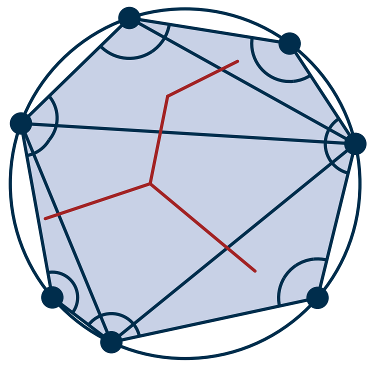
Computational Geometry

Exercise 6 *Assignment 5, 6 and Greedy Routing in Hyperbolic Geometry*

Jean-Pierre, Marcus, Wendy

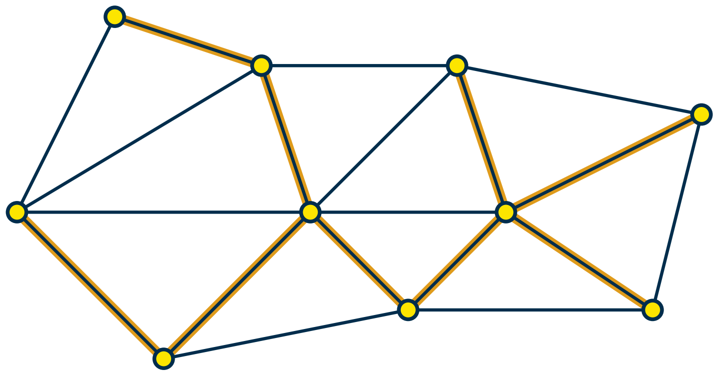
Assignment 5

Triangulation of Co-Circular Points



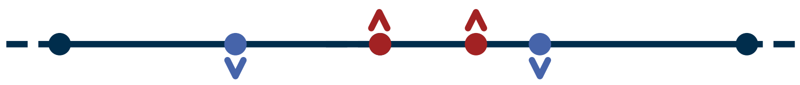
- Find optimal triangulation (smallest angle vector)

MST ⊆ Delaunay



- Prove that an MST is a subset of the Delaunay triangulation

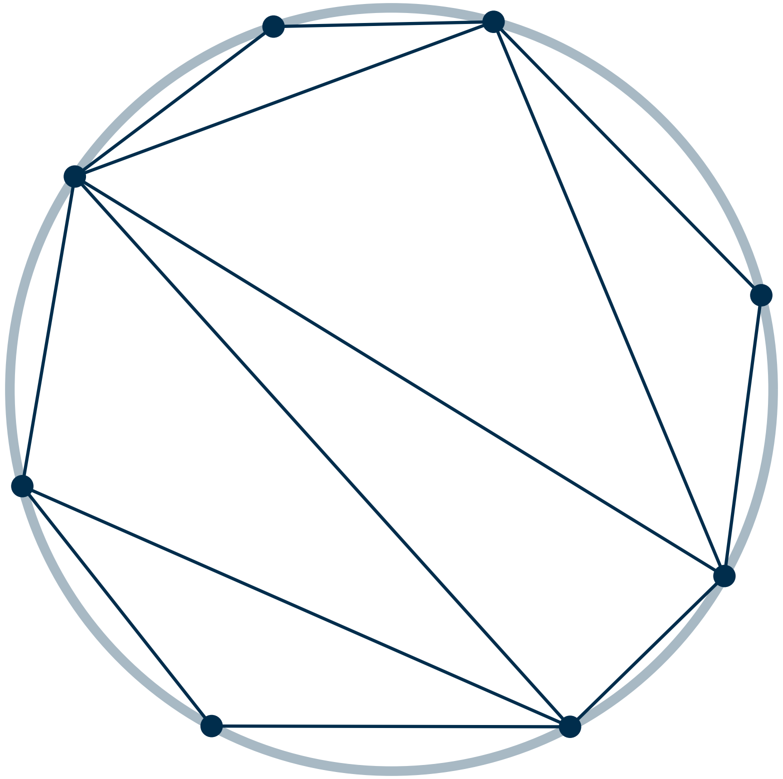
Foldability of Mountain/Valley Patterns



- test foldability in $O(n)$

Triangulation of Co-Circular Points

Vector of angles is optimized ⇔ Vector of lengths is optimized



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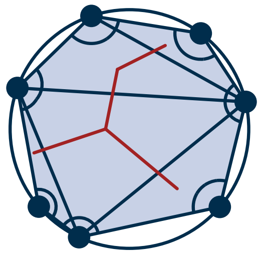
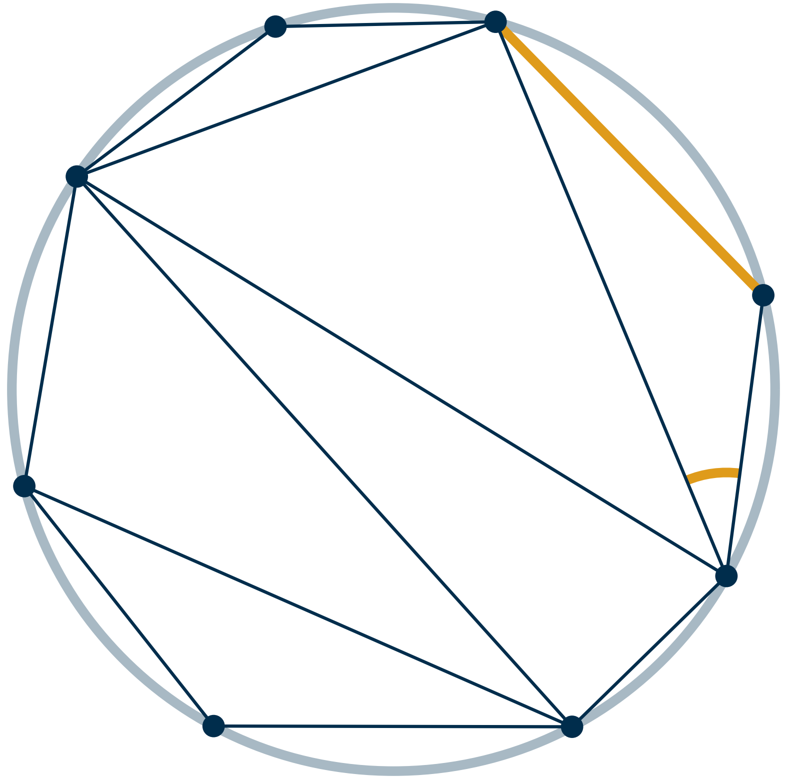


Vector of angles is optimized ⇔ Vector of lengths is optimized

- Every polygon edge has one opposite angle



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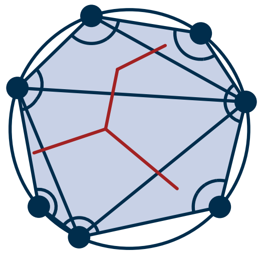

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- Every polygon edge has one opposite angle
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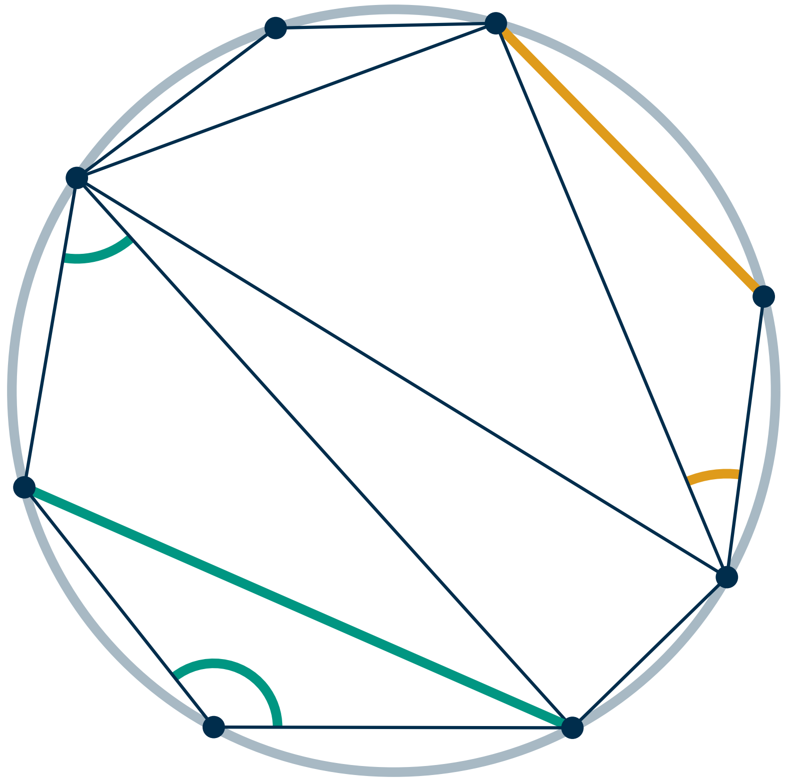

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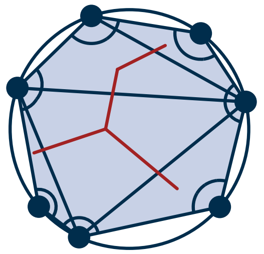

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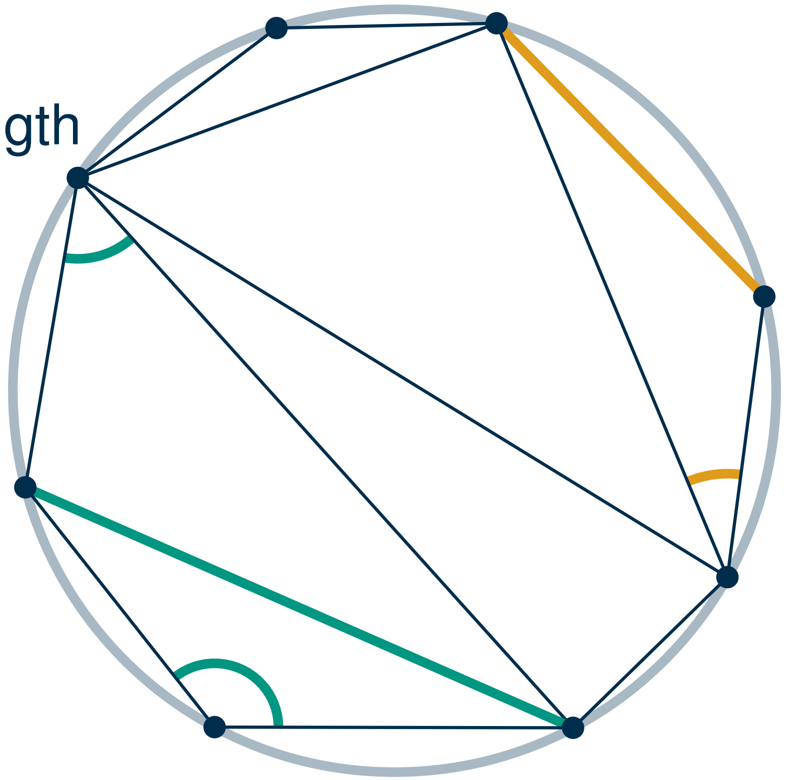

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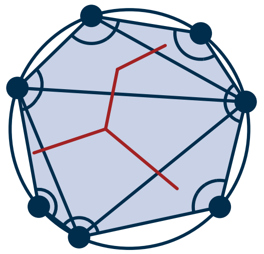

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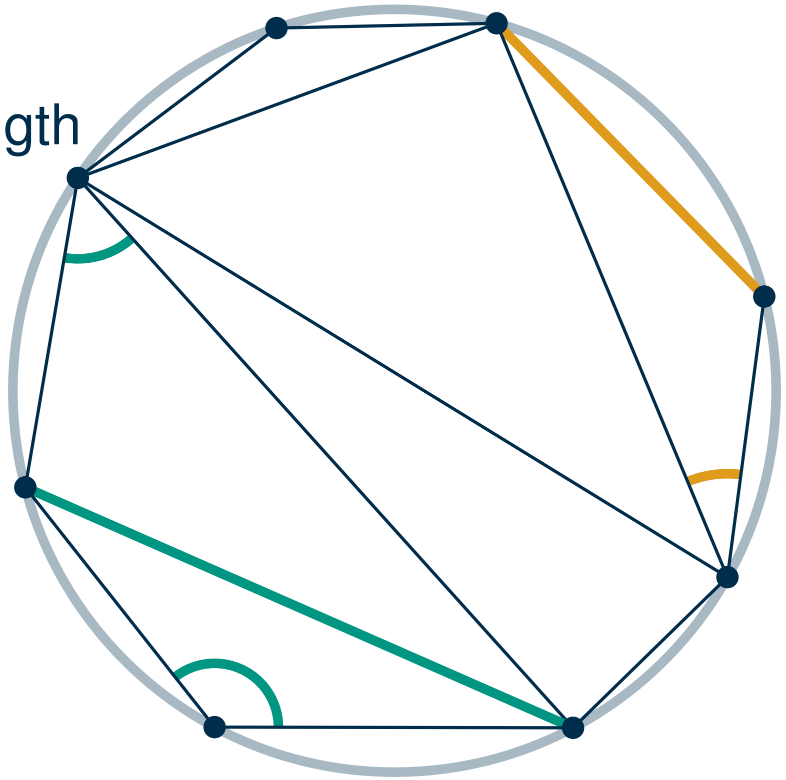

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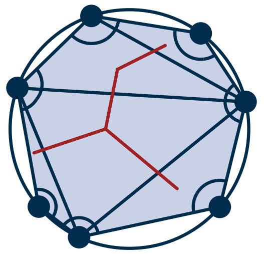

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- Polygon edges are irrelevant.
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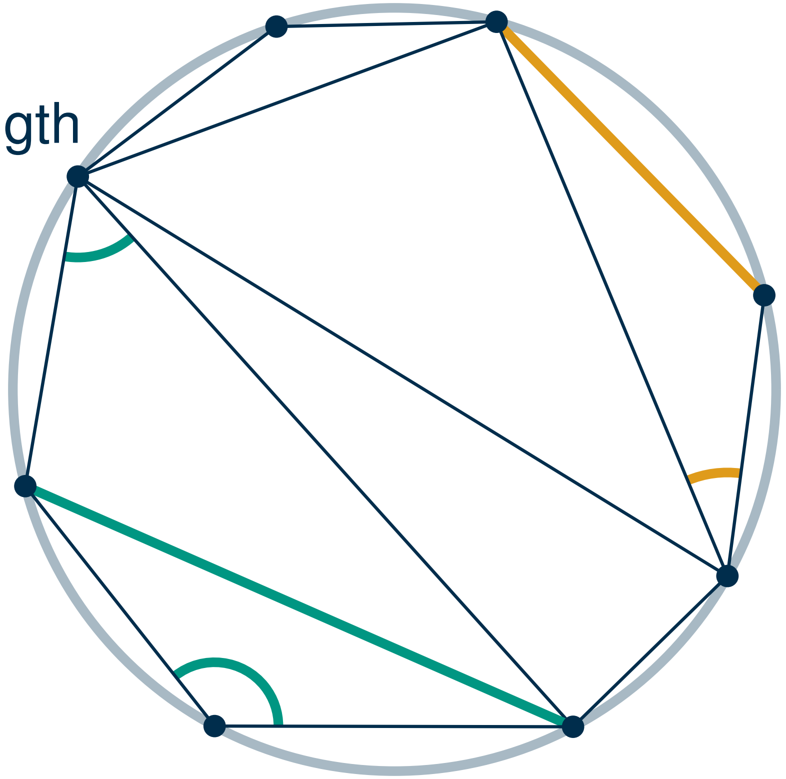

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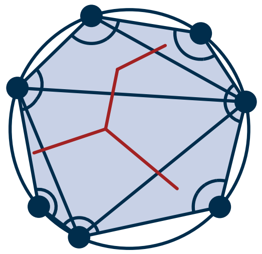

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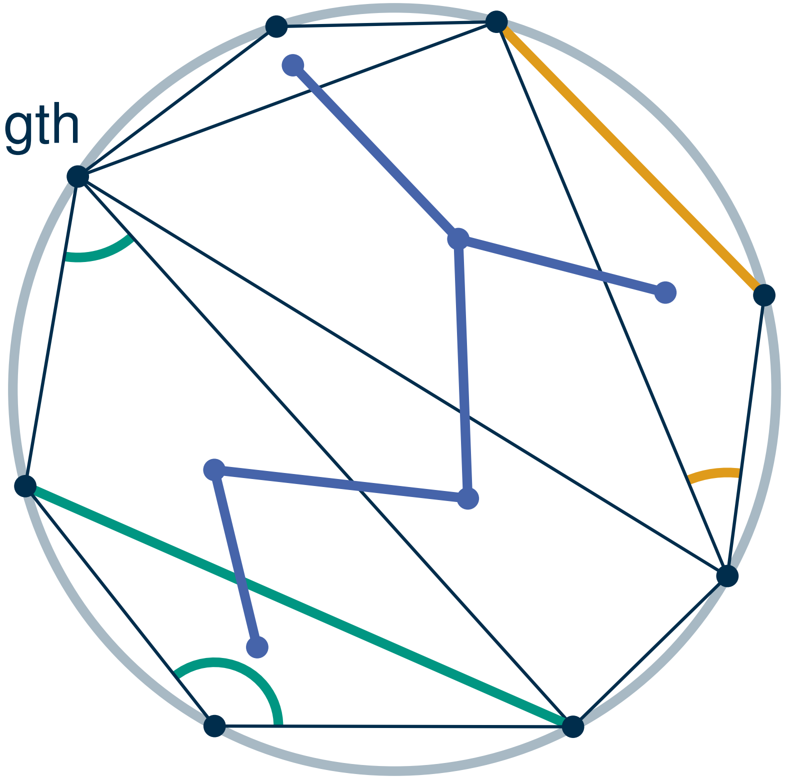

Weak dual is path

- Triangluation has $n-2$ triangles and $n-3$ cords $\Rightarrow$ Tree

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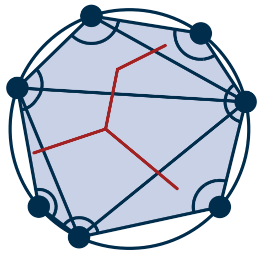

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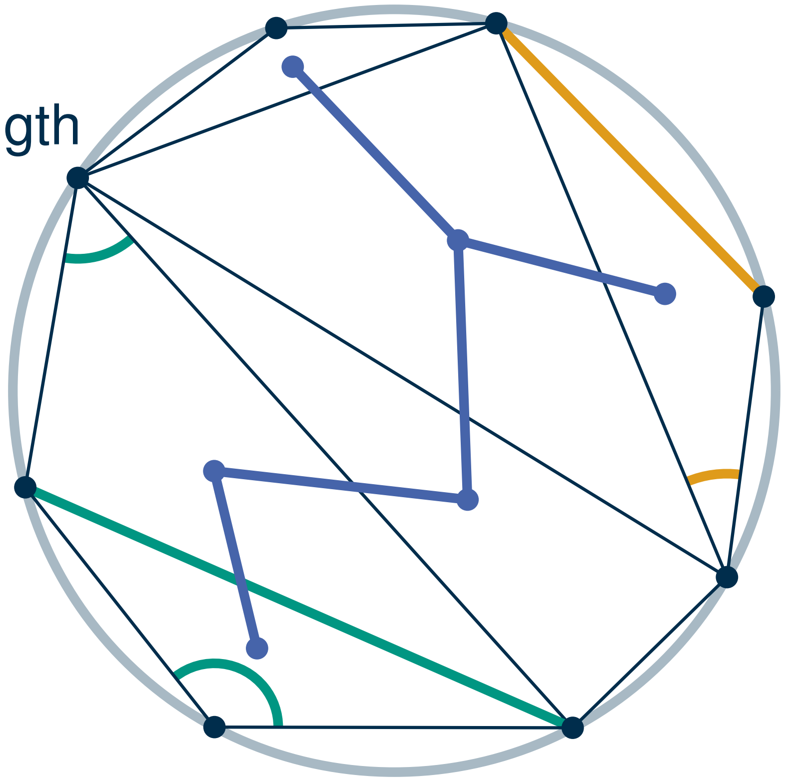

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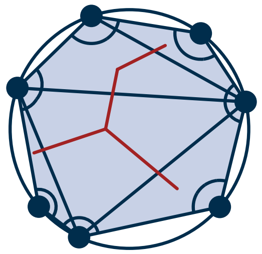

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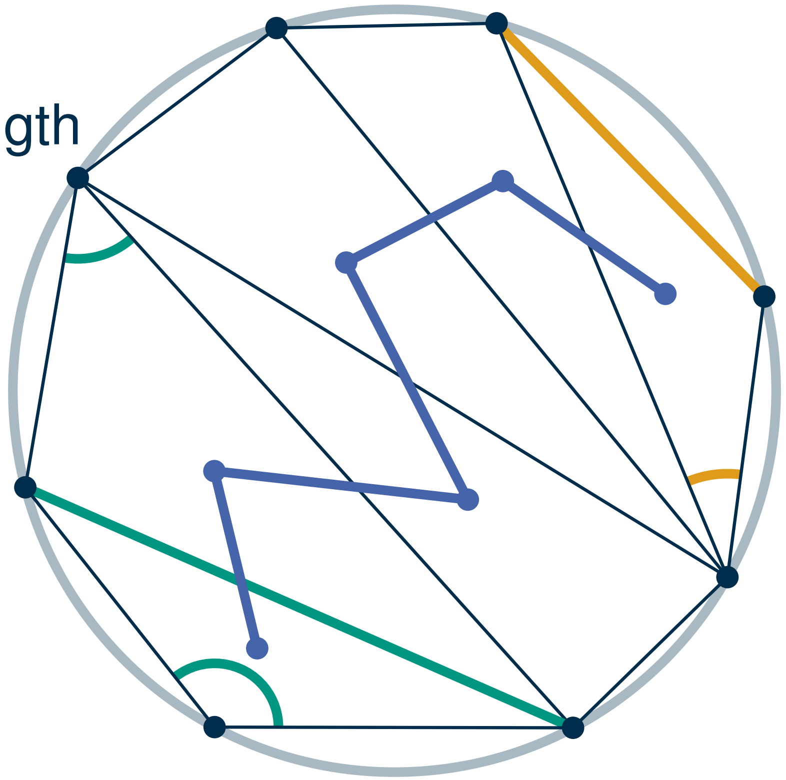

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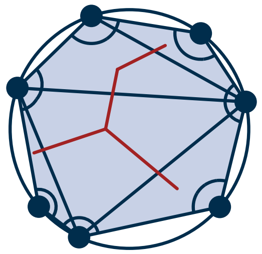

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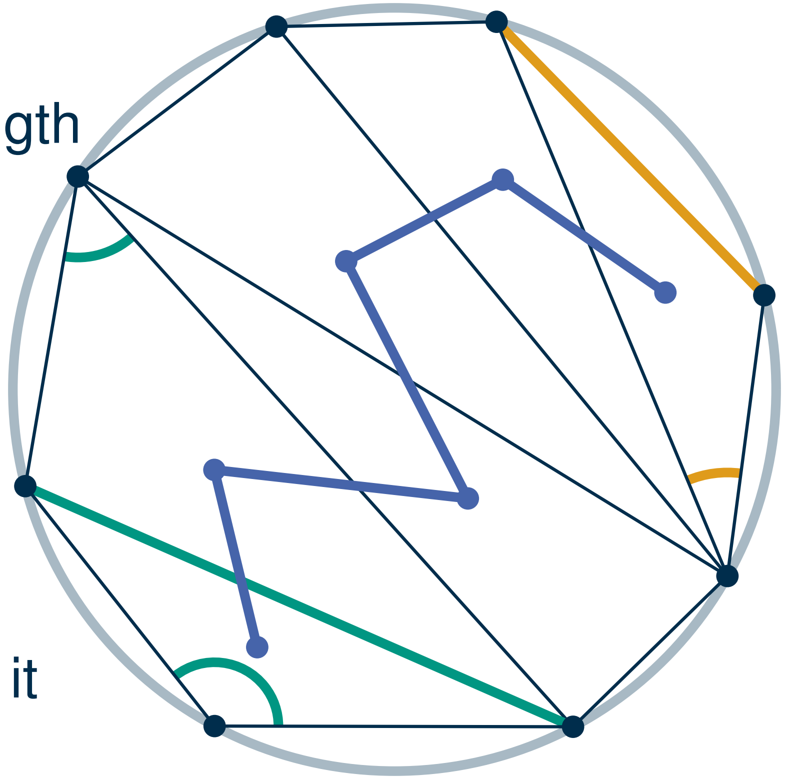

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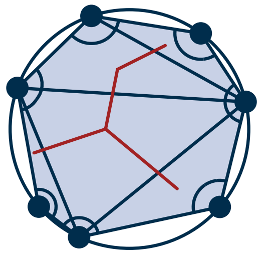

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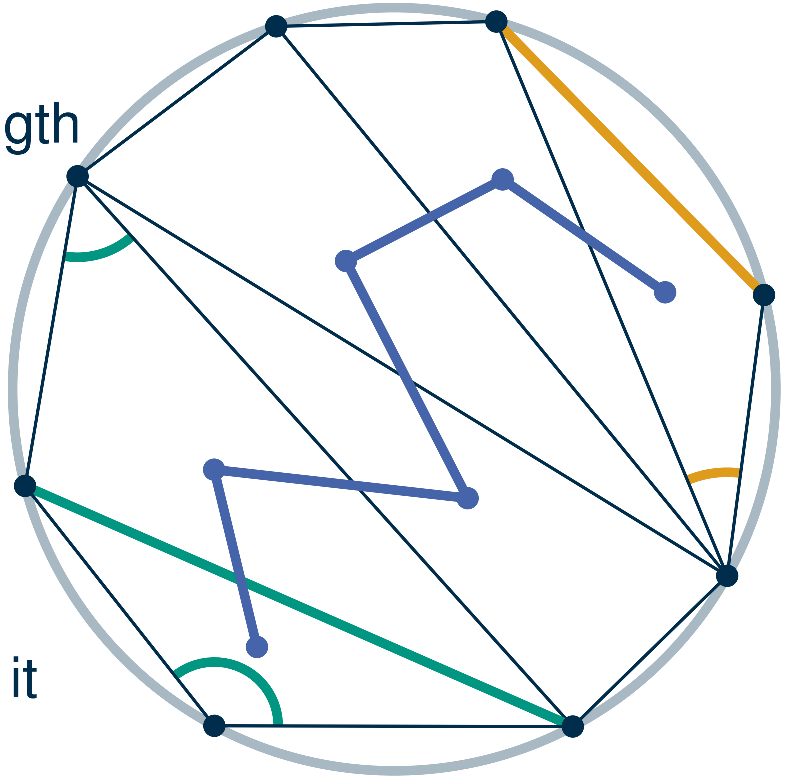
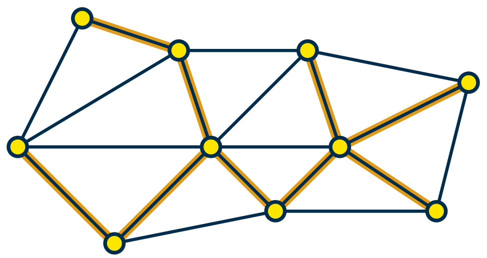


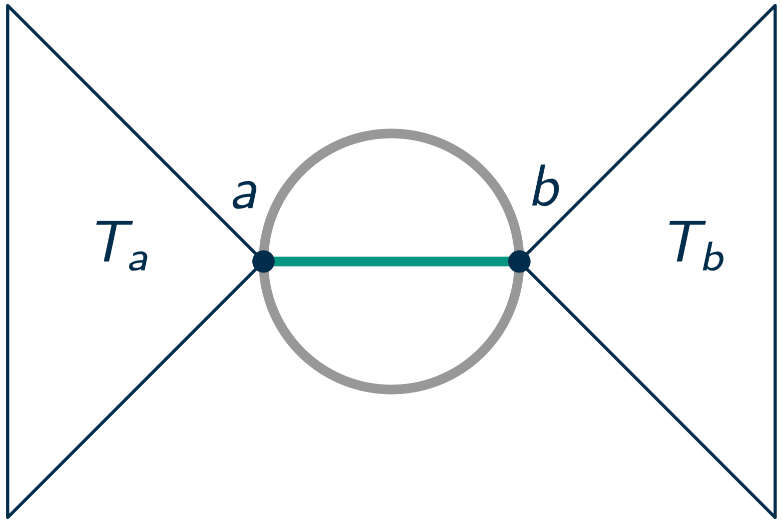
Weak dual is path

- Triangluation has $n - 2$ triangles and $n - 3$ cords $\Rightarrow$ Tree
- Vertex with degree 3 $\Rightarrow$ find larger internal edges
- If the starting points of the path are know, it is easy to extend it
- The starting points will be two out of the three largest ears
  - an *ear* is an internal edge, that touches two polygon edges

MST $\subseteq$ Delaunay



- Let ab be an edge of the MST, consider the smallest circle touching a and b



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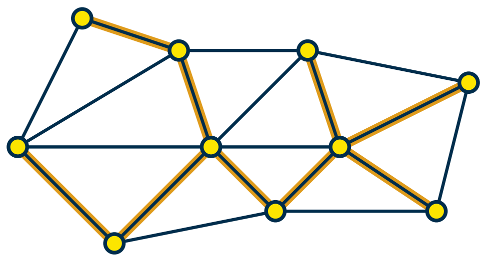

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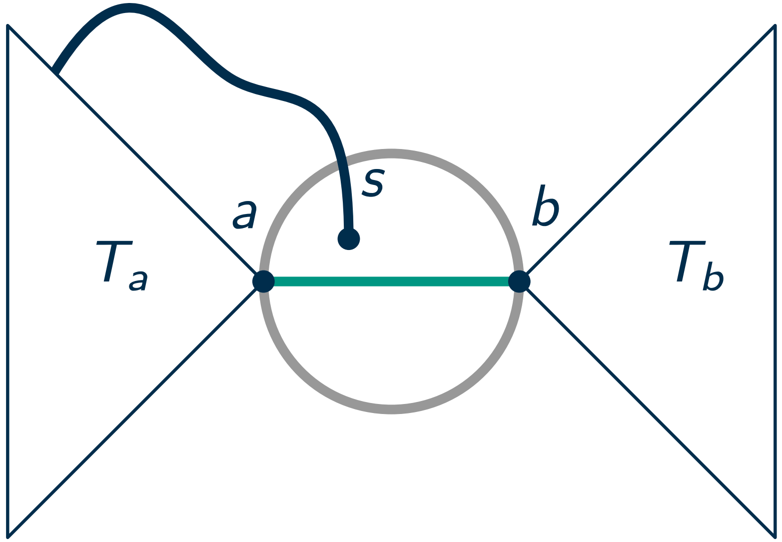
- The circle is empty, otherwise we can find a smaller MST
  - assume s is in the circle and $s \in T_a$



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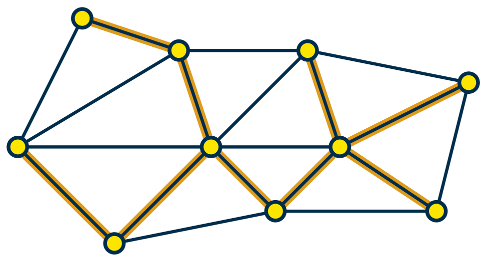

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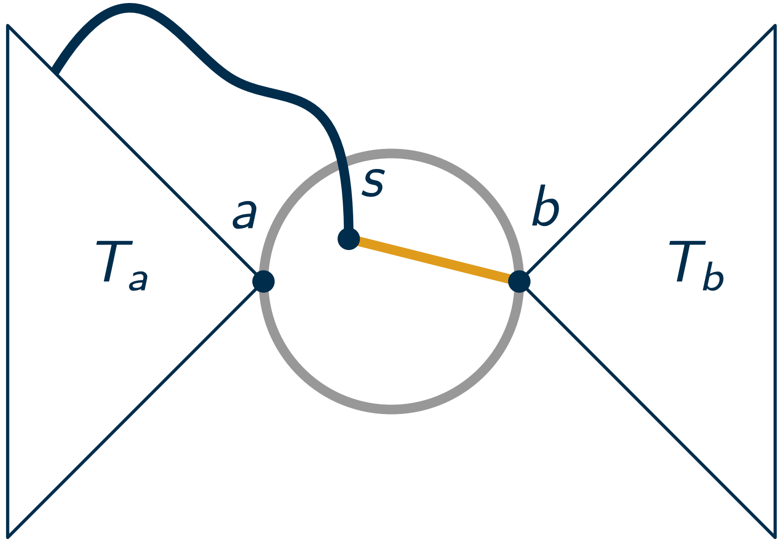

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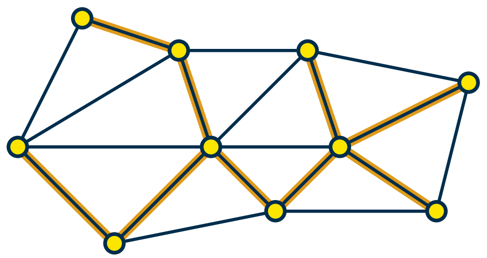

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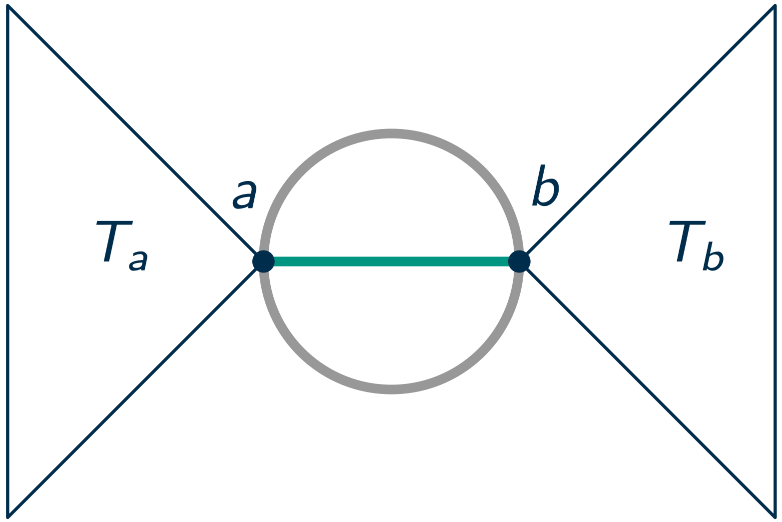
- Empty circle $\Rightarrow$ ab is in delauney triangulation
  - blow up circle until third point is hit



Your Submissions

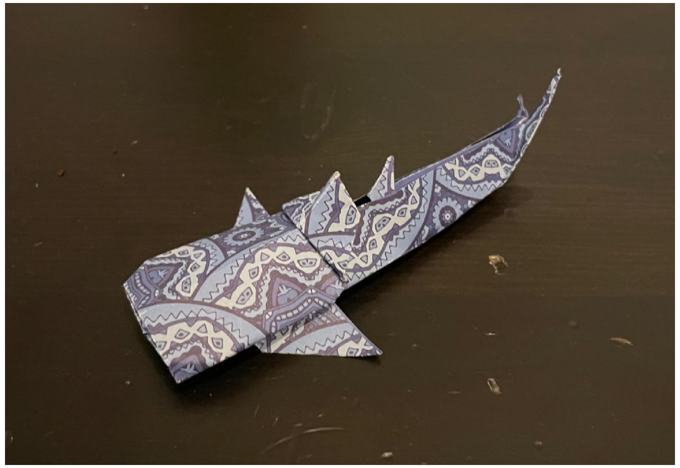
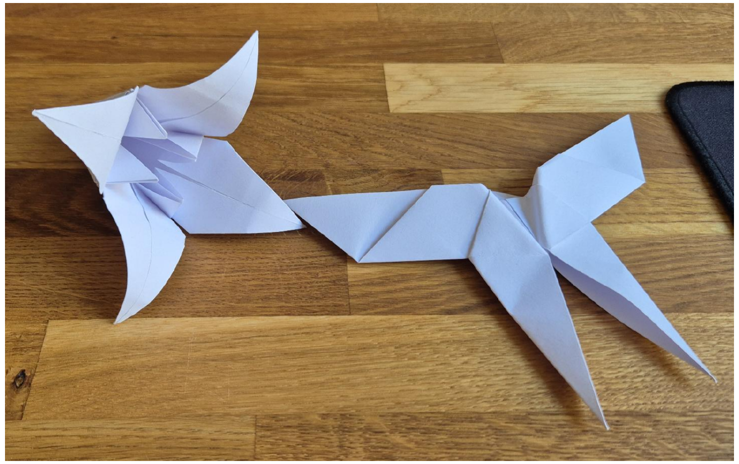
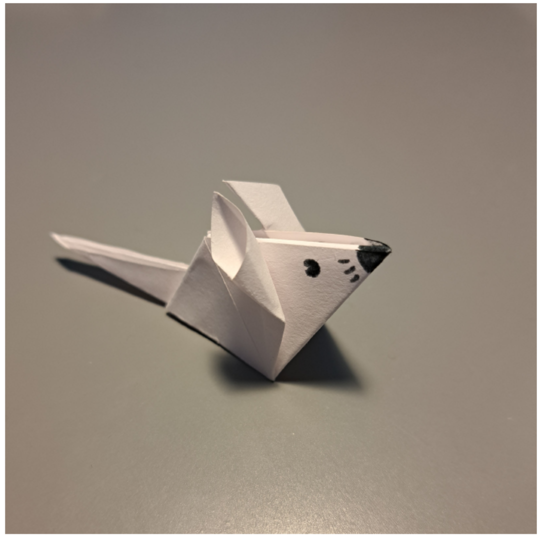


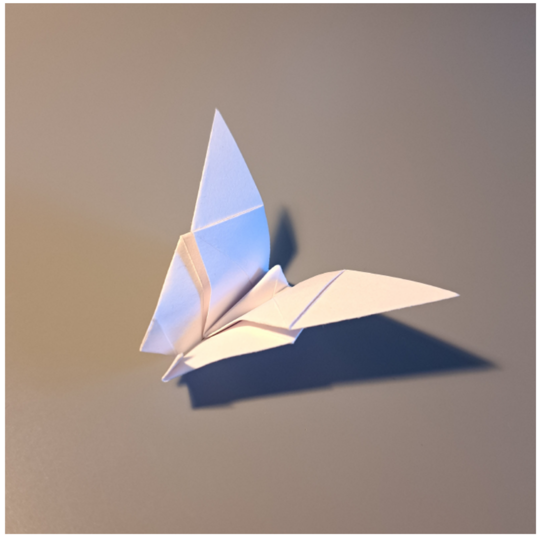
Figure 4: Even the paper pelican can hold things in its beak.

Your Submissions





(a) Mouse

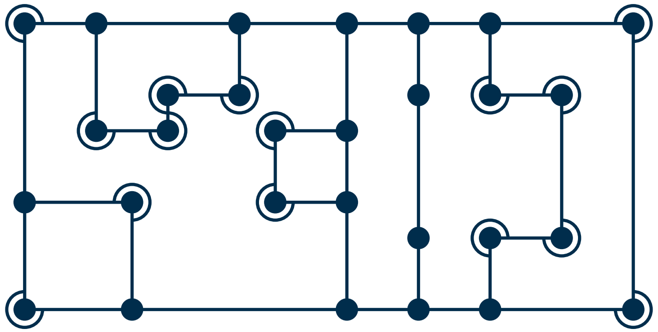


(b) Butterfly

Figure 3: Origami eines Otters

Assignment 6

Bend and Behold



- Show that it is NP-hard to decide, whether a drawing exists with few $\frac{3}{2}\pi$ angles

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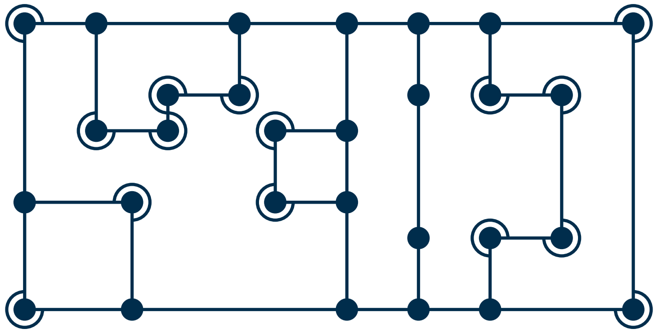

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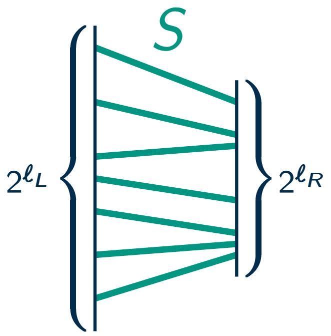
Nice (h, b)-Decomposition



- Find $B \subseteq S$ of size $\mathcal{O}(b)$
- Endpoints from elements in B are close or few segments inbetween
- *Discrete* version $\tilde{B}$ (enpoints are multiples of $2^{\ell_L - h}$)

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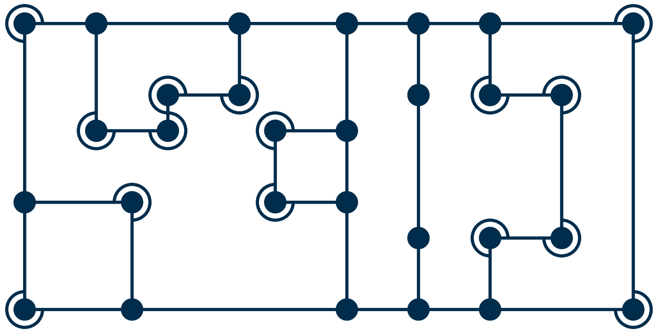

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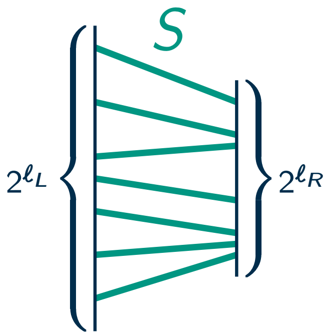

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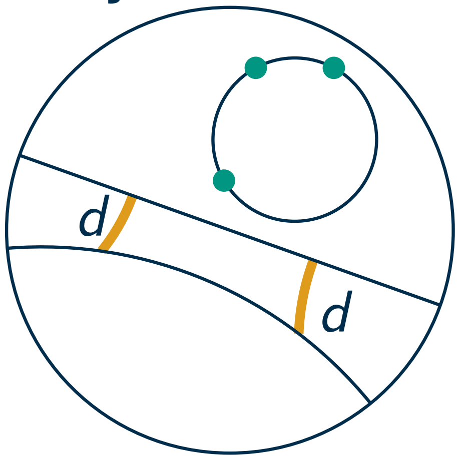
Geometry



- Draw line with distance d to other line
- How many circles lie on 3 points?
- For points A, B, C there is no point D in ABC^+ with same distance to A and B

Graph Embeddings

Many graphs have some form of hidden geometry

- Embedding: Input is a Graph; output is a position for every vertex
- Finding this geometry, is useful for various scenarios

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Road network

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Road network **obvious geometry**

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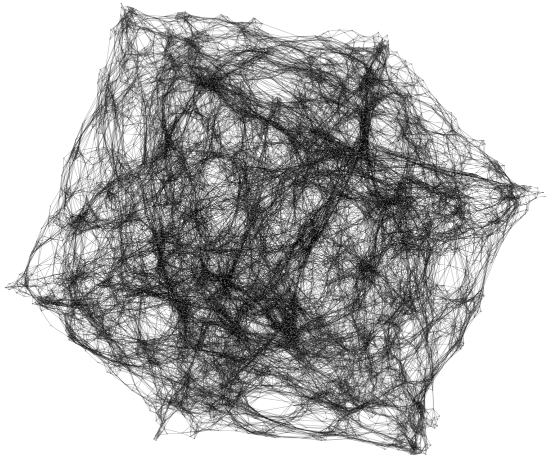
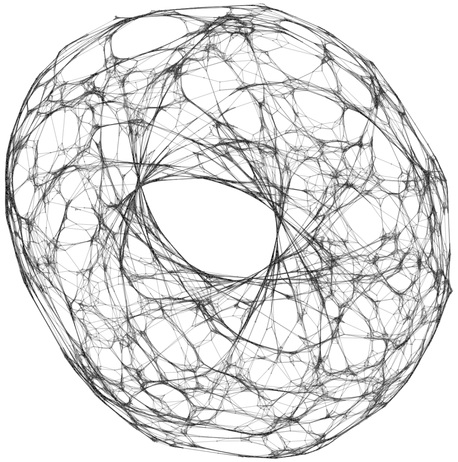
Many important questions

- What is the right embedding space/dimension?
- How can we measure the quality of an embedding?
- What problems can our embedding solve?

Some Examples

Generated Graphs





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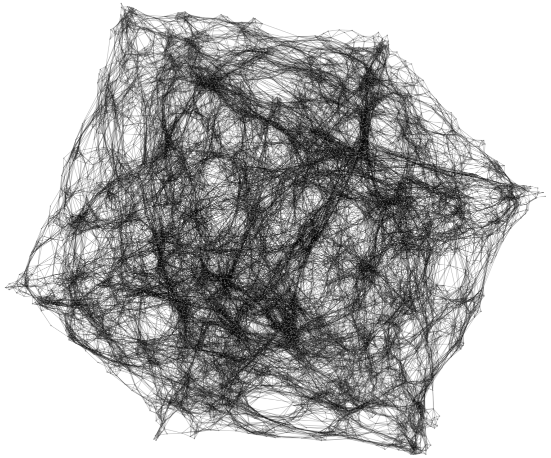
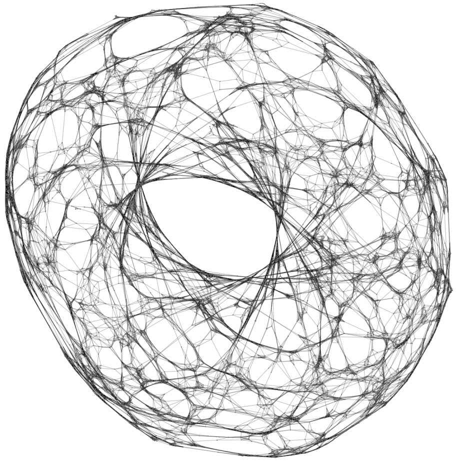
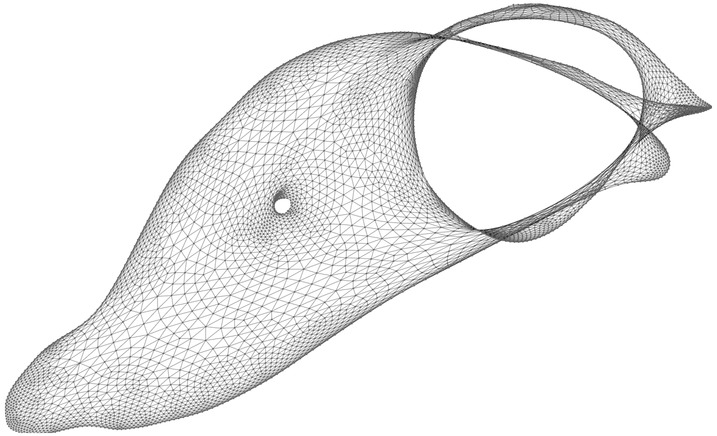
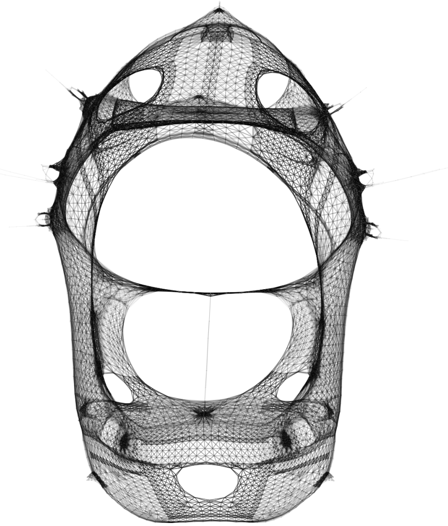




Real World Graphs





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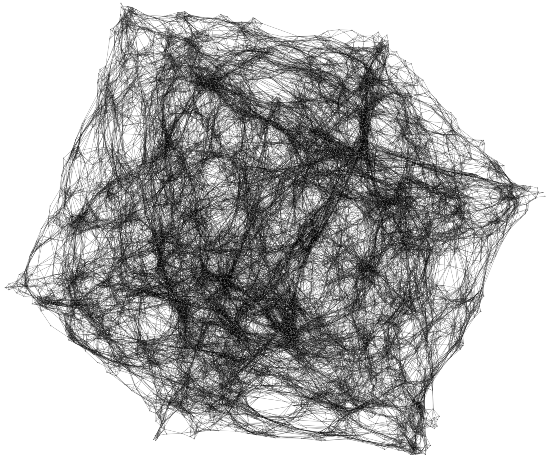
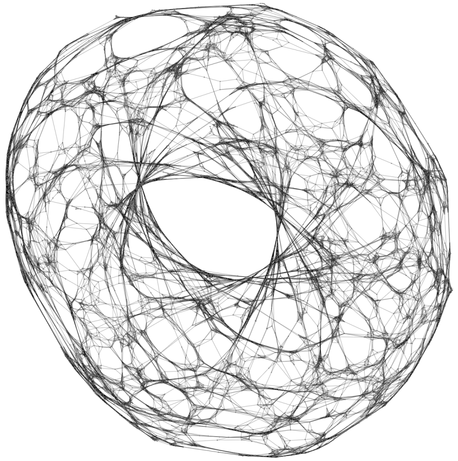
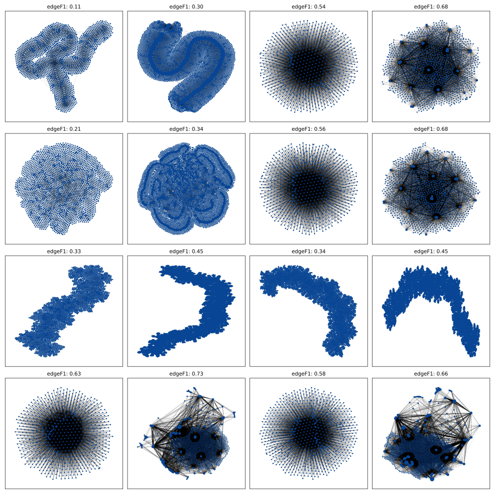
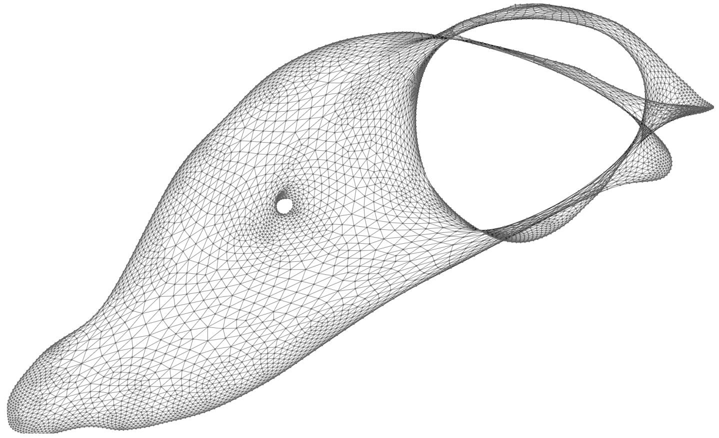
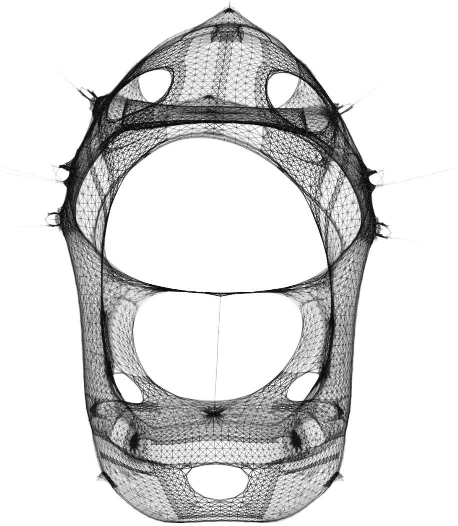




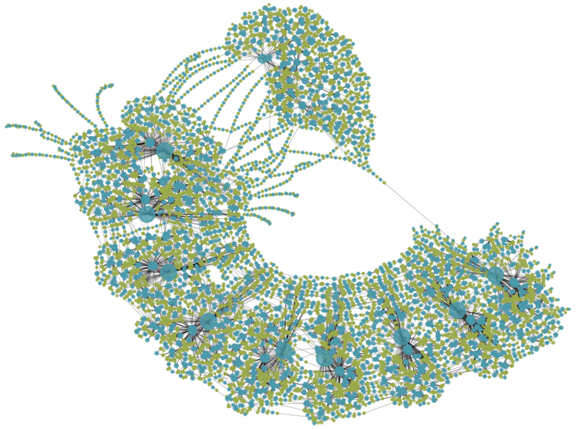


Real World Graphs



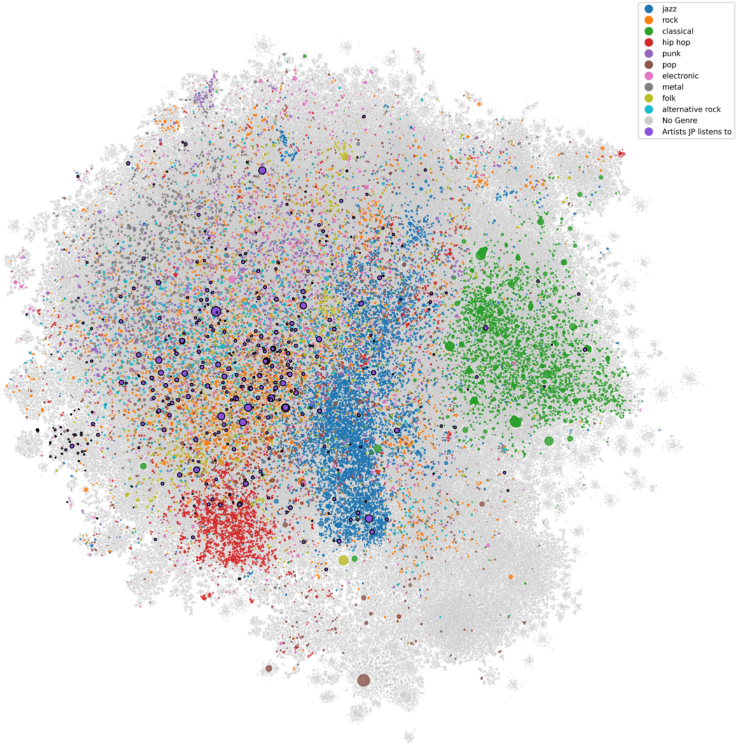


SAT-Instances



Some Examples

Music Artists



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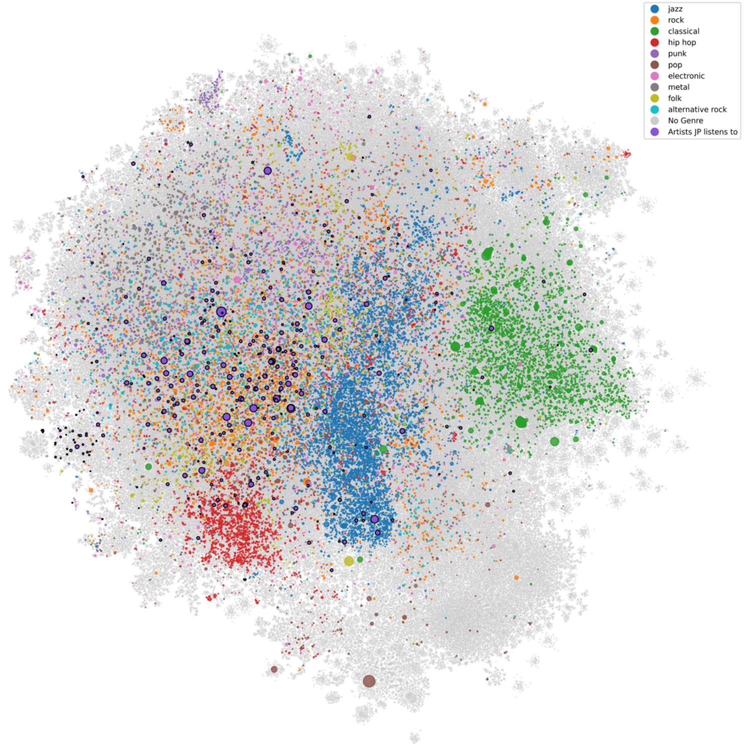
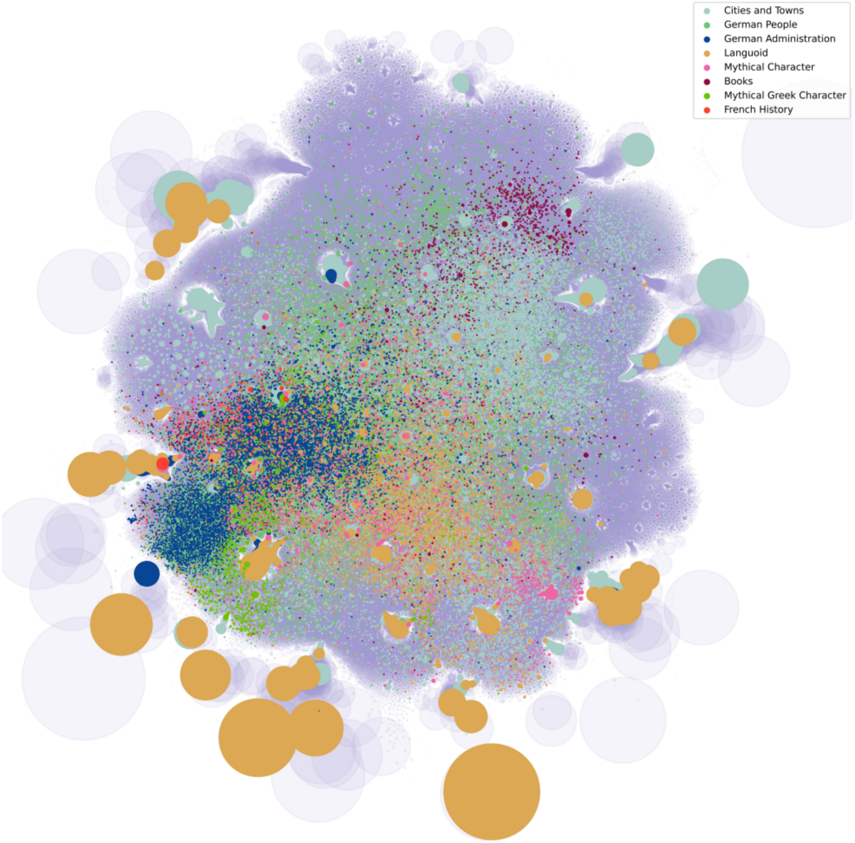


Wikipedia



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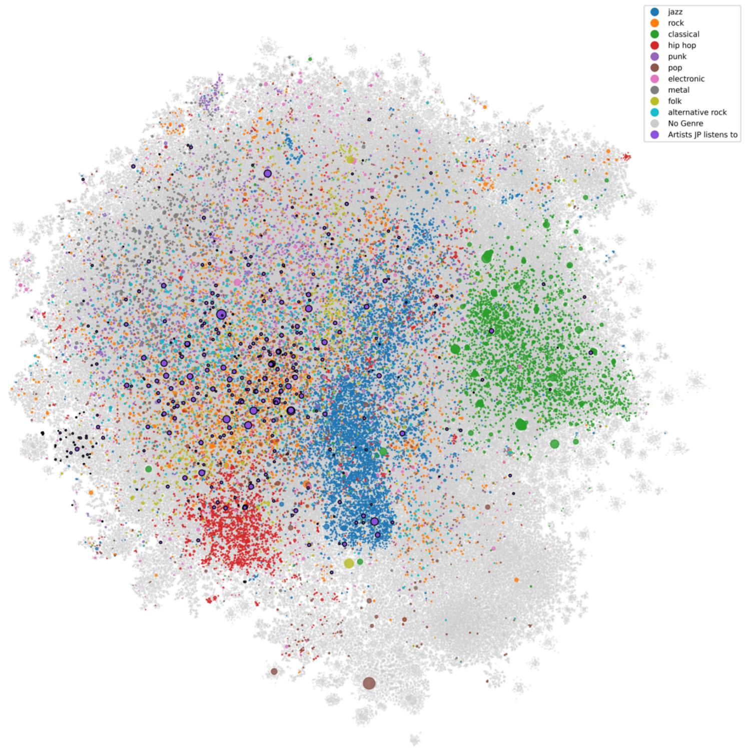

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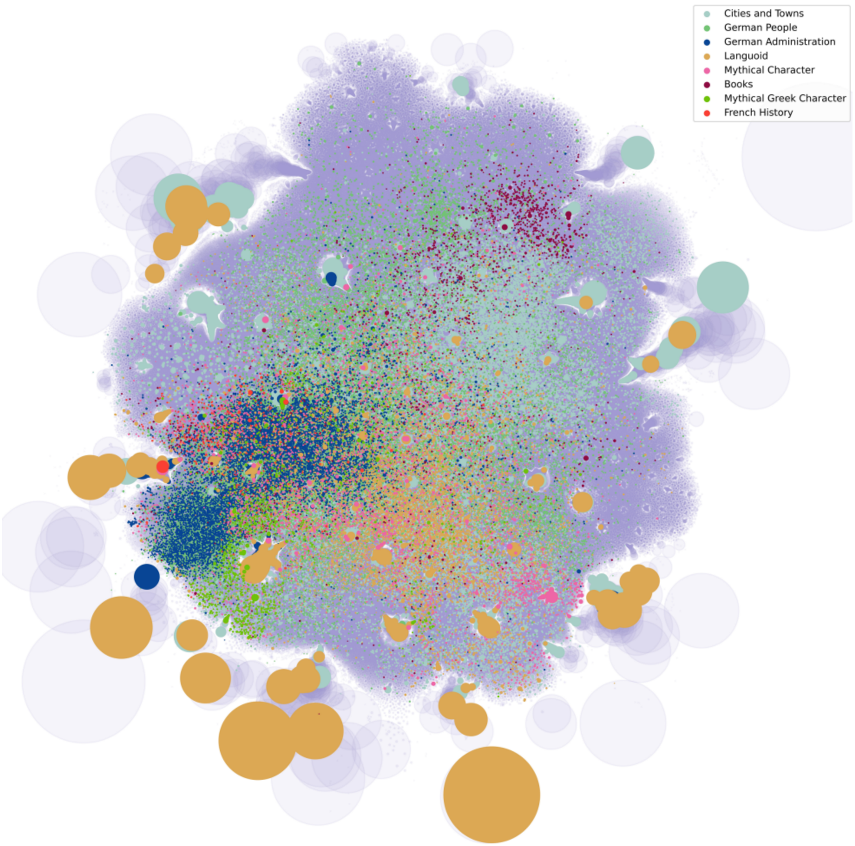
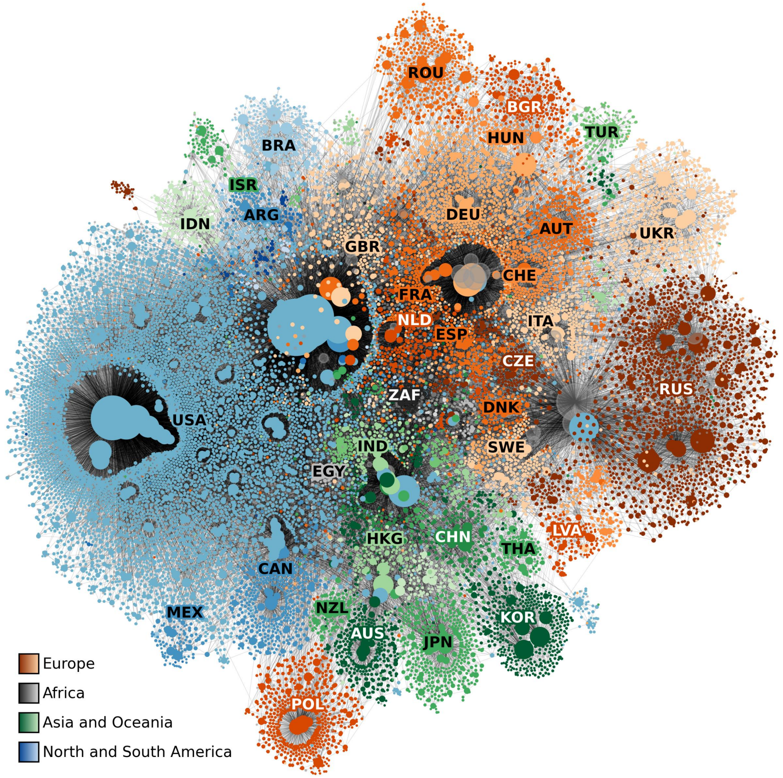
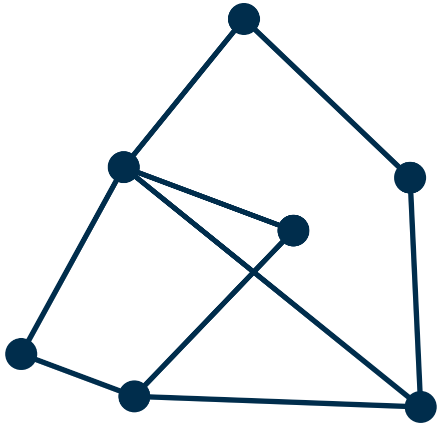


Internet Graph



Greedy Routing



Greedy Routing

- **Given**: $2d$ drawing of G, **Goal**: move from s to t

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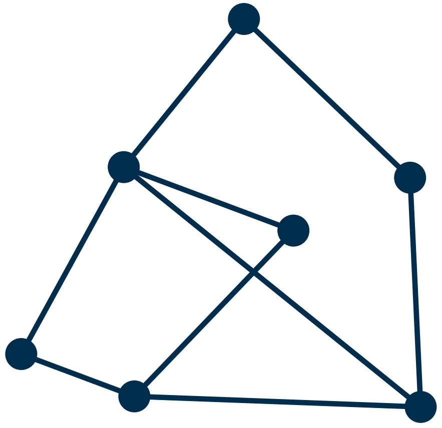

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- **Given**: $2d$ drawing of G, **Goal**: move from s to t
- Strategy: at every step, select neighbour that minimizes the Euclidean distance to t the most
- successful, if we eventually reacht t
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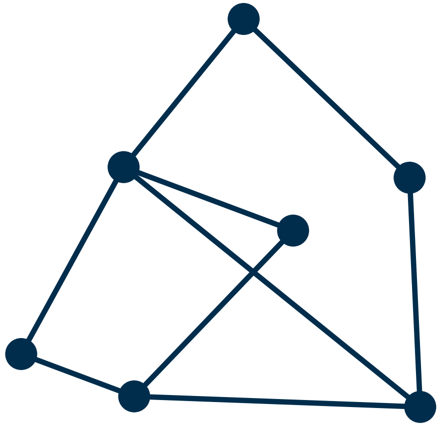

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- $2d$ drawing of a graph
- for every pair of vertices, greedy routing is successful

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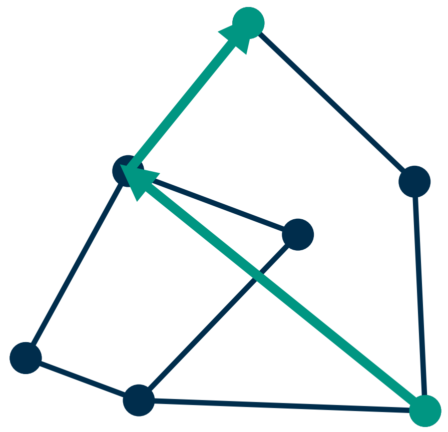

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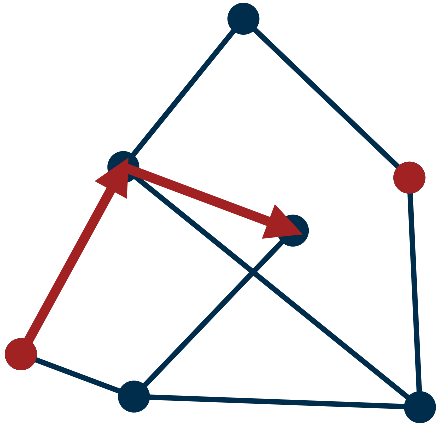

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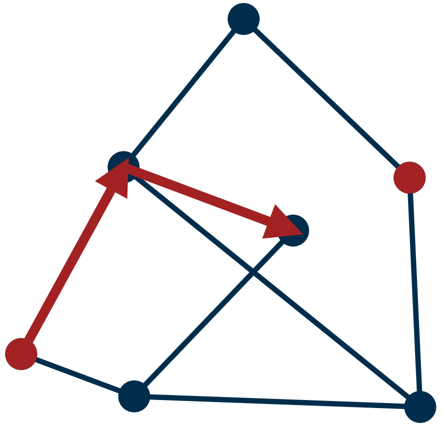

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Does every graph have a greedy embedding in the Euclidean plane?

How about the hyperbolic plane?

Can you find counterexamples?

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A greedy embedding is permissable if and only if for every $t \neq s$, there is a neighbor of s that is closer to t then s.

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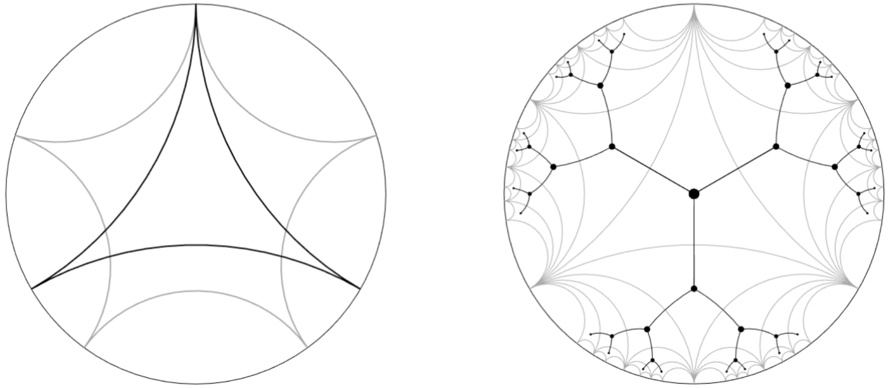

Hyperbolic Embeddings for Near-Optimal Greedy Routing

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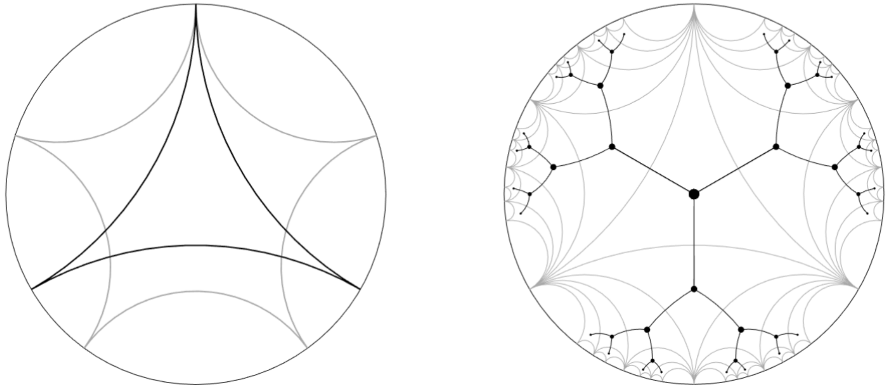

Hyperbolic Embeddings for Near-Optimal Greedy Routing

Is every graph embeddable in hyperbolic space?