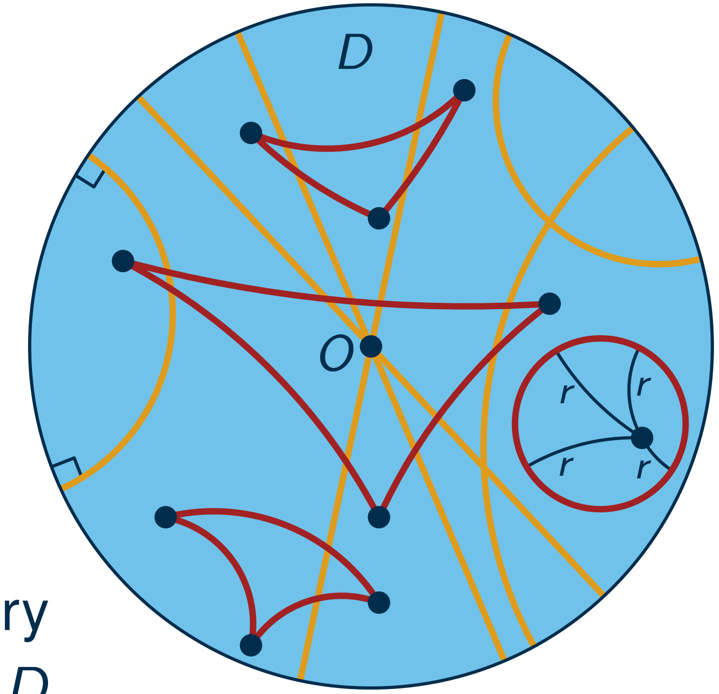
Computational Geometry

Geometric Graphs – Euclidean and Hyperbolic

Thomas Bläsius

Recap: Poincaré Disk



Points

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- let $\mathcal{P}$ be the set of points in the interior of the disk

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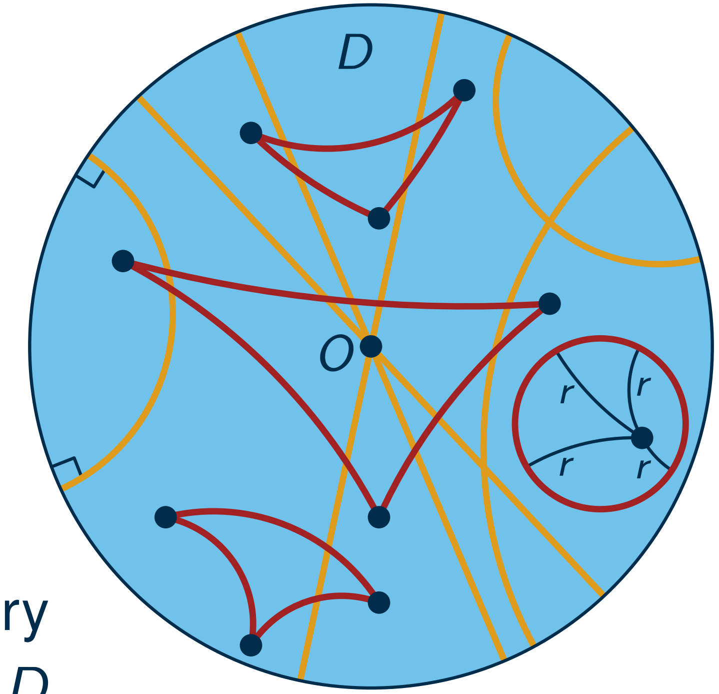

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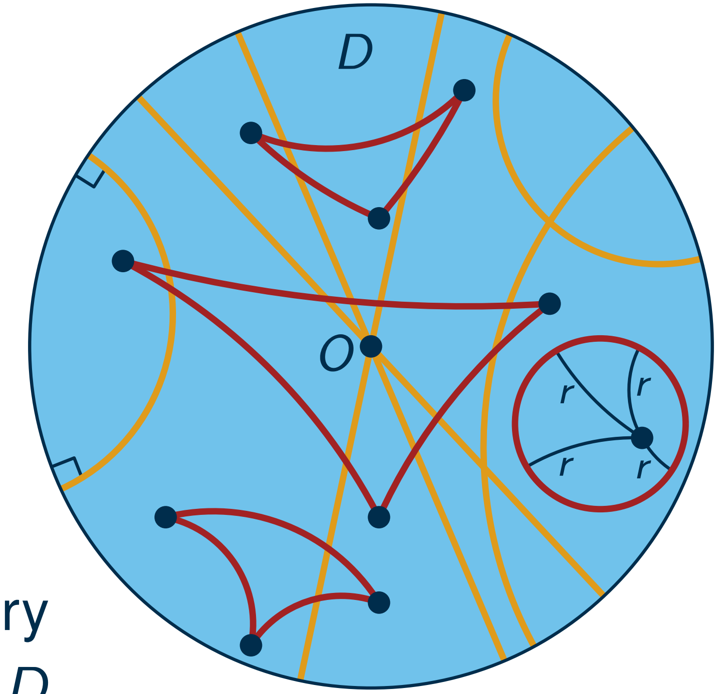

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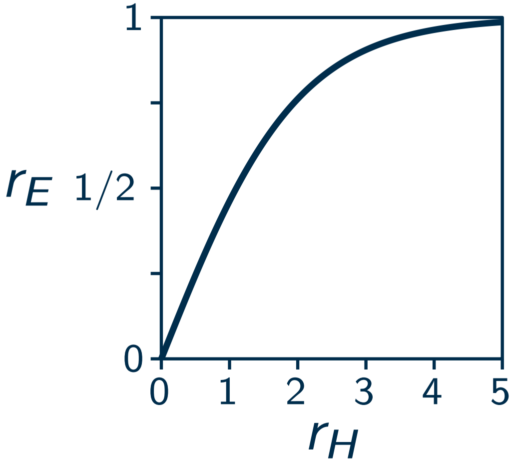
- to use specifics of the hyperbolic plane, we need to go far away from O
- problem: we quickly approach the boundary of D
- different radii become hard to distinguish



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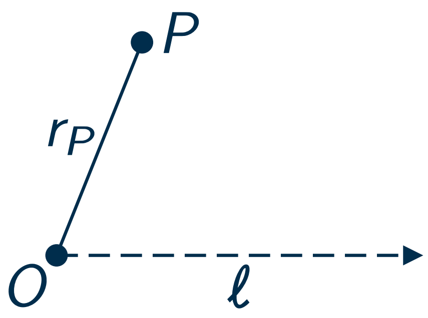

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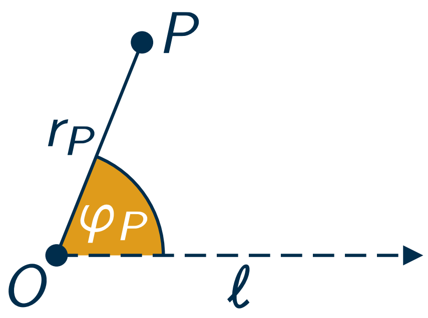

(counterclockwise)

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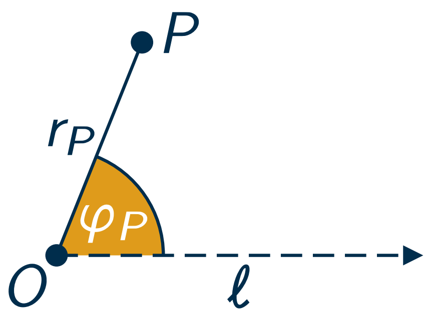

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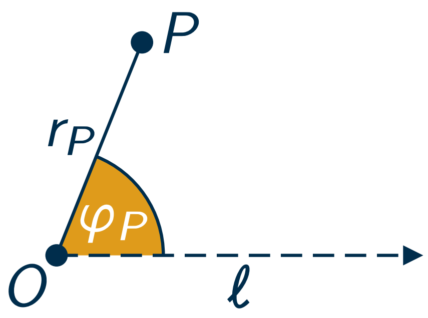

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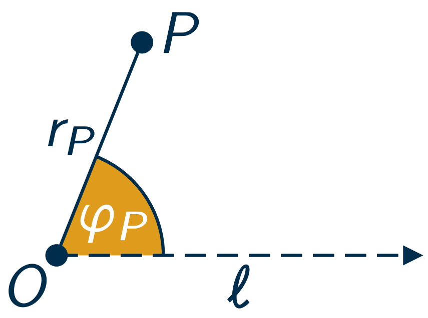

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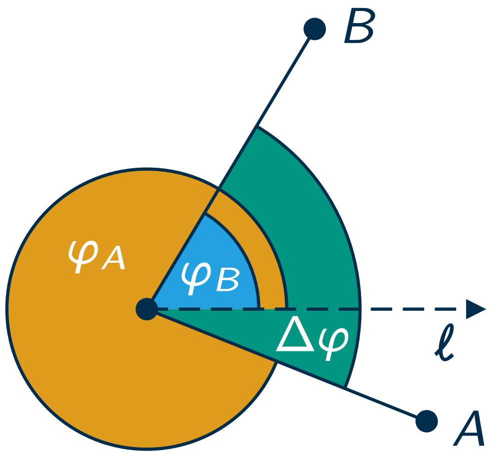

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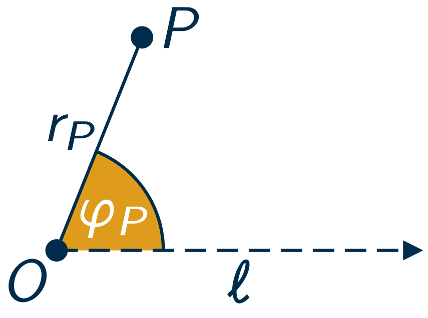

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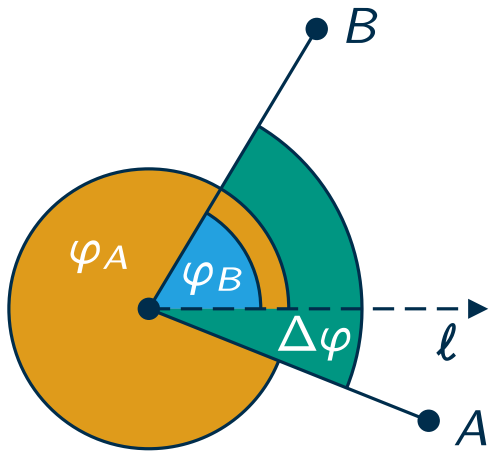

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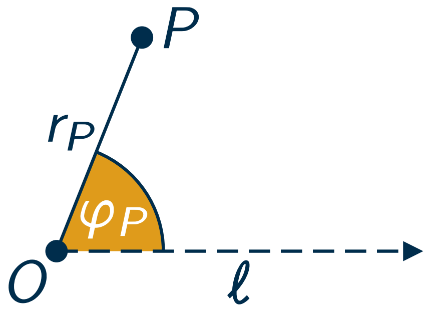

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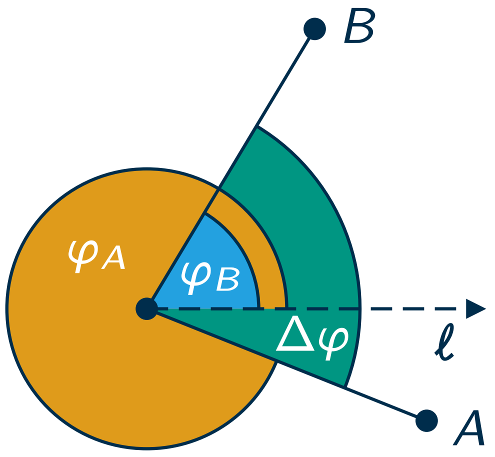

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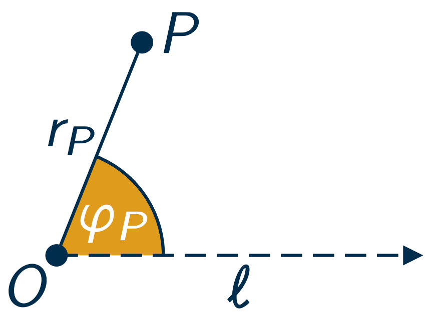

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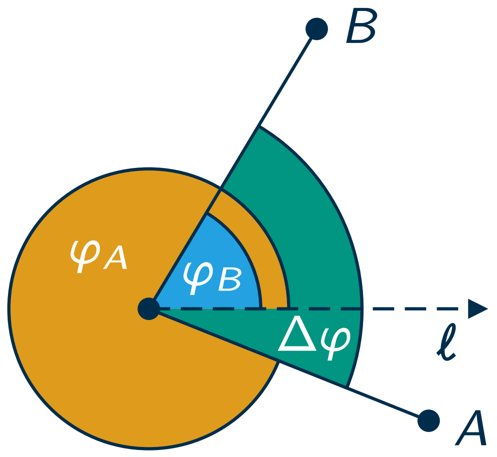

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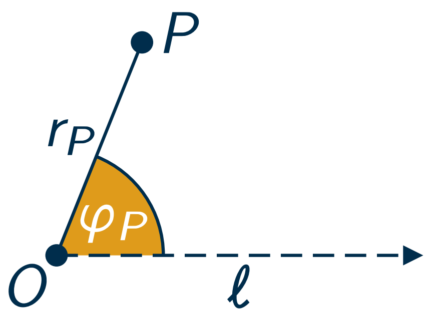

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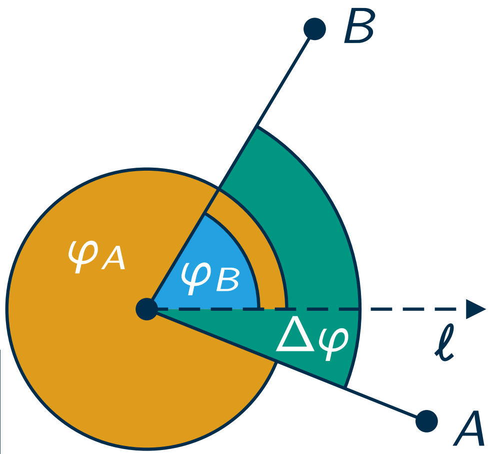

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- use native polar coordinate for every point in the hyperbolic plane
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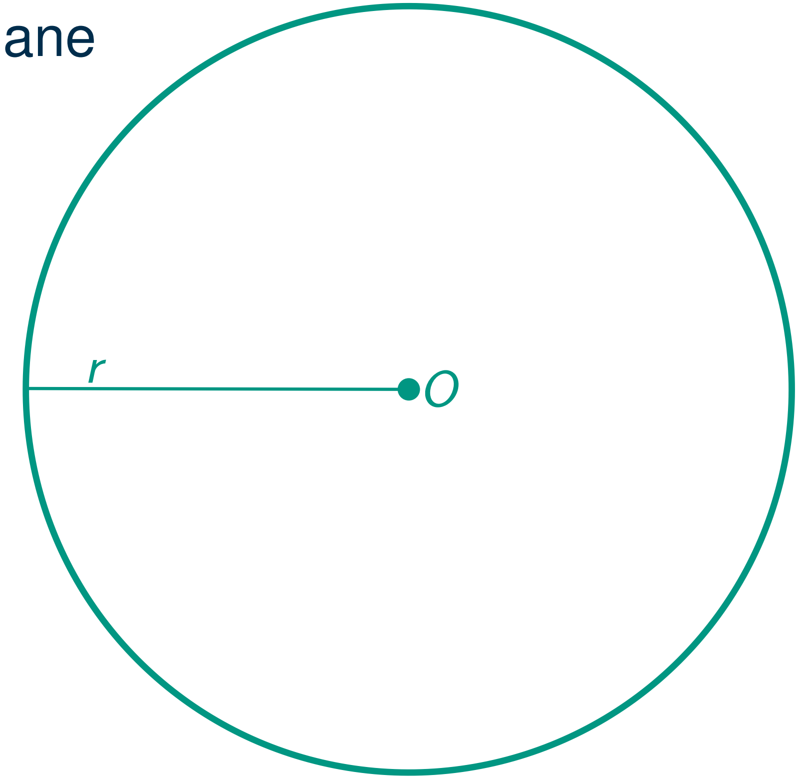

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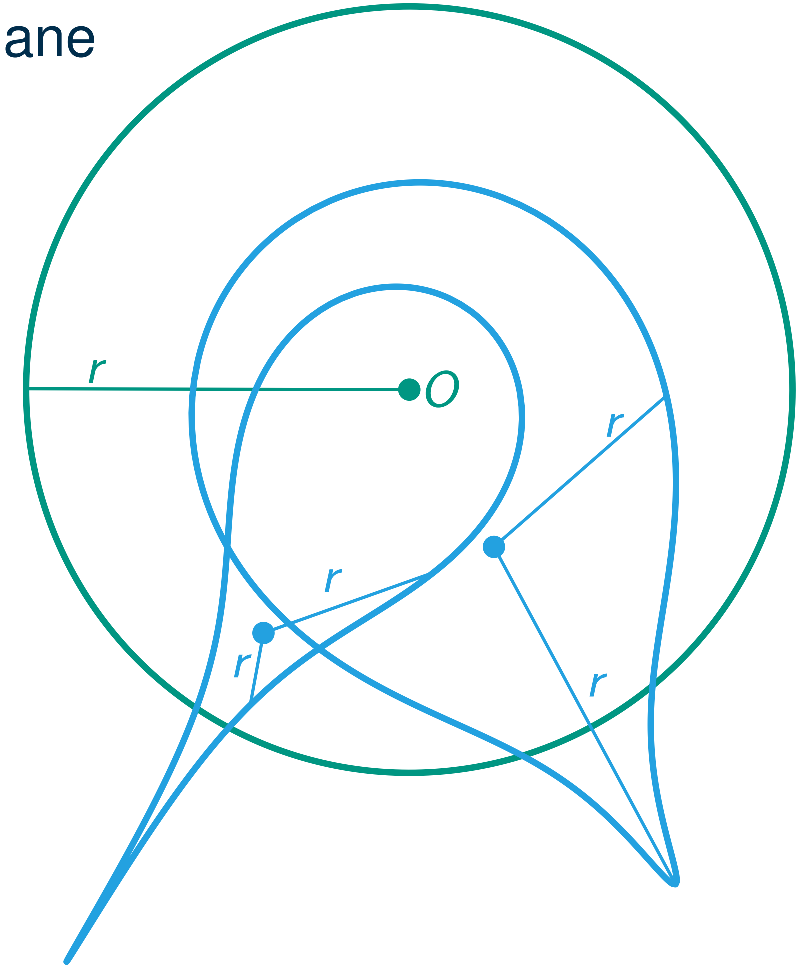

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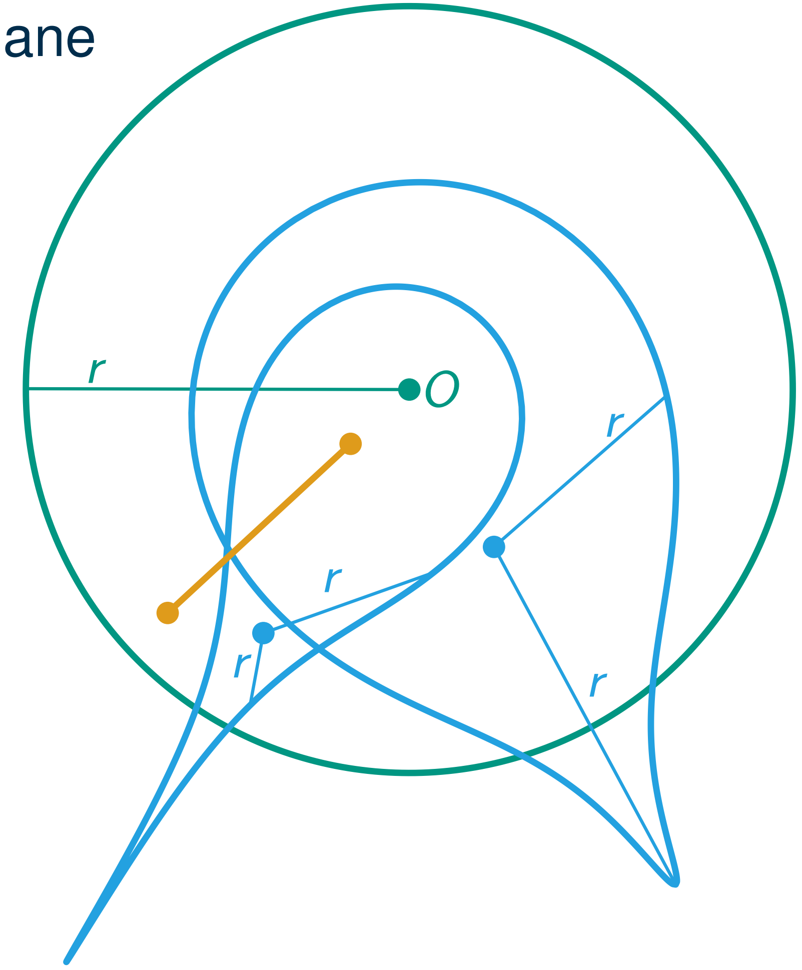

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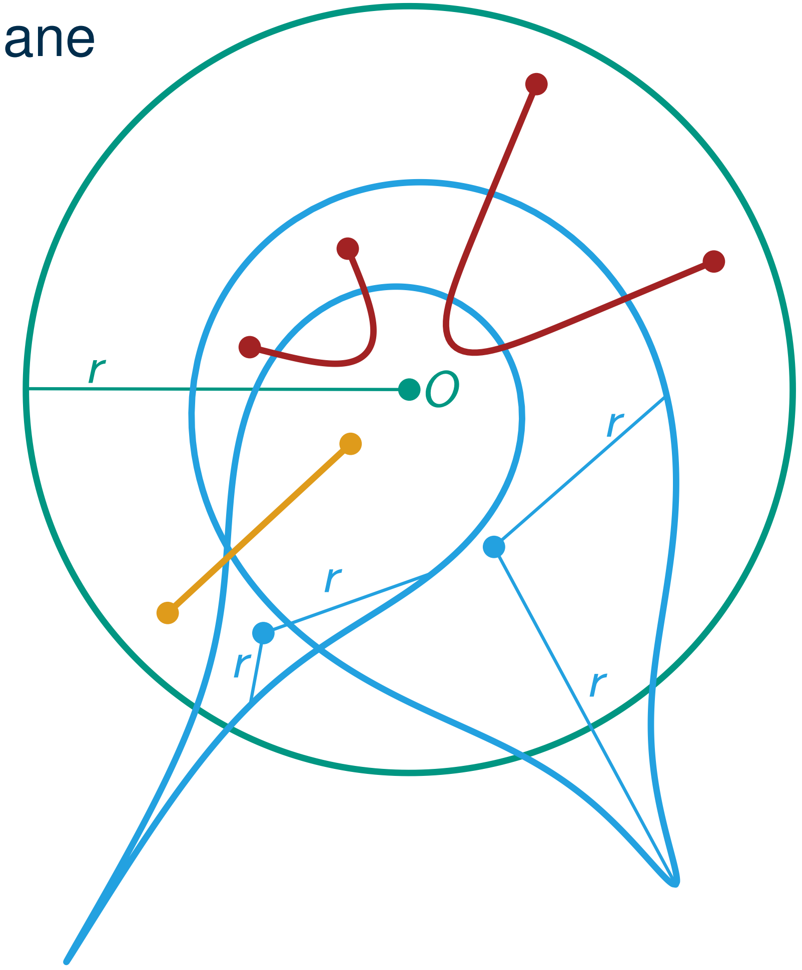

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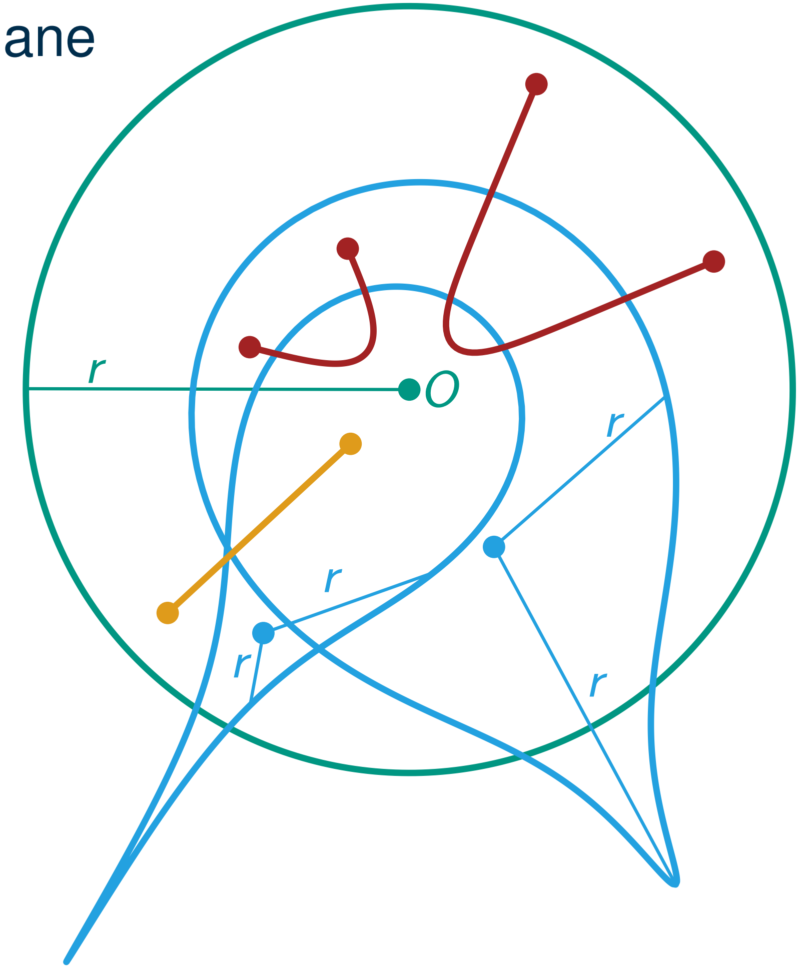

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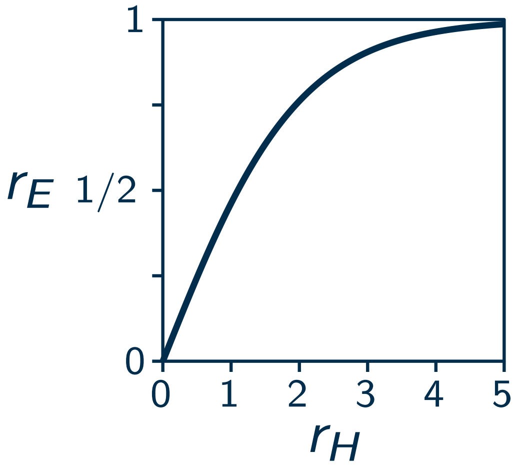

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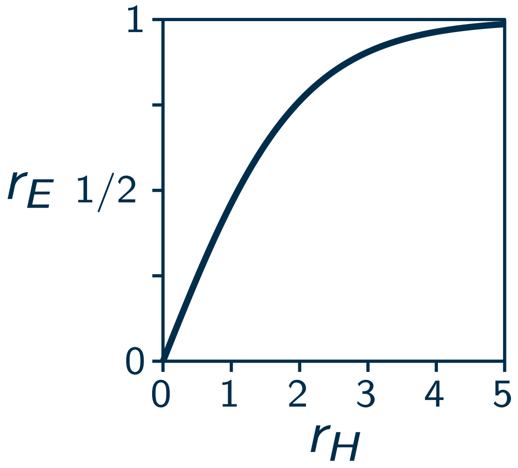

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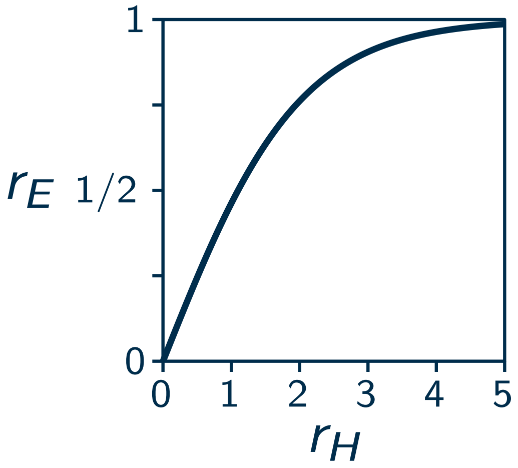

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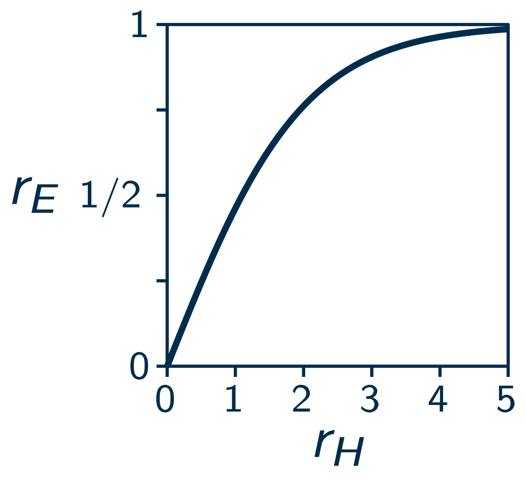
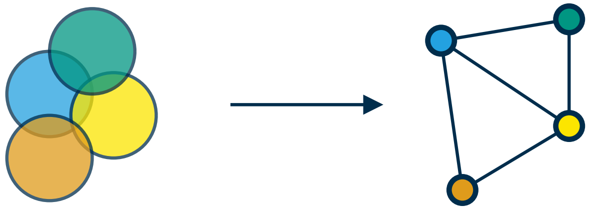


Heuristic For Choosing A Model

- visual representation of hyperbolic data $\rightarrow$ native model
- computations on coordinates $\rightarrow$ native model (or also: hyperboloid)
- thinking about and proving stuff $\rightarrow$ Poincaré Disk (or also: upper half-plane, Beltrami–Klein)

Unit-Disk Graphen

Definition
A graph is a **unit disk graph** if it is the intersection graph of disks of radius 1.



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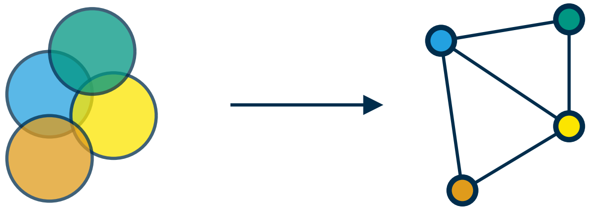

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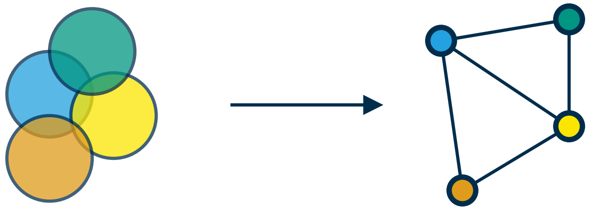

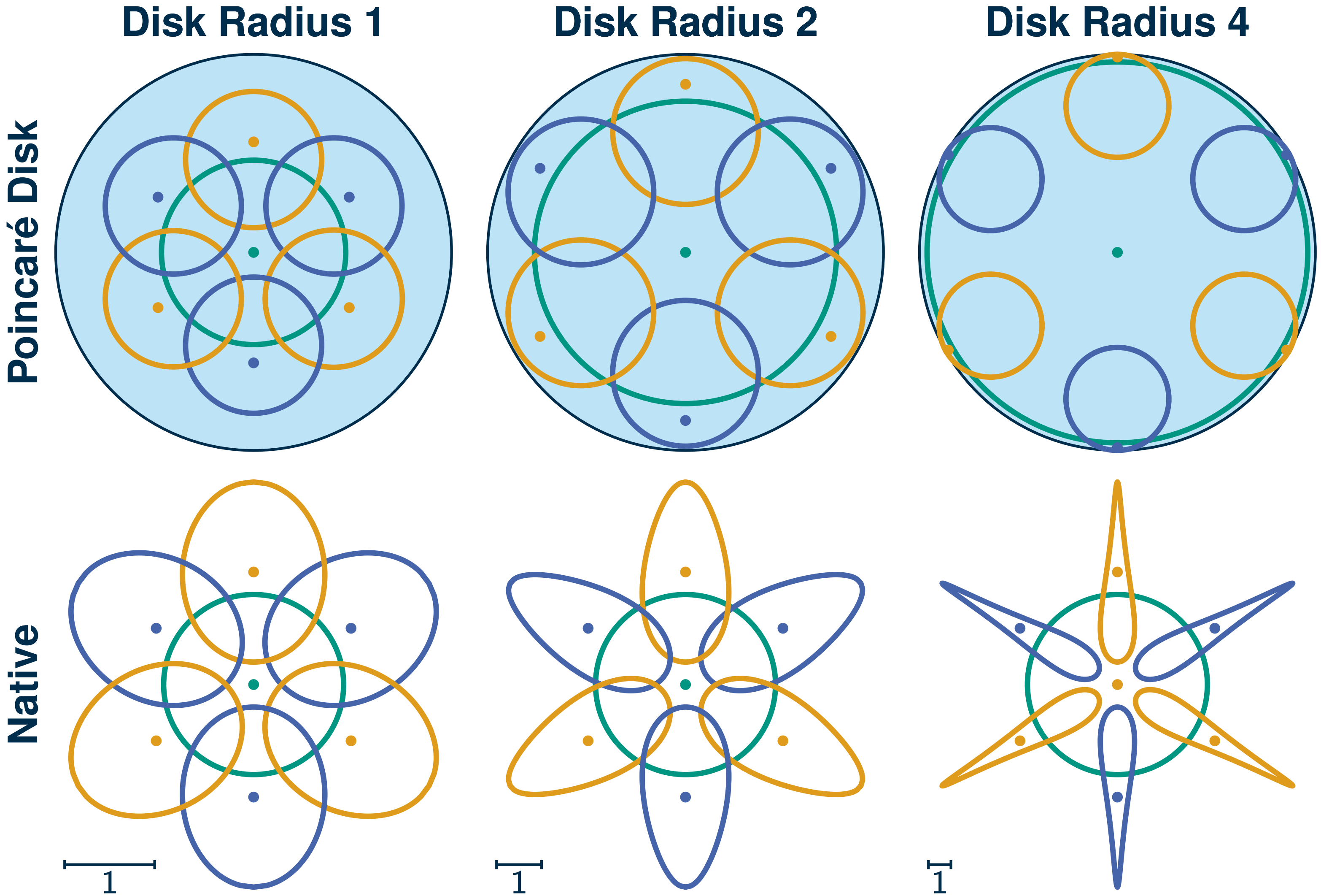

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- can be defined analogously
- but: the radius matters

Hyperbolic Uniform Disk Graphs

Definition

Note: the disks in the intersection representation have radius $t/2$

$G = (V, E)$ is a **hyperbolic uniform disk graph** if it there are vertex positions $p\colon V \to \mathbb{H}^2$ and a threshold t such that $uv \in E \Leftrightarrow d(p(u), p(v)) \leq t$.

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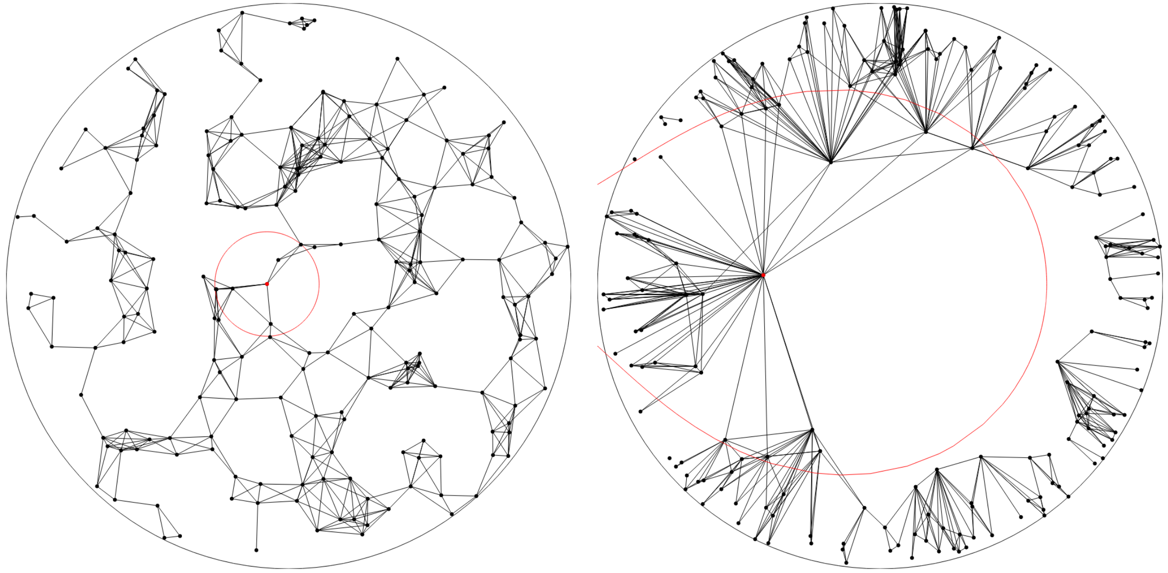
Do I Get Different Graph Structures Depending On t?

- small threshold t
  - similar to the Euclidean setting
  - regular / homogeneous
  - grid-like
- large t
  - irregular / heterogeneous
  - hierarchical / tree-like



https://thobl.github.io/hyperbolic-unit-disk-graph/

How Hyperbolic Should It Be?

small threshold t → large threshold t

hyperbolic uniform disk graphs

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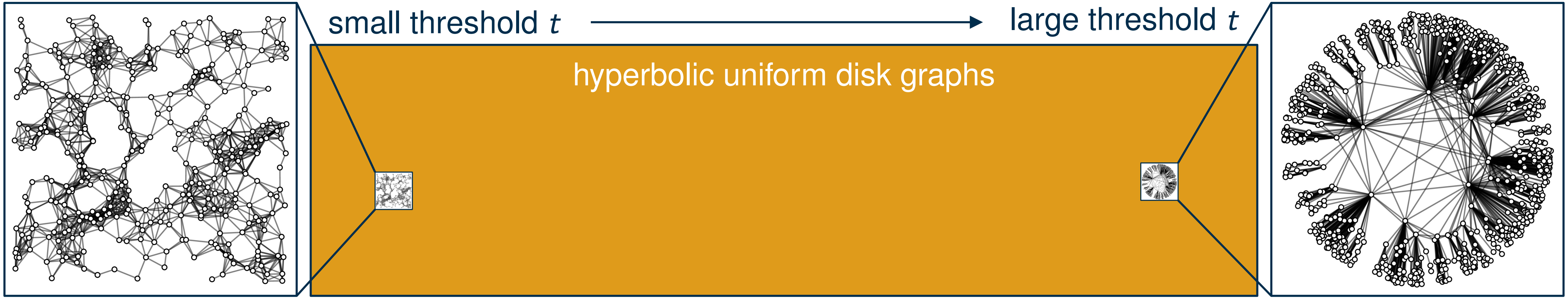

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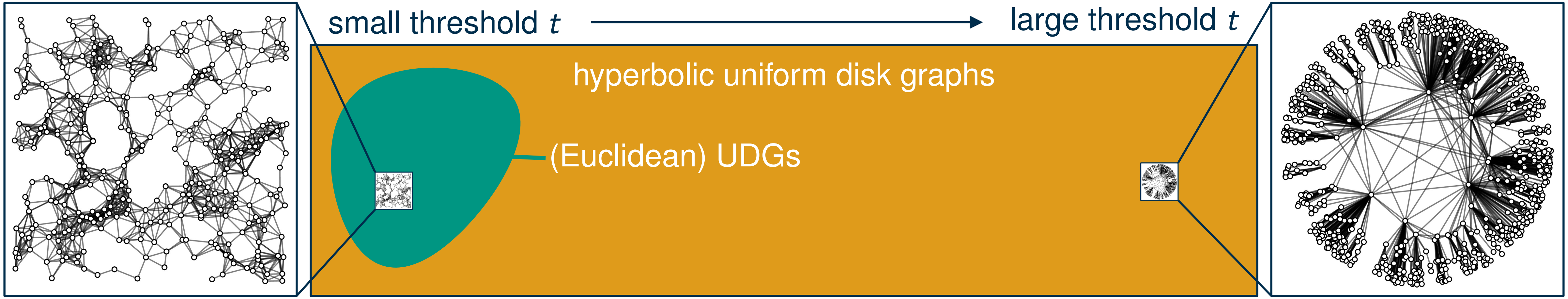

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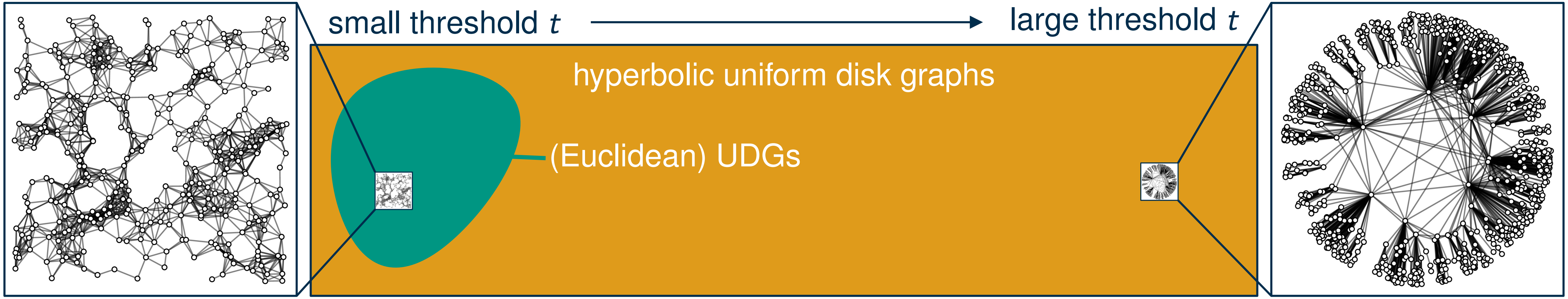

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- many hyperbolic UDGs are not very hyperbolic

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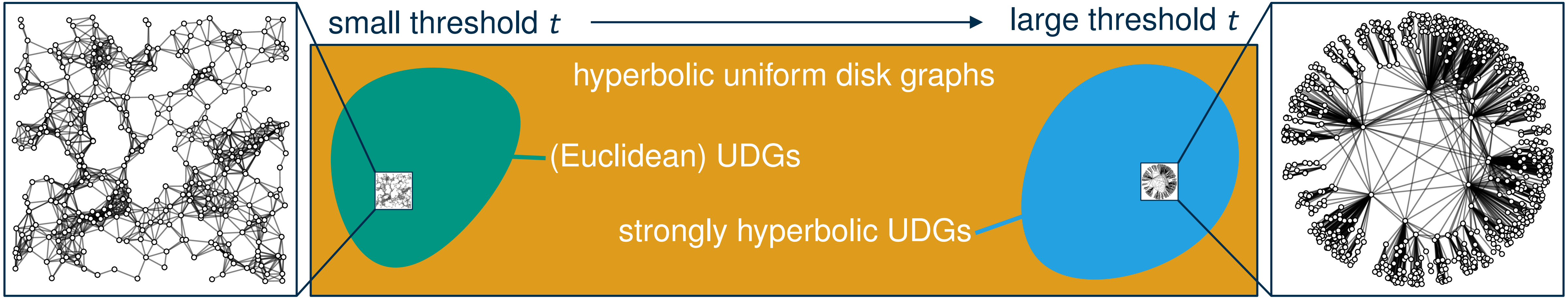

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Strongly Hyperbolic Uniform Disk Graphs

- goal: complement to Euclidean UDGs
- with hierarchical / heterogeneous structure
- How do we formalize this? How large is large enough for t?

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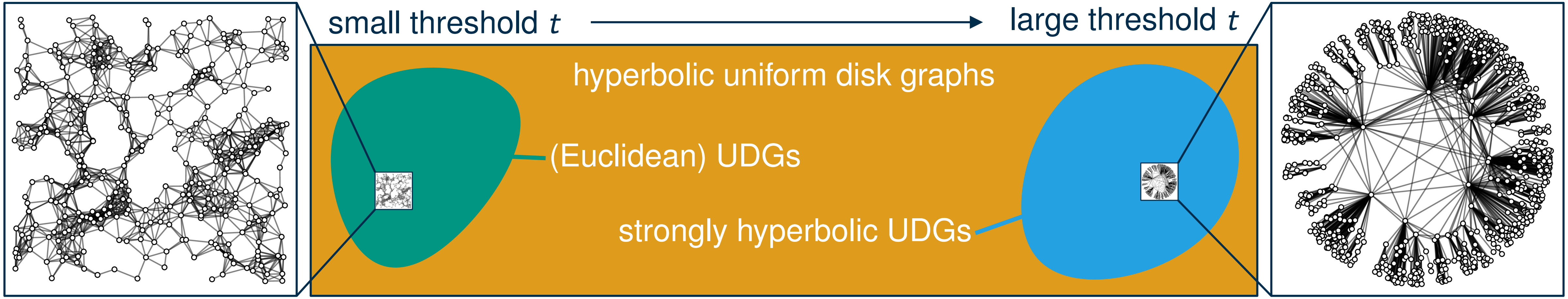

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- many hyperbolic UDGs are not very hyperbolic

Strongly Hyperbolic Uniform Disk Graphs

- goal: complement to Euclidean UDGs
- with hierarchical / heterogeneous structure
- How do we formalize this? How large is large enough for t?

There are different answers to this.
We look at only one of them.

Strongly Hyperbolic Uniform Disk Graphs

Definition

Note: the disks in the intersection representation have radius $t/2$

$G = (V, E)$ is a **hyperbolic uniform disk graph** if it there are vertex positions $p\colon V \to \mathbb{H}^2$ and a threshold t such that $uv \in E \Leftrightarrow d(p(u), p(v)) \leq t$.
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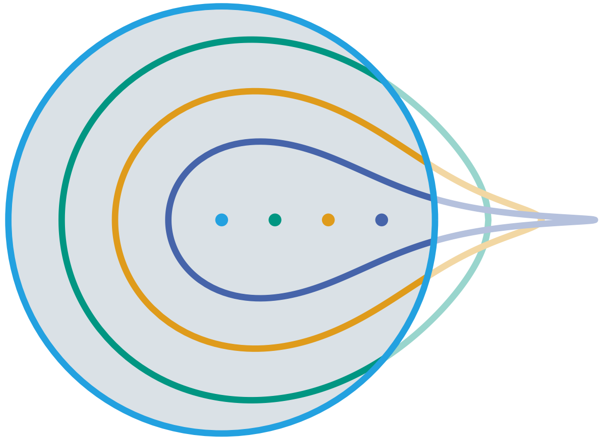
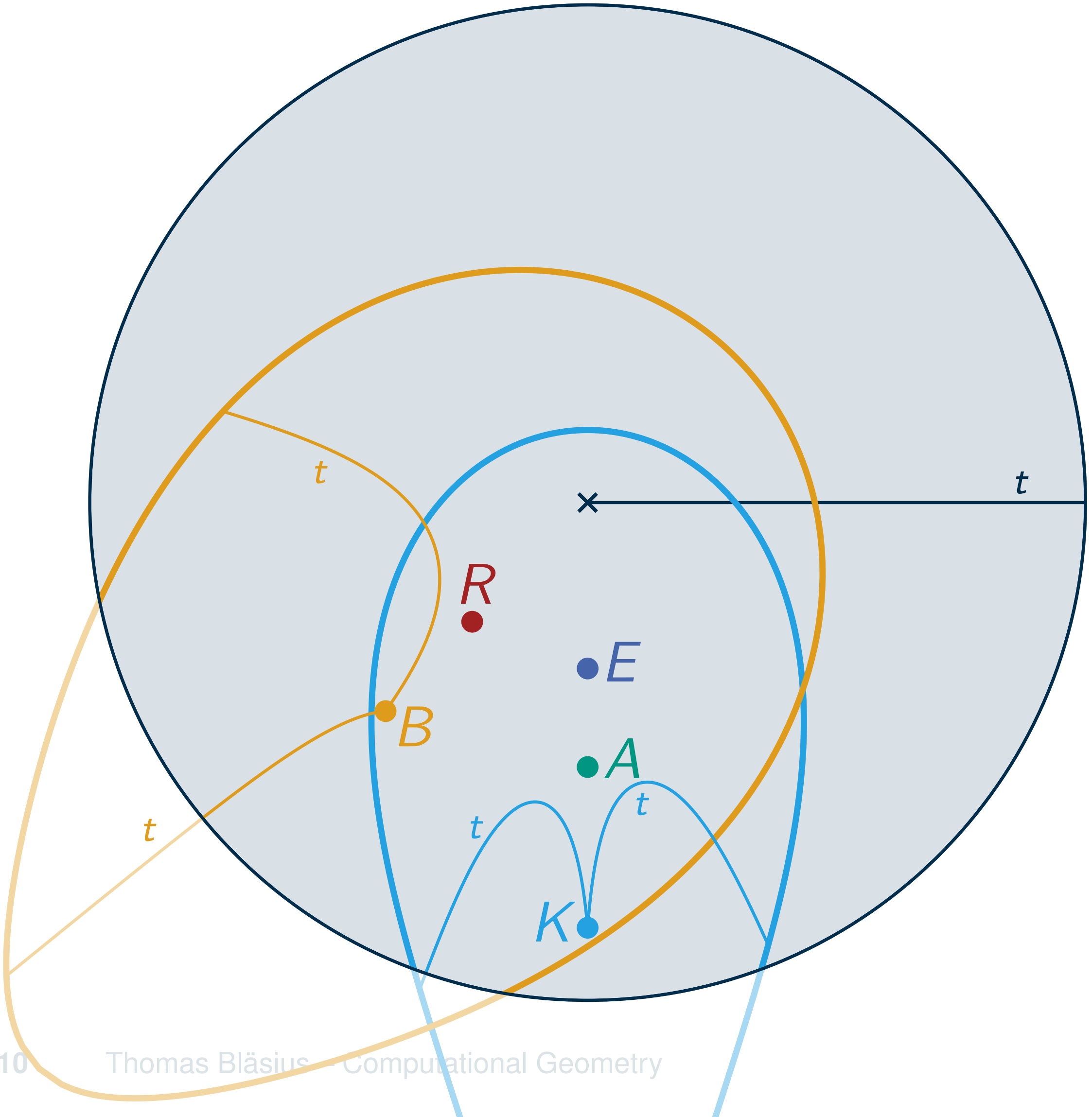
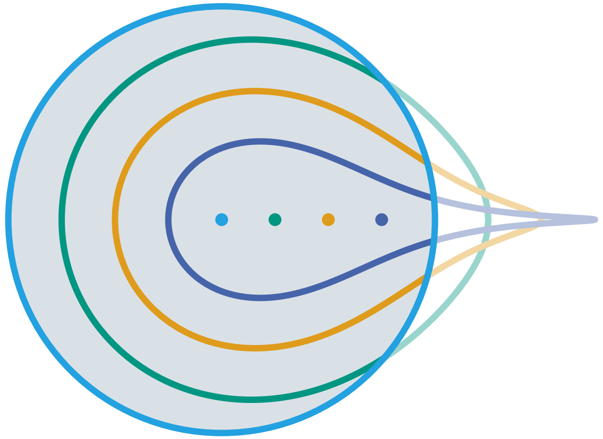
Observations

- a vertex at the origin is universal: adjacent to all other vertices
- the further out the vertex, the smaller its area of neighbors
- maximal heterogeneity: every distance from the origin yields differently sized area in which neighbors lie



Which Edges Exist?





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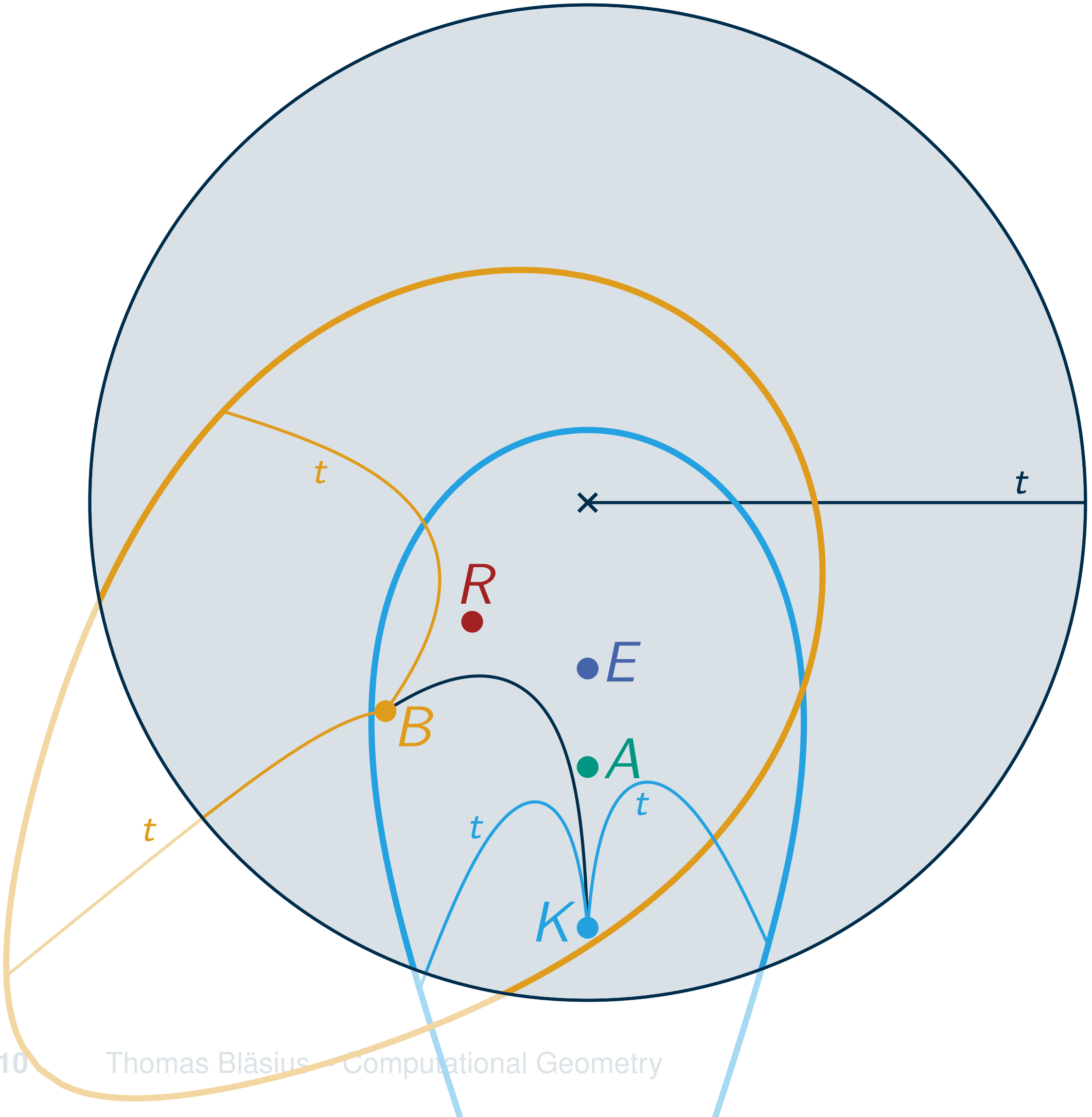

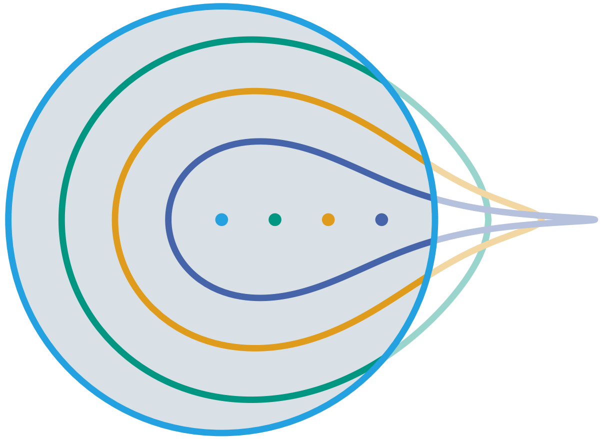

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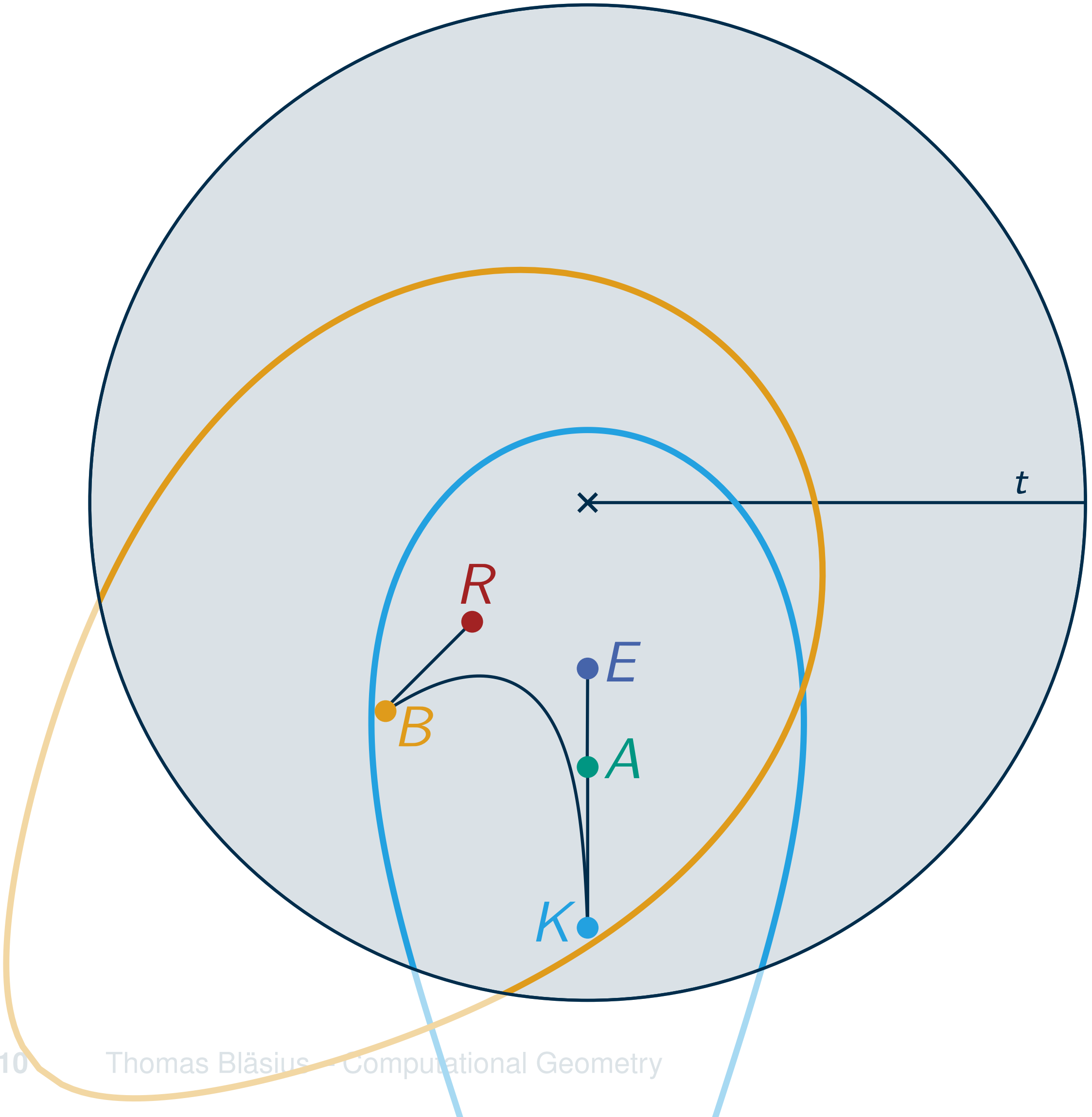

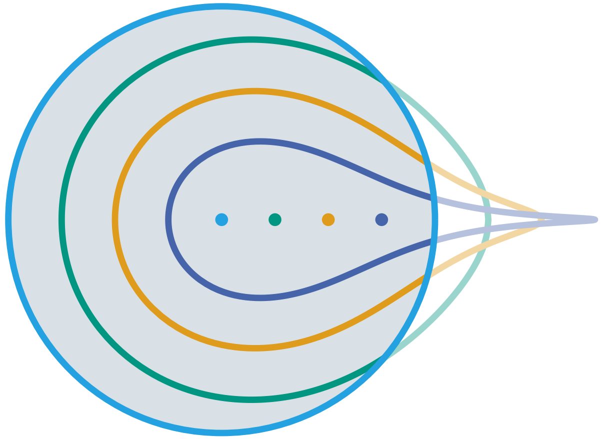

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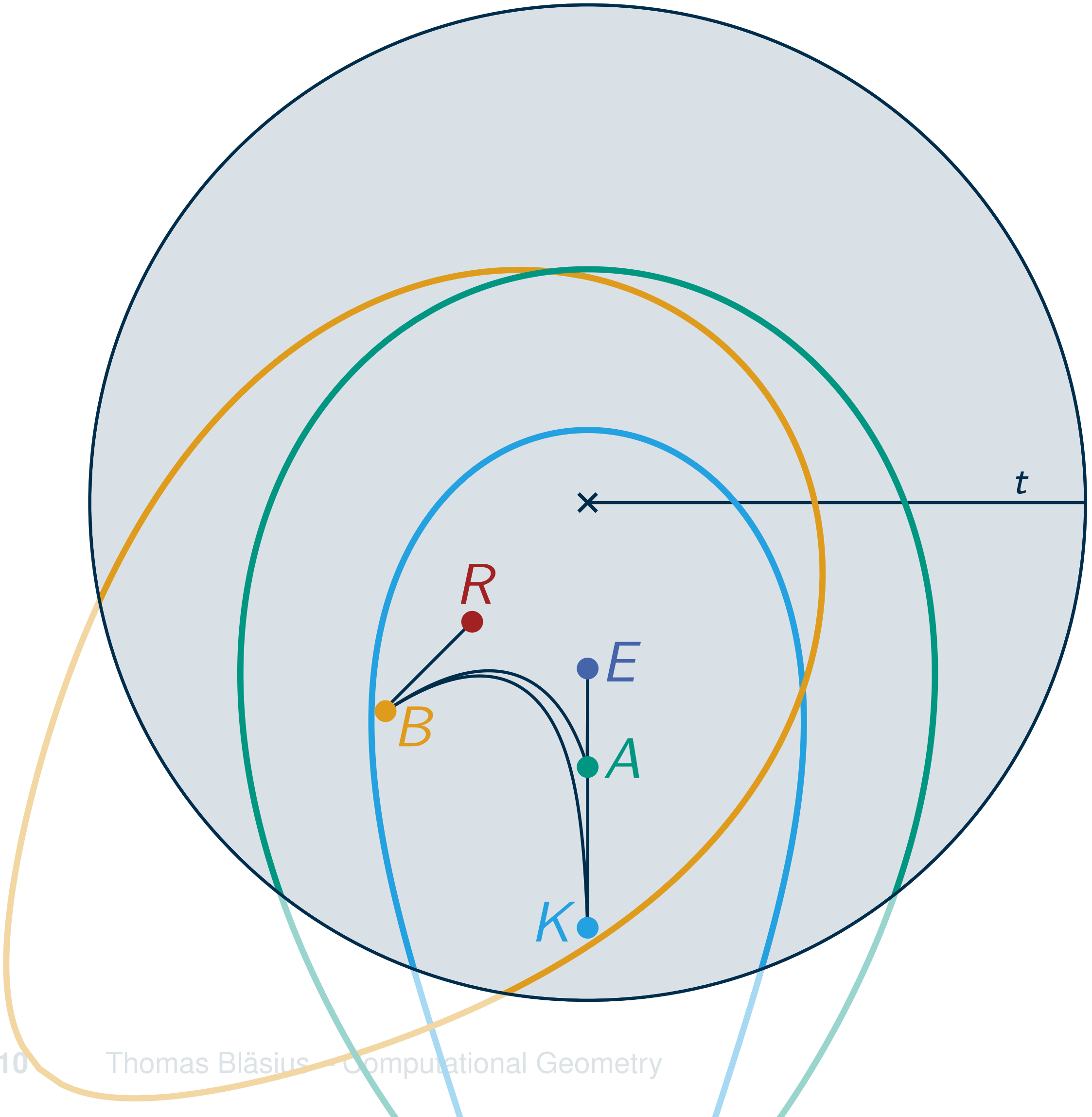

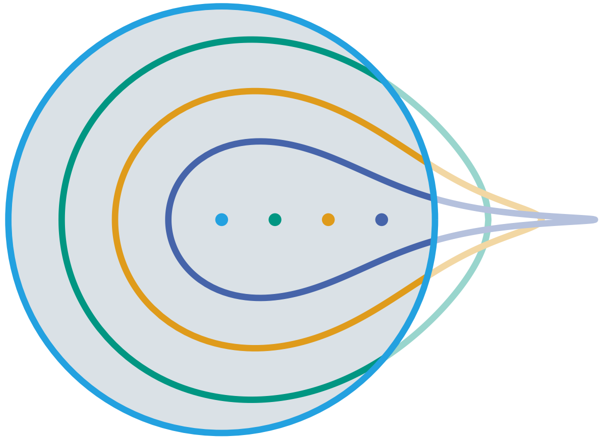

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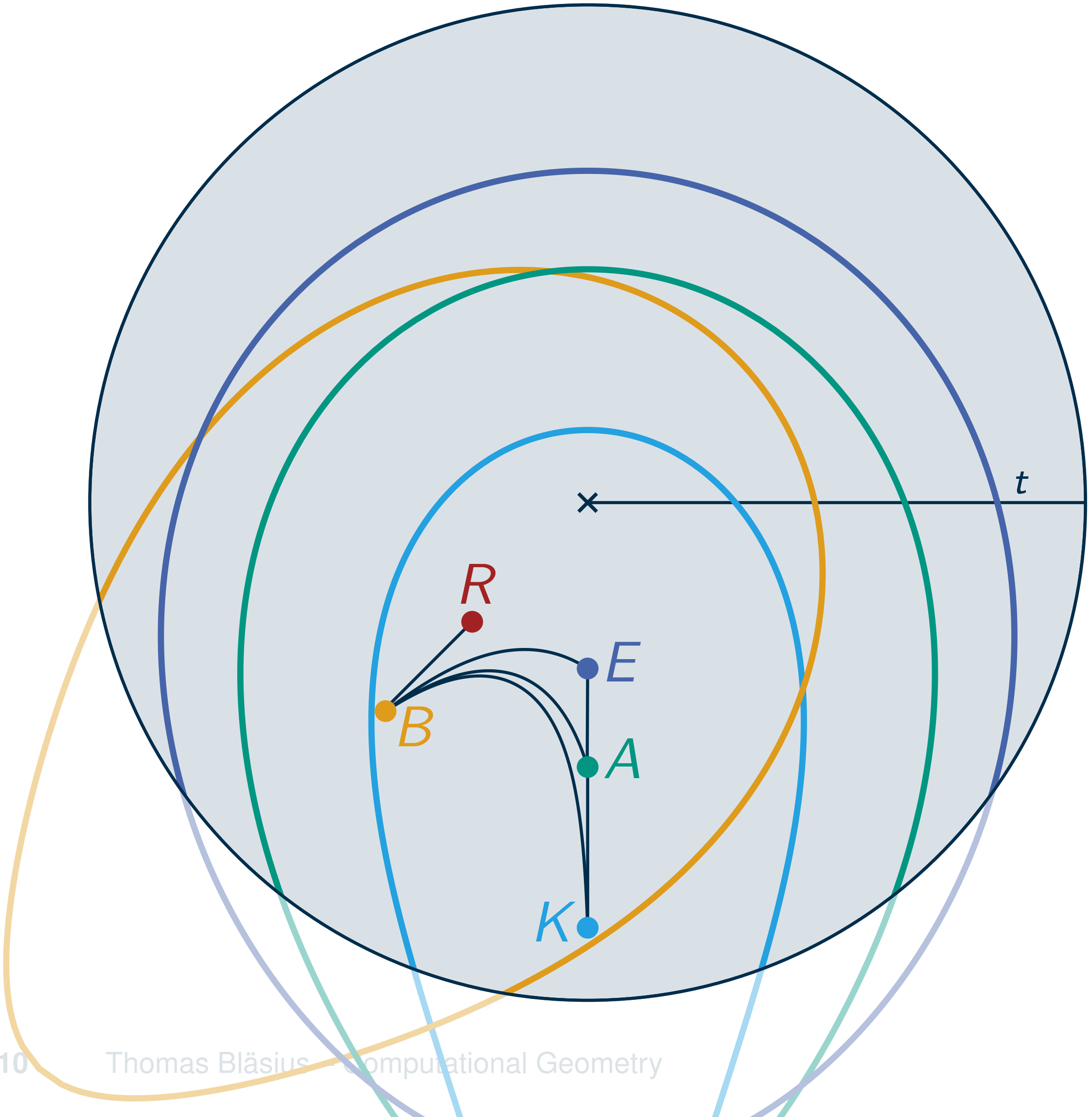

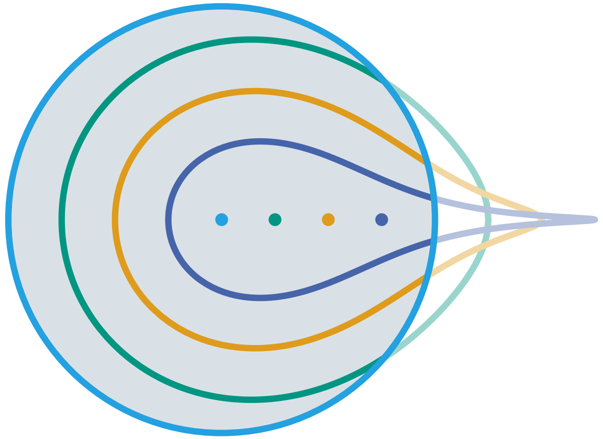

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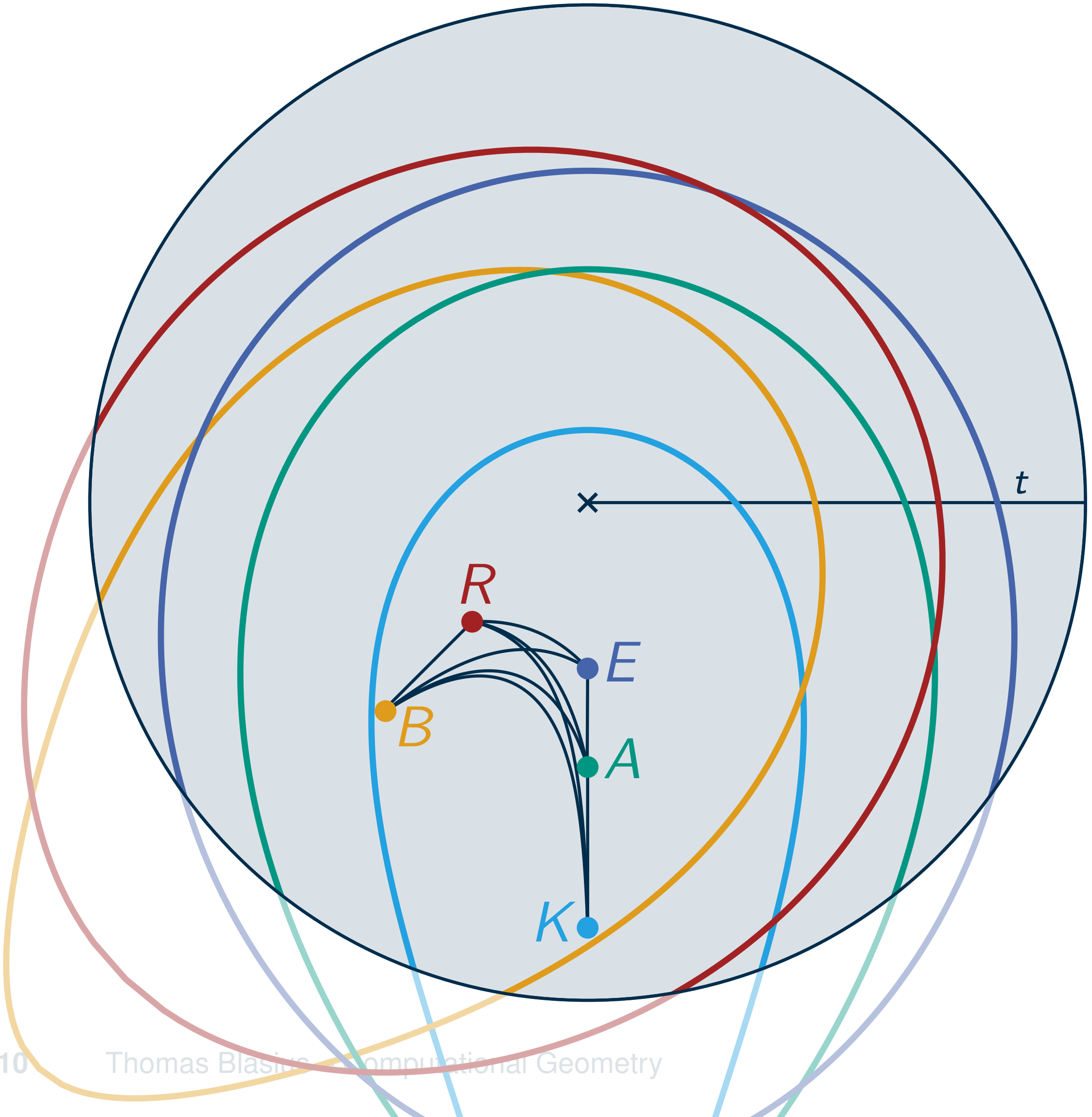

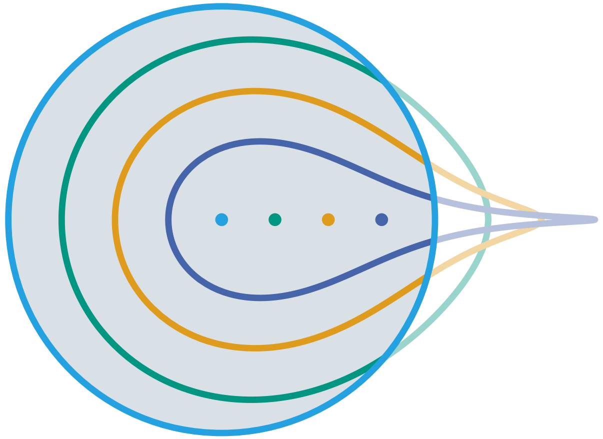

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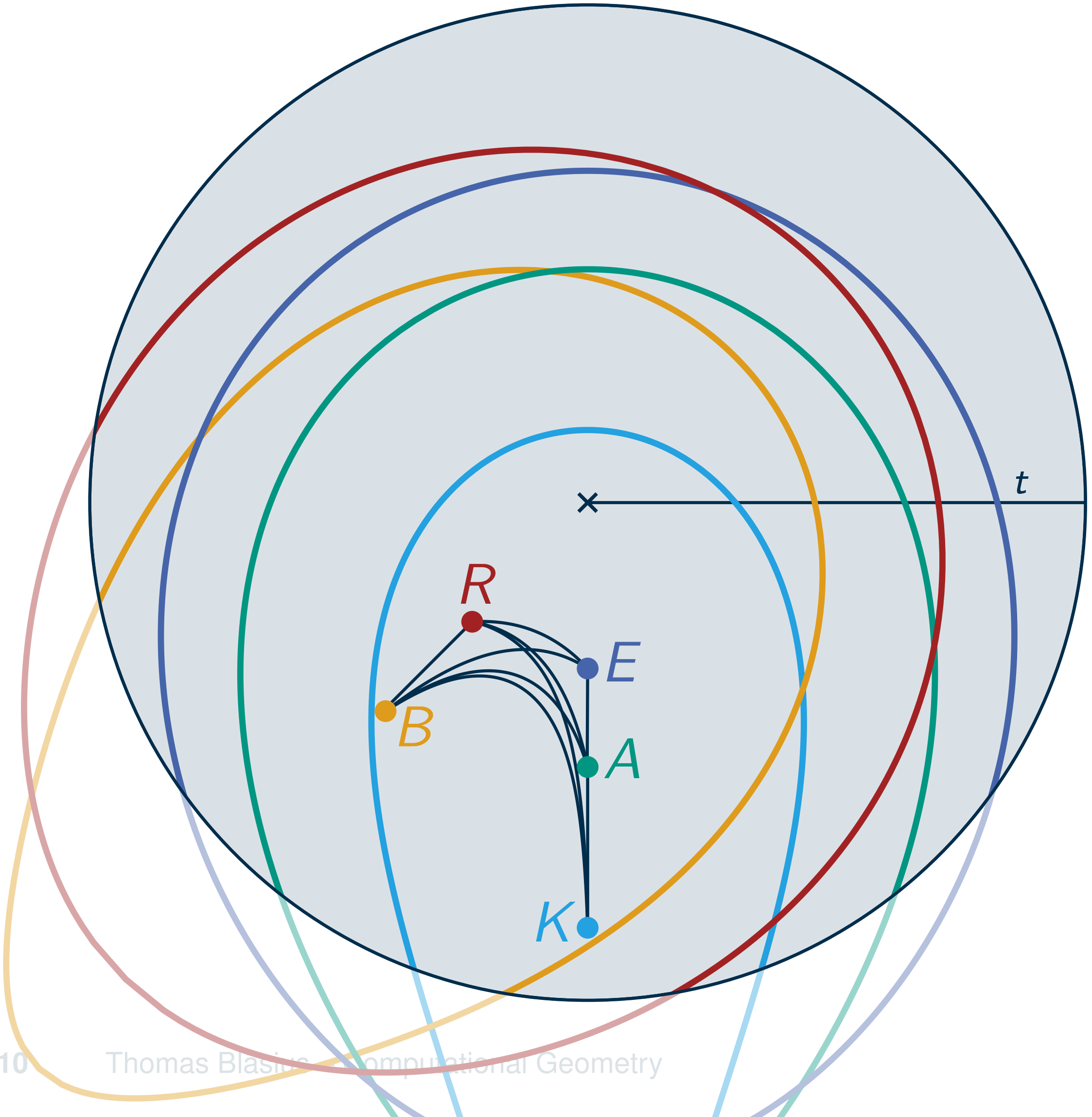

Observation
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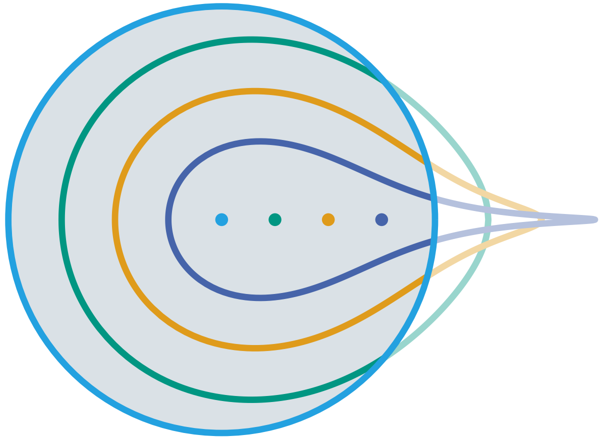

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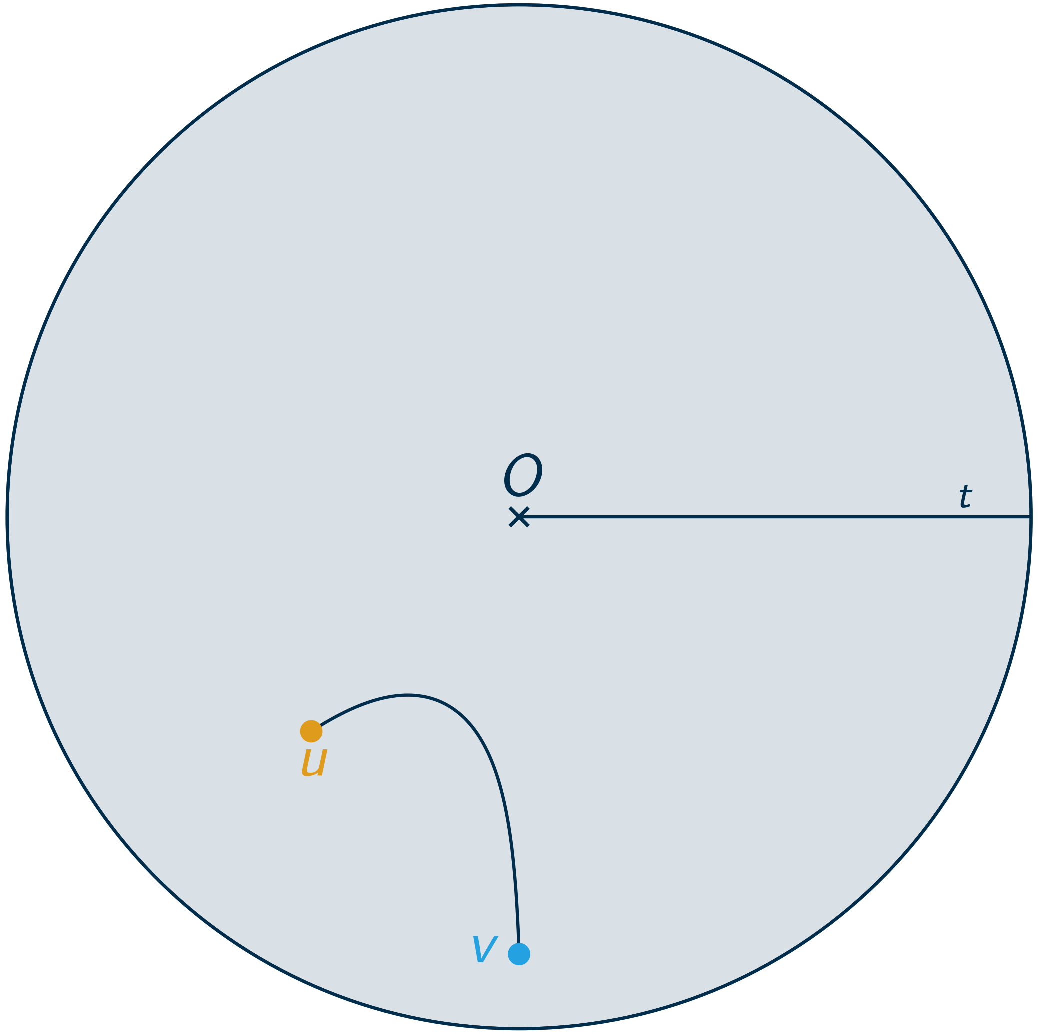

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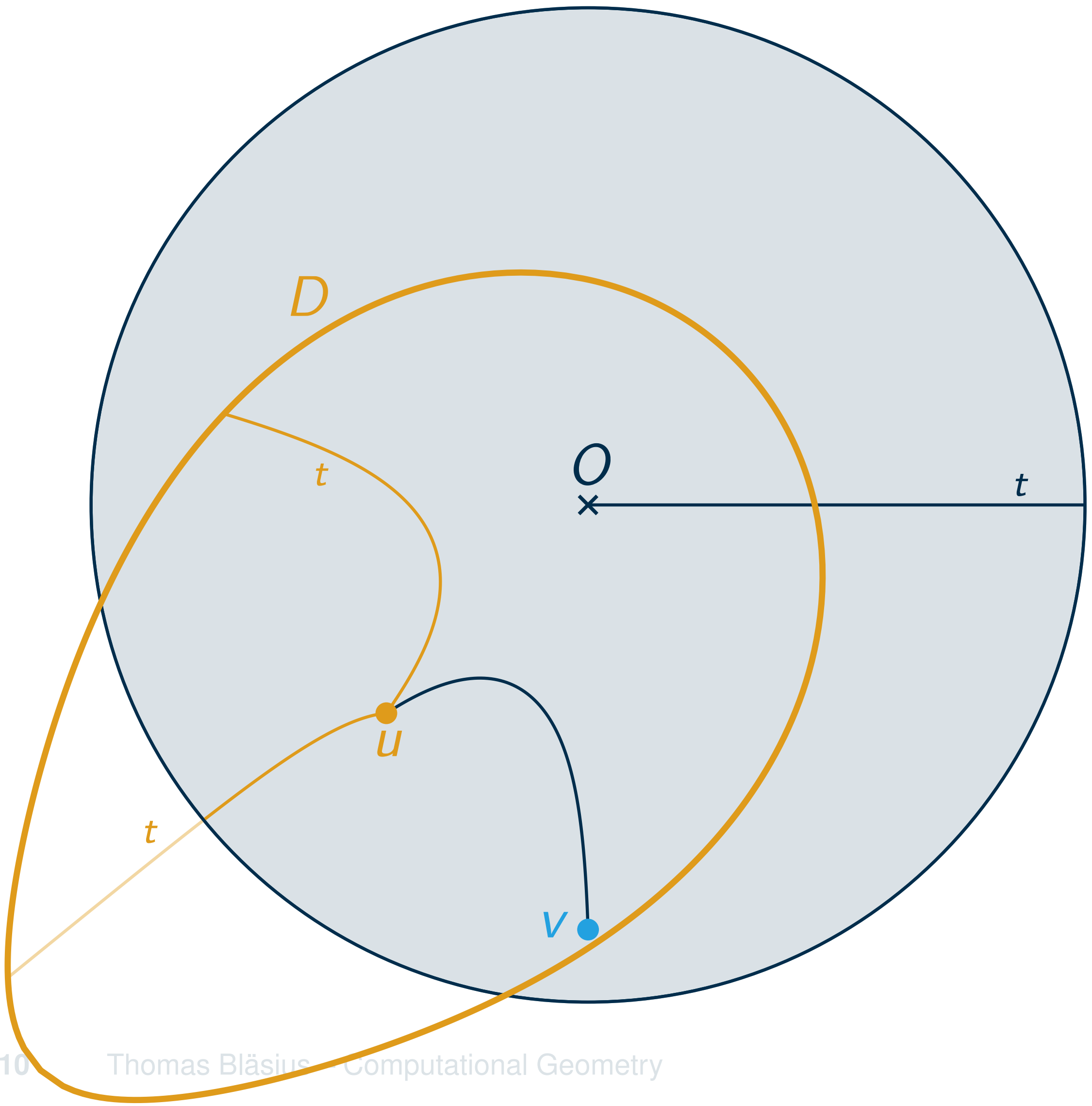

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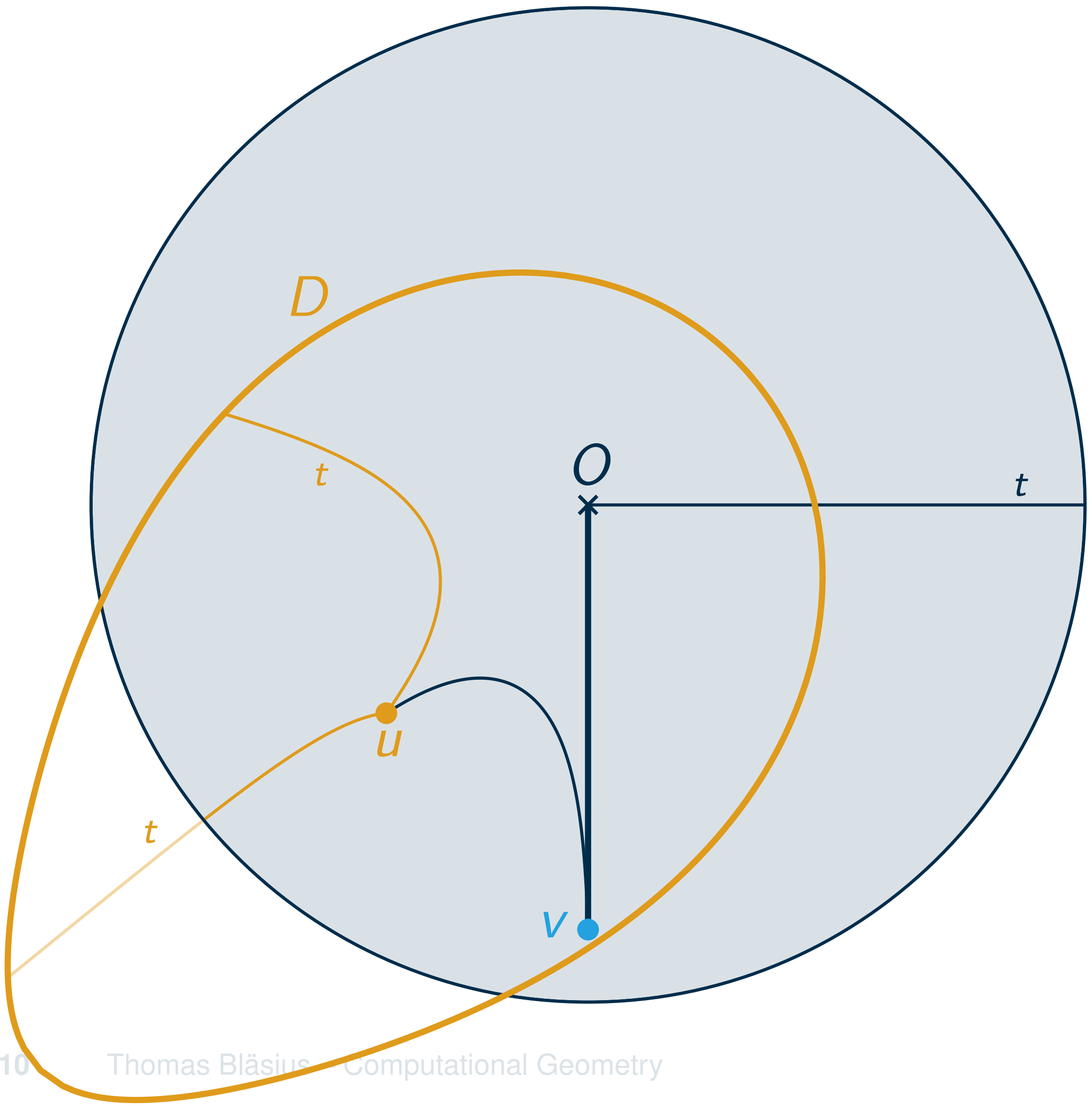

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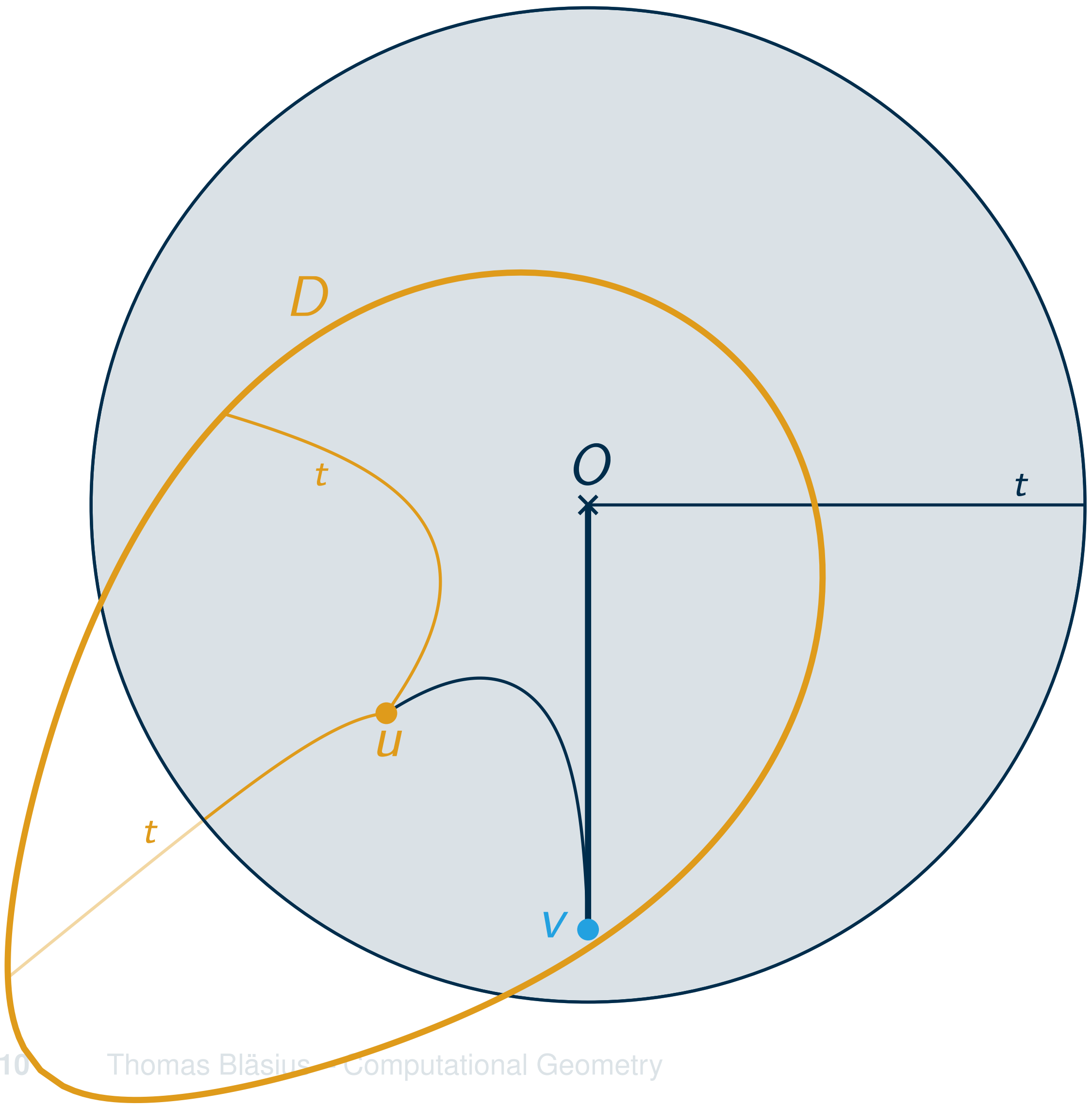

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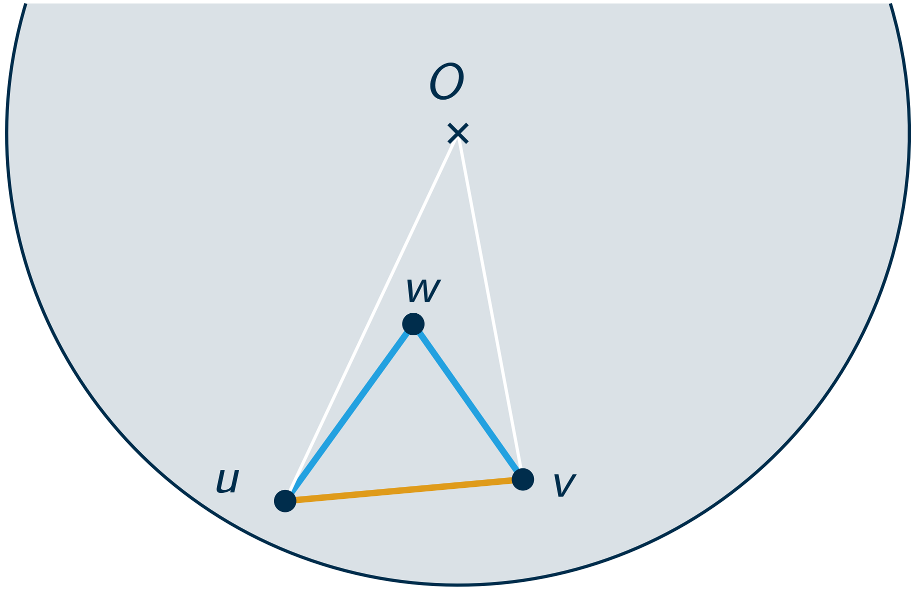
- $d(u, v') \leq t$ for every v' on $\overline{vO}$

Forced Edges (Strongly Hyperbolic)

Situation

- strongly hyperbolic UDG-representation
- w lies between u and v (w.r.t. angle)
- w lies closer to the origin than u and v



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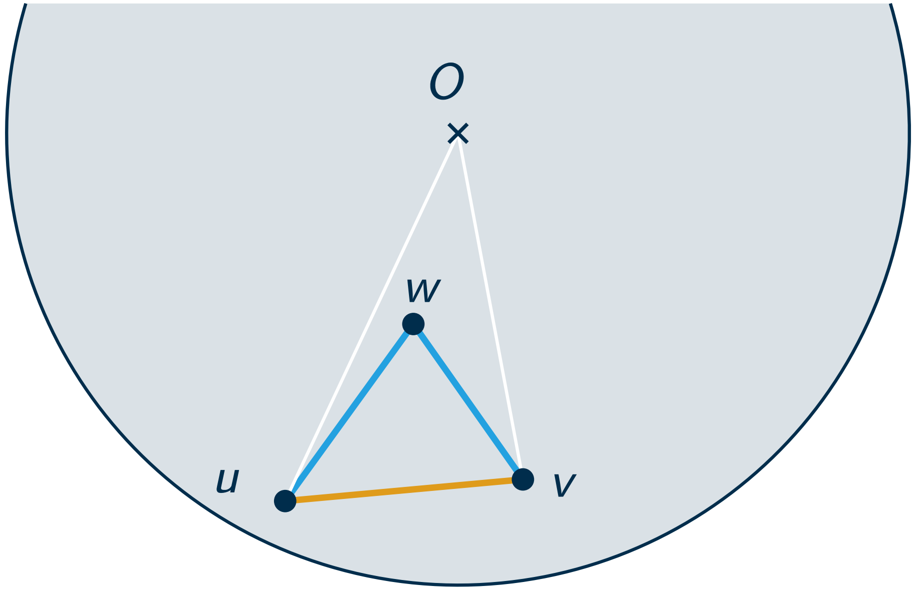

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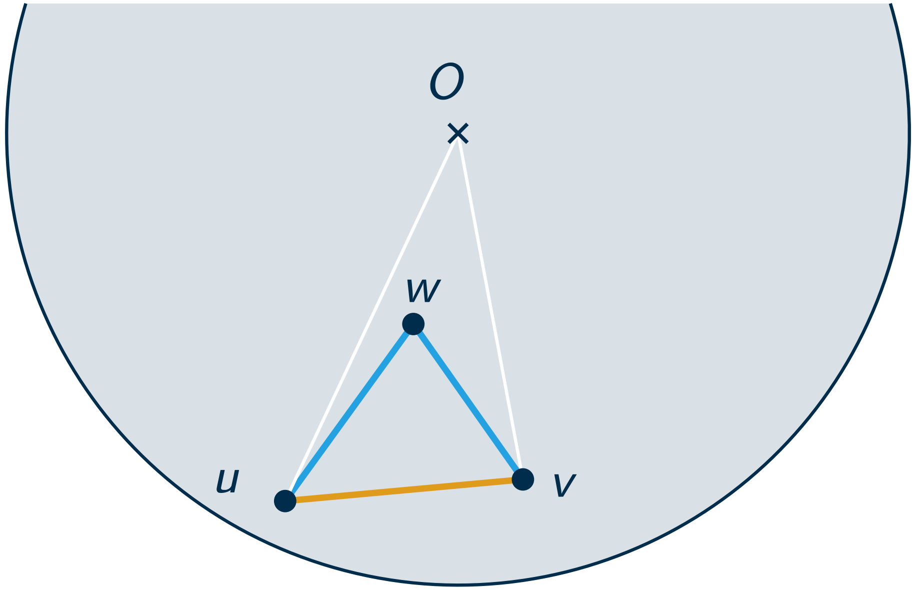

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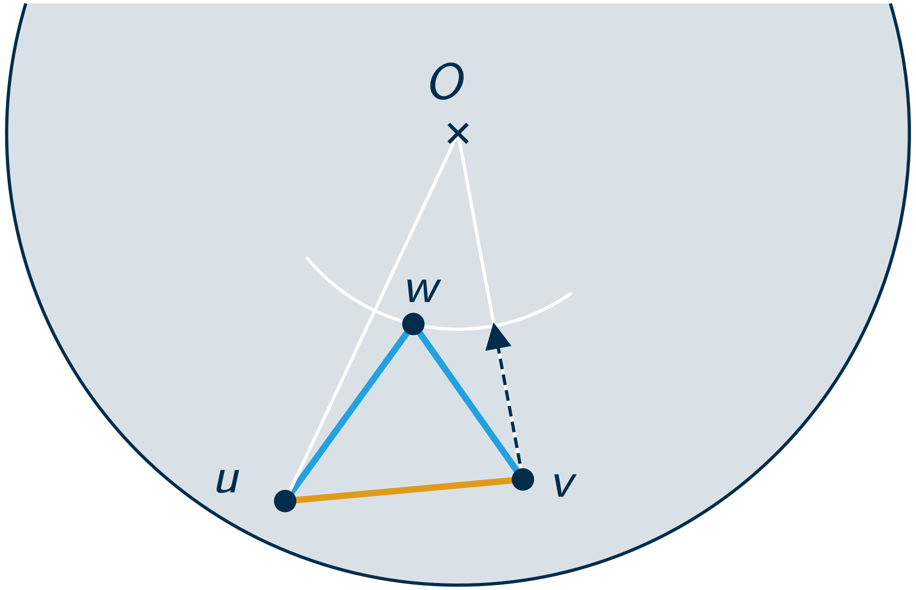

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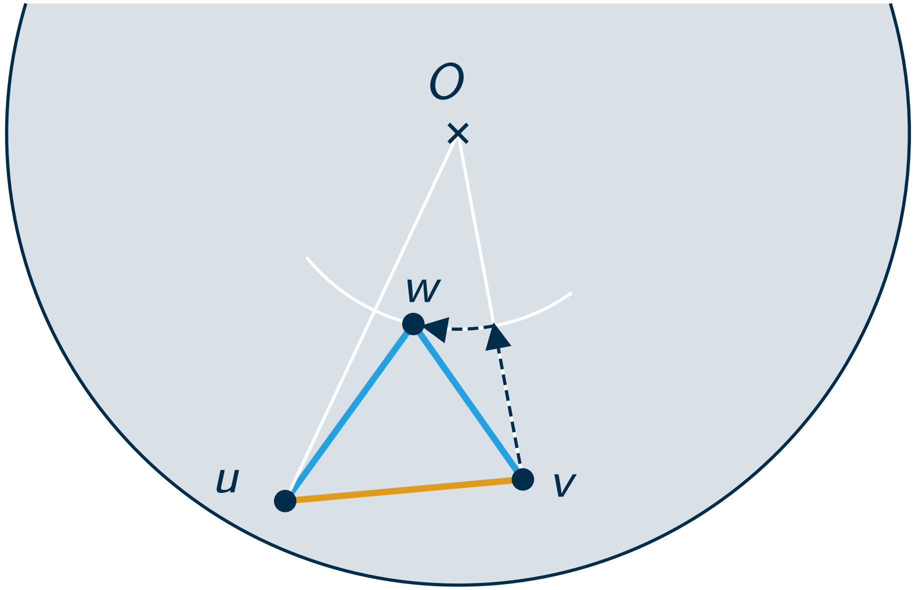

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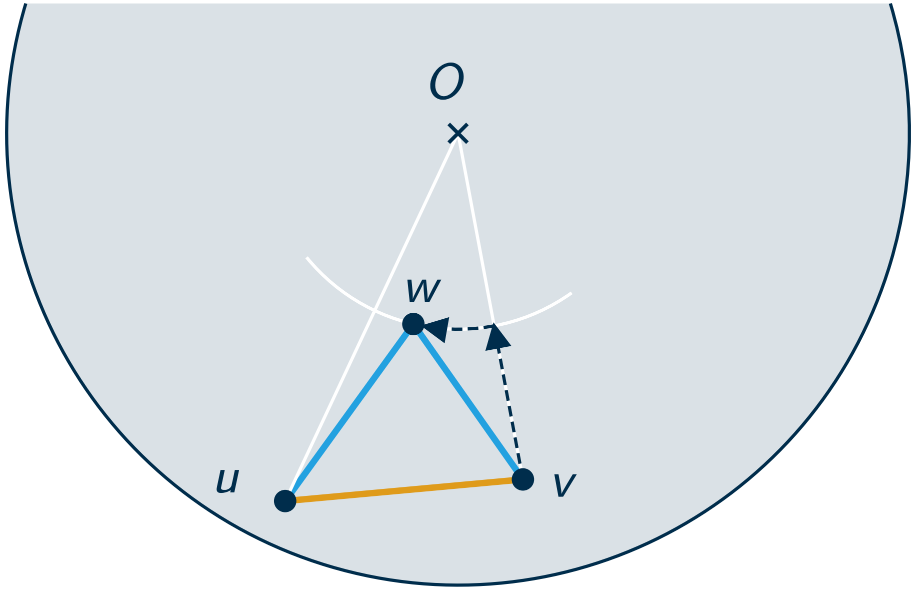

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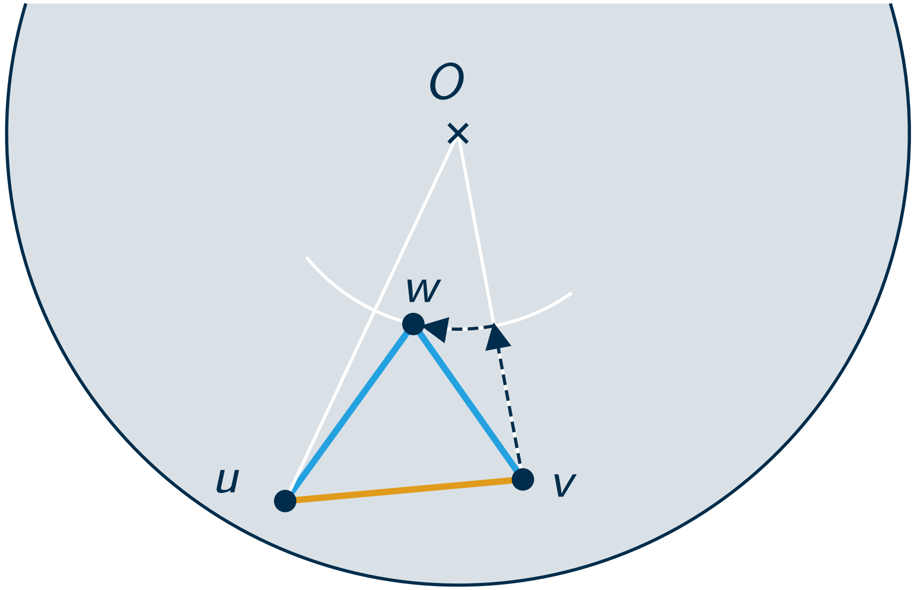

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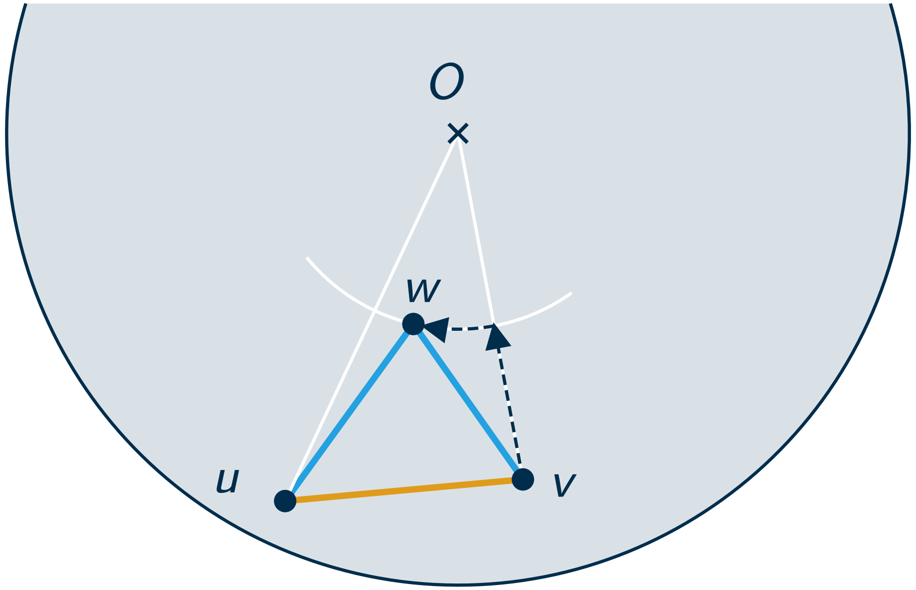

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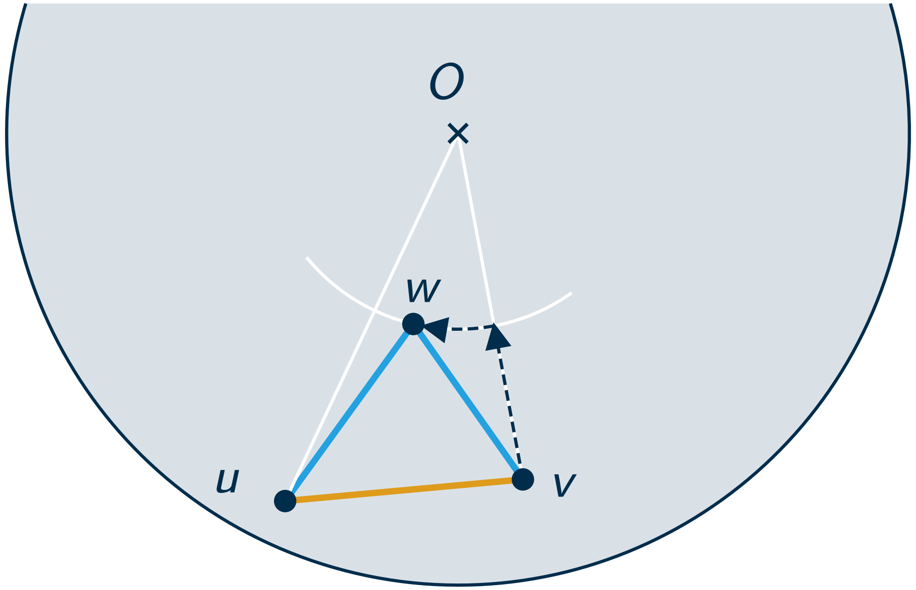

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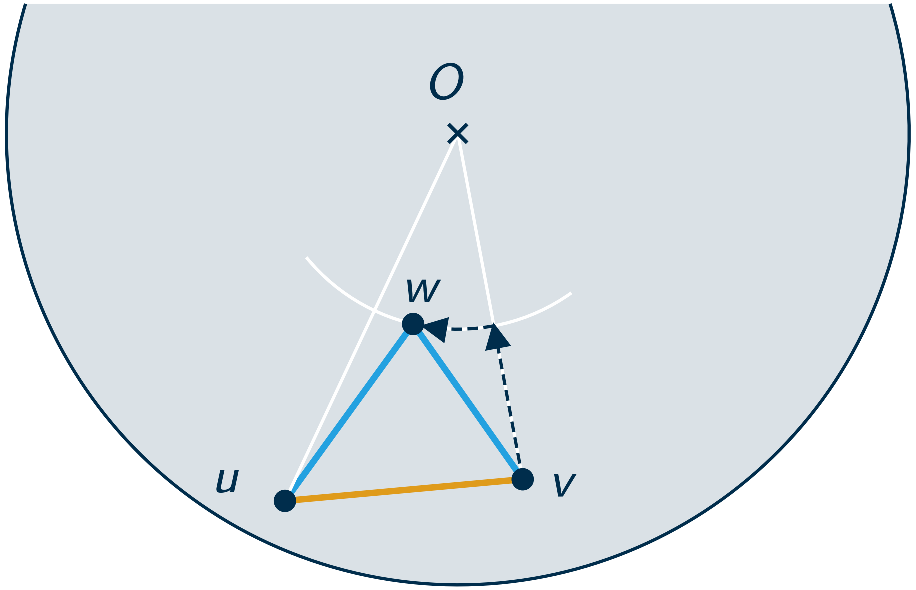

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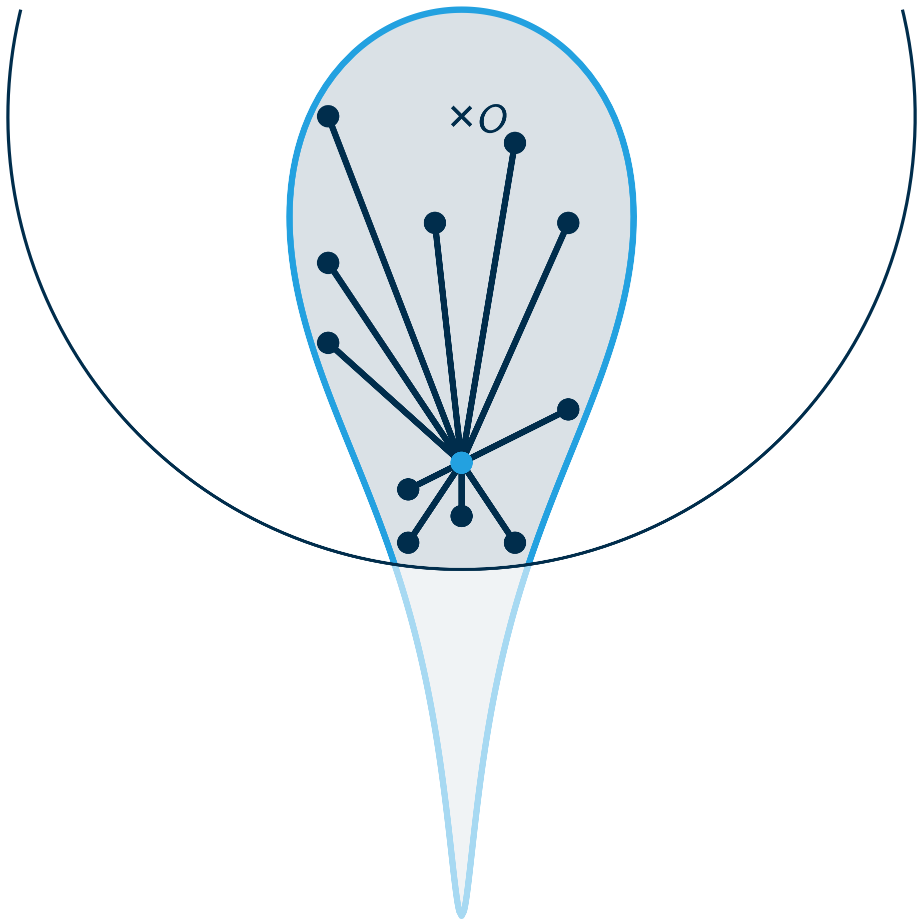
Notes

- you can't sneak past a vertex that lies closer to the origin without connecting to it
- hierarchical structure: the closer to the origin, the higher up in the hierarchy

Inner Neighborhood (Strongly Hyperbolic)



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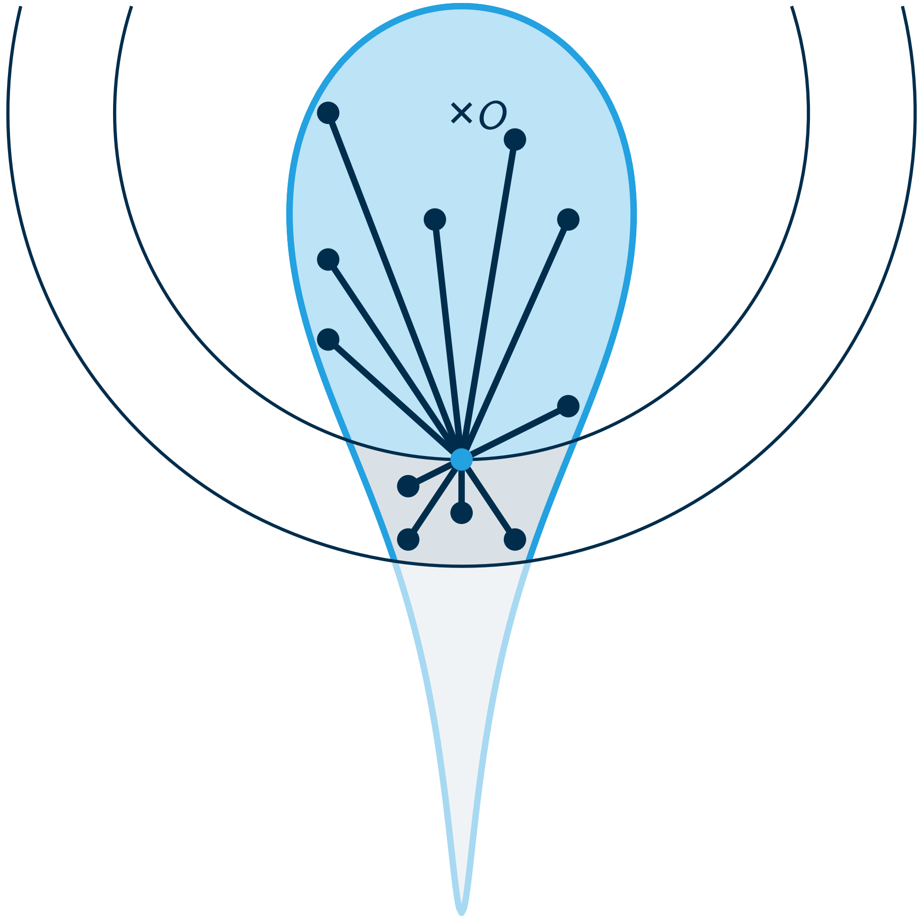

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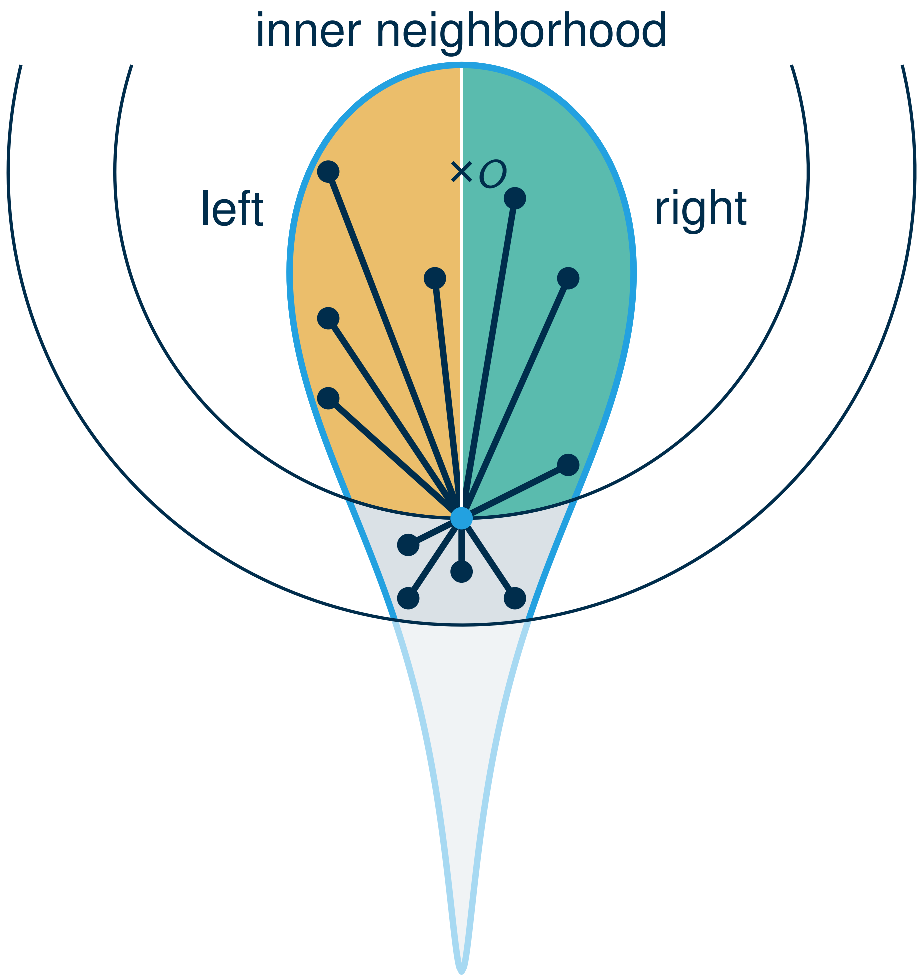

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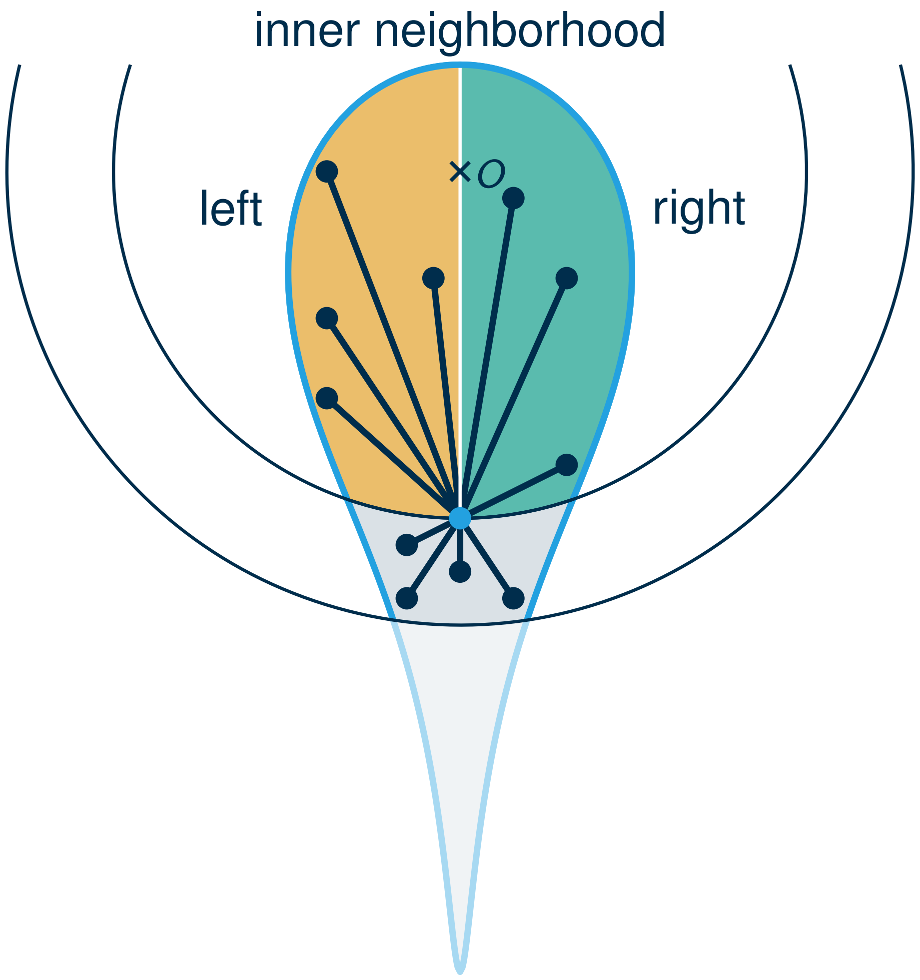
Theorem
The left and right inner neighborhood each from a clique.



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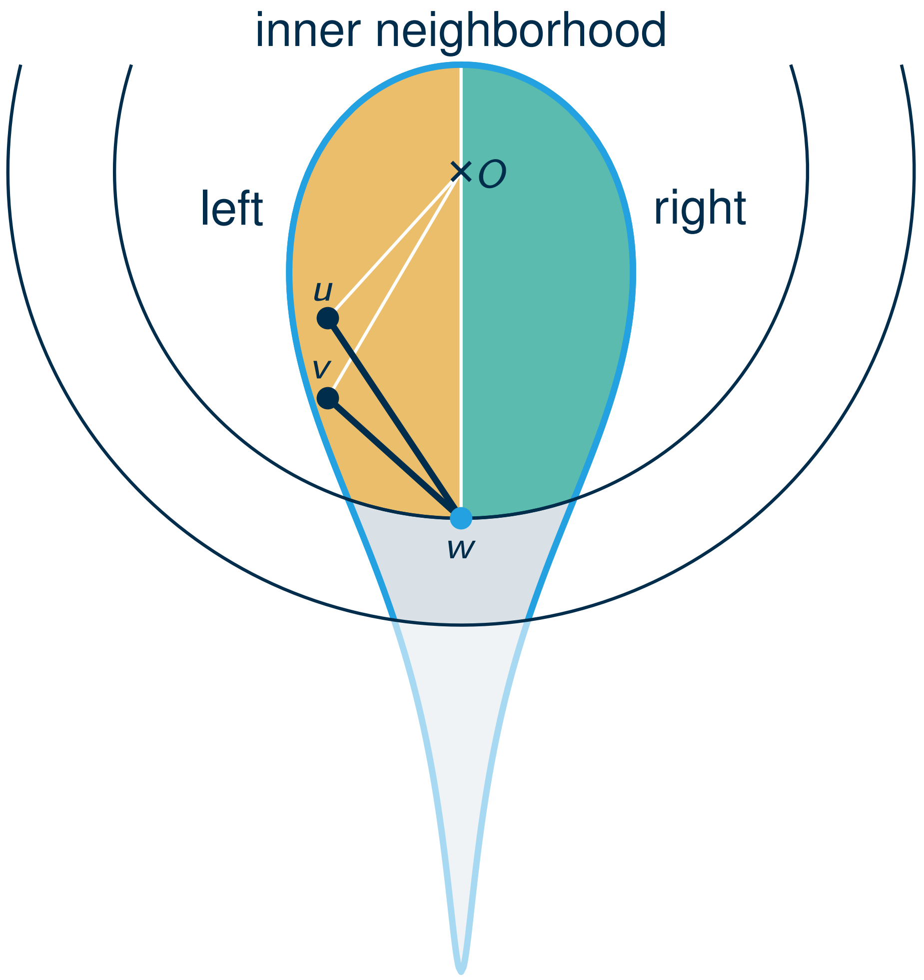

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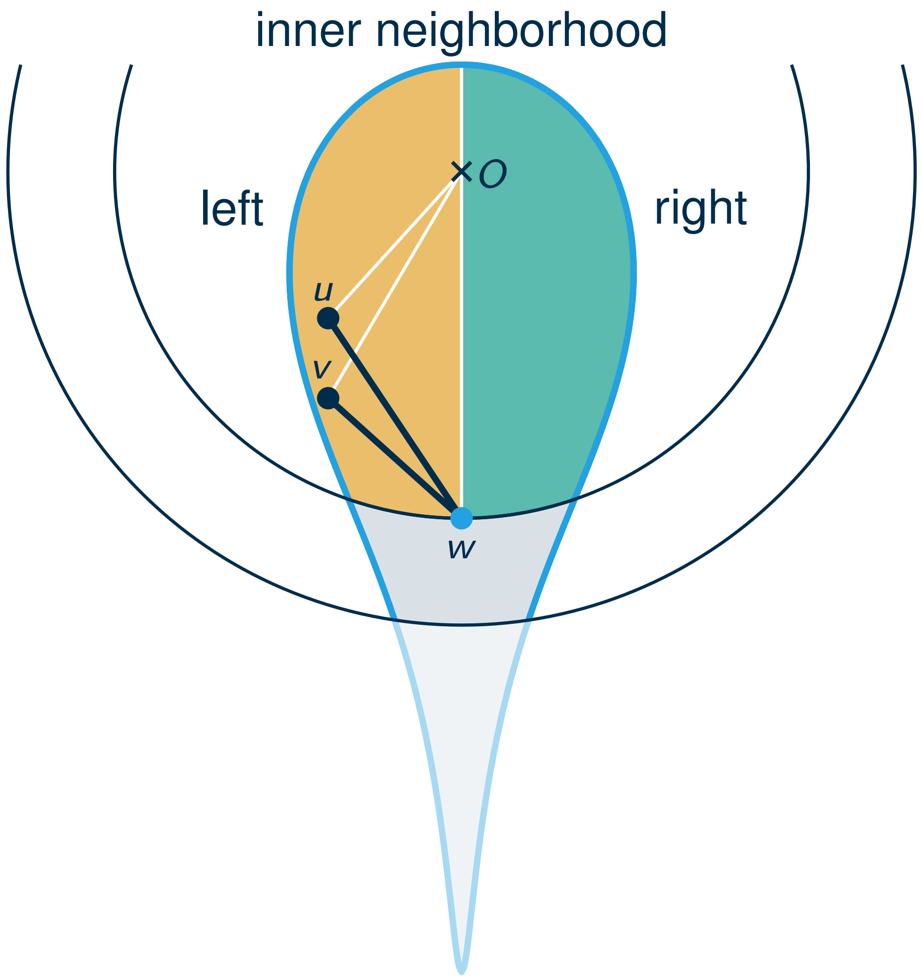

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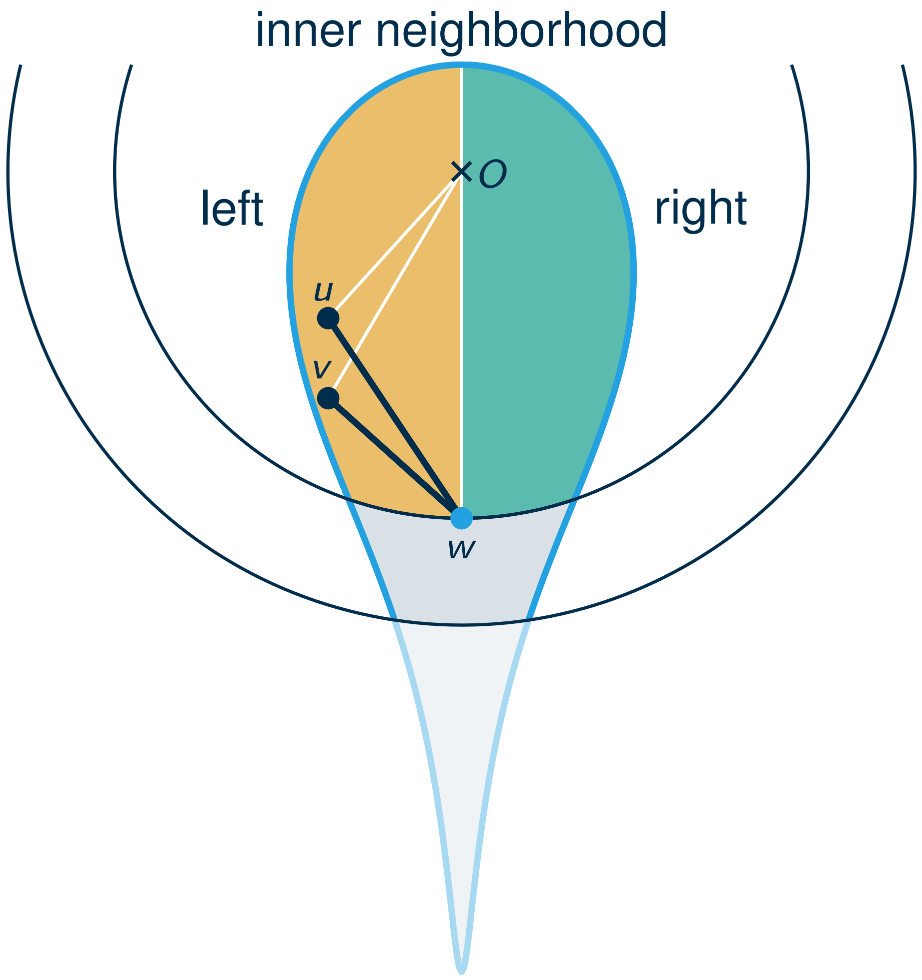

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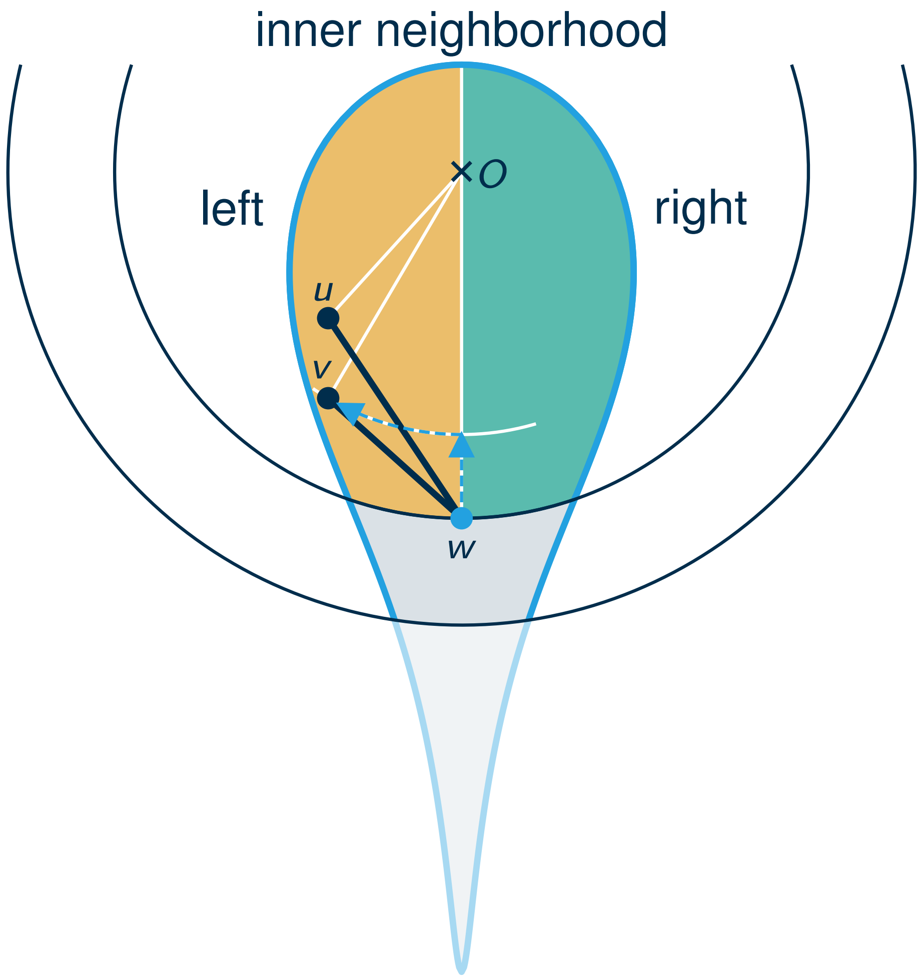

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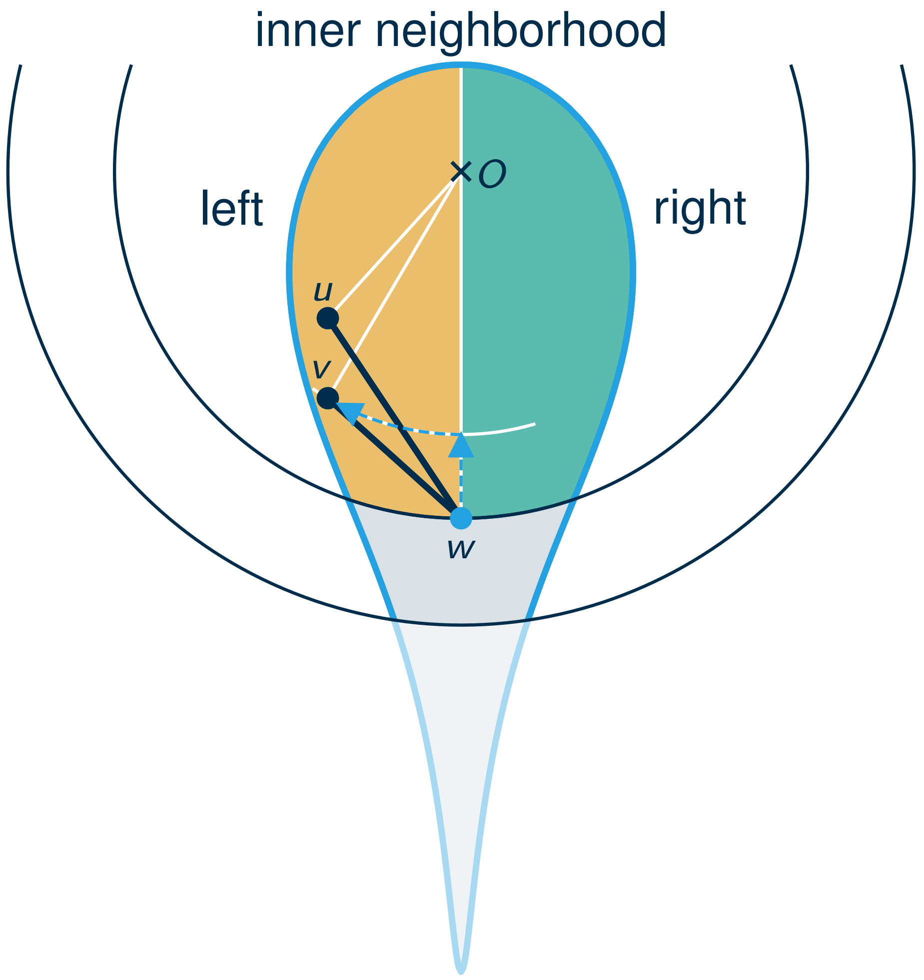

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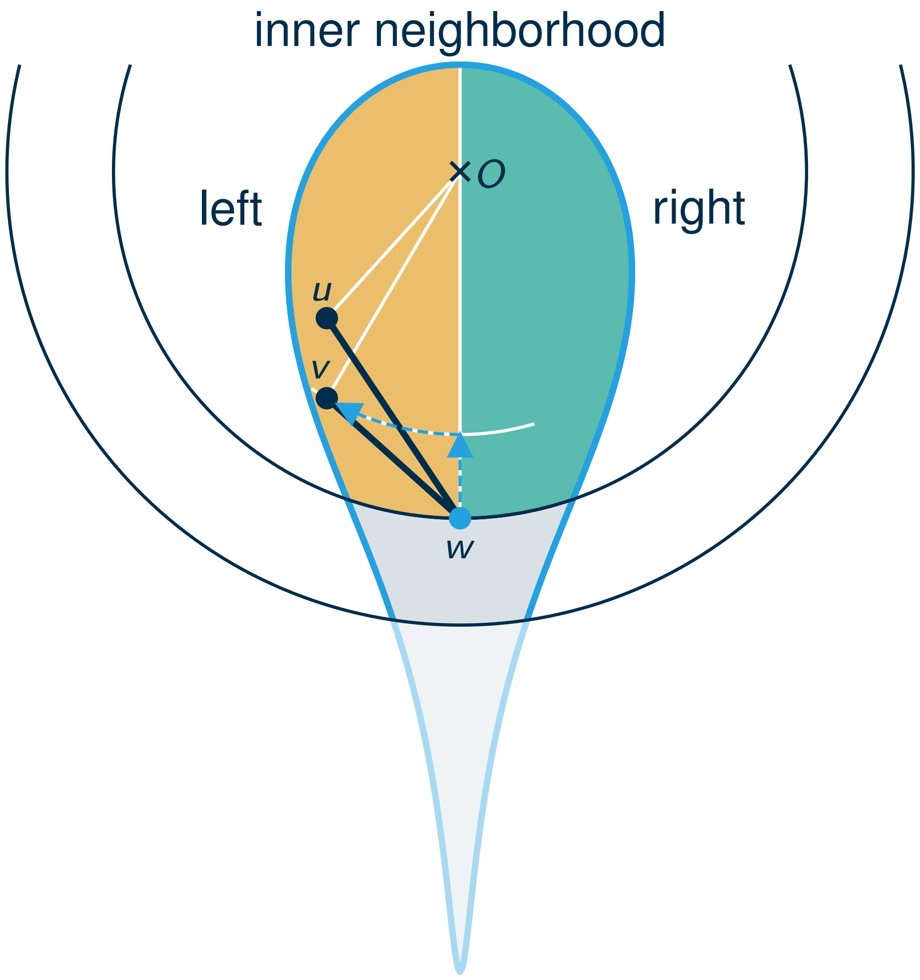

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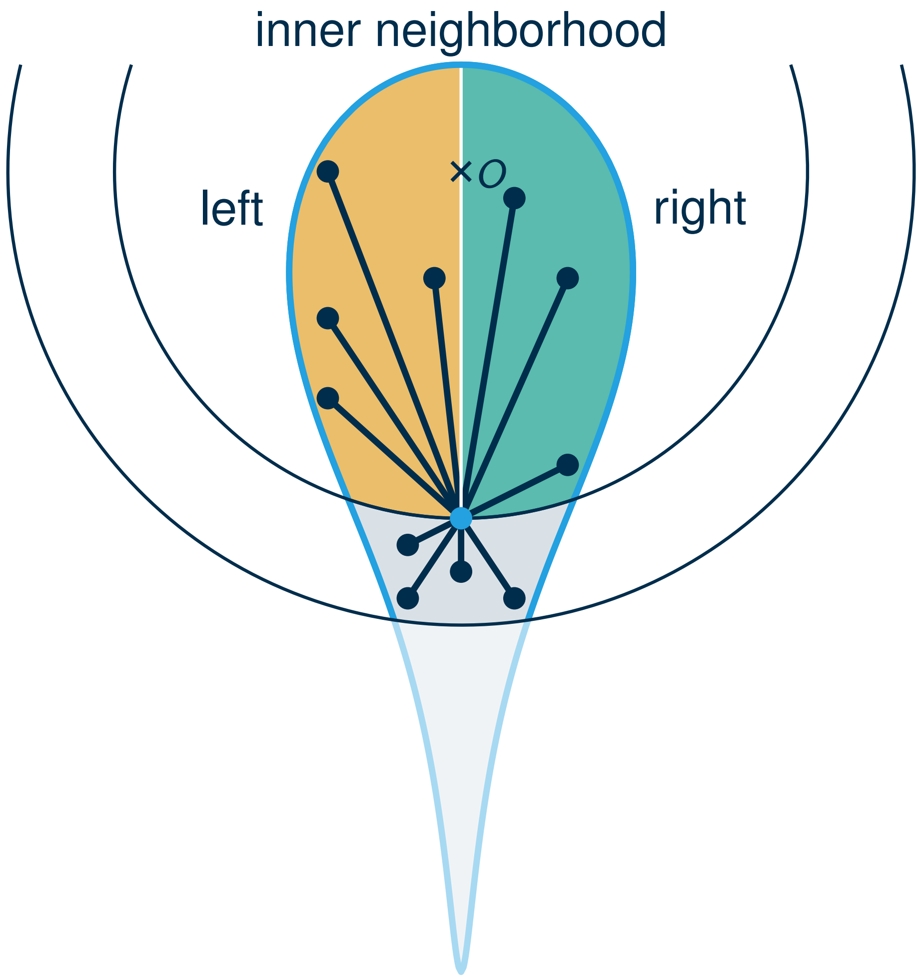

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- sort vertices $v_1, \ldots, v_n$ from large to small radius
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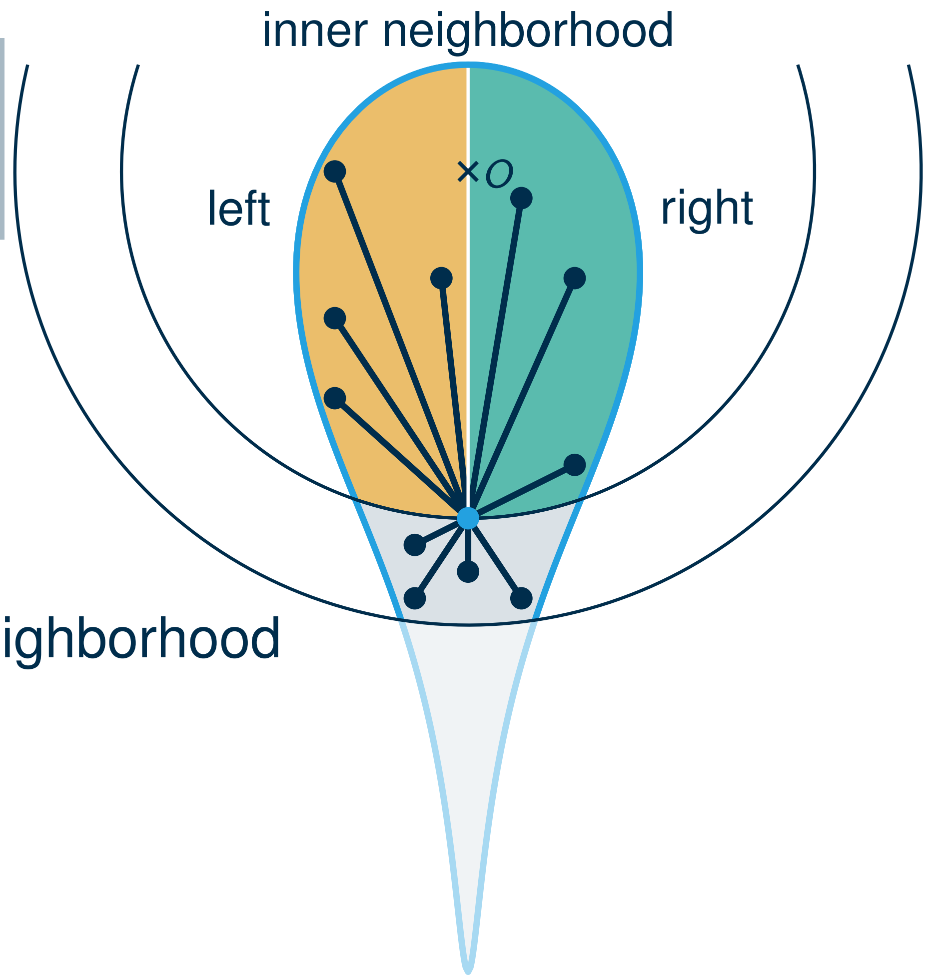

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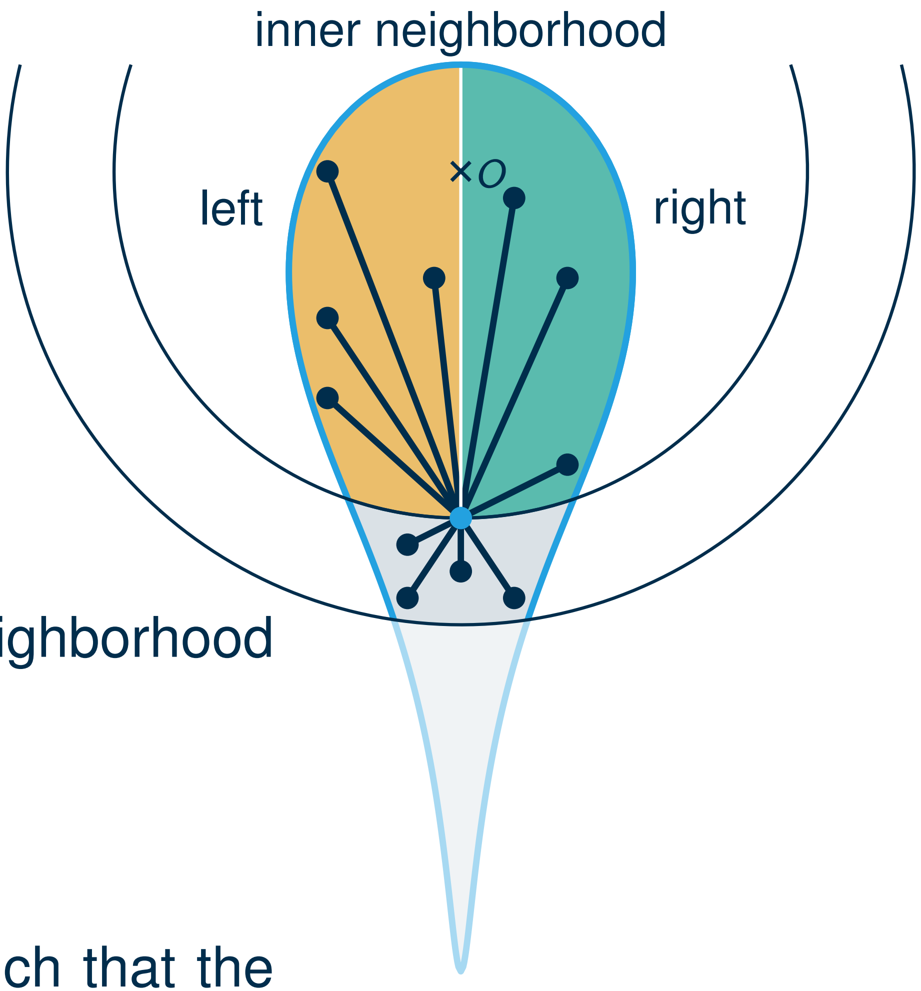

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Comparison: Chordal Graphs

- graph chordal $\Leftrightarrow$ perfect elimination scheme
- perfect elimination scheme: iteratively delete vertices such that the neighborhood of the deleted vertex forms clique

What Can We Do With This?

Corollary
Let $G = (V, E)$ be a strongly hyperbolic UDG. There is a vertex order $V = \{v_1, \dots, v_n\}$ such that the neighborhood of v_i in $G_i = G[\{v_i, \dots, v_n\}]$ can be covered by two cliques ($\forall i \in [n]$).

short break

Can we efficiently test whether a graph can be covered with two cliques?

Does such an order help to find the largest clique in G?

What Can We Do With This?

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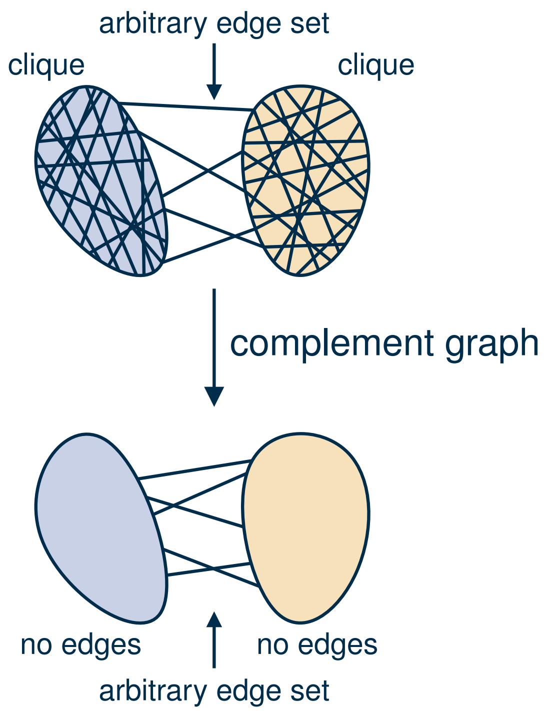
Graphs That Can Be Covered With Two Cliques

- complement can be covered with two independent sets (aka bipartite)



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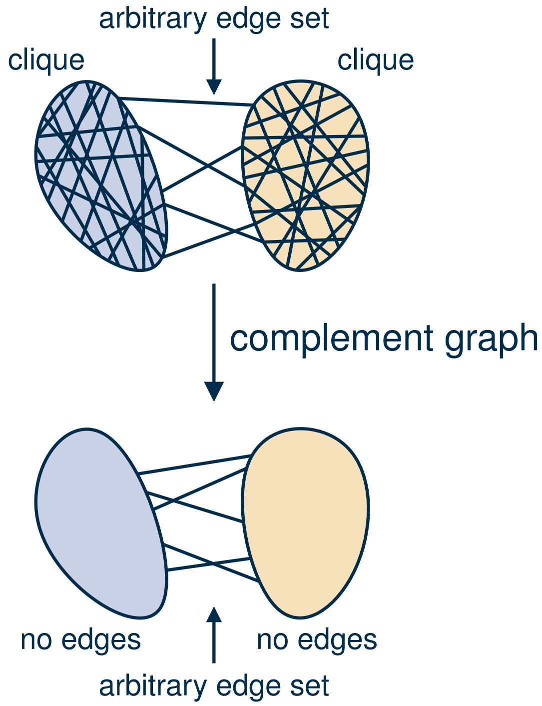

Can we also check efficiently, whether a graph can be covered with three cliques?

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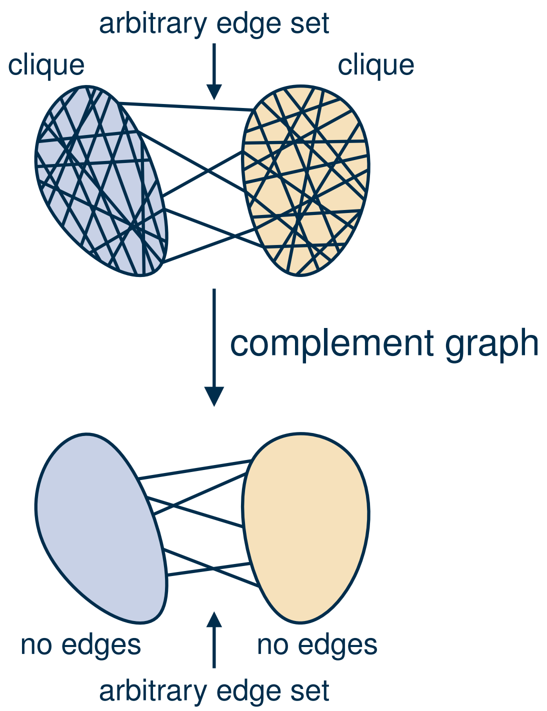

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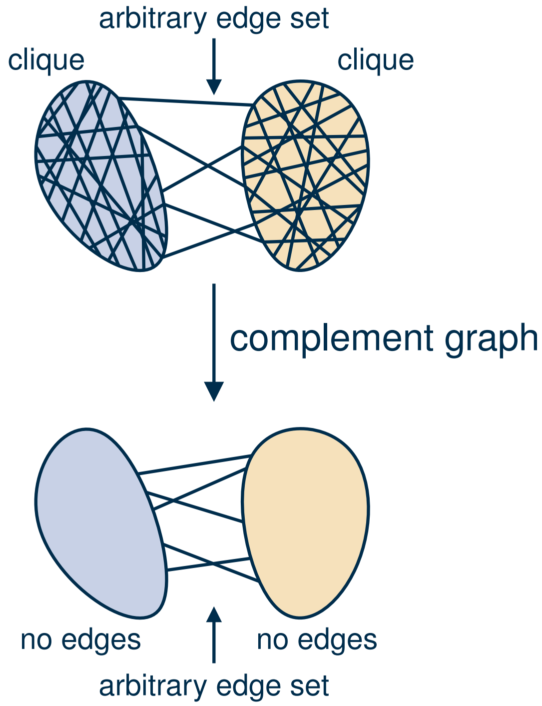

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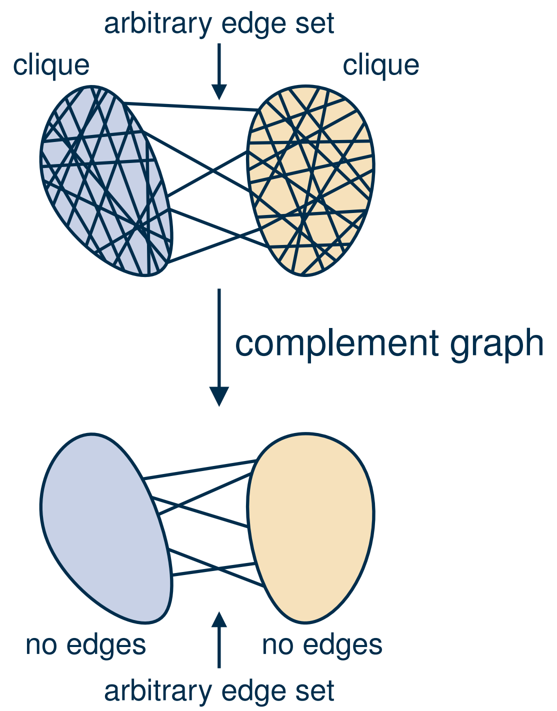

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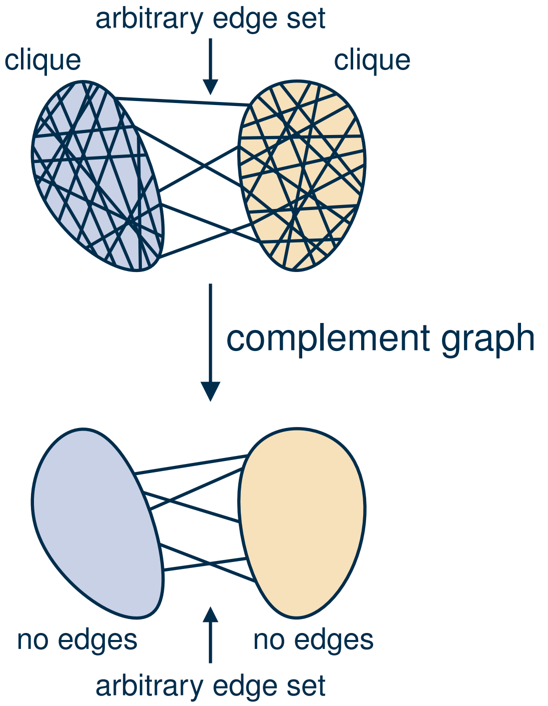

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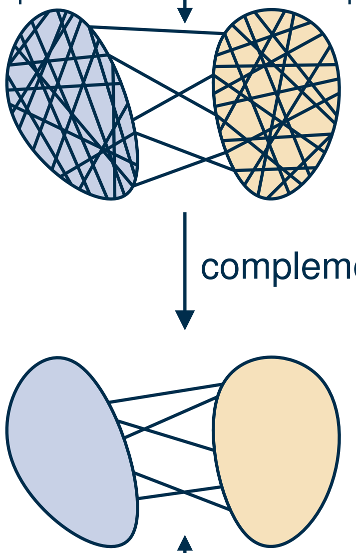

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Finding A Maximum Clique in G

- the maximum clique has a first vertex (smallest i)
- for every i: find maximum clique of G_i that contains v_i
- maximum over all i yields largest clique in G

Wrap-UP

Seen Today

- hyperbolic uniform disk graphs as generalization of Euclidean UDGs
- finding the maximum clique in strongly hyperbolic uniform disk graphs in polynomial time

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  - good benchmarks instances for evaluating algorithms (efficient generator available)

Eval

Evaluation

- overall very positive :-)

1.20) Nicht gefallen hat mir insbesondere:

- -
- Das in der Vorlesung zur Delauny Triangulation die 5 min Pause gefehlt haben :(
- Die geometrischen Argumente könnten für mich öfter noch detailreicher ausfallen, also ein bisschen näher an Axiomen. Oder man baut am Anfang mal eine geometrische Toolbox auf.

  Die Kästchen mit "Why?" finde ich grundsätzlich gut, aber sind manchmal etwas extrem. Für komplexere Argumente lieber nicht nutzen.
- Für die Nachbereitung zuhause sind die Folien manchmal nicht so aufschlussreich, wenn man in der Vorlesung mal nicht mitgekommen ist. Zwar sind die Beispiele während der Vorlesung selbst sehr anschaulich und hilfreich, für die Nachbereitung würden aber manchmal 1-2 mehr erklärende Sätze auf den Folien doch auch nochmal weiterhelfen
- Geometrie
- Mir fällt die Vorlesung sehr schwer, ich kann häufig nicht allen Schritten auf den Folien folgen. Ich wüsste aber auch nicht, was man da anders machen könnte. Ich glaub ich muss einfach mehr Vor- und Nachbereiten
- Wenn alle Menschen im Raum in der Lage sind die Veranstaltung auf deutsch abzuhalten (was zumindest laut einer Umfrage in der Übung der Fall war), ist das vielleicht doch auch ganz angenehm (:

Thursday & Wednesday

Last Meeting On Thursday

- I can give a overview over all topics
- let me know, if you have topics I should repeat in more detail
- some infos concerning the exam

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Picnic On Wednesday

- starting at 5pm (17:00) at “Wiese am Fasanengarten” (or indoors with board games if the whether is bad)
- also see: `https://cloud.iti.kit.edu/index.php/s/WKXecGDdcZCtrQF`
- please bring your own cutlery and plate