Computational Geometry

Geometry

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Back To School: Congruence Theorems

Theorem **(Congruence Theorem SSS)**
ΔABC and $\Delta A'B'C'$ with $|\overline{AB}| = |\overline{A'B'}|$, $|\overline{BC}| = |\overline{B'C'}|$, and $|\overline{CA}| = |\overline{C'A'}|$ are congruent.

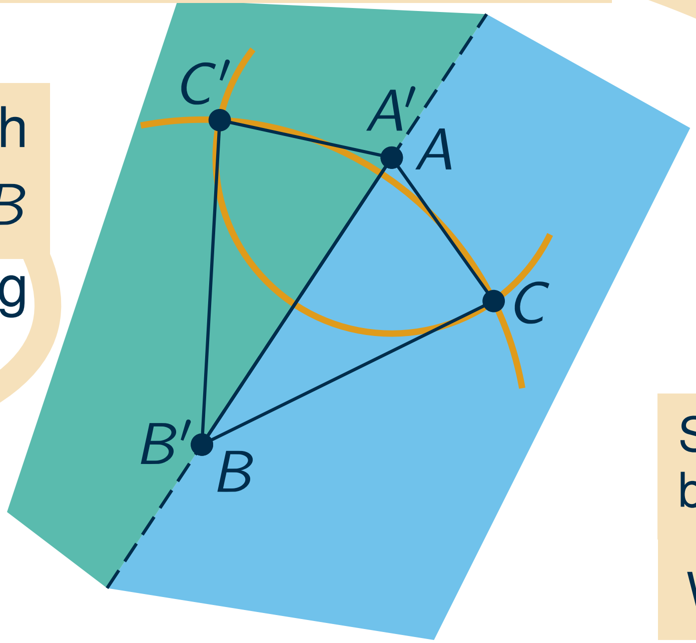
Proof

- $|\overline{AB}| = |\overline{A'B'}| \Rightarrow$ there is a motion m, with $m(A) = A'$ and $m(B) = B'$
- $A'B'$ defines two half planes
- in each half plane, there is only one point with distance $|\overline{AC}|$ from A and distance $|\overline{BC}|$ from B
- either m or m together with a reflection along $A'B'$ map C to C' $\Rightarrow$ triangles are congruent

Is this really true? If so, why?

Follows in the end from the triangle inequality.

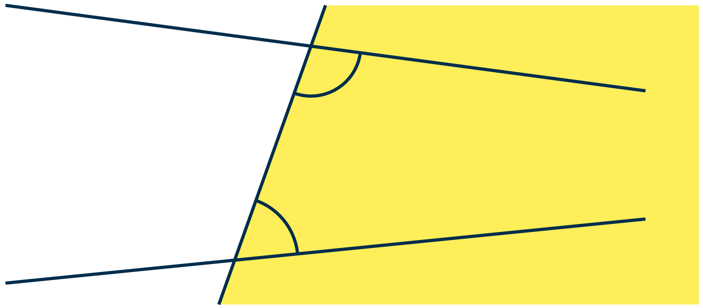
And how do we prove the triangle inequality?



Seems obvious...
but does m really exist?

Who guarantees this?

Euclid’s Axiomatic Approach



Euclid (around 300 BC)

- fix certain ground truths (postulates and axioms)
- everything else should follow without using intuition

The Five Axioms

- Things that are equal to the same thing are also equal to one another.
- If equals are added to equals, then the wholes are equal.
- If equals are subtracted from equals, then the differences are equal.
- Things that coincide are equal.
- The whole is greater than the part.

The Five Postulates Require That

- one can draw a straight line from any point to any point;
- one can extend every segment to a straight line;
- one can draw a circle around every center with every radius;
- all right angles are equal;
- if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which the angles are less than two right angles.

the last postulate is called **parallel postulate**

Trivia

Euclid's Elements (around 300 BC)

- a series of books on everything known in math at the time with more or less consistent notation
- follows the previously mentioned deductive reasoning (based on axioms)
- includes: Pythagorean theorem, sum of angles in a triangle, first binomial formula, Thales' theorem, first intercept theorem, similarity theorems for triangles, Euclidean algorithm (GCD), infinitely many primes, ...
- after the bible the most edited, commented, and translated book

Parallel Postulate

- over thousands of years: people tried to deduce it based on the other axioms
- turns out: you cannot deduce it (we will see this later)

Modern Perspective

- Euclid did not always manage to avoid using intuition completely
- definitions like: point – that which has no part; straight line – that which lies evenly with the points on itself
- Hilbert followed the deductive reasoning approach more rigorously (*Grundlagen der Geometrie*, 1899)
- Hilbert (1891): »Man muss jederzeit an Stelle von "Punkte, Geraden, Ebenen" "Tische, Stühle, Bierseidel" sagen können.«

Modern Axiomatic Perspective

Basic Building Blocks

- **basic terms** that are initially meaningless (“point” and “table” are interchangeable)
- the basic terms gain meaning from a set of **axioms** (properties that we wish our basic terms to have)
- **theorems**, that can be deduced from the axioms
- **definitions** are just abbreviations that simplify notation

Desirable Properties For A System Of Axioms

- consistency (free of contradictions)
- independence (no axiom can be deduced from the others)
- completeness (every formulatable statement is (dis)provable)
- you don’t always get what you want (see Gödel’s incompleteness theorem)

Plan For Today

- axiomatic system for geometry (with five groups of axioms I–V)
- we follow the axiomatic system of Kolmogorov (1977) (equivalent to Hilbert’s system)
- we assume to already have basic stuff like numbers (Peano) and set theory (ZFC)

Basic Terms: Points & Lines

Definition
Let $\mathcal{P}$ and $\mathcal{L}$ be disjoint sets. We call their elements **points** and **lines**, respectively. Then $I \subseteq \mathcal{P} \times \mathcal{L}$ is called an **incidence structure**. If $(P, \ell) \in I$, we say that P and ℓ are **incident**.

Example
- $\mathcal{P}$ is the set of chairs and $\mathcal{L}$ the set of tables in a restaurant
- $(P, \ell) \in I$, if the chair P stands at the table ℓ

Definition
An incidence structure is called a **geometry** if no two lines are incident to the same points.

Example
- $\mathcal{L} \subseteq 2^{\mathcal{P}}$ and $(P, \ell) \in I \Leftrightarrow P \in \ell$ (we denote it with $(\mathcal{P}, \mathcal{L}, \in)$)
- easy to show: every geometry is isomorphic to $(\mathcal{P}, \mathcal{L}, \in)$ (for the canonical definition of *isomorphic*)

Absolute/Euclidean Geometry & Incidence Axioms

Definition
An incidence structure $(\mathcal{P}, \mathcal{L}, \in)$ together with a map $d\colon \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ that satisfies axiom groups I–IV is called **absolute geometry**. If it satisfies axiom groups I–V, it is called **Euclidean geometry**. For $A, B \in \mathcal{P}$, $d(A, B)$ is called the **distance** between A and B.

Axiom Group I: Axioms of Incidence

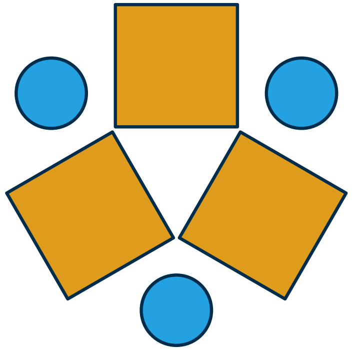
(1) For every two points $A \neq B$, there is exactly one line ℓ with $A \in \ell$ and $B \in \ell$. (we denote it as: $\ell = AB$)

(2) Every line contains at least two points.

(3) There are three points that do not lie on the same line.

Example

- stools are points
- tables are lines
- incidence: stands next to



Does this satisfy I?

Theorem
Two different lines share at most one point.

Axioms of Distance

Definition: Points that lie on the same line are called **collinear**.

Axiom Group II: Axioms of Distance

(1) For all points A, B: $d(A, B) \geq 0$ and $d(A, B) = 0 \Leftrightarrow A = B$.

(2) For all points A, B: $d(A, B) = d(B, A)$.

(3) For all points A, B, C, it holds that $d(A, B) + d(B, C) \geq d(A, C)$. Moreover, A, B, C are collinear if and only if

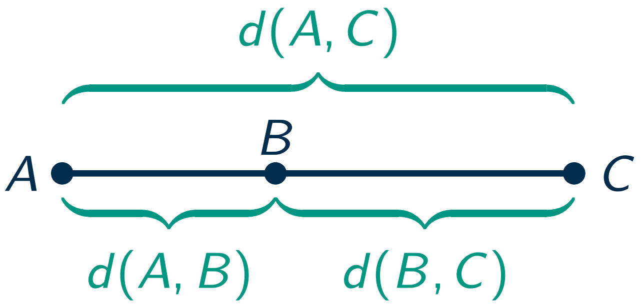
$$d(A, B) + d(B, C) = d(A, C),$$
$$d(A, C) + d(C, B) = d(A, B), \text{ or}$$
$$d(B, A) + d(A, C) = d(B, C).$$

Note

- axiom group II turns a set of points into a metric space
- three points on a line $\Leftrightarrow$ it is not a detour to visit one of them on the way between the others
- we will give the one a name in a moment: it lies **between** the others

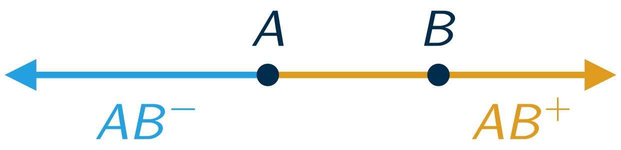
Line Segments, Rays, and Convexity

Definition
B lies **between** A and C if $d(A, B) + d(B, C) = d(A, C)$ and $B \notin \{A, C\}$.



Definition
Let $A, B \in \mathcal{P}$ with $A \neq B$. We call $(AB) = \{P \in \mathcal{P} \mid \text{P lies between A and B}\}$ the **open segment** and $\overline{AB} = (AB) \cup \{A, B\}$ the **segment** between A and B. A and B are the **end points** of $\overline{AB}$ and $d(A, B)$ is its **length**.

Definition
Let $A, B \in \mathcal{P}$ with $A \neq B$. Define $AB^+ = \{P \mid P \in \overline{AB} \text{ or } B \in \overline{AP}\}$ and $AB^- = \{P \mid A \in \overline{PB}\}$. The sets AB^+ and AB^- are called **rays** starting at A.



Definition
A set $M \subseteq \mathcal{P}$ is **convex**, if $\overline{AB} \subseteq M$ for all $A, B \in M$.

Axioms of Order and Half Planes

Axiom Group III: Axioms of Order

(1) For every point A and every number $a \in \mathbb{R}^+$, every ray starting at A contains exactly one point B with $d(A, B) = a$.

(2) Every line ℓ partitions the set $\mathcal{P} \backslash \ell$ in two non-empty subsets such that for every $A, B \in \mathcal{P} \backslash \ell$, the segment $\overline{AB}$ intersects ℓ if and only if A and B are in different subsets.

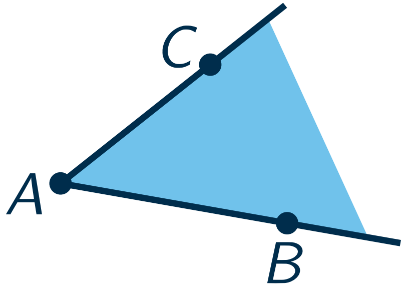
Definition

The two sets are called **open half planes** with **boundary line** ℓ. The union with ℓ yields a **half plane**. The half plane with boundary line $\ell = AB$ that contains a point $C \notin \ell$ is denoted with ABC^+. The other half plane with ABC^-.

Note

- (1) ensures that we have infinitely many points
- (2) in particular tells us that half planes are convex

Angles and Motions



Definition
The union of two rays is an **angle** $\angle BAC = AB^+ \cup AC^+$, with two **arms** AB^+ and AC^+. It is **straight** if $\angle BAC = AB$ and a **zero angle** if $AB^+ = AC^+$. $ABC^+ \cap ACB^+$ is its **interior**.

Definition: A surjective map $m\colon \mathcal{P} \to \mathcal{P}$ is called **motion** if it preserves distances.

Properties That More Or Less Directly Follow

- a motion is also injective as $d(A, B) = 0 \Leftrightarrow A = B$
- motions preserve the between-relation, segments, lines, rays, half planes, angles, etc.
- for $d(A, B) = d(A', B') > 0$, there exist at most two motions that map A to A' and B to B'

Axiom Group IV: Axiom of Motion
For $d(A, B) = d(A', B') > 0$ there are at least two motions that map A to A' and B to B'.

Parallel Axiom

Definition: Two lines that do not intersect are called **parallel**.

Axiom Group V: Euclidean Parallel Axiom
For every line ℓ and every point $P \notin \ell$, there is at most one line through P that is parallel to ℓ.

Recap

Axiom Group I: Incidence
two points define a line; every line contains two points; there are three non-collinear points

Axiom Group II: Distance
distance is a metric; tightness of triangle inequality if and only if collinear

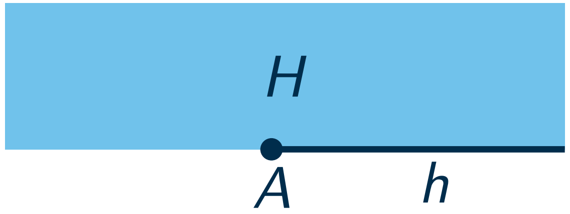
Axiom Group III: Order
there is a point in every direction with every distance; lines split the plane into half planes

Axiom Group IV: Motion
two motions that map segments of equal length onto each other (preserving orientation)

Definition
An incidence structure $(\mathcal{P}, \mathcal{L}, \in)$ together with a map $d : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ that satisfies axiom groups I–IV is called **absolute geometry**. If it satisfies axiom groups I–V, it is called **Euclidean geometry**. For $A, B \in \mathcal{P}$, $d(A, B)$ is called the **distance** between A and B.

Absolute Geometry: Flags and Special Motions

Definition
Let $h = AB^+$ be a ray and H be a half plane with boundary line AB. The triple (A, h, H) is called **flag**.



Theorem
For any two flags (A, h, H) and (A', h', H'), there is exactly one motion that maps (A, h, H) to (A', h', H') (i.e., $m(A) = A'$, $m(h) = h'$, $m(H) = H'$).

Definition
The motion that maps (A, h, H) to (A', h', H') is called **translation** if $A \neq A'$, $h' \subseteq h$ and $H = H'$. **(Point) reflection** and **Rotation** can be defined similarly.

Absolute Geometry: Triangles and Congruence

Definition
Let A, B, C be non-collinear points. Then $\Delta ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$ is the **triangle** with **sides** $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

Definition
Two sets of points S and S' are **congruent** ($S \cong S'$) if there is a motion m with $m(S) = S'$.

Theorem (Congruence)
ΔABC and $\Delta A'B'C'$ are congruent if $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, and $\overline{CA} \cong \overline{C'A'}$. **(SSS)**

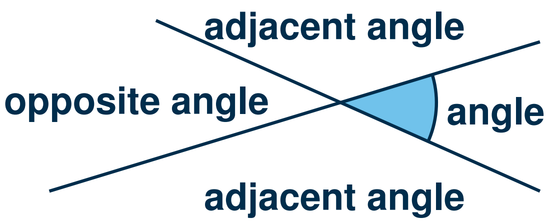
…if $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, and $\angle BAC \cong \angle B'A'C'$. **(SAS)**

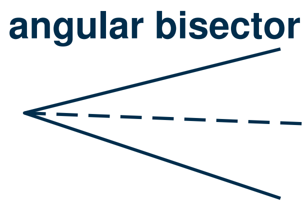
…if $\overline{AB} \cong \overline{A'B'}$, $\angle BAC \cong \angle B'A'C'$, and $\angle ABC \cong \angle A'B'C'$. **(ASA)**

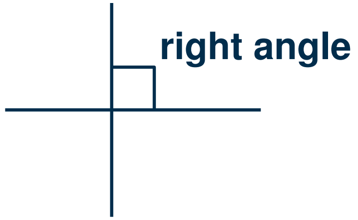
Absolute Geometry: Miscellaneous

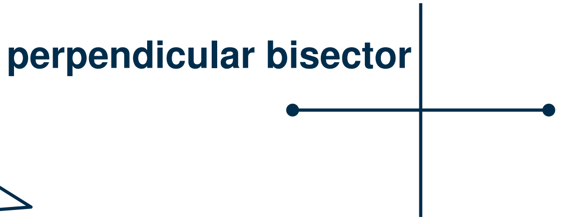
Disclaimer

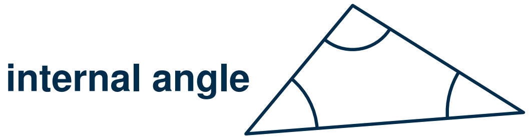
- we now use terms that we have not defined formally
- you probably know them and have an intuition for them
- a definition (without intuition) is usually straight-forward

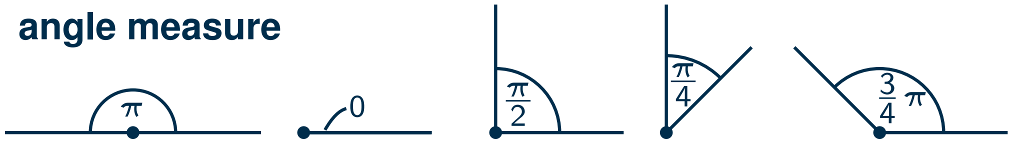












Theorems

- every angle is congruent to its opposite angle
- the following are unique: the angular bisector, the midpoint, the perpendicular bisector
- the perpendicular bisector of $\overline{AB}$ is the set of all points with equal distance to A and B
- sum of all internal angles (measure) in a triangle $\leq \pi$
- if there is a triangle with internal angle sum π, then every triangle has internal angle sum π

Seen So Far

Axiom Group I: Incidence
two points define a line; every line contains two points; there are three non-collinear points

Axiom Group II: Distance
distance is a metric; tightness of triangle inequality if and only if collinear

Axiom Group III: Order
there is a point in every direction with every distance; lines split the plane into half planes

Definition
An incidence structure $(\mathcal{P}, \mathcal{L}, \in)$ together with a map $d\colon \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ that satisfies axiom groups I–IV is called **absolute geometry**. If it satisfies axiom groups I–V, it is called **Euclidean geometry**. For $A, B \in \mathcal{P}$, $d(A, B)$ is called the **distance** between A and B.

Axiom Group IV: Motion
two motions that map segments of equal length onto each other (preserving orientation)

Axiom Group V: Euclidean Parallel Axiom
line ℓ and point $P \notin \ell \Rightarrow$ at most one line through P parallel to ℓ

What Happens If We Negate The Parallel Axiom?

Axiom Group V': Hyperbolic Parallel Axiom
There is a line ℓ and a point $P \notin \ell$ such that there are two lines through P parallel to ℓ.

Definition
An incidence structure (with d) satisfying axiom groups I–IV, V' is called **hyperbolic geometry**.

The Hyperbolic Plane

What Happens If We Negate The Parallel Axiom?

- the axioms remain consistent
- we get a second model that satisfies the axioms of the absolute plane (next to the Euclidean plane)
- all theorems for the absolute plane also hold in the hyperbolic plane

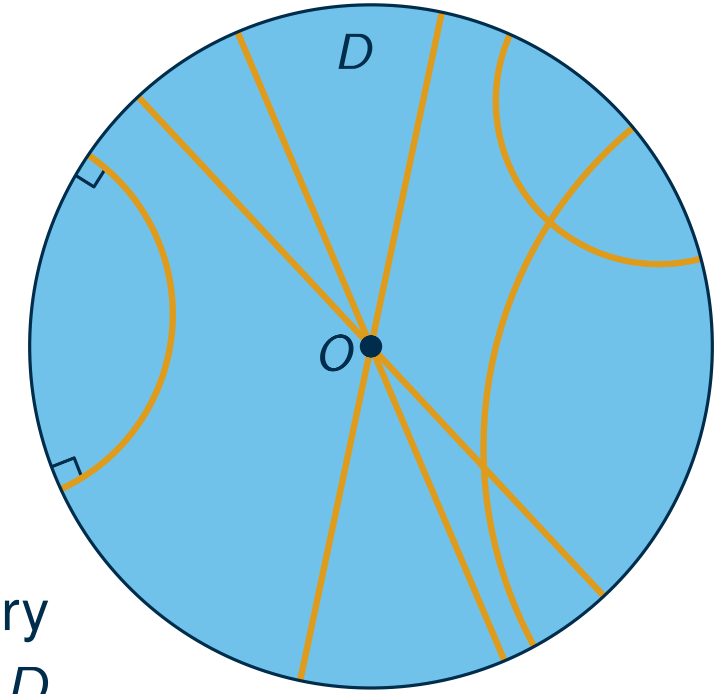
Theorem (Hyperbolic Plane)
For every line ℓ and every point $P \notin \ell$, there are infinitely many lines parallel to ℓ through P.

The sum of interior angles in a triangle is less than π. There are not rectangles.

This All Seems Somewhat Strange

- so far: consider Euclidean geometry without intuition $\rightarrow$ naturally yields hyperbolic geometry
- we are somehow lacking an intuition for the hyperbolic plane
- helpful: models that represent the hyperbolic plane

Poincaré Disk Model



Points

- consider a (Euclidean) disk D with radius 1 around the point O
- let $\mathcal{P}$ be the set of points in the interior of the disk

Lines

- let $\mathcal{L}$ be the union of:
  - set of open segments through O with endpoints on D's boundary
  - set of open circular arcs in D perpendicular to the boundary of D

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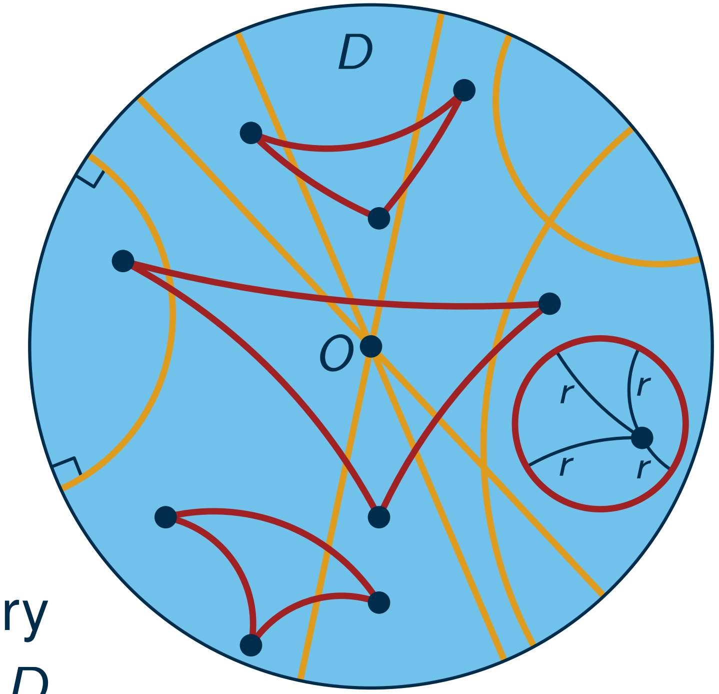
Short Break

Can you verify that the model satisfies some of the axioms?

Poincaré Disk Model



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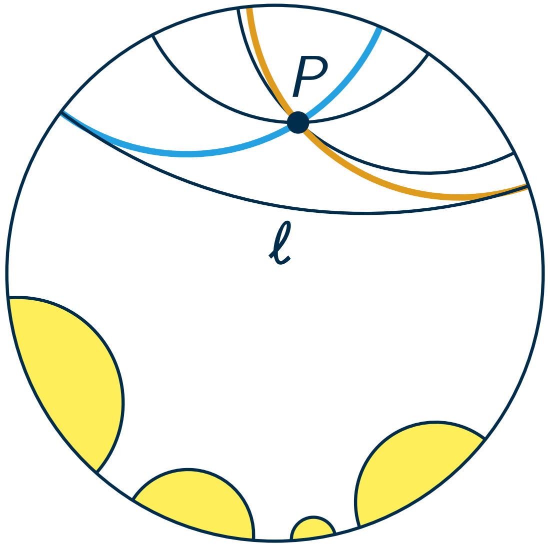
Lines

- let $\mathcal{L}$ be the union of:
  - set of open segments through O with endpoints on D's boundary
  - set of open circular arcs in D perpendicular to the boundary of D

It Holds That

- $(\mathcal{P}, \mathcal{L}, \in)$ together with an appropriate distance function satisfies axiom groups I–IV and V'
- the model is angle preserving
- this makes it "intuitively obvious" that the sum of internal angles in a triangle is less than π
- distances are distorted, but hyperbolic circles look like circles (with different center)
- points on the boundary of D are **not** part of the hyperbolic plane

Poincaré Disk – Parallels, Half Plans, Ideal Points

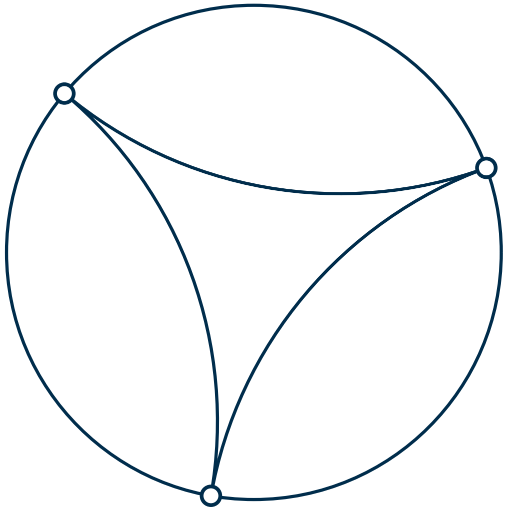


Parallel Lines Through One Point

- you can easily find multiple lines parallel to ℓ through P
- two lines are only barely parallel
  - they "intersect" ℓ at the boundary of the Poincaré Disk
  - those are the **limiting parallels** of ℓ through P

Half Planes

- there are infinitely many disjoint half planes (being pairwise congruent)
- so compared to the Euclidean plane, there is somehow more space



Ideal Points

- points on the disk's boundary are called **ideal points** (they are not points!)
- each line "ends" in two ideal points
- ideal triangle: three lines that "connect" three ideal points (generalizes to n-gons)

Poincaré Disk – Distances

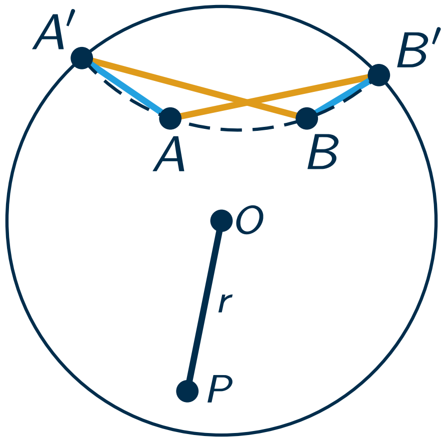
Distances

- A and B: two points in the Poincaré disk
- A' and B': ideal points of the line AB

hyperbolic distance

Euclidean distance (in the Poincaré disk)

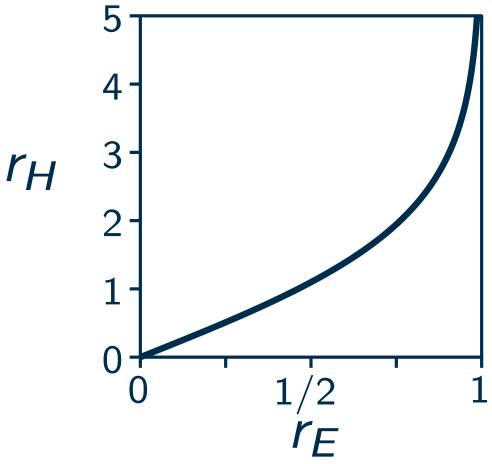
$$d_H(A, B) = \log \frac{d_E(A, B') \cdot d_E(A', B)}{d_E(A, A') \cdot d_E(B, B')}$$

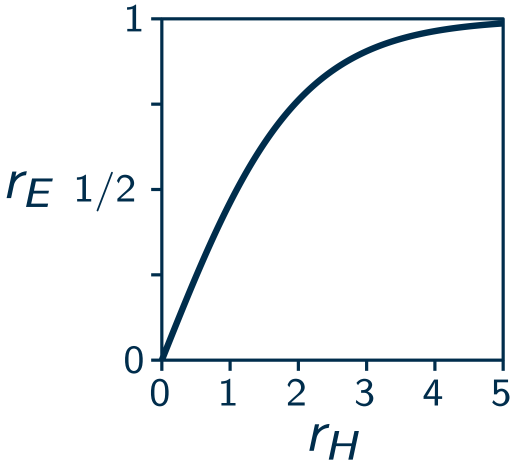


Distance To The Origin

- let O be the origin and P a point with $r_E = d_E(O, P)$ und $r_H = d_H(O, P)$
- then:

$$r_H = \log \frac{1 + r_E}{1 - r_E} = 2\,\mathrm{arctanh}(r_E) \quad \text{and} \quad r_E = \frac{e^{r_H} - 1}{e^{r_H} + 1} = \tanh\left(\frac{r_H}{2}\right)$$

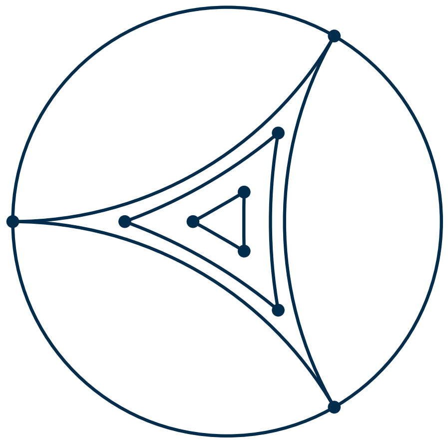




Area In The Hyperbolic Plane

Area Of A Triangle

- π − (sum of internal angles)
- all triangles have area strictly below π
- an ideal triangle has area π

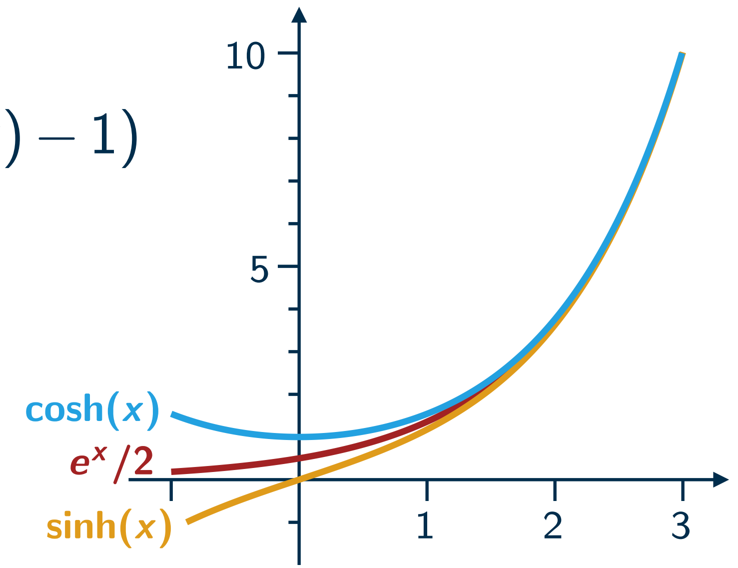


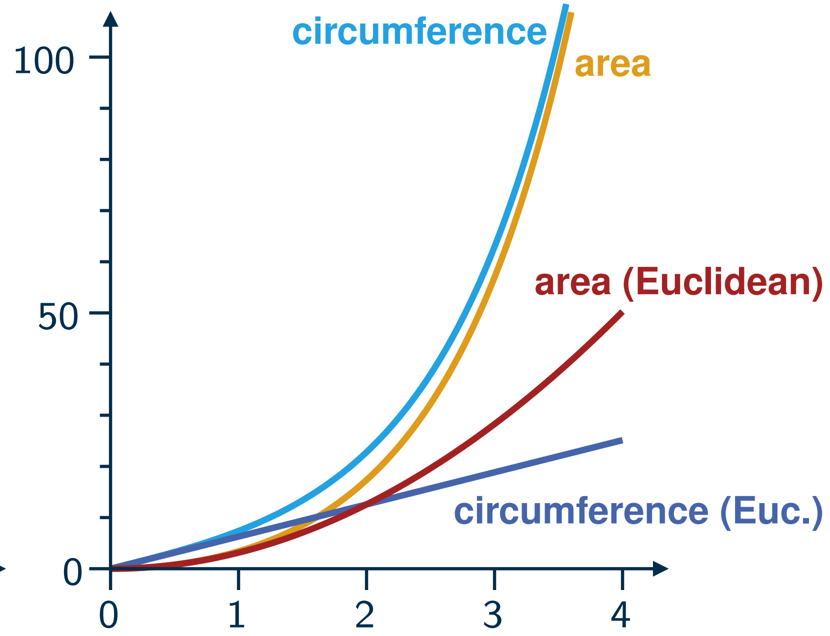
Disks With (Hyperbolic) Radius *r*

- circumference: $2\pi \sinh(r)$
- area: $4\pi \sinh^2(r/2) = 2\pi(\cosh(r) - 1)$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$





Wrap-Up

Seen Today

- axiomatic construction of geometry: defining and proving without intuition
- the parallel axiom and the hyperbolic plane
- basic similarities and differences between Euclidean and hyperbolic geometry
- Poincaré Disk helps the intuition

What Else Is There?

- several other models: upper half-plane (Poincaré half plane), hyperboloid, Beltrami-Klein, native polar coordinates, . . .
- different coordinate systems
- different applications in the hyperbolic plane

Some Useful Ressources

"yellow pages": many useful infos and formulas http://www.maths.gla.ac.uk/wws/cabripages/hyperbolic/hyperbolic0.html

Ipelets

- Poincaré disk https://github.com/thobl/ipelets/tree/master/poincare
- native polar coordinates https://github.com/maxkatzmann/native-hyperbolic-ipelet

Hipe hyperbolic Ipe (native polar) https://github.com/maxkatzmann/Hipe

HManim: hyperbolic extension of Manim https://maxkatzmann.github.io/hmanim/

Hyperbolic Games

- HyperRogue http://www.roguetemple.com/z/hyper/
- Hyperbolica https://www.youtube.com/playlist?list=PLh9DXIT3m6N4qJK9GKQB3yk61tVe6qJvA
- hyperbolic Sokoban https://sokyokuban.com/

Bonus: Hyperbolic Fish (more or less)

M. C. Escher
Circle Limit III

Bonus: Photographic Tiling

http://poincare.sourceforge.net/