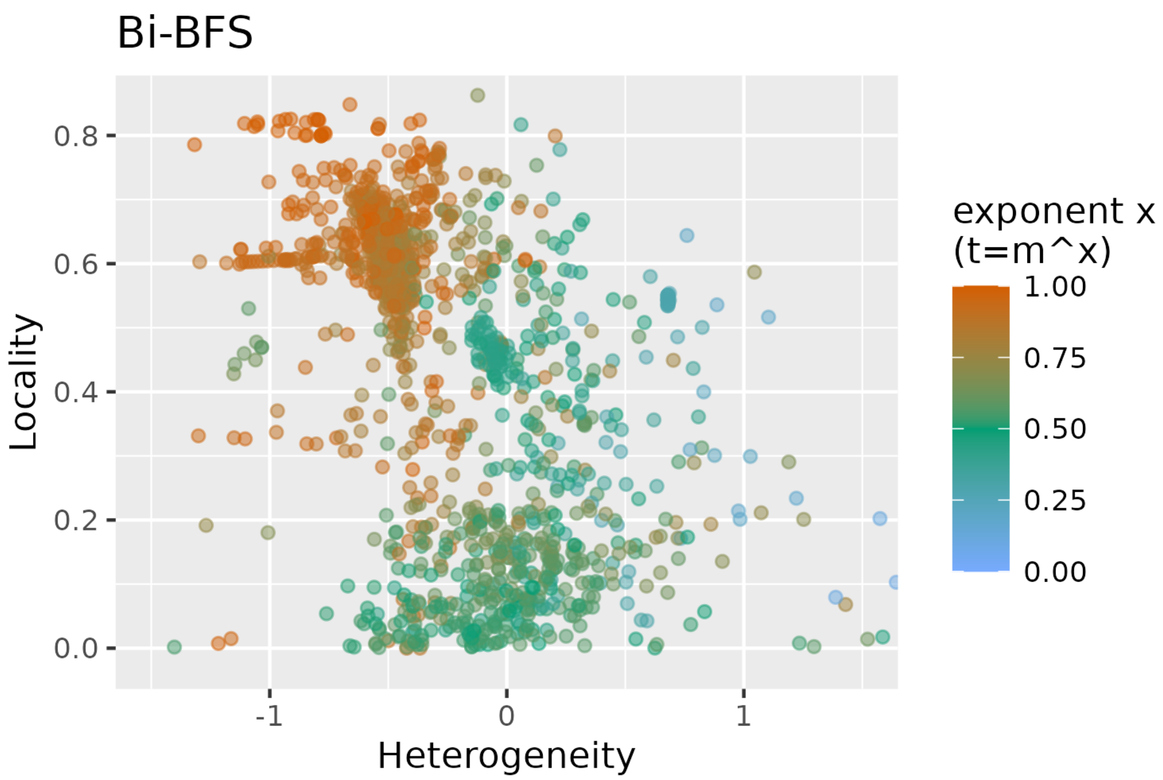
Beating the Worst Case

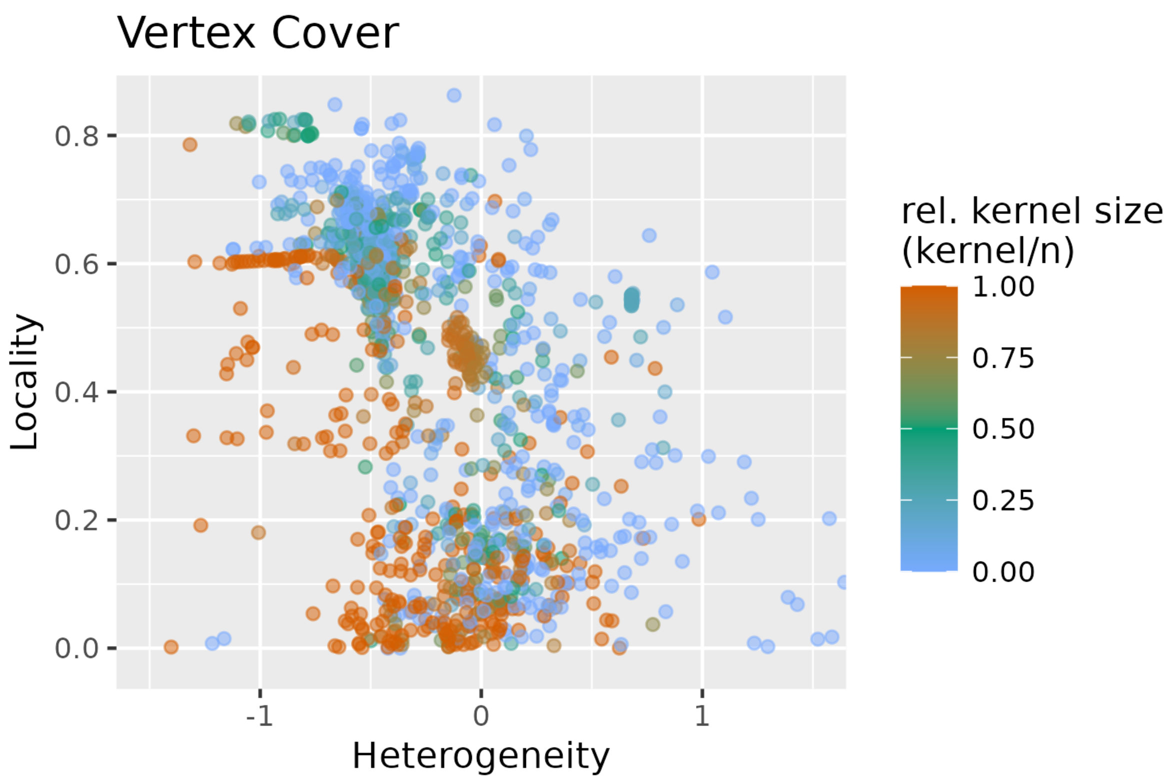
Practical Course – 7th meeting

Jean-Pierre, Marcus

Exercise Sheet 3

Solution Exercise Sheet 3





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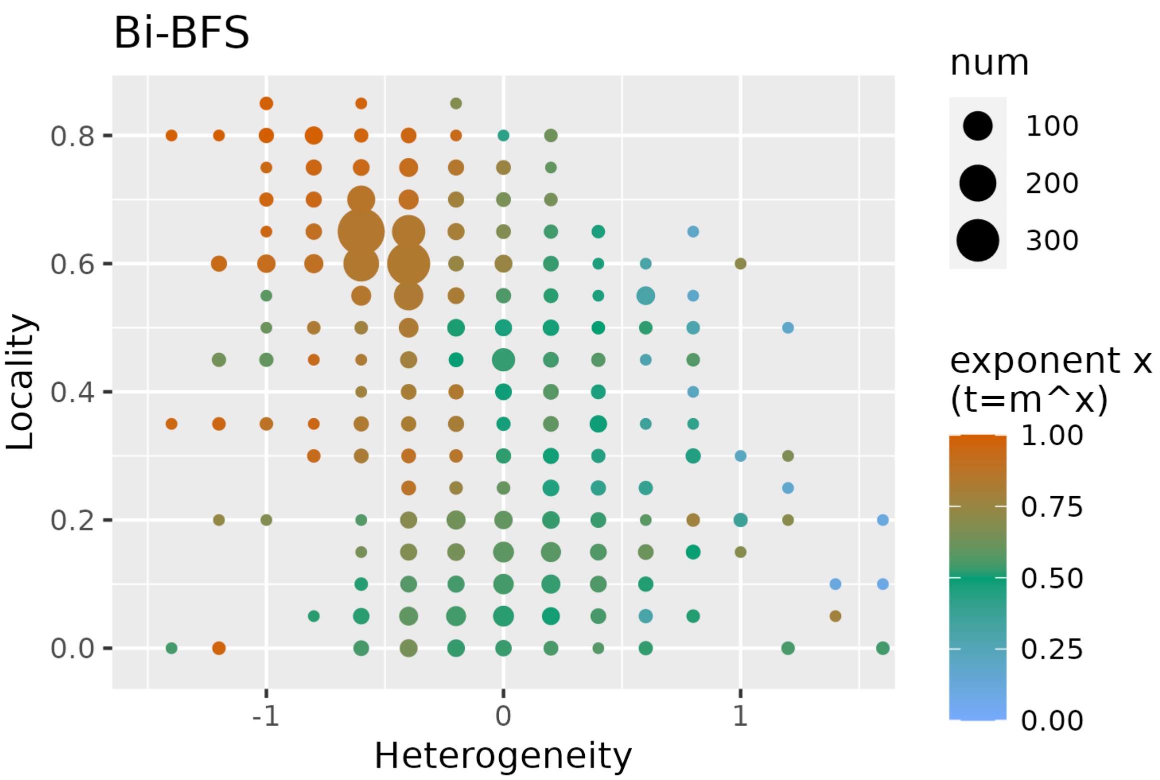

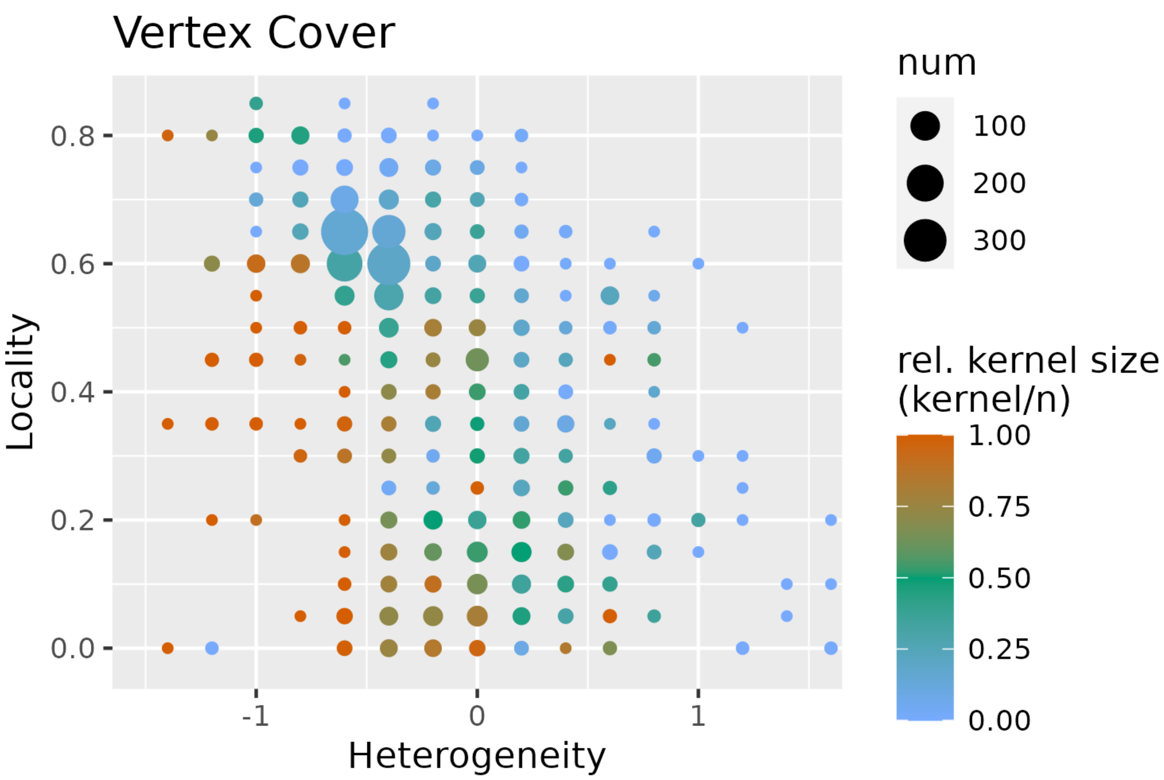

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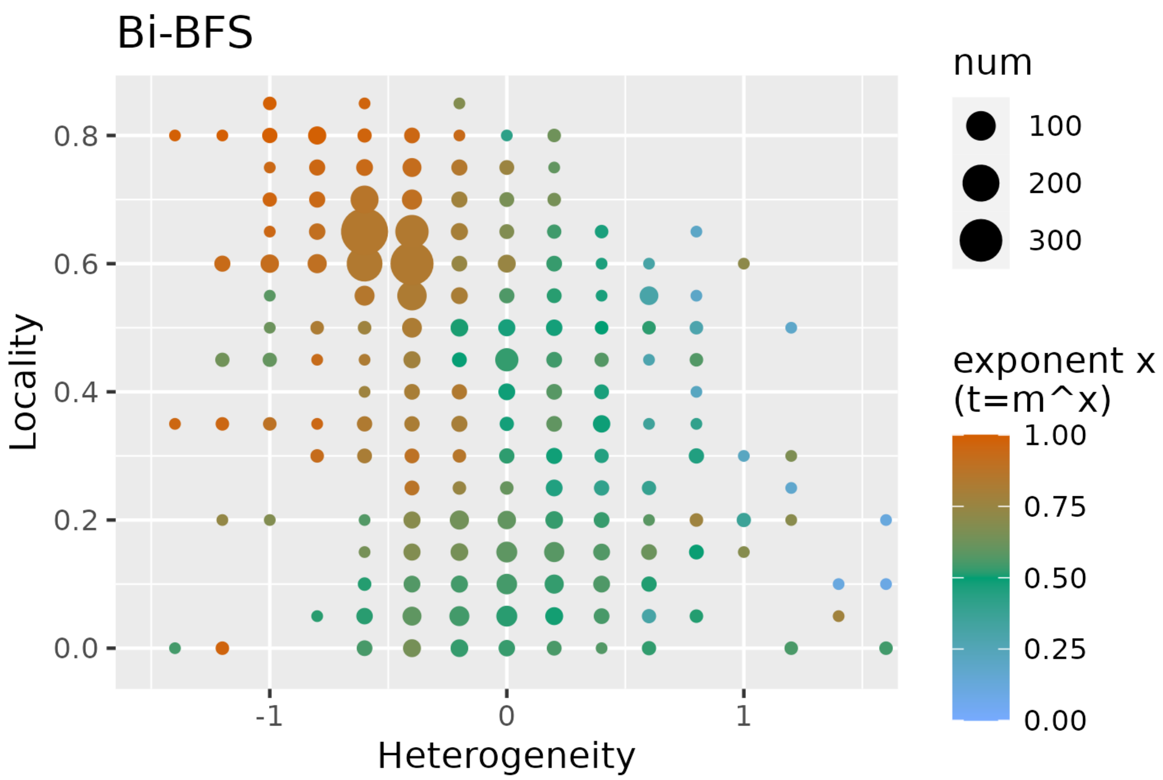

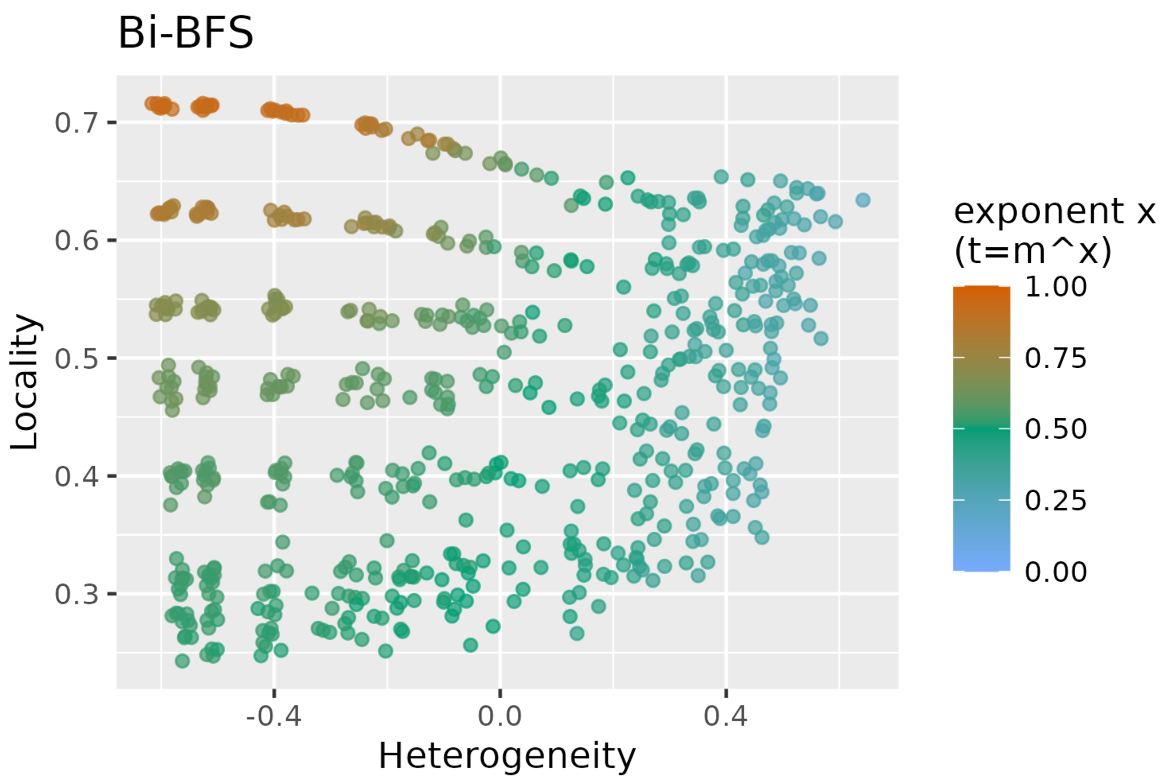

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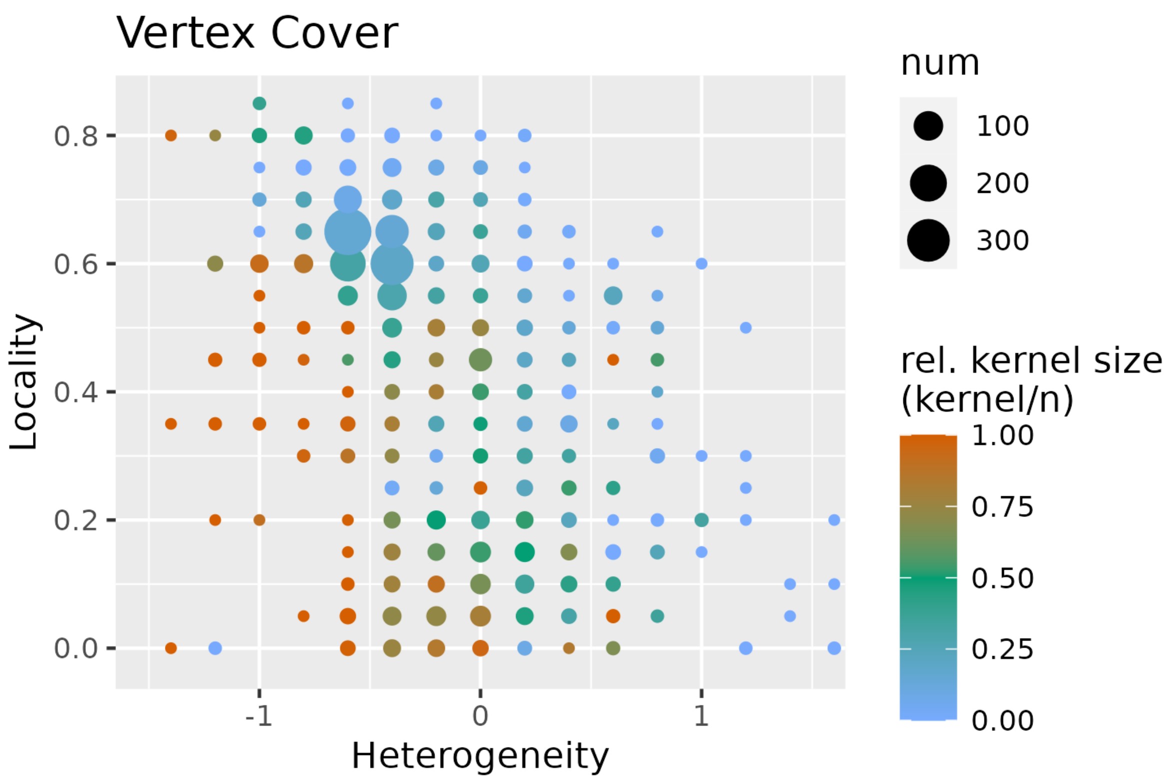
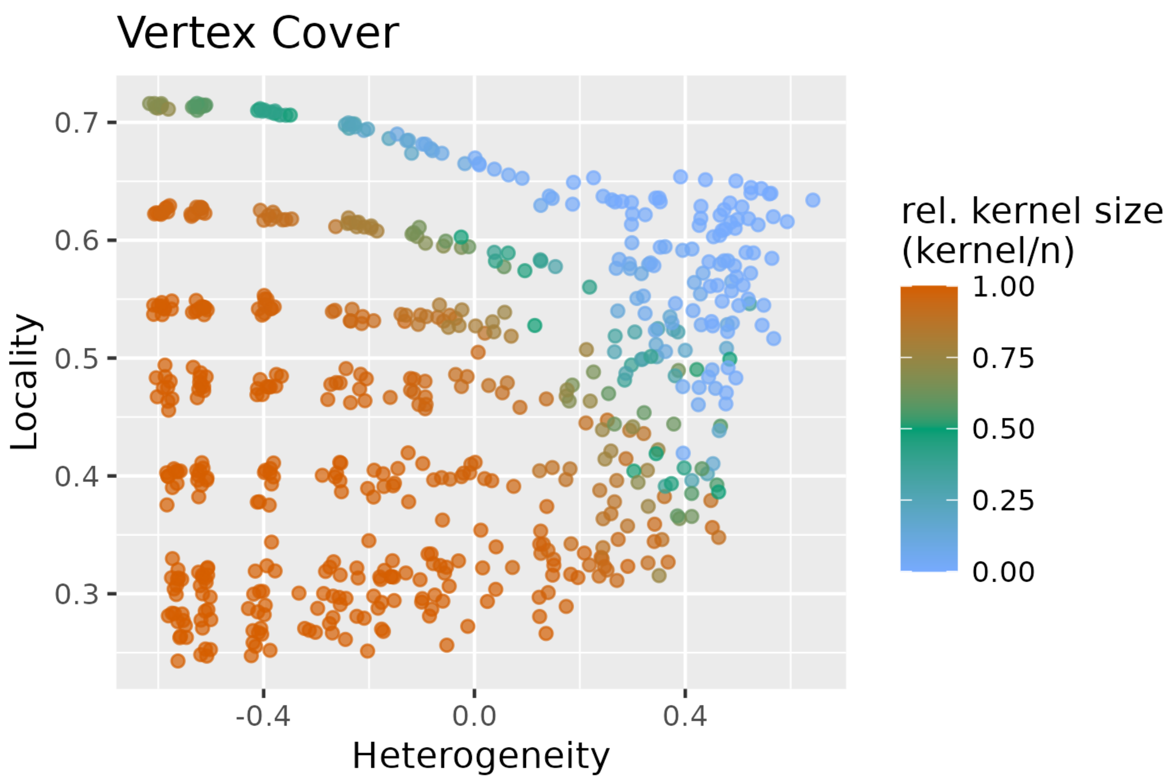
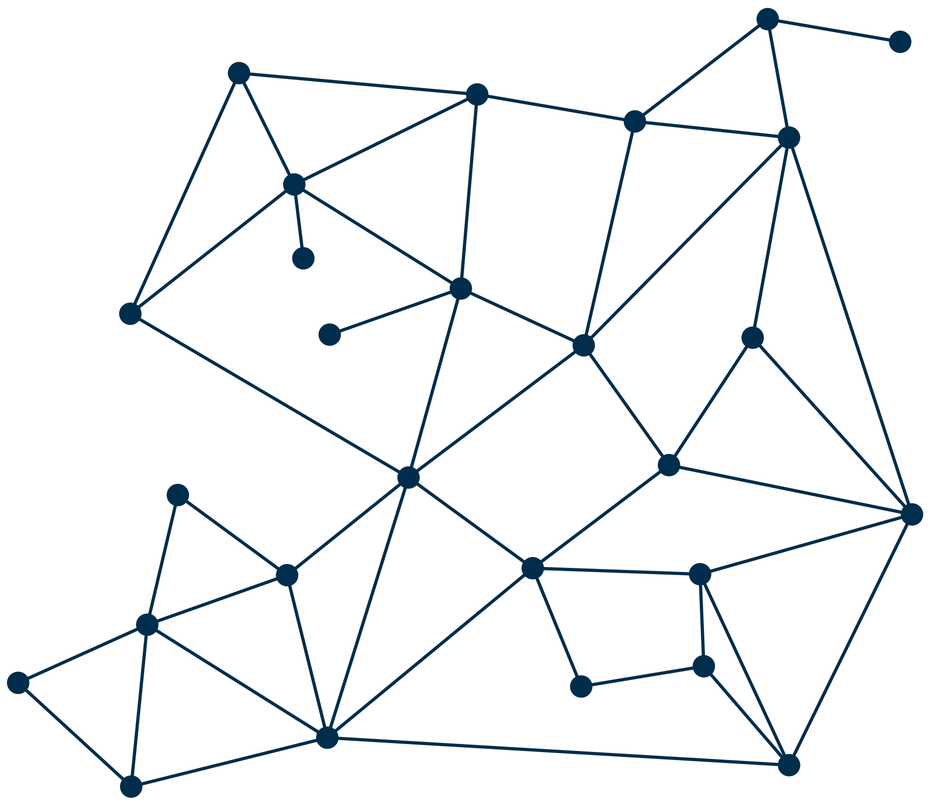




Eccentricity and Diameter



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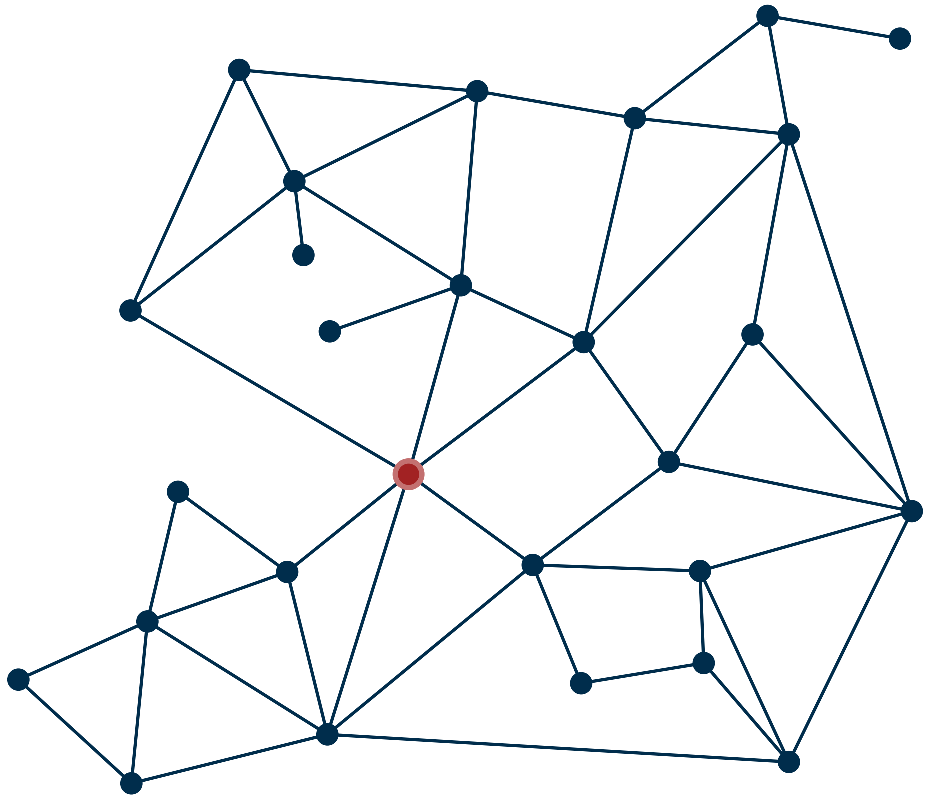

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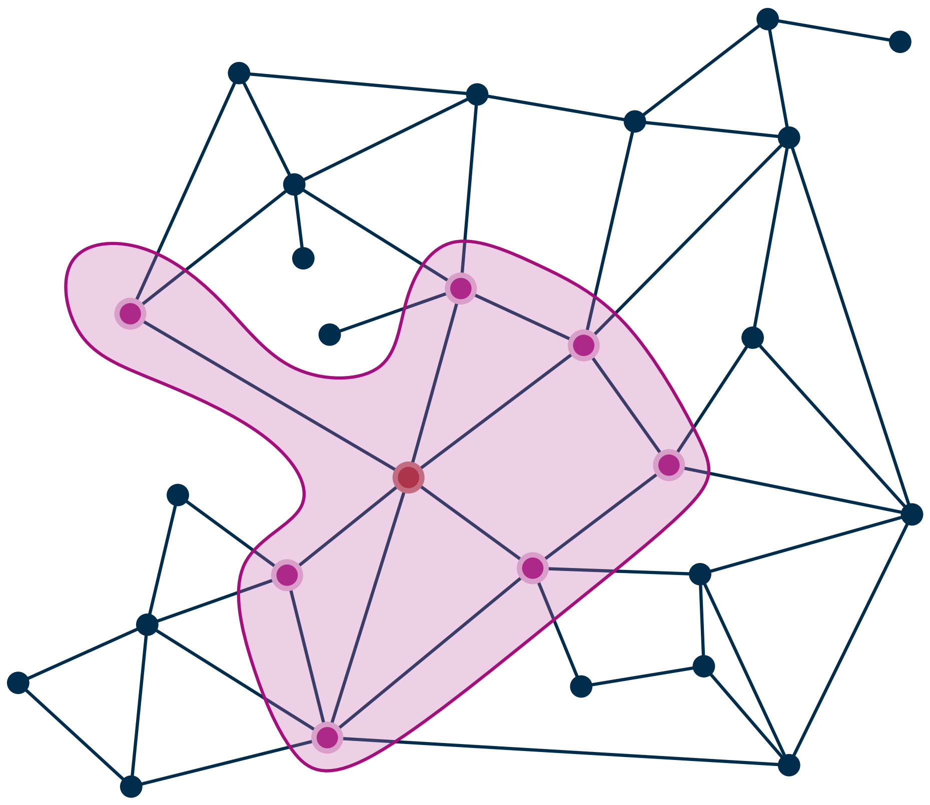

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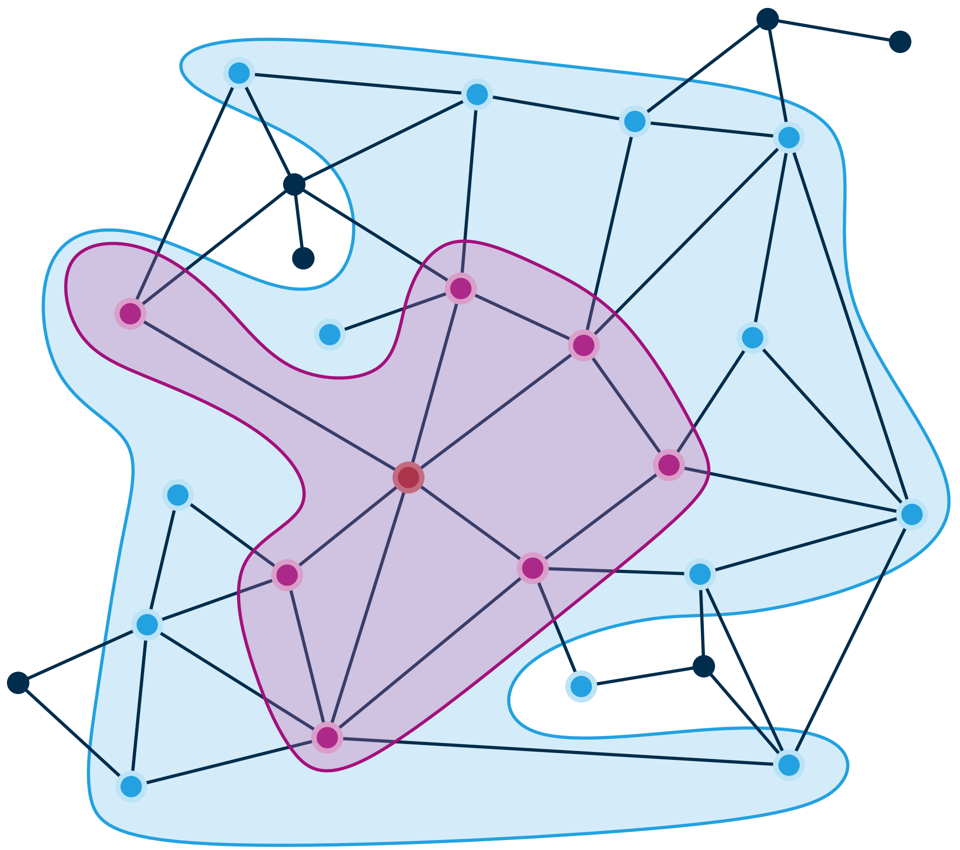

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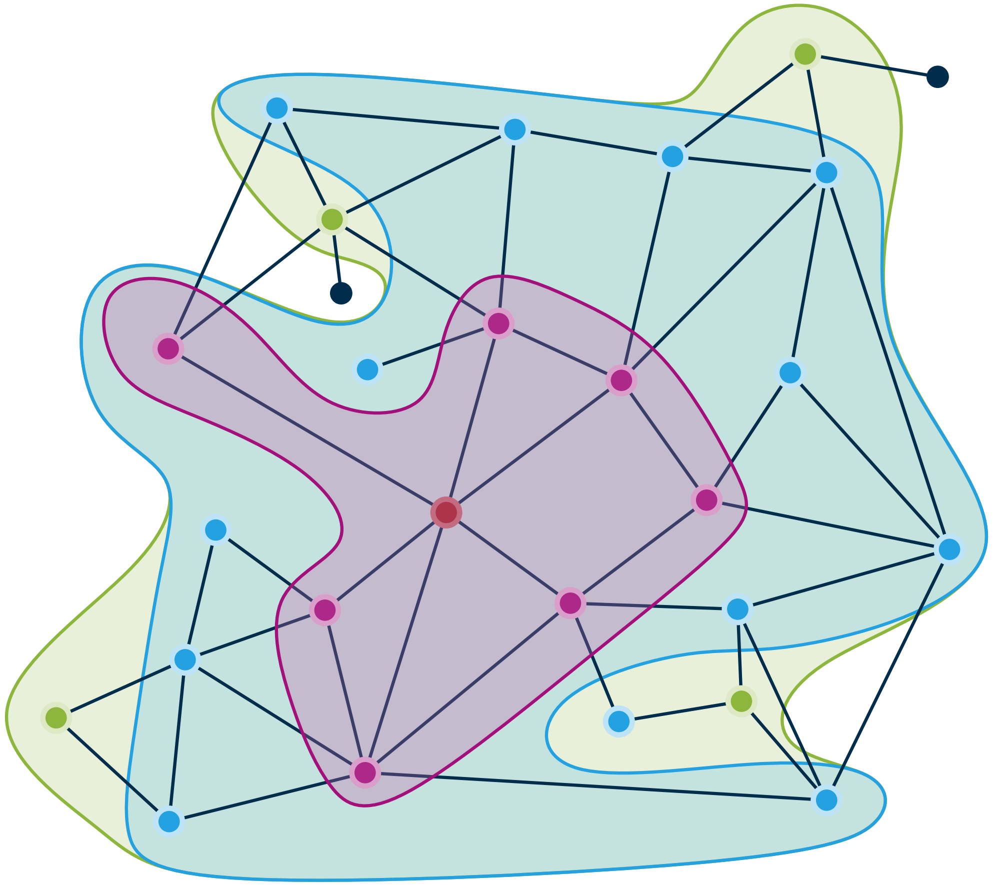

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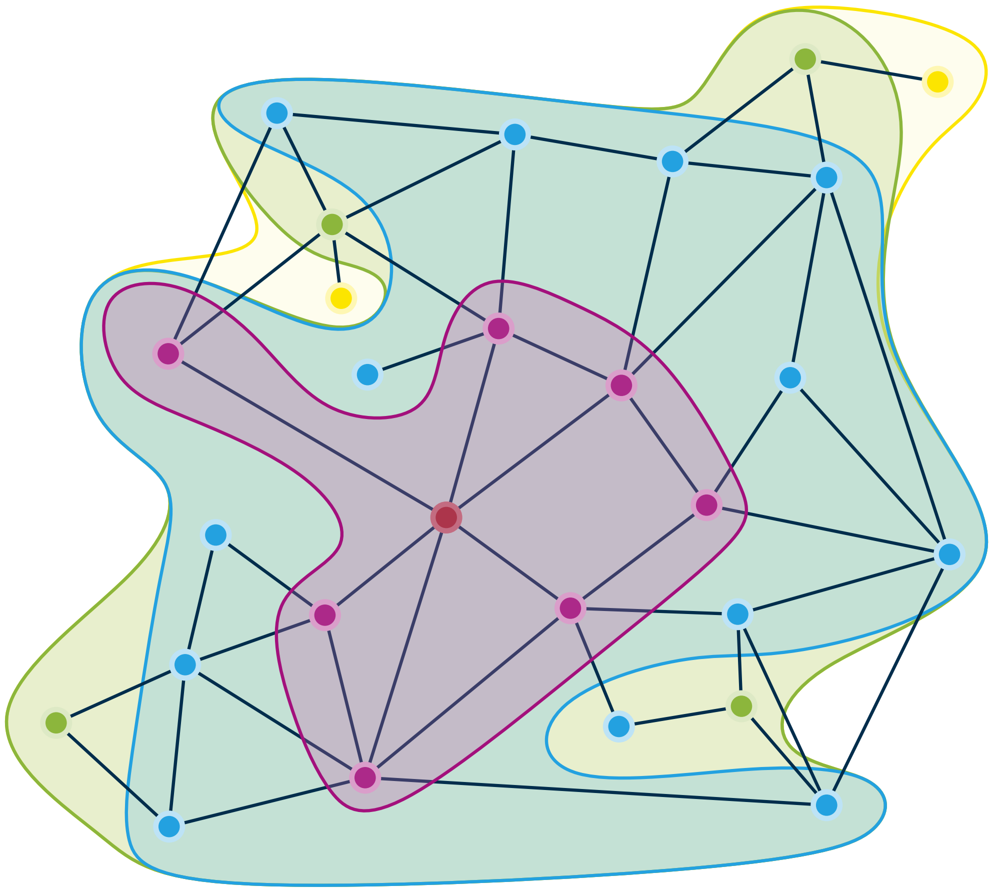

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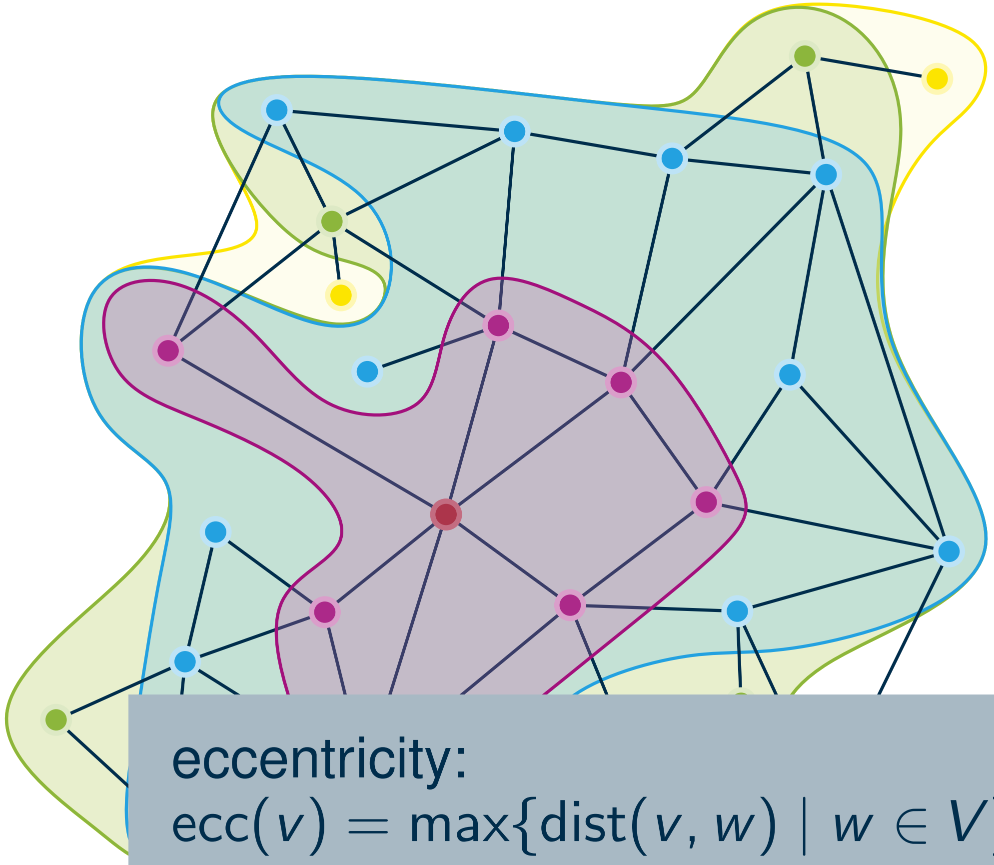
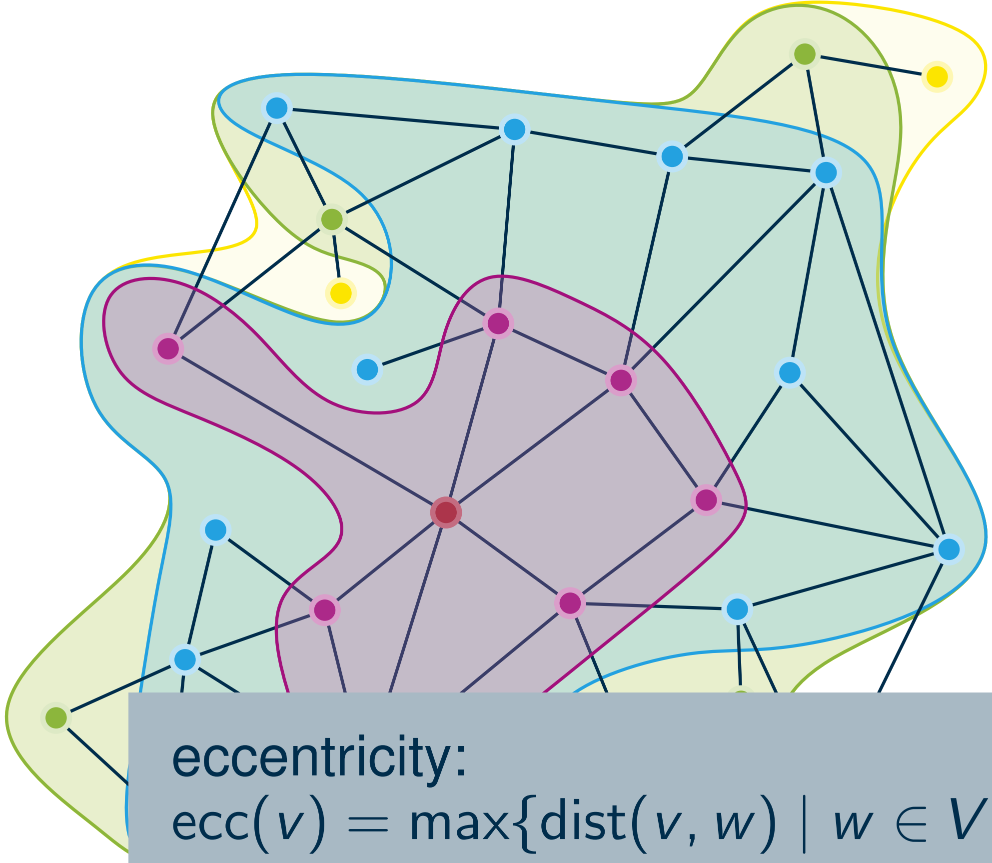


eccentricity:
$\mathrm{ecc}(v) = \max\{\mathrm{dist}(v, w) \mid w \in V\}$

Eccentricity and Diameter



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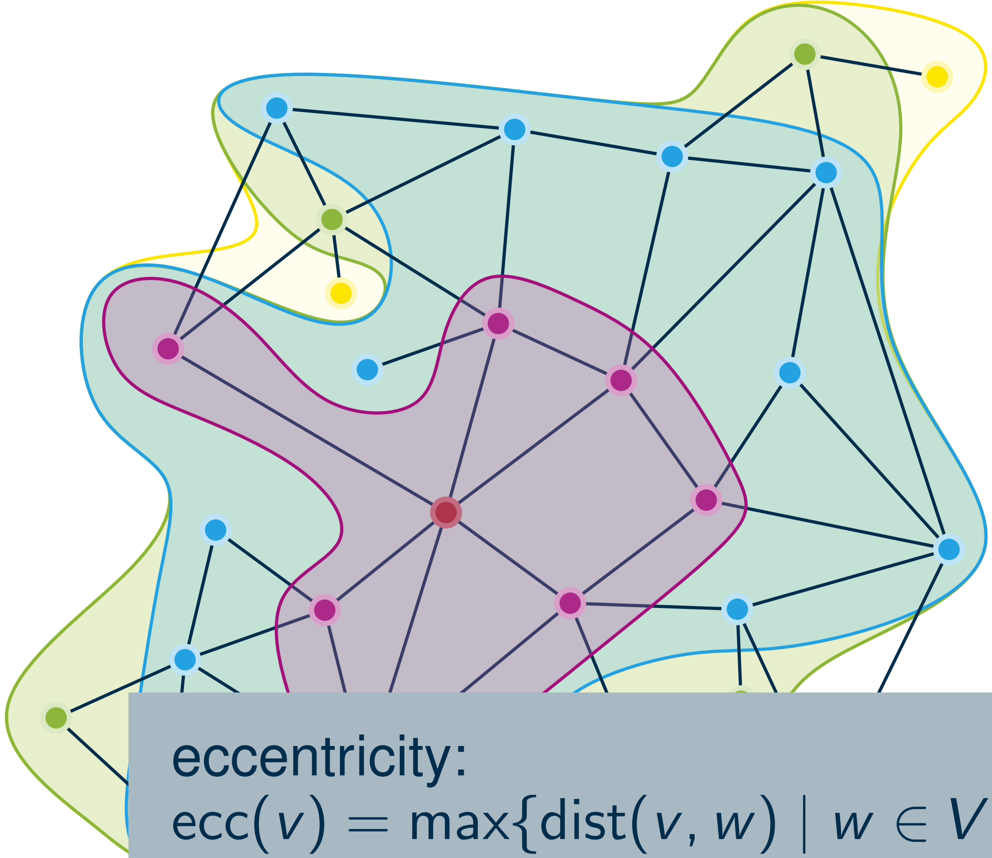
$\mathrm{diam}(G) = \max\{\mathrm{ecc}(v) \mid v \in V\}$

Eccentricity and Diameter

How to compute the diameter of a graph?



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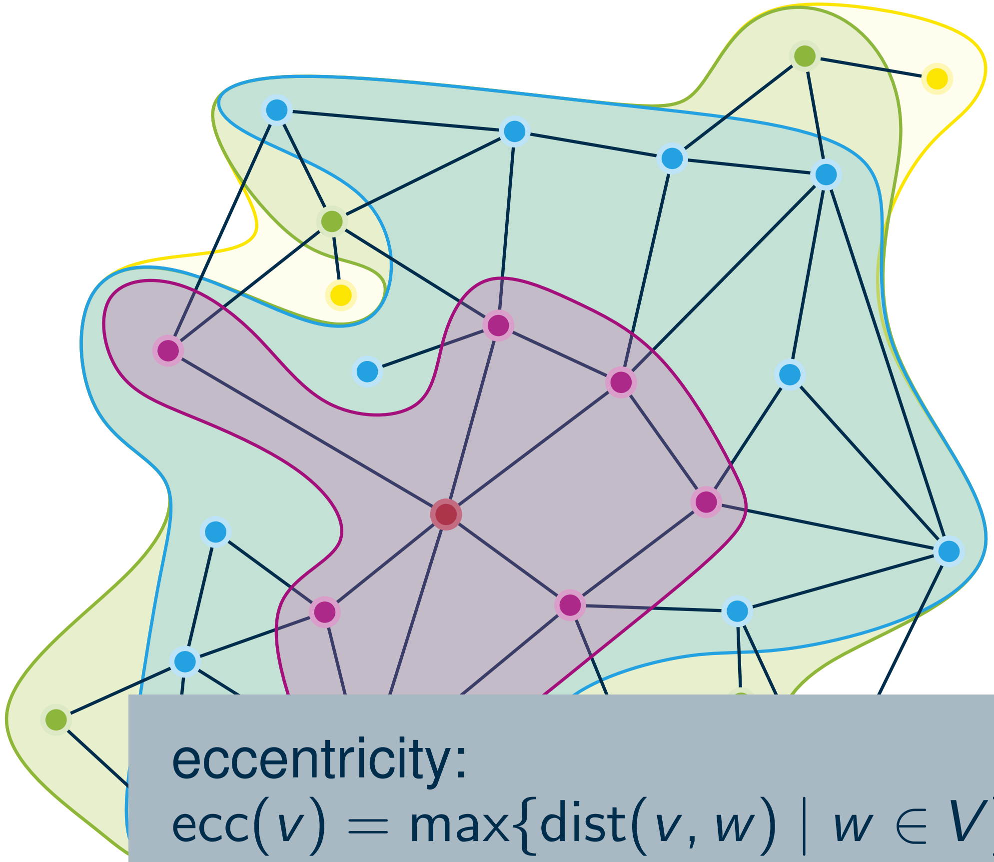
- naive: n times BFS



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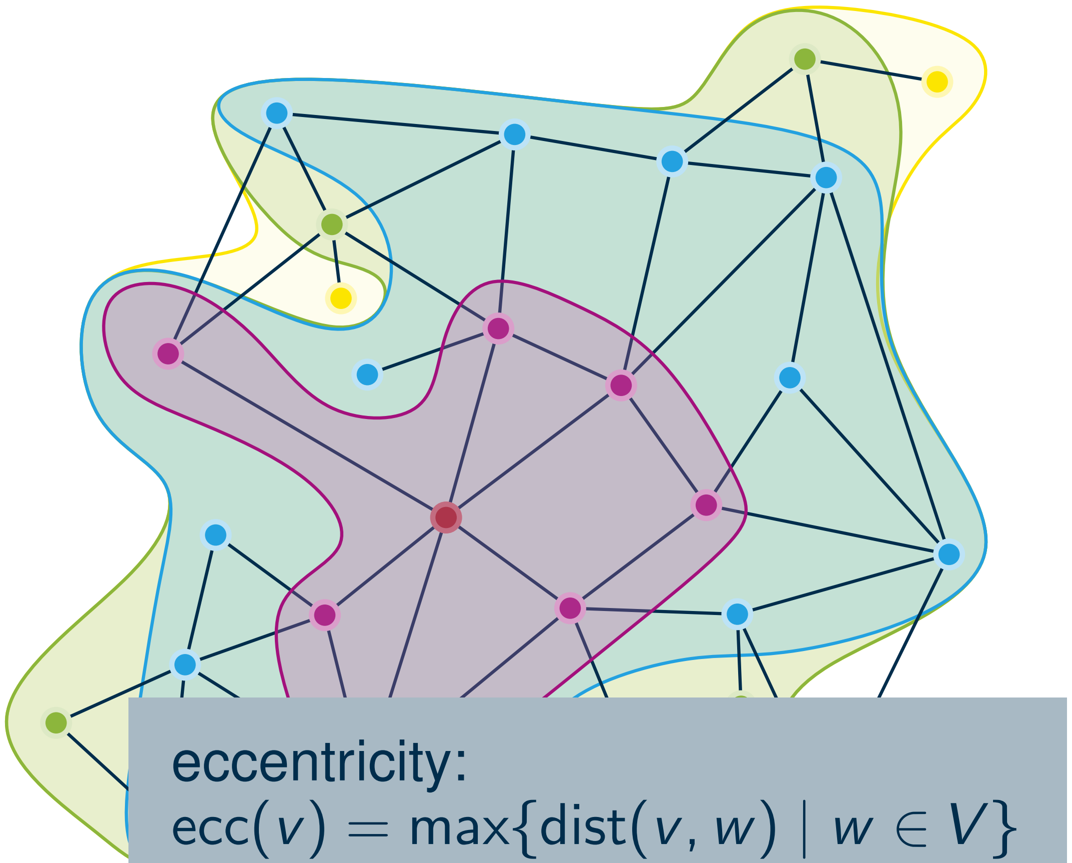
- naive: n times BFS
- can we do better?



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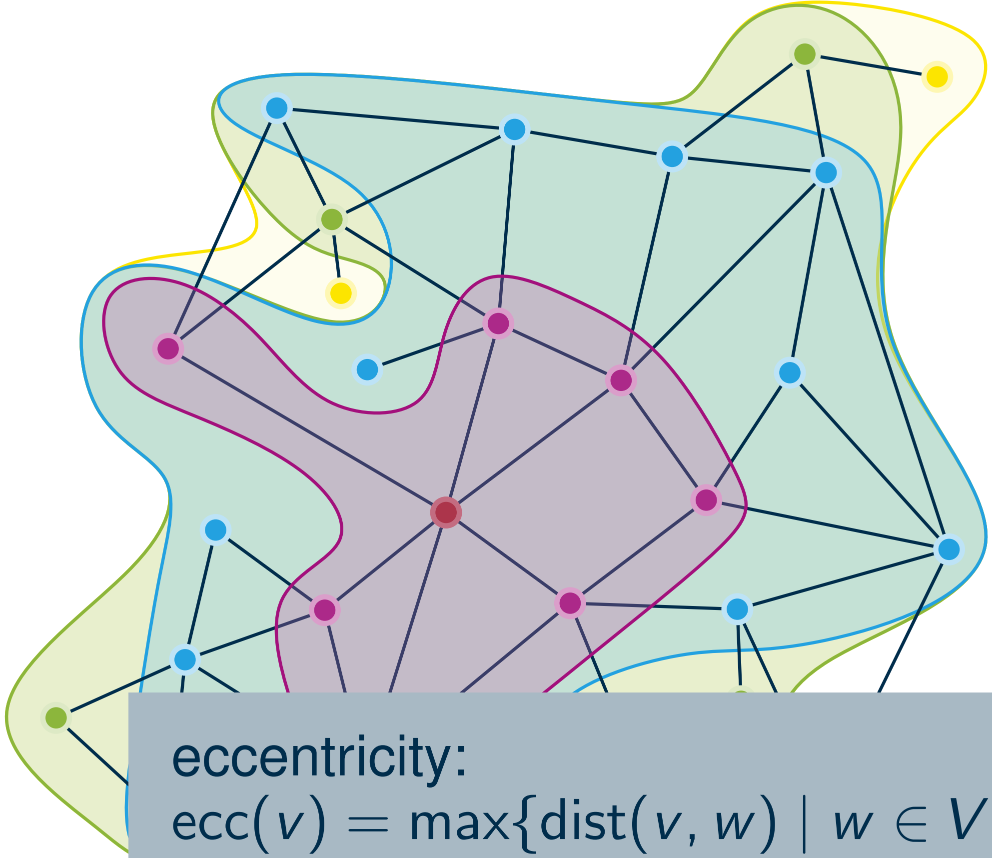
Eccentricity and Diameter

How to compute the diameter of a graph?

- naive: n times BFS
- can we do better?
  - no: (probably)

> “Any $O(m^{2-\varepsilon})$ time algorithm that distinguishes whether the diameter of a given undirected unweighted graph is 2 or at least 3 would imply an improved CNF-SAT algorithm. [Liam, Williams – STOC’13] ”



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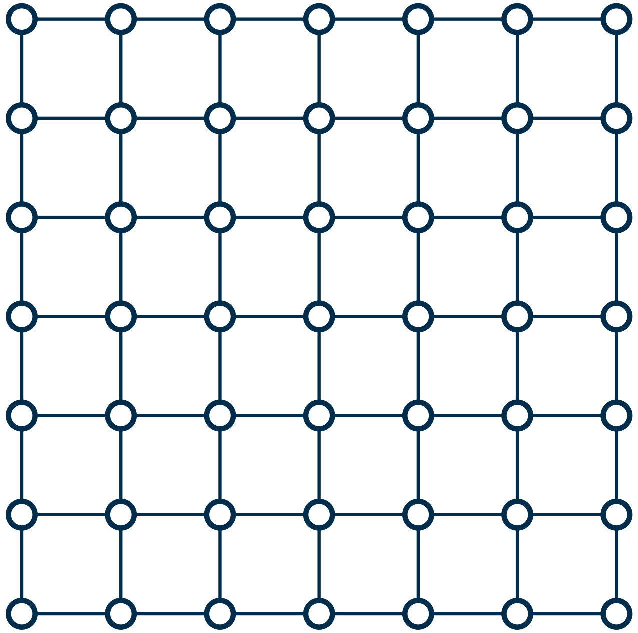
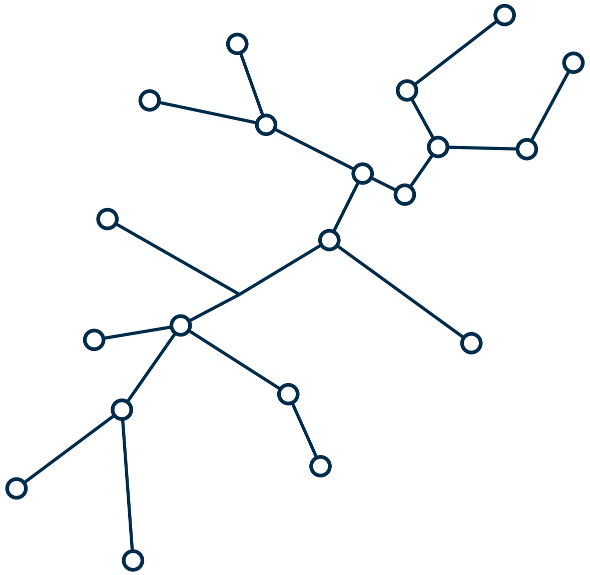
Easy inputs?





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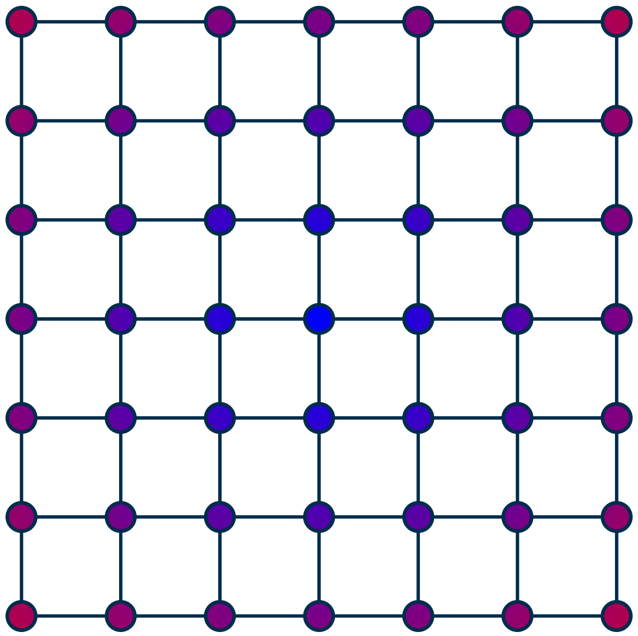

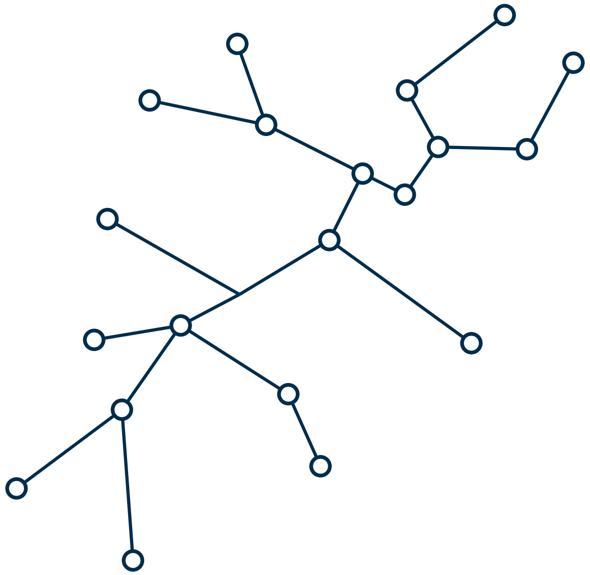

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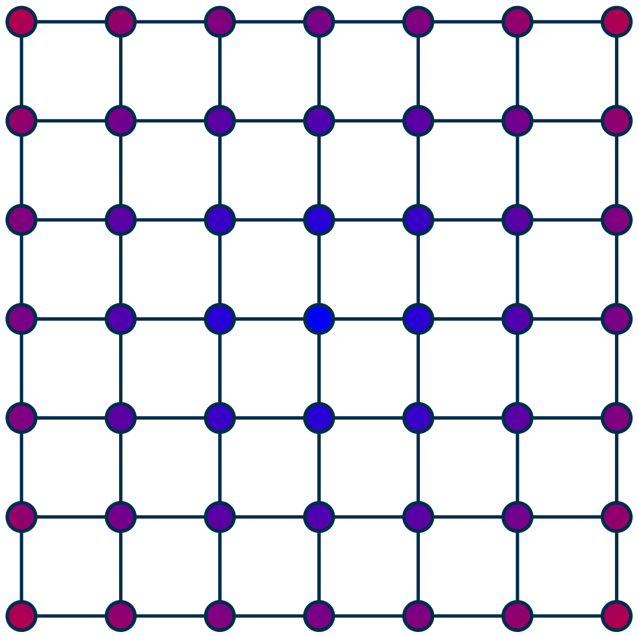

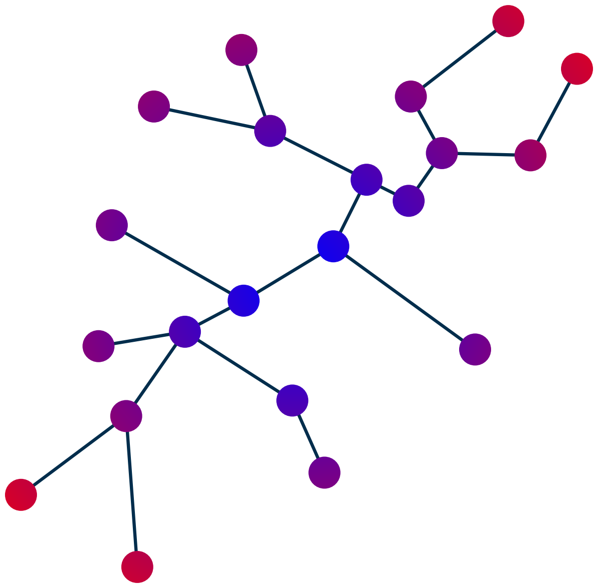

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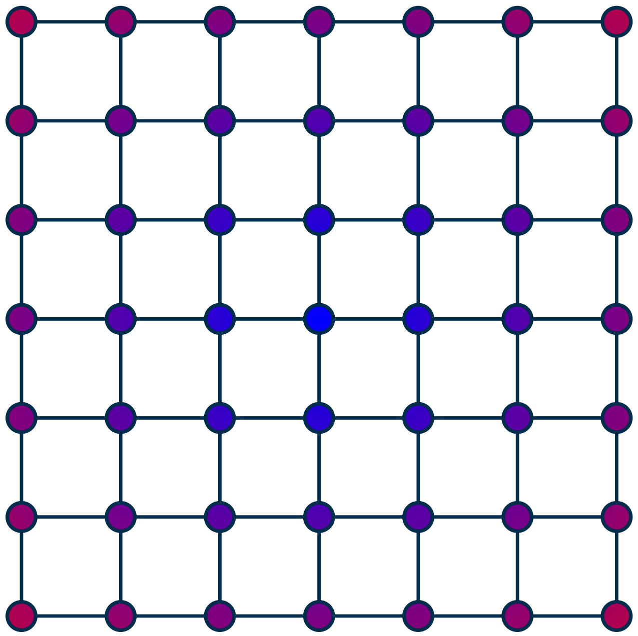

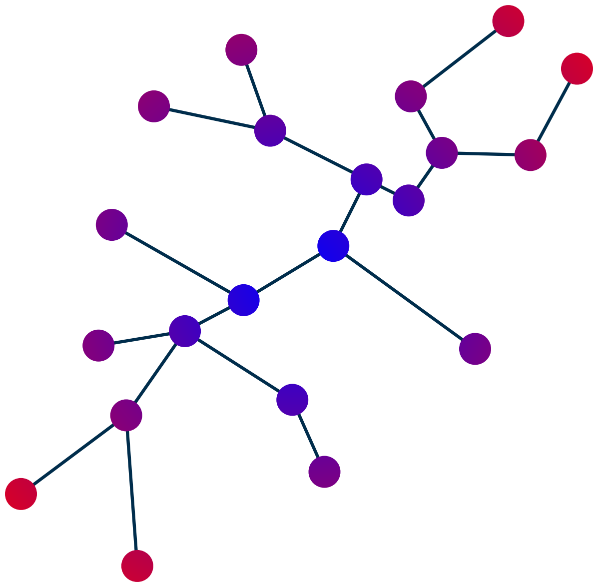

Idea: only run BFS from peripheral vertices

iFUB algorithm

(iterative fringe upper bound, [Crescenzi et al. 2013])

iFUB Algorithm

- select *central* vertex c

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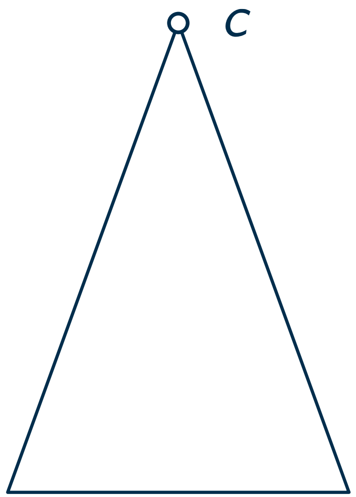

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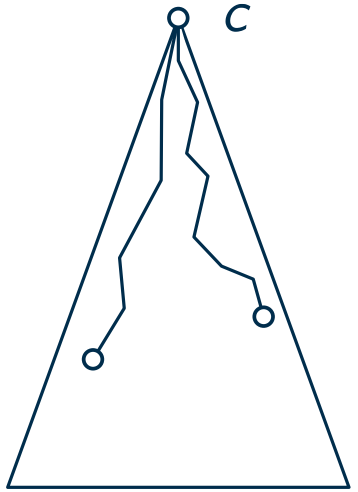

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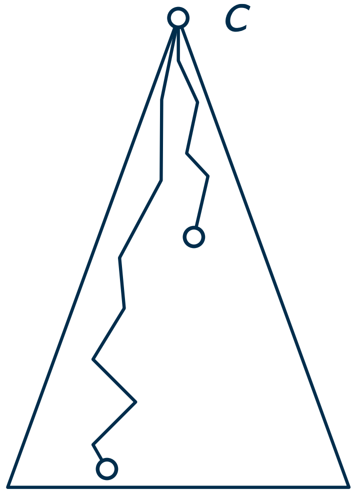

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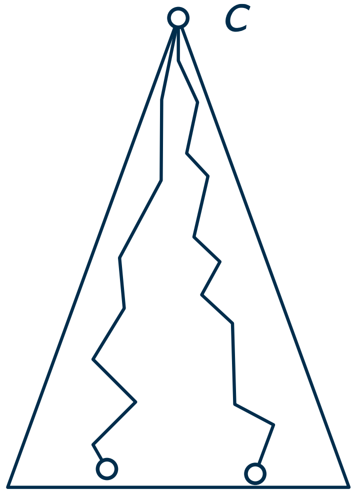

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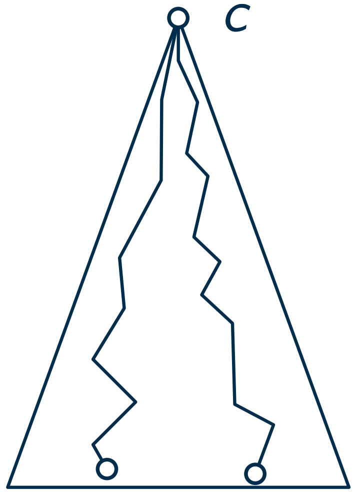
Observation: there is a diametrical vertex w with distance at least $\mathrm{diam}(G)/2$ from c.



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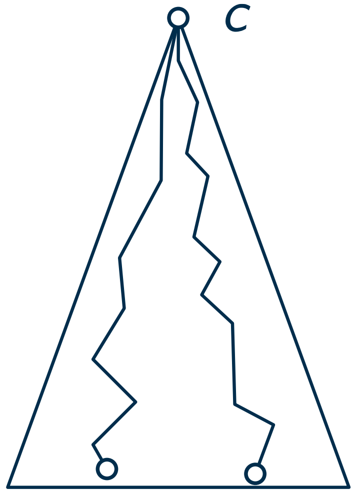

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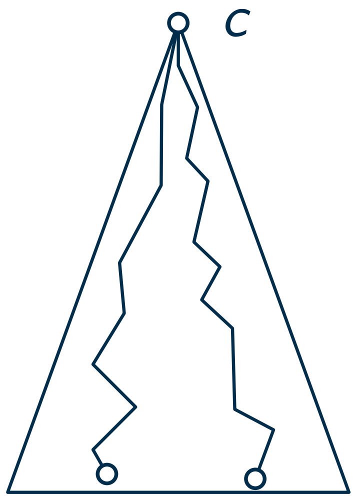

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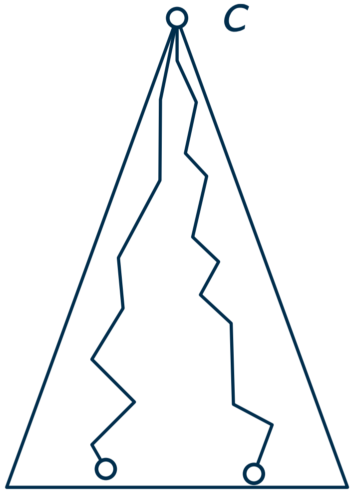

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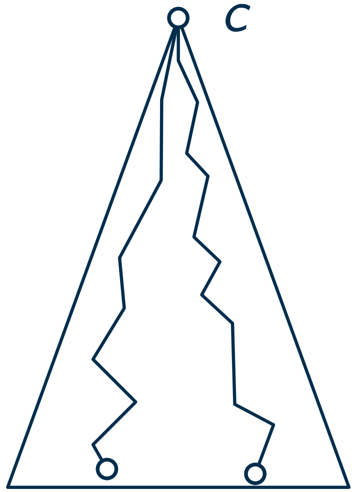

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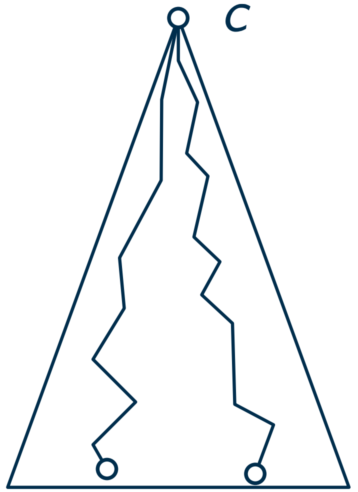

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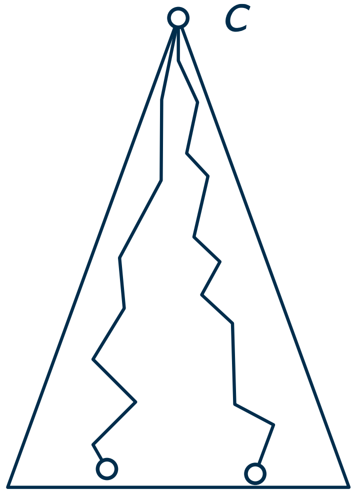

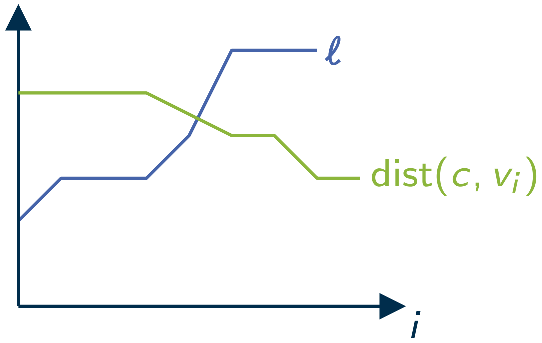

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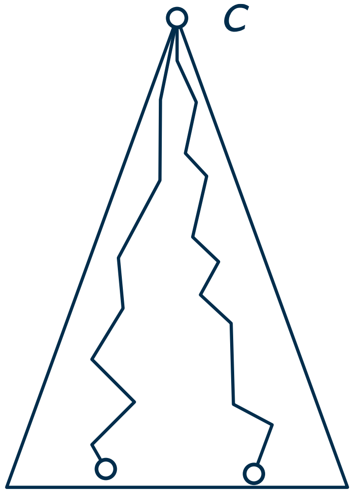

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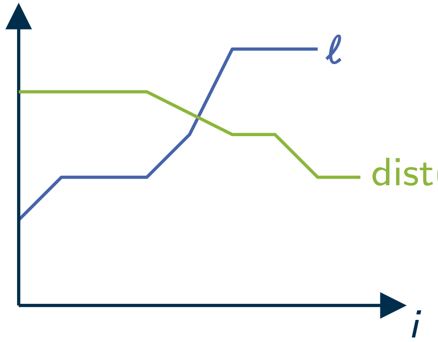

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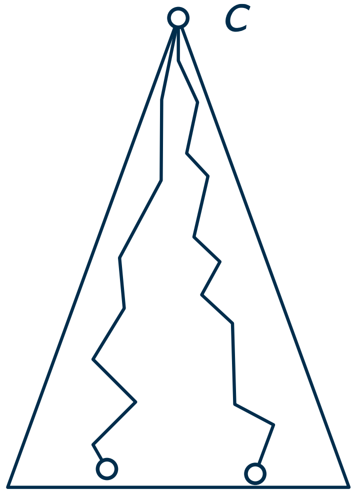

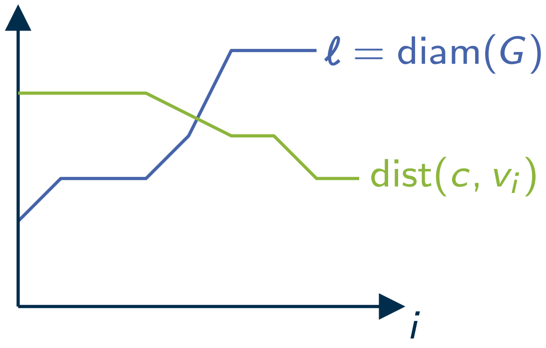

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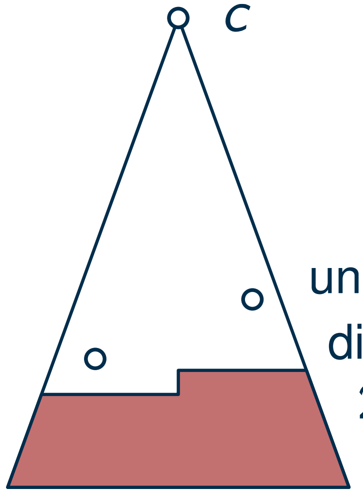

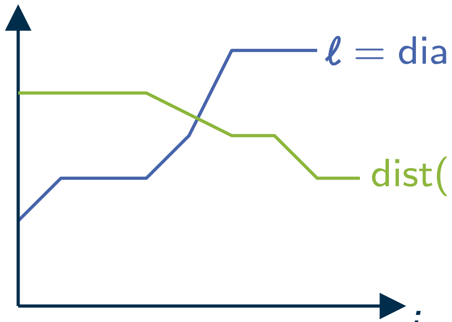

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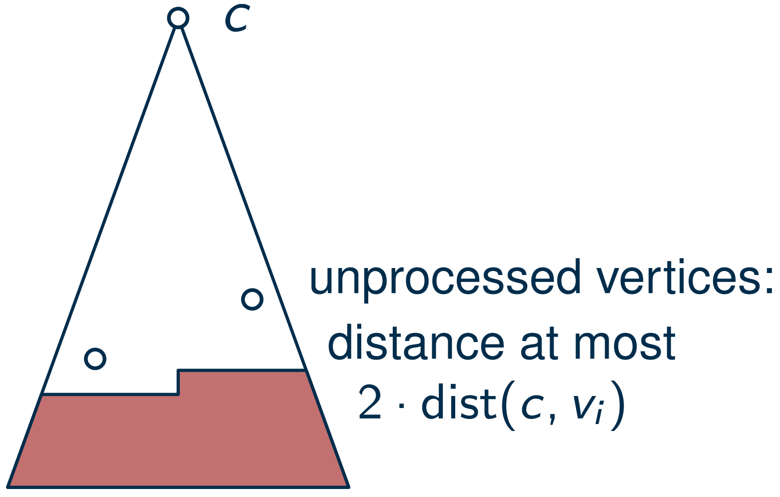
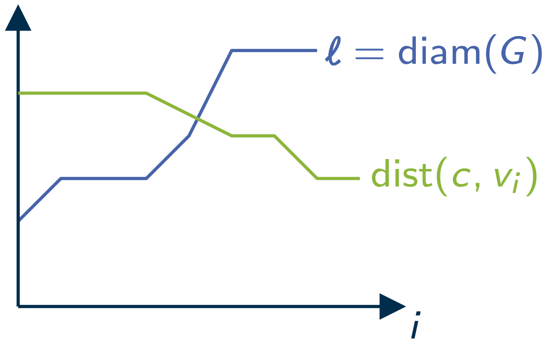


$\Rightarrow 2 \cdot \mathrm{dist}(c, v_i) \leq \ell$ already implies $\ell = \mathrm{diam}(G)$



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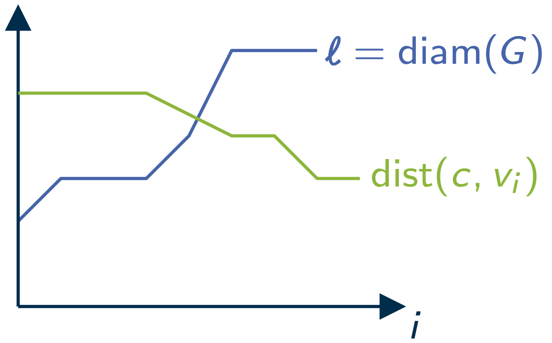

Question: How to choose c?

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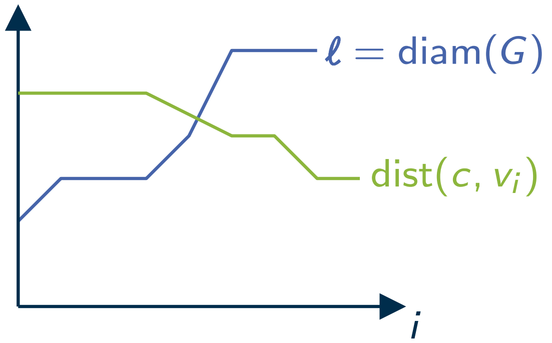

Question: How to choose c?

- heterogeneous graph: maximum degree

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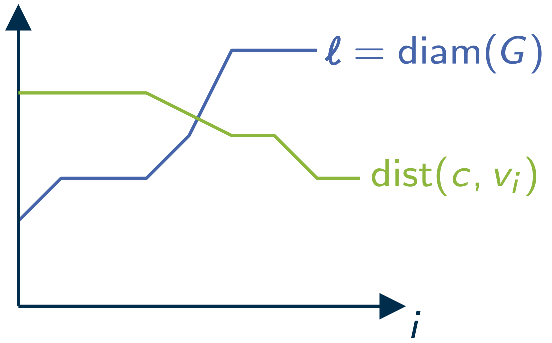

Question: How to choose c?

- heterogeneous graph: maximum degree
- otherwise: *2-sweep*

iFUB Algorithm

- select *central* vertex c

Observation: there is a diametrical vertex w with distance at least $\mathrm{diam}(G)/2$ from c.

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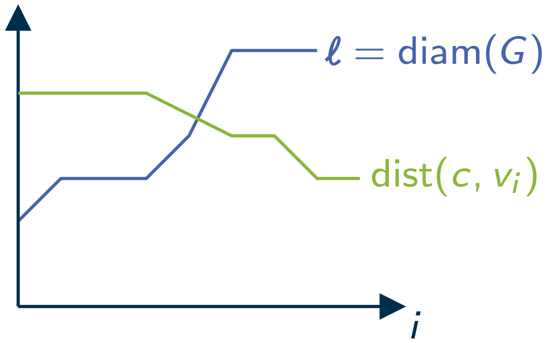

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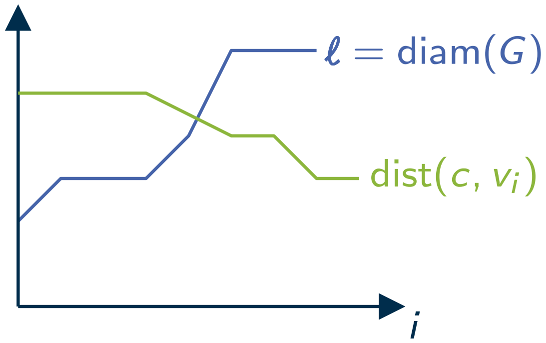

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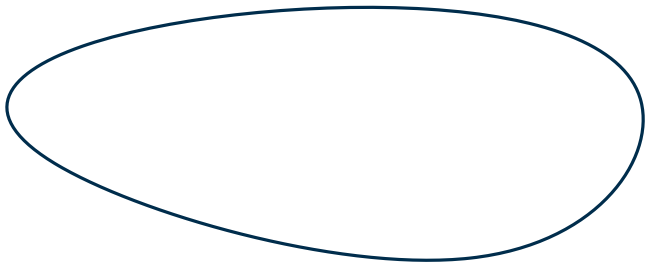

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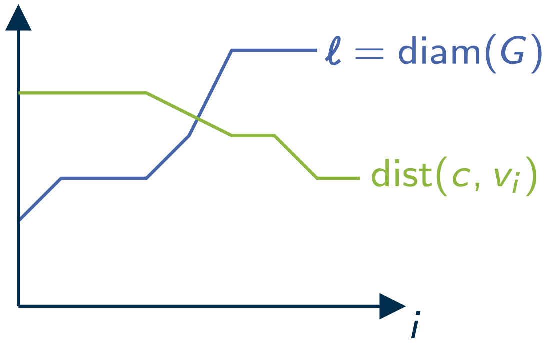

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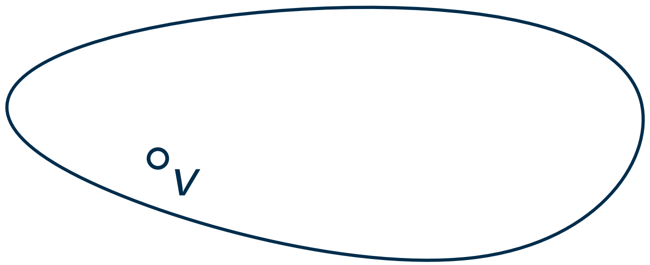

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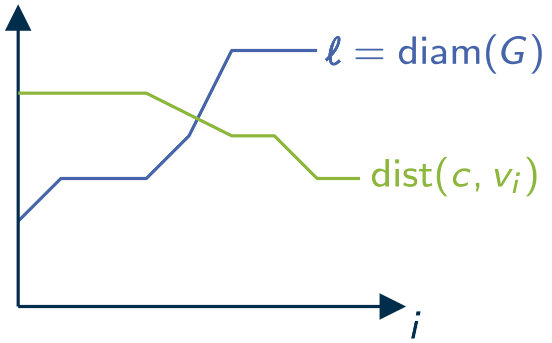

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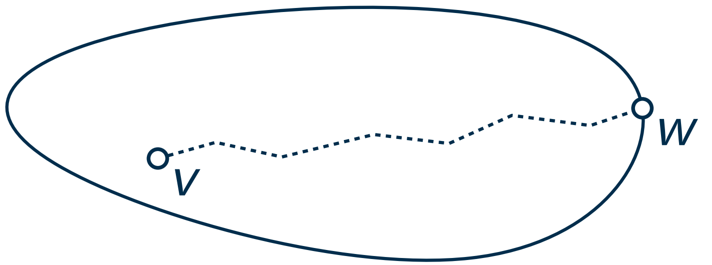

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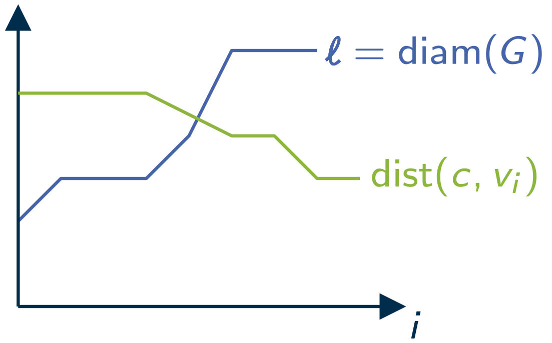

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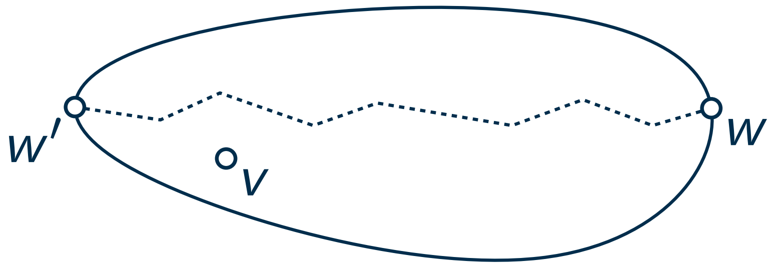

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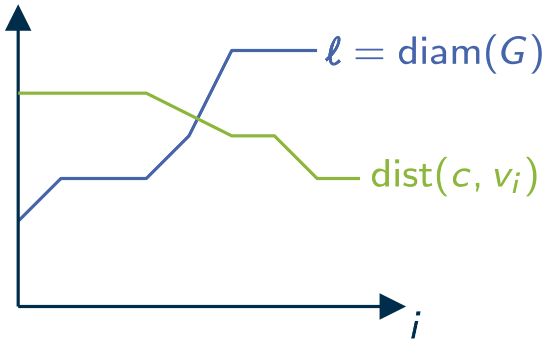

Question: How to choose c?

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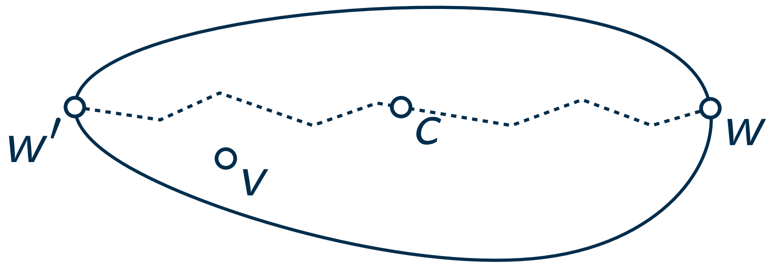
  - choose c in middle of shortest path between w and w'



Exercise Sheet 4 & Project

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- Time frame: only *one* week

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Afterwards: Project (6 weeks)

- Goal: investigate and answer a research question
- Inspiration:
  - On the External Validity of Average-Case Analyses of Graph Algorithms [B., F. 2022]
  - Deterministic Performance Guarantees for Bidirectional BFS on Real-World Networks [B., W. 2022]
  - Understand / improve algorithms (e.g., diameter, hard problems, . . .)