Computational Geometry

Geometry

Thomas Bläsius

Back To School: Congruence Theorems

Theorem **(Congruence Theorem SSS)**
ΔABC and $\Delta A'B'C'$ with $|\overline{AB}| = |\overline{A'B'}|$, $|\overline{BC}| = |\overline{B'C'}|$, and $|\overline{CA}| = |\overline{C'A'}|$ are congruent.

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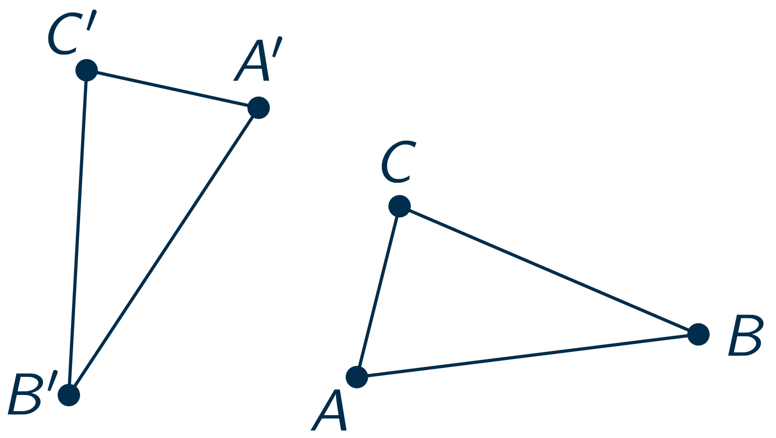

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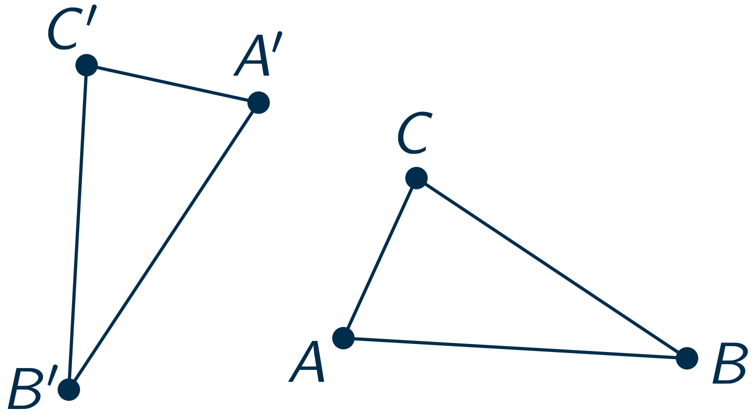

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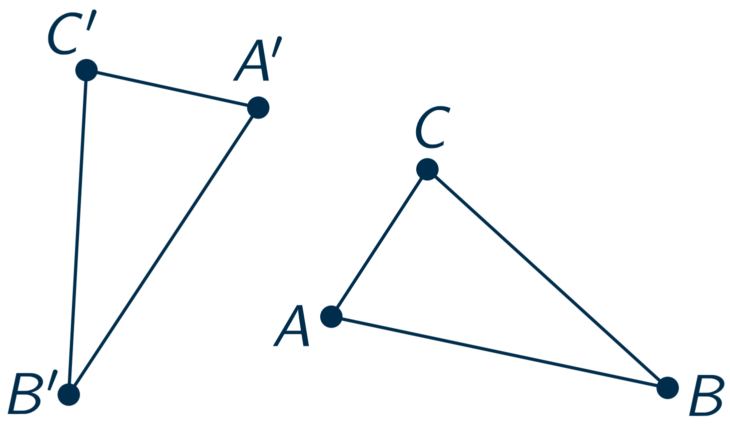

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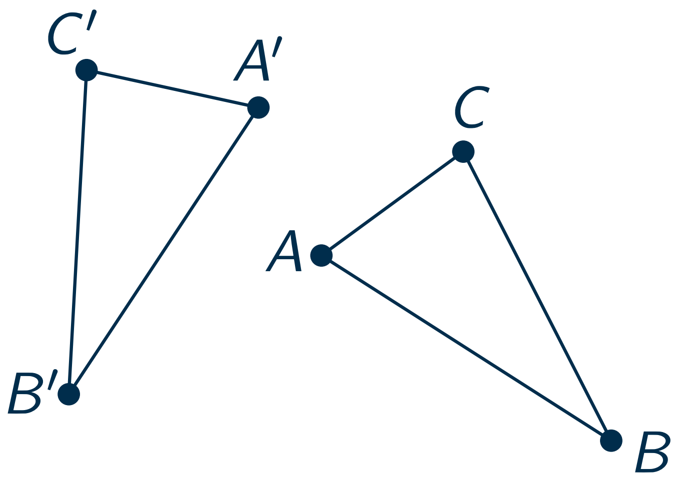

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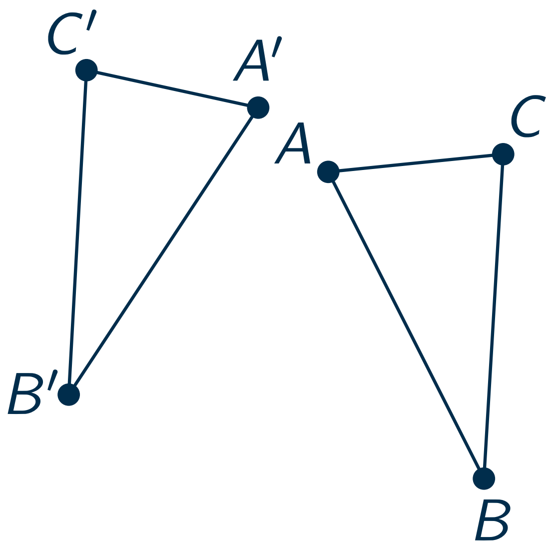

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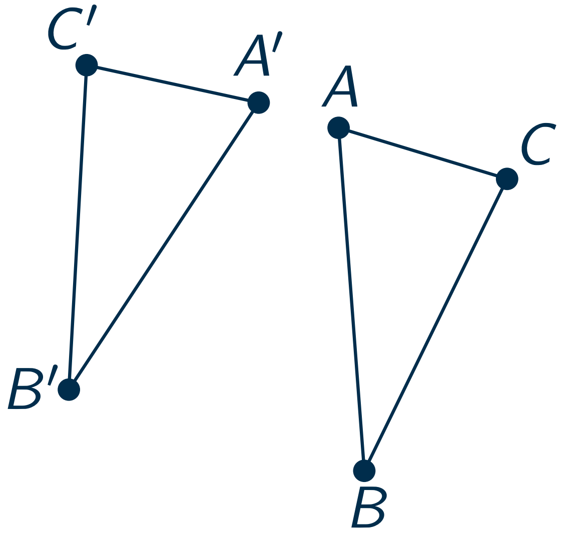

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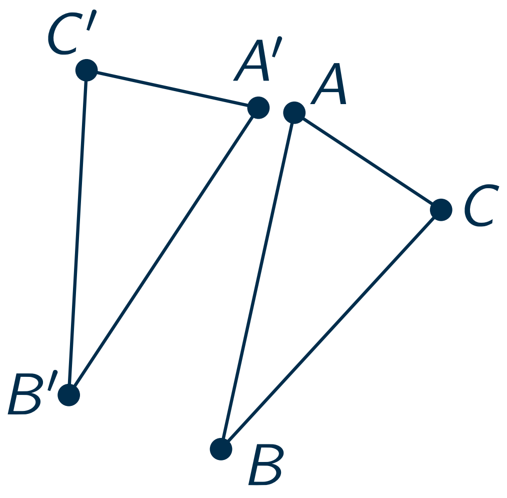

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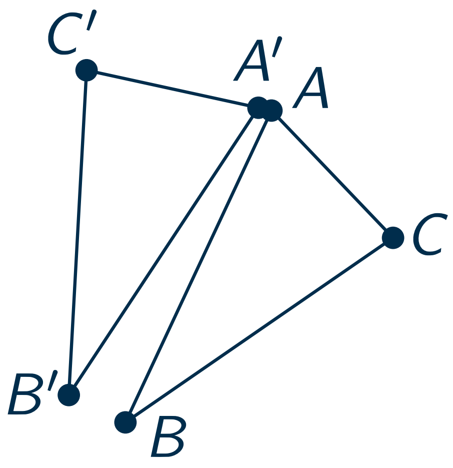

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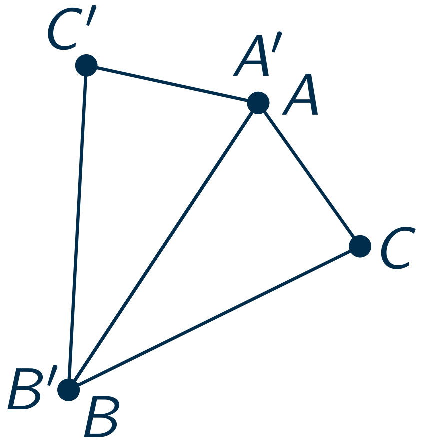

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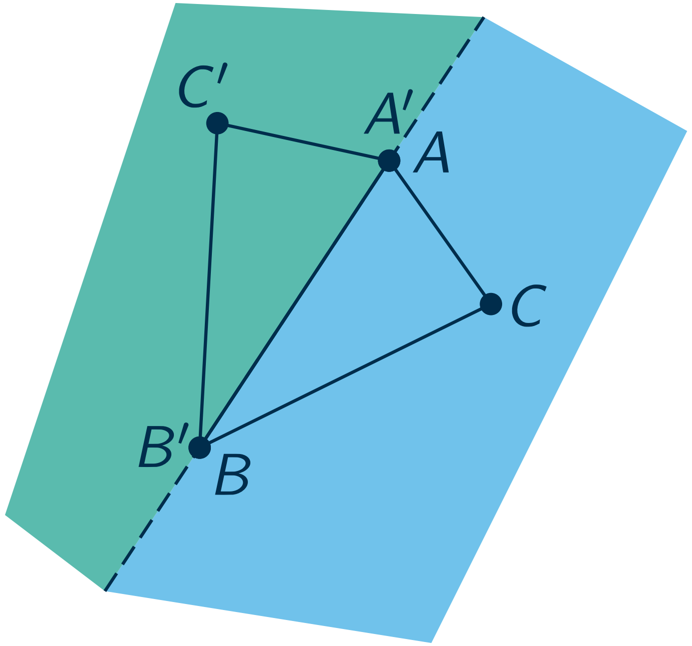

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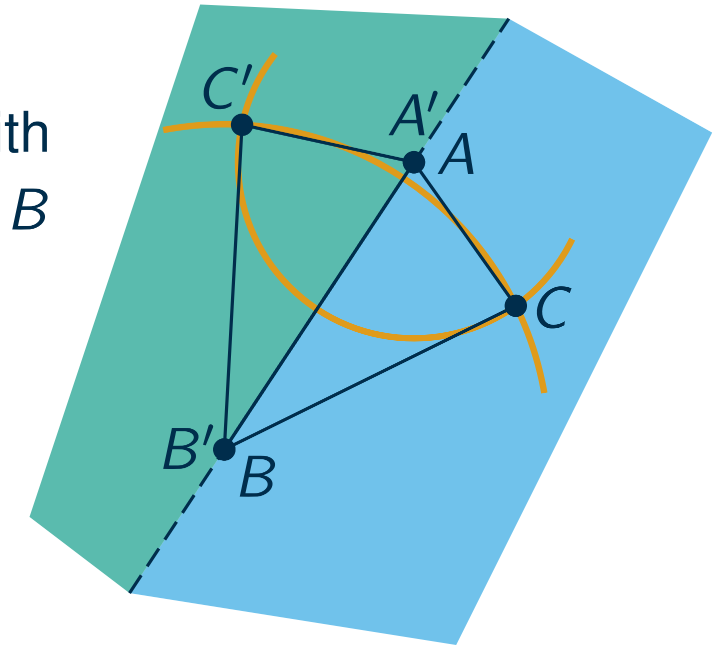

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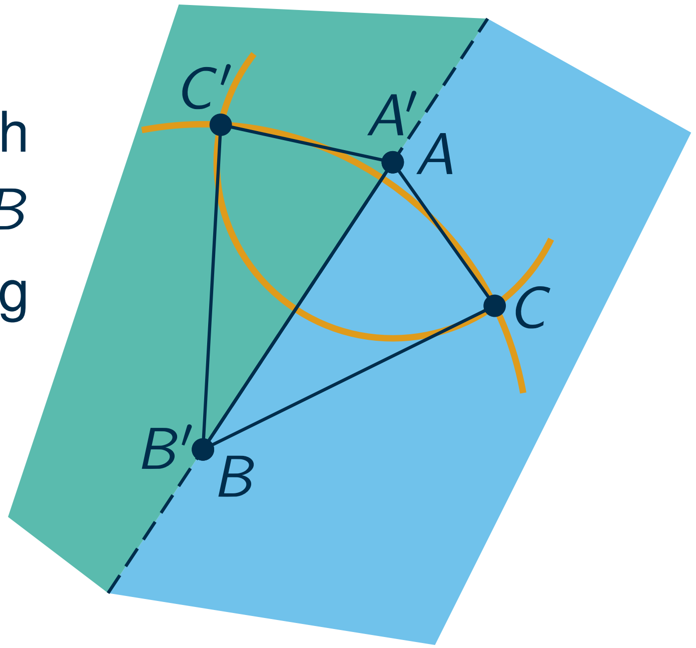

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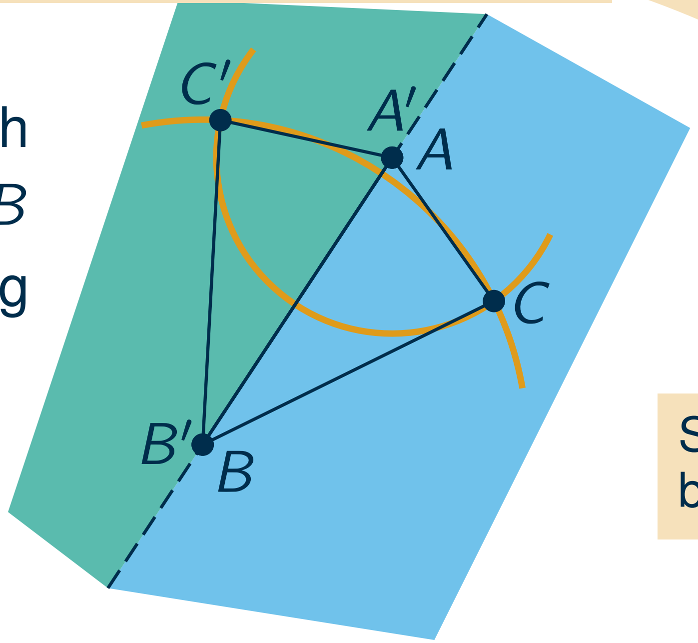

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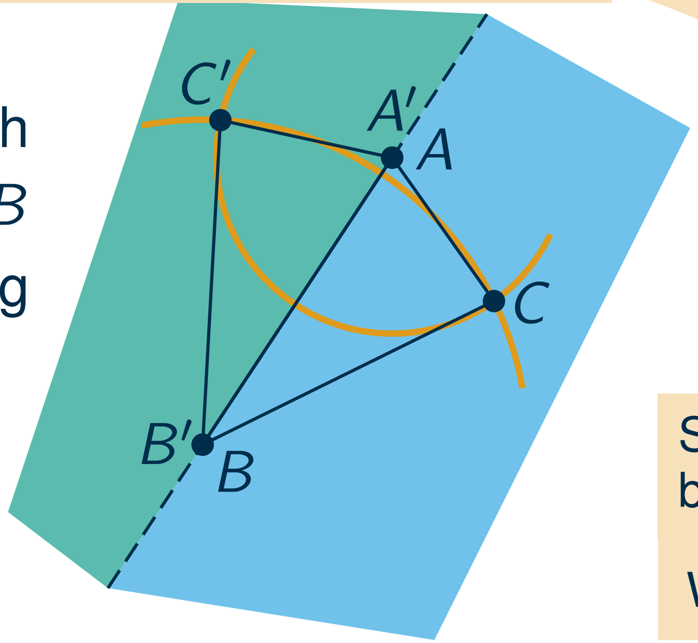

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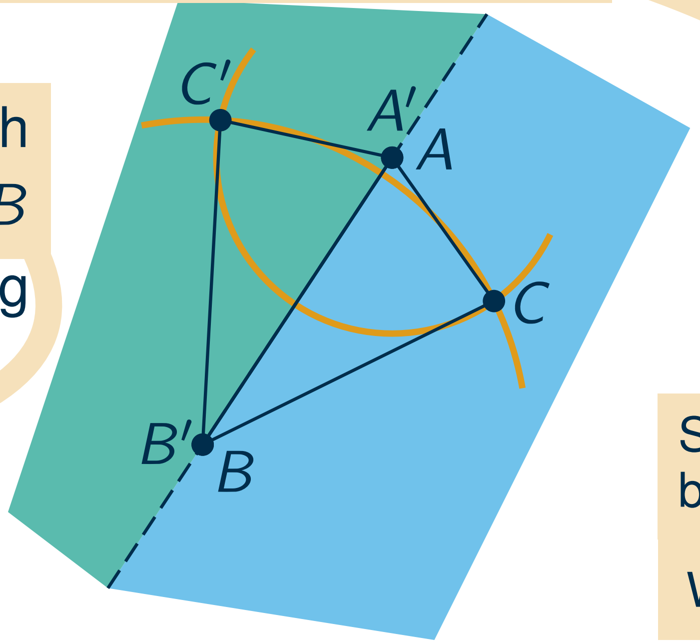

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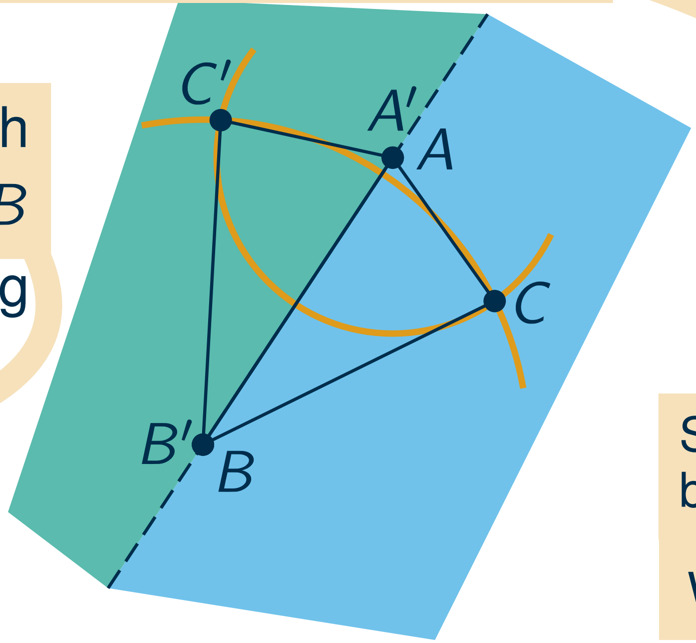

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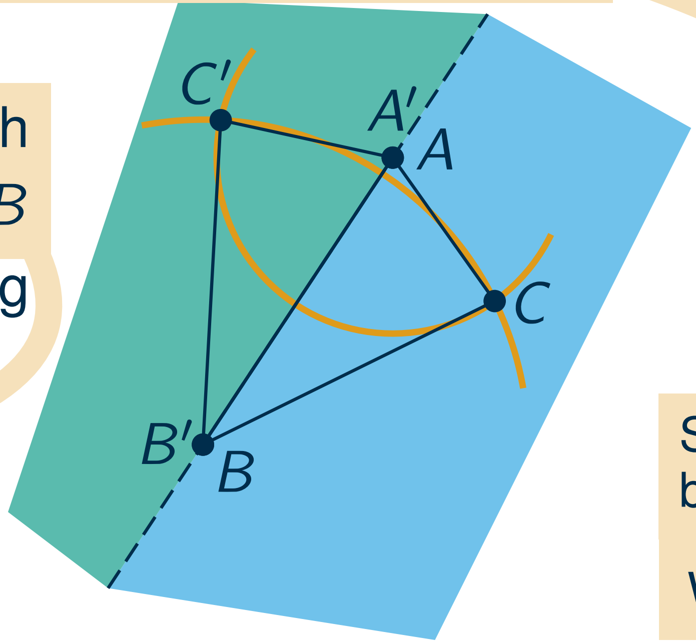

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Euclid's Axiomatic Approach

Euclid (around 300 BC)

- fix certain ground truths (postulates and axioms)
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The Five Axioms

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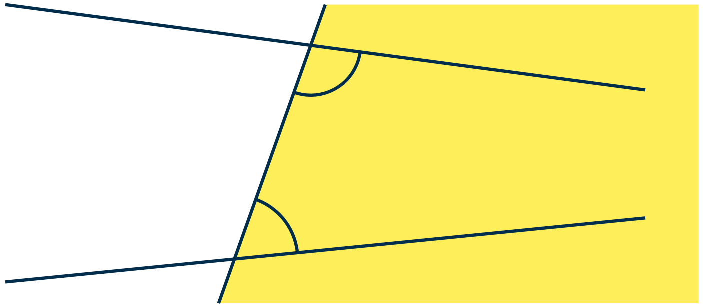

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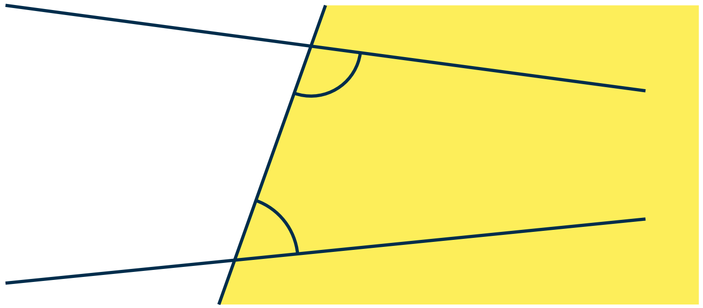

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the last postulate is called **parallel postulate**

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- Hilbert (1891): »Man muss jederzeit an Stelle von "Punkte, Geraden, Ebenen" "Tische, Stühle, Bierseidel" sagen können.«

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- the basic terms gain meaning from a set of **axioms** (properties that we wish our basic terms to have)
- **theorems**, that can be deduced from the axioms
- **definitions** are just abbreviations that simplify notation

Desirable Properties For A System Of Axioms

- consistency (free of contradictions)
- independence (no axiom can be deduced from the others)
- completeness (every formulatable statement is (dis)provable)
- you don’t always get what you want (see Gödel’s incompleteness theorem)

Modern Axiomatic Perspective

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Plan For Today

- axiomatic system for geometry (with five groups of axioms I–V)
- we follow the axiomatic system of Kolmogorov (1977) (equivalent to Hilbert’s system)
- we assume to already have basic stuff like numbers (Peano) and set theory (ZFC)

Basic Terms: Points & Lines

Definition
Let $\mathcal{P}$ and $\mathcal{L}$ be disjoint sets. We call their elements **points** and **lines**, respectively. Then $I \subseteq \mathcal{P} \times \mathcal{L}$ is called an **incidence structure**. If $(P, \ell) \in I$, we say that P and ℓ are **incident**.

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- $\mathcal{P}$ is the set of chairs and $\mathcal{L}$ the set of tables in a restaurant
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- $\mathcal{L} \subseteq 2^{\mathcal{P}}$ and $(P, \ell) \in I \Leftrightarrow P \in \ell$ (we denote it with $(\mathcal{P}, \mathcal{L}, \in)$)
- easy to show: every geometry is isomorphic to $(\mathcal{P}, \mathcal{L}, \in)$ (for the canonical definition of *isomorphic*)

Absolute/Euclidean Geometry & Incidence Axioms

Definition
An incidence structure $(\mathcal{P}, \mathcal{L}, \in)$ together with a map $d\colon \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ that satisfies axiom groups I–IV is called **absolute geometry**. If it satisfies axiom groups I–V, it is called **Euclidean geometry**. For $A, B \in \mathcal{P}$, $d(A, B)$ is called the **distance** between A and B.

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Axiom Group I: Axioms of Incidence

(1) For every two points $A \neq B$, there is exactly one line ℓ with $A \in \ell$ and $B \in \ell$. (we denote it as: $\ell = AB$)

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- stools are points
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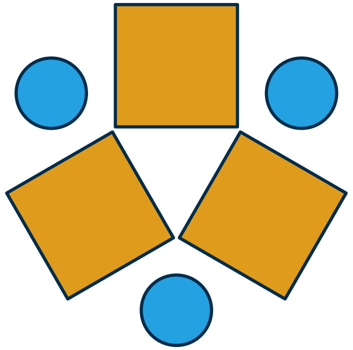

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Theorem
Two different lines share at most one point.

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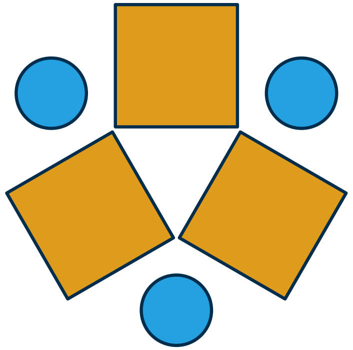

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(3) For all points A, B, C, it holds that $d(A, B) + d(B, C) \geq d(A, C)$. Moreover, A, B, C are collinear if and only if

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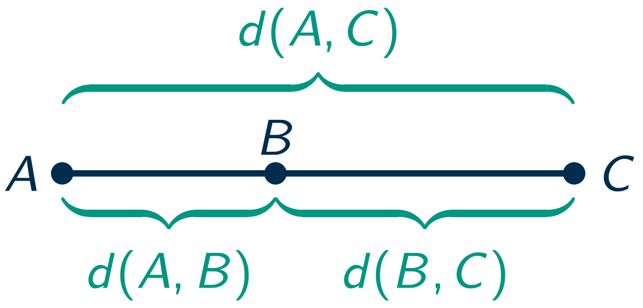
- we will give the one a name in a moment: it lies **between** the others

Line Segments, Rays, and Convexity

Definition
B lies **between** A and C if $d(A, B) + d(B, C) = d(A, C)$ and $B \notin \{A, C\}$.



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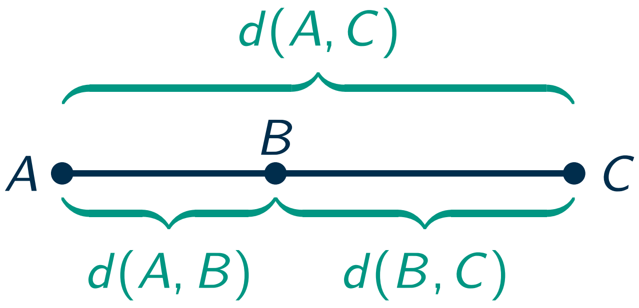

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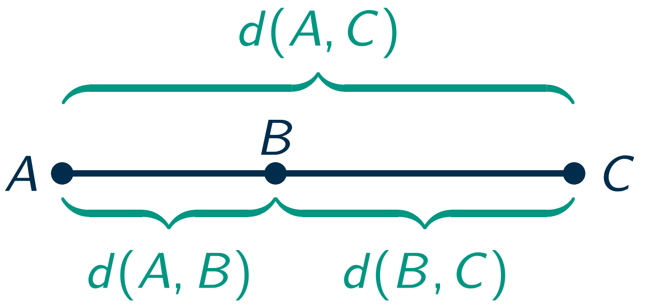

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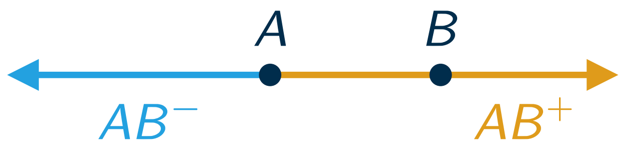

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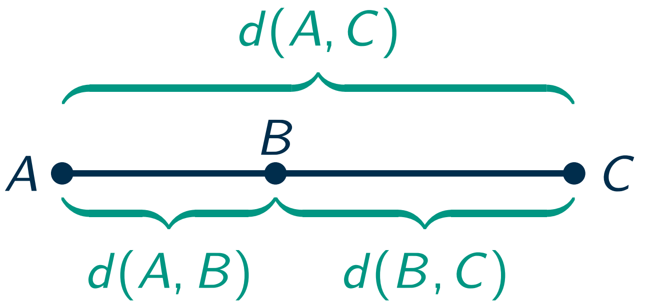

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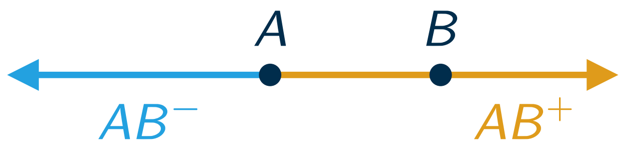

Definition
A set $M \subseteq \mathcal{P}$ is **convex**, if $\overline{AB} \subseteq M$ for all $A, B \in M$.

Axioms of Order and Half Planes

Axiom Group III: Axioms of Order

(1) For every point A and every number $a \in \mathbb{R}^+$, every ray starting at A contains exactly one point B with $d(A, B) = a$.

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The two sets are called **open half planes** with **boundary line** ℓ. The union with ℓ yields a **half plane**. The half plane with boundary line $\ell = AB$ that contains a point $C \notin \ell$ is denoted with ABC^+. The other half plane with ABC^-.

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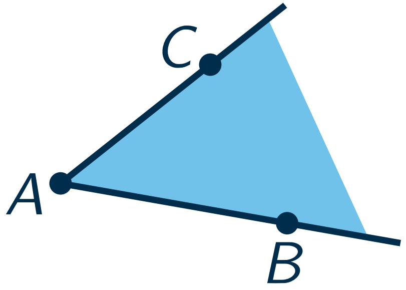
Note

- (1) ensures that we have infinitely many points
- (2) in particular tells us that half planes are convex

Angles and Motions



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The union of two rays is an **angle** $\angle BAC = AB^+ \cup AC^+$, with two **arms** AB^+ and AC^+. It is **straight** if $\angle BAC = AB$ and a **zero angle** if $AB^+ = AC^+$. $ABC^+ \cap ACB^+$ is its **interior**.

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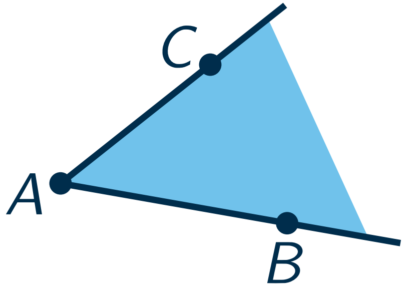

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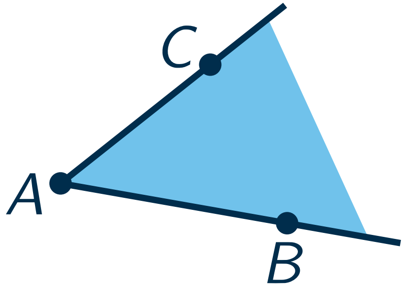

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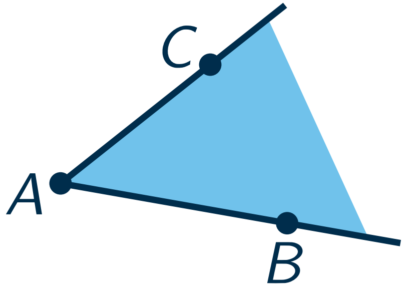

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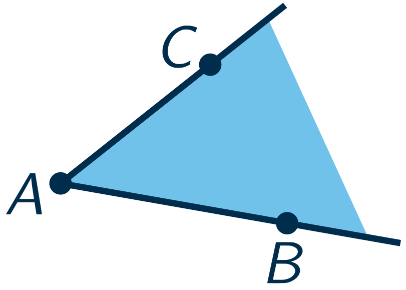

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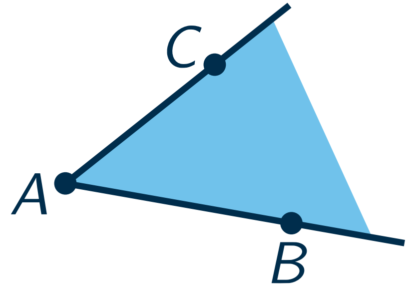

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Axiom Group IV: Axiom of Motion
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Axiom Group V: Euclidean Parallel Axiom
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Recap

Axiom Group I: Incidence
two points define a line; every line contains two points; there are three non-collinear points

Axiom Group II: Distance
distance is a metric; tightness of triangle inequality if and only if collinear

Axiom Group III: Order
there is a point in every direction with every distance; lines split the plane into half planes

Axiom Group IV: Motion
two motions that map segments of equal length onto each other (preserving orientation)

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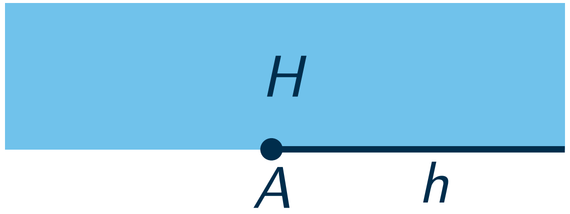
Absolute Geometry: Flags and Special Motions

Definition
Let $h = AB^+$ be a ray and H be a half plane with boundary line AB. The triple (A, h, H) is called **flag**.



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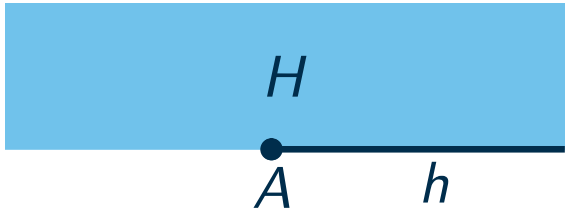

Theorem
For any two flags (A, h, H) and (A', h', H'), there is exactly one motion that maps (A, h, H) to (A', h', H') (i.e., $m(A) = A'$, $m(h) = h'$, $m(H) = H'$).

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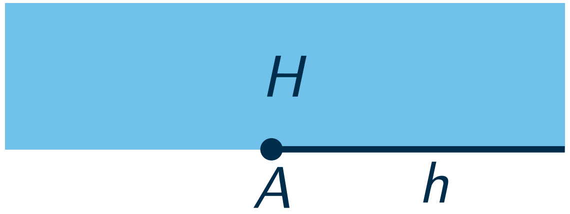
Let $h = AB^+$ be a ray and H be a half plane with boundary line AB. The triple (A, h, H) is called **flag**.



Theorem
For any two flags (A, h, H) and (A', h', H'), there is exactly one motion that maps (A, h, H) to (A', h', H') (i.e., $m(A) = A'$, $m(h) = h'$, $m(H) = H'$).

Definition
The motion that maps (A, h, H) to (A', h', H') is called **translation** if $A \neq A'$, $h' \subseteq h$ and $H = H'$. **(Point) reflection** and **Rotation** can be defined similarly.

Absolute Geometry: Triangles and Congruence

Definition
Let A, B, C be non-collinear points. Then $\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$ is the **triangle** with **sides** $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

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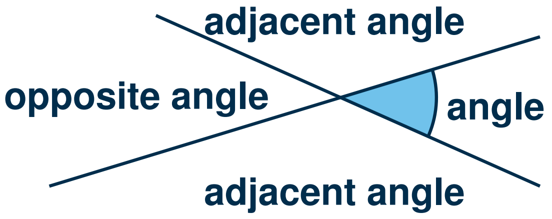

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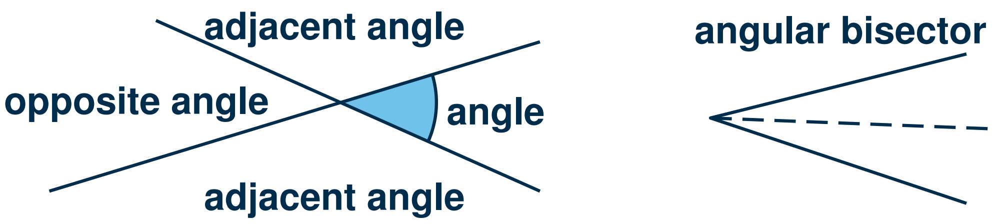

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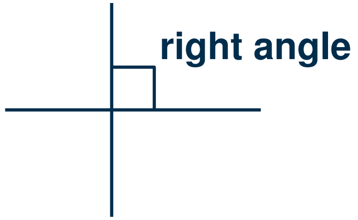

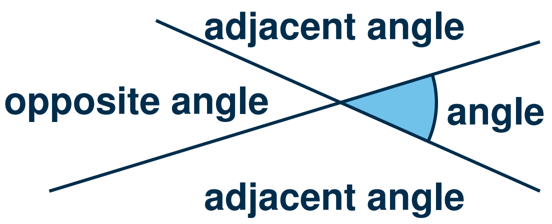

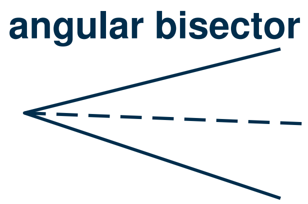

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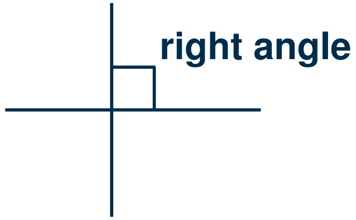

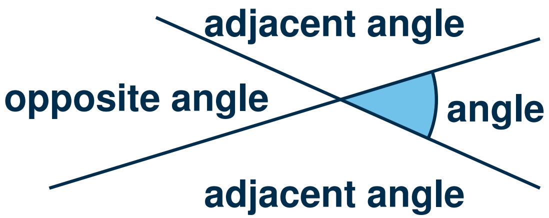

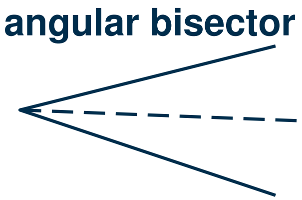

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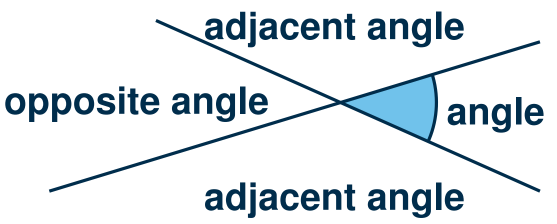

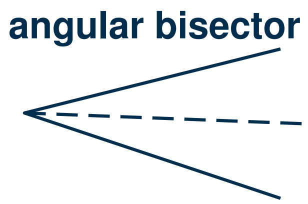

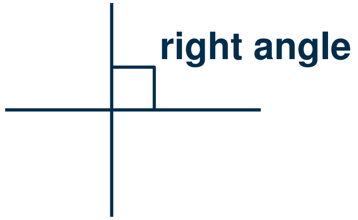

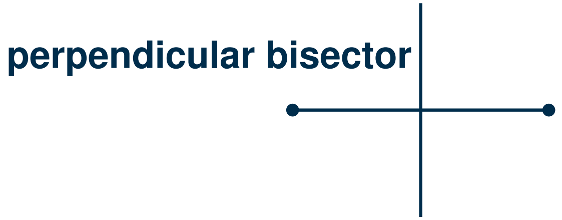

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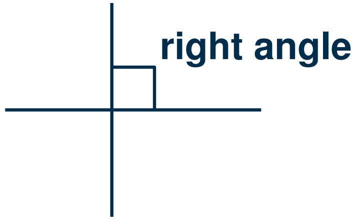

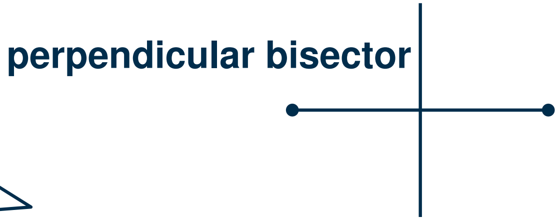

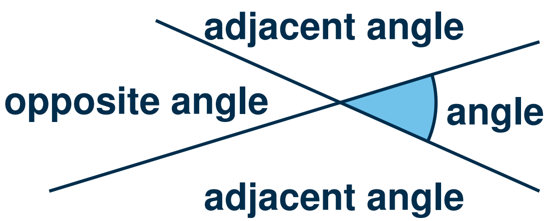

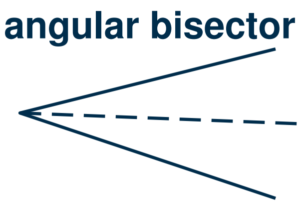

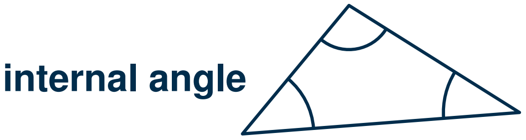

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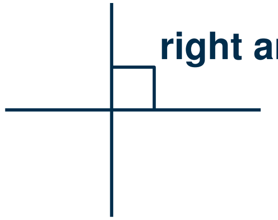

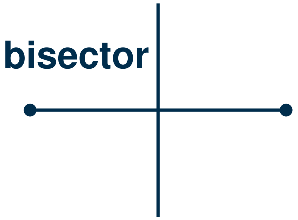

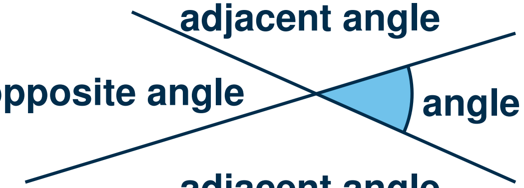

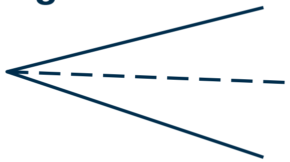

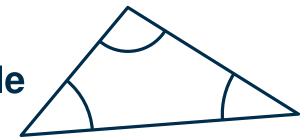

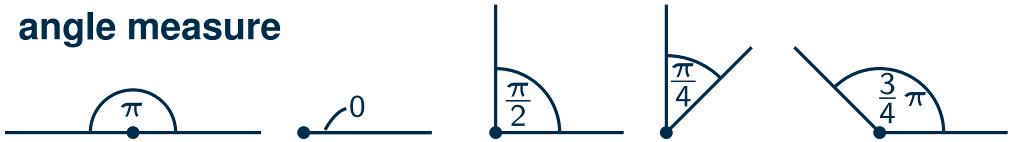

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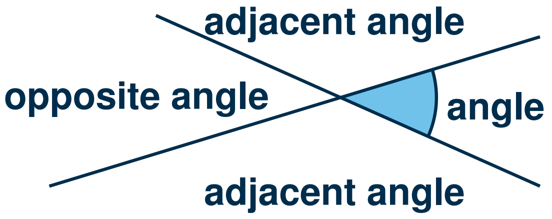

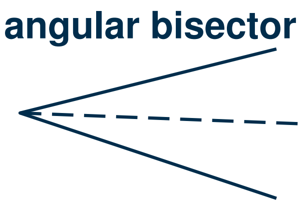

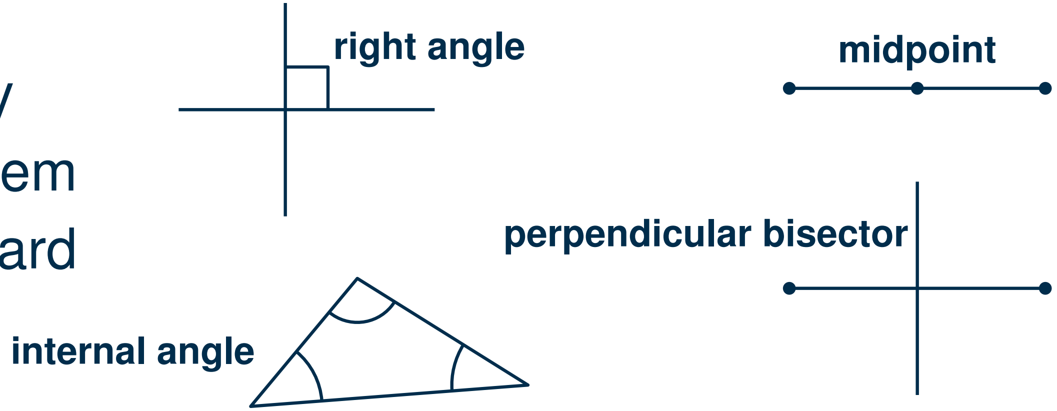

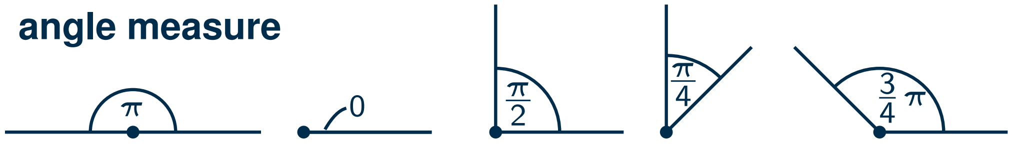

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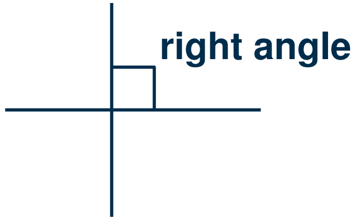

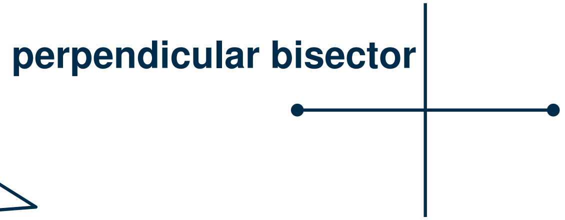

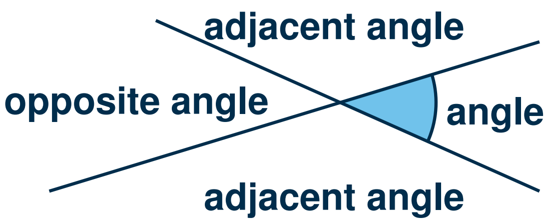

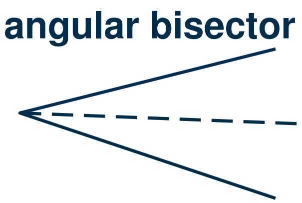

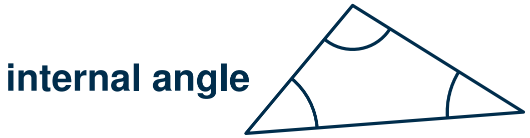

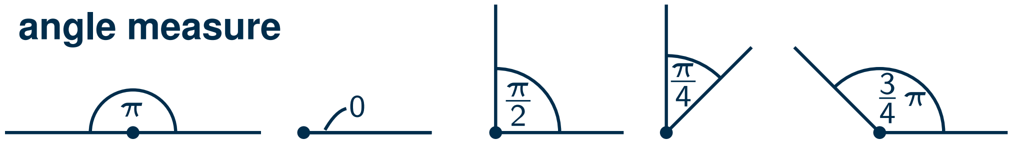

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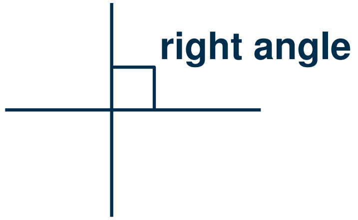

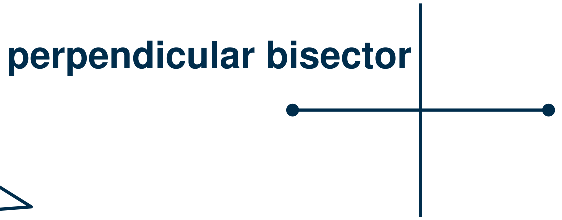

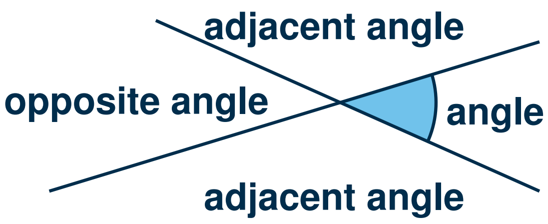

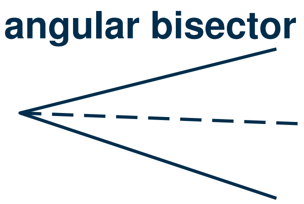

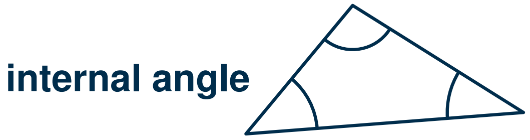

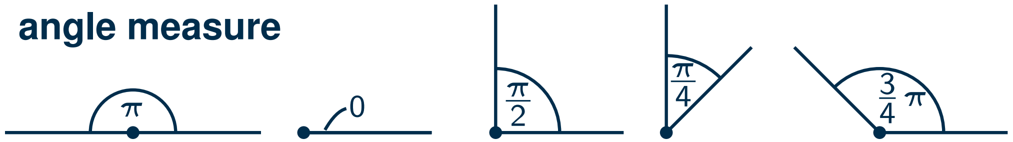

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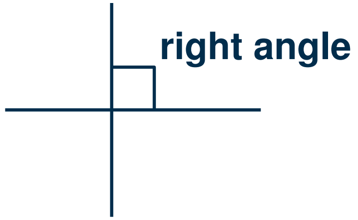

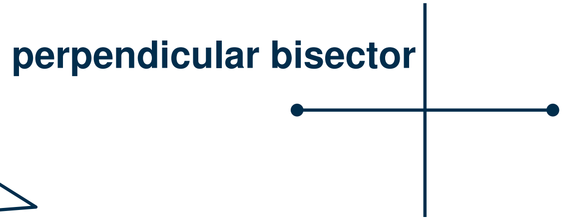

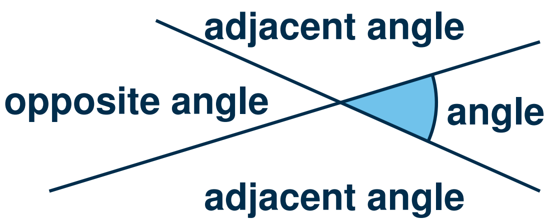

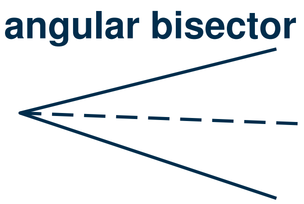

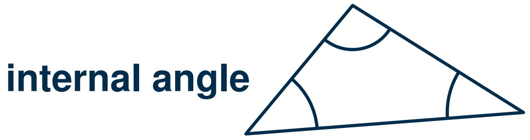

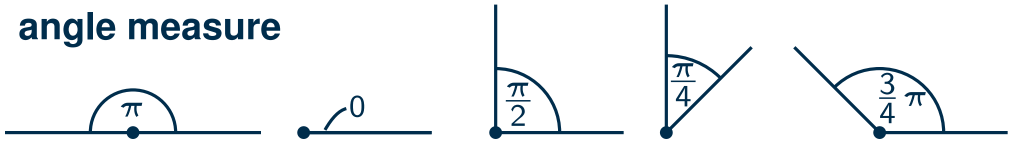

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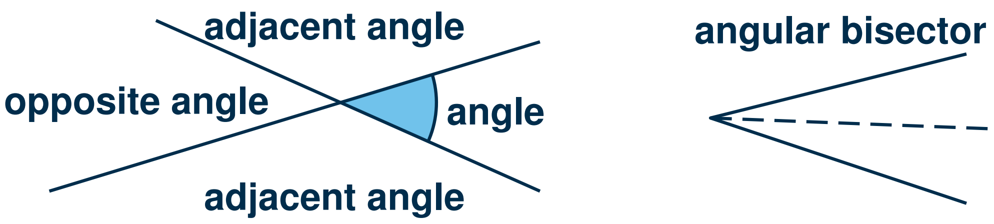

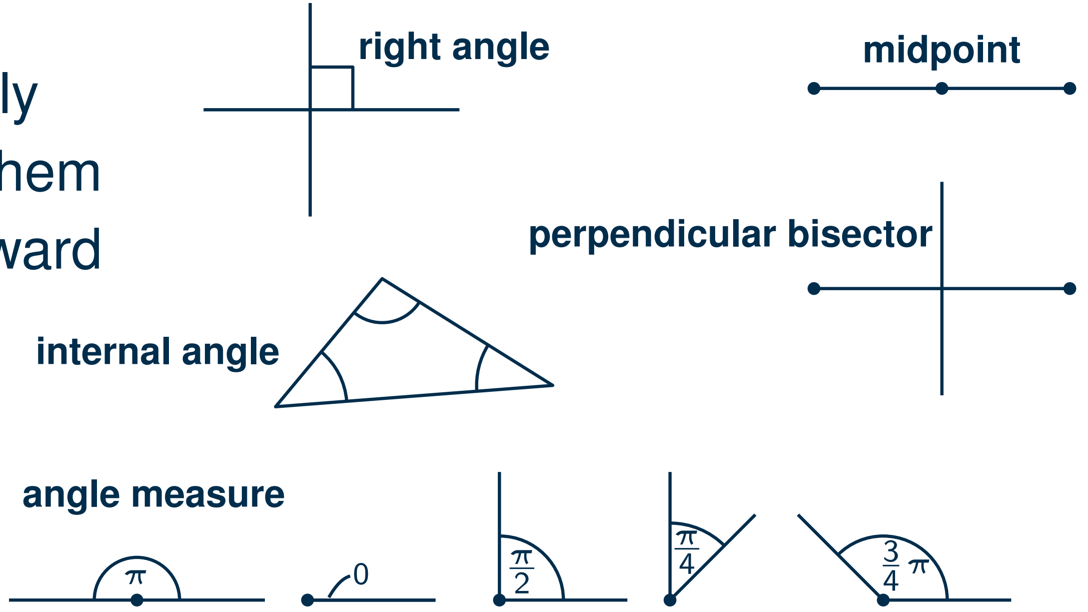

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- if there is a triangle with internal angle sum π, then every triangle has internal angle sum π

Seen So Far

Axiom Group I: Incidence
two points define a line; every line contains two points; there are three non-collinear points

Axiom Group II: Distance
distance is a metric; tightness of triangle inequality if and only if collinear

Axiom Group III: Order
there is a point in every direction with every distance; lines split the plane into half planes

Definition
An incidence structure $(\mathcal{P}, \mathcal{L}, \in)$ together with a map $d\colon \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ that satisfies axiom groups I–IV is called **absolute geometry**. If it satisfies axiom groups I–V, it is called **Euclidean geometry**. For $A, B \in \mathcal{P}$, $d(A, B)$ is called the **distance** between A and B.

Axiom Group IV: Motion
two motions that map segments of equal length onto each other (preserving orientation)

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line ℓ and point $P \notin \ell \Rightarrow$ at most one line through P parallel to ℓ

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Axiom Group V': Hyperbolic Parallel Axiom
There is a line ℓ and a point $P \notin \ell$ such that there are two lines through P parallel to ℓ.

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Axiom Group V': Hyperbolic Parallel Axiom
There is a line ℓ and a point $P \notin \ell$ such that there are two lines through P parallel to ℓ.

Definition
An incidence structure (with d) satisfying axiom groups I–IV, V' is called **hyperbolic geometry**.

The Hyperbolic Plane

What Happens If We Negate The Parallel Axiom?

- the axioms remain consistent
- we get a second model that satisfies the axioms of the absolute plane (next to the Euclidean plane)
- all theorems for the absolute plane also hold in the hyperbolic plane

The Hyperbolic Plane

What Happens If We Negate The Parallel Axiom?

- the axioms remain consistent
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For every line ℓ and every point $P \notin \ell$, there are infinitely many lines parallel to ℓ through P.

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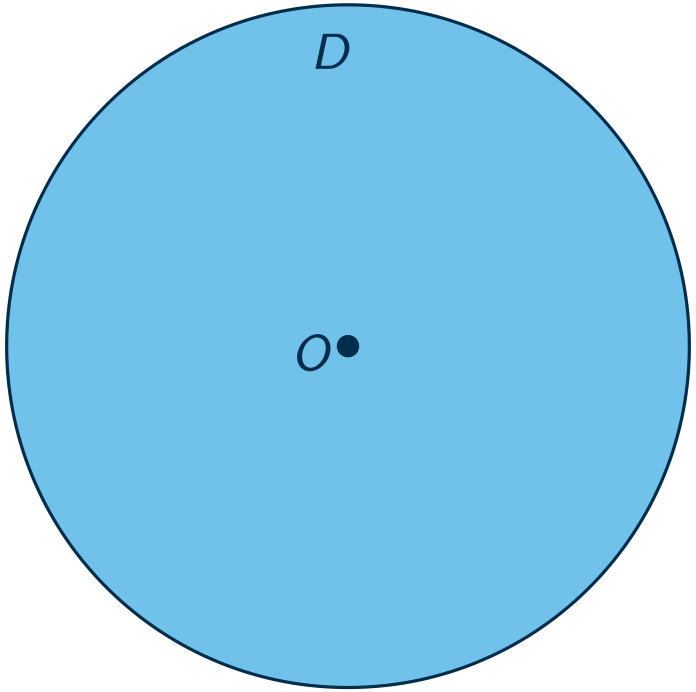
- we are somehow lacking an intuition for the hyperbolic plane
- helpful: models that represent the hyperbolic plane

Poincaré Disk Model

Points

- consider a (Euclidean) disk D with radius 1 around the point O
- let $\mathcal{P}$ be the set of points in the interior of the disk



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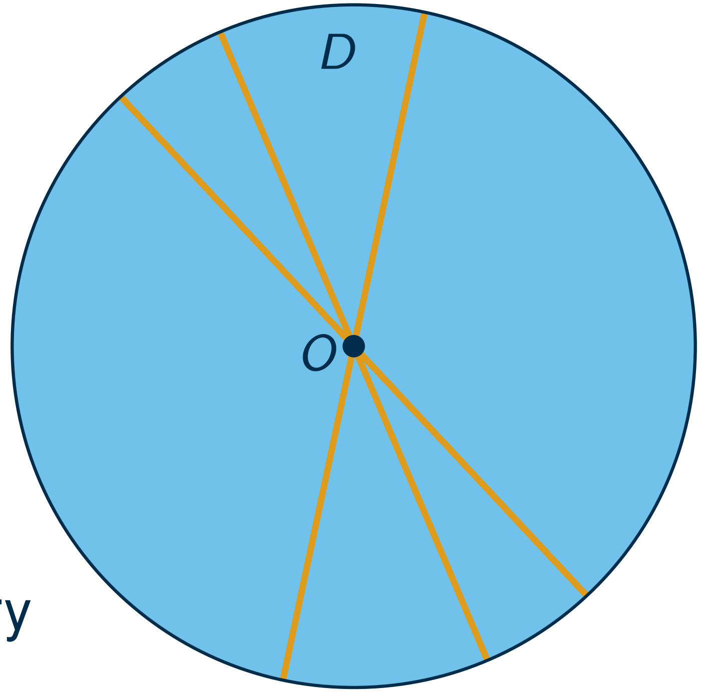

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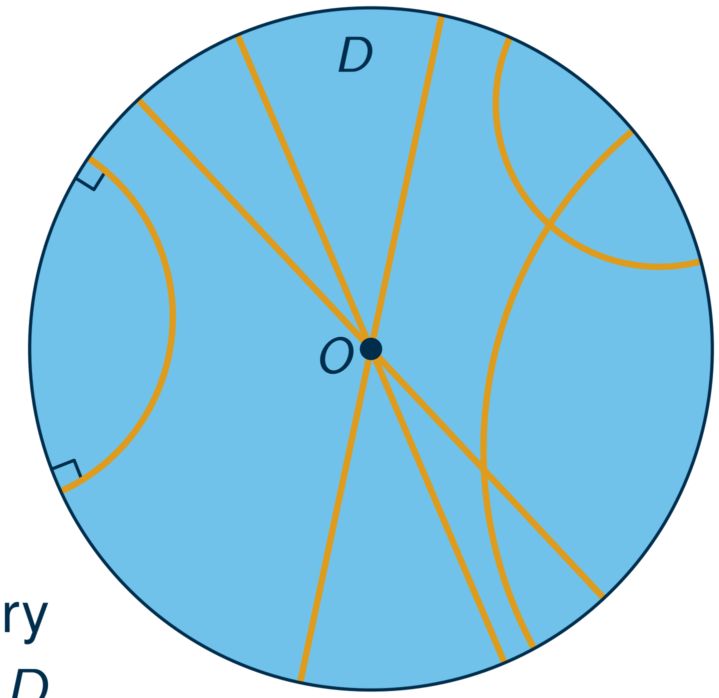

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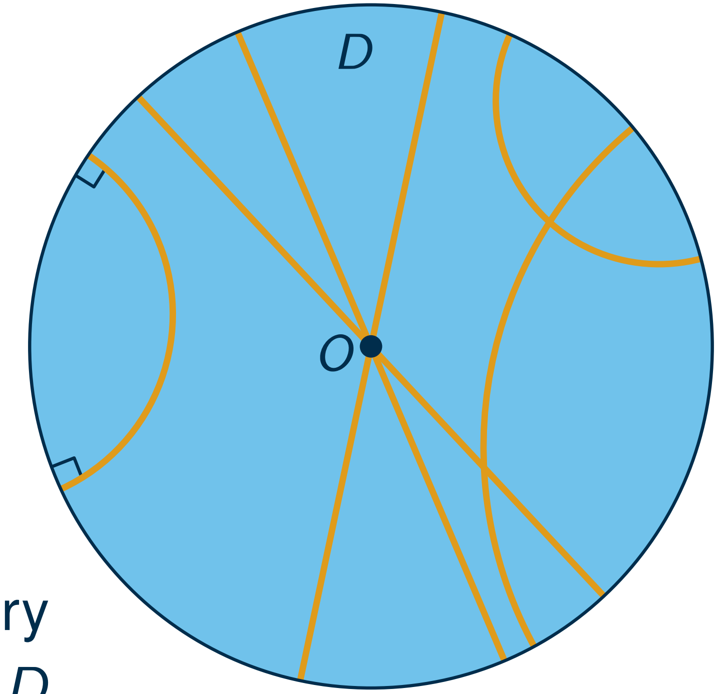

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Axiom Group I: Incidence
two points define a line; every line contains two points; there are three non-collinear points

Axiom Group II: Distance
distance is a metric; tightness of triangle inequality if and only if collinear

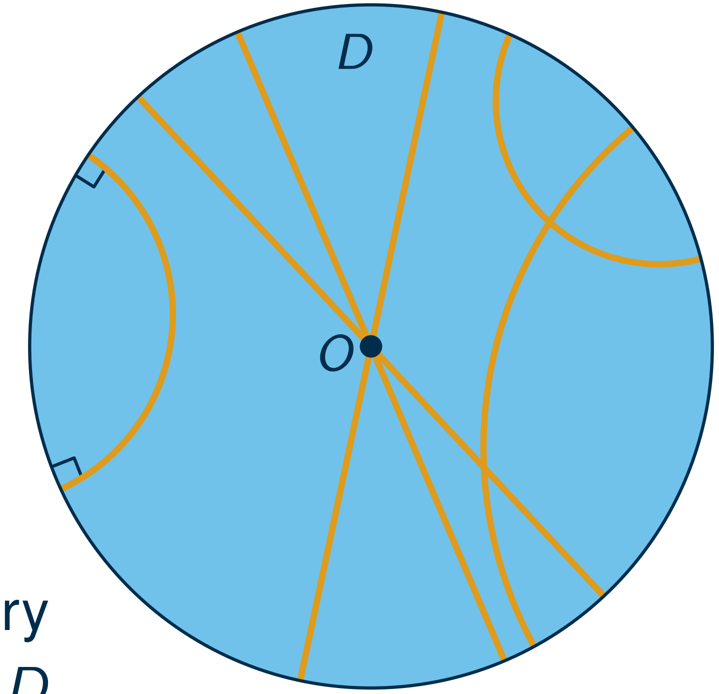
Axiom Group III: Order
there is a point in every direction with every distance; lines split the plane int half planes

Axiom Group IV: Motion
two motions that map segments of equal length onto each other (preserving orientation)

Axiom Group V': Hyperbolic Parallel Axiom
There is a line ℓ and a point $P \notin \ell$ such that there are two lines through P parallel to ℓ.

Short Break
Can you verify that the model satisfies some of the axioms?

Poincaré Disk Model



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It Holds That

- $(\mathcal{P}, \mathcal{L}, \in)$ together with an appropriate distance function satisfies axiom groups I–IV and V'

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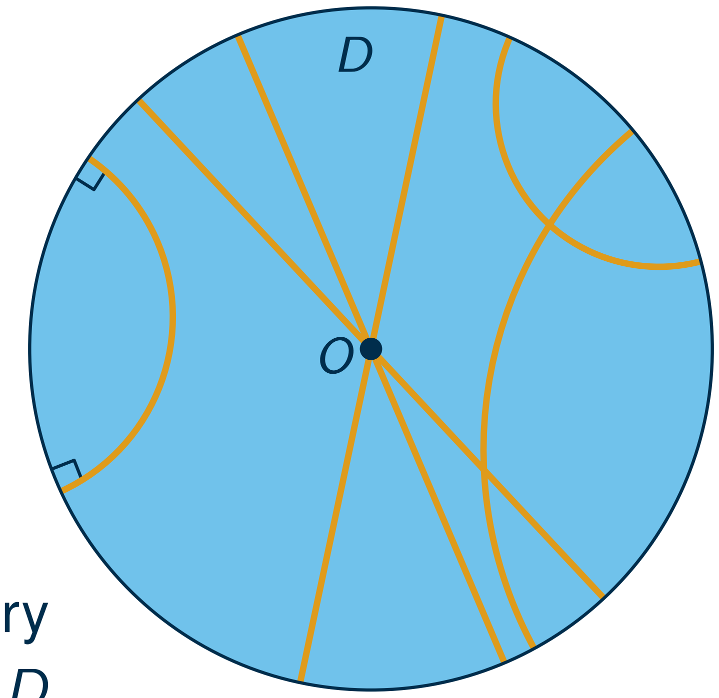

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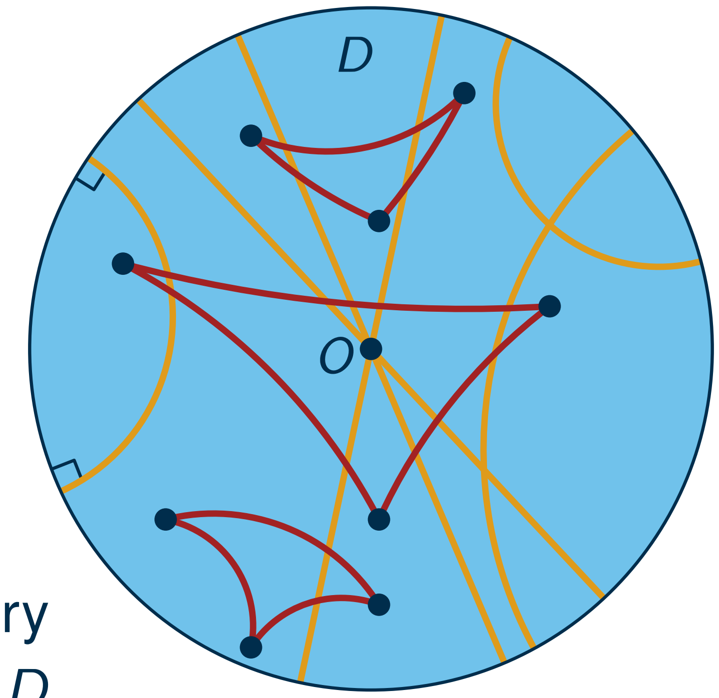

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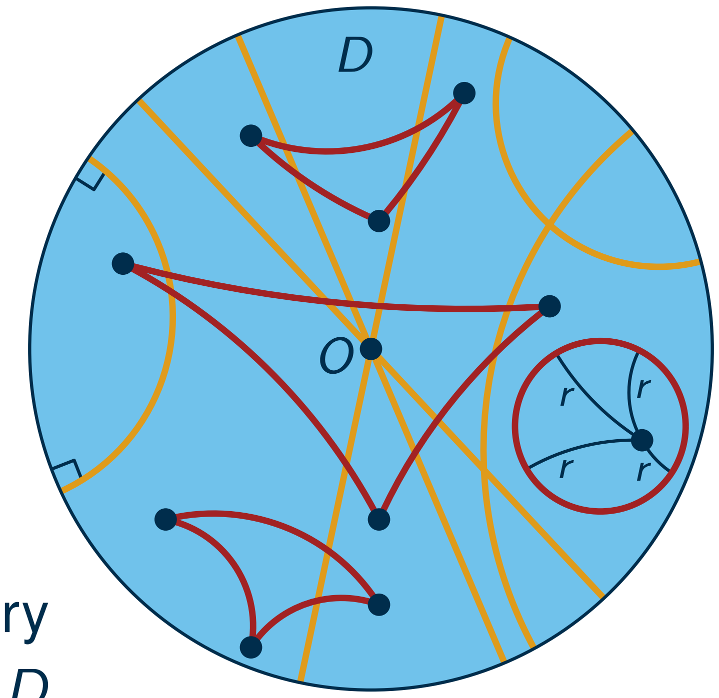

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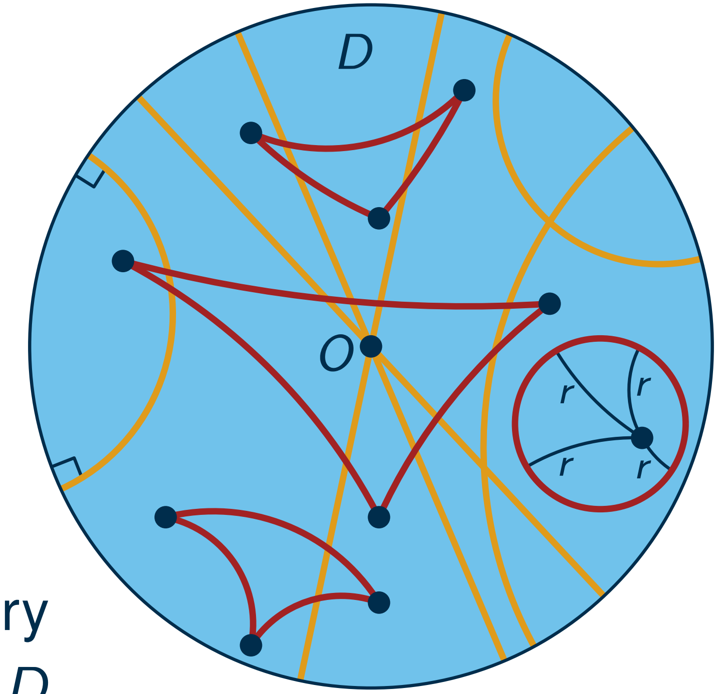

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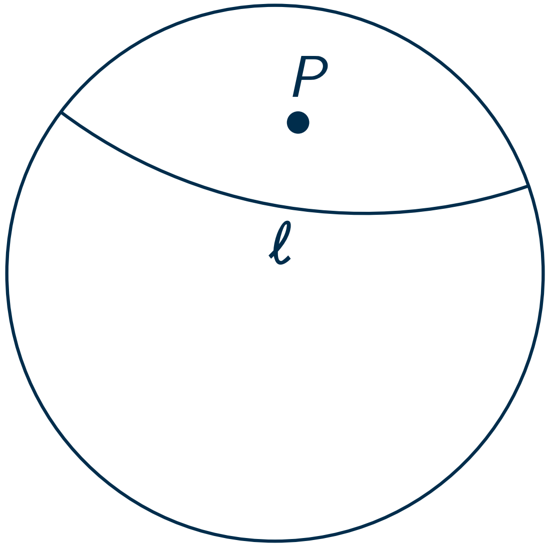
- distances are distorted, but hyperbolic circles look like circles (with different center)
- points on the boundary of D are **not** part of the hyperbolic plane

Poincaré Disk – Parallels, Half Plans, Ideal Points

Parallel Lines Through One Point



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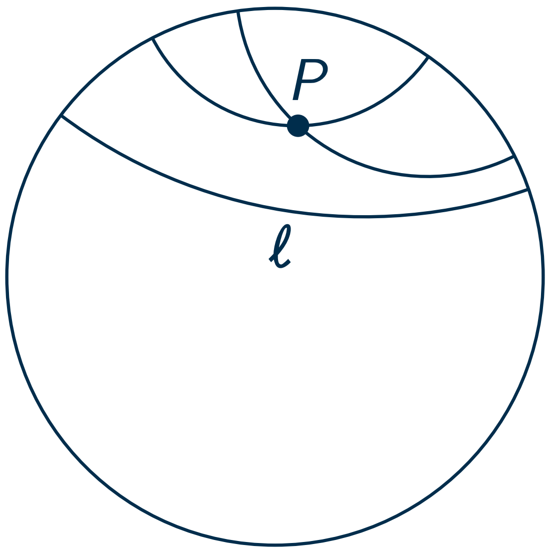

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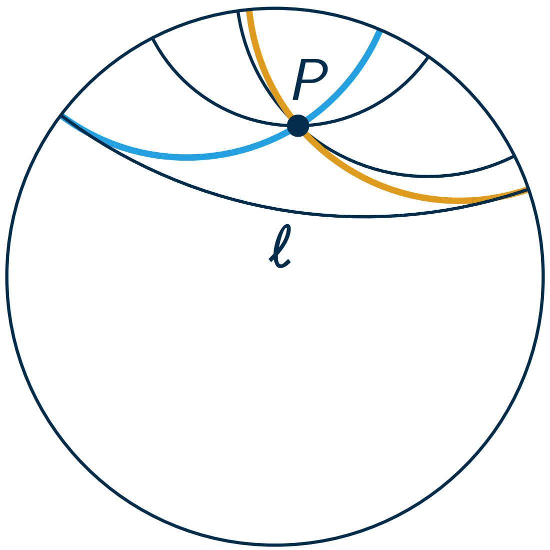
  - those are the **limiting parallels** of ℓ through P



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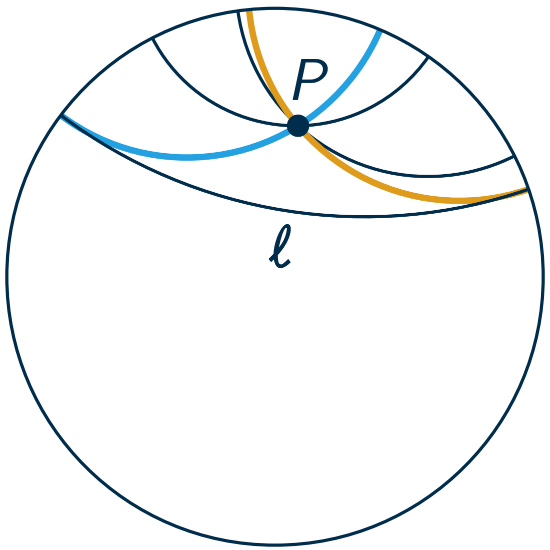

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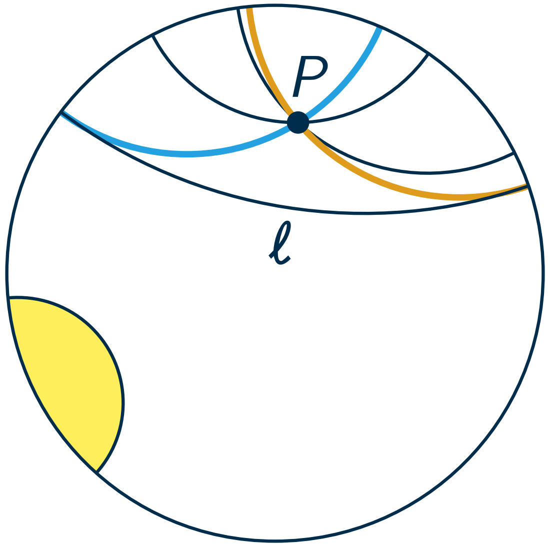

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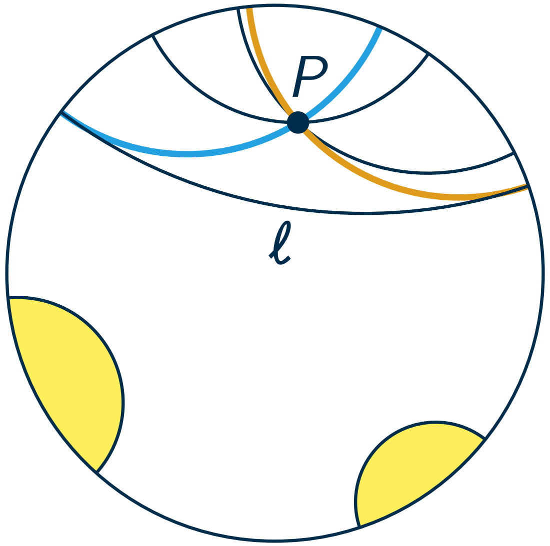

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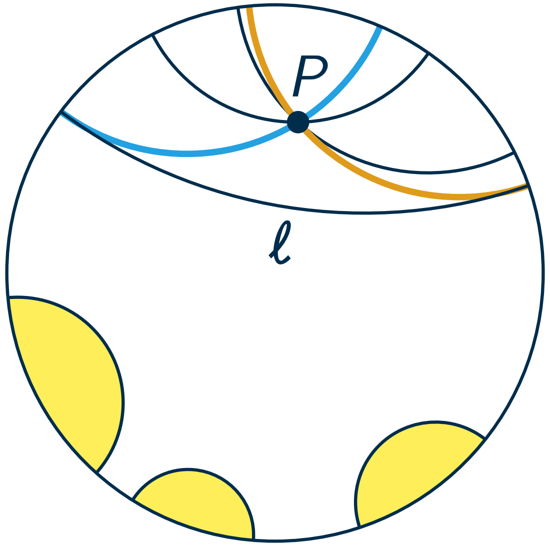

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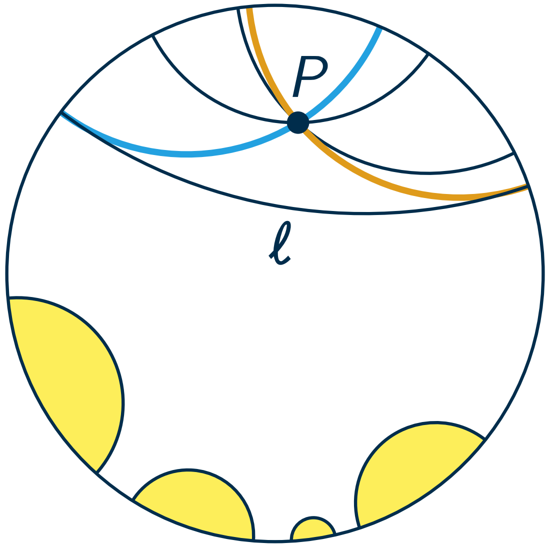

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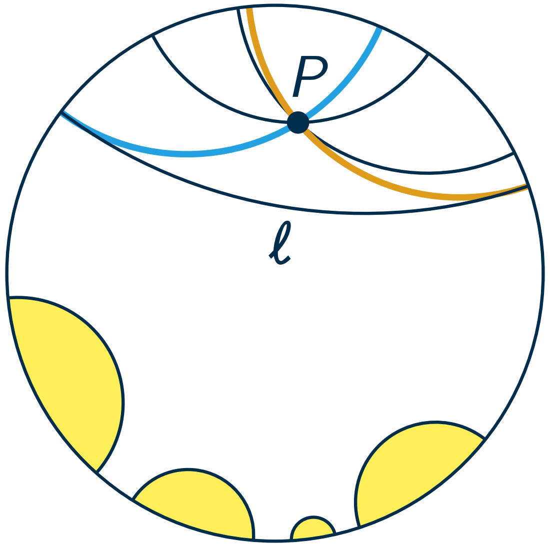

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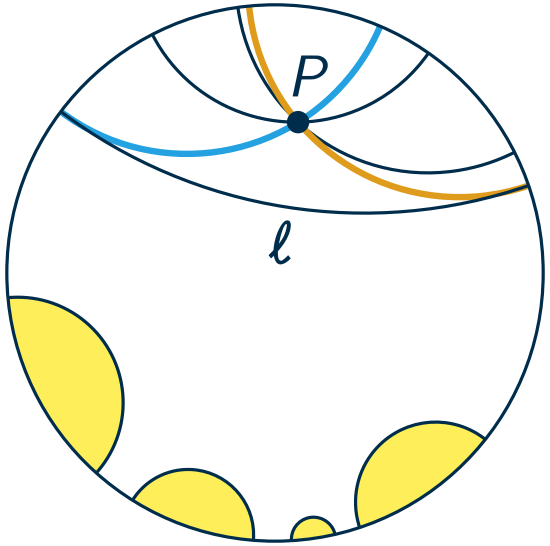

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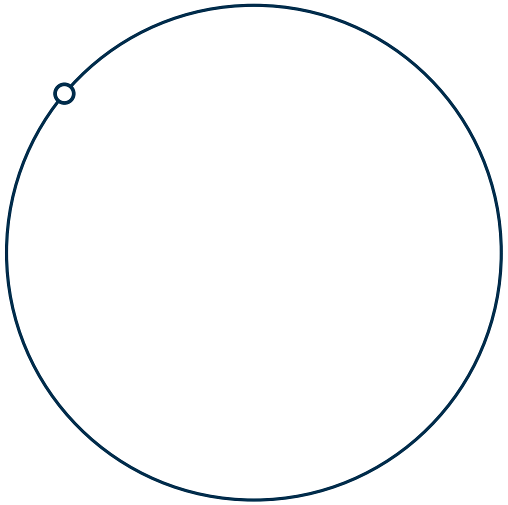

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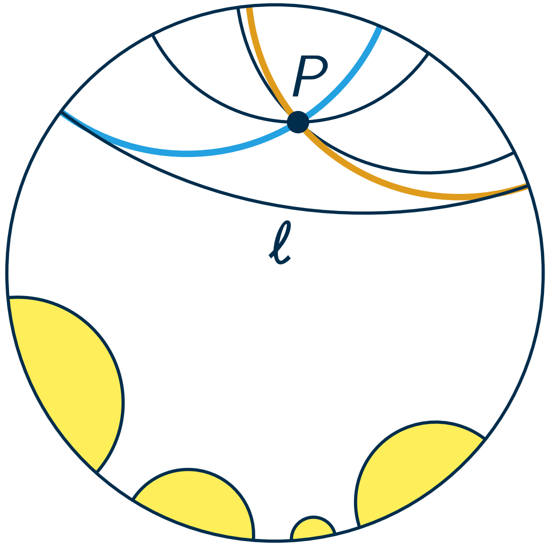

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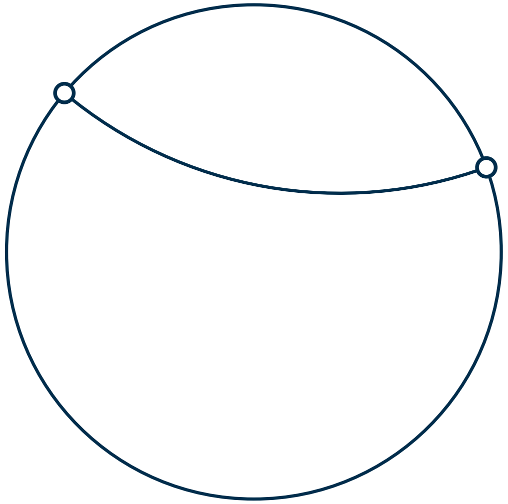

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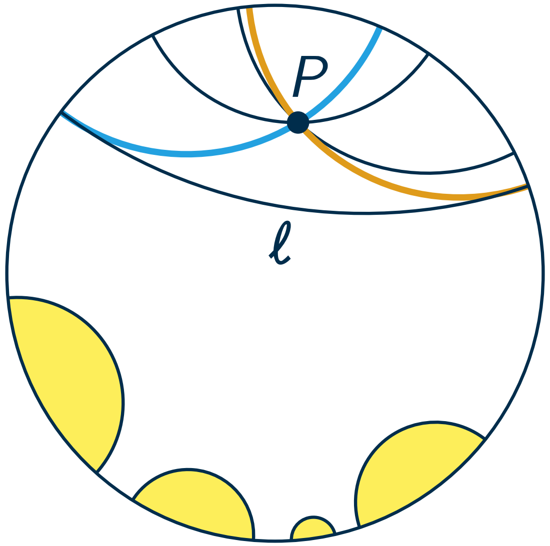

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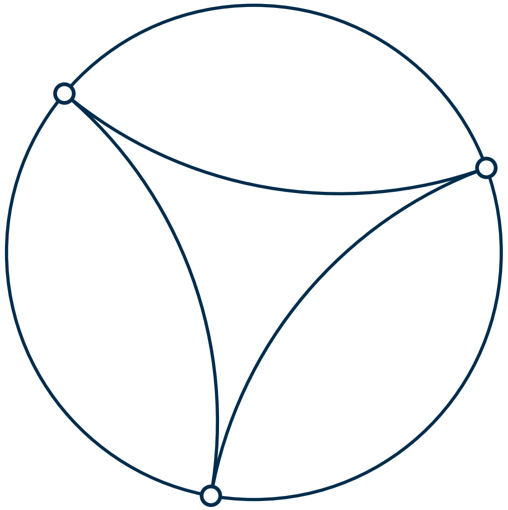

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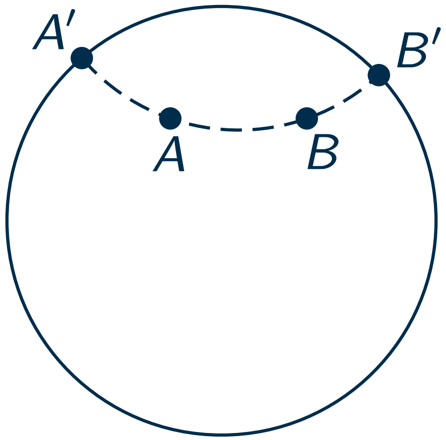
- ideal triangle: three lines that “connect” three ideal points (generalizes to n-gons)

Poincaré Disk – Distances

Distances

- A and B: two points in the Poincaré disk
- A' and B': ideal points of the line AB



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hyperbolic distance

Euclidean distance (in the Poincaré disk)

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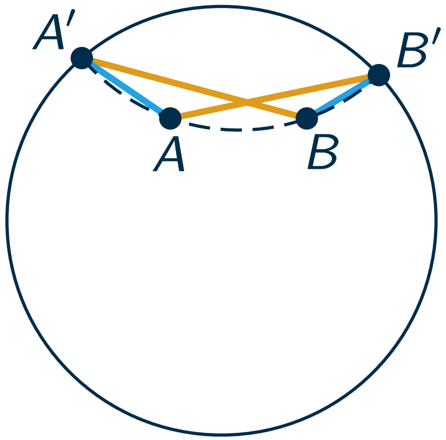

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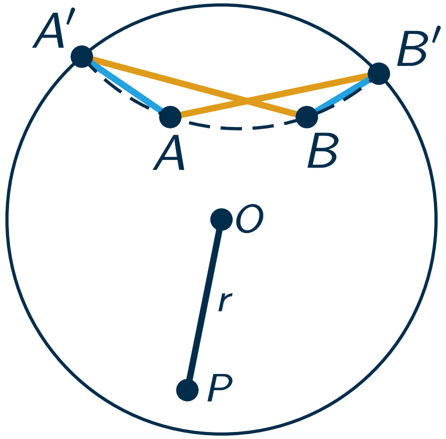
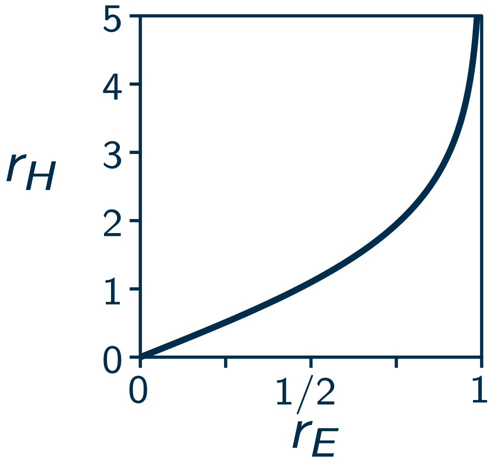
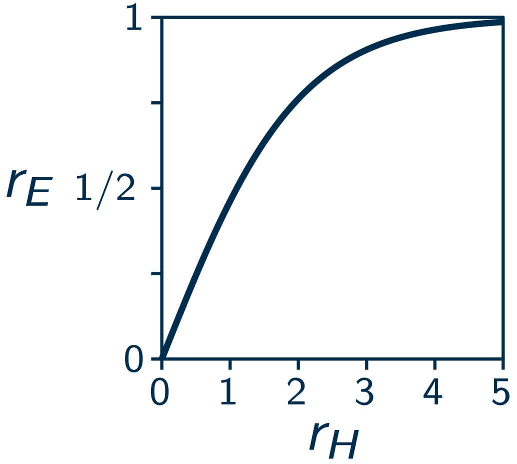


Distance To The Origin

- let O be the origin and P a point with $r_E = d_E(O, P)$ und $r_H = d_H(O, P)$
- then:

$$r_H = \log \frac{1 + r_E}{1 - r_E} \quad \text{and} \quad r_E = \frac{e^{r_H} - 1}{e^{r_H} + 1}$$





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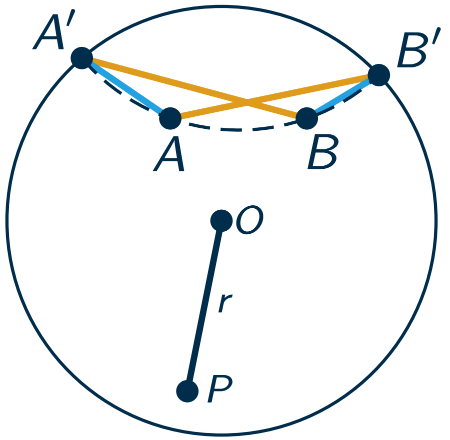

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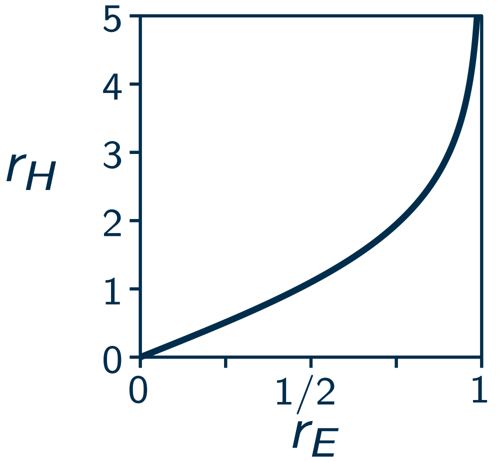
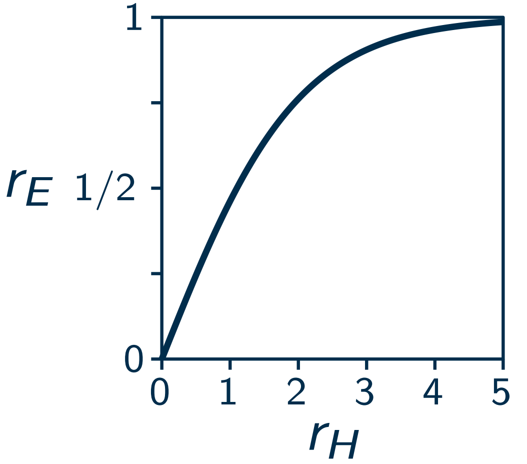
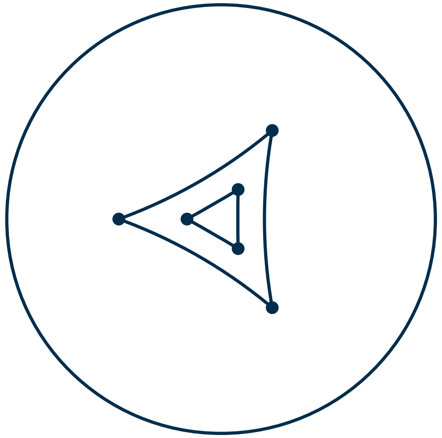




Area In The Hyperbolic Plane



Area Of A Triangle

- $\pi -$ (sum of internal angles)
- all triangles have area strictly below π

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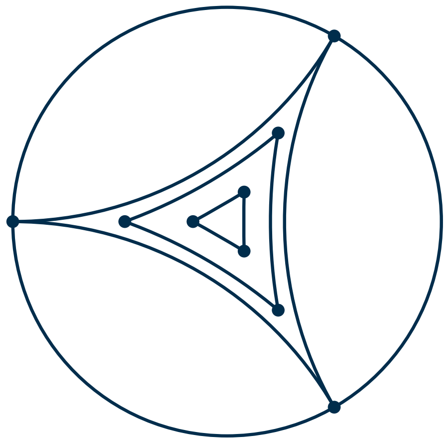

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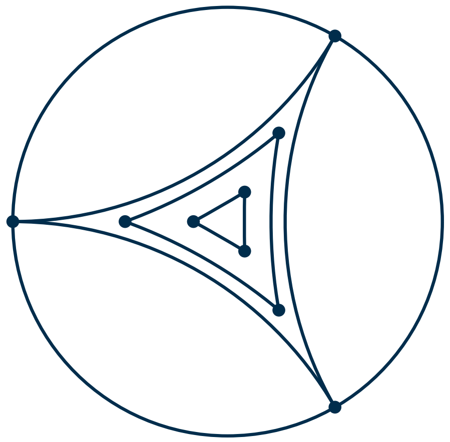

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Disks With (Hyperbolic) Radius r

- circumference: $2\pi \sinh(r)$
- area: $4\pi \sinh^2(r/2) = 2\pi(\cosh(r) - 1)$

Area In The Hyperbolic Plane

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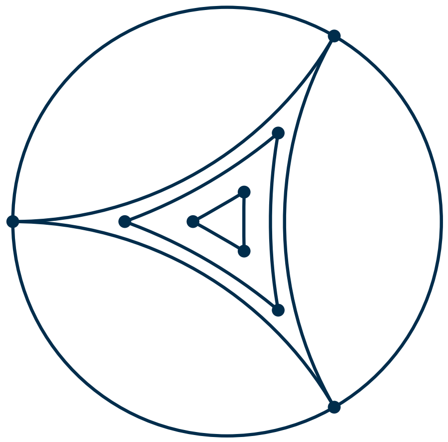
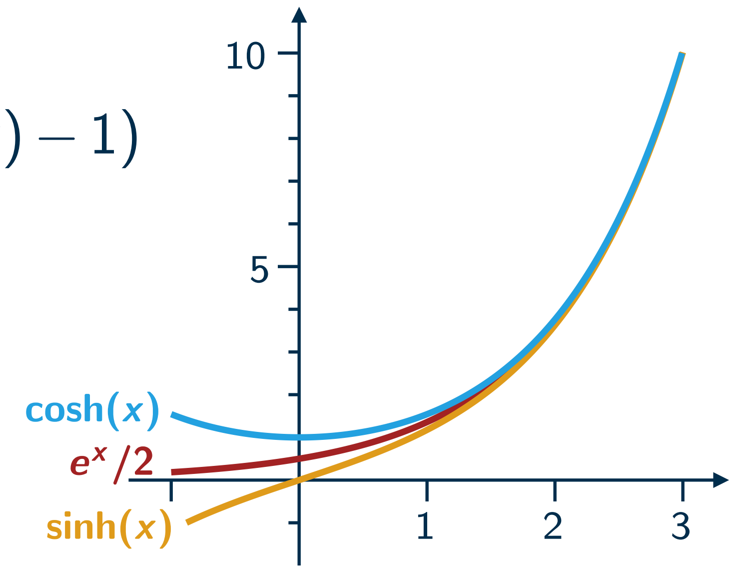


Disks With (Hyperbolic) Radius *r*

- circumference: $2\pi \sinh(r)$
- area: $4\pi \sinh^2(r/2) = 2\pi(\cosh(r) - 1)$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

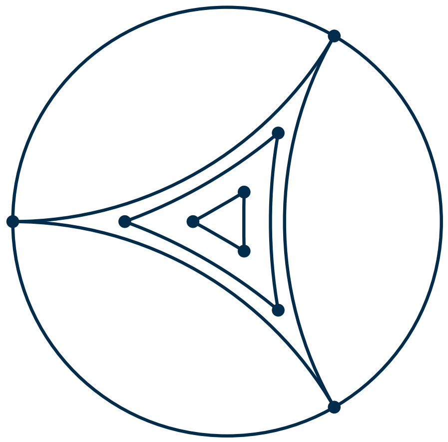
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$



Area In The Hyperbolic Plane

Area Of A Triangle

- $\pi-$ (sum of internal angles)
- all triangles have area strictly below π
- an ideal triangle has area π



Disks With (Hyperbolic) Radius r

- circumference: $2\pi\sinh(r)$
- area: $4\pi\sinh^2(r/2)=2\pi(\cosh(r)-1)$

$$\sinh(x)=\frac{e^x-e^{-x}}{2}$$
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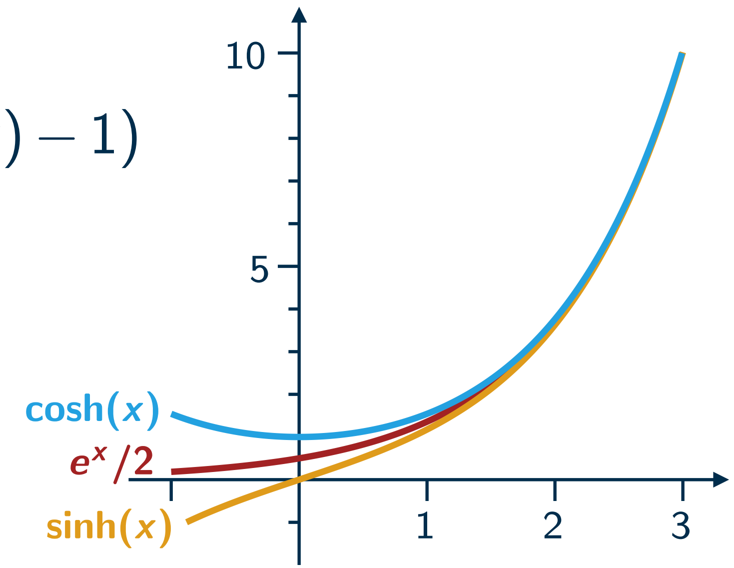

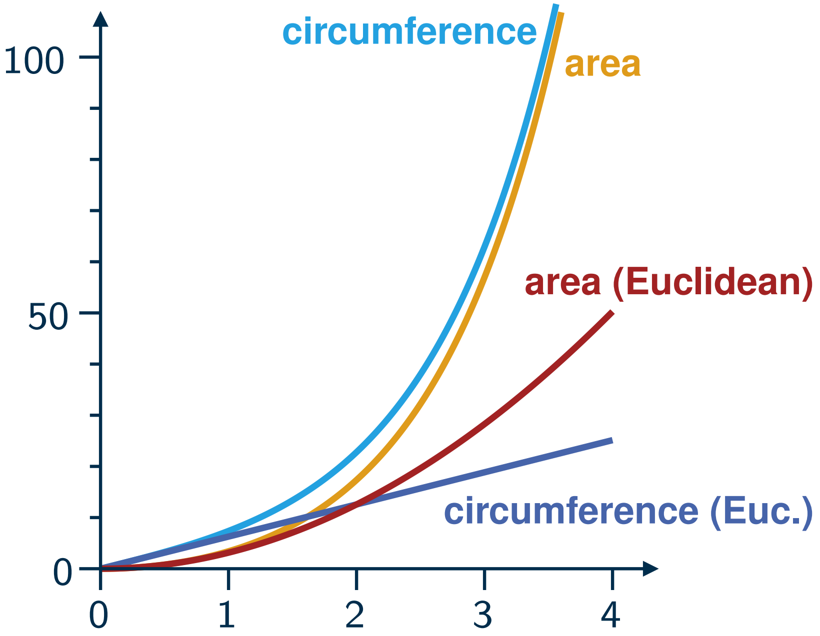

Wrap-Up

Seen Today

- axiomatic construction of geometry: defining and proving without intuition

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What Else Is There?

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What Else Is There?

- several other models: upper half-plane (Poincaré half plane), hyperboloid, Beltrami-Klein, native polar coordinates, . . .
- different coordinate systems
- different applications in the hyperbolic plane

Some Useful Ressources

“yellow pages”: many useful infos and formulas `http://www.maths.gla.ac.uk/wws/cabripages/hyperbolic/hyperbolic0.html`

Ipelets

- Poincaré disk `https://github.com/thobl/ipelets/tree/master/poincare`
- native polar coordinates `https://github.com/maxkatzmann/native-hyperbolic-ipelet`

Hipe hyperbolic Ipe (native polar) `https://github.com/maxkatzmann/Hipe`

HManim: hyperbolic extension of Manim `https://maxkatzmann.github.io/hmanim/`

Hyperbolic Games

- HyperRogue `http://www.roguetemple.com/z/hyper/`
- Hyperbolica `https://www.youtube.com/playlist?list=PLh9DXIT3m6N4qJK9GKQB3yk61tVe6qJvA`
- hyperbolic Sokoban `https://sokyokuban.com/`

Bonus: Hyperbolic Fish

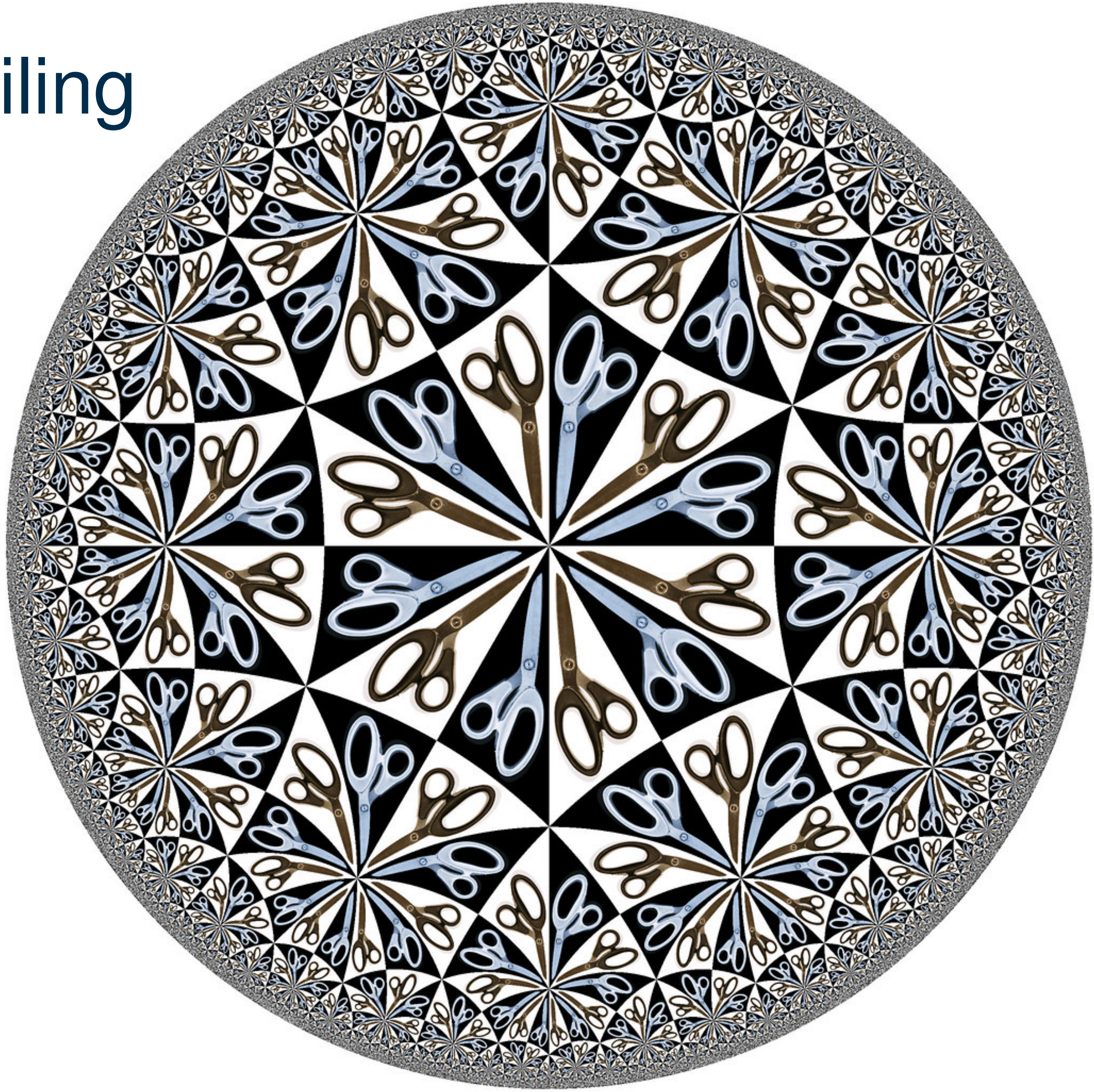
M. C. Escher
Circle Limit III

Bonus: Hyperbolic Fish (more or less)

M. C. Escher
Circle Limit III

Bonus: Photographic Tiling

http://poincare.sourceforge.net/



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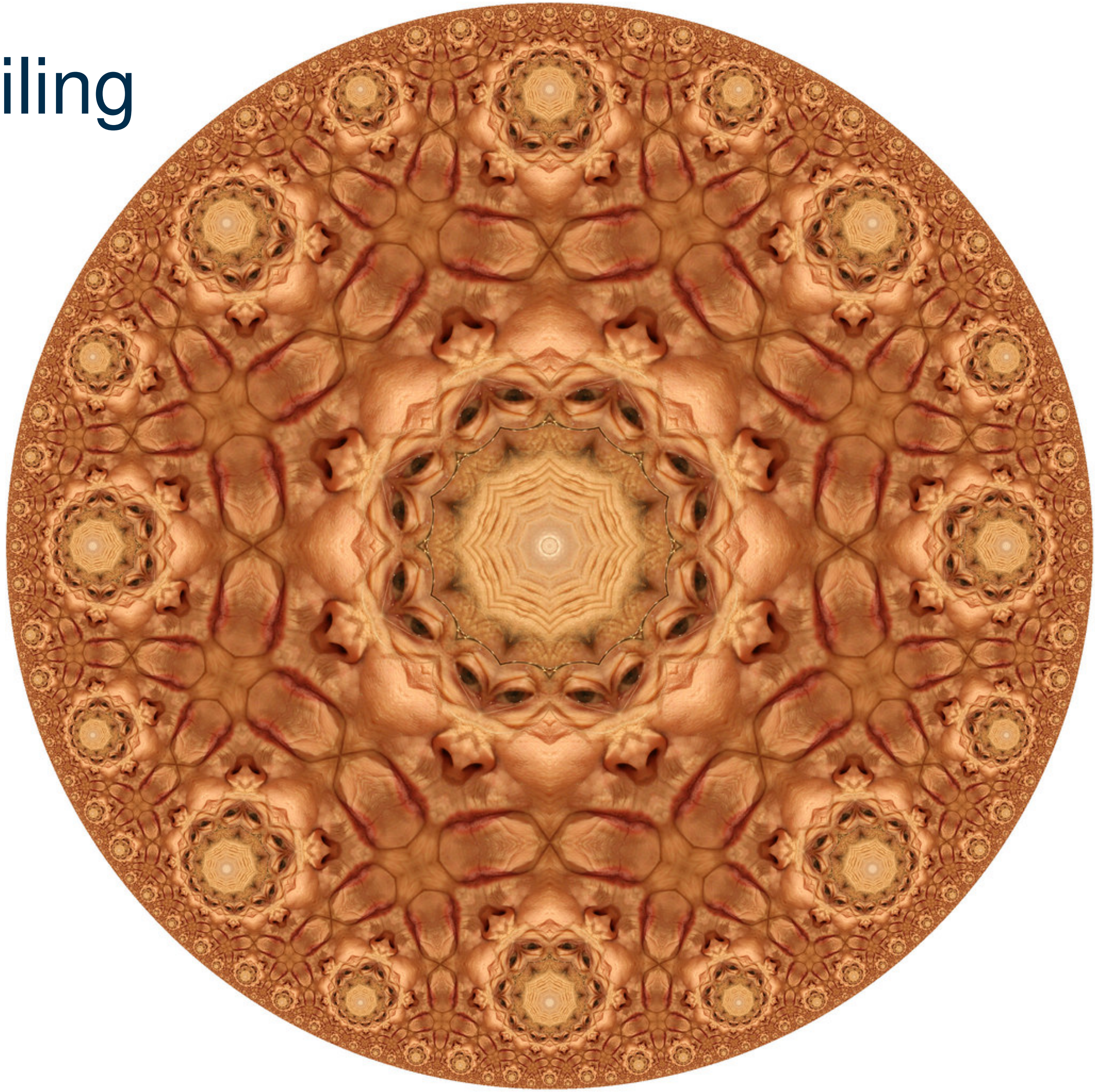

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