Computational Geometry

Real RAM, Word RAM, Point Location
What is a computer?

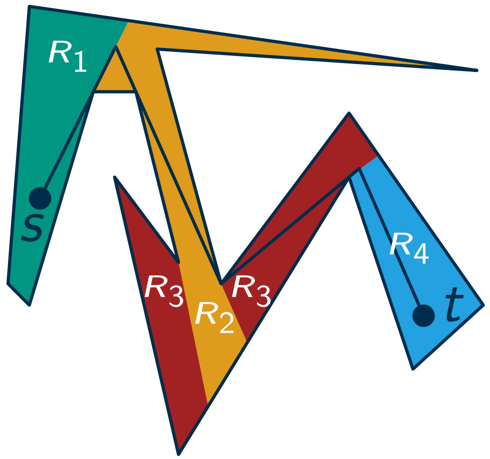
Thomas Bläsius

What Can Your Computer Do?

Model Of Computation

- RAM (random access machine): memory access in $O(1)$ via an address
- real RAM
  - every memory cell holds a real number (of arbitrary size/precision)
  - arithmetic operations $(+, -, \cdot, /)$ in $O(1)$
  - rounding to integers is **not** allowed (otherwise, you can do broken things)
  - common model in computational geometry $\rightarrow$ abstracts away precision issues
  - potential problem: sometimes too powerful (more powerful than your computer)
- word RAM
  - every memory cell holds a **word** consisting of w bits
  - w is sufficiently large ($\geq \log n$, but potentially much larger)
  - arithmetic operations on integers (of size up to 2^w) in $O(1)$
  - bit-wise logical operations and bit shifts in $O(1)$

Minimum Link Path In Polygons



Problem: Minimum Link Path
Given a polygon P as well as points s and t in P, compute an st-path inside P with the minimum number of segments.

Theorem **(without proof)**
The minimum link path between two points in a polygon of size n can be computed in $O(n)$ time.

Theorem **(without proof)**
There are instances encodable with $\Theta(n \log n)$ bits such that representing the polygons $R_1, \dots, R_n$ requires $\Theta(n^2 \log n)$ bits.

General Idea

- R_i = set of points reachable from s with i links
- iteratively compute R_{i+1} from R_i

Which Of Them Is True?

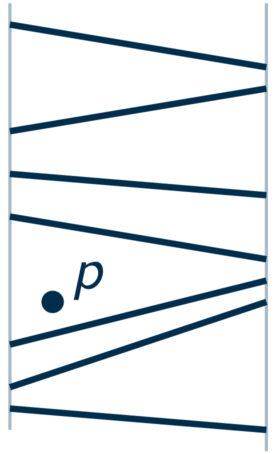
- first theorem assumes a real RAM
- an implementations (e.g., with doubles for coordinates) is maybe not robust

Point-Location In A Vertical Slab (word RAM)

Problem
Let S be a set of disjoint segments between two vertical lines. Build a data structure that can answer between which two segments a query point p lies. (with acceptable memory consumption)

Lets Start With One Dimension

- predecessor search in a sequence of numbers
- default solution: binary search provides $\Theta(\log n)$ queries
- goal: queries in $o(\log n)$
- use properties of the word RAM
  - numbers are integers
  - numbers lie in the interval $[0, 2^w)$
  - arithmetic operations, bit-wise logical operations, and bit-shifts on words of length w in $O(1)$



Beating The $\log n$ Lower Bound

Binary Search Basics

- walks down a decision tree
- decision for left/right subset: one comparison
- per step: number of integers is halved
- recursion depth: $\log_2(n)$

Ideas For Improvement

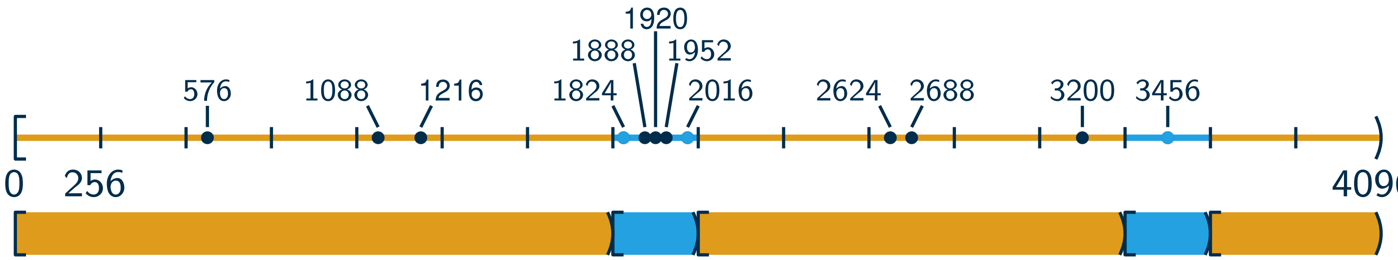
- wider branching: shrink number of possible integers by the factor b in each step
  - recursion depth: $\log_b(n)$ (sub-logarithmic if b super-constant)
  - problem: decision for the correct subinterval too expensive
- shrinking the interval (in the beginning, all numbers lie in $[0, 2^w)$)
  - shrink the interval by the factor 2^h
  - recursion depth: $\frac{w}{h}$
  - problem: deciding whether a subinterval still contains integers
- combining both ideas
  - in each step: shrink the number of integers or the interval
  - decision for the correct interval: $O(1)$ with bit magic

Subdivision Into Subintervals

word size: w
number of integers: n
interval: $[0, 2^\ell)$
number of cells: 2^h
branching width: b

One Subdivision Step

- divide interval of size 2^ℓ into 2^h cells of size $2^{\ell-h}$
- mark each $\frac{n}{b}$th integer
- mark cells that contain marked integers
- define partition into subintervals
  - each marked cell is a subinterval
  - each maximal sequence of unmarked cells is a subinterval



$n = 12$
$\ell = 12$
$h = 4$
$b = 3$

Properties Of This Subdivision

- we have $O(b)$ subintervals
- the boundaries of each subinterval is a multiple of $2^{\ell-h}$
- each subinterval has length $2^{\ell-h}$ or contains at most $\frac{n}{b}$ integers

Recursive Subdivision

word size: w
number of integers: n
interval: $[0, 2^\ell)$
number of cells: 2^h
branching width: b

Lemma **(nice (h, b)-subdivision)**

Given n integers in $I = [0, 2^\ell)$. I can be subdivided into $O(b)$ subintervals such that:

- each subinterval has length $2^{\ell-h}$ or contains at most $\frac{n}{b}$ integers
- the boundaries of the subintervals are multiples of $2^{\ell-h}$

Recursive Decision Tree

- one child for each subinterval in the (h, b)-subdivision
- stop recursion if only few integers are left
- height of the tree: at most $\log_b(n) + \frac{w}{h}$

Why?

Note: In the recursive calls, we may need to shift the subinterval to 0 and increase it to the next power of 2, such that it has the form $[0, 2^\ell)$.

Searching For The Successor Of An Integer q

- find subinterval containing q at most $\log_b(n) + \frac{w}{h}$ times
- bit magic: each step runs in $O(1)$ if $\frac{w}{h} \geq \frac{b}{c}$ (for a constant c)

$$\left.\right\} \begin{array}{l} O(\log_b(n) + b) \text{ for } \frac{w}{h} = \frac{b}{c} \\ O(\frac{\log n}{\log\log n}) \text{ for } b = \sqrt{\log n} \end{array}$$

How Many Bits Do You Need?

How many bits are necessary in total to encode the numbers b, r, e, a, k?

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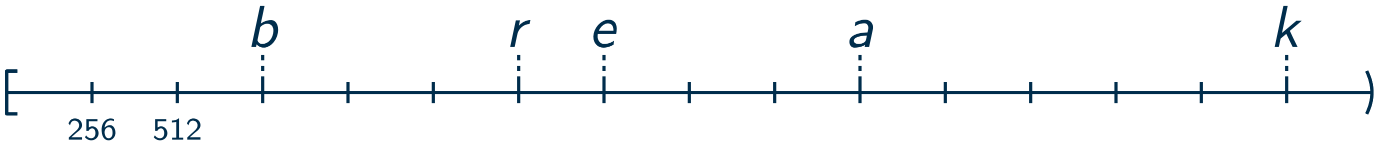


How many bits are necessary in total to encode the numbers b, r, e, a, k?

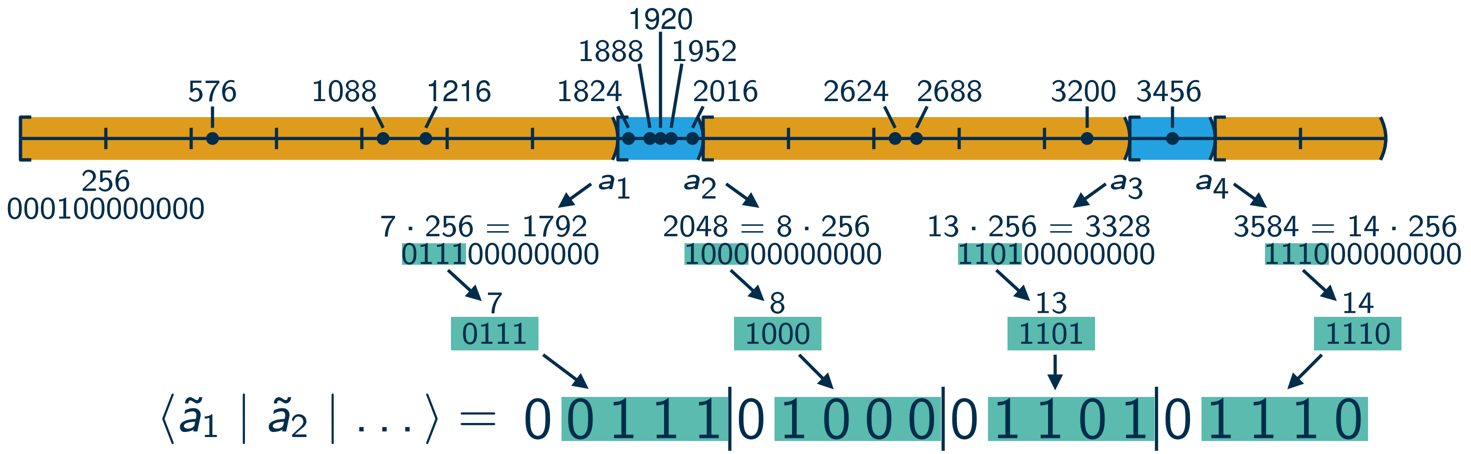
$$b = 3 \cdot 256 \longrightarrow \boxed{0011}00000000$$
$$r = 6 \cdot 256 \longrightarrow \boxed{0110}00000000$$
$$e = 7 \cdot 256 \longrightarrow \boxed{0111}00000000$$
$$a = 10 \cdot 256 \longrightarrow \boxed{1010}00000000$$
$$k = 15 \cdot 256 \longrightarrow \boxed{1111}00000000$$

0011 | 0110 | 0111 | 1010 | 1111

20 bits

Representing The Subintervals

word size: w | $n = 12$
number of integers: n | $\ell = 12$
interval: $[0, 2^\ell)$ | $h = 4$
number of cells: 2^h | $b = 3$
branching width: b

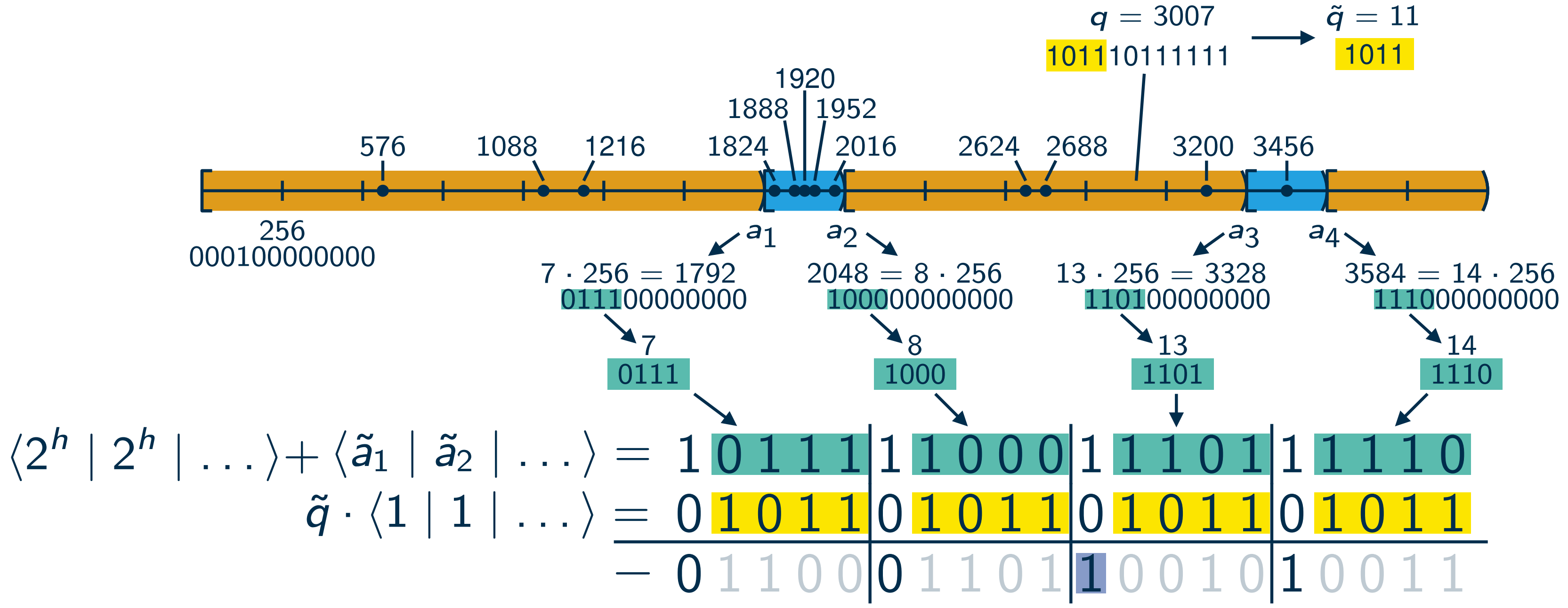


Representing The $\bar{b}$ Interval Boundaries $a_1, \ldots, a_{\bar{b}}$

- boundaries are multiples of $2^{\ell - h}$
- divide by $2^{\ell - h}$ $\rightarrow$ result is $\leq 2^h$
- concatenate binary representation: $h \cdot \bar{b}$ bits
- spend one extra bit per boundary $\rightarrow$ $(h + 1) \cdot \bar{b}$ bits
- assume $(h + 1) \cdot \bar{b} \leq w$ $\rightarrow$ requires just one word (recall: we assumed things to work if $\frac{w}{h} \geq \frac{b}{c}$ for a constant c)
- store the resulting word $\langle \tilde{a}_1 \mid \tilde{a}_2 \mid \ldots \rangle$ at the corresponding node in the recursion tree

Finding The Right Subintervall

word size: w — $n = 12$
number of integers: n — $\ell = 12$
interval: $[0, 2^\ell)$ — $h = 4$
number of cells: 2^h — $b = 3$
branching width: b



Query: Which Subinterval Contains q?

- find $\tilde{q} = \lfloor q / 2^{\ell-h} \rfloor$ in $\{\tilde{a}_1, \tilde{a}_2, \dots\}$ instead (division by $2^{\ell-h}$ preserves order)
- it holds that: $\tilde{a}_i < \tilde{q} \Leftrightarrow (2^h + \tilde{a}_i - \tilde{q}) \,\&\, 2^h = 0$
- compute: $(\langle 2^h \mid 2^h \mid \dots \rangle + \langle \tilde{a}_1 \mid \tilde{a}_2 \mid \dots \rangle - \tilde{q} \cdot \langle 1 \mid 1 \mid \dots \rangle) \,\&\, \langle 2^h \mid 2^h \mid \dots \rangle$
- most of it can be precomputed $\rightarrow O(1)$ ($1\times$ subtraction, $1\times$ multiplication, $1\times$ bit-wise &)
- the most significant 1-bit yields i, such that $a_{i-1} < q \leq a_i$ (MS1B computable with $O(1)$ elementary operations)

Why?

Recap

word size: w
number of integers: n
interval: $[0, 2^\ell)$
number of cells: 2^h
branching width: b

Lemma **(nice (h, b)-subdivision)**

Given n integers in $I = [0, 2^\ell)$. I can be subdivided into $O(b)$ subintervals such that:

- each subinterval has length $2^{\ell-h}$ or contains at most $\frac{n}{b}$ integers
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1D Search $\rightarrow$ 2D Search

1D: In Each Node Of The Searchtree — **Why Does This Help?**

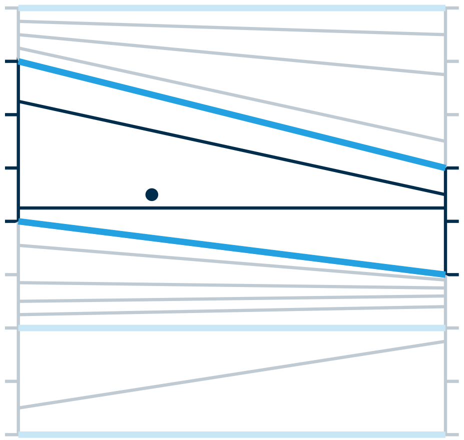
- $O(b)$ boundaries, each a multiple of $2^{\ell-h}$ — h bits per boundary $\rightarrow$ fit in just one word
- subintervals: small ($2^{\ell-h}$) or few integers ($\frac{n}{b}$) — sufficient progress in each node

1D: Query

- search neighboring boundary — works in $O(1)$
- continue in corresponding subtree — happens $\leq \log_b(n) + \frac{w}{h}$ times

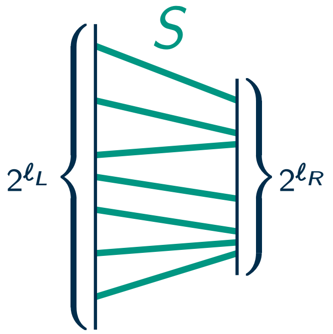
Extension To Point Location In A Slab

- choose boundaries with similar properties
- query a point (x, y)
  - 1D search with respect to y at the position x
  - round all coordinates $\rightarrow$ h bits per coordinate
- recursive call between boundaries



Problem: What if boundaries intersect the segments?

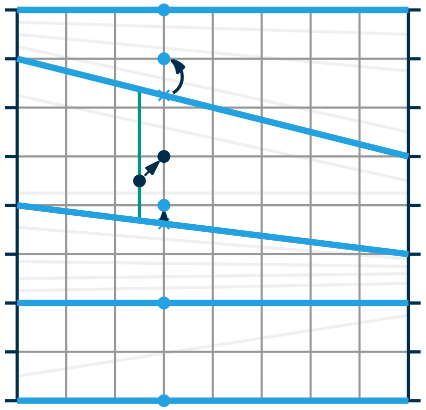
Nice Subdivision



Lemma **(nice (h, b)-subdivision)**
There are $O(b)$ boundary segments $s_0, s_1, \dots \in S$ such that
- between s_i and s_{i+1} there are $\leq \frac{n}{b}$ segments from S or $y_L(s_{i+1}) - y_L(s_i) < 2^{\ell_L - h}$ or $y_R(s_{i+1}) - y_R(s_i) < 2^{\ell_R - h}$
- $\tilde{s}_0 \prec s_0 \prec \tilde{s}_2 \prec s_2 \prec \cdots$ for "rounded" boundaries $\tilde{s}_i$

Proof: exercise

Input
- two vertical line segments that have length 2^{ℓ_L} and 2^{ℓ_R}
- n disjoint segments S between the two vertical segments



Notes On Running The Query For A Point $q = (x, y)$
- left/right endpoints of all $\tilde{s}_i$ can be packed into just one word
- the (rounded) intersections of all $\tilde{s}_i$ with a vertical line at $\tilde{x}$ are computable with $O(1)$ operations
- searching on the rounded intersections for $\tilde{y}$ as in the 1D case
- un-rounding: position of $\tilde{q}$ in the $\tilde{s}_i$ $\rightarrow$ position of q in the s_i (only $O(1)$ additional comparisons)

Wrap-Up

Seen Today

- models of computation: word RAM and real RAM
- real RAM often useful for computational geometry (but high precision sometimes unrealistic)
- bounded precision of the word RAM can be useful:
  $\rightarrow$ search and point location in vertical slab in $O(\frac{\log n}{\log \log n})$

What Else Is There?

- general point location with $o(\log n)$ queries in the word RAM
- problems with $o(n \log n)$ solutions on the word RAM:
  - 3D convex hull
  - Voronoi diagram
  - Euclidean MST
  - triangulation of polygons
  - line segment intersection

Literature

- **Transdichotomous Results in Computational Geometry, I: Point Location in Sublogarithmic Time** (2009)
  Timothy Chan, Mihai Pătraşcu
  https://doi.org/10.1137/07068669X

- **Transdichotomous Results in Computational Geometry, II: Offline Search** (2010)
  Timothy Chan, Mihai Pătraşcu
  https://arxiv.org/abs/1010.1948

- **On the bit complexity of minimum link paths: Superquadratic algorithms for problem solvable in linear time** (1999)
  Simon Kahana, Jack Snoeyink
  https://doi.org/10.1016/S0925-7721(98)00041-8