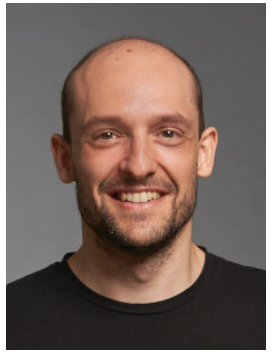
Parameterized Algorithms

FPT-Basics

Thomas Bläsius

Before we start



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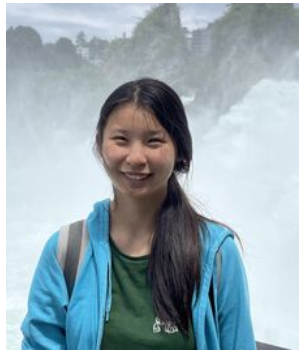

Wendy

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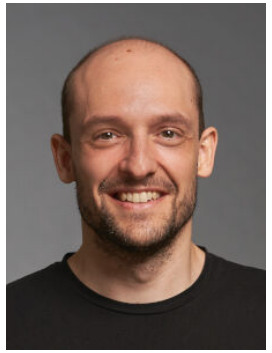

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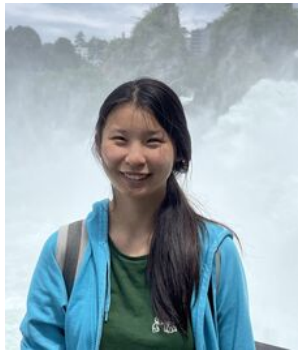

Wendy

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You

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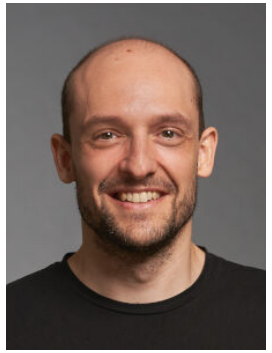
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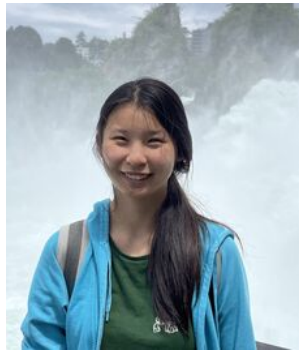
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Materials & infos

- slides, exercise sheets on our homepage: `https://scale.iti.kit.edu/teaching/2026ss/param_algo/start`

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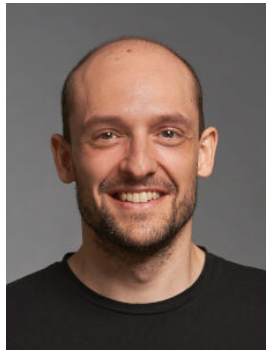
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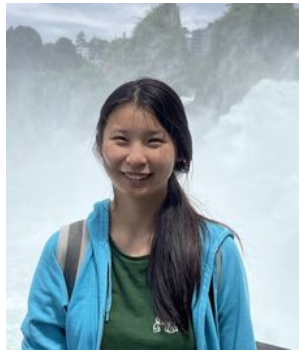
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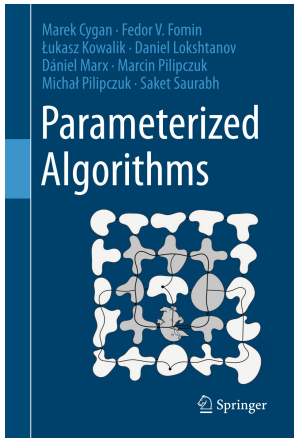

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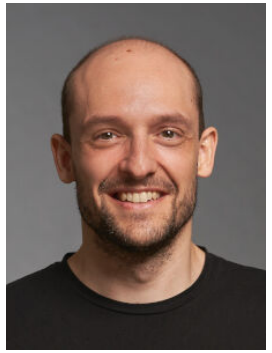
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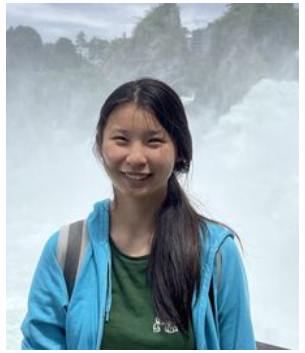
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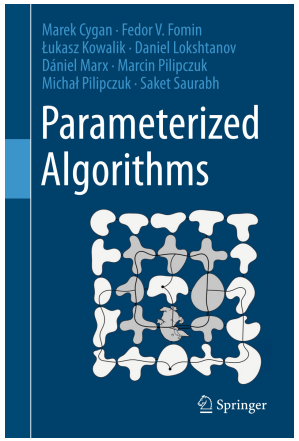

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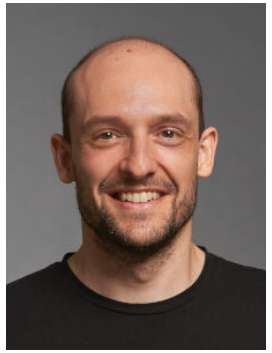
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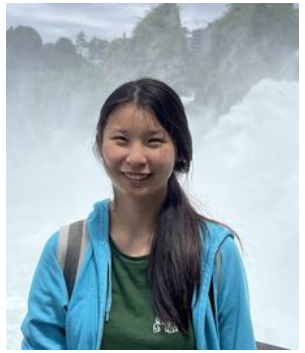
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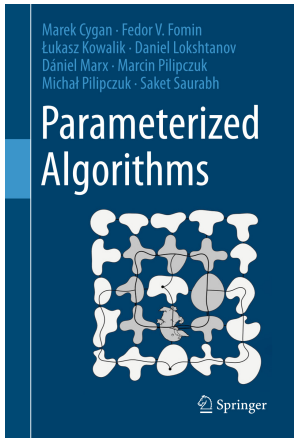
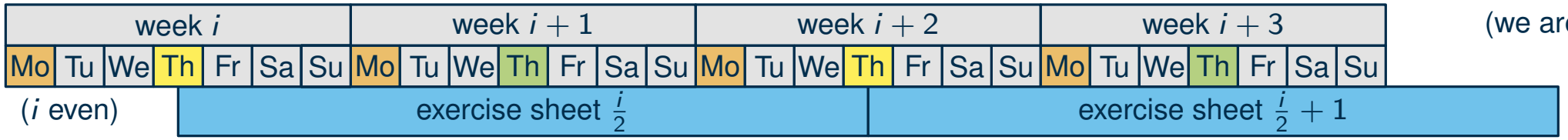


Requirements

- good algorithmic understanding
- no (little) prior knowledge

Rough Schedule



(we are in week 1 right now)

Lecture

- lecture with slides
- new topics

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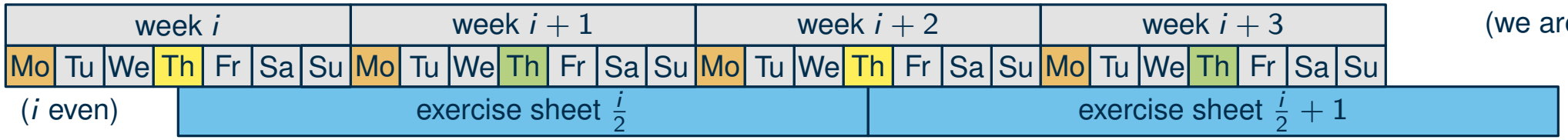

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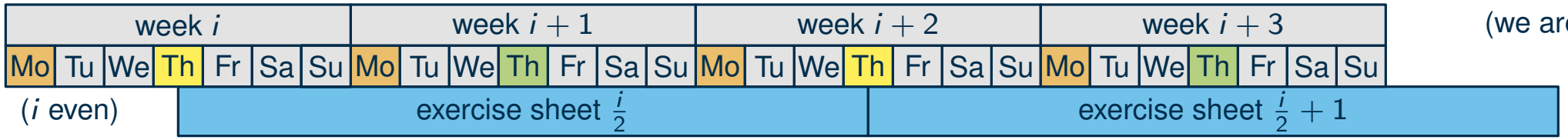

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- with Wendy, Jean-Pierre, and Elly
- recap
- support solving exercise sheets
- additional exercises

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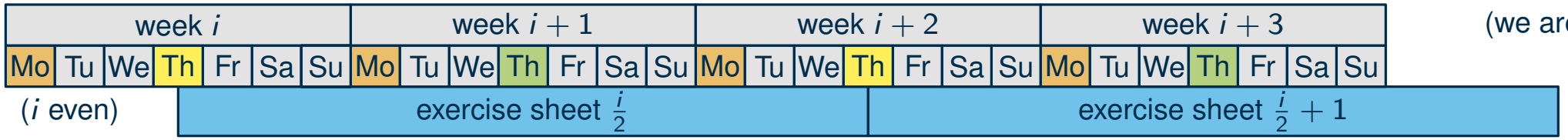

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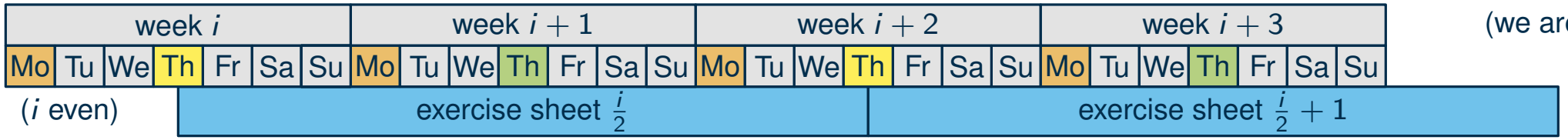

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Exam
- oral exam (20 min)
- admission only with exercise certificate

Exercise certificate

hand in here:
https://ilias.studium.kit.edu/ilias.php?baseClass=ilrepositorygui&ref_id=2902870

Goal: $\frac{1}{2}$ of the points in total **and** $\frac{1}{4}$ on every exercise sheet

Exercise certificate

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What if I don't find the solution?

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we don't want to make your life hard and we also don't bite
we just want you to learn something and have fun doing so

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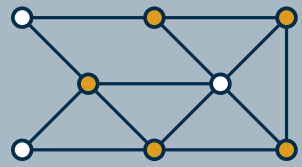
Our goal

- you spend some time with the content of the lecture and write down your solution
- then, the exercise certificate should not be a big obstacle

Parameterized perspective

Problem: k-Vertex Cover
Does the graph $G = (V, E)$ have a vertex cover of size k?



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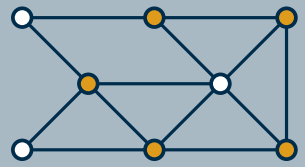

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- k-VERTEX COVER is NP-complete

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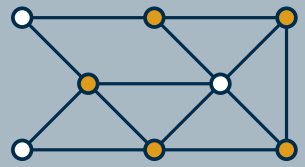

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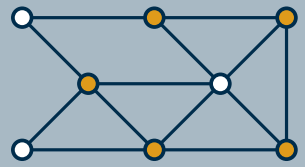

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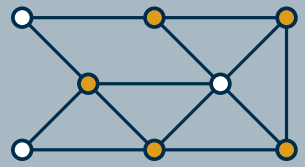

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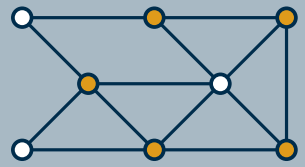

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- What if G has small maximum degree?
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- What if G has small chromatic number?

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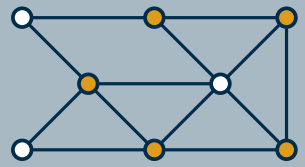

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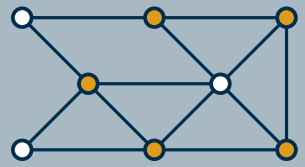

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Goal

- analyze running time not only depending on input size, but also take parameter into account

Fundamental definition: FPT

Definition
A **parameterized problem** is a language $L \subseteq \Sigma^\star \times \mathbb{N}$. An algorithm solves L if it correctly decides $(x, k) \in L$ for every instance $(x, k) \in \Sigma^\star \times \mathbb{N}$. The **parameter** of (x, k) is k.

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Which running times belong to FPT-algorithms?

$A = 2^k(n^4 + k^2)$ $\quad$ $D = k^2 n!$ $\quad$ $G = k! n \log n$

$B = k^{\log n}$ $\quad$ $E = n^{17} k^{k!}$ $\quad$ $H = n^k n^2$

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Observation for FPT algorithms

- k constant $\Rightarrow$ polynomial running time
- k constant $\Rightarrow$ asymptotic running time independent of k

  This second point is important! It does not hold for n^k.

Example: Parameterized Vertex Cover

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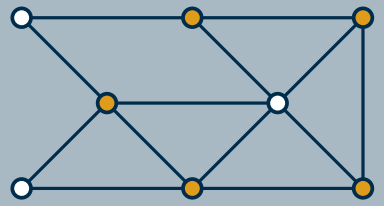

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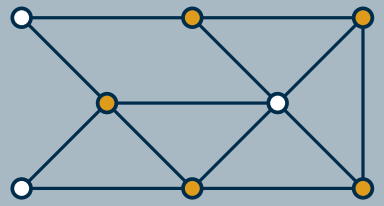

(vertex set $V' \subseteq V$ with $e \cap V' \neq \emptyset$ for all $e \in E$)

Complexity

- Vertex Cover is NP-complete

Example: Parameterized Vertex Cover

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A **parameterized problem** is a language $L \subseteq \Sigma^\star \times \mathbb{N}$. An algorithm solves L if it correctly decides $(x, k) \in L$ for every instance $(x, k) \in \Sigma^\star \times \mathbb{N}$. The **parameter** of (x, k) is k.

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A parameterized problem is **fixed parameter tractable (FPT)** if it can be solved in $O(f(k)n^c)$ time, where f is a computable function, n is the input size, and c is a constant.

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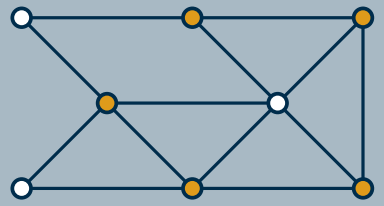

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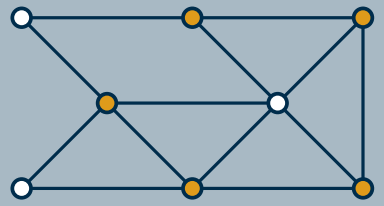

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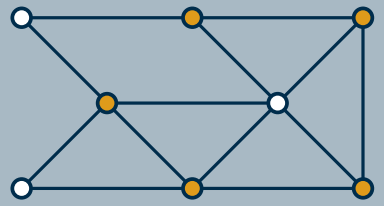

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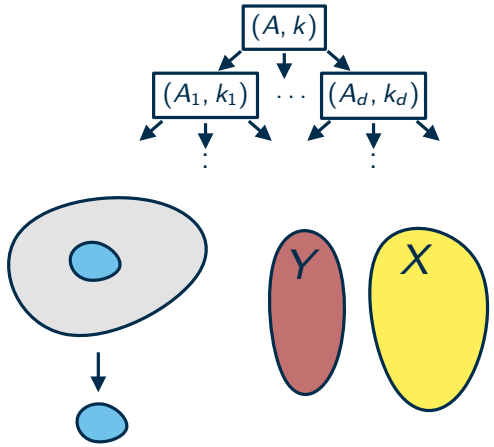
- is Vertex Cover $\in$ FPT?

poly for constant k

Content

Basic toolbox

- bounded search trees
- kernelization
- iterative compression



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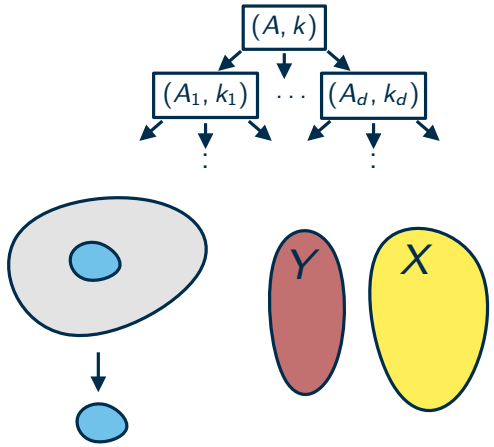
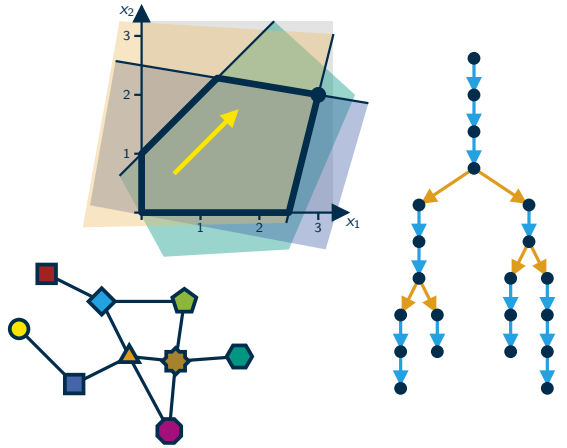


Extended toolbox

- linear programs
- branch-and-reduce
- color coding



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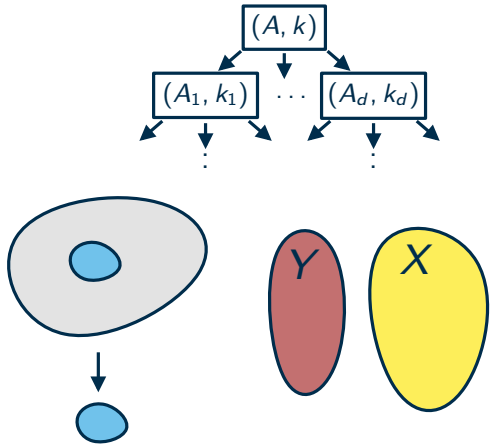

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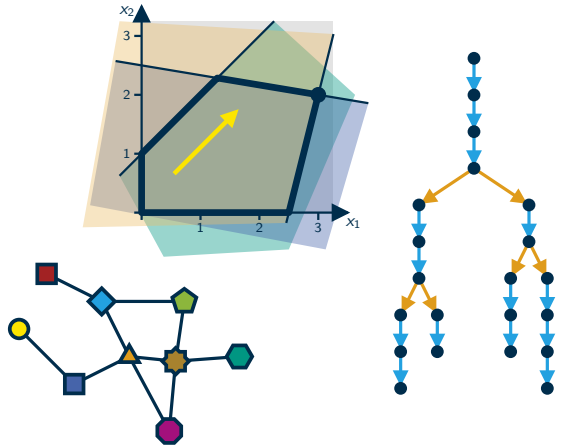

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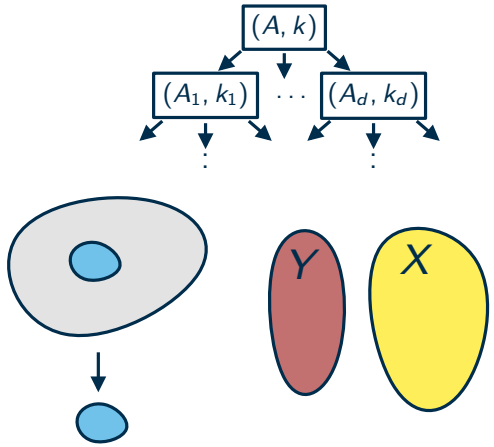

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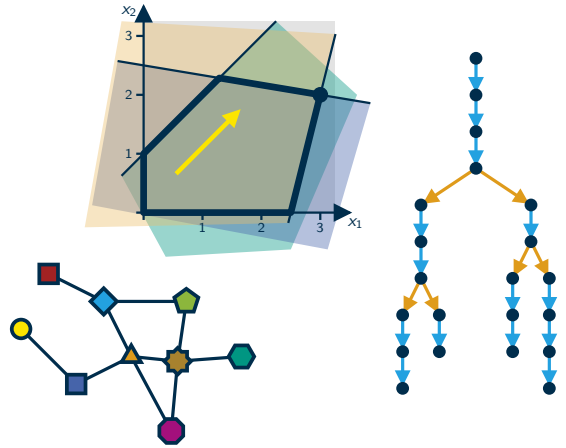

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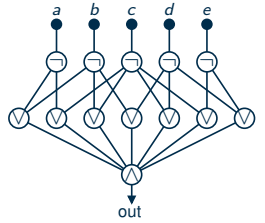

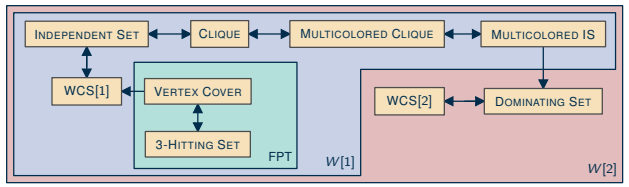

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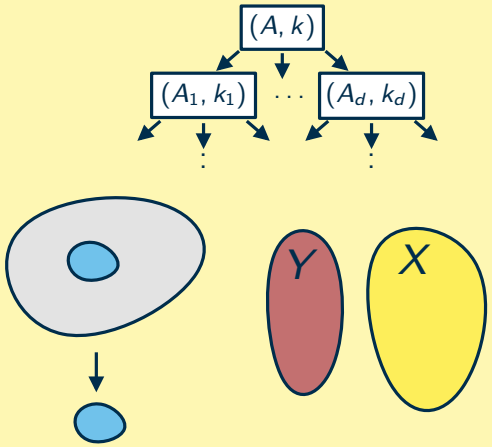

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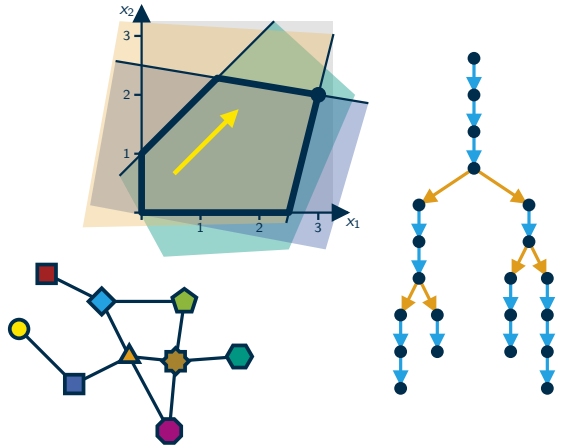

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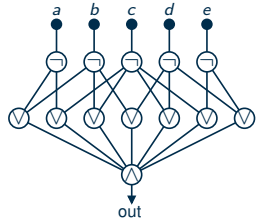

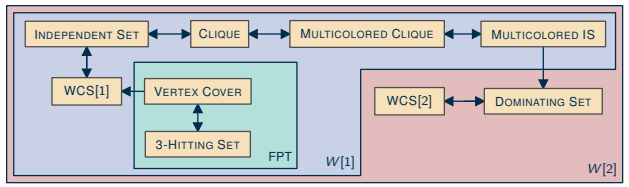

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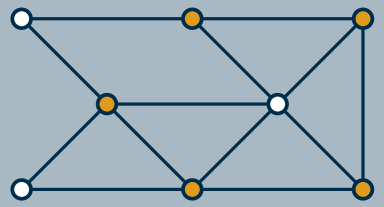

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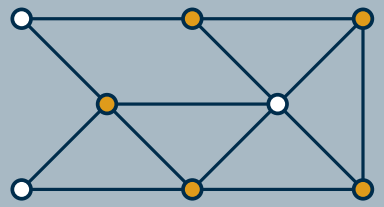

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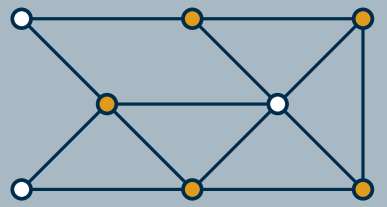

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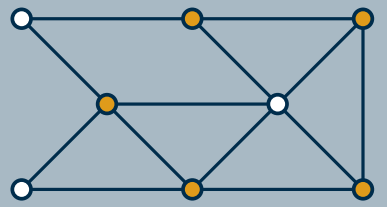

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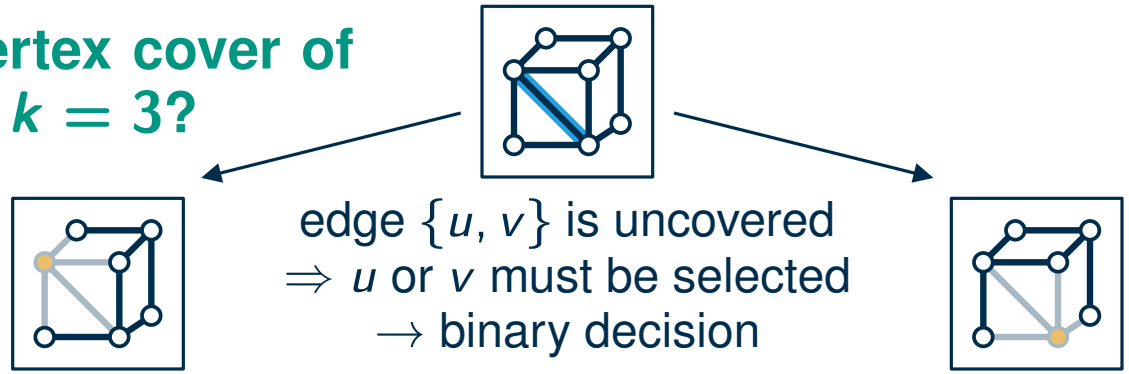

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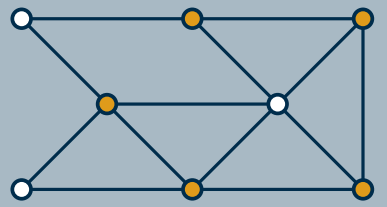

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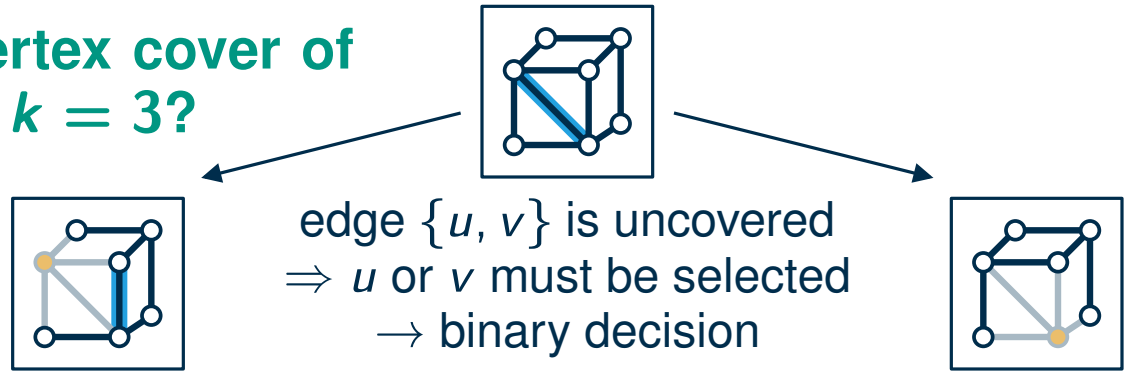

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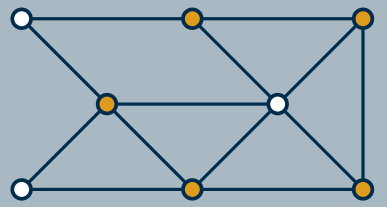

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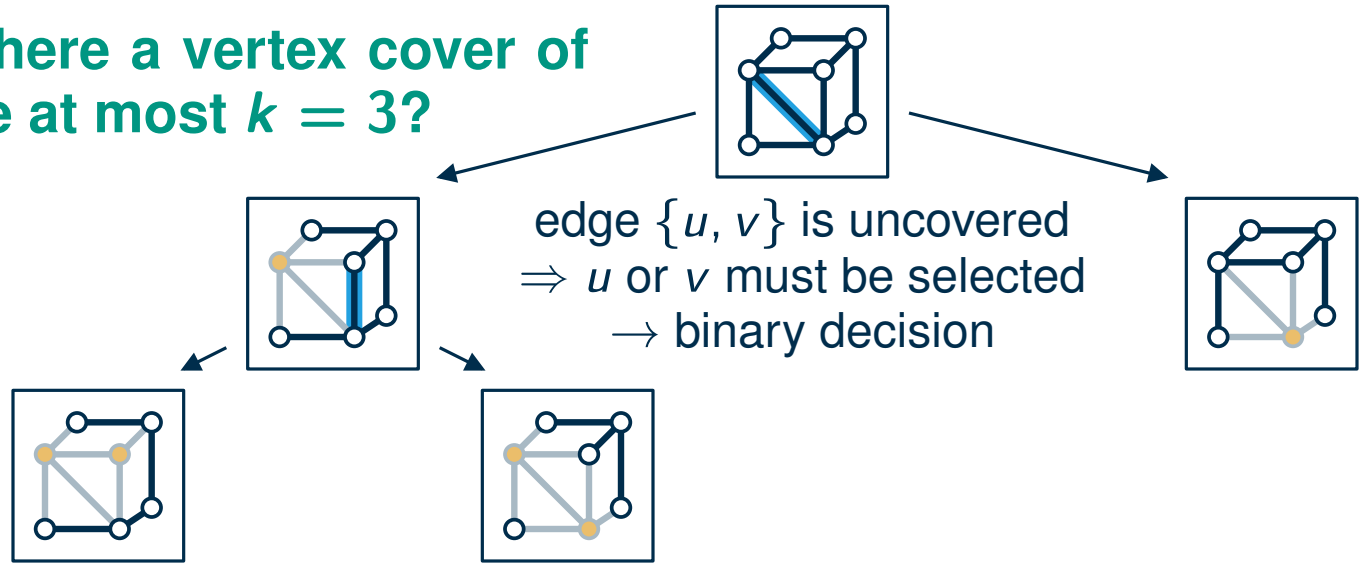

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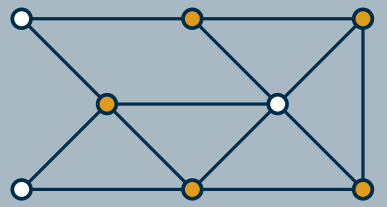

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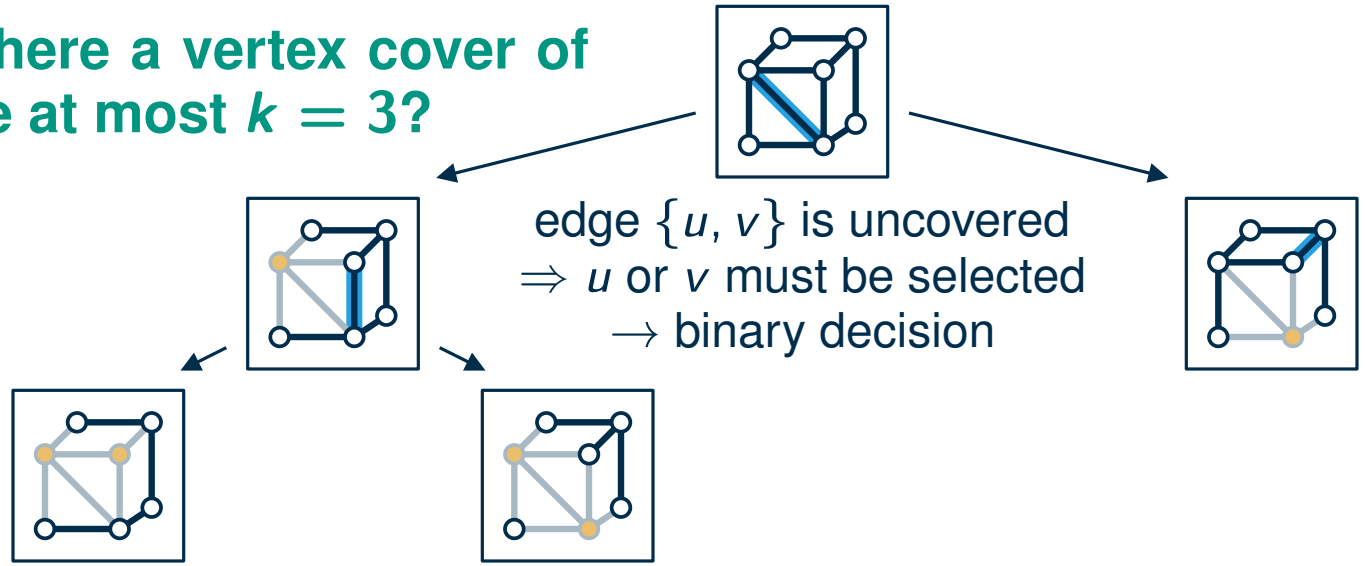

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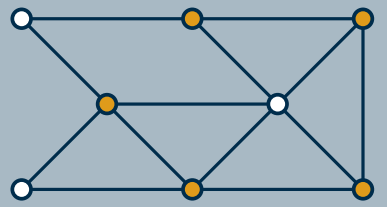

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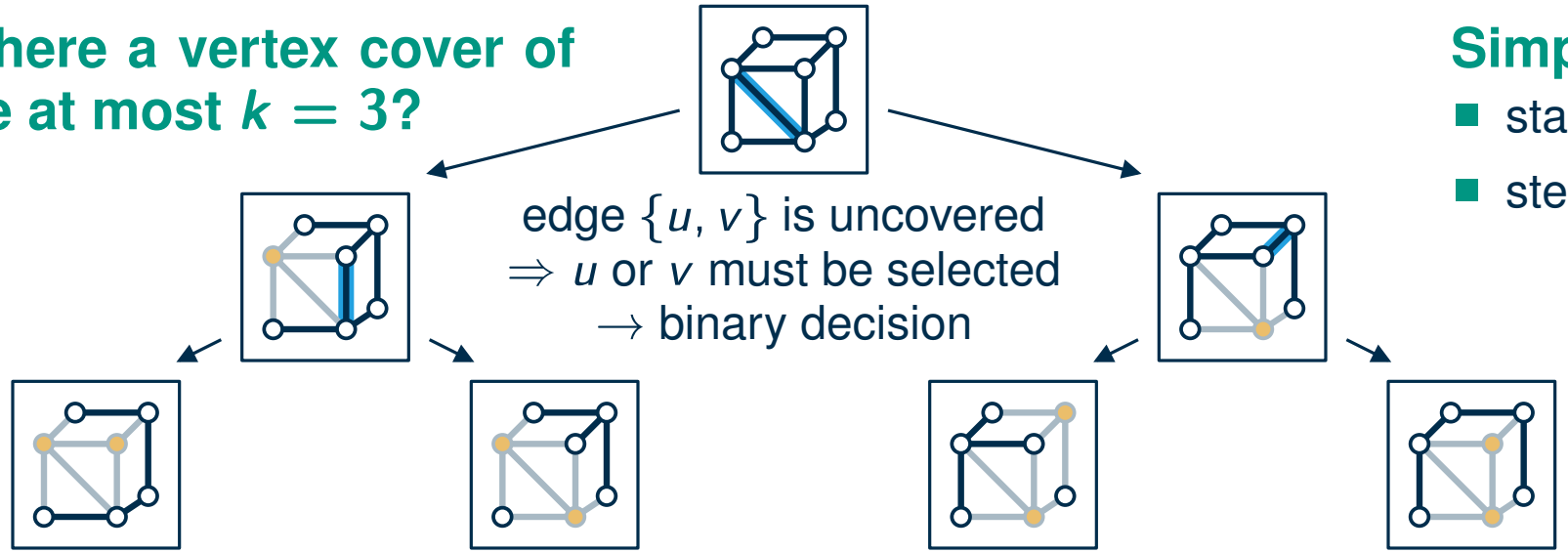

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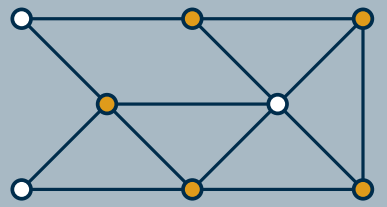

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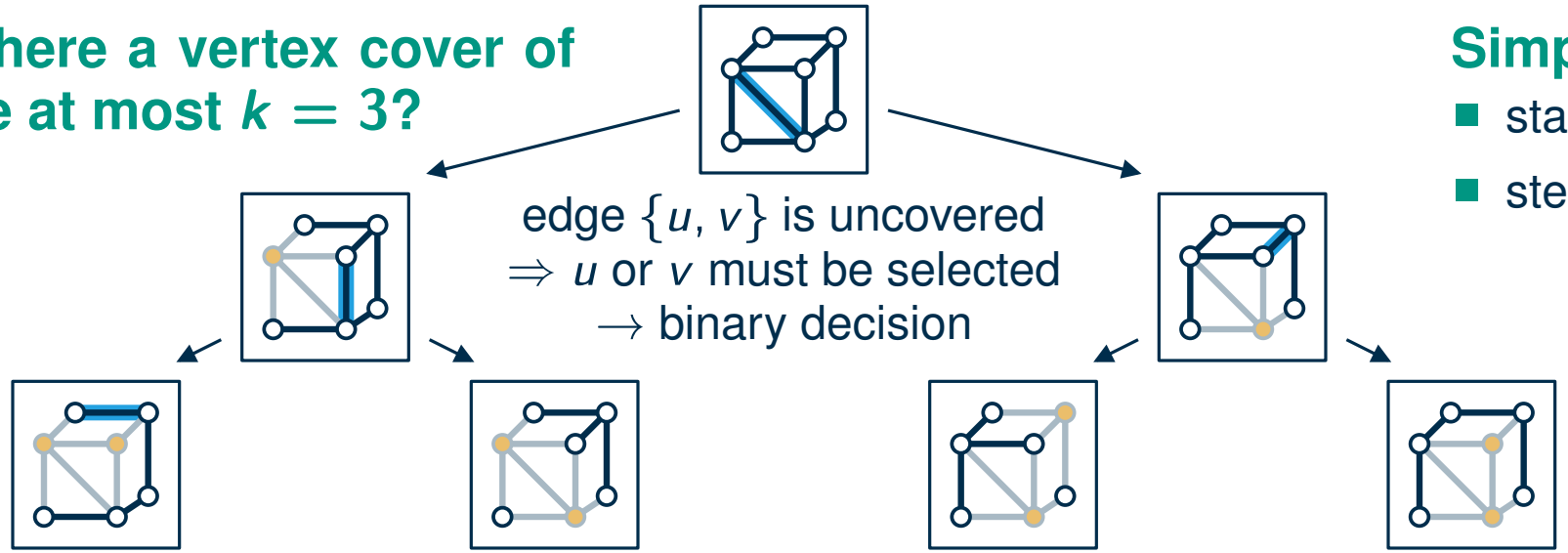

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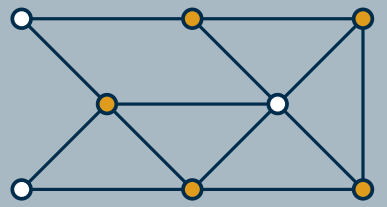

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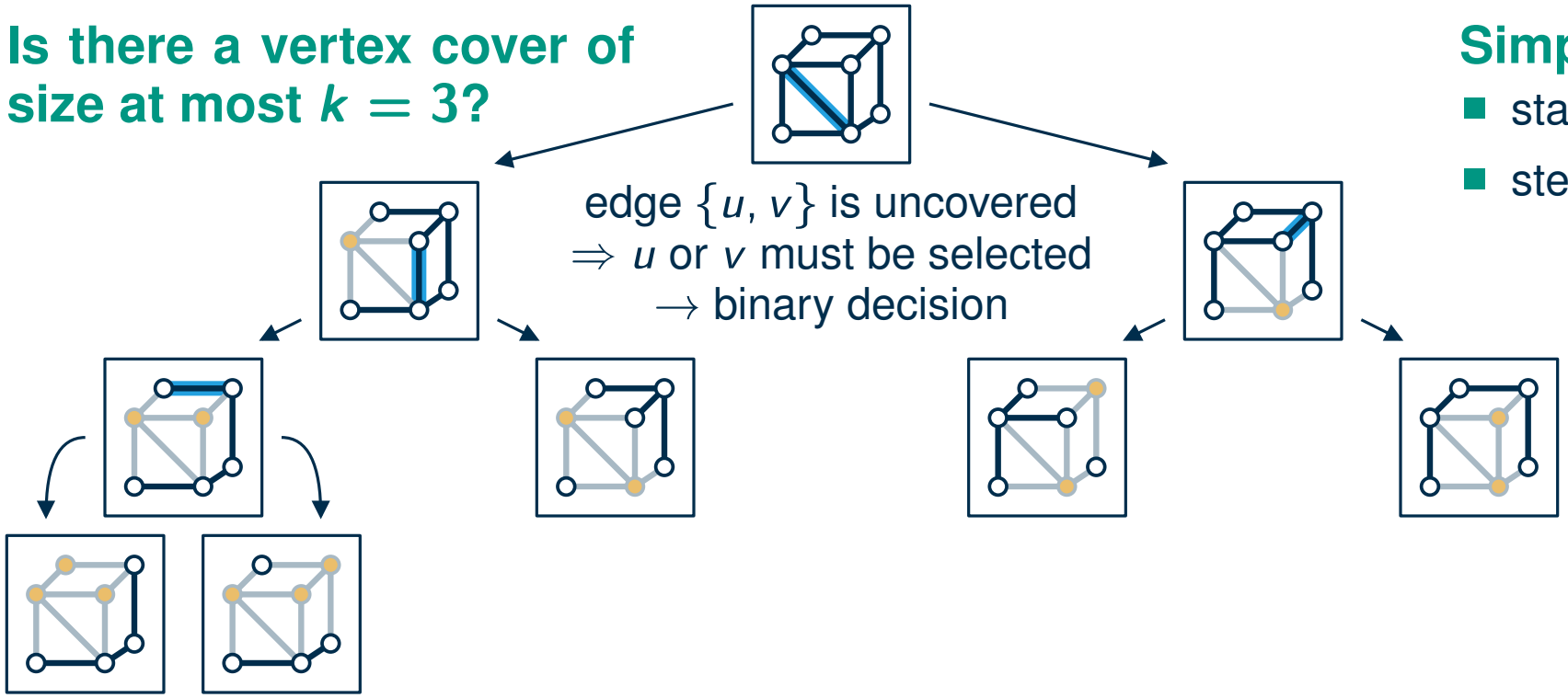

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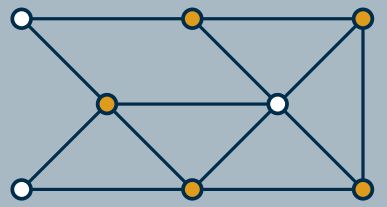

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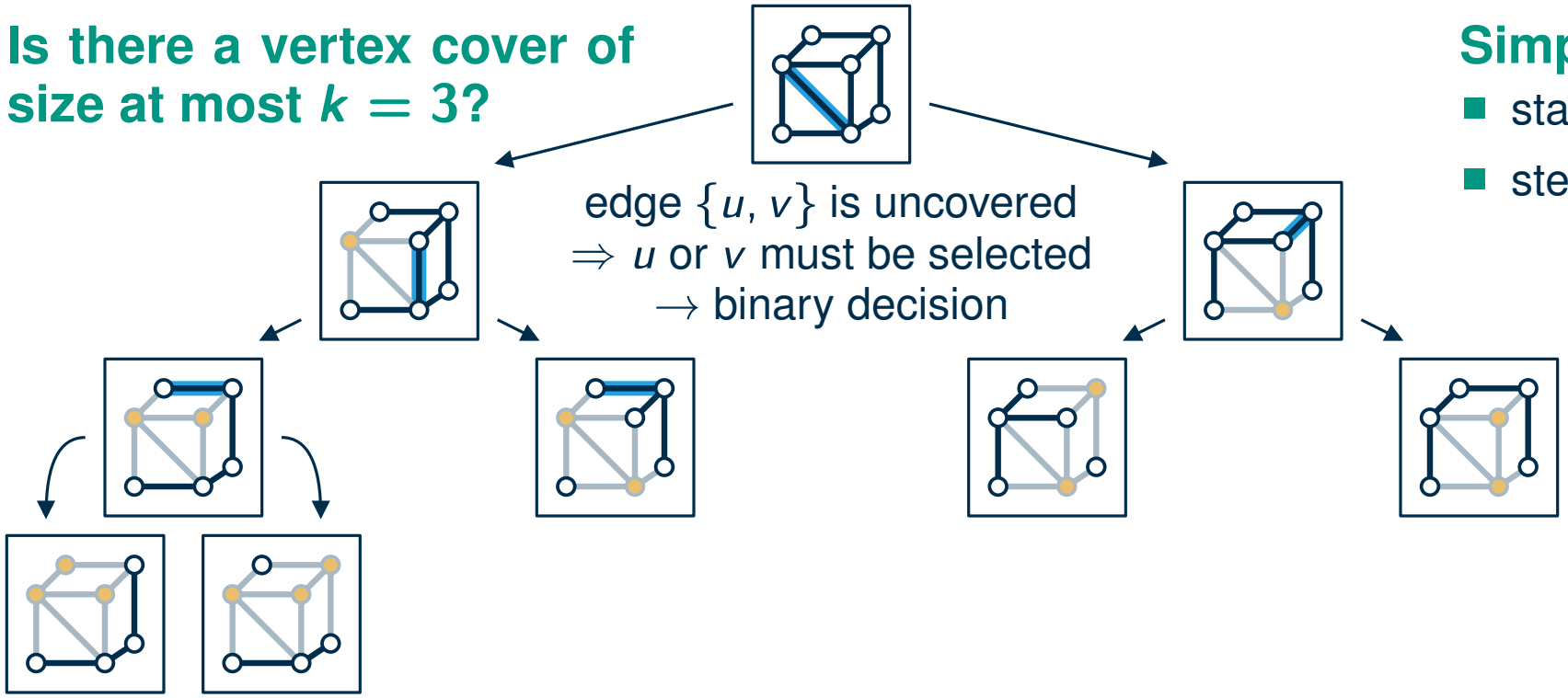

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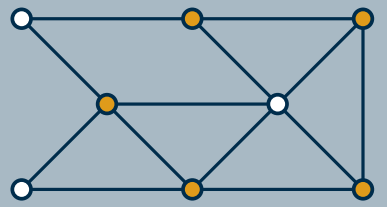

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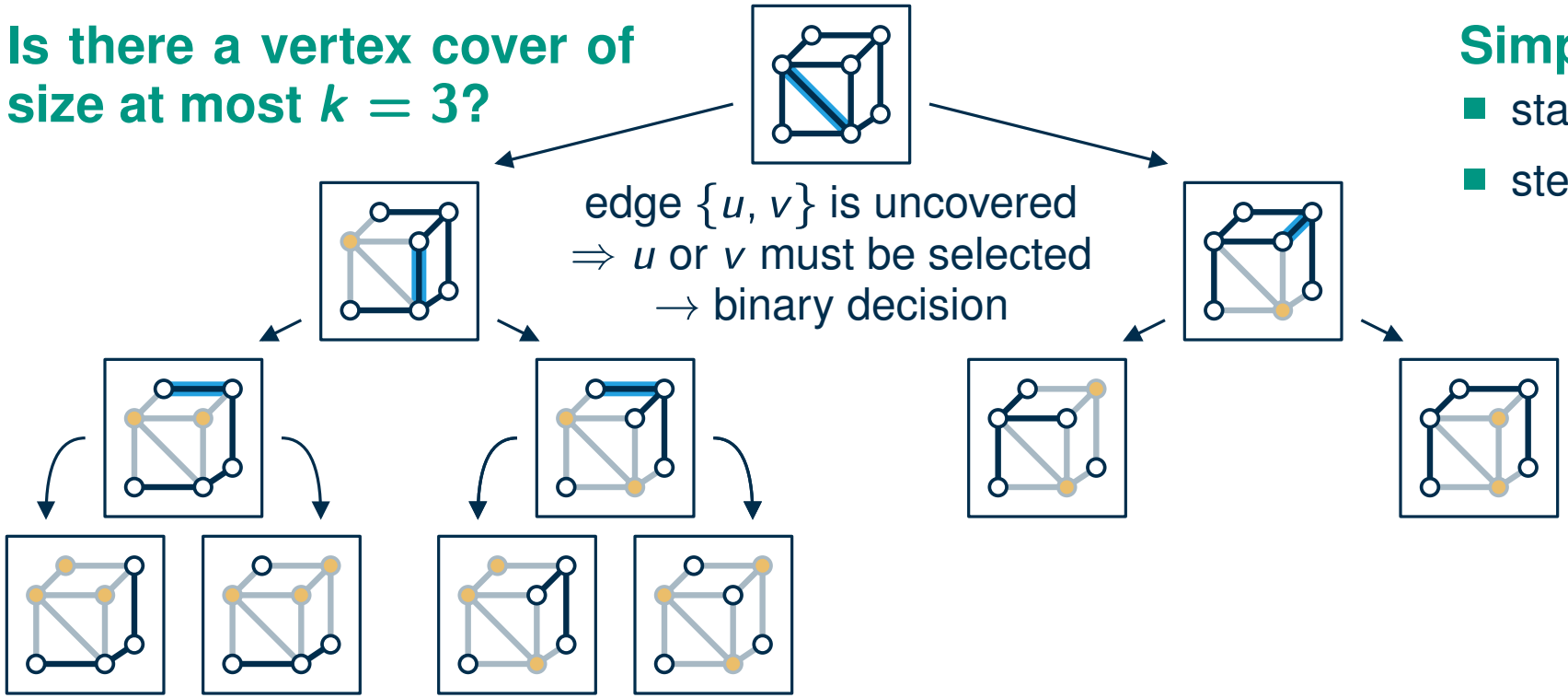

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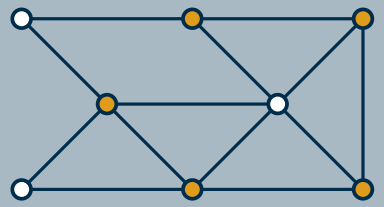

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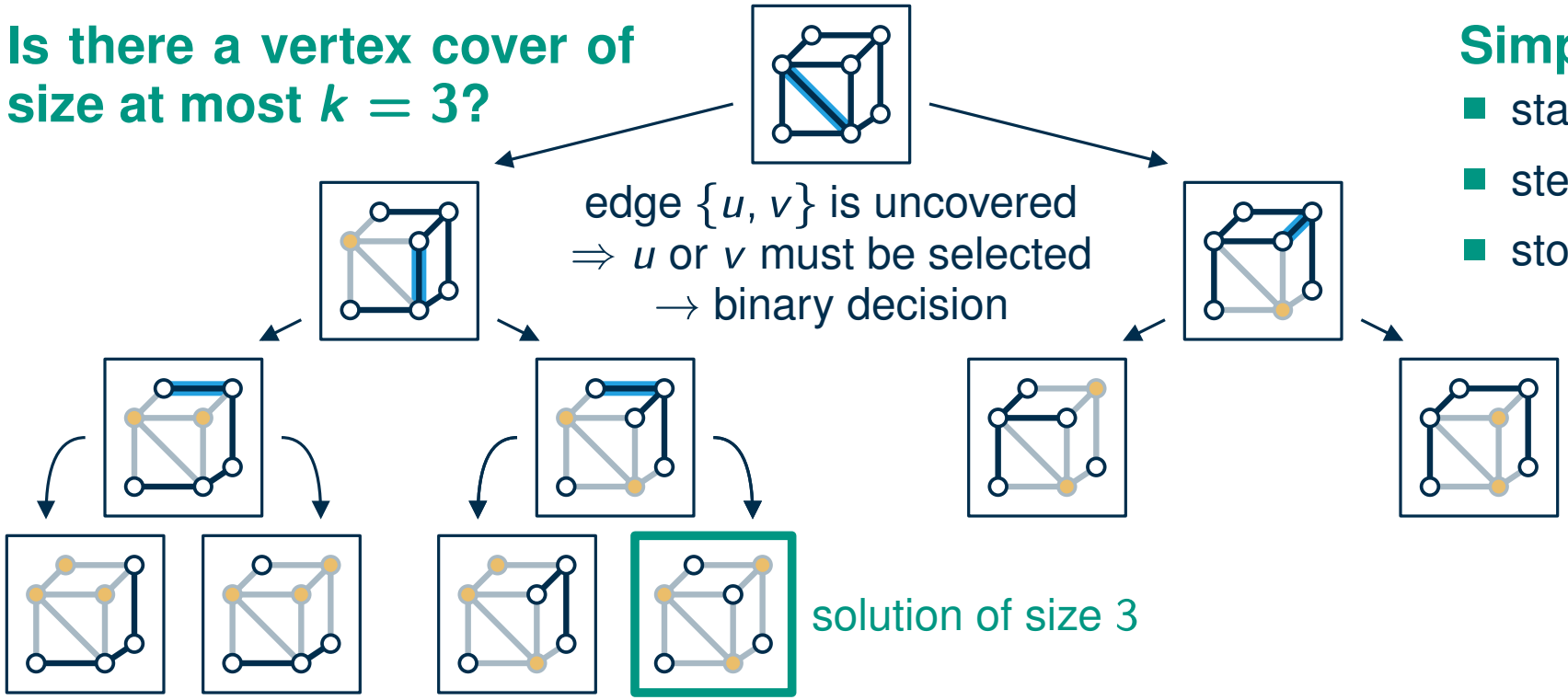


edge $\{u, v\}$ is uncovered
$\Rightarrow$ u or v must be selected
$\rightarrow$ binary decision

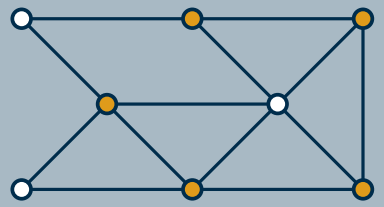
solution of size 3

Simple branching algorithm

- start: $V' = \emptyset \rightarrow$ no edge covered
- step: branch on uncovered edge
- stop if: all edges covered or already k vertices selected

Bounded search tree

Problem: VERTEX COVER
Given a graph $G = (V, E)$ and a parameter k. Does G have a vertex cover of size k?



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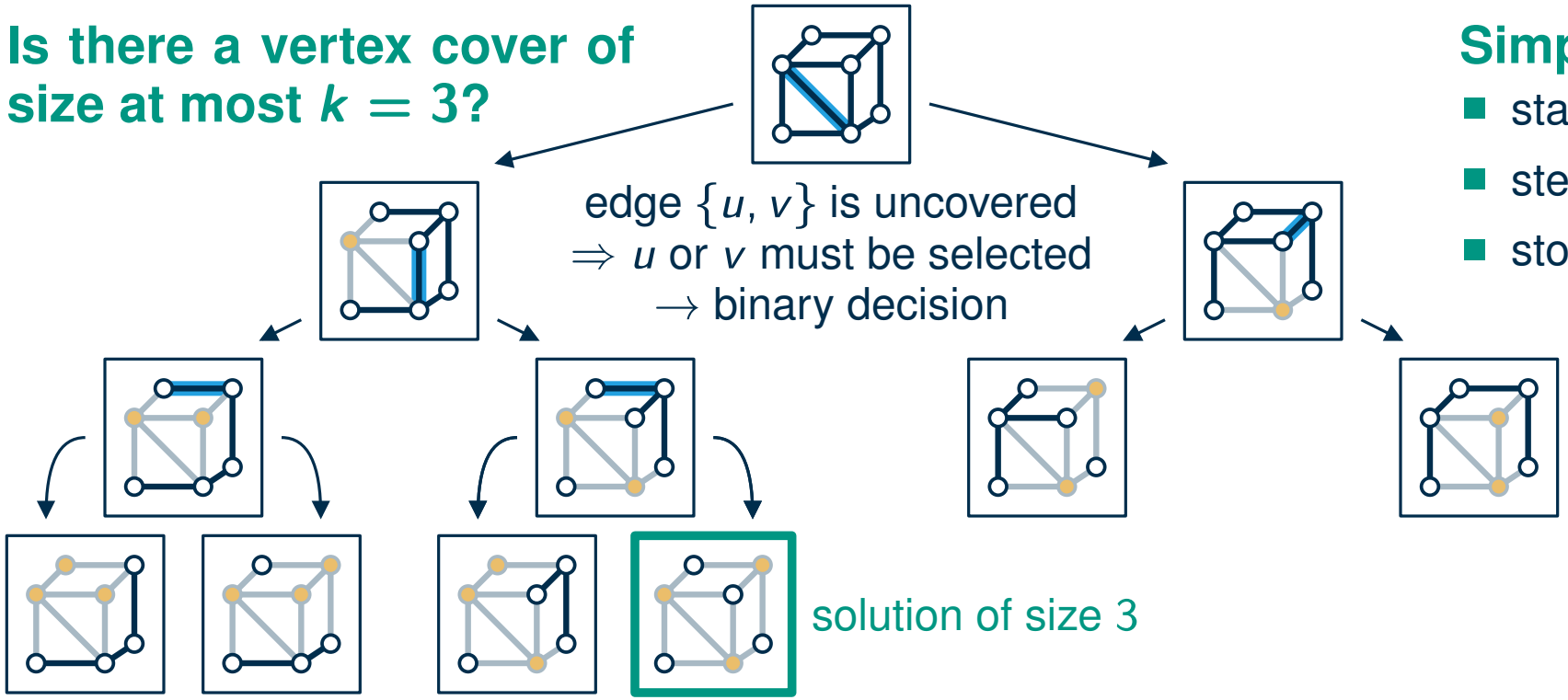

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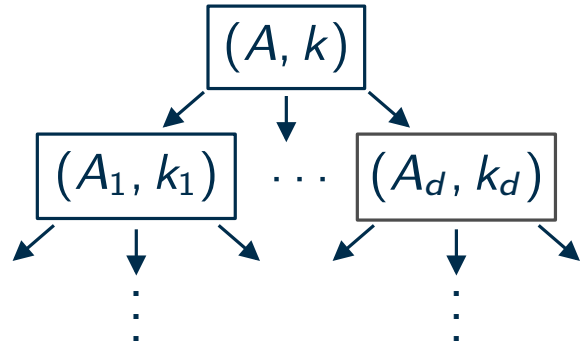
running time: $O(2^k m)$

Basic FPT-techniques

Bounded search tree

- for instance (A, k) compute in time n^c instances $(A_1, k_1), \ldots, (A_d, k_d)$, such that:
  (A, k) yes instance $\Leftrightarrow$ (A_i, k_i) yes instance for an $i \in [1, d]$



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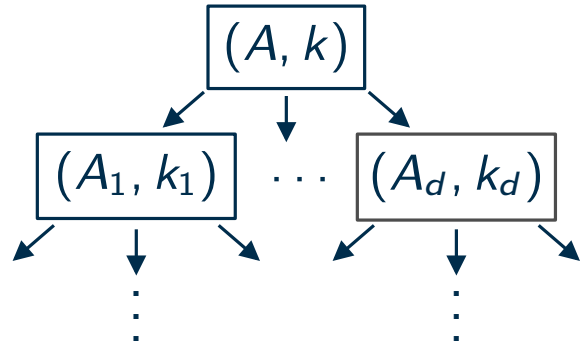

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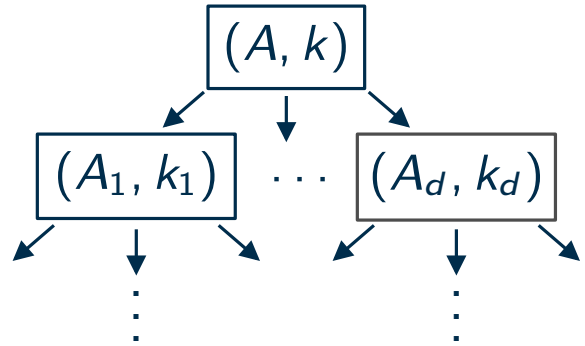

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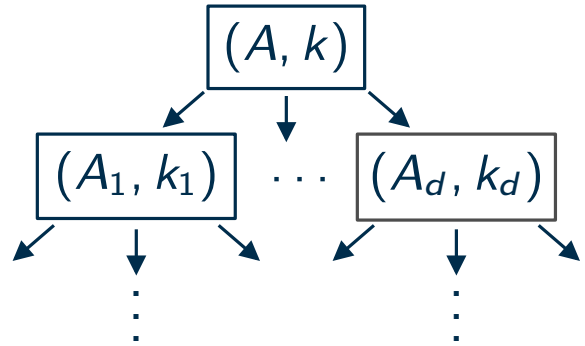
- bound d by a function $f_1(k)$
- bound tree height by $f_2(k)$ (example: parameter shrinks in each step)
- $\Rightarrow$ FPT-algo with running time $O(f_1(k)^{f_2(k)} n^c)$



Kernelization for Vertex Cover

Reduction rule 1: delete an isolated vertex

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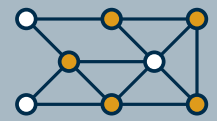

Kernelization for Vertex Cover

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- add a vertex v with $\deg(v) > k$ to the VC
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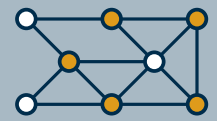

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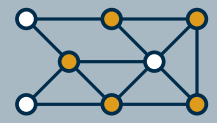

Lemma
Reduction rule 1 is safe. (new instance is YES-instance $\Leftrightarrow$ old instance is YES-instance)

Proof: obvious

Kernelization for Vertex Cover

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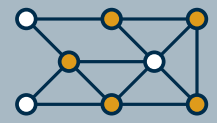

Lemma
Reduction rule 2 is safe. (new instance is YES-instance $\Leftrightarrow$ old instance is YES-instance)

Proof

- either v or every neighbor of v must be part of the vertex cover
- if we don't choose v, the vertex cover will be too large ($\deg(v) > k$)

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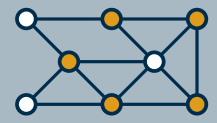
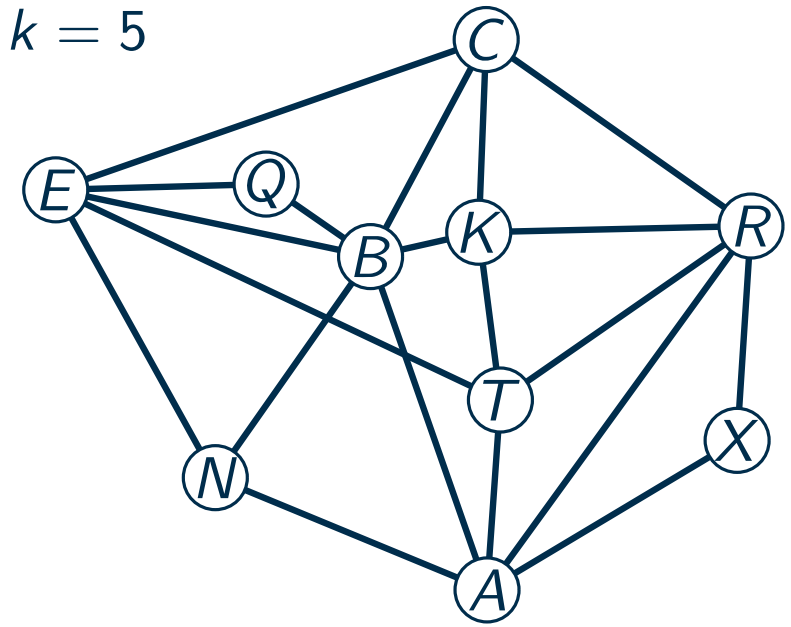


Apply reduction rule 2 exhaustively



Solution:

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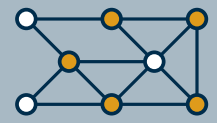

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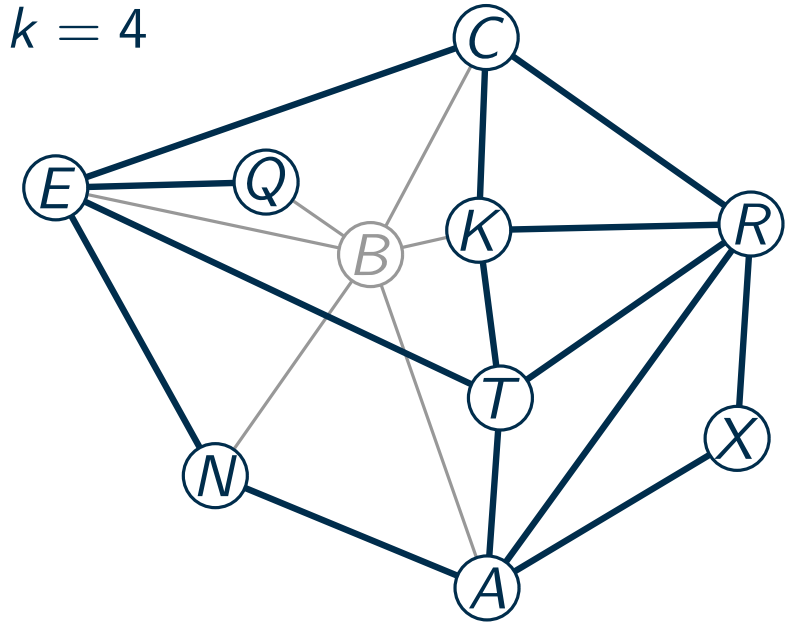

Solution: Ⓑ

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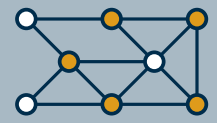

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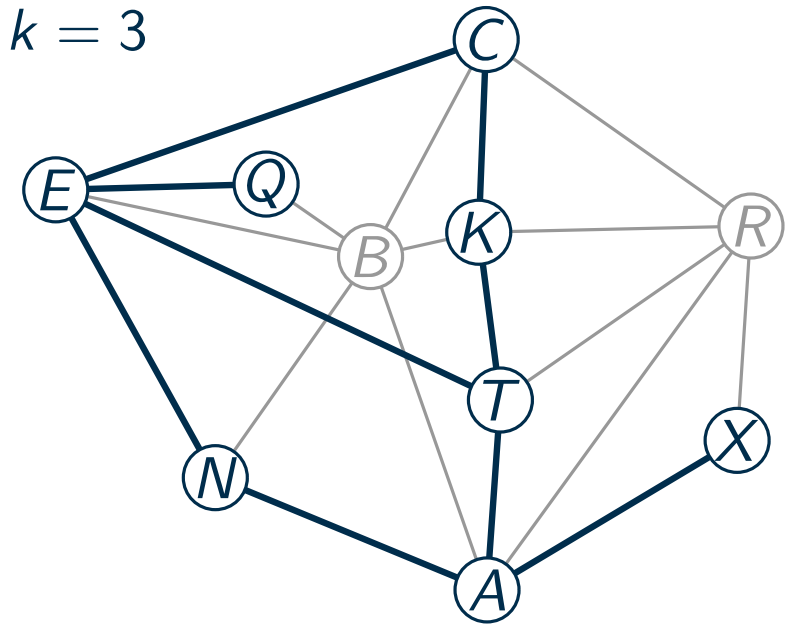

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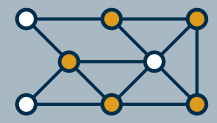

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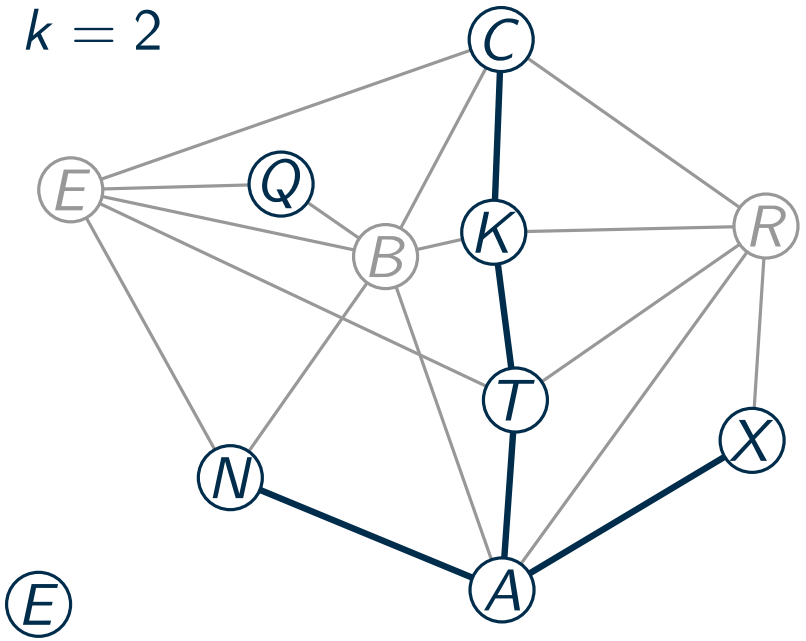

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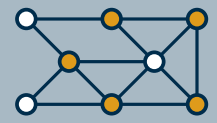

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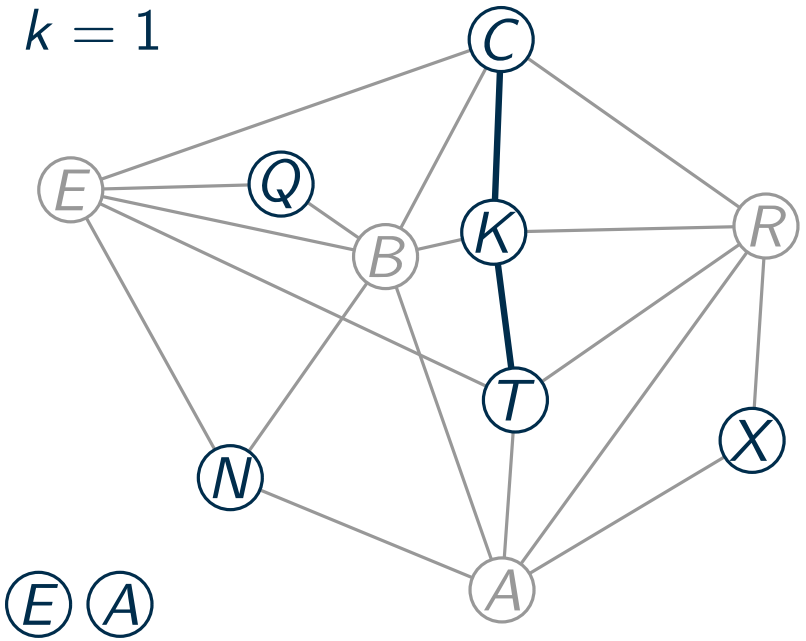

Solution: B R E A

Kernelization for VERTEX COVER

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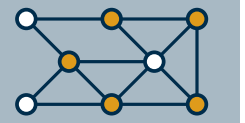

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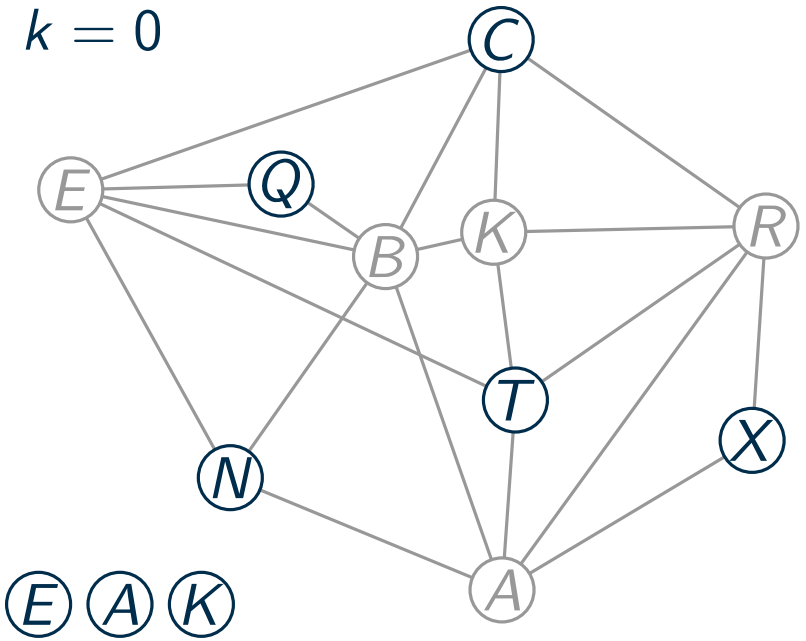

Solution: (B) (R) (E) (A) (K)

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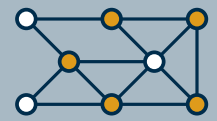

Reduction rule 3

- rules 1 and 2 not applicable and $m > k^2$: reduce to NO-instance of constant size

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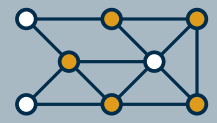

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Lemma
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Proof

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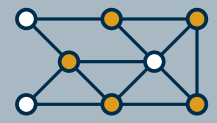

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- rule 2 not applicable $\Rightarrow \deg(v) \leq k$ for all $v \in V$

Kernelization for VERTEX COVER

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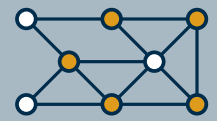

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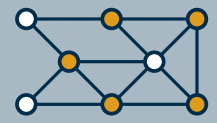

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Proof

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- every vertex covers at most k edges
- k vertices cover at most k^2 edges $\Rightarrow$ NO-instance, as $m > k^2$

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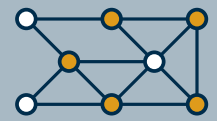

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Kernelization

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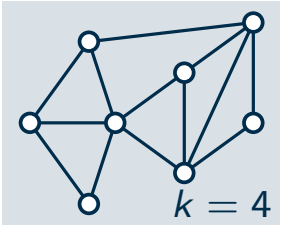

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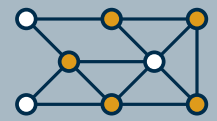

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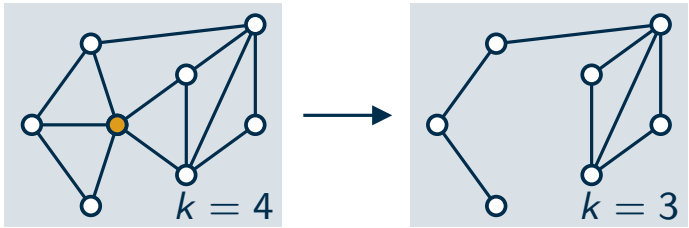

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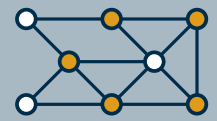

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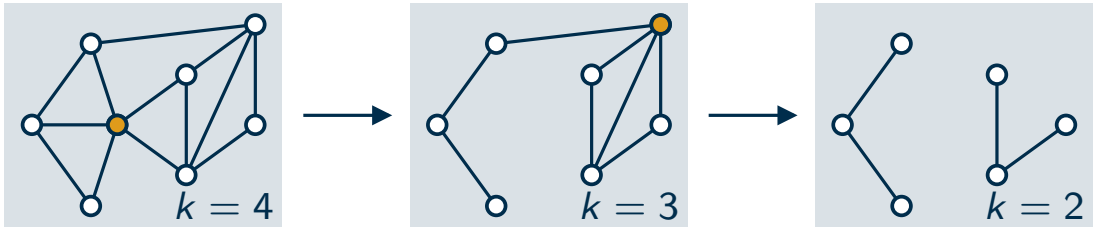

Kernelization for Vertex Cover

Reduction rule 1: delete an isolated vertex

Reduction rule 2

- add a vertex v with $\deg(v) > k$ to the VC
- new instance: remove v, reduce k by 1

Problem: Vertex Cover
Given a graph $G = (V, E)$ and a parameter k.
Does G have a vertex cover of size k?
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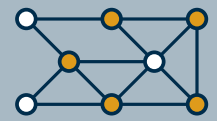
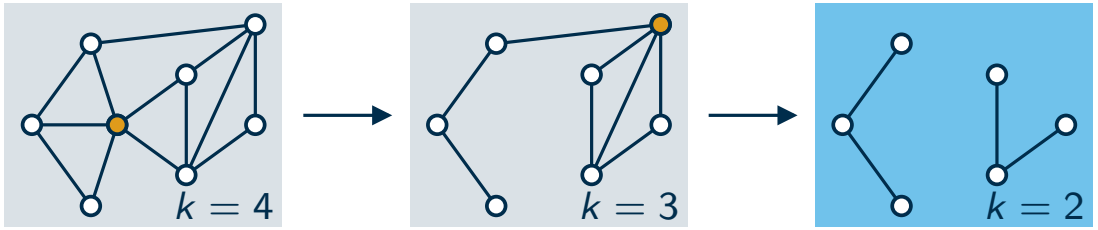


Reduction rule 3

- rules 1 and 2 not applicable and $m > k^2$: reduce to NO-instance of constant size

Kernelization

- apply reduction rules exhaustively
- it remains a problem core of size $O(k^2)$



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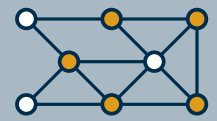

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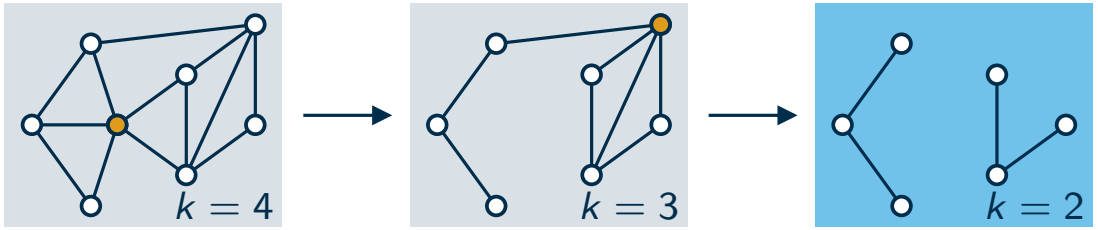

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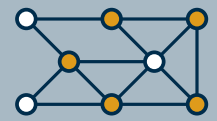

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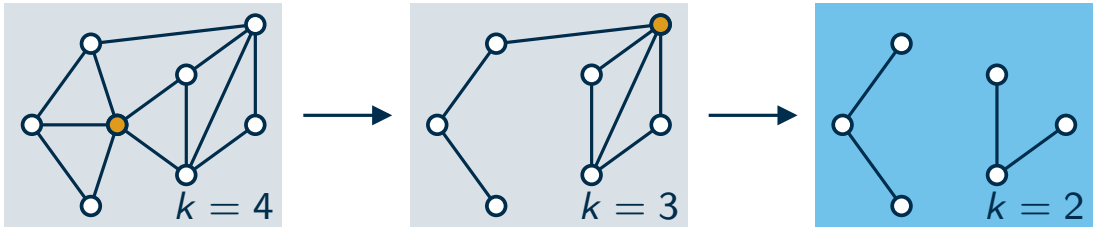

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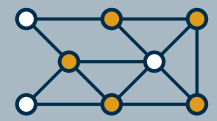

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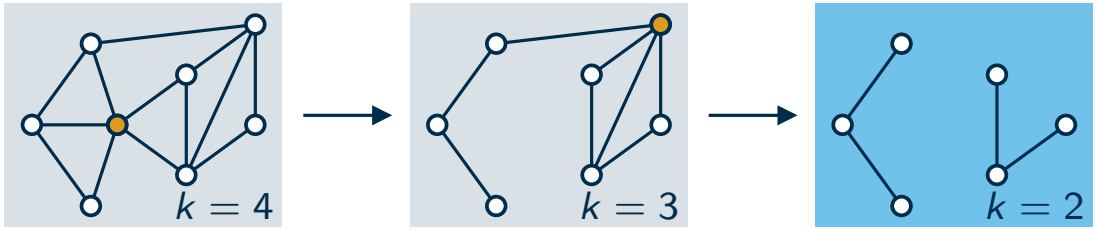

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FPT algorithm

- brute-force on the kernel: $O(2^{k^2} k^2)$
- or better: search tree on the kernel

Basic FPT-techniques

Bounded search tree

- for instance (A, k) compute in time n^c instances $(A_1, k_1), \ldots, (A_d, k_d)$, such that: (A, k) yes instance $\Leftrightarrow$ (A_i, k_i) yes instance for an $i \in [1, d]$
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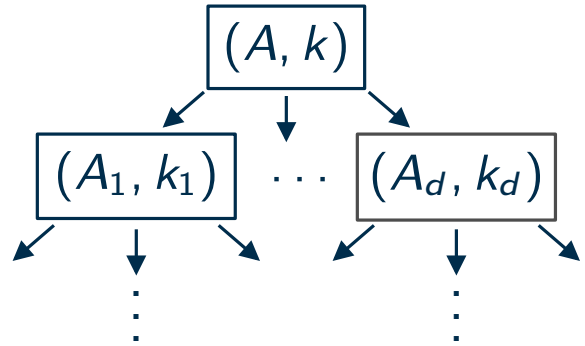

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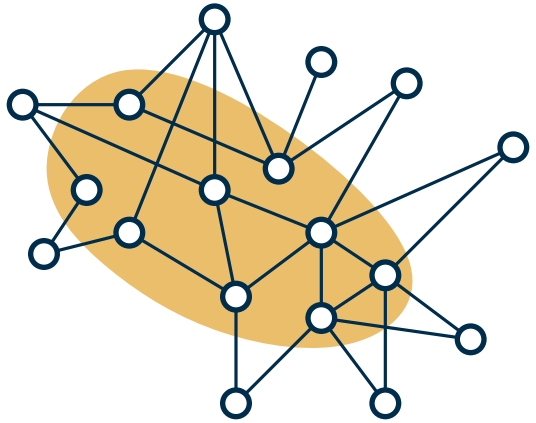

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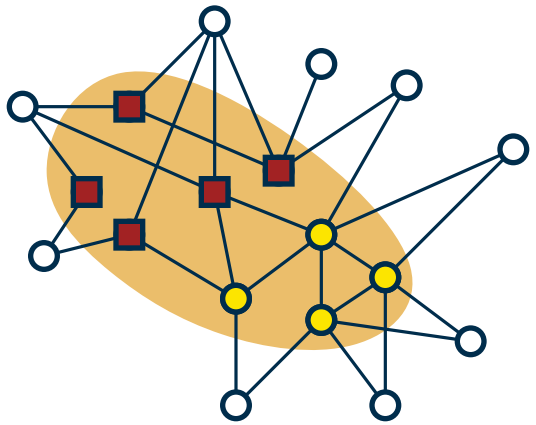

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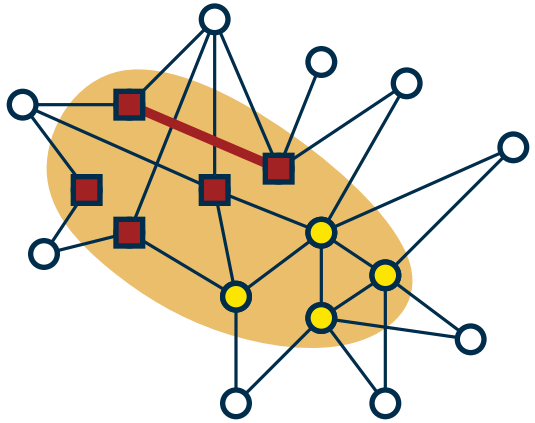

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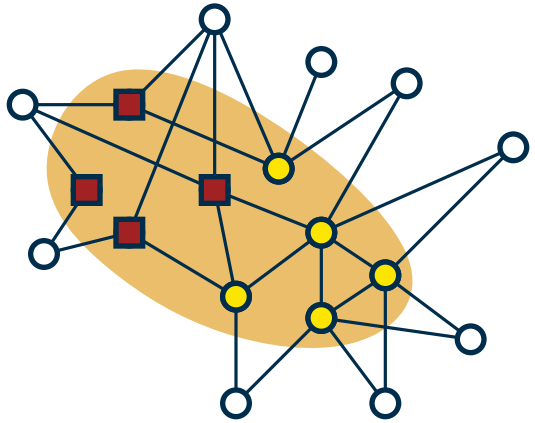

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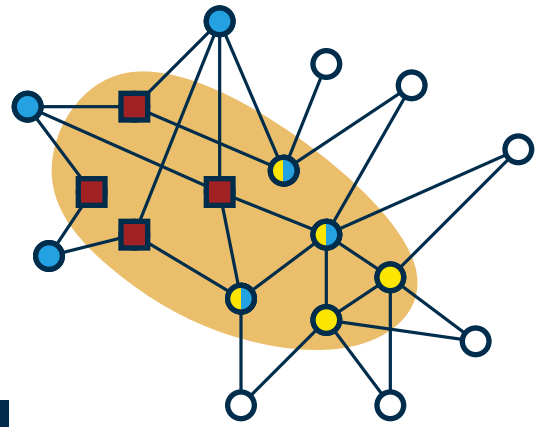

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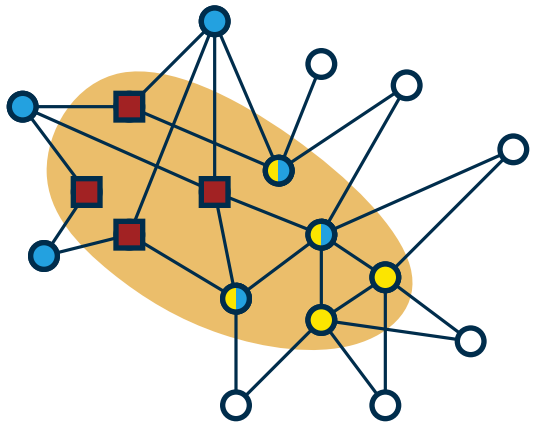

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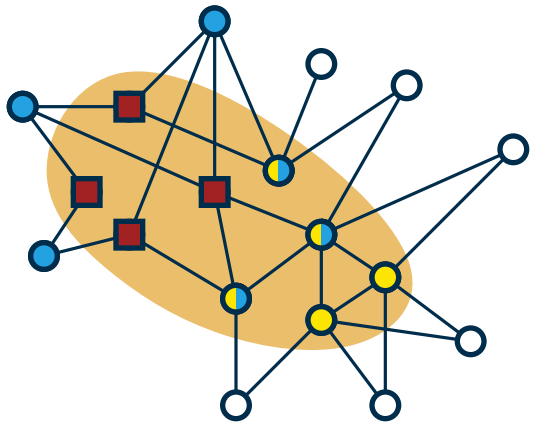

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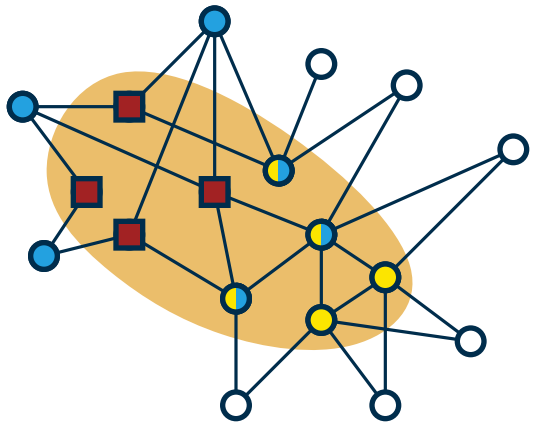

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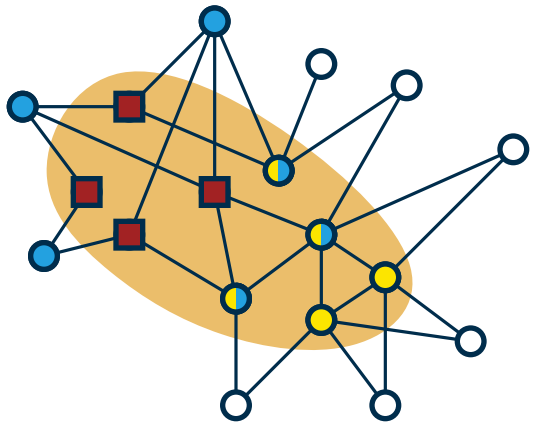

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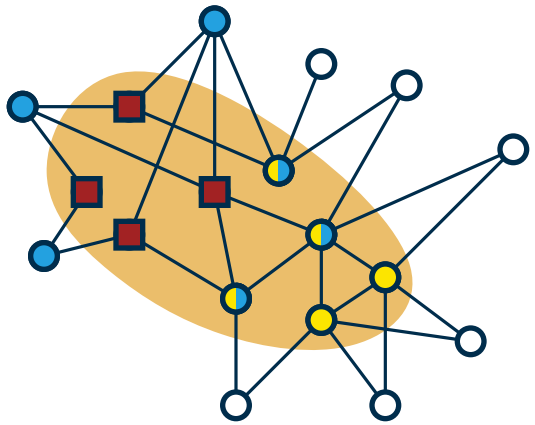

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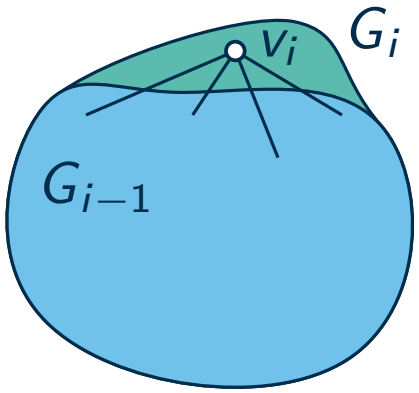

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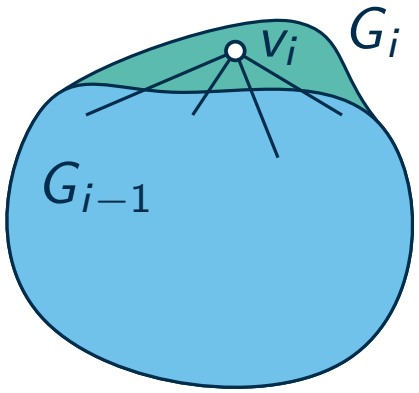

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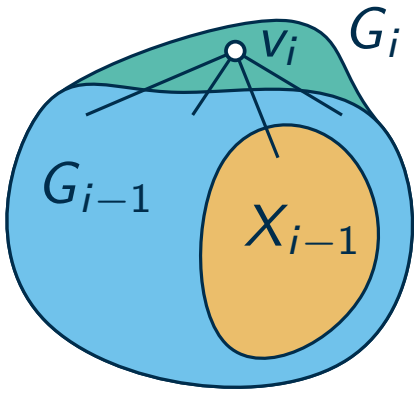

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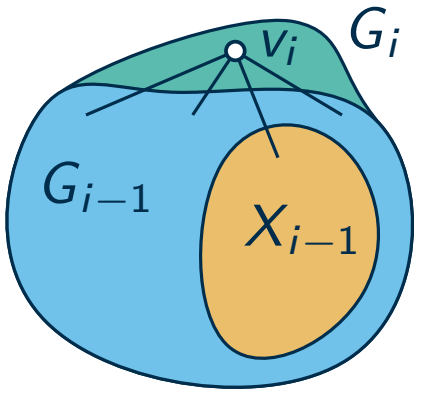

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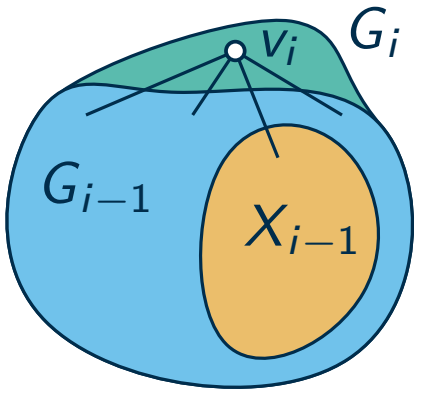

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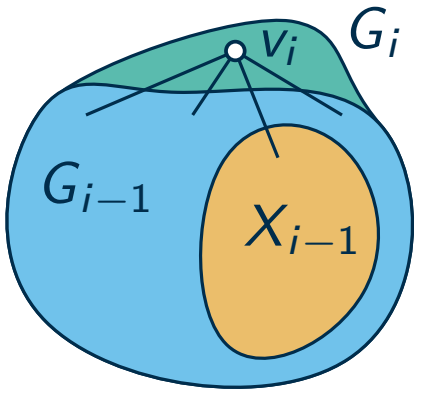

Running time?

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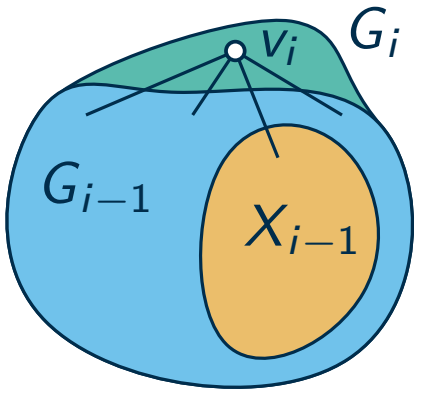

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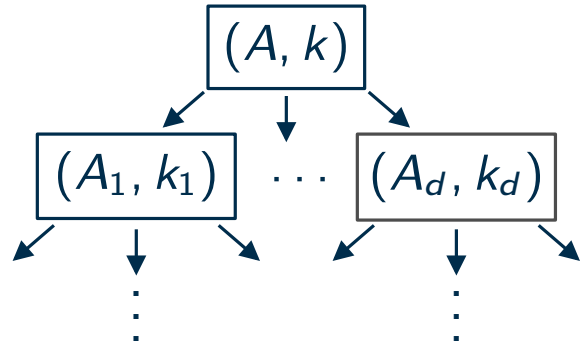

Kernelization

- successively apply safe reduction rules
- show: what remains is a core, whose size only depends on k

Iterative compression

- compression algo: solve the problem under the assumption of knowing a slightly larger solution
- grow the instance step by step and maintain a small solution via repeated compression

Wrap-Up

$$\text{VERTEX COVER} \in \text{FPT}$$

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$\textsc{Vertex Cover} \in \text{FPT}$

(when parameterized with the solution size!)

FAQ: How does FPT relate to NP?

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Formally correct answer: It does not

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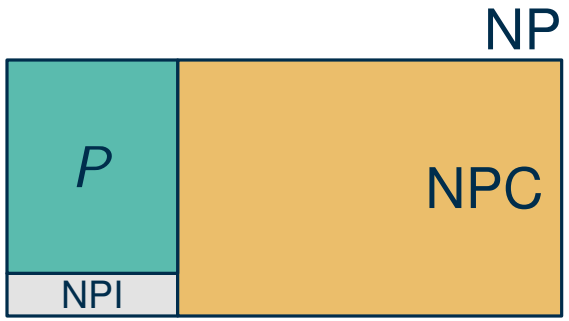

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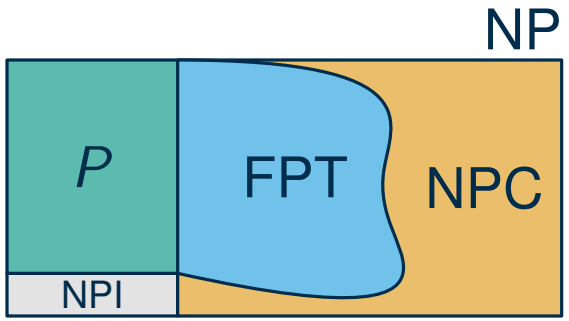

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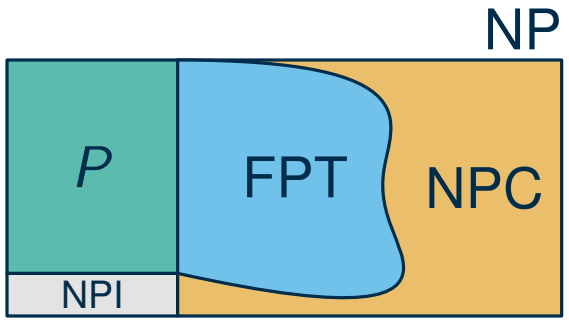

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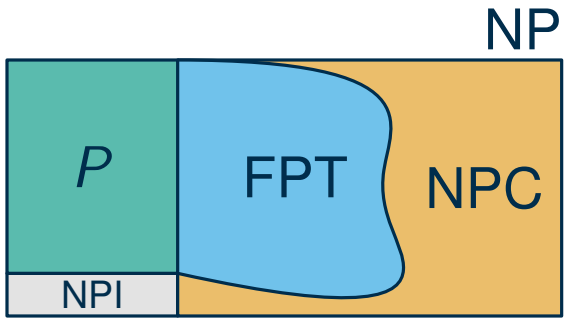

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Bonus info: para-NP

- para-NP is the equivalent of NP for parameterized problems
- FPT relates to para-NP like P to NP

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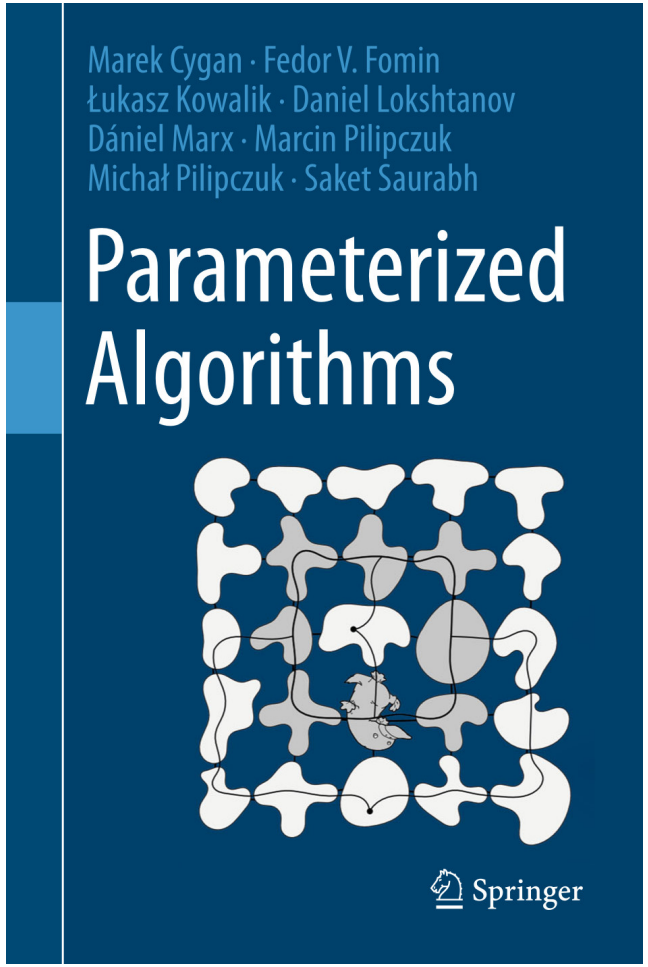
Weaker variants of FPT

- f not being computable makes your result slightly weaker
- even weaker: non-uniform (i.e., different algorithms for different k)
- also see: Section 2.2 of Downey & Fellows book

Recommended Books



Parameterized Algorithms

- excellent book
- (almost) all discussed topics can be found there
- free for KIT students: `link.springer.com/book/10.1007/978-3-319-21275-3`

Fundamentals of Parameterized Complexity

- slightly older book
- also free for KIT students:
  `https://link.springer.com/book/10.1007/978-1-4471-5559-1`