Computational Geometry

Geometric Graphs – Euclidean and Hyperbolic

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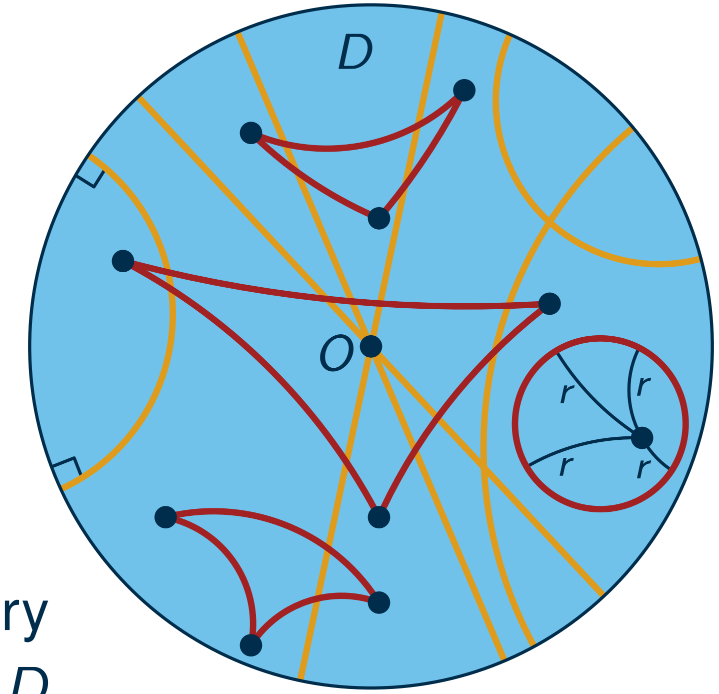
Recap: Poincaré Disk



Points

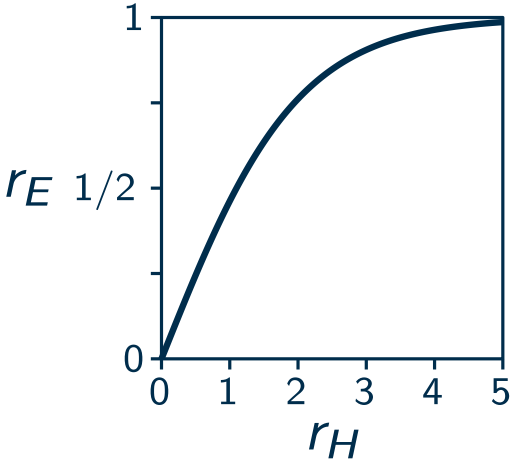
- consider a (Euclidean) disk D with radius 1 around the point O
- let $\mathcal{P}$ be the set of points in the interior of the disk

Lines

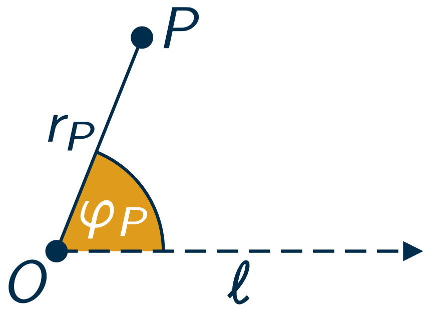
- let $\mathcal{L}$ be the union of:
  - set of open segments through O with endpoints on D's boundary
  - set of open circular arcs in D perpendicular to the boundary of D

Observations

- close to O: very similar to the Euclidean plane
- to use specifics of the hyperbolic plane, we need to go far away from O
- problem: we quickly approach the boundary of D
- different radii become hard to distinguish



(Native) Polar Coordinates



(counterclockwise)

Polar Coordinates

- reference: origin O, ray ℓ starting at O
- **radius** r_P of P: $d(O, P)$
- **angle** φ_P of P: angle between ℓ and $OP+$

Native Polar Coordinates

- d is the hyperbolic distance (i.e., not the Euclidean distance in the Poincaré disk)

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

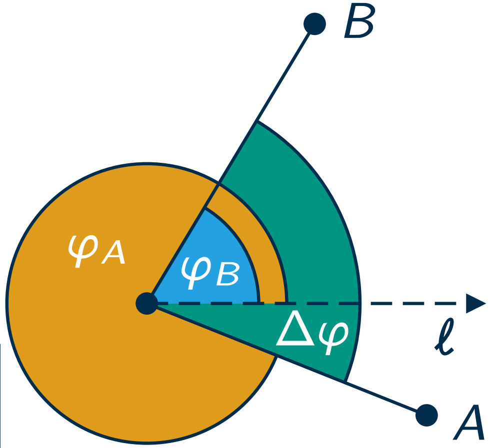
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Distance Between Points In Polar Coordinates

- consider two points $A = (r_A, \varphi_A)$ and $B = (r_B, \varphi_B)$
- let $\Delta\varphi$ be their angular distance: $\Delta\varphi = \min\{|\varphi_A - \varphi_B|, 2\pi - |\varphi_A - \varphi_B|\}$
- then: $d(A, B) = \operatorname{arcosh}\left[\cosh(r_A)\cosh(r_B) - \sinh(r_A)\sinh(r_B)\cos(\Delta\varphi)\right]$

$$\approx \log\left[2 \cdot (e^{r_A}/2 \cdot e^{r_B}/2 - e^{r_A}/2 \cdot e^{r_B}/2 \cdot \cos(\Delta\varphi))\right]$$

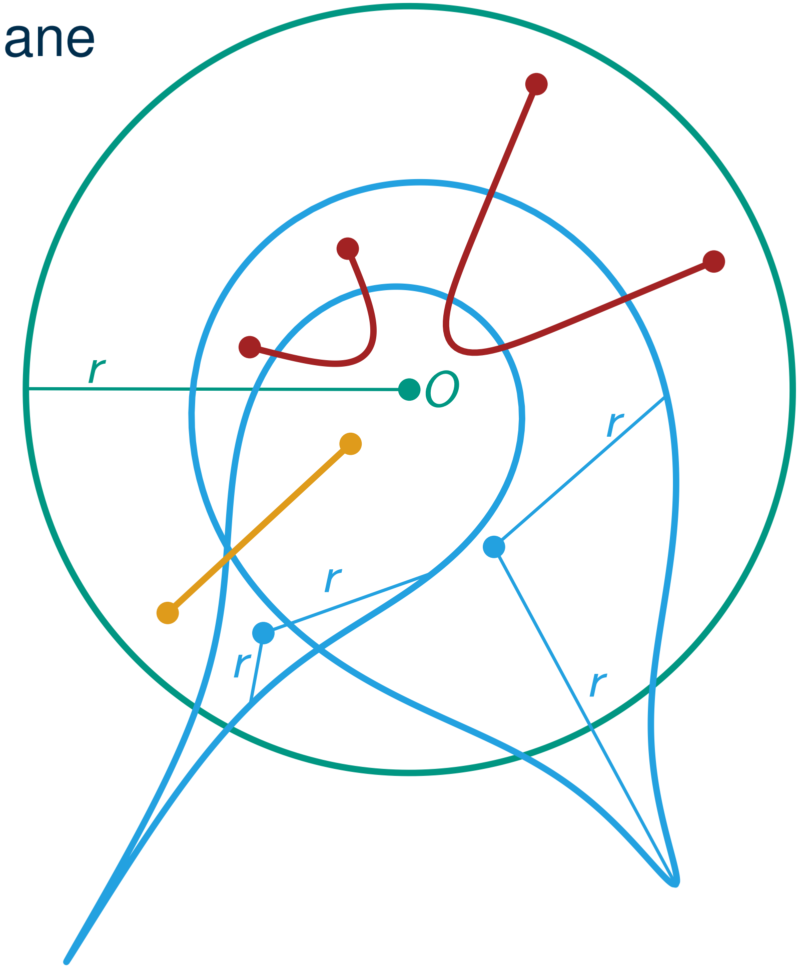
$$= \log\left[e^{r_A + r_B} \cdot (1 - \cos(\Delta\varphi))/2\right] = r_A + r_B - \log\left[\frac{2}{1 - \cos(\Delta\varphi)}\right]$$



The Native Model

Native Model

- use native polar coordinate for every point in the hyperbolic plane
- pretend that they are just Euclidean polar coordinates
- circles around the origin are circles
- circles with different centers are tear-shaped
- segments on lines through the origin are segments
- their visible length is the correct hyperbolic length
- other segments are bent towards the origin
- representation does **not** preserve angles



$$d(A,B) \approx r_A + r_B - \log\left[\frac{2}{1-\cos(\Delta\varphi)}\right]$$

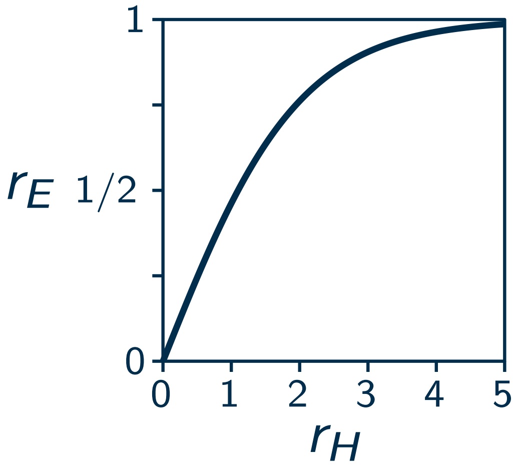
Poincaré vs. Native

Advantages Of The Poincaré Disk

- representation preserves angles
- lines are Euclidean circular arcs, circles are Euclidean circles $\rightarrow$ these are familiar objects
- we can use our Euclidean intuition to gain insights on the hyperbolic plane

Disadvantages Of The Poincaré Disk

- points quickly boundary for growing hyperbolic distance
- can’t see anything except for very close to the origin
- prone to numeric issues

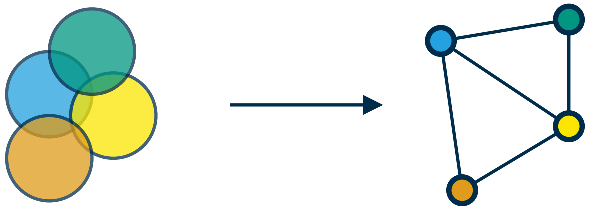


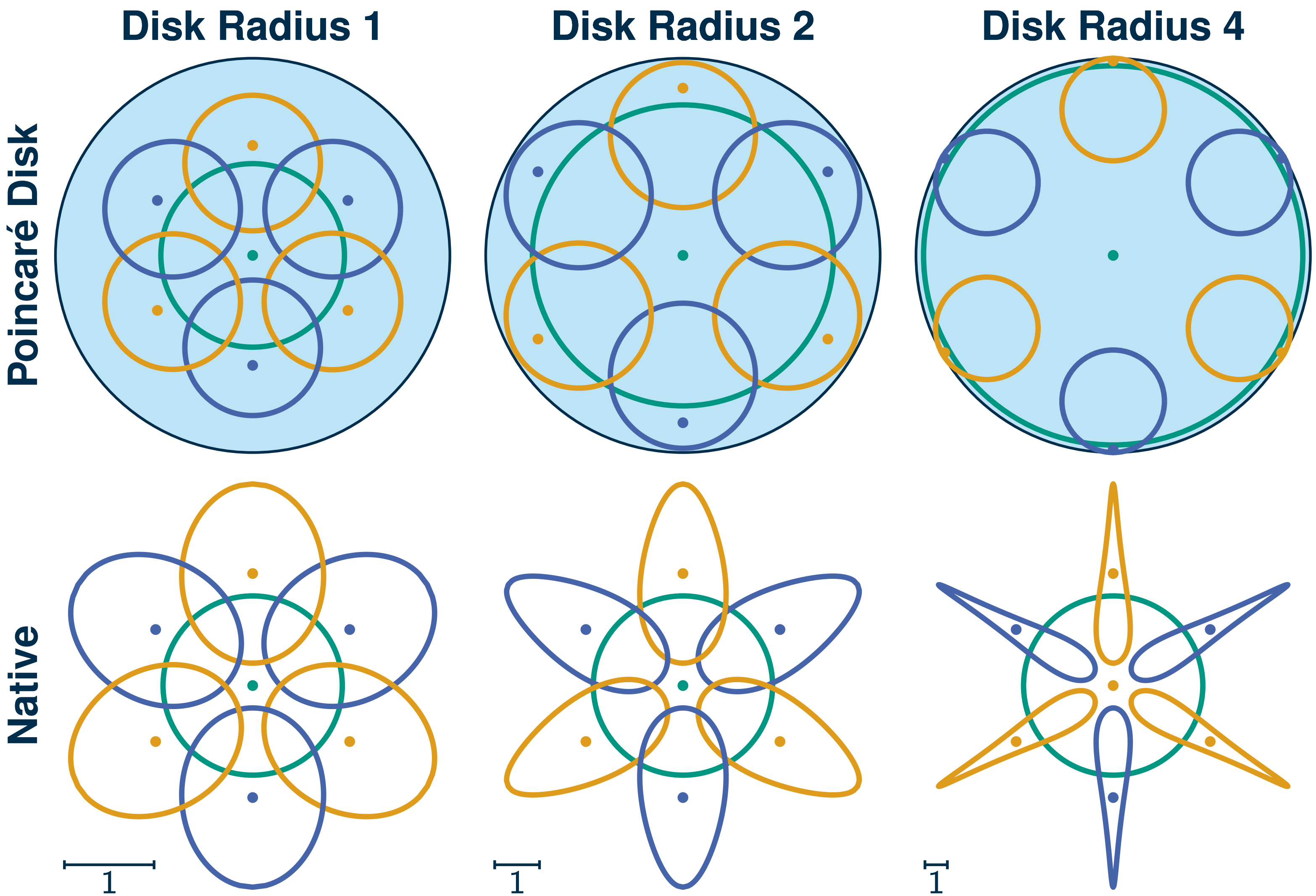
Heuristic For Choosing A Model

- visual representation of hyperbolic data $\rightarrow$ native model
- computations on coordinates $\rightarrow$ native model (or also: hyperboloid)
- thinking about and proving stuff $\rightarrow$ Poincaré Disk (or also: upper half-plane, Beltrami–Klein)

Unit-Disk Graphen

Definition
A graph is a **unit disk graph** if it is the intersection graph of disks of radius 1.





And Now Hyperbolic

- can be defined analogously
- but: the radius matters

Hyperbolic Uniform Disk Graphs

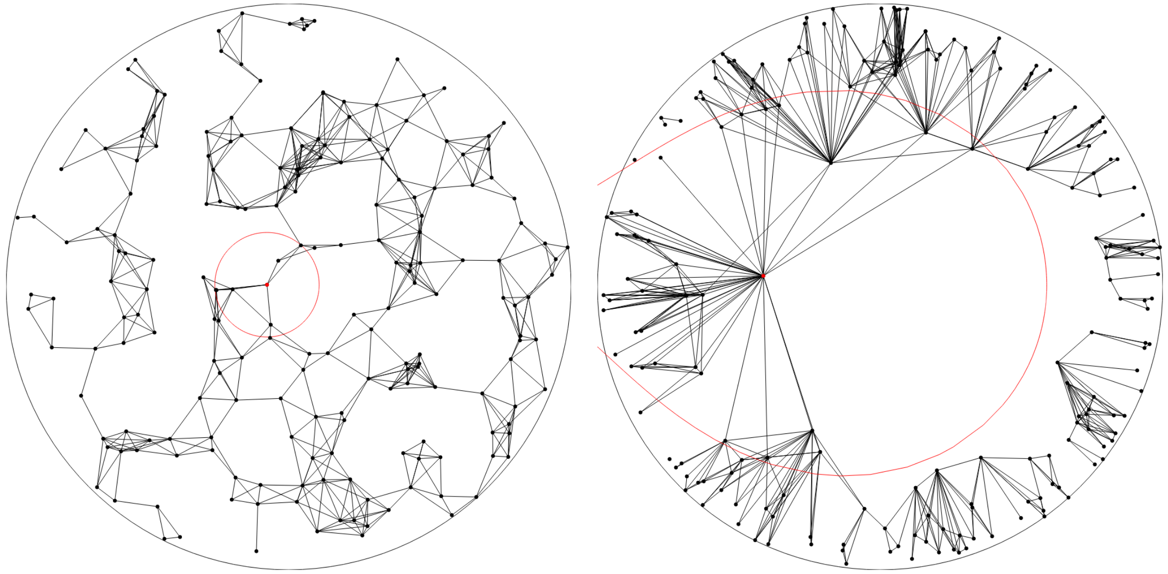
Definition

Note: the disks in the intersection representation have radius $t/2$

$G = (V, E)$ is a **hyperbolic uniform disk graph** if it there are vertex positions $p\colon V \to \mathbb{H}^2$ and a threshold t such that $uv \in E \Leftrightarrow d(p(u), p(v)) \leq t$.

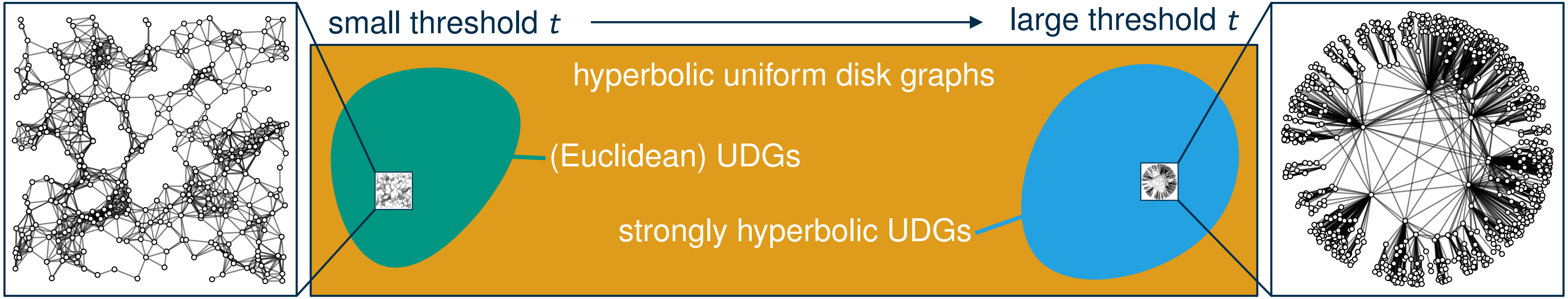
Do I Get Different Graph Structures Depending On t?

- small threshold t
  - similar to the Euclidean setting
  - regular / homogeneous
  - grid-like
- large t
  - irregular / heterogeneous
  - hierarchical / tree-like



https://thobl.github.io/hyperbolic-unit-disk-graph/

How Hyperbolic Should It Be?



Situation

- Euclidean UDGs are a subclass of hyperbolic UDGs
- many hyperbolic UDGs are not very hyperbolic

Strongly Hyperbolic Uniform Disk Graphs

- goal: complement to Euclidean UDGs
- with hierarchical / heterogeneous structure
- How do we formalize this? How large is large enough for t?

There are different answers to this.
We look at only one of them.

Strongly Hyperbolic Uniform Disk Graphs

Definition

Note: the disks in the intersection representation have radius $t/2$

$G = (V, E)$ is a **hyperbolic uniform disk graph** if it there are vertex positions $p \colon V \to \mathbb{H}^2$ and a threshold t such that $uv \in E \Leftrightarrow d(p(u), p(v)) \leq t$.
It is a **strongly hyperbolic UDG** if p maps all vertices into a disk of radius t.

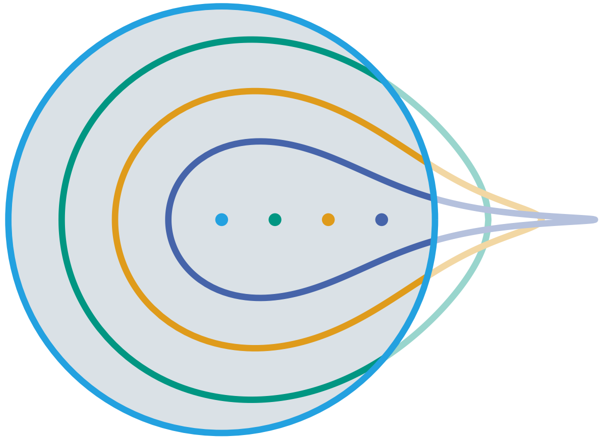
https://thobl.github.io/hyperbolic-unit-disk-graph/

Visualization

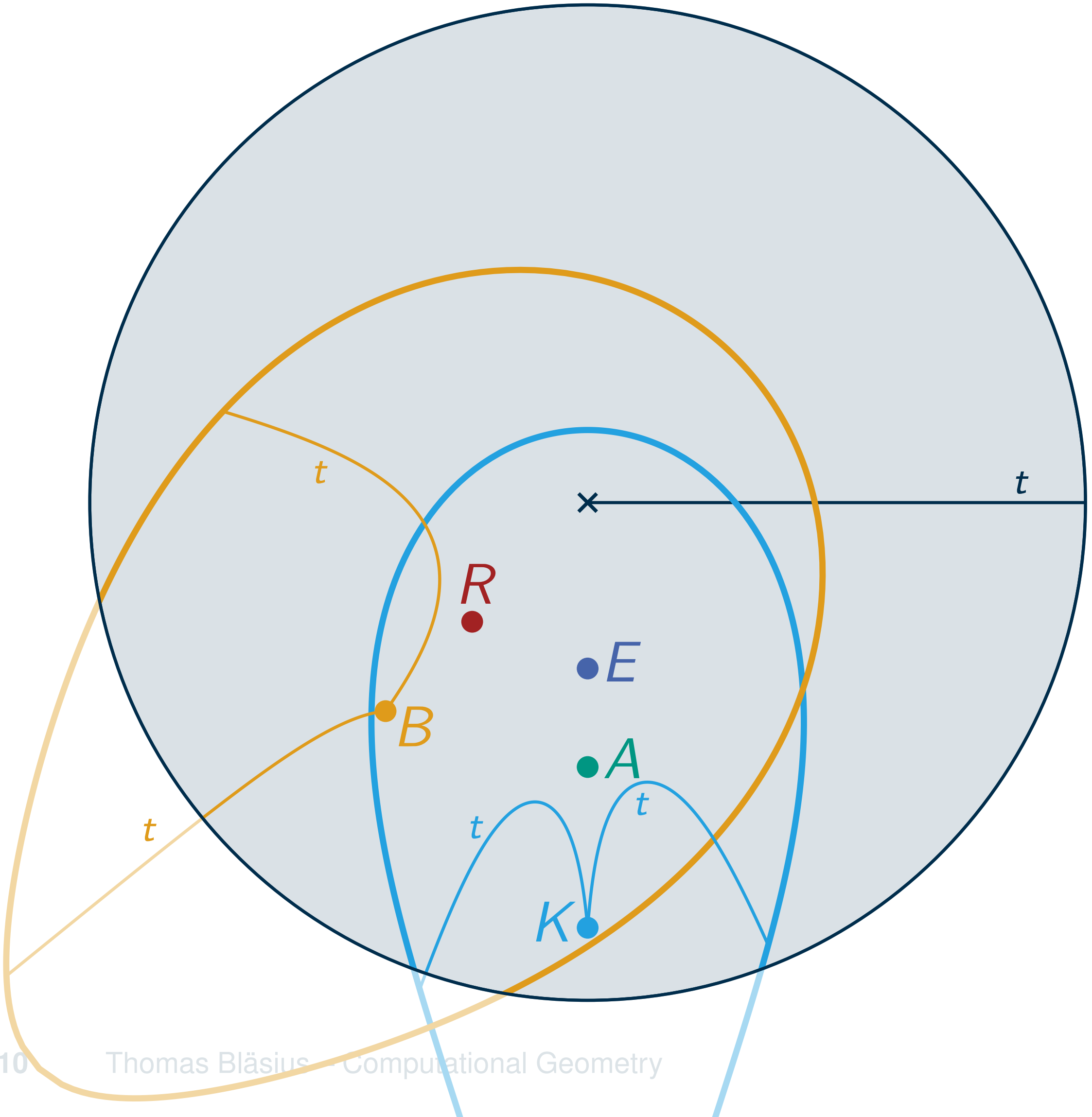
- choose the center of this disk as origin for the native polar coordinates
- **note:** not to be confused with the Poincaré Disk

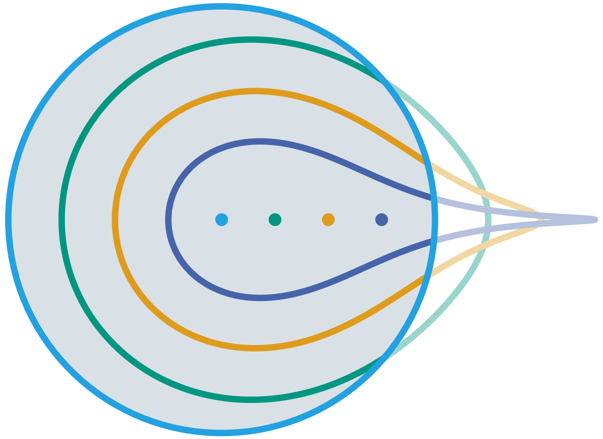
Observations

- a vertex at the origin is universal: adjacent to all other vertices
- the further out the vertex, the smaller its area of neighbors
- maximal heterogeneity: every distance from the origin yields differently sized area in which neighbors lie



Which Edges Exist?





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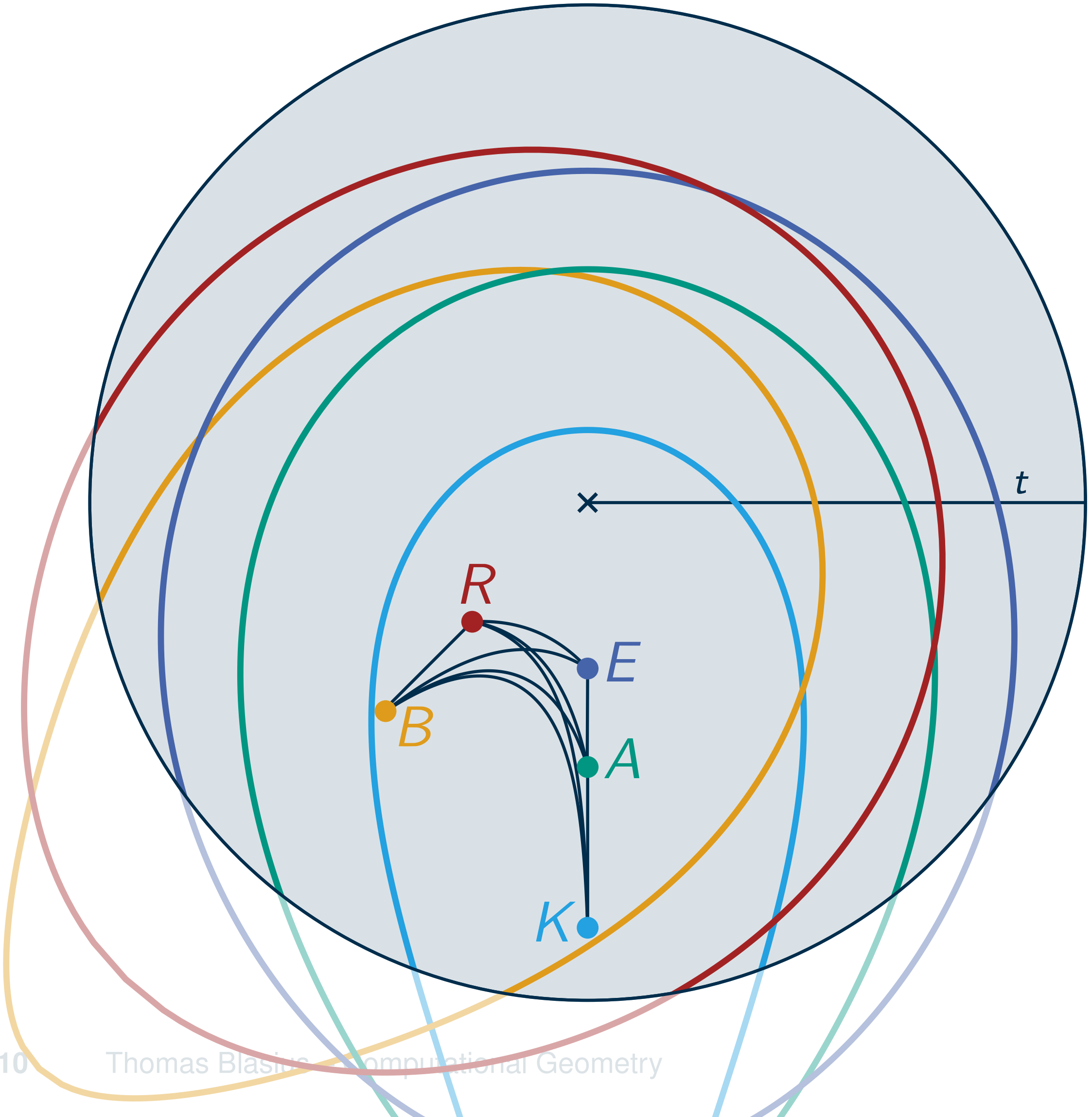
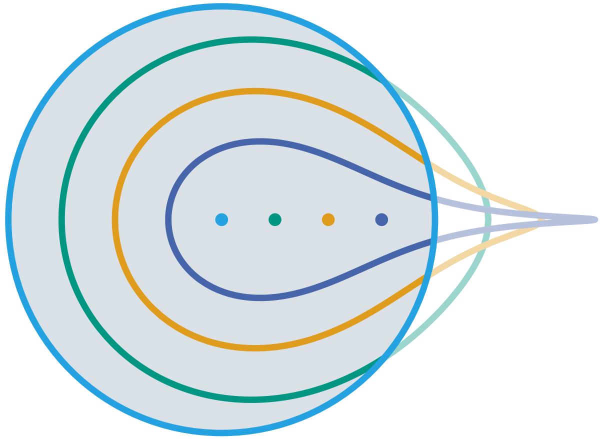


Observation
Let u and v lie in a disk of radius t with center O such that $d(u, v) \leq t$. Then $d(u, v) \leq t$ remains true when moving v closer to O.

(note: $d(u, v)$ might increase, but not above t)



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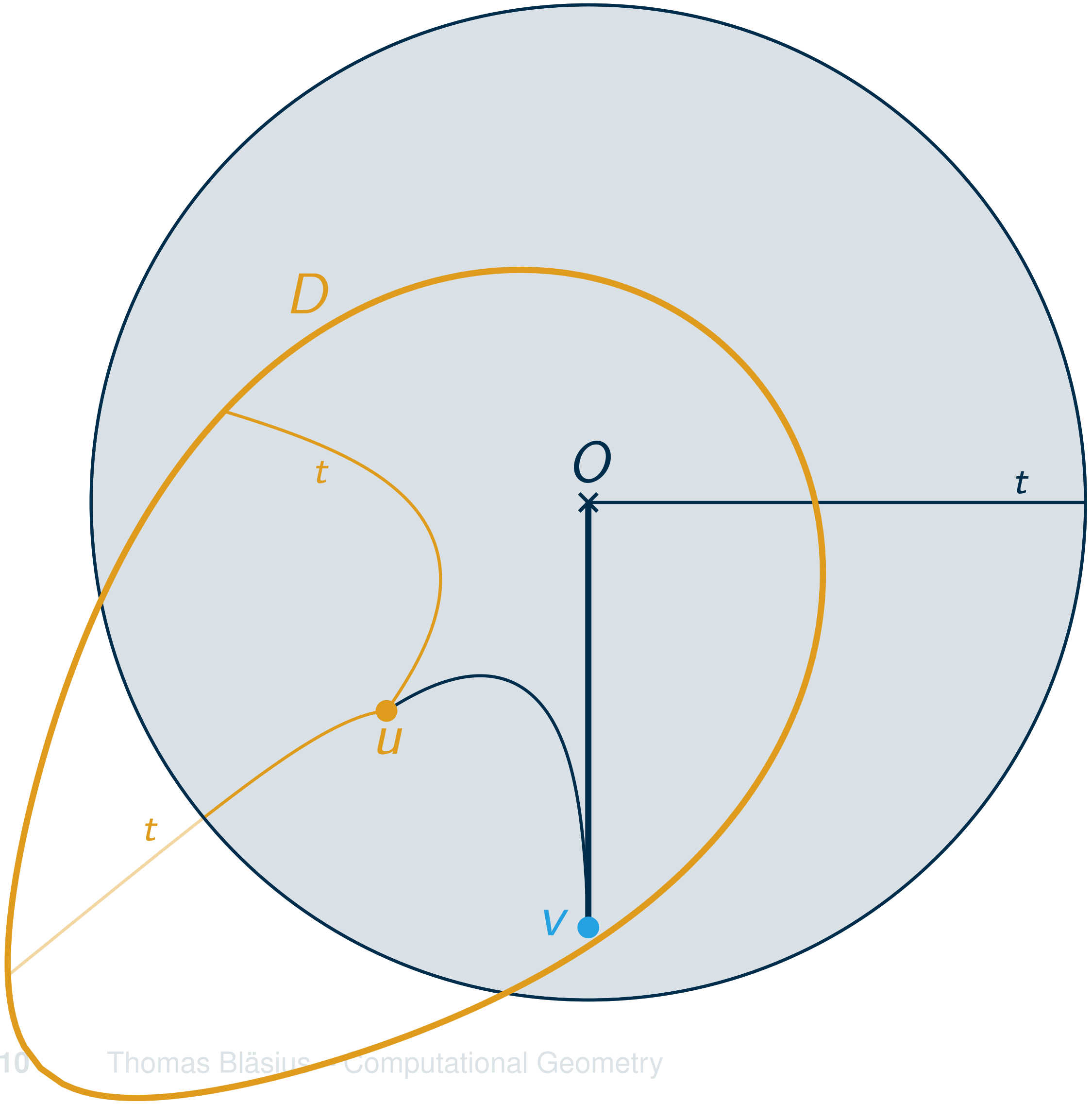

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Let u and v lie in a disk of radius t with center O such that $d(u, v) \leq t$. Then $d(u, v) \leq t$ remains true when moving v closer to O.

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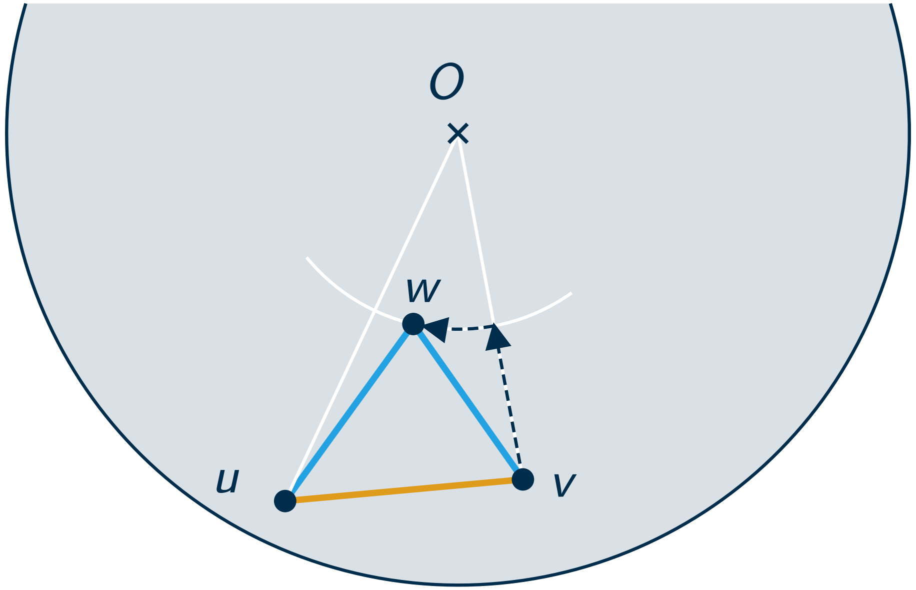
Proof (more rigorous than looking at a picture)

- consider disk D around u with radius t
- v and the origin O are contained in D
- D convex $\Rightarrow$ segment $\overline{vO}$ contained in D
- $d(u, v') \leq t$ for every v' on $\overline{vO}$

Forced Edges (Strongly Hyperbolic)

Situation

- strongly hyperbolic UDG-representation
- w lies between u and v (w.r.t. angle)
- w lies closer to the origin than u and v



Theorem
If $uv \in E$, then $uv, vw \in E$.

Proof: move v to w in two steps:

- set $r_v = r_w$ $\rightarrow$ moves v towards the origin O $\Rightarrow d(u, v) \leq t$ remains true
- set $\varphi_v = \varphi_w$ $\rightarrow$ decreases angle difference $\Rightarrow d(u, v) \leq t$ remains true

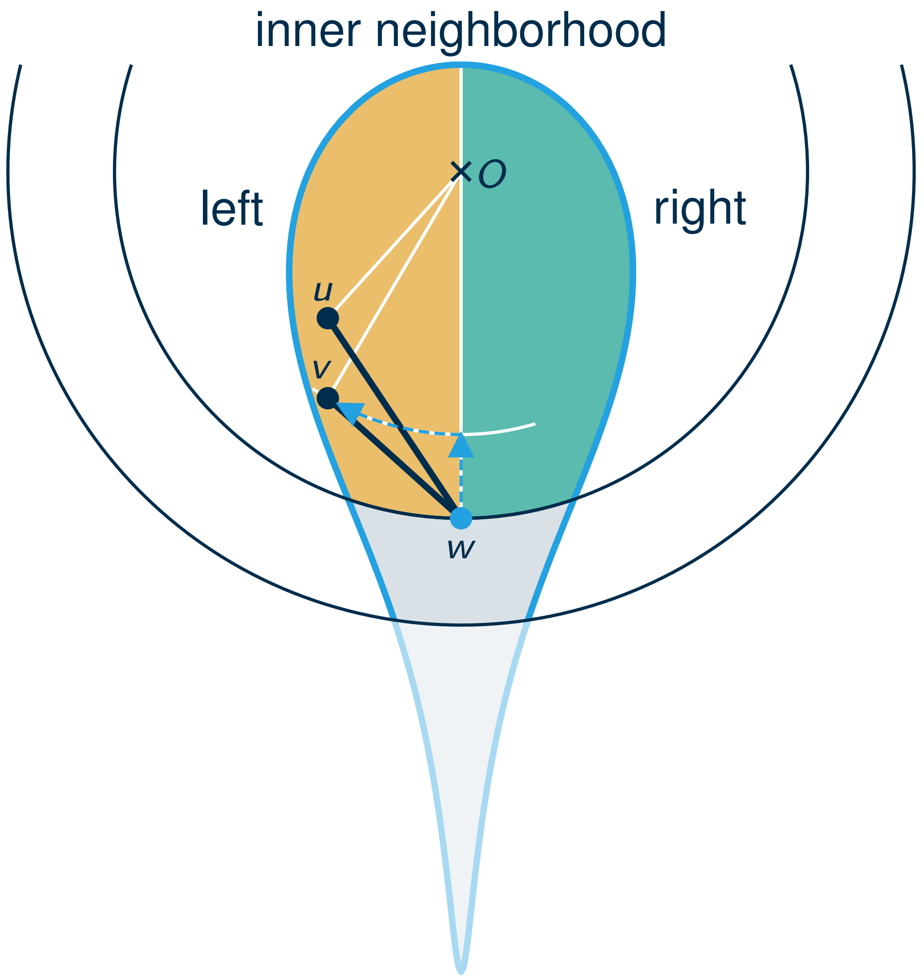
$\Rightarrow d(u, w) \leq t$ (same for $d(v, w)$)

Notes

- you can't sneak past a vertex that lies closer to the origin without connecting to it
- hierarchical structure: the closer to the origin, the higher up in the hierarchy

Inner Neighborhood (Strongly Hyperbolic)

Theorem
The left and right inner neighborhood each from a clique.

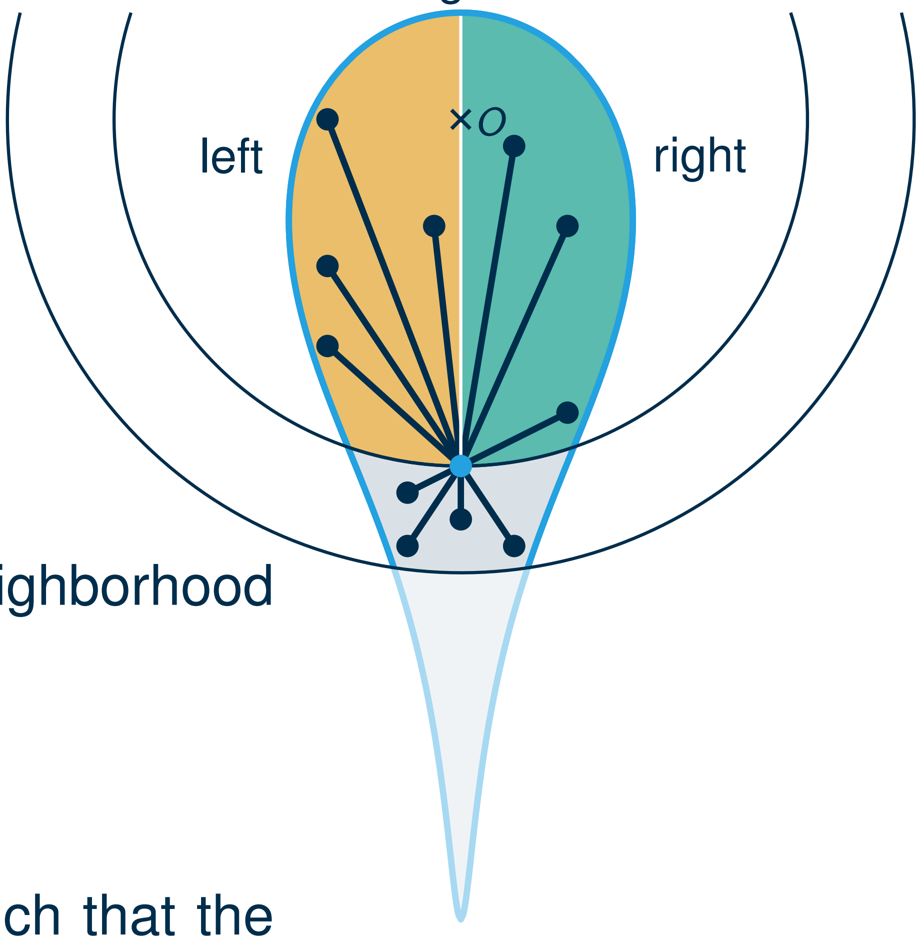


Proof

- show: u and v are connected
- w.l.o.g.: v has smaller angle difference to w
- move w to v in two steps:
  - set $r_w = r_v$ $\rightarrow$ moves w towards the origin O
  - set $\varphi_w = \varphi_v$ $\rightarrow$ decreases angle difference
- $d(u, w) \leq t$ remains true $\Rightarrow d(u, v) \leq t \Rightarrow uv \in E$

Inner Neighborhood (Strongly Hyperbolic)

Theorem
The left and right inner neighborhood each from a clique.



Corollary: Almost Perfect Elimination Scheme

- sort vertices $v_1, \ldots, v_n$ from large to small radius
- delete vertices one after another: $G_i = G[\{v_i, \ldots, v_n\}]$
- v_i has only inner neighbors in G_i
- elimination scheme: iteratively delete vertices such the neighborhood of the deleted vertex can be covered with two cliques

Comparison: Chordal Graphs

- graph chordal $\Leftrightarrow$ perfect elimination scheme
- perfect elimination scheme: iteratively delete vertices such that the neighborhood of the deleted vertex forms clique

What Can We Do With This?

Corollary
Let $G = (V, E)$ be a strongly hyperbolic UDG. There is a vertex order $V = \{v_1, \ldots, v_n\}$ such that the neighborhood of v_i in $G_i = G[\{v_i, \ldots, v_n\}]$ can be covered by two cliques ($\forall i \in [n]$).

short break

Can we efficiently test whether a graph can be covered with two cliques?

Does such an order help to find the largest clique in G?

Wrap-UP

Seen Today

- hyperbolic uniform disk graphs as generalization of Euclidean UDGs
- finding the maximum clique in strongly hyperbolic uniform disk graphs in polynomial time

What Else Is There

- other results for hyperbolic uniform disk graphs
  - recognition is $\exists\mathbb{R}$-complete (open for strongly hyperbolic UDGs)
  - strongly hyperbolic setting: local routing (similar to greedy routing) with low stretch
  - large threshold $t \Rightarrow$ balanced separators coverable with few cliques, algo for independent set
- model for real-world networks: real-world networks often have an underlying geometry
- geometric graphs with random vertex positions (e.g., random geometric graphs)
  - possible model for average case analysis (shortest paths, vertex cover (approx), SAT)
  - good benchmarks instances for evaluating algorithms (efficient generator available)

Eval

Evaluation

- overall very positive :-)

1.20) Nicht gefallen hat mir insbesondere:

- -
- Das in der Vorlesung zur Delauny Triangulation die 5 min Pause gefehlt haben :(
- Die geometrischen Argumente könnten für mich öfter noch detailreicher ausfallen, also ein bisschen näher an Axiomen. Oder man baut am Anfang mal eine geometrische Toolbox auf.

  Die Kästchen mit "Why?" finde ich grundsätzlich gut, aber sind manchmal etwas extrem. Für komplexere Argumente lieber nicht nutzen.
- Für die Nachbereitung zuhause sind die Folien manchmal nicht so aufschlussreich, wenn man in der Vorlesung mal nicht mitgekommen ist. Zwar sind die Beispiele während der Vorlesung selbst sehr anschaulich und hilfreich, für die Nachbereitung würden aber manchmal 1-2 mehr erklärende Sätze auf den Folien doch auch nochmal weiterhelfen
- Geometrie
- Mir fällt die Vorlesung sehr schwer, ich kann häufig nicht allen Schritten auf den Folien folgen. Ich wüsste aber auch nicht, was man da anders machen könnte. Ich glaub ich muss einfach mehr Vor- und Nachbereiten
- Wenn alle Menschen im Raum in der Lage sind die Veranstaltung auf deutsch abzuhalten (was zumindest laut einer Umfrage in der Übung der Fall war), ist das vielleicht doch auch ganz angenehm (:

Thursday & Wednesday

Last Meeting On Thursday

- I can give a overview over all topics
- let me know, if you have topics I should repeat in more detail
- some infos concerning the exam

Picnic On Wednesday

- starting at 5pm (17:00) at “Wiese am Fasanengarten” (or indoors with board games if the whether is bad)
- also see: `https://cloud.iti.kit.edu/index.php/s/WKXecGDdcZCtrQF`
- please bring your own cutlery and plate