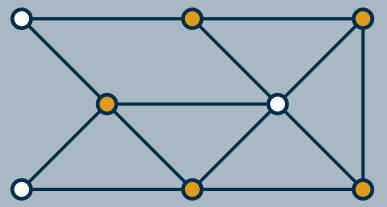

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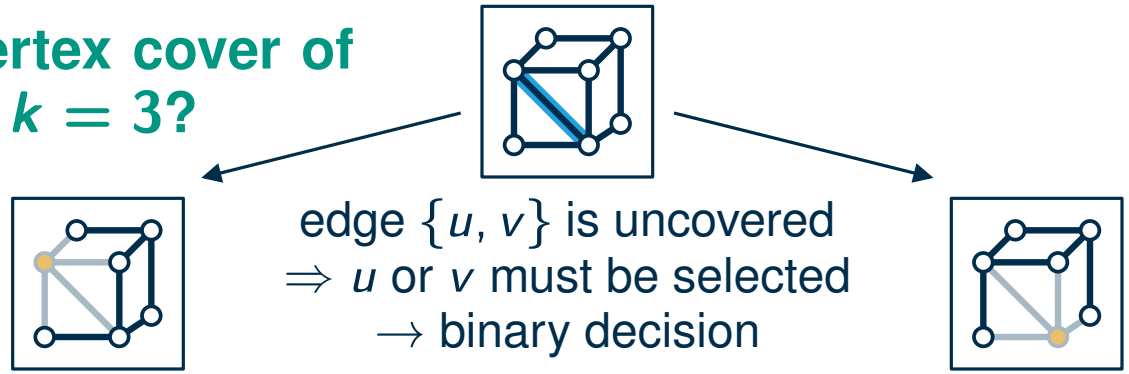

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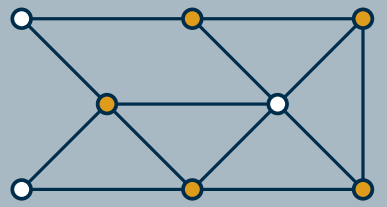

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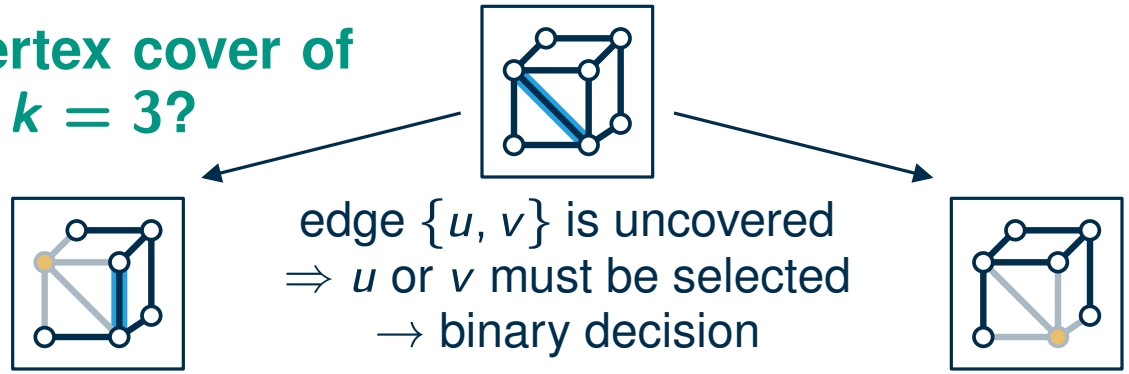

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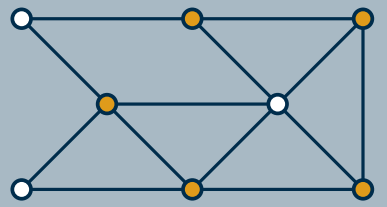

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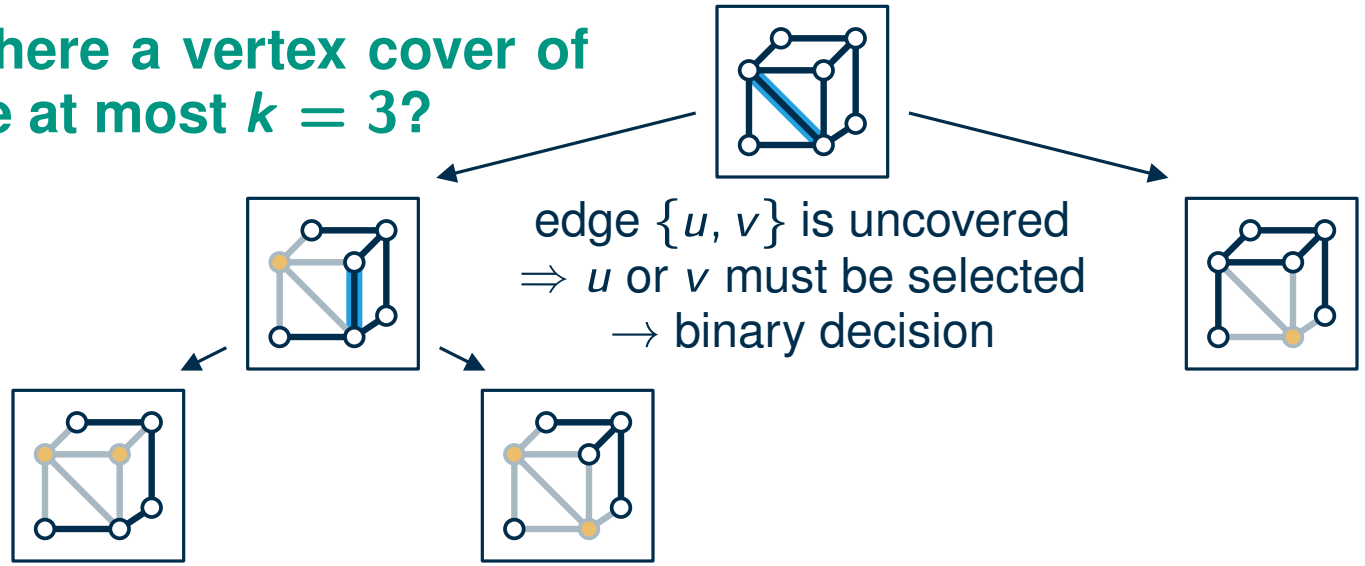

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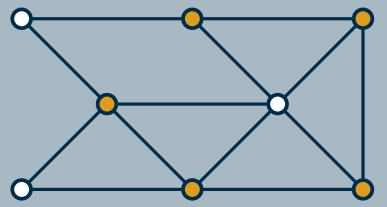

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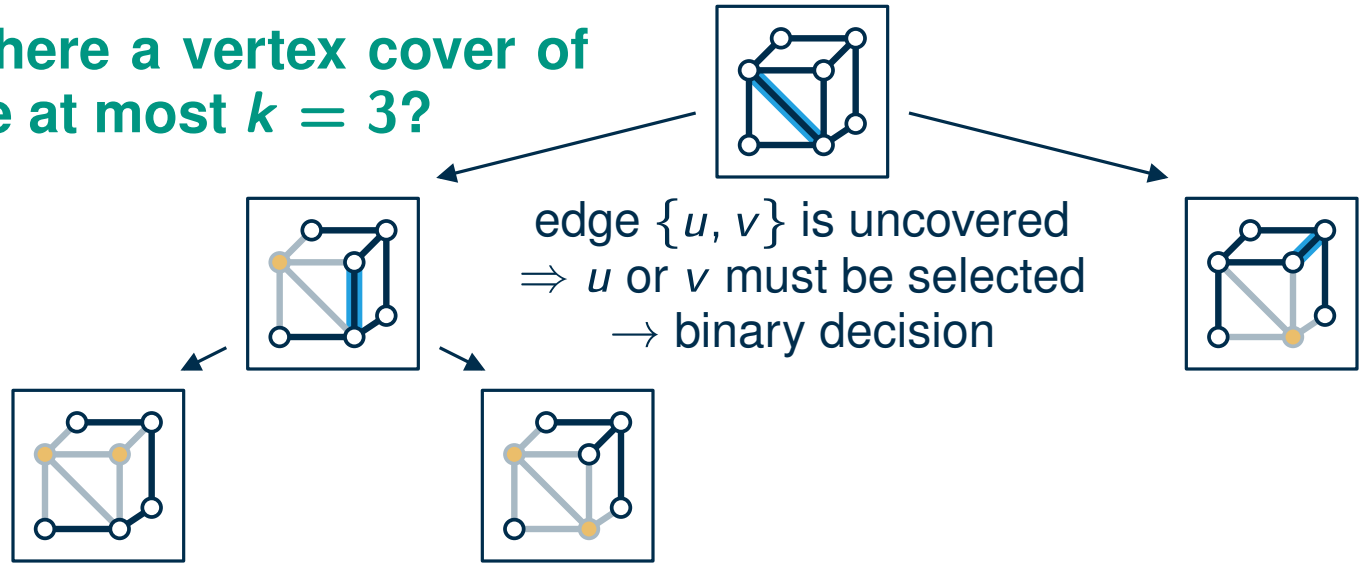

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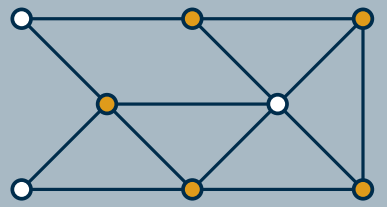

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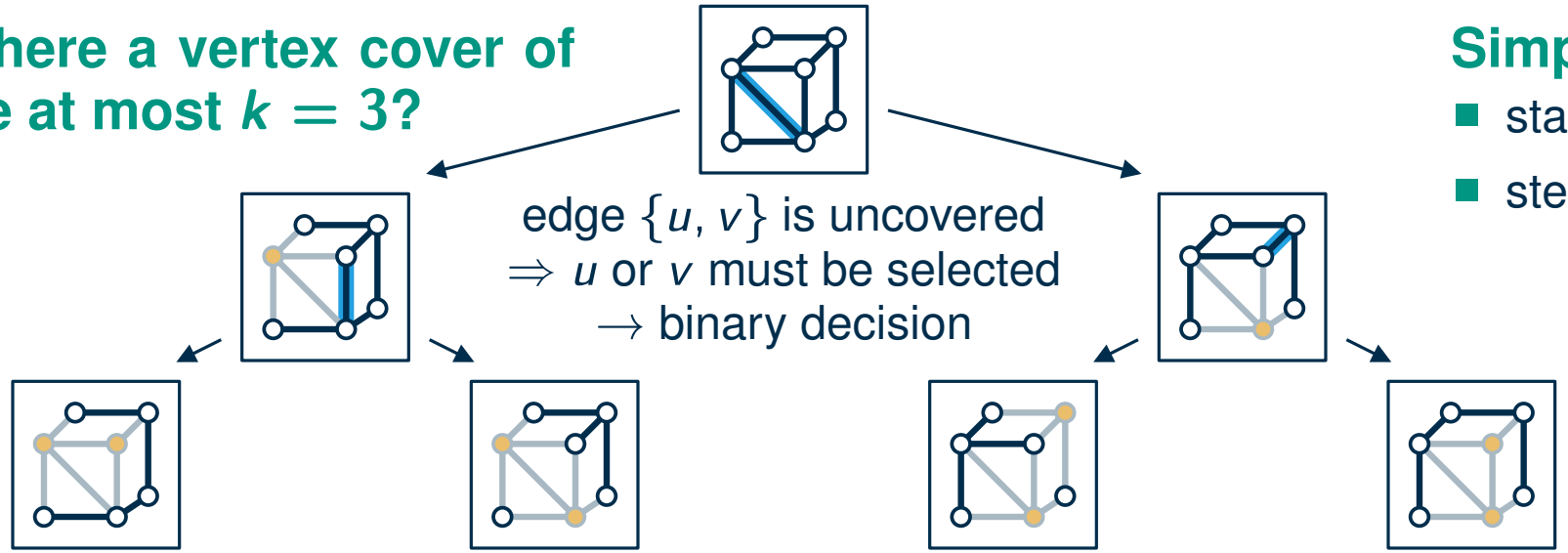

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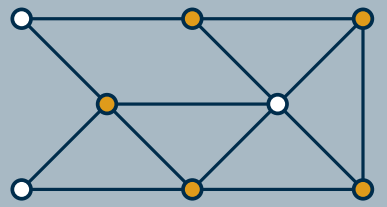

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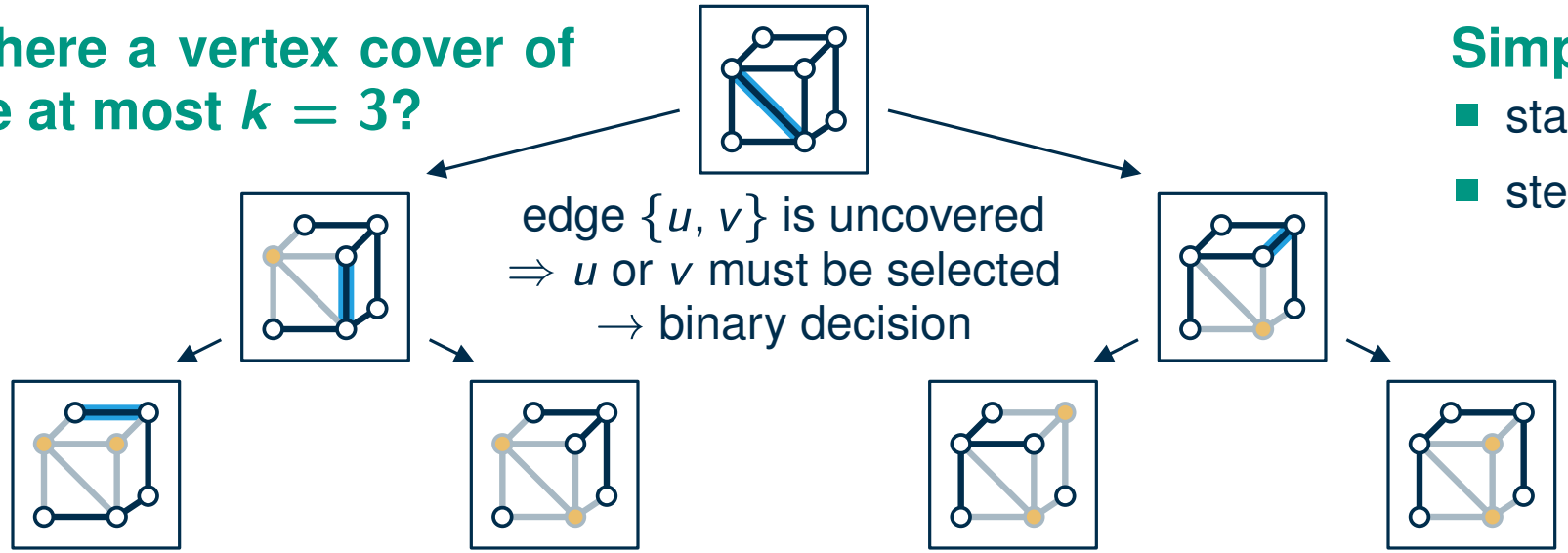

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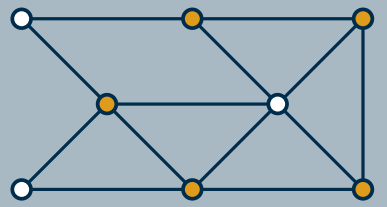

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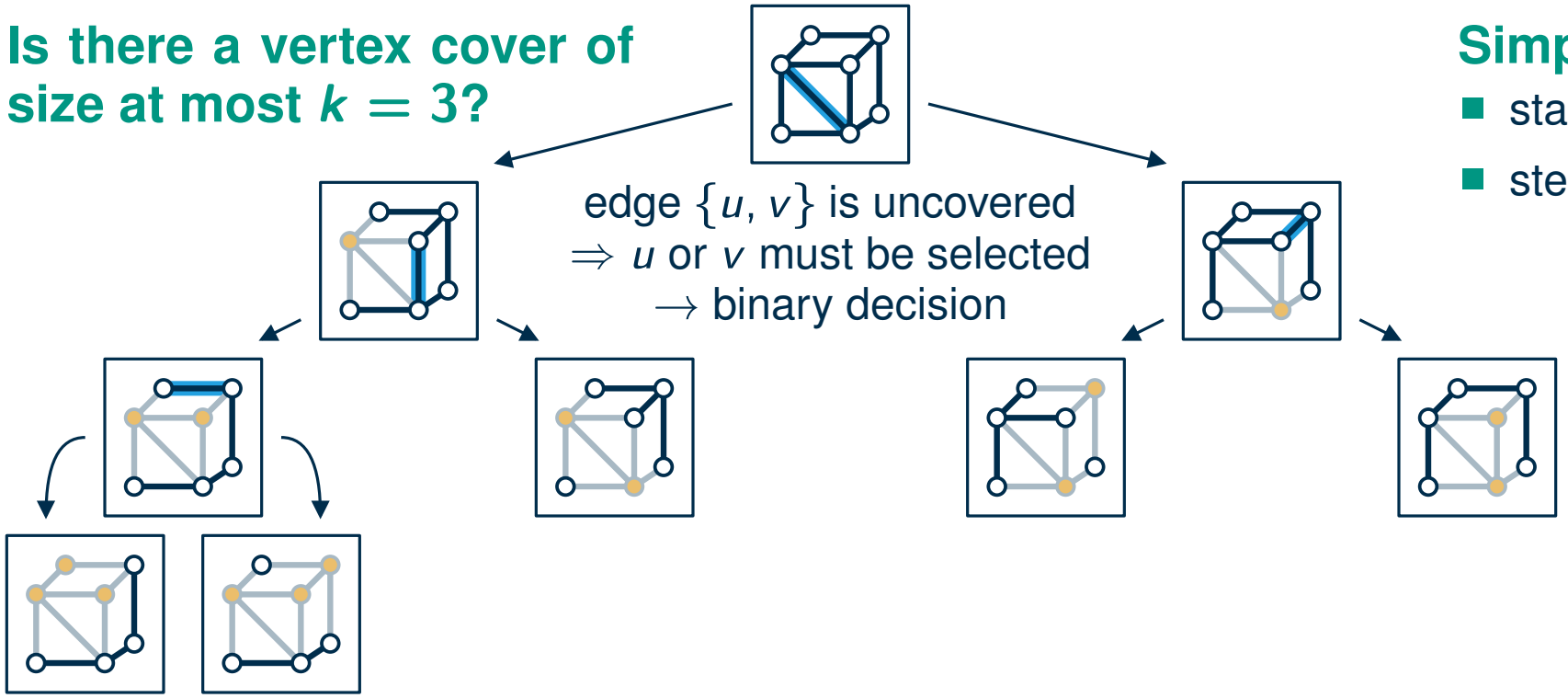

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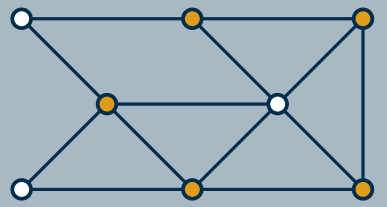

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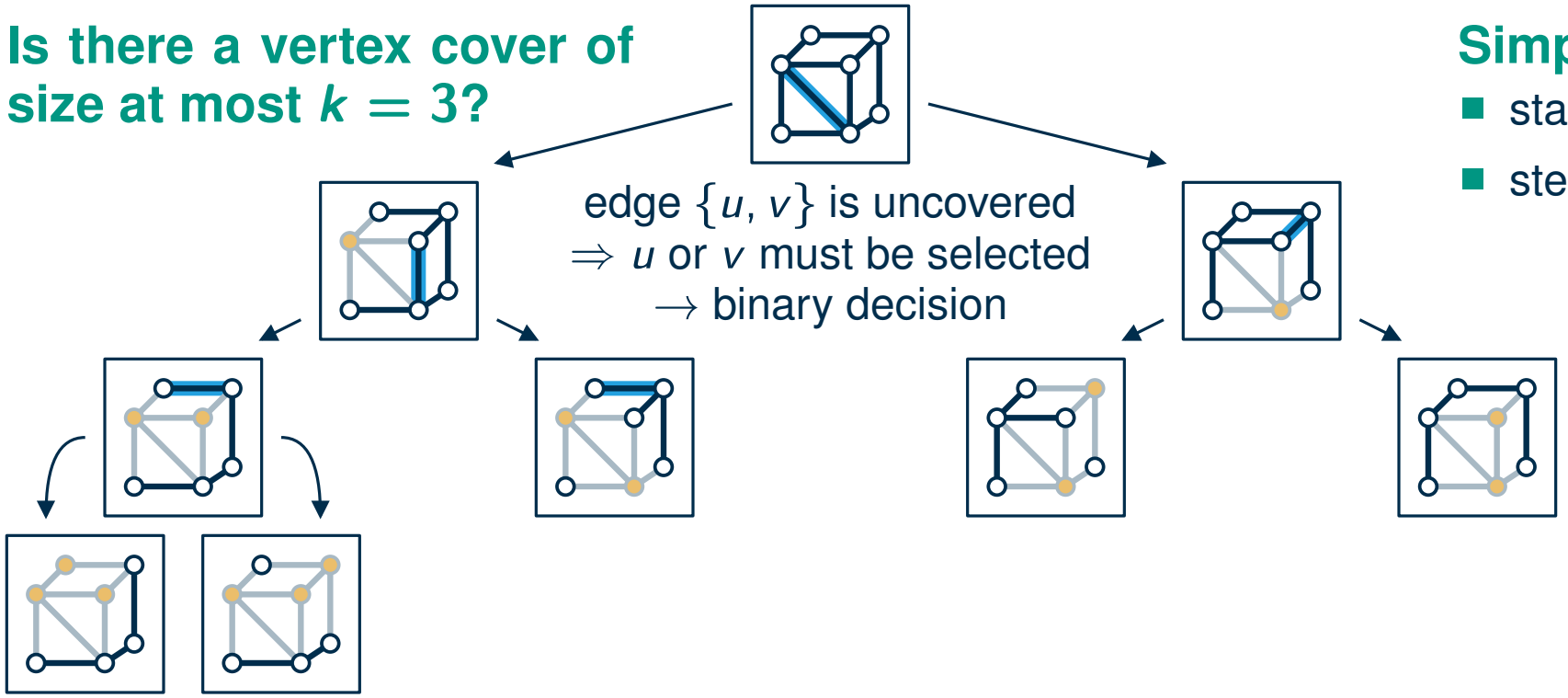

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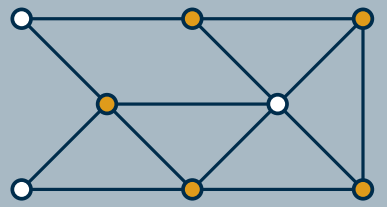

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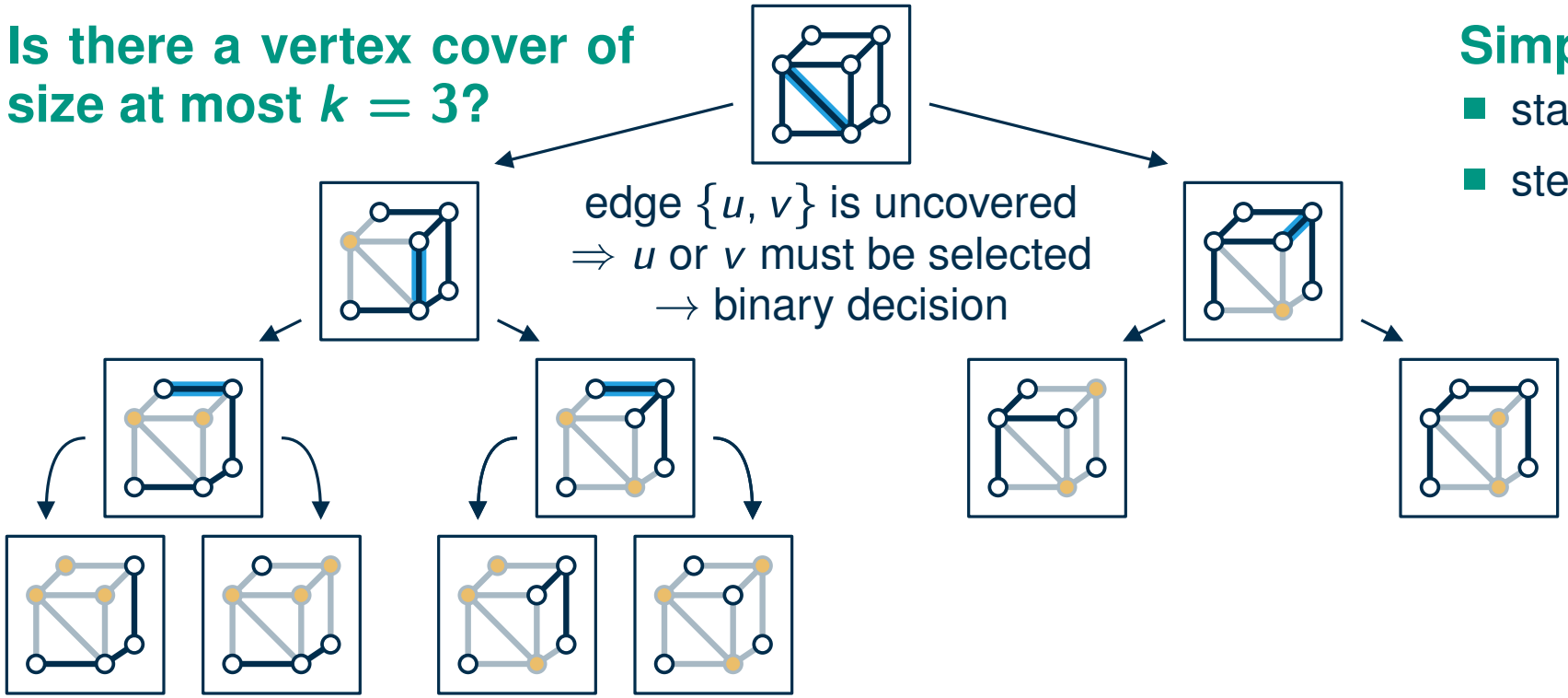

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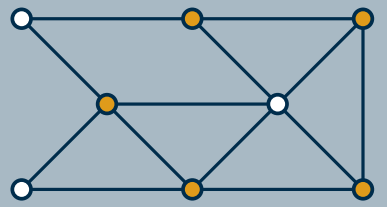

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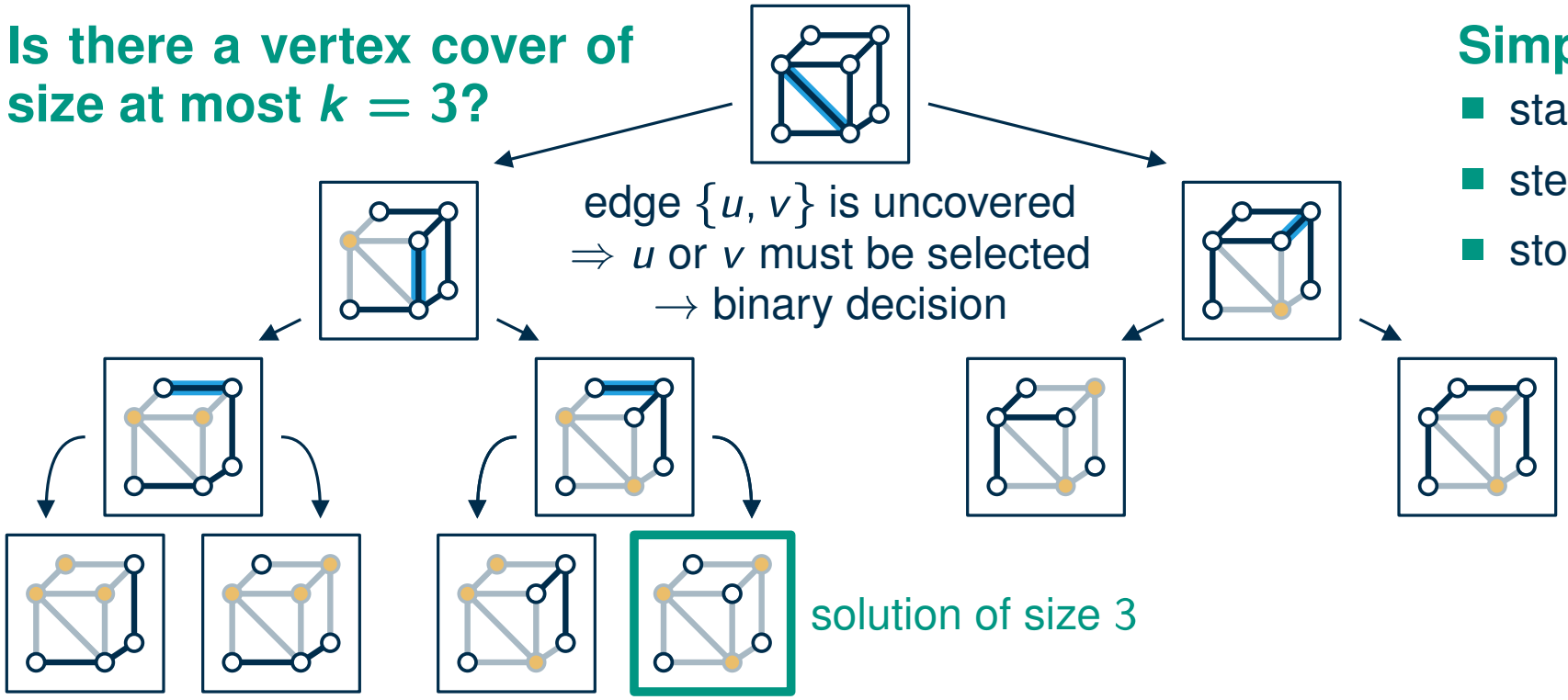

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solution of size 3

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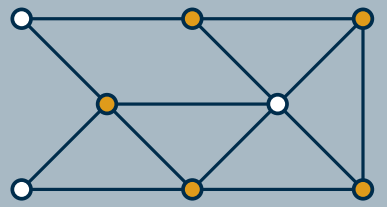

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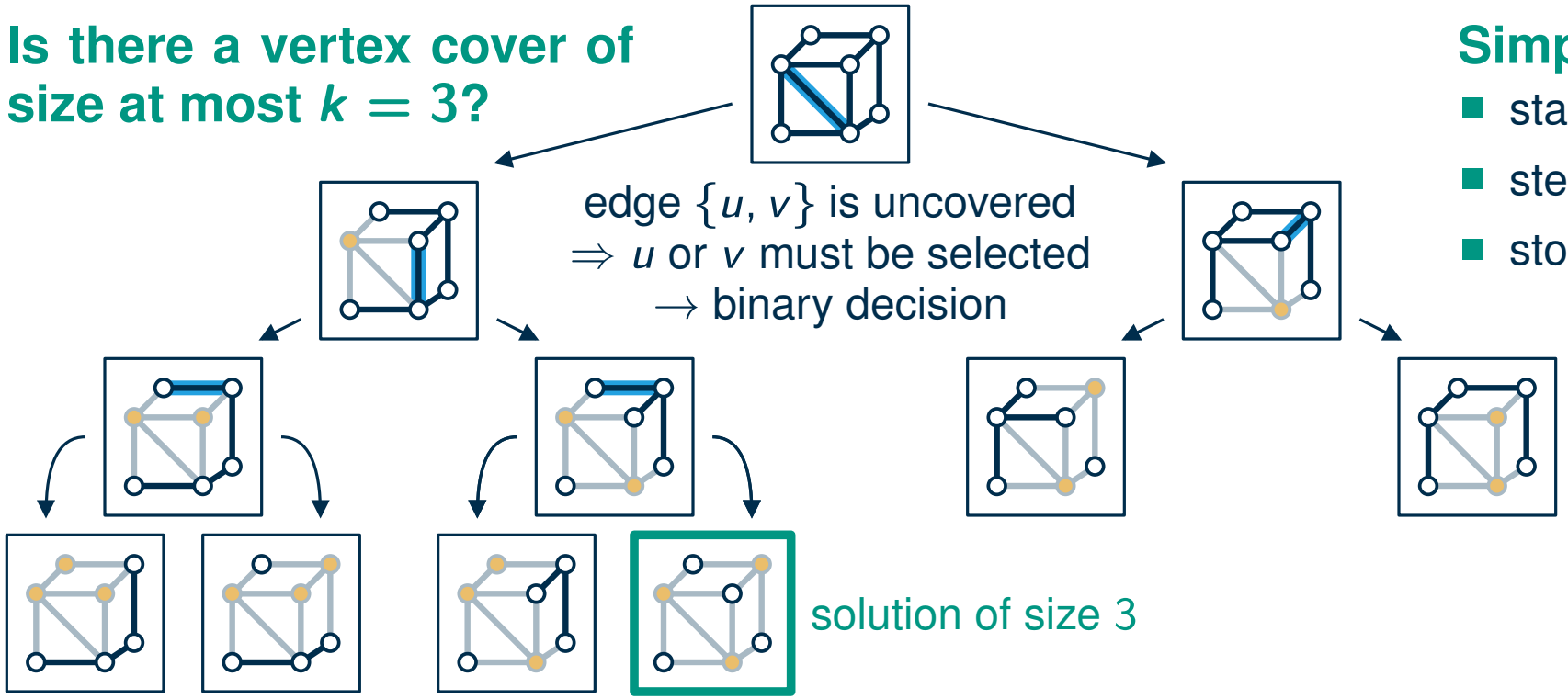

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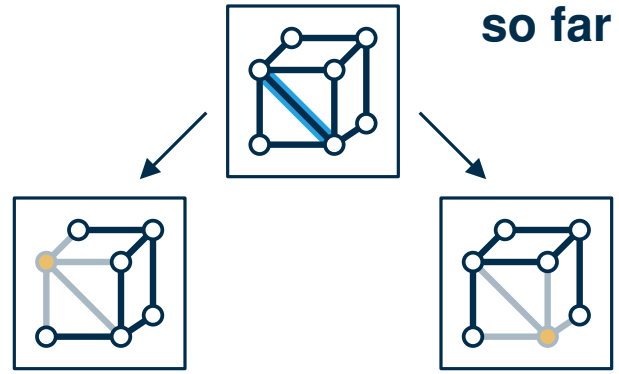
running time: $O(2^k m)$

Search tree with stronger bounds

Can we be faster?

- goal: reduce the size of the search tree (so far: 2^k leaves)
- branching rule so far: for edge $\{u, v\}$ choose either u or v



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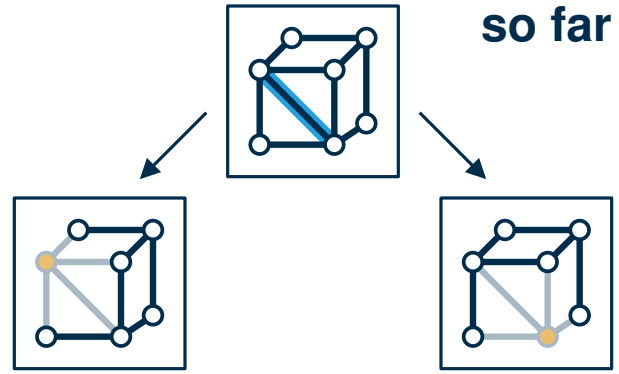

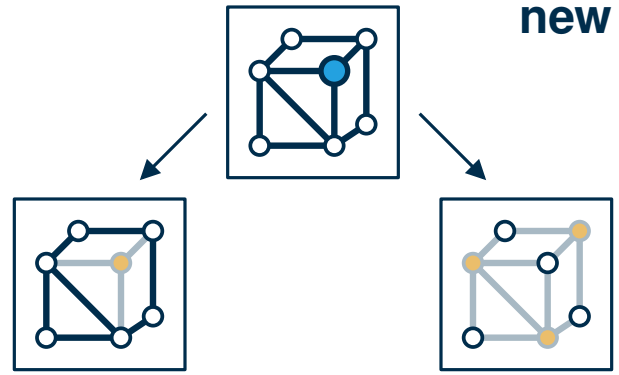

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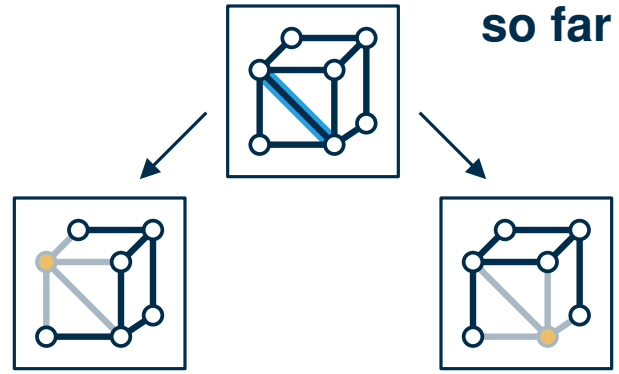

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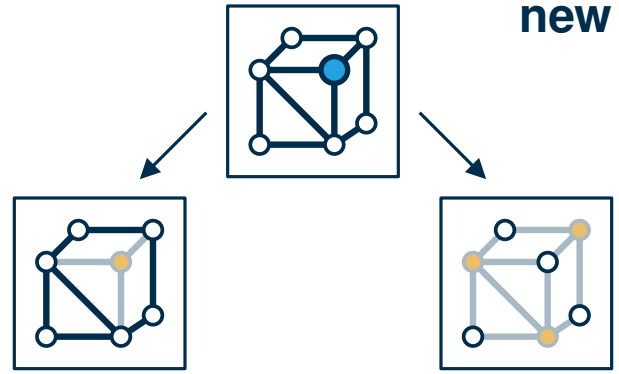

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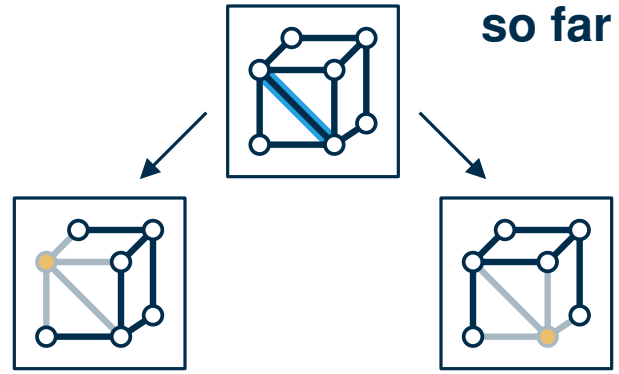
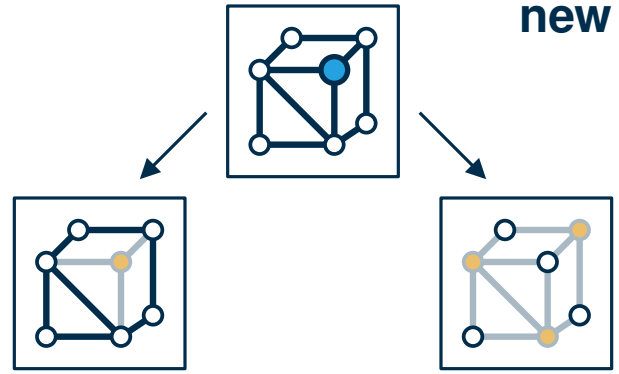


How many leaves does the resulting tree have?

- always choose a vertex v with degree ≥ 2



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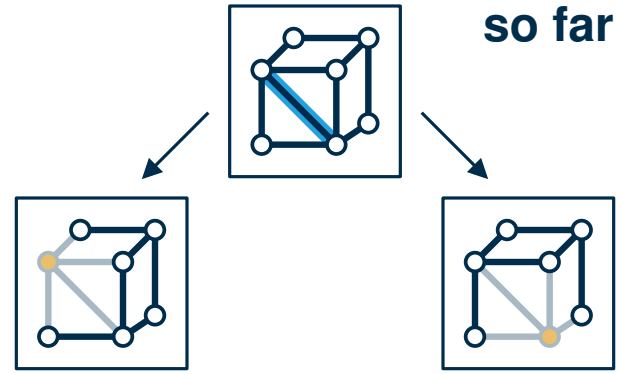
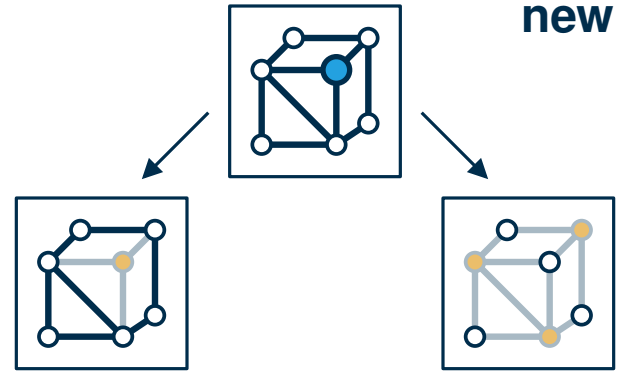


How many leaves does the resulting tree have?

- always choose a vertex v with degree ≥ 2
- upper bound for number of leaves:

$$T(k) =$$



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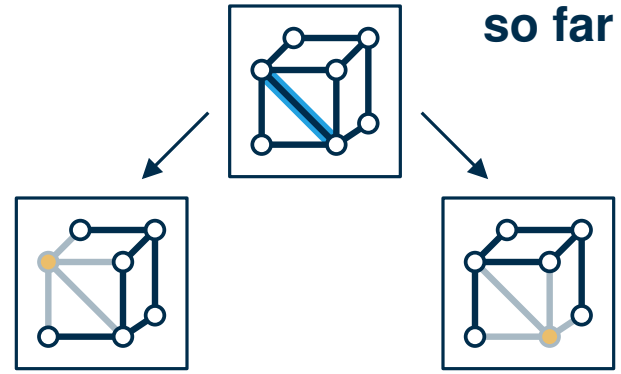

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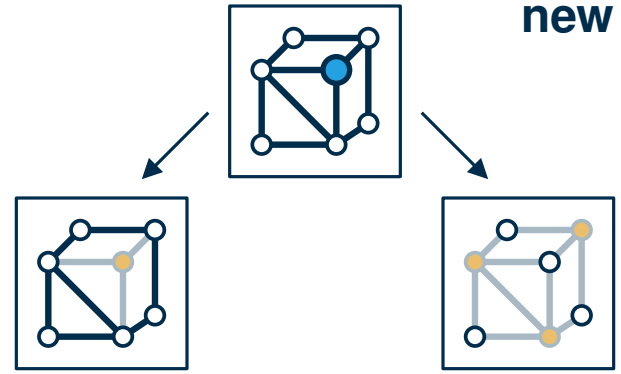
$$T(k) = \begin{cases} T(k-1) + T(k-2), & \text{if } k \geq 2 \\ 1, & \text{otherwise} \end{cases}$$



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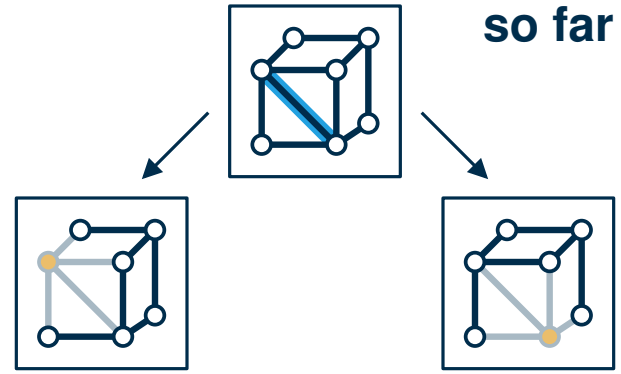

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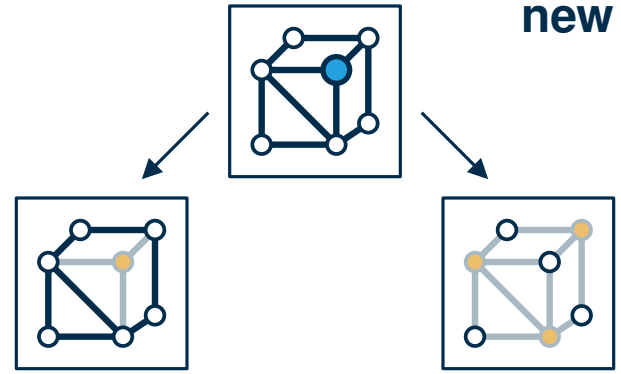

Search tree with stronger bounds

Can we be faster?

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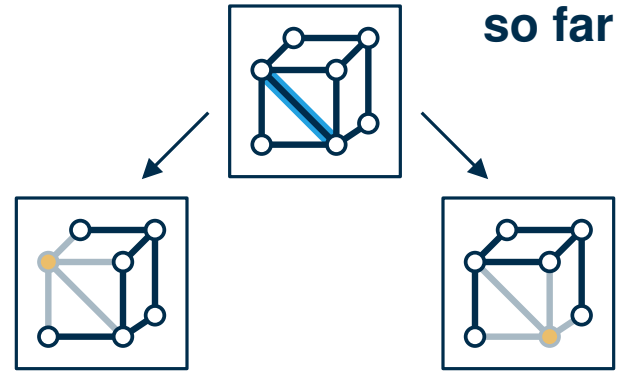

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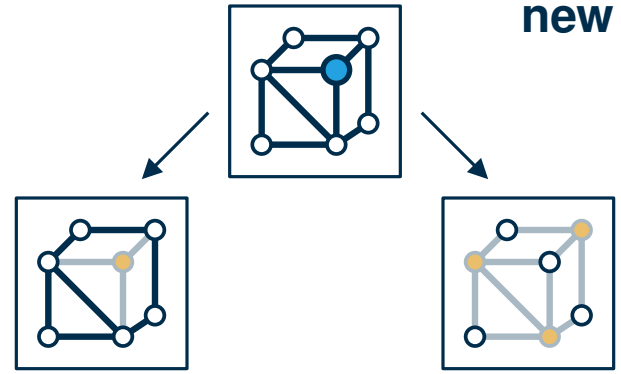

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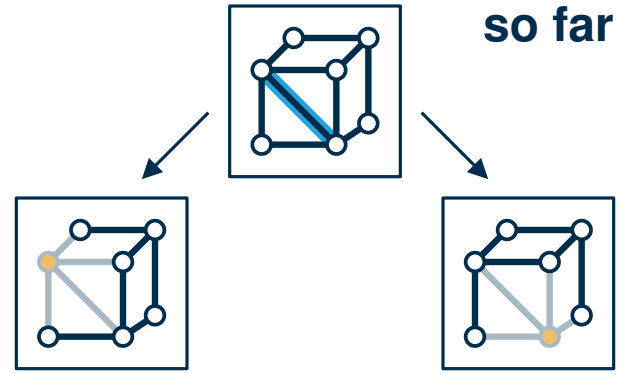

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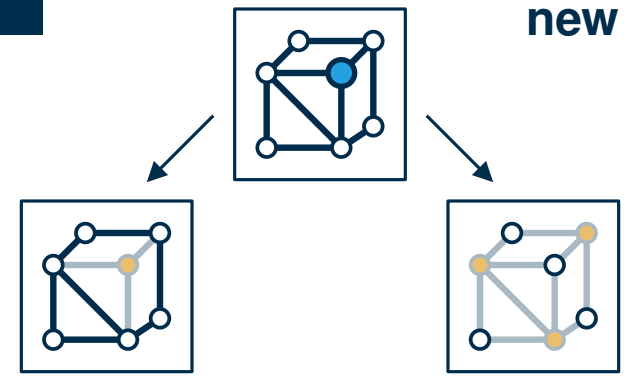
Is it ok, to just count the leaves?



Branching vectors

Branching in general

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- guess the base λ
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How can we guess the base?

- find root of a polynomial (more details in two moments)
- WolframAlpha helps with finding the root

Tree size

Example: branching vector (1, 2)

- recurrence:

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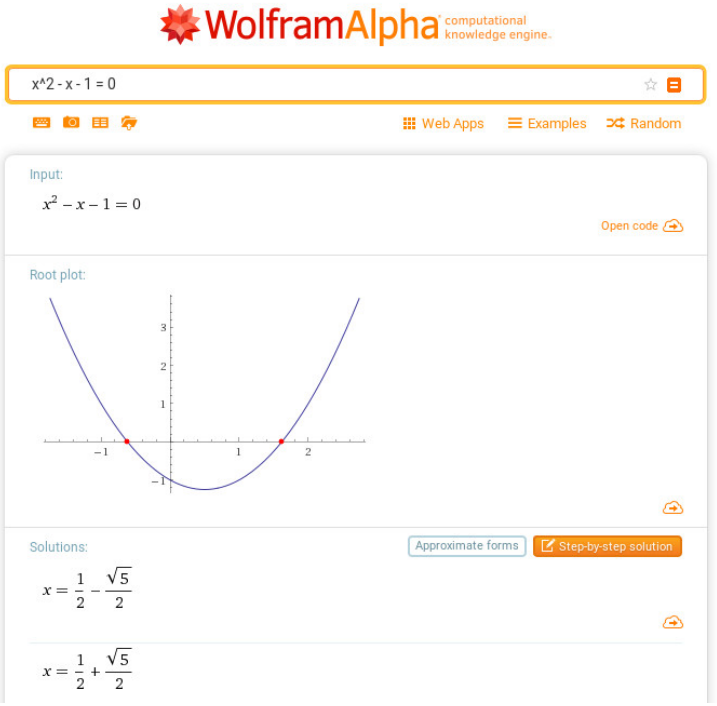

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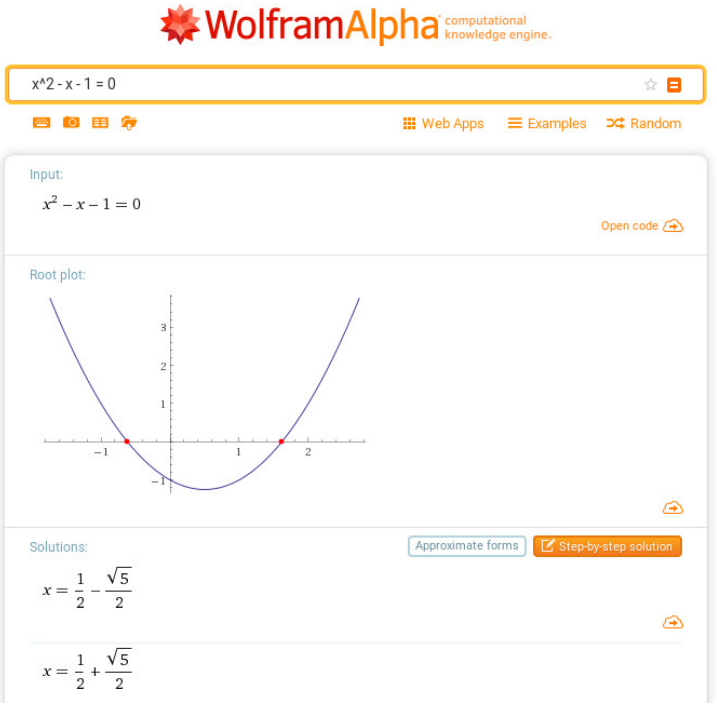

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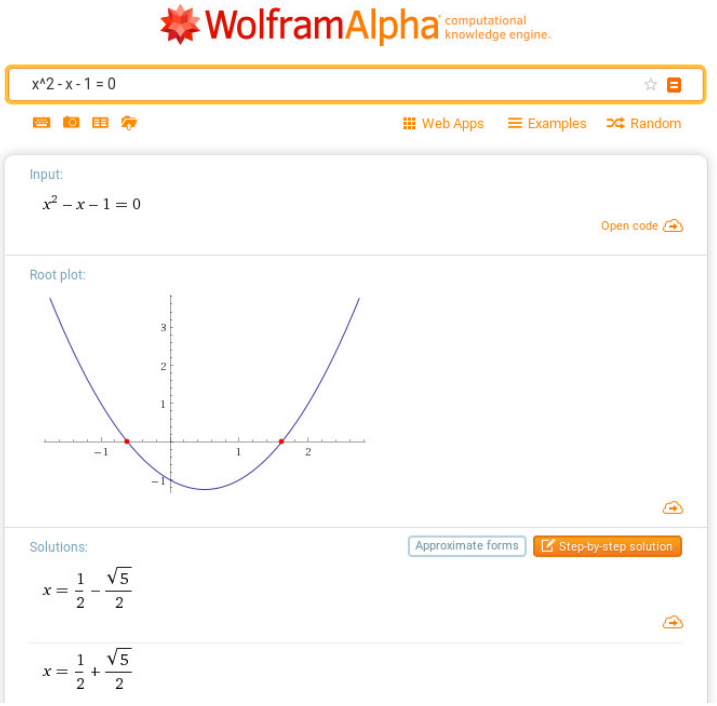

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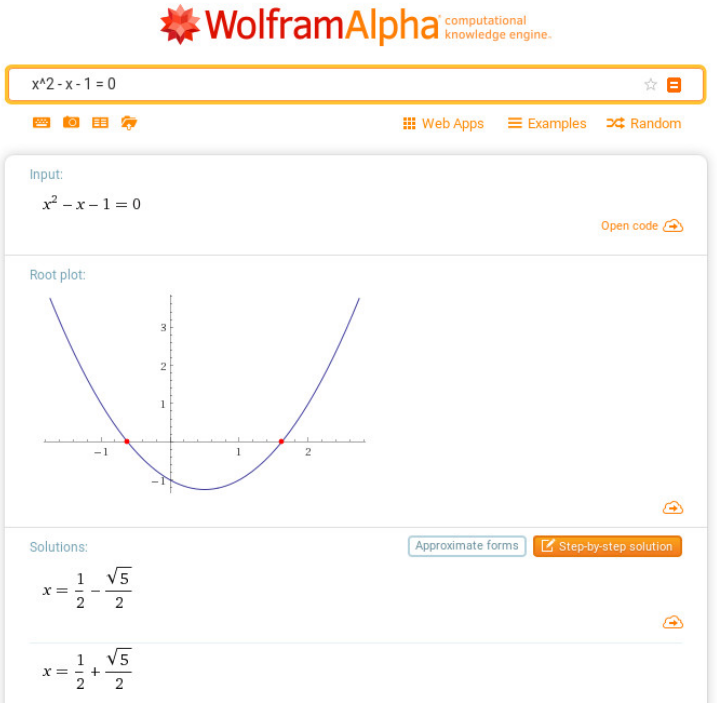

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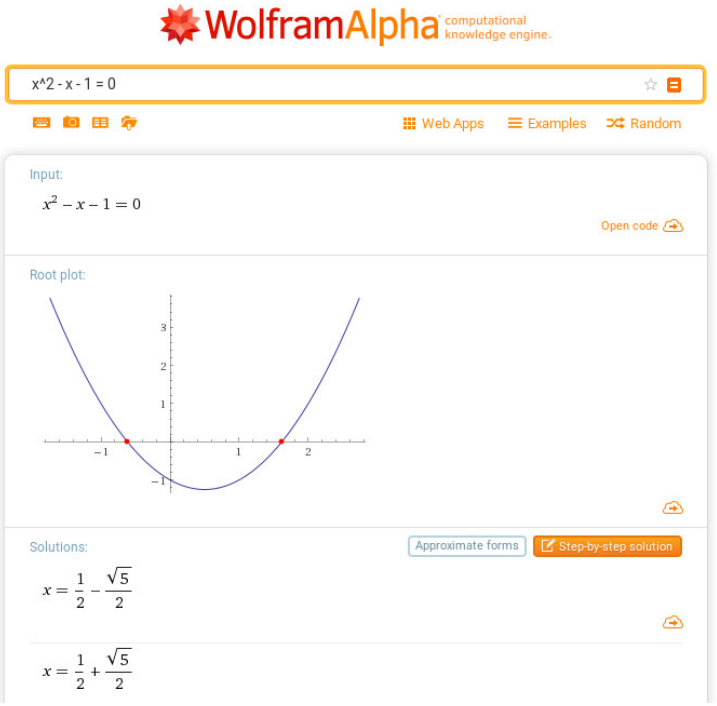

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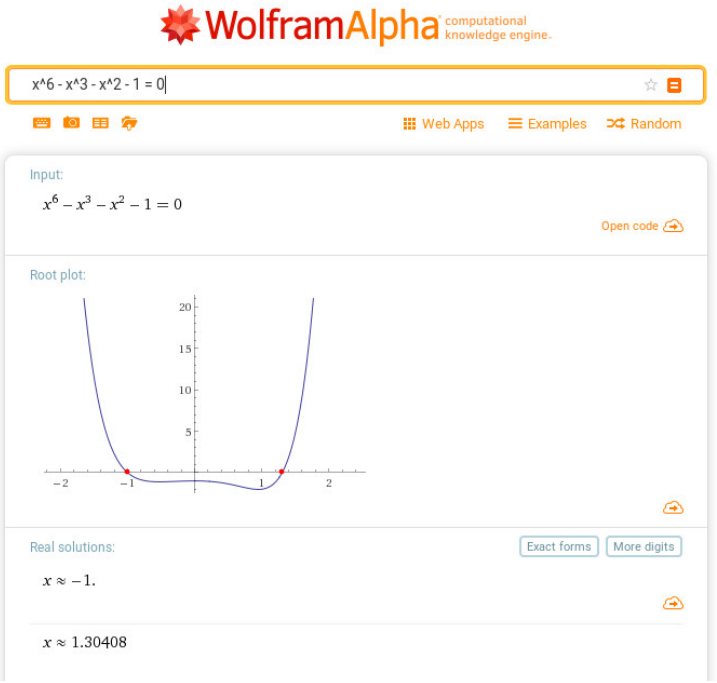

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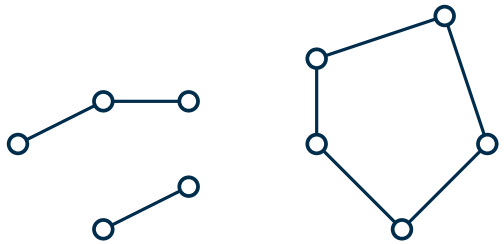

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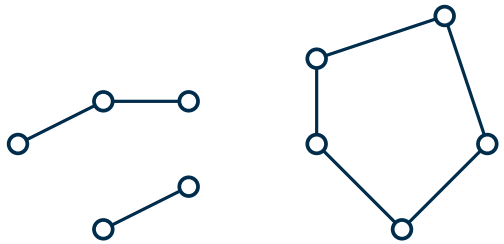

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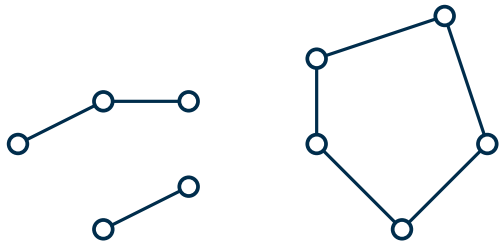

- better branching: for a vertex v with $\deg(v) \geq 3$, choose v or $N(v) \rightarrow$ vector $(1,3)$

Back to Vertex Cover

branching vector	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,3)$	$(3,3)$	$(3,3,6)$	$(3,4,6)$
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Seen so far

- for an edge uv, choose u or $v \rightarrow$ branching vector $(1,1)$
- for a vertex v with $\deg(v) \geq 2$, choose v or $N(v) \rightarrow$ vector $(1,2)$

Why is there always a vertex v with $\deg(v) \geq 2$?

Can we improve?

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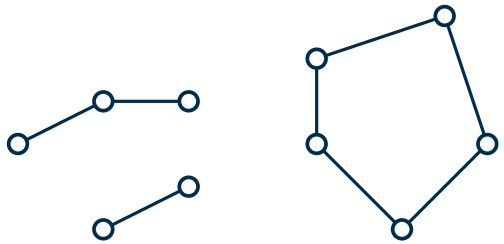

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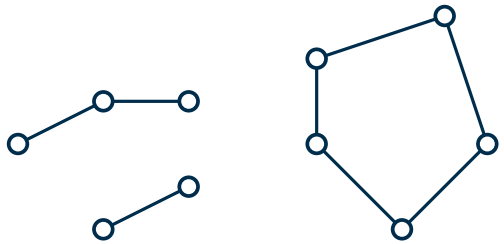

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- Vertex Cover is NP-hard for graphs with maximum degree 3

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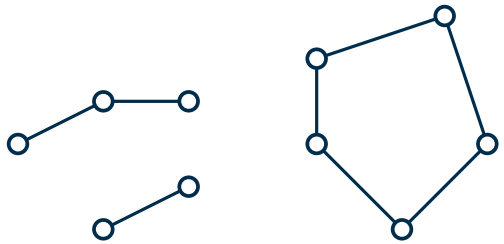

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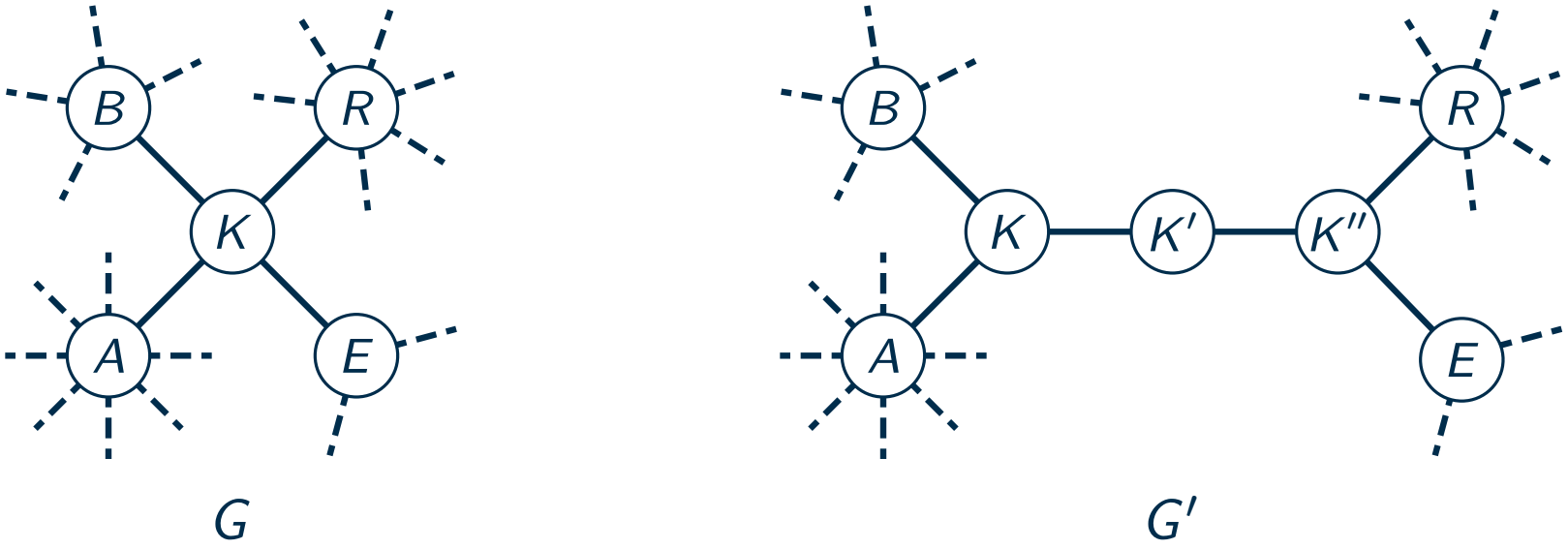
Can we improve further?

- Vertex Cover is NP-hard for graphs with maximum degree 3
- branching vector $(1,4)$ cannot be achieved as easily

What is happening here?

What does the following construction achieve?



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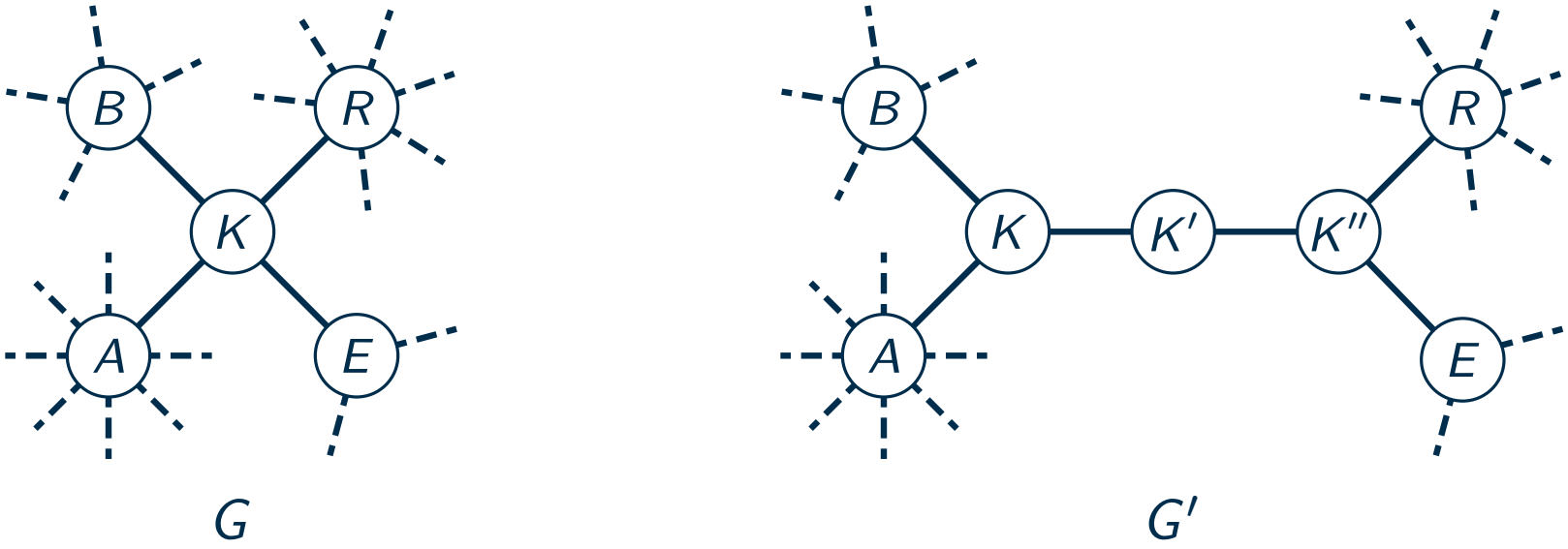

Answer

- G has a VC of size $k \Leftrightarrow G'$ has a VC of size $k + 1$

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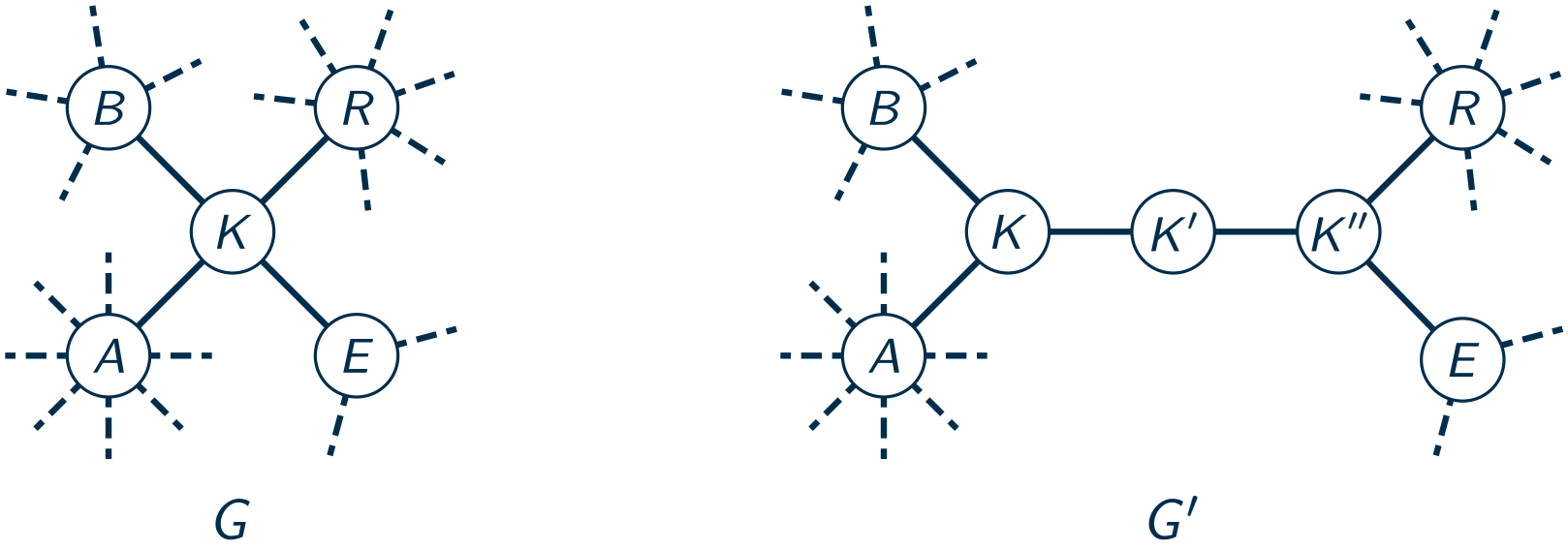

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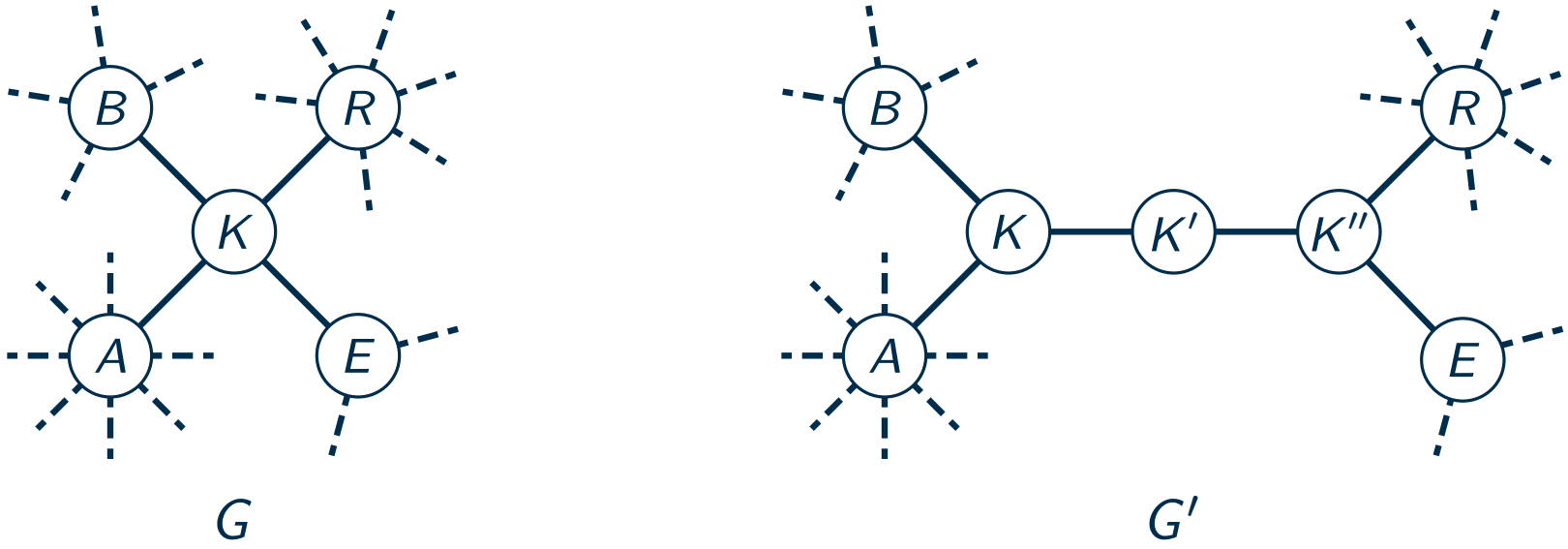

Answer

- G has a VC of size $k \Leftrightarrow G'$ has a VC of size $k+1$
- degree of K is reduced from 4 to 3
- generalizes to higher degrees $\rightarrow$ shows that VERTEX COVER is NP-hard for max-deg-3 graphs

Eliminate $>$ Ignore

Vertices with low degree

- so far: ignore vertex v if $\deg(v) = 1$ or $\deg(v) = 2$
- now: get completely rid of such vertices

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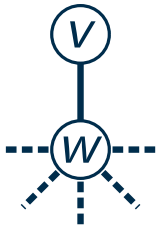

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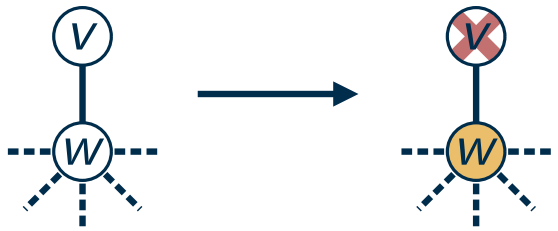

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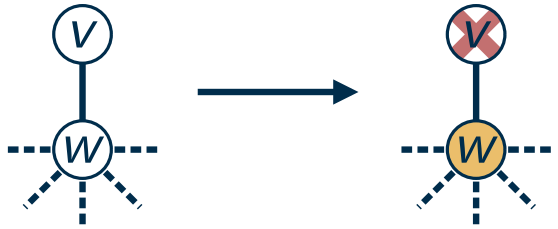

Does this count as branching?

- this is actually a reduction rule (or a branching rule with vector (1))
- no negative impact on the base in the running time

Degree-2 vertices

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,3)$	$(3,3)$	$(3,3,6)$	$(3,4,6)$
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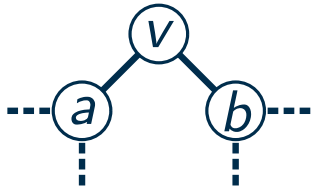

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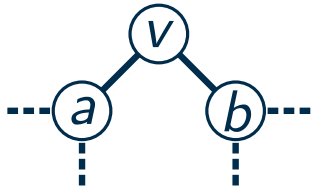

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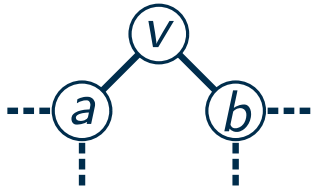

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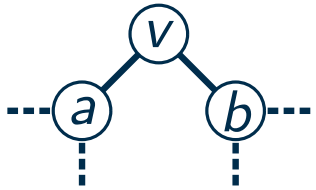
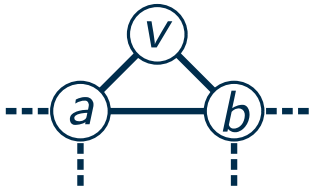


Case 1: $ab \in E$



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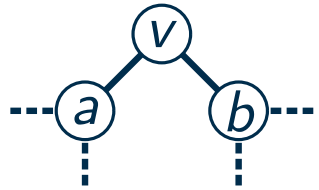

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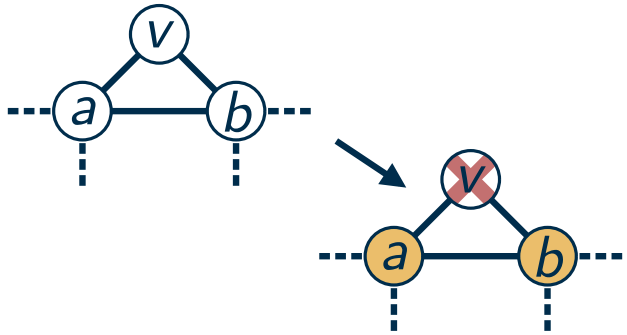

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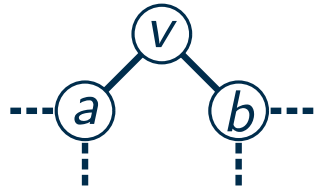

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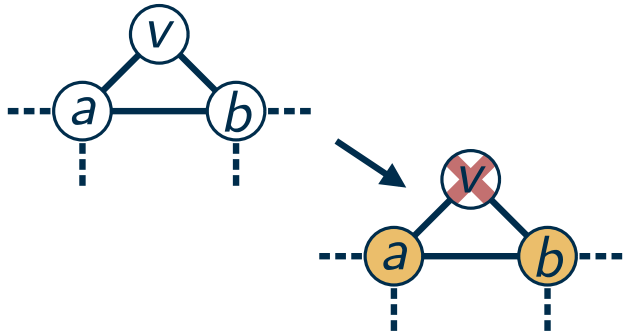

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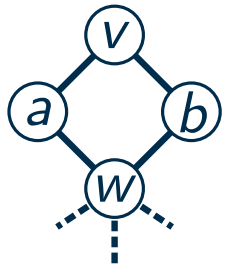

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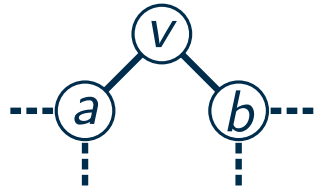

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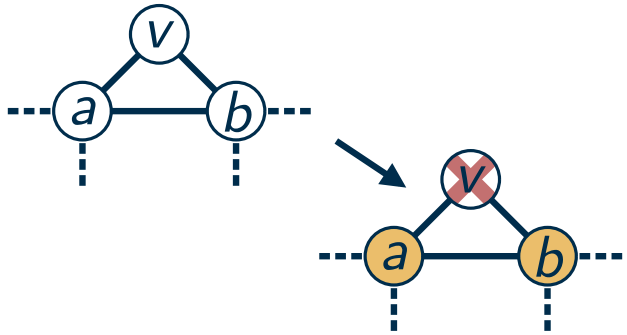

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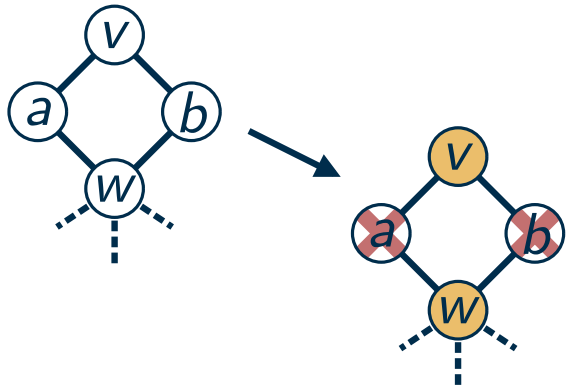

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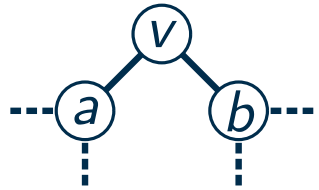

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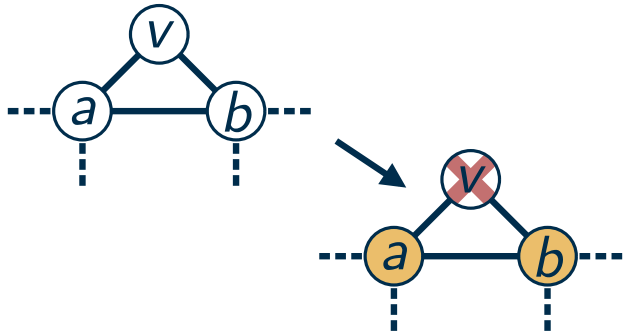

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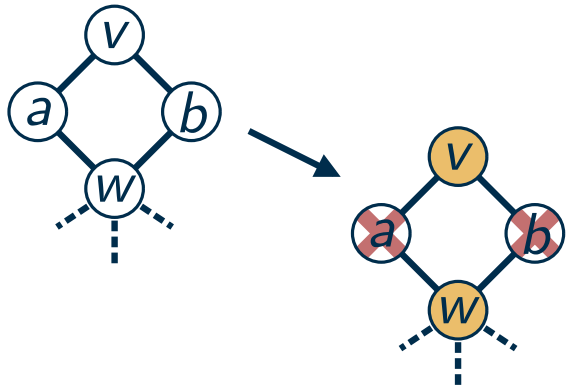
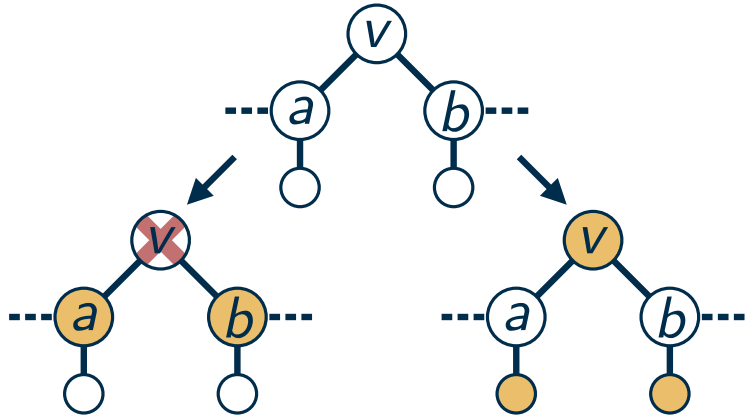


Case 3: $|N(a) \cup N(b)| \geq 3$

- branch with vector $(2, 3)$



Useful case-distinction

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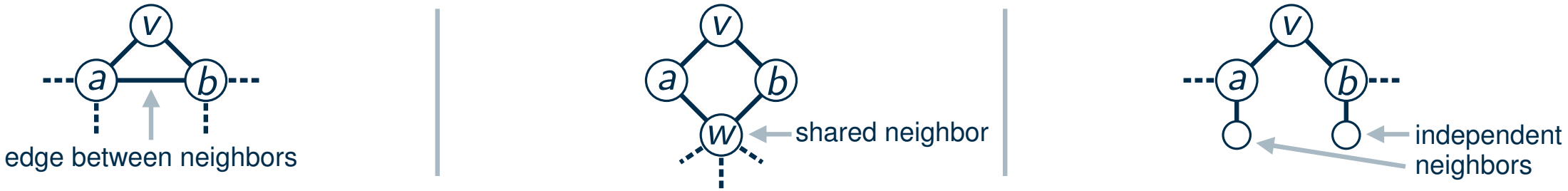
Case distinction for degree-2 vertices

- cases in a sense capture, how independent the neighborhood of v is



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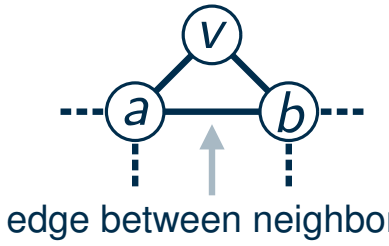

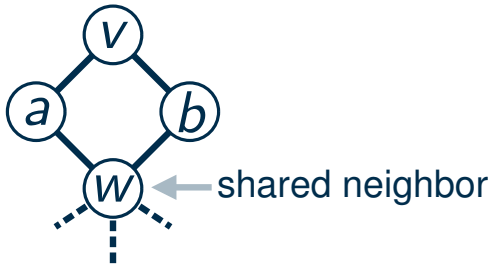

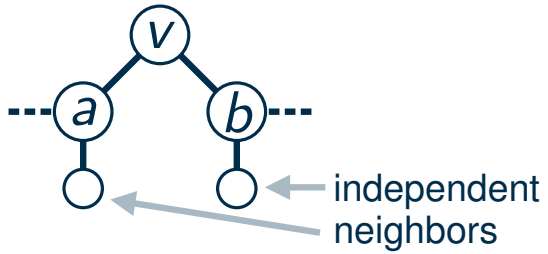

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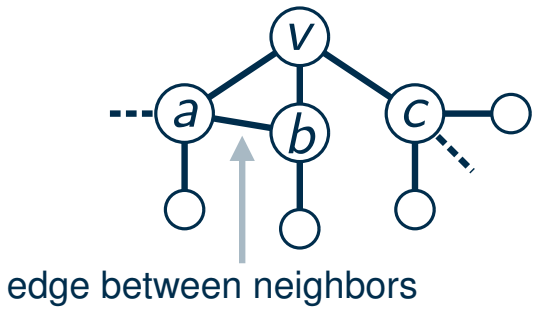
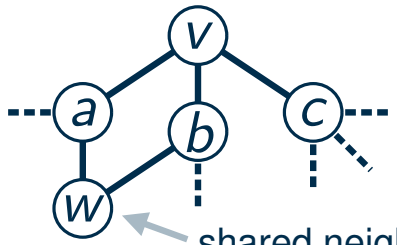
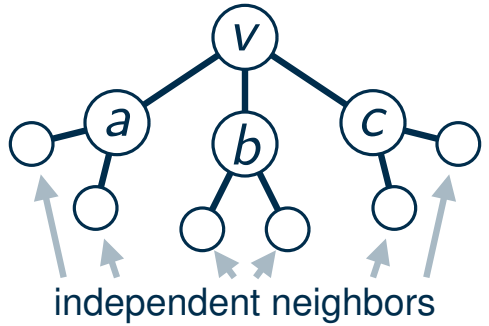
- dependent neighborhood $\Rightarrow$ more structure $\Rightarrow$ can be argued away

Goal for degree-3 vertices: similar case distinction but use that $\deg(u) \geq 3$ for all $u \in V$



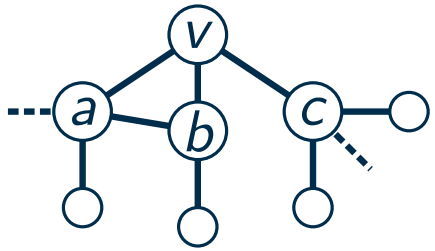




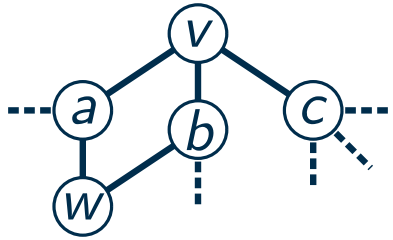
Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
2	1.618	1.466	1.381	1.325	1.325	1.26	1.342	1.305

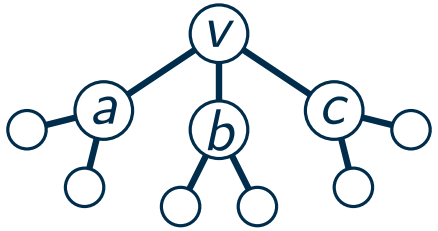
Case 1: $ab \in E$



Case 2: $w \in N(a) \cup N(b)$



Case 3: no edge among a, b, c and v is only common neighbor



Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
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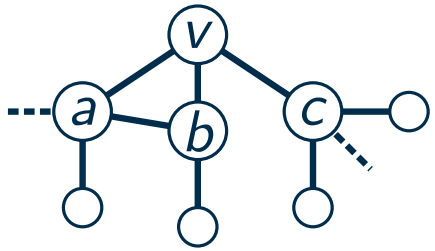
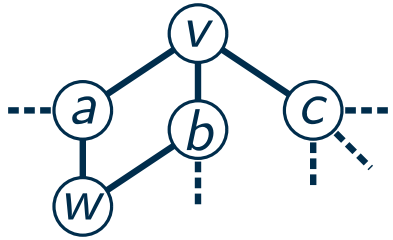
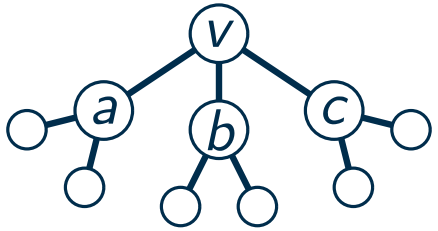


Case 2: $w \in N(a) \cup N(b)$



Case 3: no edge among a, b, c and v is only common neighbor

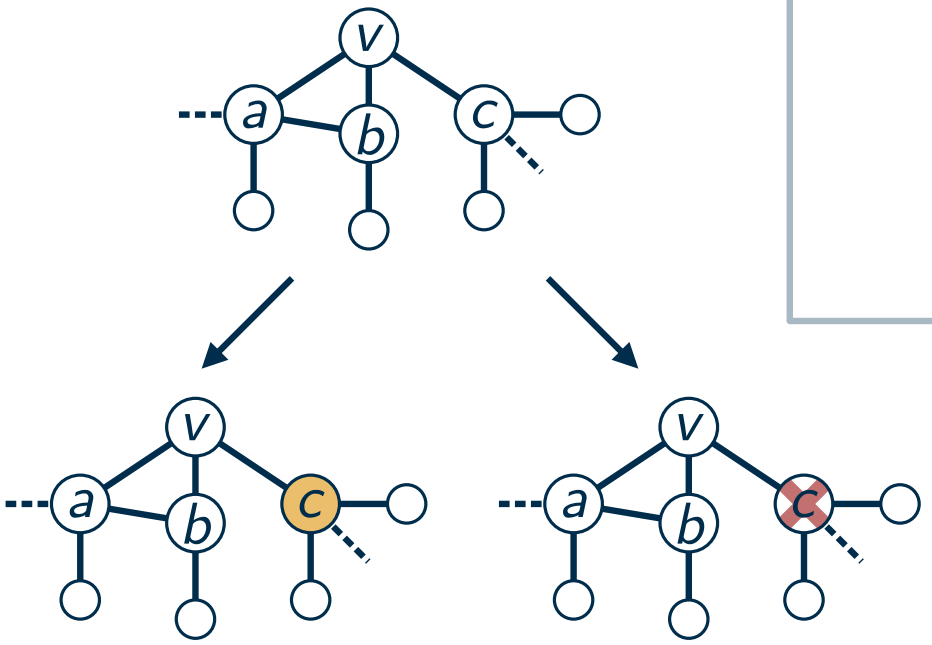


Does this cover all cases?

Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
2	1.618	1.466	1.381	1.325	1.325	1.26	1.342	1.305

Case 1: $ab \in E$



Case 2: $w \in N(a) \cup N(b)$

Case 3: no edge among a, b, c and v is only common neighbor

Branching for case 1

- select or exclude c

Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
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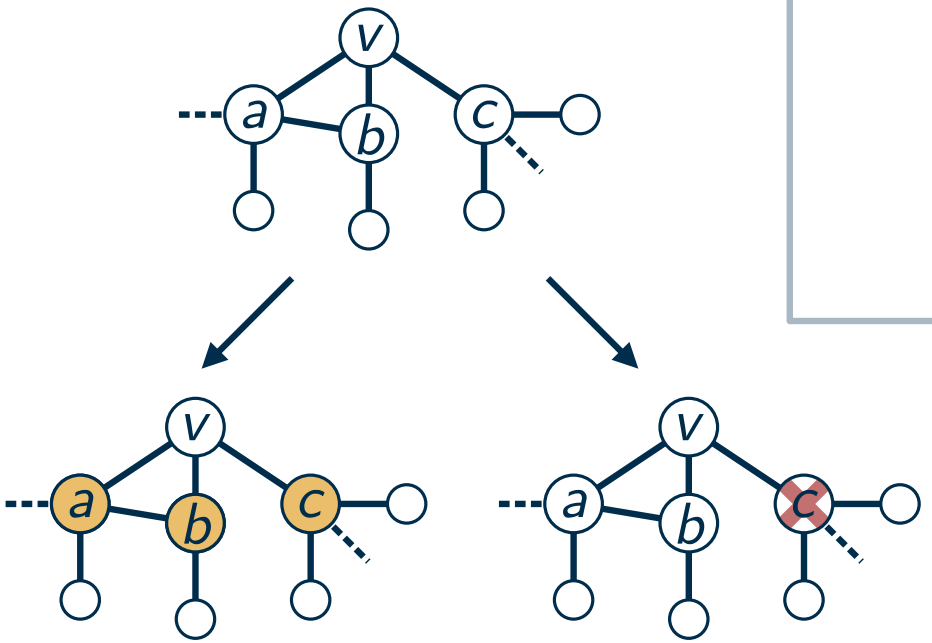

Case 2: $w \in N(a) \cup N(b)$

Case 3: no edge among a, b, c and v is only common neighbor

Branching for case 1

- select or exclude c
- selecting c yields degree-2 situation $\Rightarrow$ select a and b

Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
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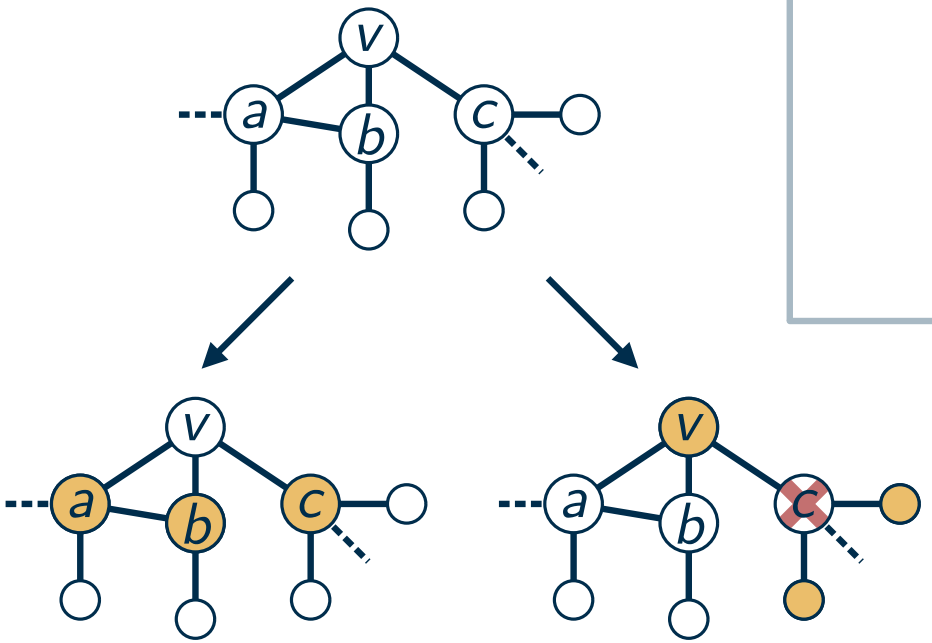

Case 2: $w \in N(a) \cup N(b)$

Case 3: no edge among a, b, c and v is only common neighbor

Branching for case 1

- select or exclude c
- selecting c yields degree-2 situation $\Rightarrow$ select a and b
- exclude $c \Rightarrow$ select $N(c)$

Degree-3 vertices

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,3)$	$(3,3)$	$(3,3,6)$	$(3,4,6)$
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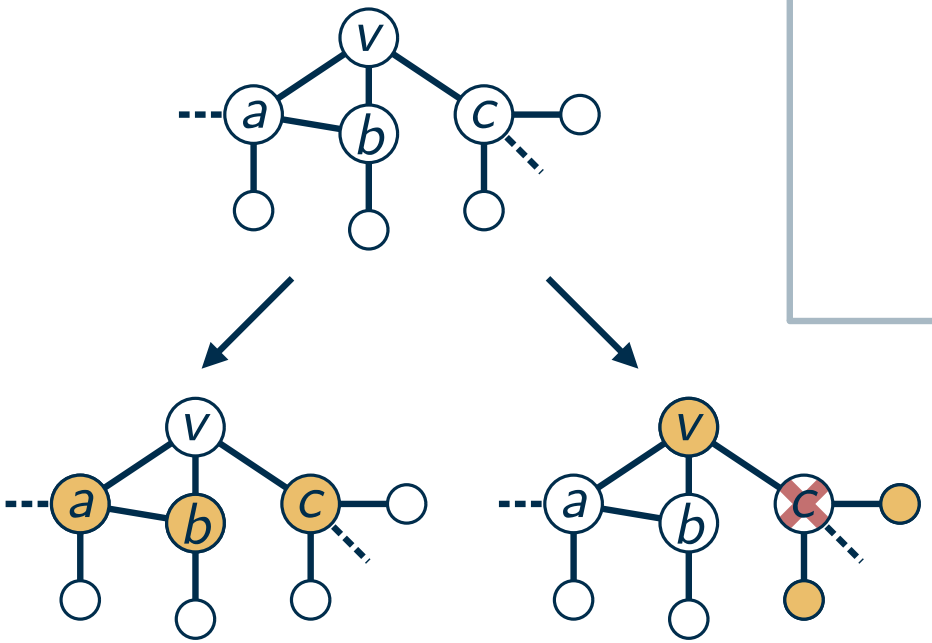

Case 2: $w \in N(a) \cup N(b)$

Case 3: no edge among a, b, c and v is only common neighbor

Branching for case 1

- select or exclude c
- selecting c yields degree-2 situation $\Rightarrow$ select a and b
- exclude $c \Rightarrow$ select $N(c)$ (note: $\deg(c) \geq 3$)
- branching vector: $(3, 3)$

Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
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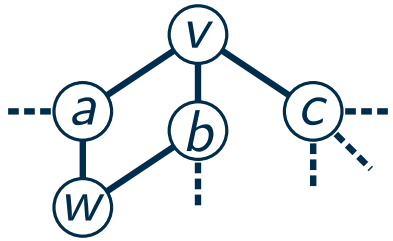
Case 1: $ab \in E$

Case 2: $w \in N(a) \cup N(b)$



Case 3: no edge among a, b, c and v is only common neighbor

Branching for case 2

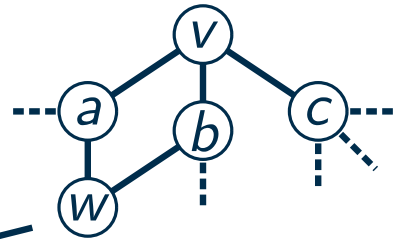
- 4-cycle $\rightarrow$ must select at least two opposite corners

Degree-3 vertices

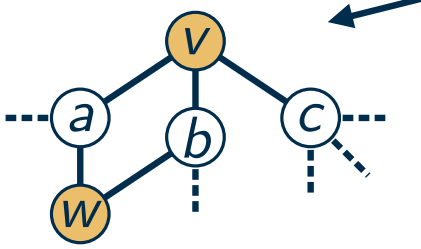
(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
2	1.618	1.466	1.381	1.325	1.325	1.26	1.342	1.305

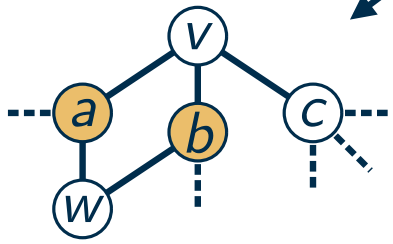
Case 1: $ab \in E$

Case 2: $w \in N(a) \cup N(b)$



Case 3: no edge among a, b, c and v is only common neighbor





Branching for case 2

- 4-cycle $\rightarrow$ must select at least two opposite corners
- branching: select $\{v, w\}$ or $\{a, b\}$

Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
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Case 1: $ab \in E$

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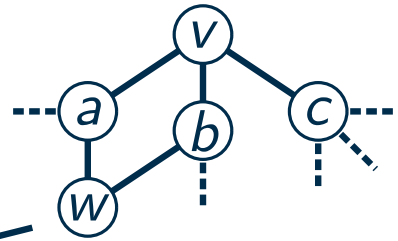
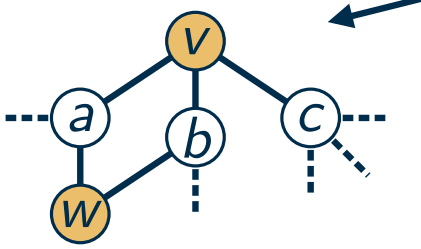
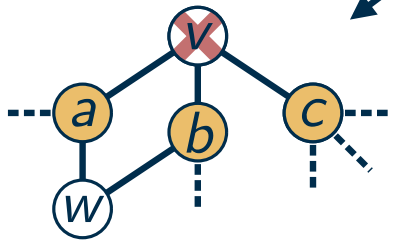


Case 3: no edge among a, b, c and v is only common neighbor





Branching for case 2

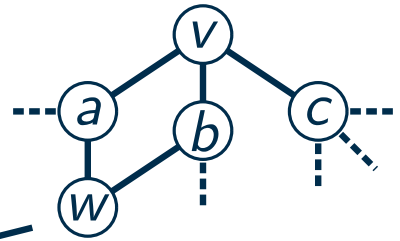
- 4-cycle $\rightarrow$ must select at least two opposite corners
- branching: select $\{v, w\}$ or $\{a, b\}$
- selecting $\{a, b\}$ yields degree-1 situation
  $\Rightarrow$ select c and exclude v

Degree-3 vertices

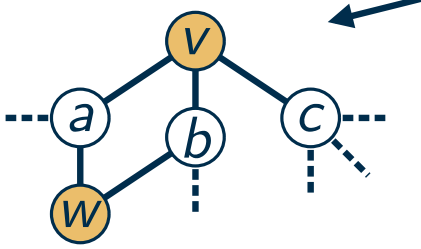
(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
2	1.618	1.466	1.381	1.325	1.325	1.26	1.342	1.305

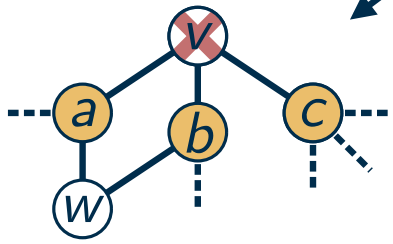
Case 1: $ab \in E$

Case 2: $w \in N(a) \cup N(b)$



Case 3: no edge among a, b, c and v is only common neighbor





Branching for case 2

- 4-cycle $\rightarrow$ must select at least two opposite corners
- branching: select $\{v, w\}$ or $\{a, b\}$
- selecting $\{a, b\}$ yields degree-1 situation
  $\Rightarrow$ select c and exclude v
- branching vector: $(2, 3)$

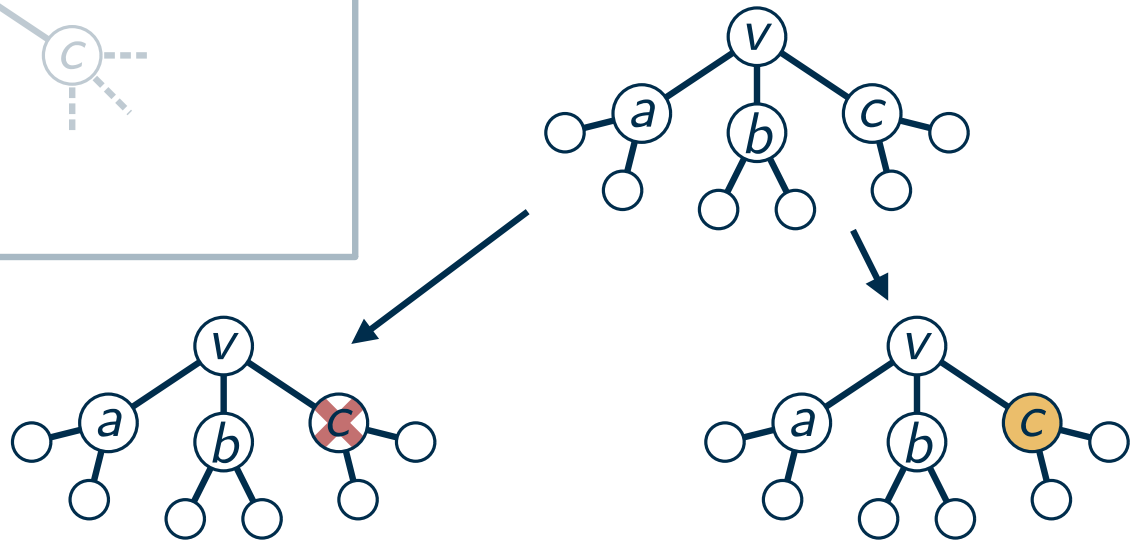
Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
2	1.618	1.466	1.381	1.325	1.325	1.26	1.342	1.305

Case 1: $ab \in E$

Case 2: $w \in N(a) \cup N(b)$

Case 3: no edge among a, b, c and v is only common neighbor



Branching for case 3

- select or exclude c

Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
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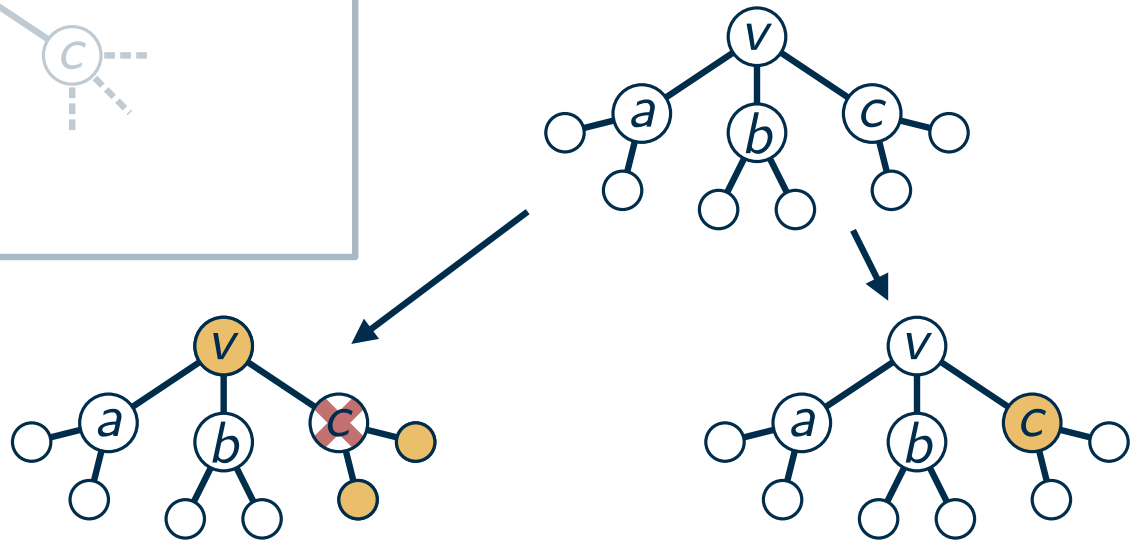
Case 1: $ab \in E$

Case 2: $w \in N(a) \cup N(b)$

Case 3: no edge among a, b, c and v is only common neighbor



Branching for case 3

- select or exclude c
- exclude $c \Rightarrow$ select $N(c)$

Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
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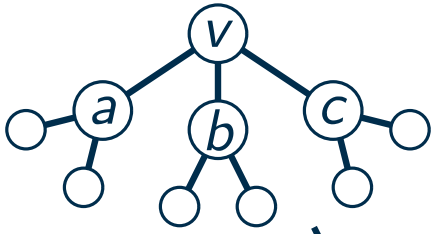

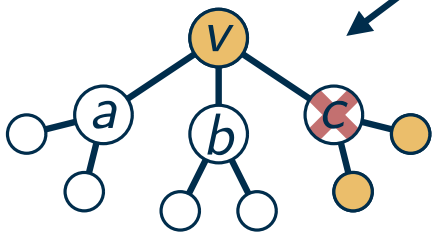

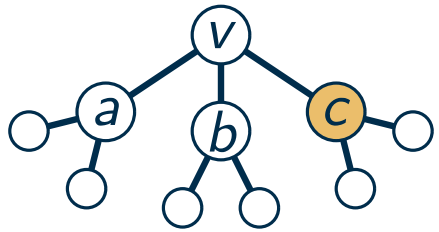

Branching for case 3

- select or exclude c
- exclude $c \Rightarrow$ select $N(c)$
- selecting c yields degree-2 situation:

Degree-3 vertices

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(2, 3)	(3, 3)	(3, 3, 6)	(3, 4, 6)
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Case 1: $ab \in E$

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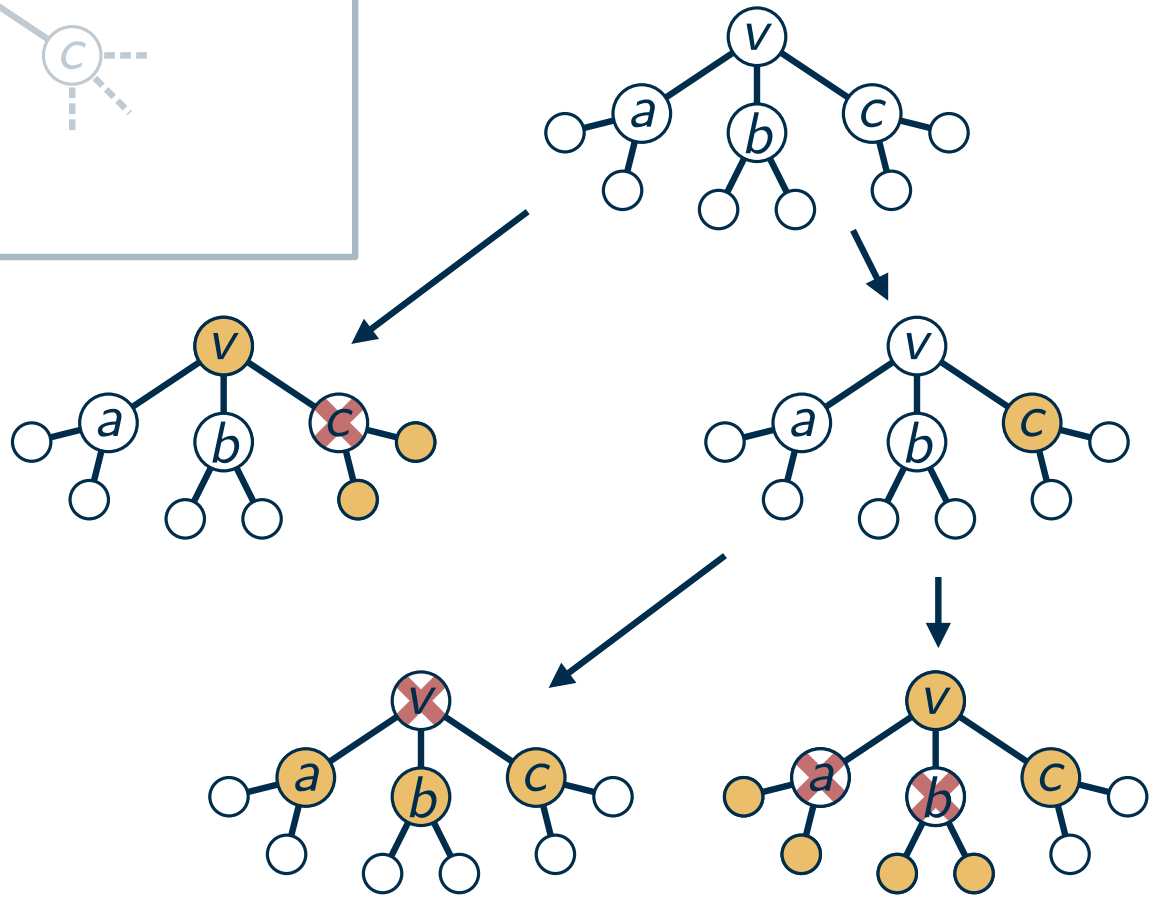

Branching for case 3

- select or exclude c
- exclude $c \Rightarrow$ select $N(c)$
- selecting c yields degree-2 situation:
  - select a and b
  - exclude a and b

Degree-3 vertices

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,3)$	$(3,3)$	$(3,3,6)$	$(3,4,6)$
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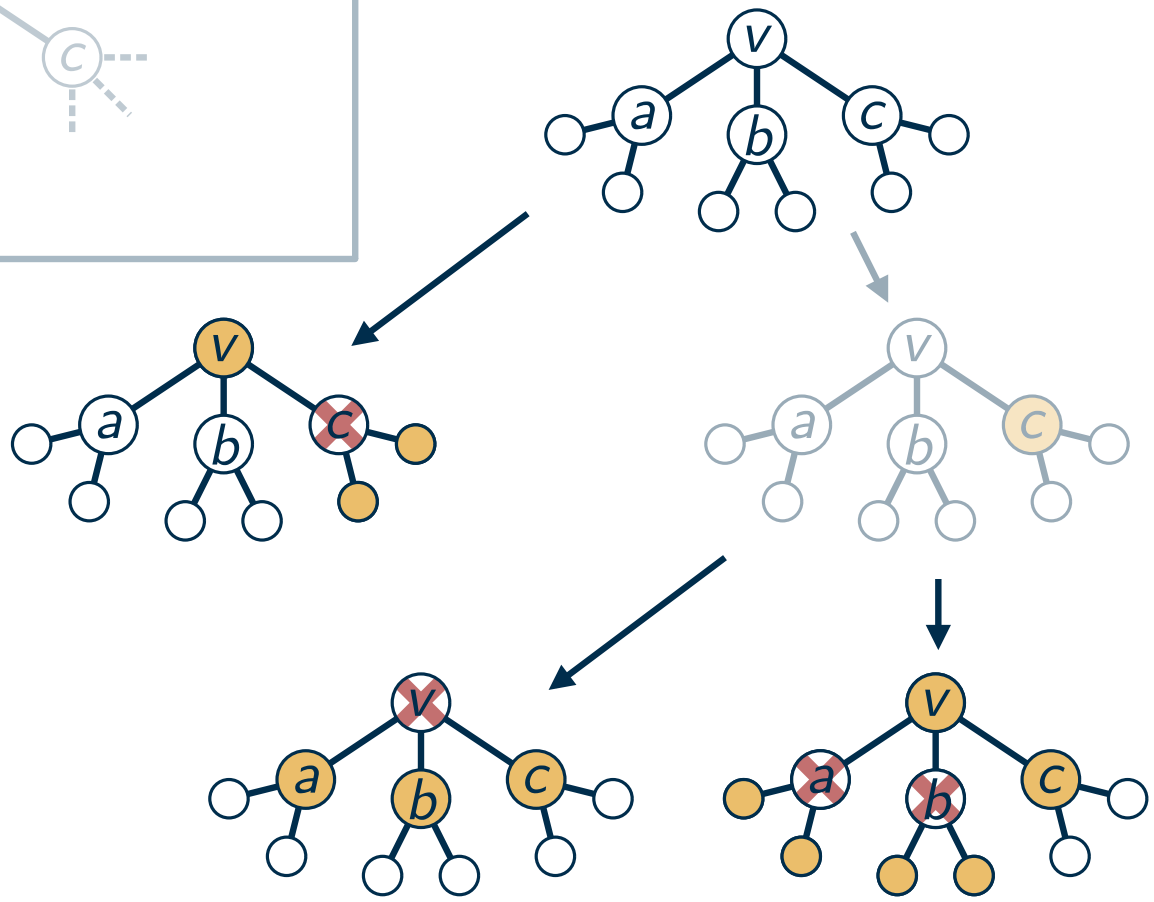


Branching for case 3

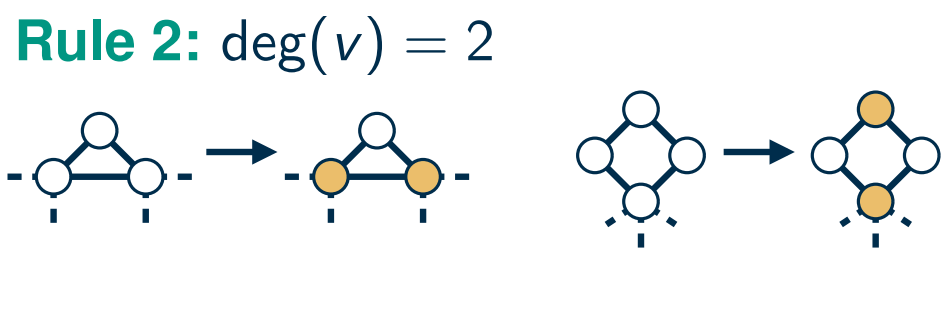
- select or exclude c
- exclude $c \Rightarrow$ select $N(c)$ (note: $\deg(c) \geq 3$)
- selecting c yields degree-2 situation:
  - select a and b
  - exclude a and b
- branching vector: $(3, 3, 6)$

Summary

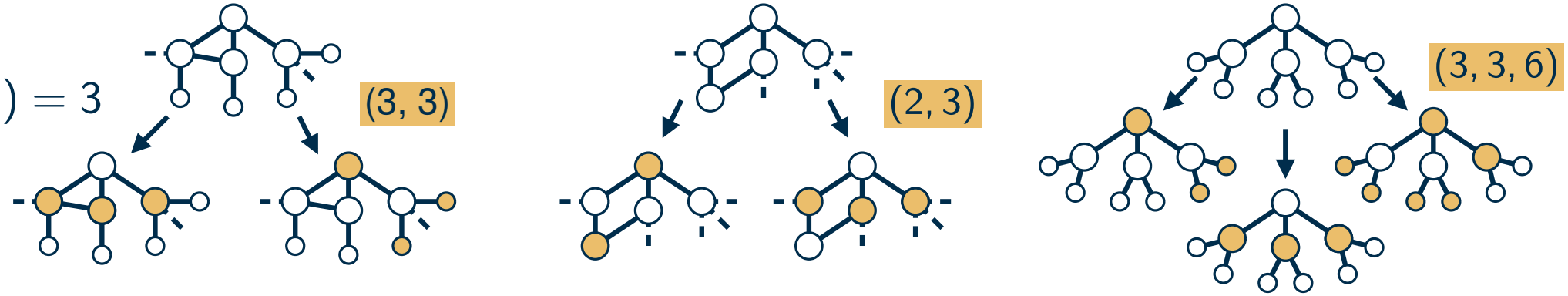
$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,3)$	$(3,3)$	$(3,3,6)$	$(3,4,6)$
2	1.618	1.466	1.381	1.325	1.325	1.26	1.342	1.305

Rule 1: $\deg(v) = 1$

Rule 2: $\deg(v) = 2$



Rule 3: $\deg(v) = 3$



Summary

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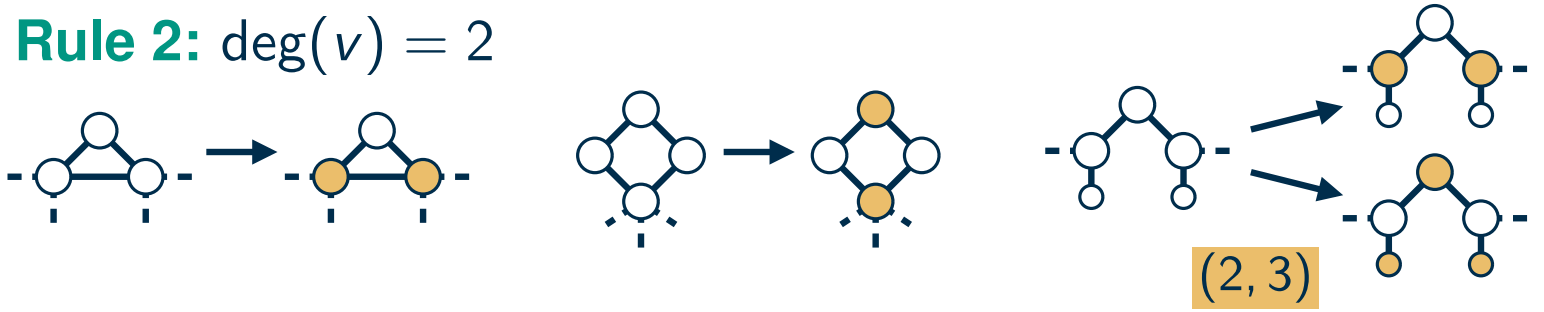
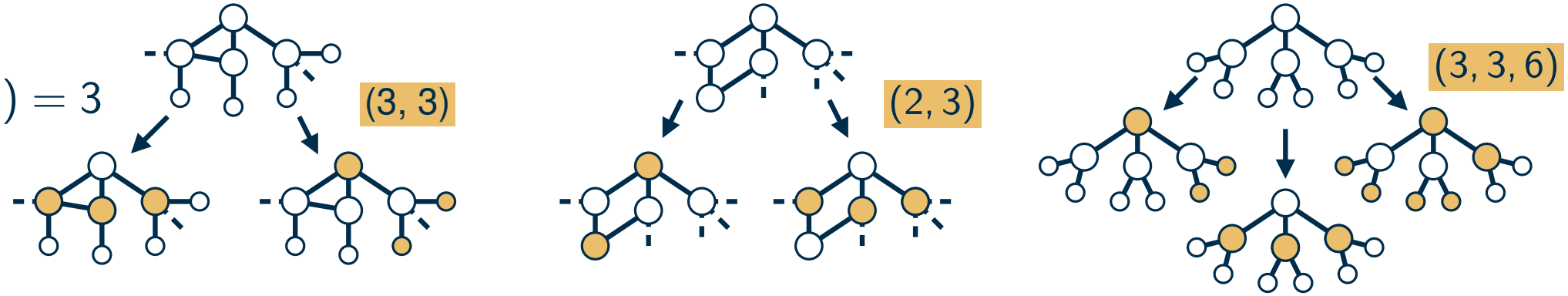
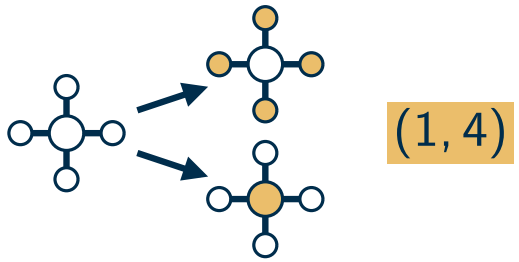


Rule 3: $\deg(v) = 3$



Rule 4: $\deg(v) = 4$



Summary

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,3)$	$(3,3)$	$(3,3,6)$	$(3,4,6)$
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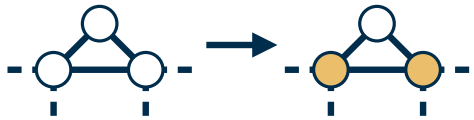
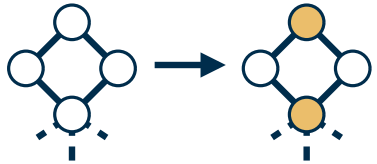
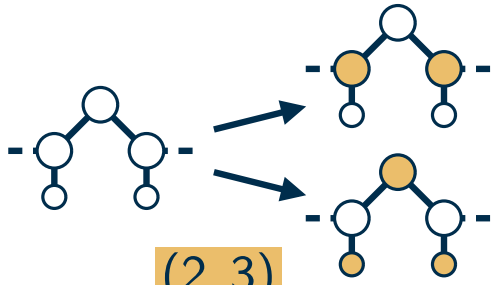
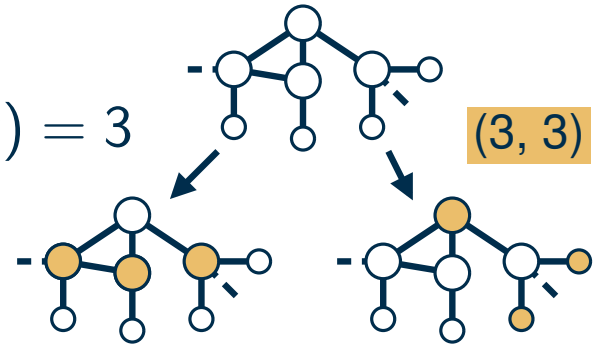




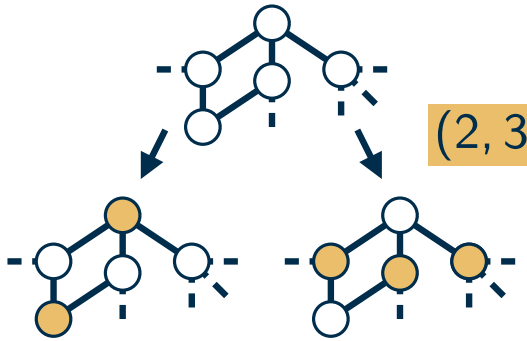


$(2,3)$

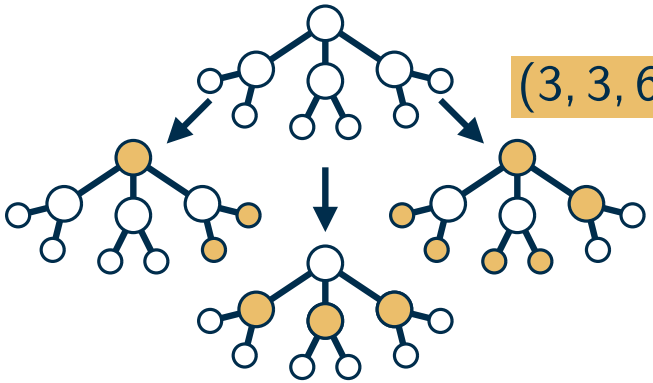
Rule 3: $\deg(v) = 3$



$(3,3)$

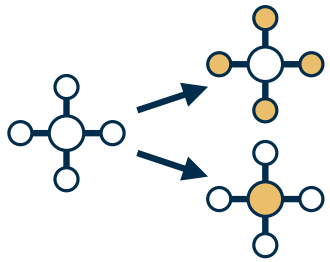


$(2,3)$



$(3,3,6)$

Rule 4: $\deg(v) = 4$



$(1,4)$

worst branching vector: $(1,4)$

$\Rightarrow$ running time: $1.381^k \cdot n^{O(1)}$

Can we do better?

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,3)$	$(3,3)$	$(3,3,6)$	$(3,4,6)$
2	1.618	1.466	1.381	1.325	1.325	1.26	1.342	1.305

Rule 1: there is $v \in V$ with $\deg(v) = 1$

Rule 2: there is $v \in V$ with $\deg(v) = 2$ $(2,3)$

Rule 3: there is $v \in V$ with $\deg(v) = 3$ $(3,3,6)$ $(3,3)$ $(2,3)$

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Rule 4: there is $v \in V$ with $\deg(v) \geq 5$ $(1, \deg(v))$

Rule 5: all vertices have degree 4 $(1,4)$

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Claim: the total running time is $1.342^k \cdot n^{O(1)}$

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Claim: the total running time is $1.342^k \cdot n^{O(1)}$

Intuitive argument

- rule 5 happens at most once on every path from the root to a leaf
- costs only a constant factor
- we can ignore the branching vector $(1,4)$

Wrap-Up & Literature

Theorem
VERTEX COVER parameterized by solution size k can be solved in $1.342^k \cdot n^{O(1)}$ time.

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Improving the base

- best since 2010: $O(1.2738^k + kn)$
  `doi.org/10.1016/j.tcs.2010.06.026`

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- also see:
  `http://fpt.wikidot.com/fpt-races`

Problem	f(k)	vertices in kernel	Reference/Comments
Vertex Cover	1.2529^k	$2k$	42
Connected Vertex Cover	2^k	no $k^{O(1)}$	26, randomized algorithm
Multiway Cut	2^k	not known	21, 38: $O(k^{s+1})$-vertex kernel with s terminals
Directed Multiway Cut	$2^{O(k^3)}$	no $k^{O(1)}$	34
Almost-2-SAT (VC-PM)	2.3146^k	$O(k^6)$	37, 38, randomized kernel
Multicut	$2^{O(k^3)}$	no $k^{O(1)}$	22, 35