Exercise Sheet 4

Submission due by 2025-07-03

Problem 1: Mission Impossible **5 points**

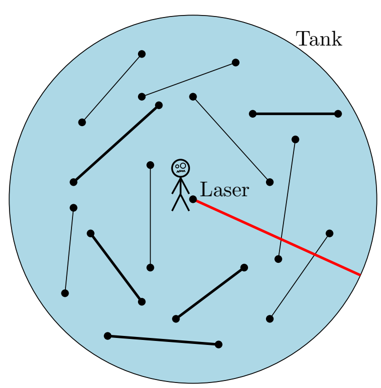
Agent Kim is in a dire situation. Her enemy, Dr. Meta, has trapped her inside a cylindrical tank filled entirely with water. Luckily, she has a laser (disguised as a whiteboard marker) that could break through the tank wall and save her.

Unfortunately, Kim is surrounded by n glass panes, each of which reduces the laser's power depending on its strength.

In a panic, Kim rotates the laser to a randomly chosen angle every $O(\log(n) + k)$ seconds, where k is the number of glass panes the laser would hit in that direction. After each reorientation, Kim must decide whether to activate the laser. Since a laser in the form of a whiteboard marker only has a very small battery, she has only one chance to fire the laser in the correct direction.

Fortunately, Agent Kim had $O(n \log n)$ preprocessing time and $O(n)$ space, during which she could study the positions of the glass panes — but not their strengths! For any given angle, she must now determine the total strength of the panes the laser would intersect in order to decide whether the beam could reach the tank wall.

How did Kim make use of her preprocessing time? And how can she now quickly determine which k panes the laser would intersect in a given direction before panic sets in and she rotates to the next direction?



Problem 2: One-and-a-half Range Queries **5 points**

Given are n points in the plane. A one-and-a-half range query is defined by $x_1, x_2, y \in \mathbb{R}$ and asks for the subset of points contained in the rectangle $[x_1, x_2] \times [y, \infty)$.

Provide an algorithm that, after $O(n \log(n))$ preprocessing time and using $O(n)$ space, can answer one-and-a-half range queries in $O(\log(n) + k)$ time, where k is the number of points contained in the queried range.

Note: Only $O(n)$ space is allowed!

Problem 3: Size of a Voronoi Diagram **3 points**

Given are $n \geq 3$ points in the plane. Prove that the number of vertices in the corresponding Voronoi diagram is at most $2n - 5$ and that the number of edges is at most $3n - 6$.

Problem 4: Lower Bound: Voronoi Diagram **2 points**

Prove that the Voronoi diagram of n points in the plane cannot be computed faster than $\Omega(n \log(n))$, unless n numbers can be sorted faster than $\Omega(n \log(n))$.

Problem 5: Fog **5 points**

To avoid collisions in poor visibility, the captains of n ships (each modeled as a point in the plane) need to determine which other ship is closest to them (if multiple ships are equally close, identifying any one of them is sufficient). Provide and algorithm that computes this information for all ships in $O(n \log(n))$ time.