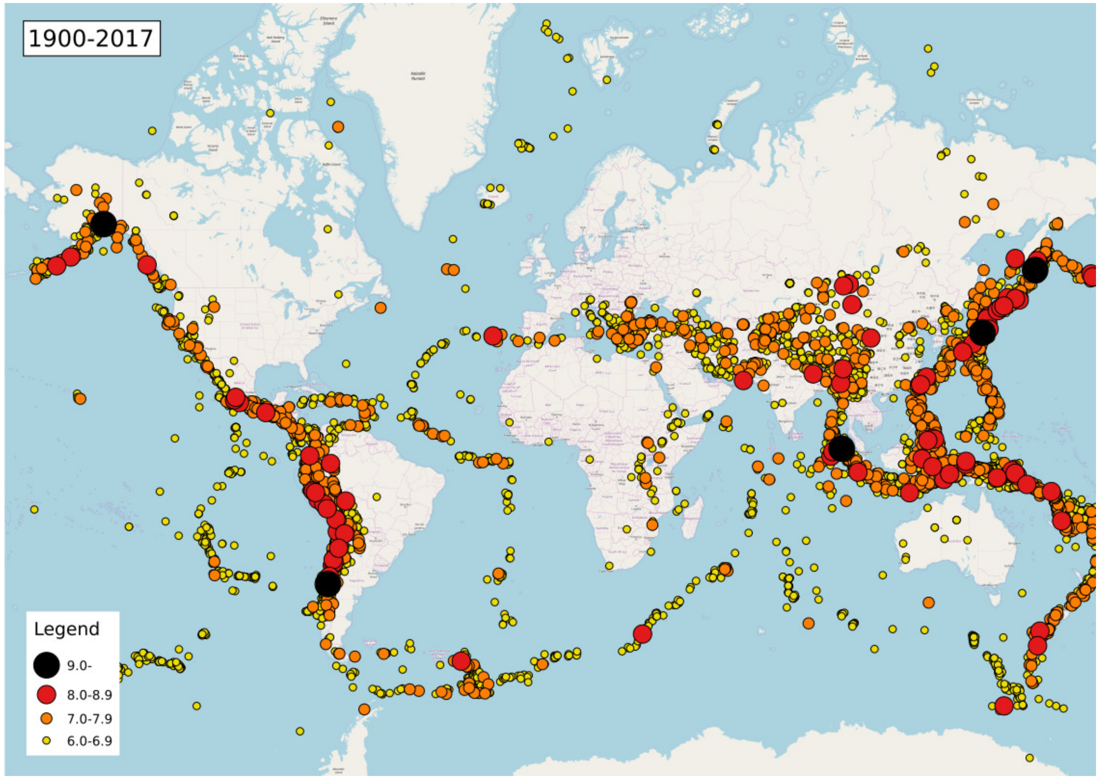
Computational Geometry
Hard Problems

Thomas Bläsius

Proportional Symbol Maps

Proportional Symbol Map (Example: Earthquakes)

- visualizing weighted points on a map
- weight represented by disk size



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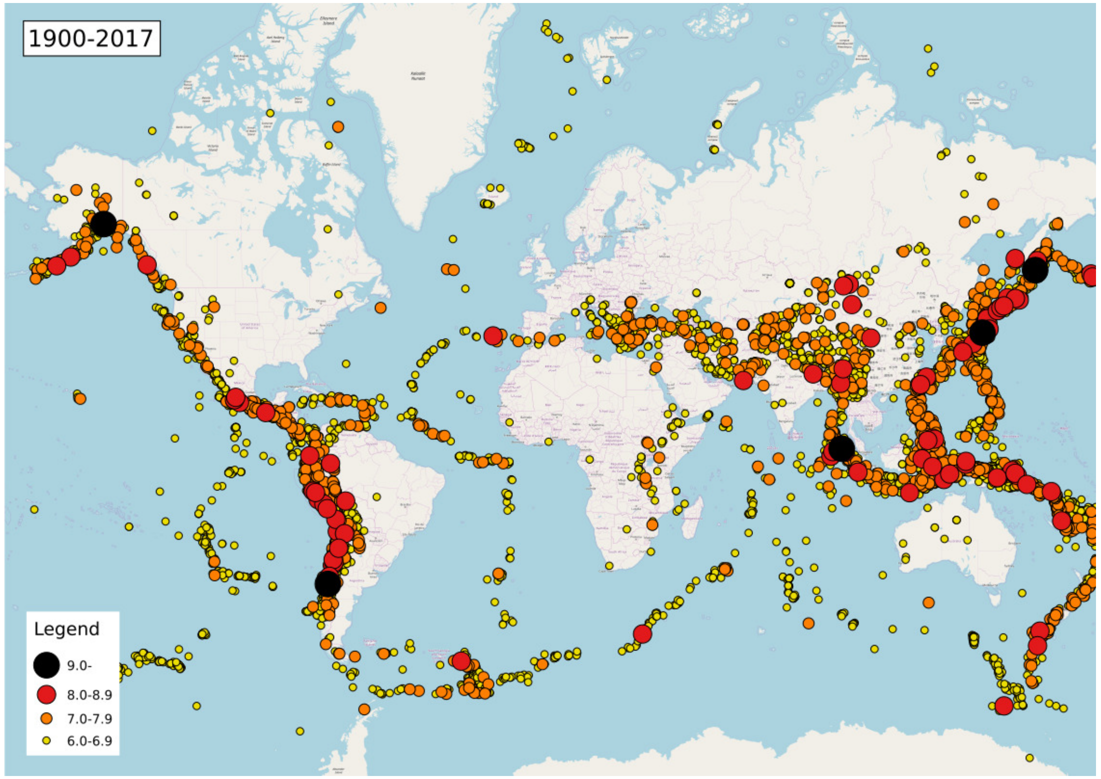

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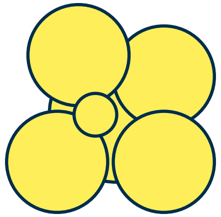

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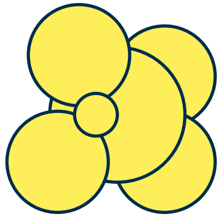

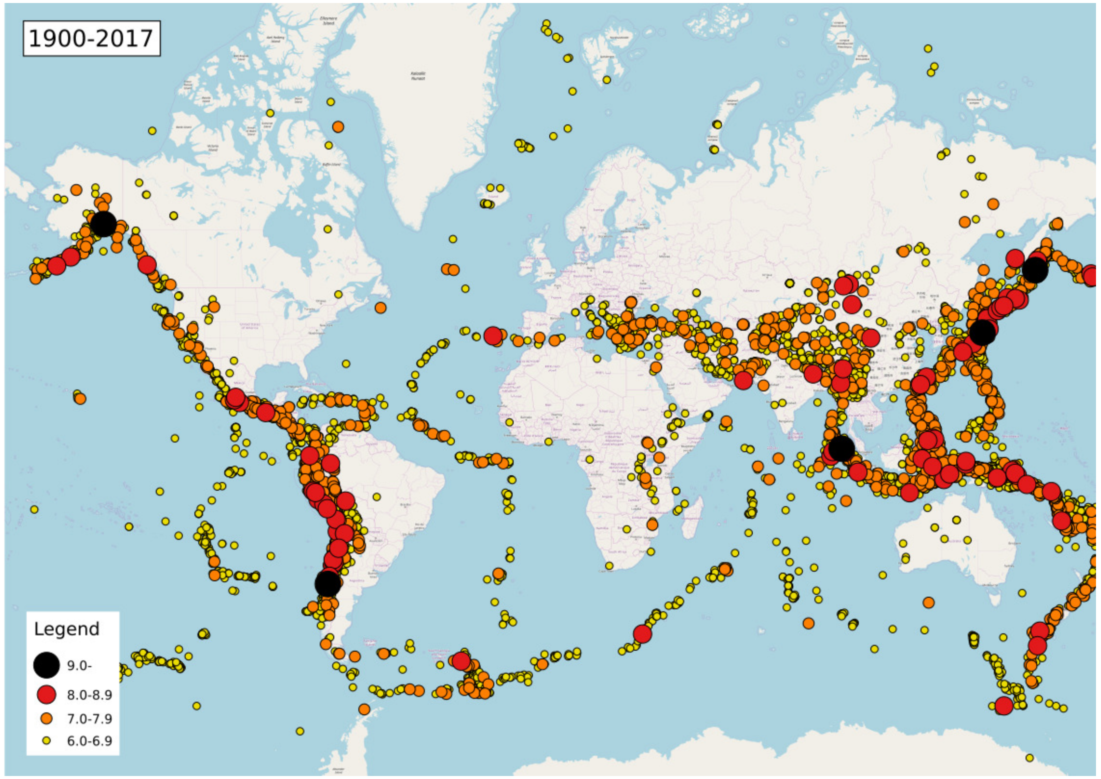

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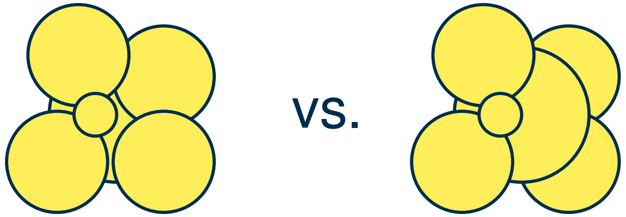

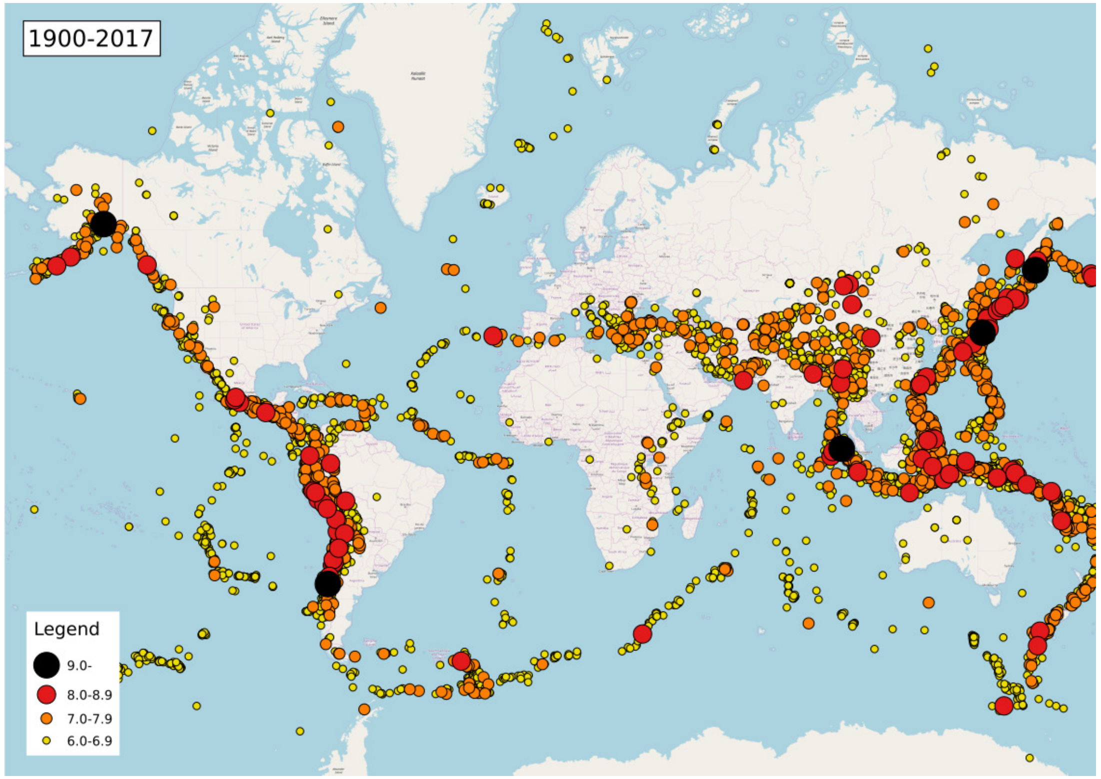

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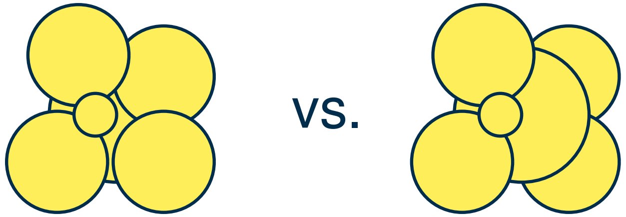

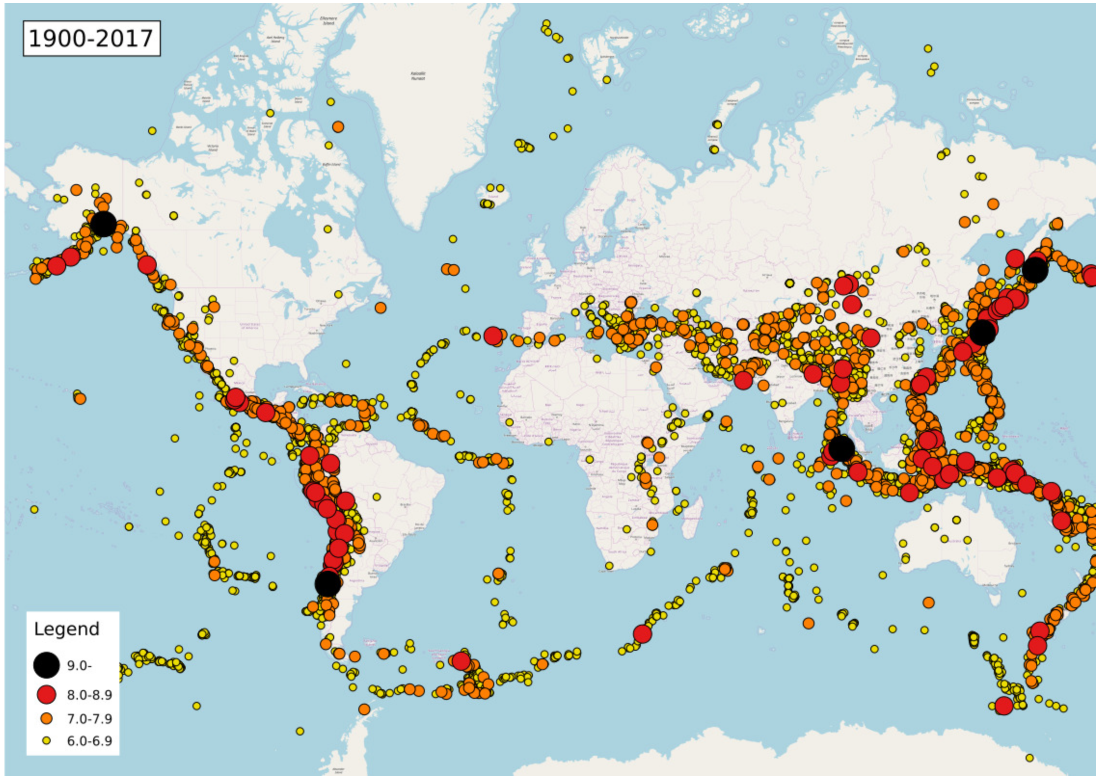

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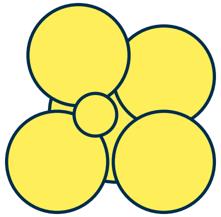

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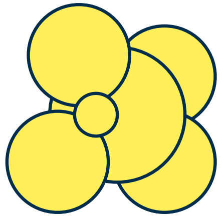
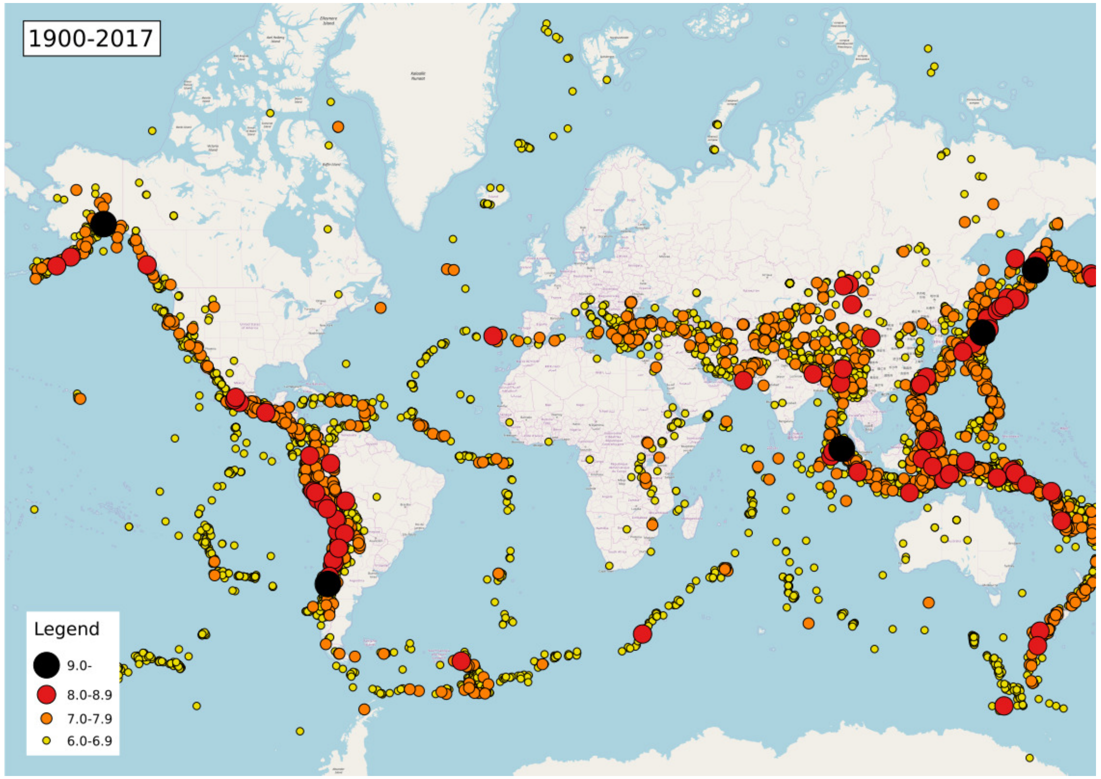




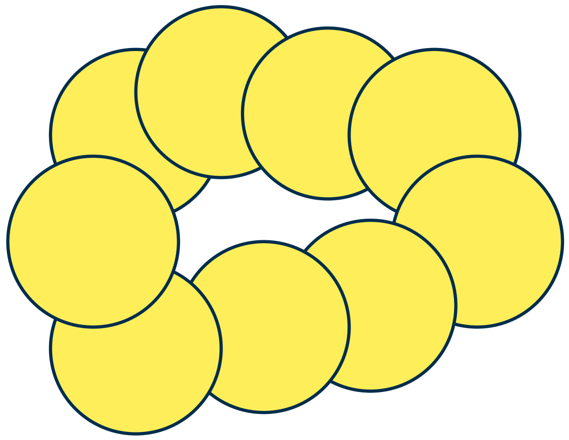
Problem

- given: set of disk with potentially different radii
- find: drawing that maximizes the visible border of each disk
- What is a valid drawing? What exactly does maximizing the visible border mean?

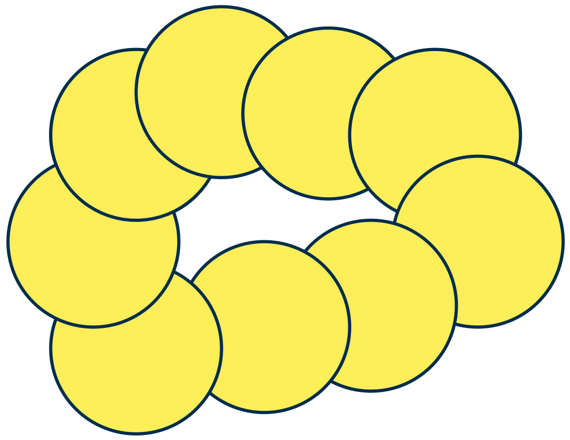
What Exactly Is The Problem?

Two Types Of Valid Drawings

- stacking: total z-order on all disks



- physically realizable: buildabel with thin coins



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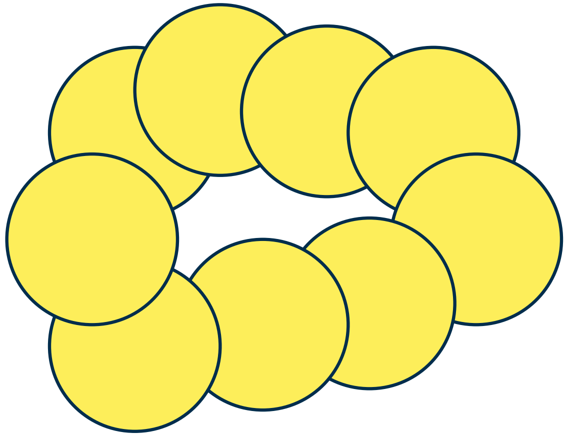

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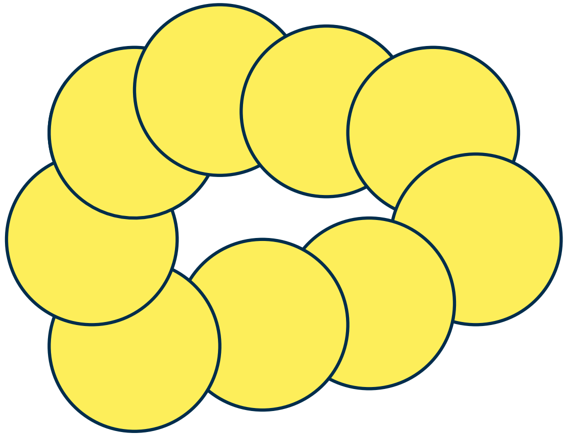

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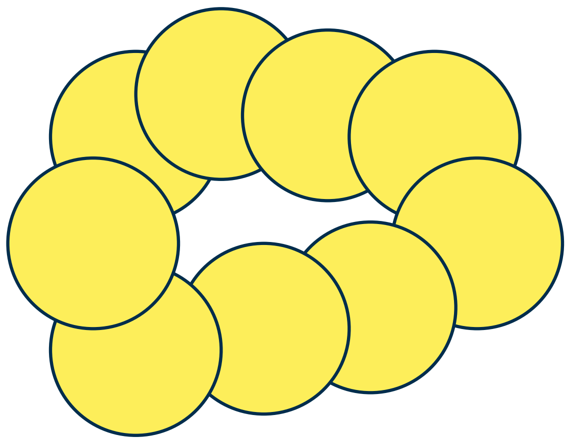

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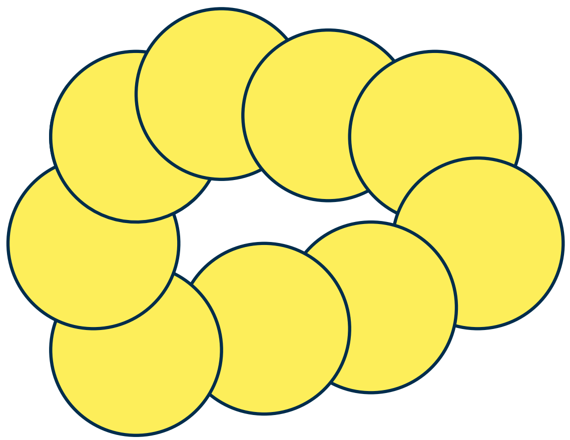

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- Max-Min: maximize minimally visible border over all disks
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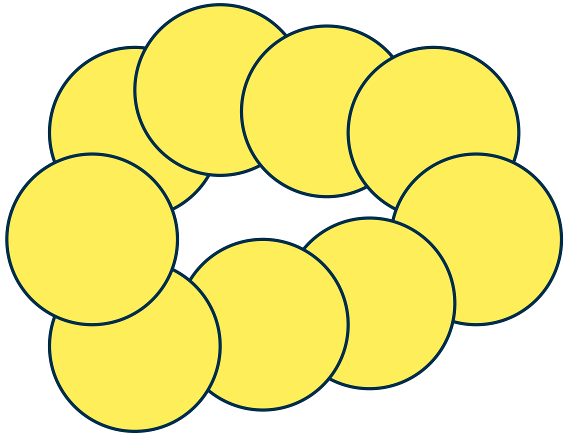

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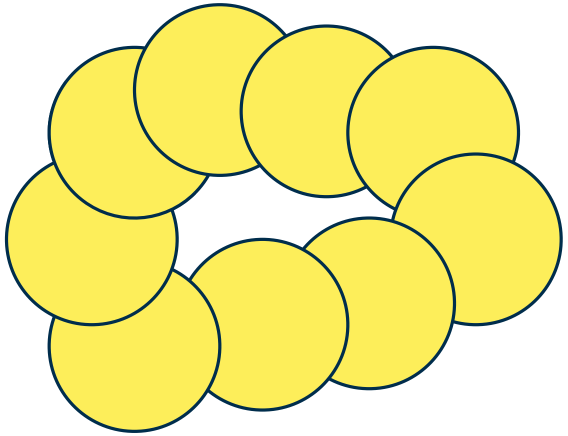

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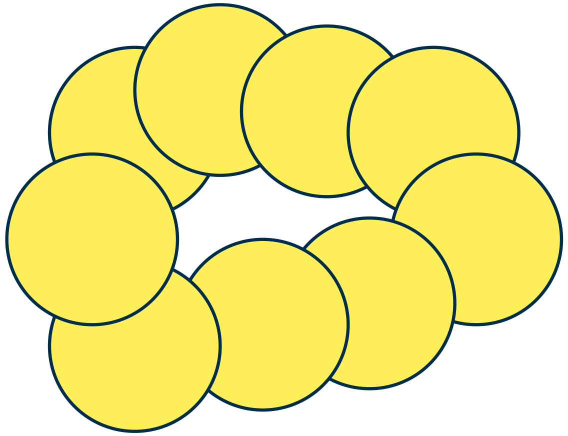

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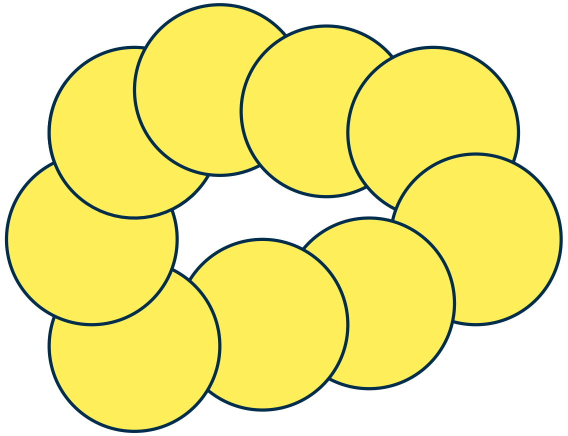

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← today

Useful NP-Hard SAT-Varints

Problem: 3-SAT
Boolean formula Φ in CNF, with ≤ 3 literals per clause. Is Φ satisfiable?

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee \neg x_4 \vee x_5) \wedge (\neg x_1 \vee x_4)$$

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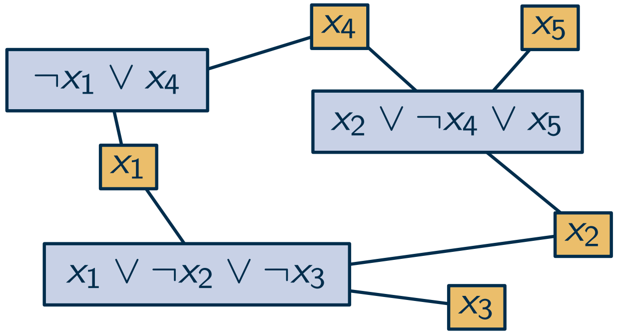
Problem: Planar 3-SAT
The clause–variable graph is planar.



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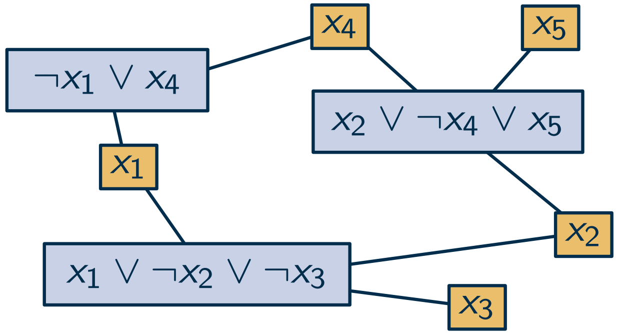

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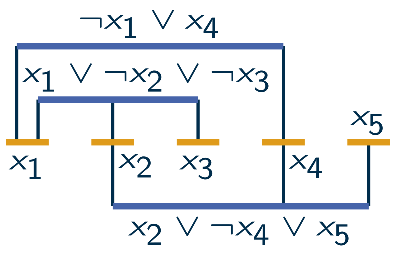
Problem: Rectilinear Planar 3-SAT
The clause–variable graph has a *rectilinear planar* drawing.



Rectilinear Planar Drawing

- vertices: horizontal segments
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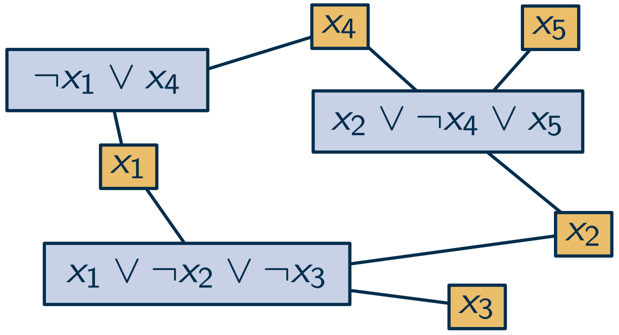

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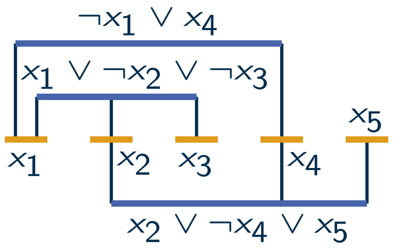

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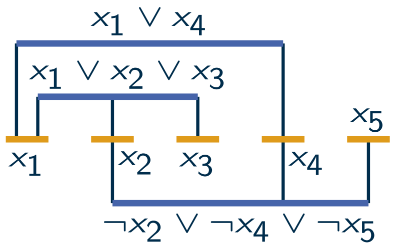

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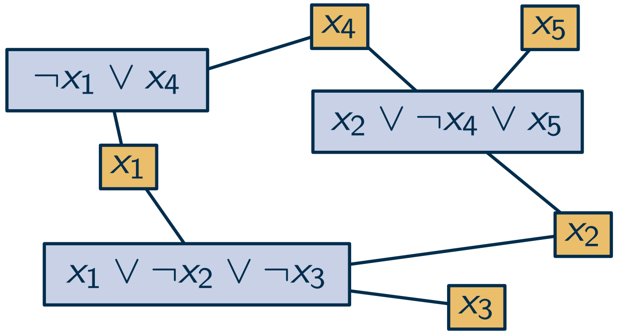

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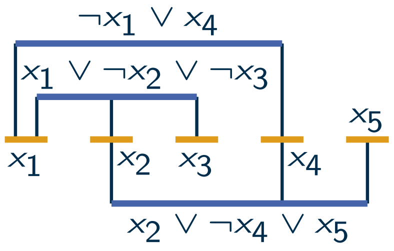

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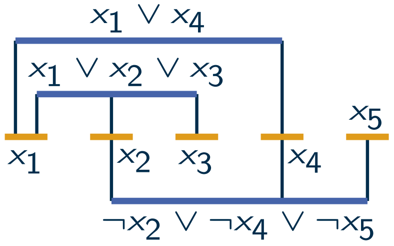

Note: allowing clauses < 2 is important here

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  - transport channel can be faulty in one direction: flip from satisfied to unsatisfied literal is ok

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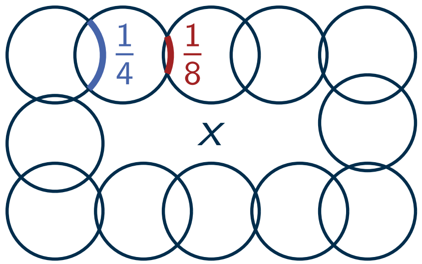

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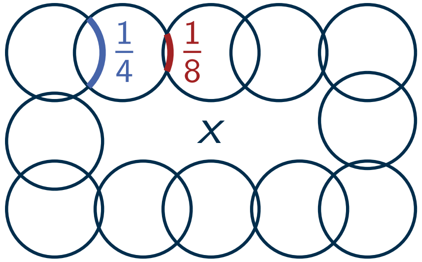

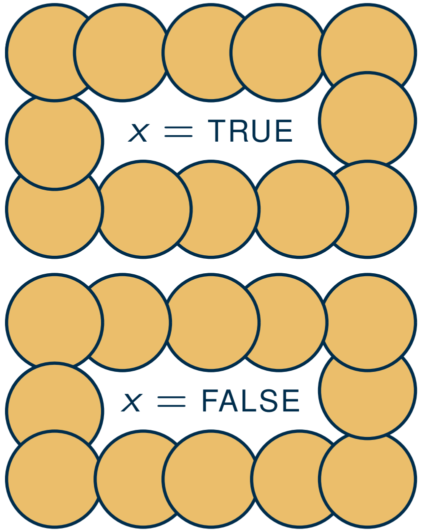

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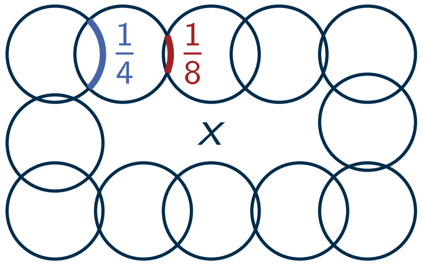

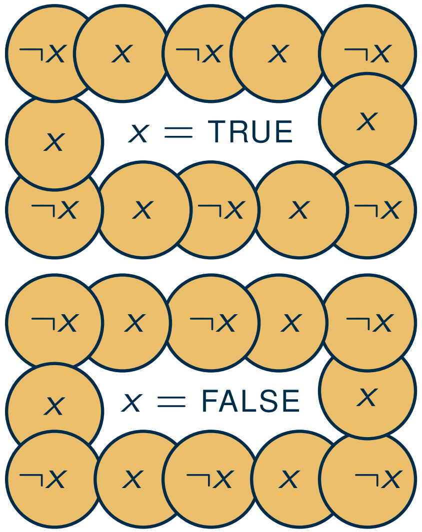

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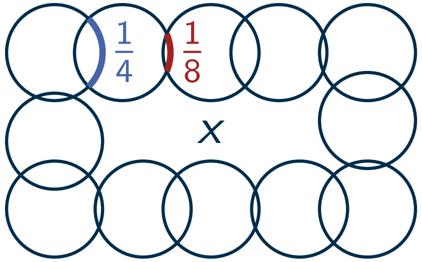

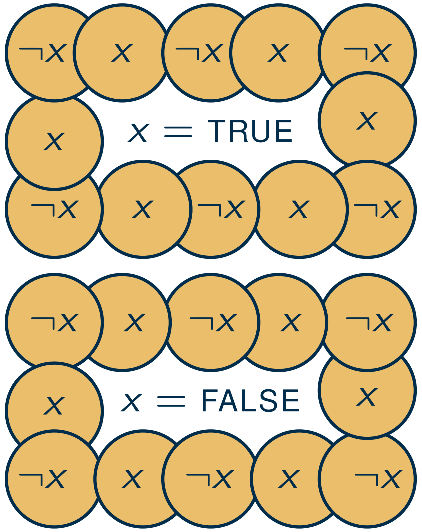

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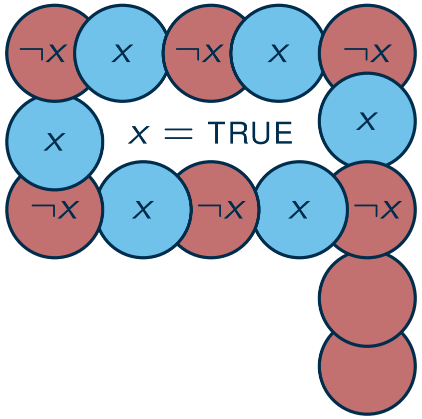

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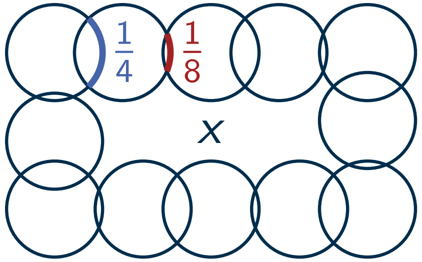

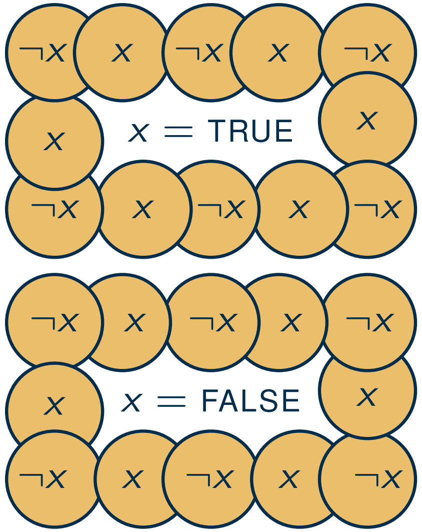

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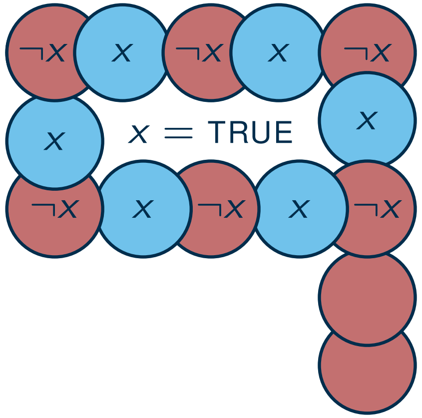

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  - every decision is forced $\rightarrow \frac{1}{4}$ of last disk covered
  - transports information $\neg x =$ FALSE to (almost) arbitrary position

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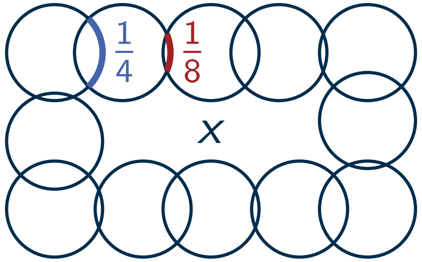

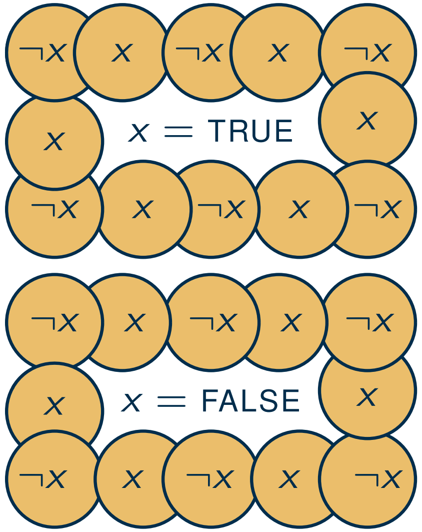

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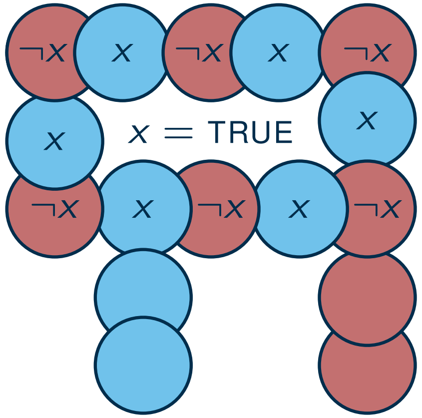

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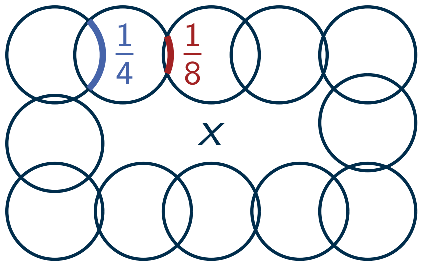
  - every decision is forced $\rightarrow \frac{1}{4}$ of last disk covered
  - transports information $\neg x = \text{FALSE}$ to (almost) arbitrary position
- chain starts at an x (instead of $\neg x$) $\rightarrow$ last disk can be completely visible

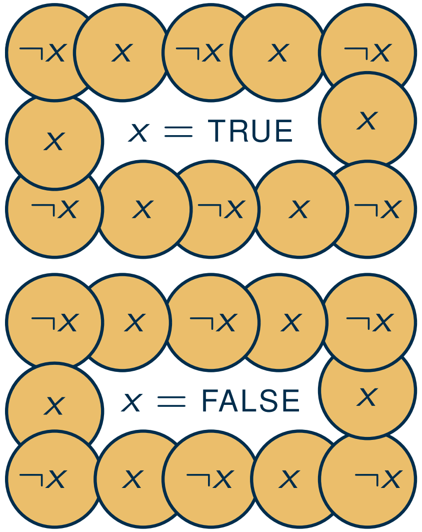
Gadgets

YES-instance: for every disk, $\geq 3/4$ of its border is visible

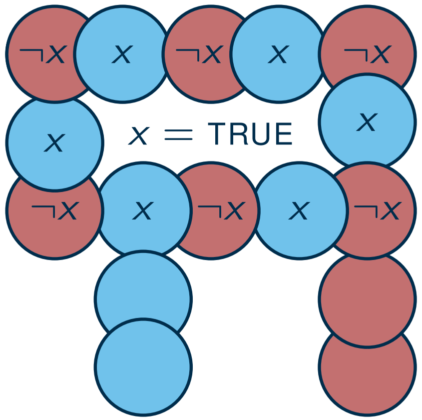
Variable Gadget

- only two configurations possible
- every different configuration covers $> 1/4$ of a disk





Transport Gadget



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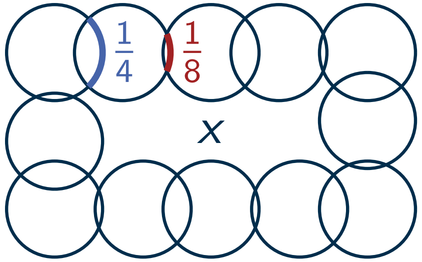

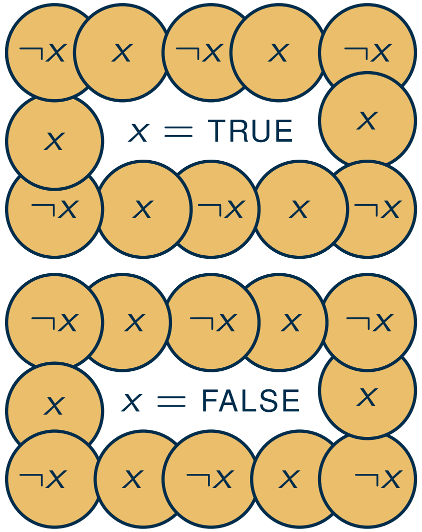

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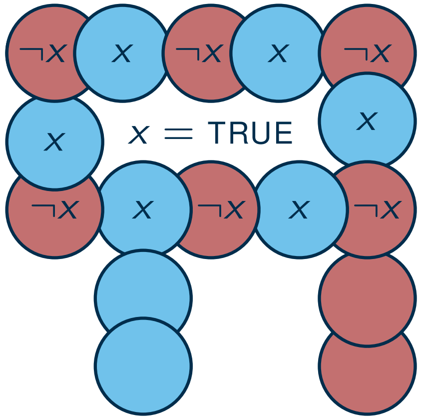

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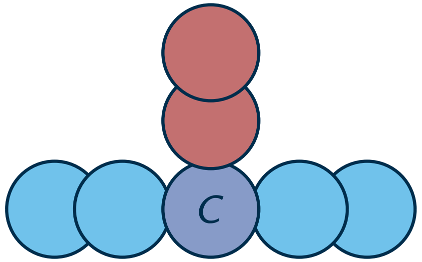
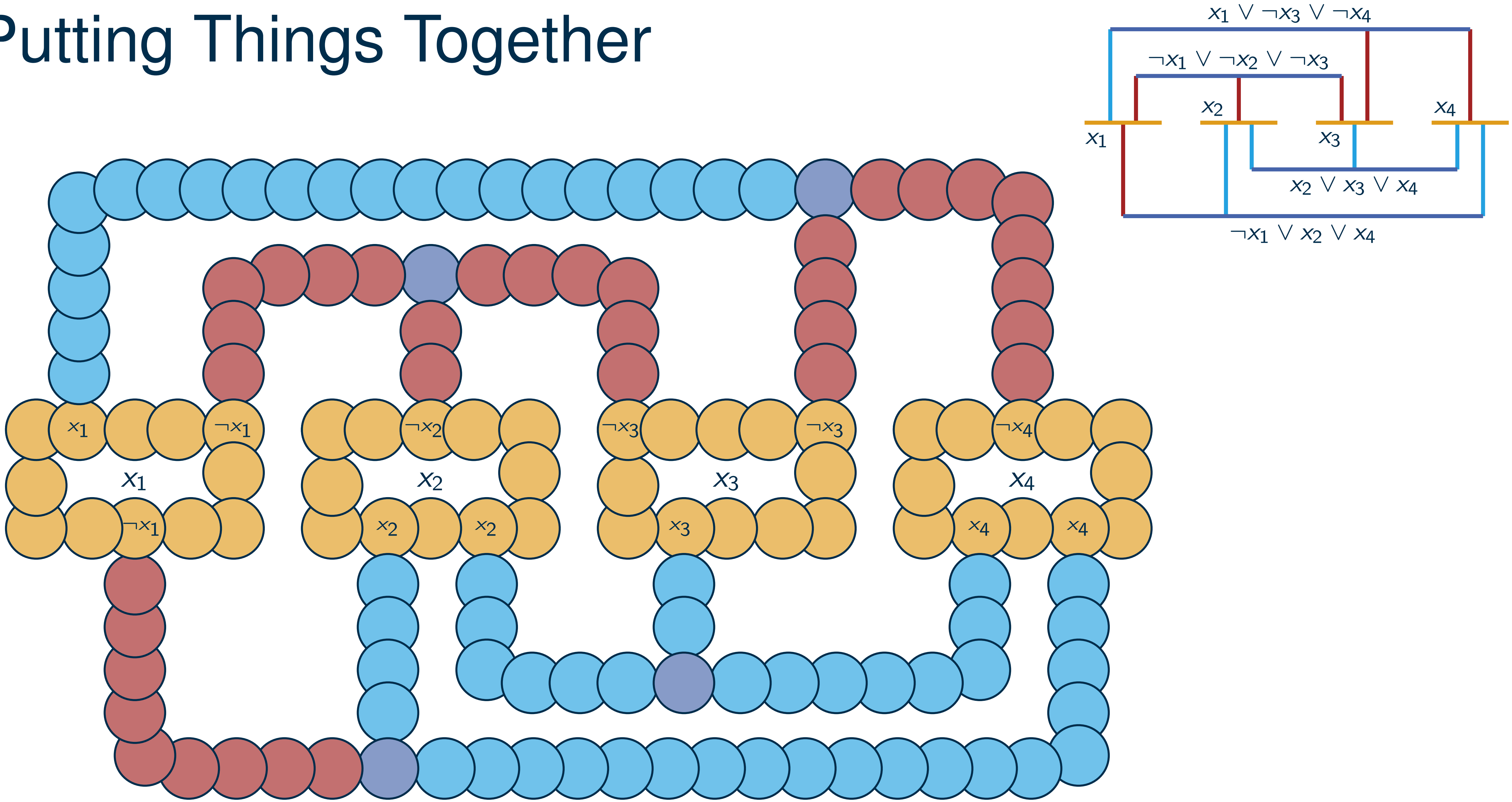
Clause Gadget

- C must overlap ≥ 1 of its neighbors $\rightarrow \geq 1$ chain comes from a `true` literal



Putting Things Together



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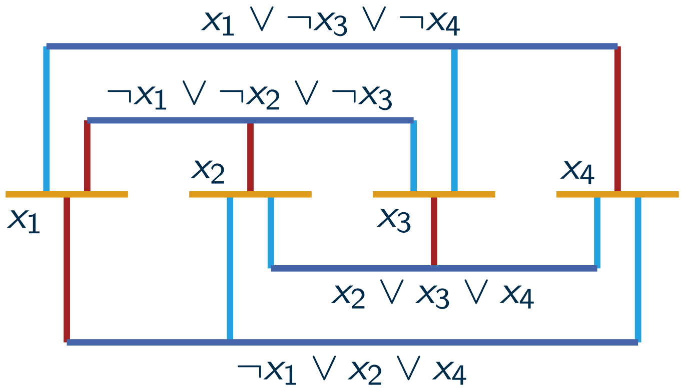

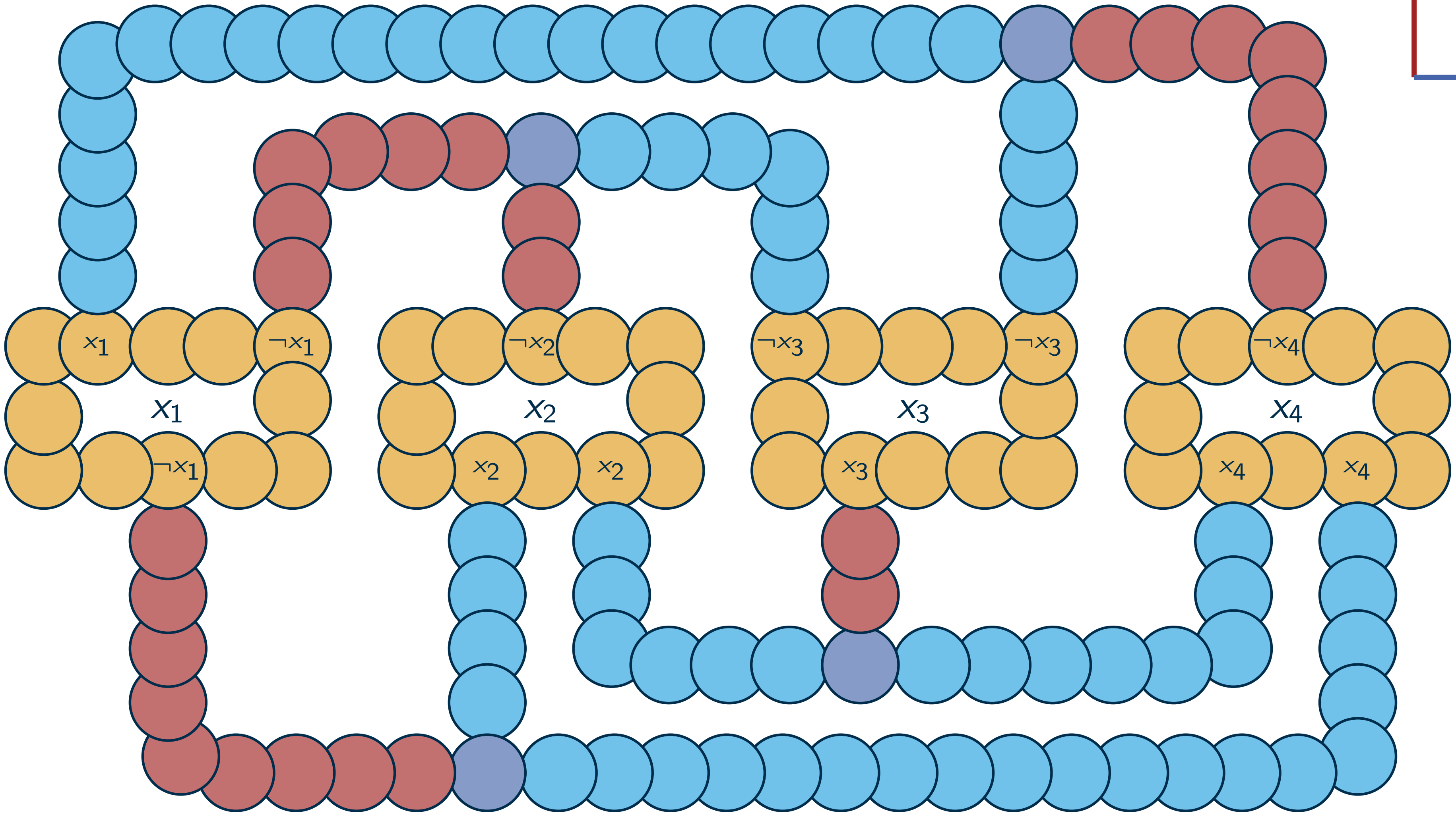

What Is Left To Show?

Theorem
Deciding whether there is a physically realizable configuration that shows $3/4$ of the border of each disk is NP-hard.

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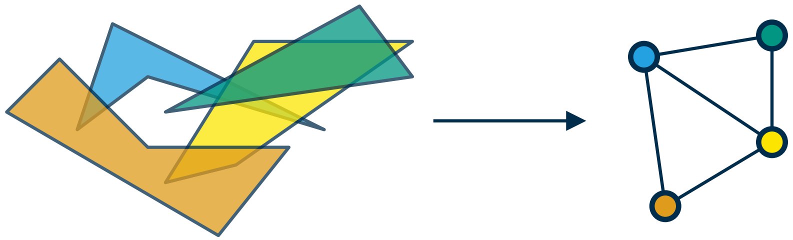
Correctness

- $3/4$ of the border of each disk visible $\Rightarrow$ formula satisfiable
- formula satisfiable $\Rightarrow$ $3/4$ of the border of each disk visible

Why?

Unit Disk Graphs

Definition
Set of geometric objects V defines **intersection graph** $G = (V, E)$ with $uv \in E \Leftrightarrow u \cap v \neq \emptyset$.



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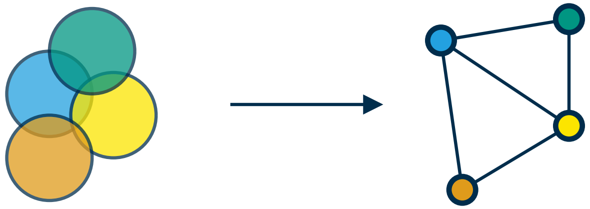

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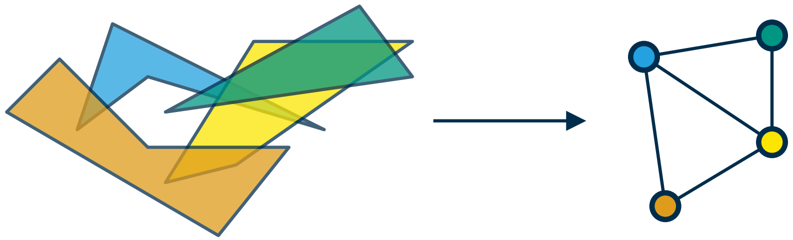

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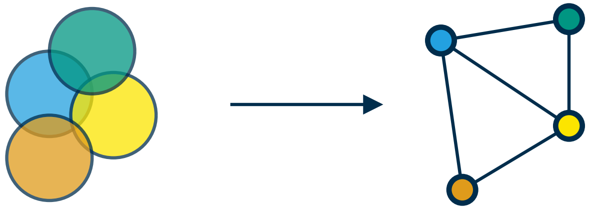
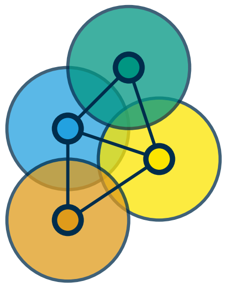


Recognition Problem
Is a given graph a unit disk graph?

Basic Observations



Goal: reduce planar monotone 3-SAT to unit disk graphs recognition

Useful Basic Observations

- equivalent: are there vertex positions such that $\text{dist}(u, v) \leq 2 \Leftrightarrow uv \in E$?

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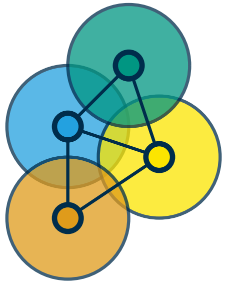

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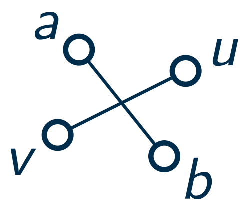

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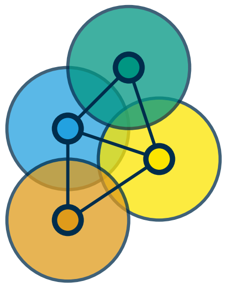

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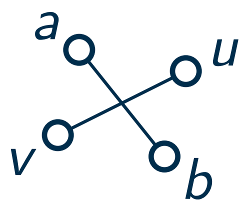
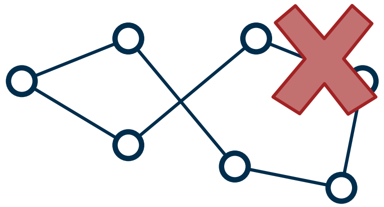


- induced cycles are planar



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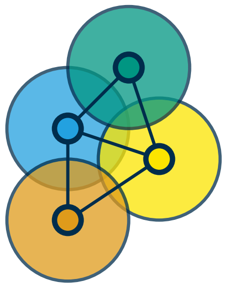

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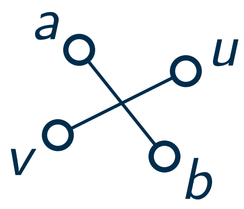

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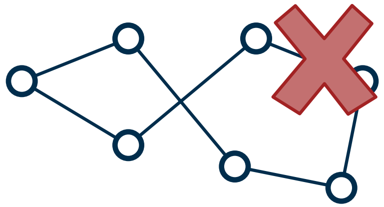
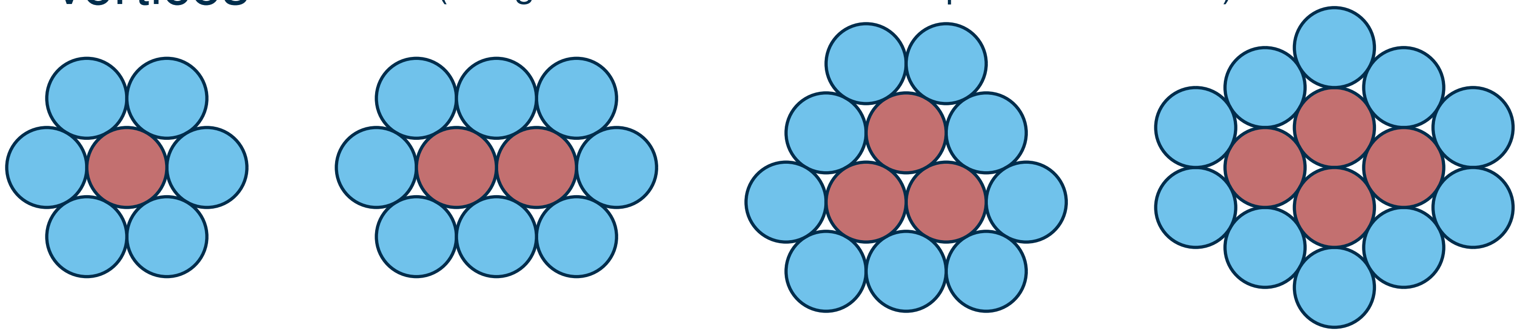
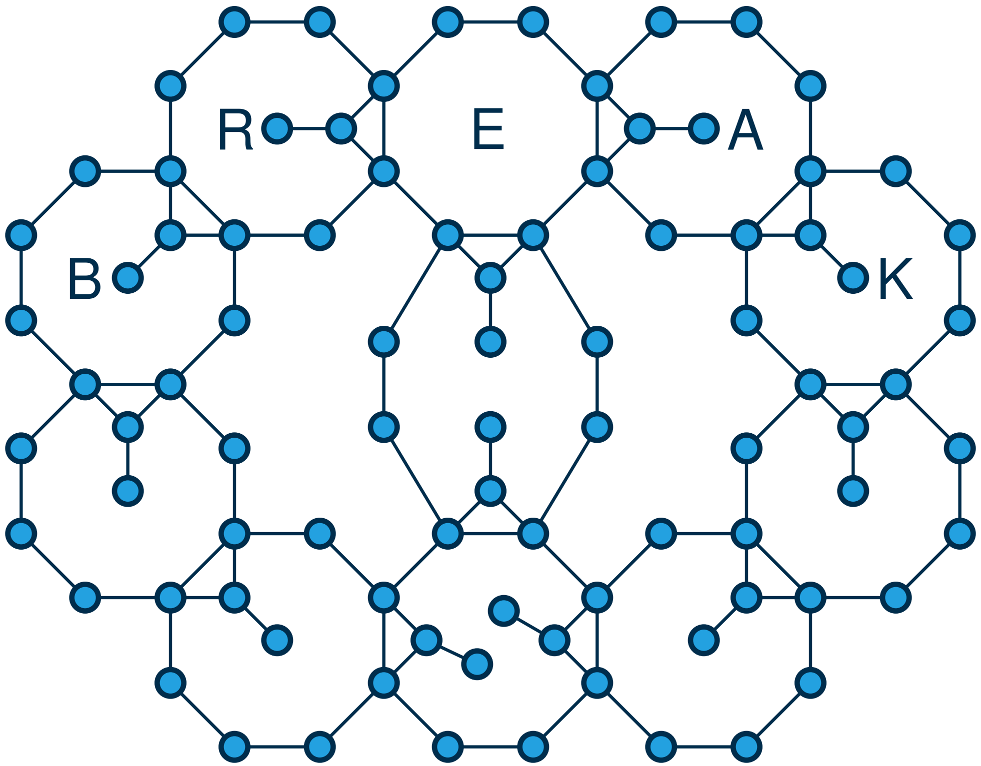


- cycles contain a limited number of independent vertices (*i*-cage contains at most *i* independent vertices)



0-cage 1-cage 2-cage 3-cage

Unit Disk Graph Or Not?



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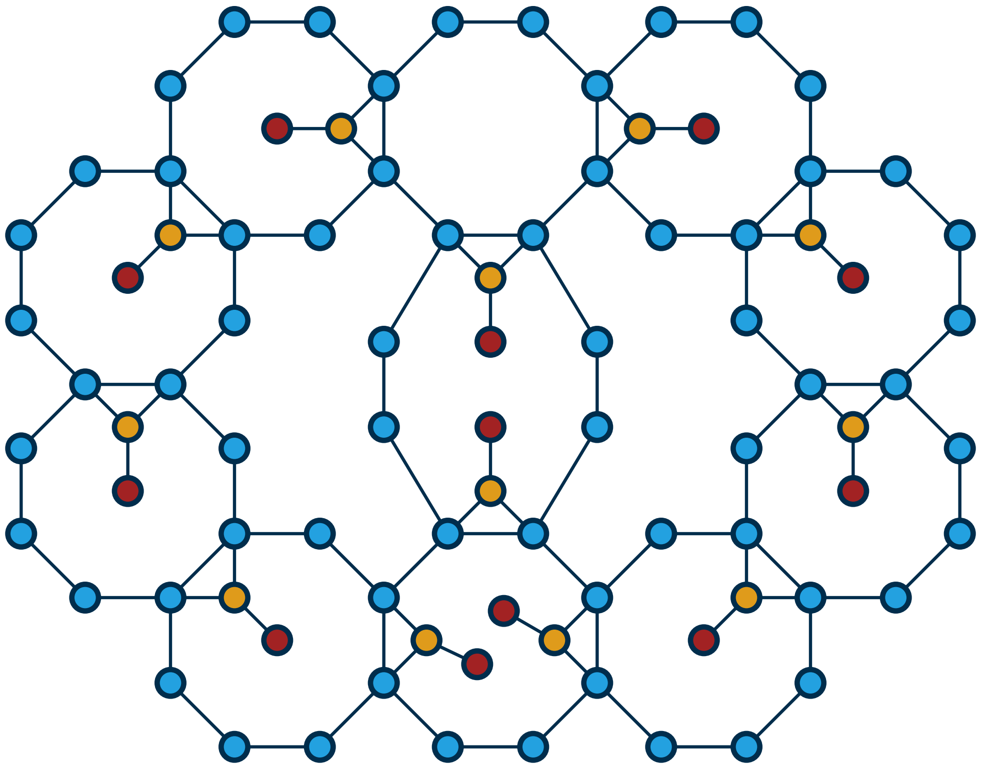

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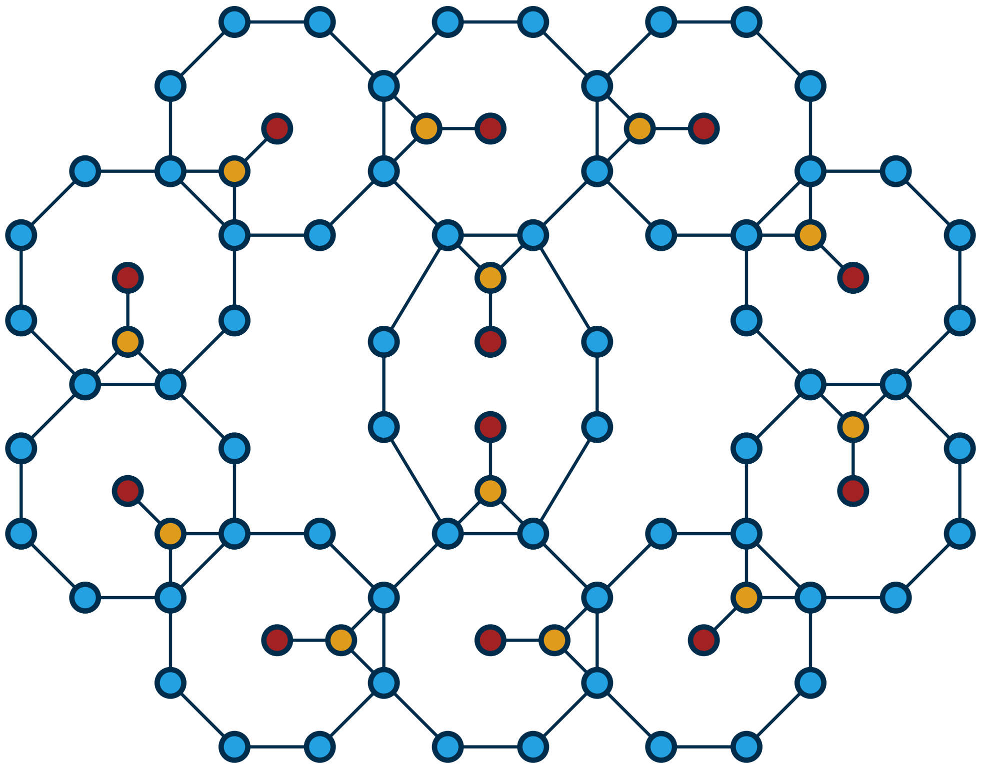

Gadgets

Gadgets We Need: variable, clause, transport

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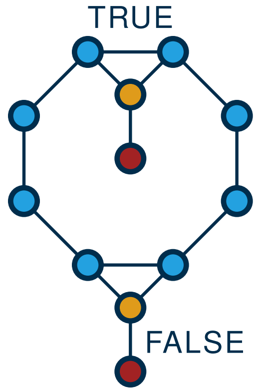
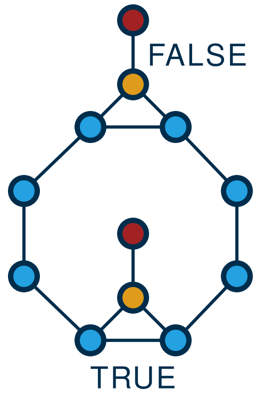
Variable Gadget





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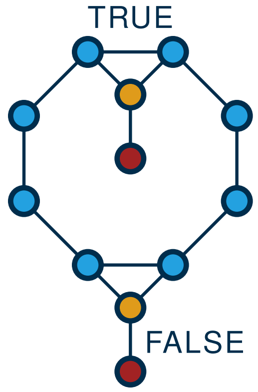

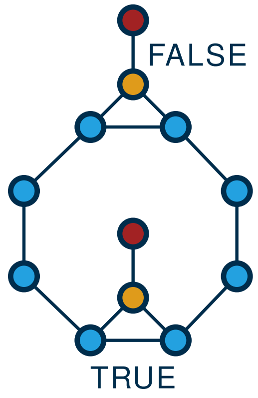

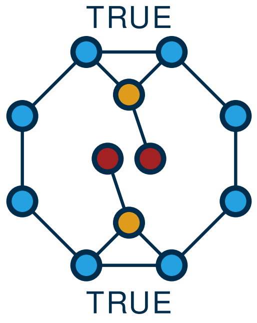

not possible (1-cage with two independent vertices)

Gadgets

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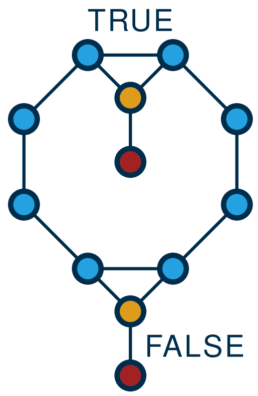

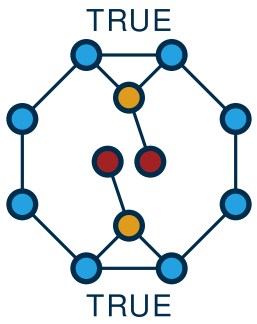

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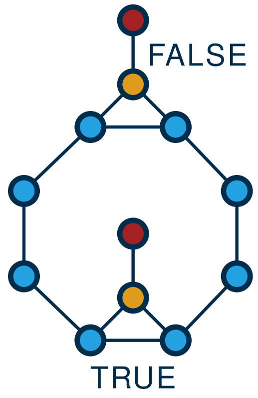

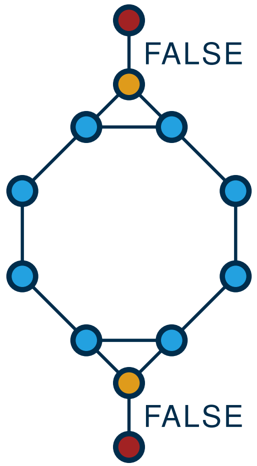

possible but never helpful (can basically be ignored)

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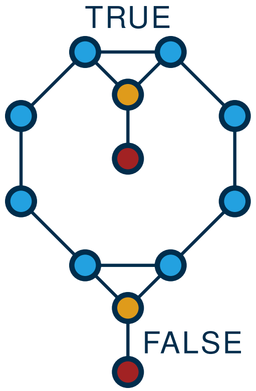
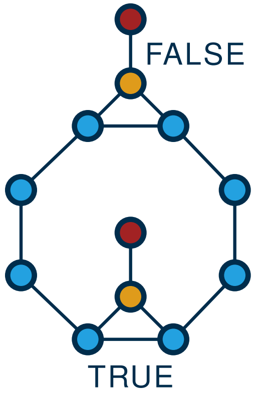
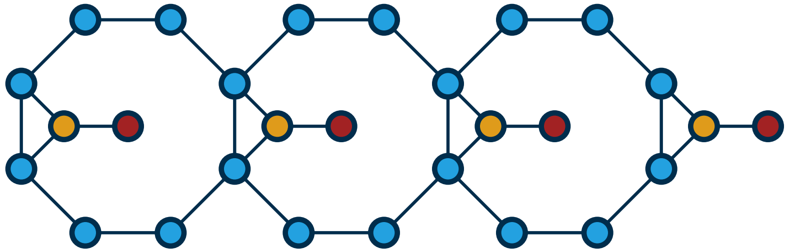




Transporting Information



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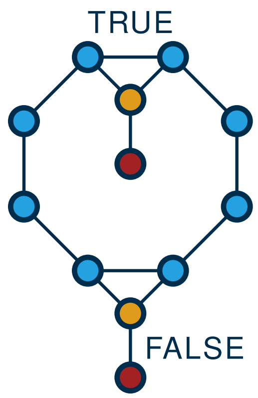

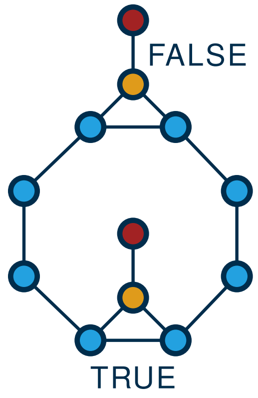

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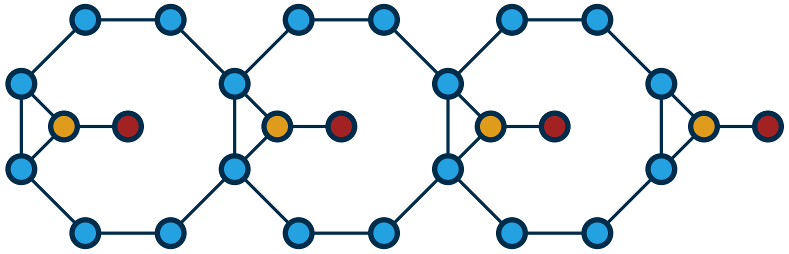

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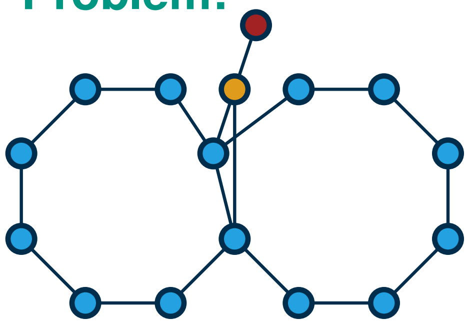

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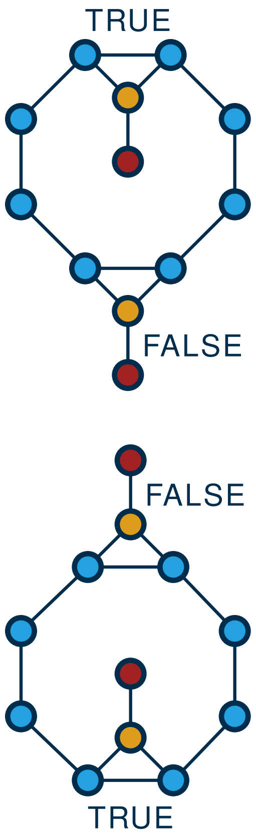

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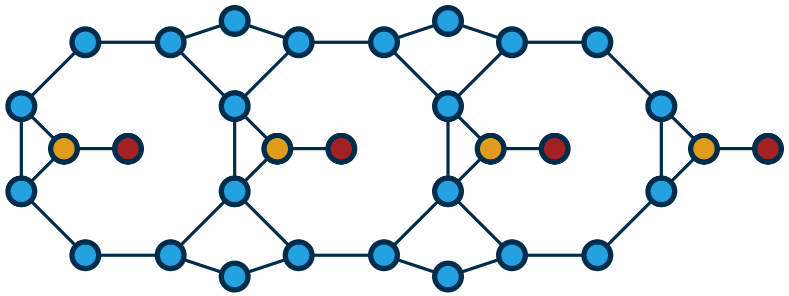

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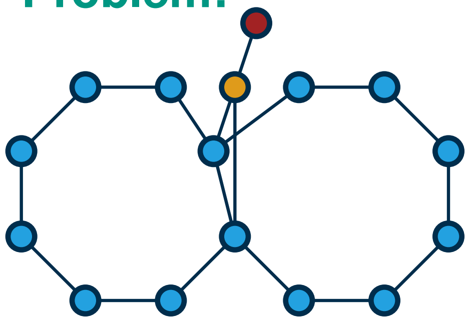
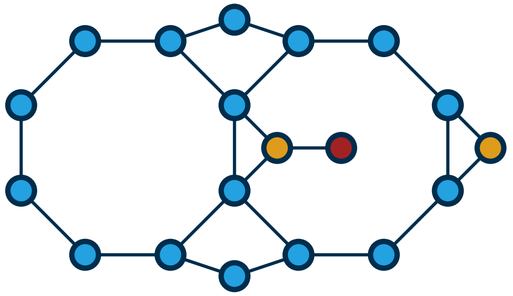


Solution:



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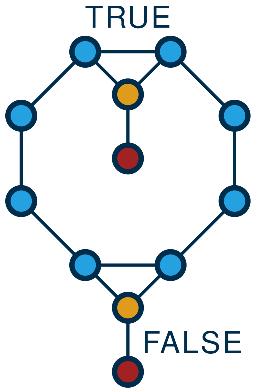

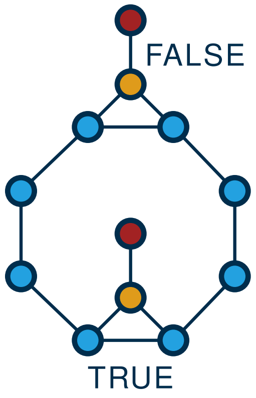

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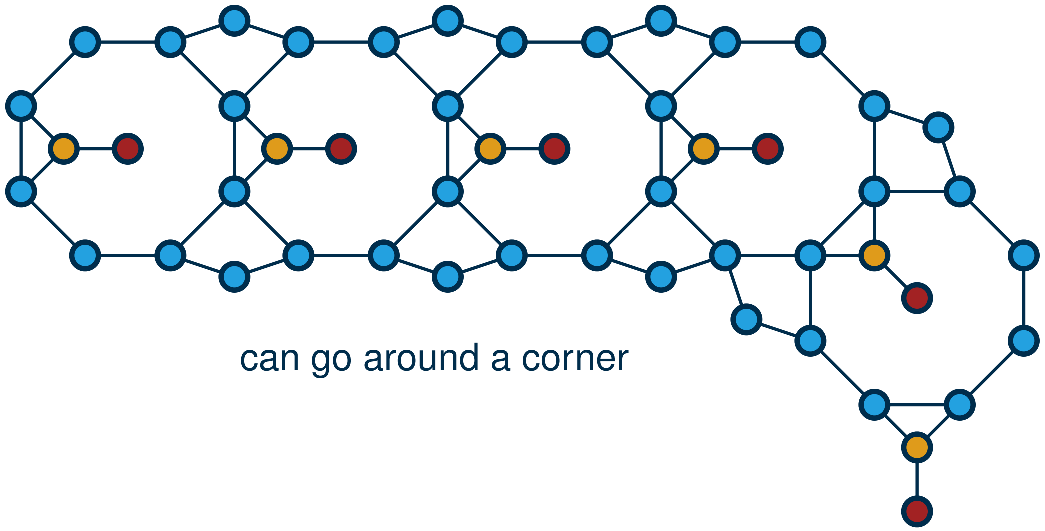

can go around a corner

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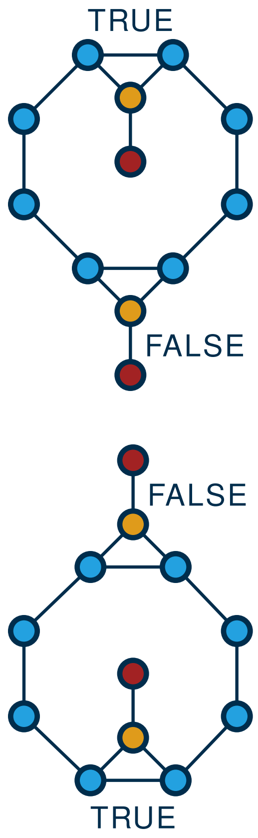

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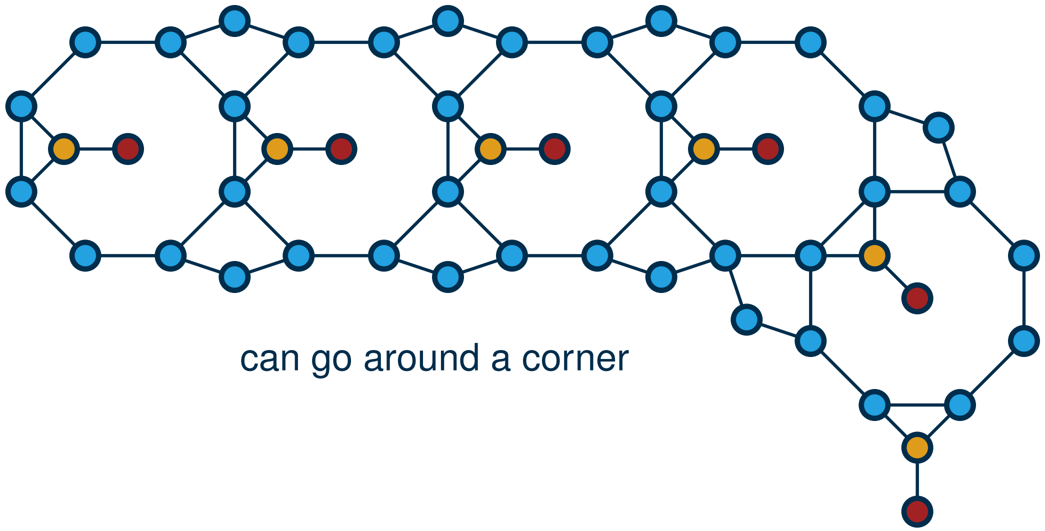
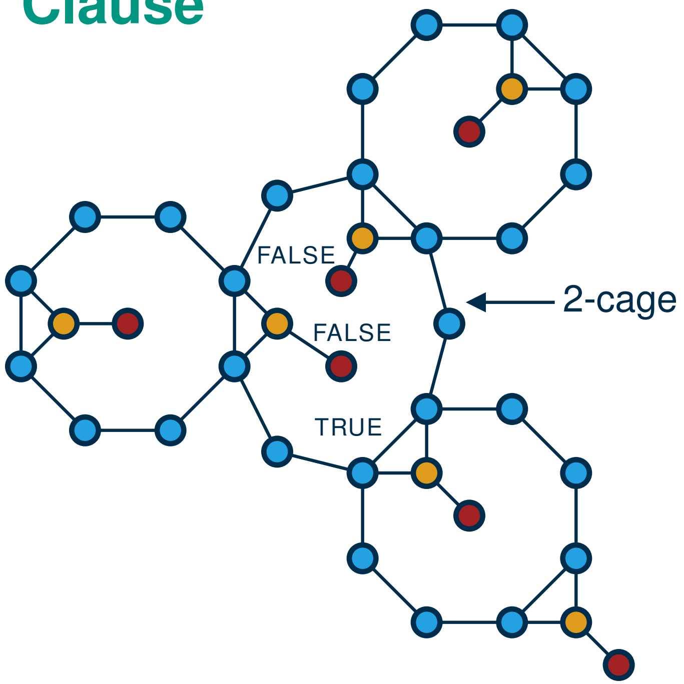


Clause



Gadgets

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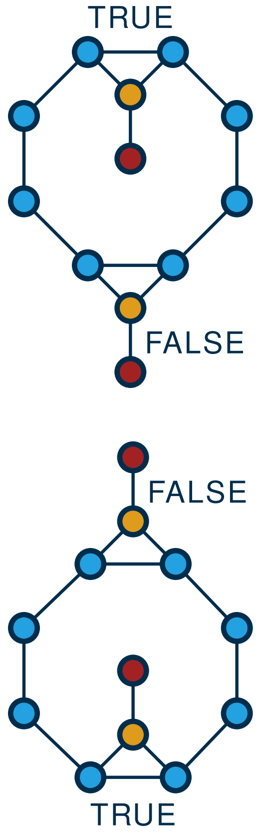

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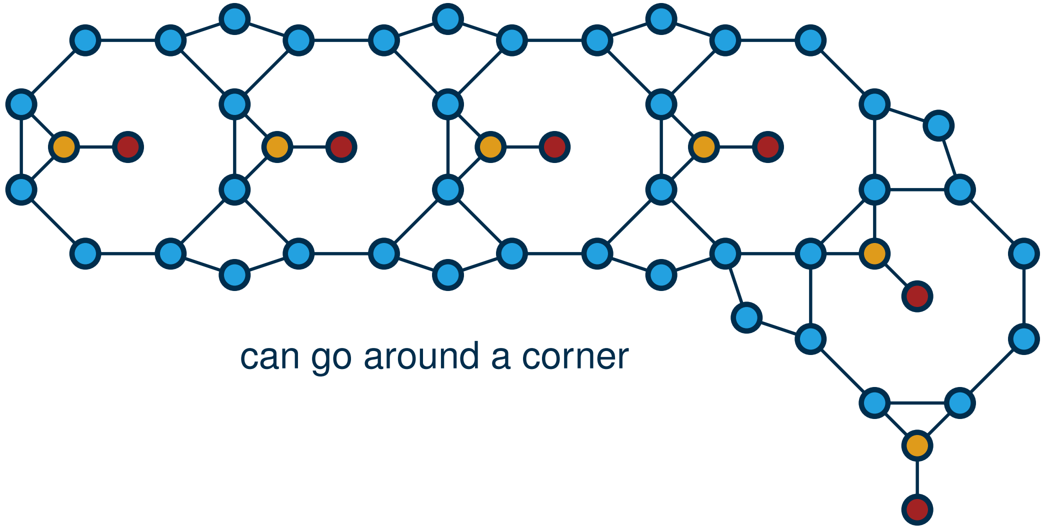
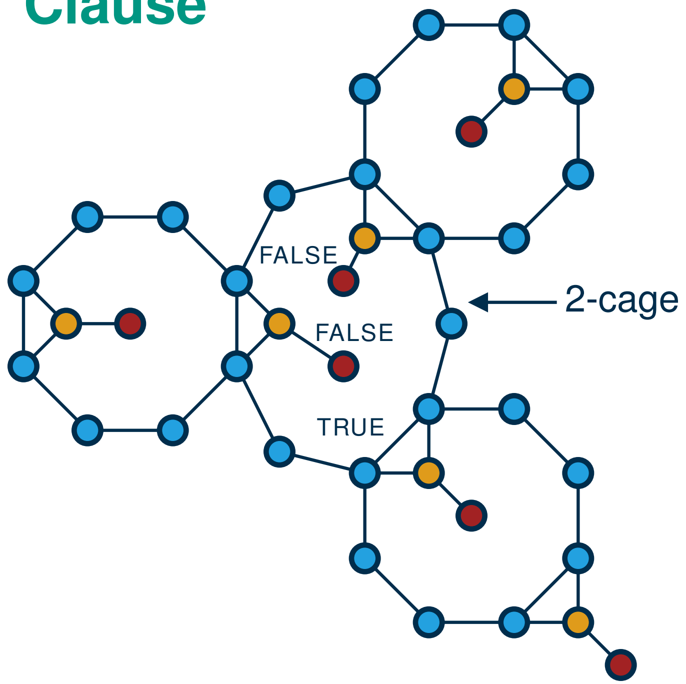


What Is Missing?

Clause



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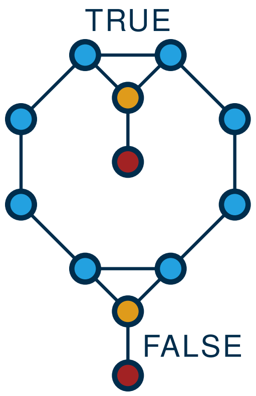
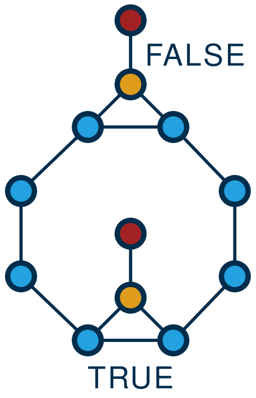
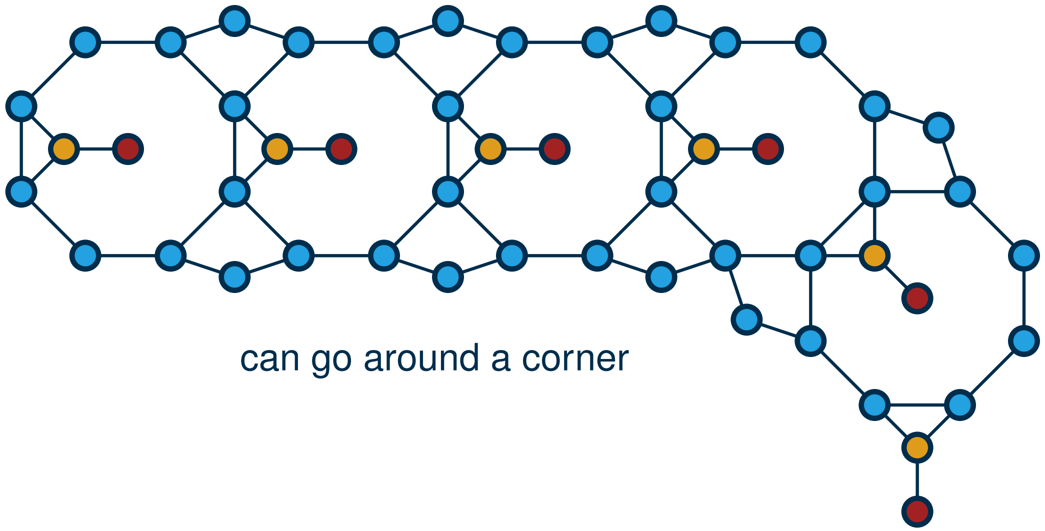
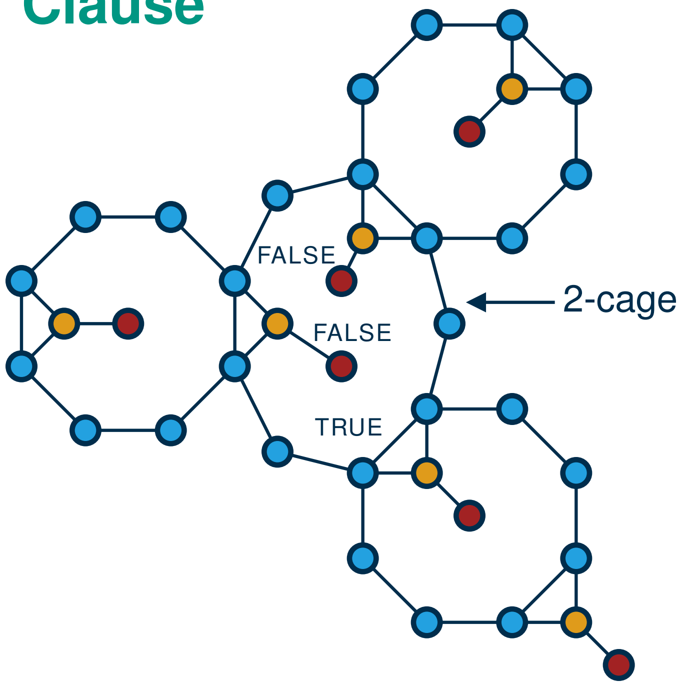




Transporting Information



Clause



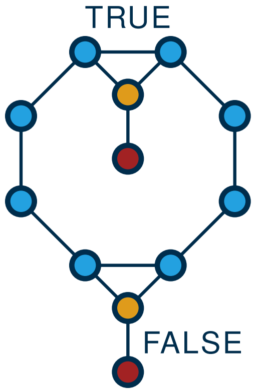
What Is Missing?

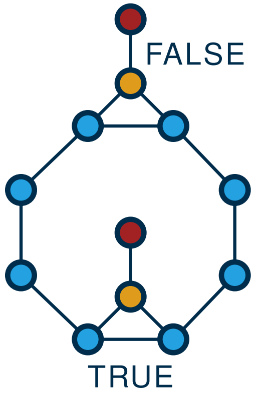
- we can transport the decision of a variable to only one clause (technically 2: one positive, one negative)
- variables are contained in multiple clauses

Gadgets

Gadgets We Need: variable, clause, transport

Variable Gadget





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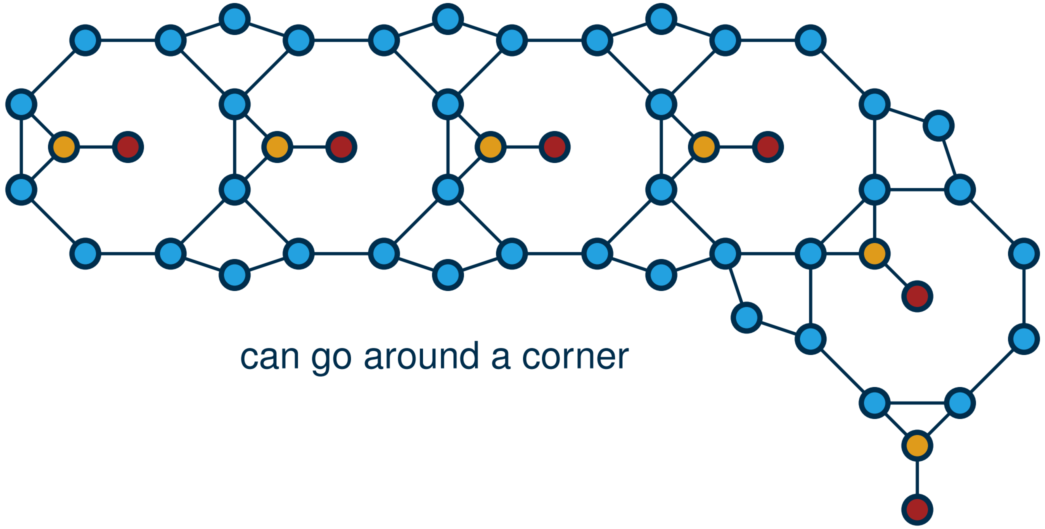

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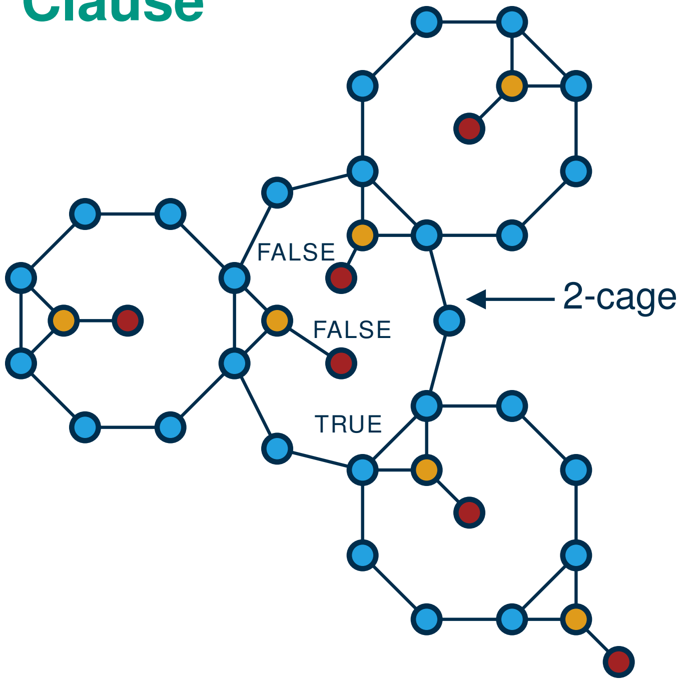

What Is Missing?

- we can transport the decision of a variable to only one clause
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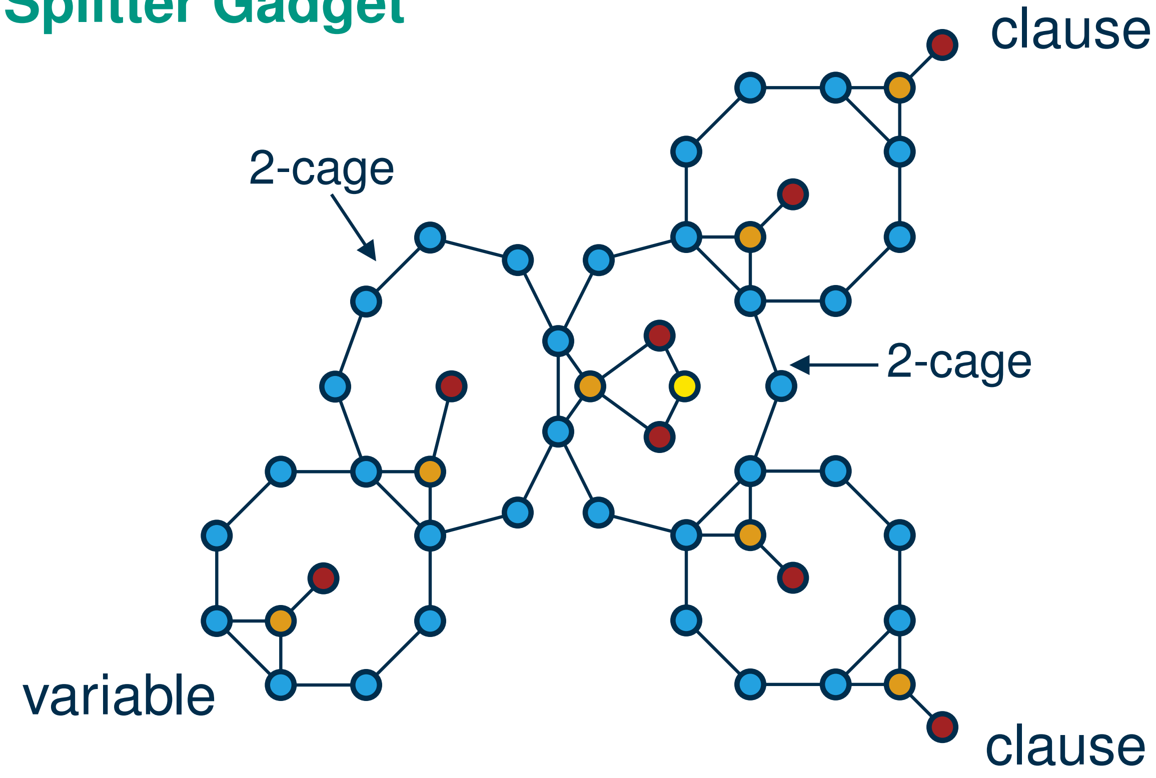
- variables are contained in multiple clauses
- we need a splitter gadget

Multiplying Information

Splitter Gadget



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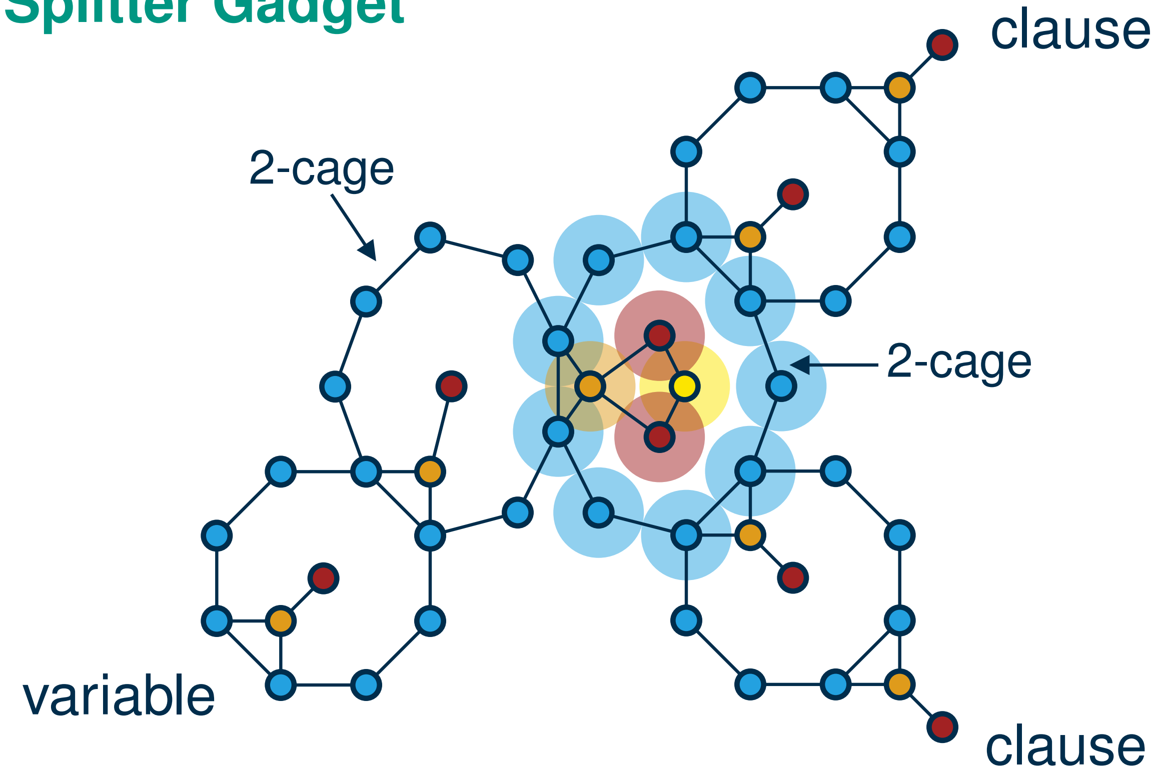

Note

- ○ forces both ● into the same 2-cage
- the 2-cage is realizable

Multiplying Information

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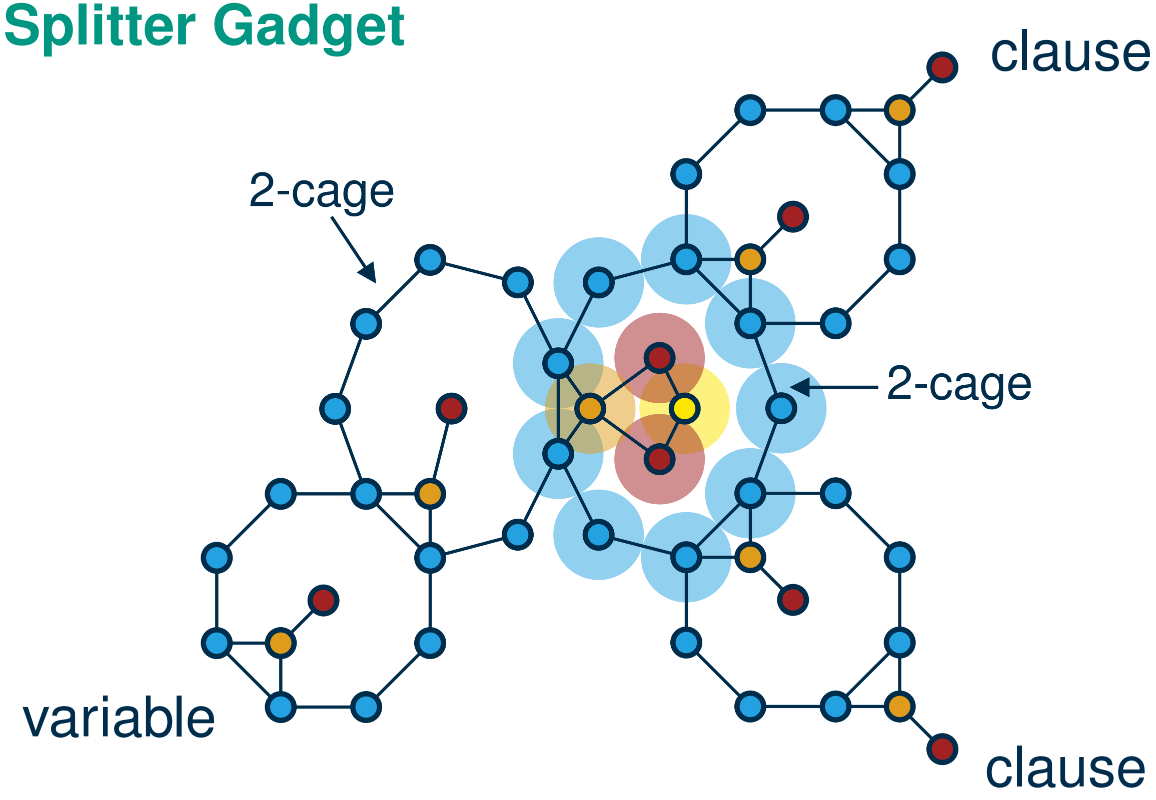

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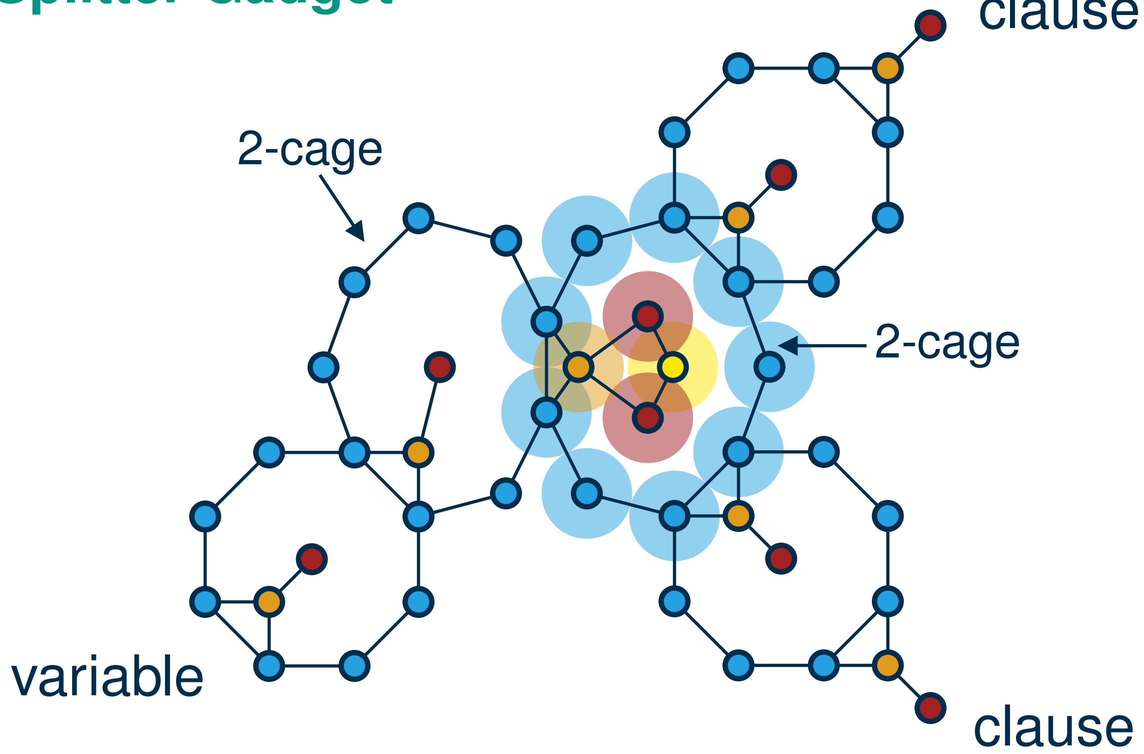

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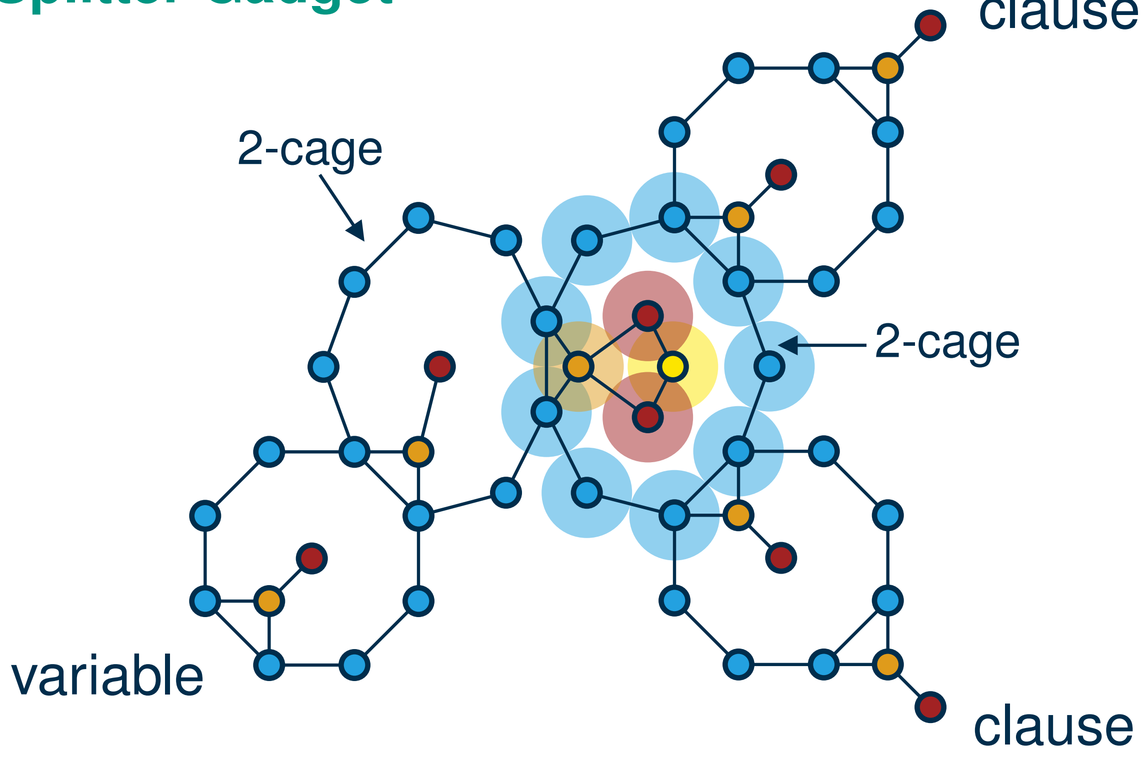

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- FALSE signal from the variable ⇒ FALSE signal to clauses (in every unit disk representation)
- TRUE signal from variable ⇒ TRUE signals to clauses possible
- gadget does what it should (flip from TRUE to FALSE is ok)

Graphs That Are Hard To Recognize

Theorem
It is NP-hard to decide whether a given graph is a unit disk graph.

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  - guess positions as certificate and check whether $uv \in E \Leftrightarrow \mathrm{dist}(u, v) \leq 2$
  - problem: this certificate sometimes needs to be exponentially large

(because we need double exponentially precise coordinates)

Existential Theory Of The Reals

Problem: Existential Theory Of The Reals
Let $F(X_1, \ldots, X_n)$ be a quantifier-free Boolean formula over (in-)equalities of real polynomials.
Is $\exists X_1 \cdots \exists X_n \; F(X_1, \ldots, X_n)$ true?

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- problem lies in $\exists\mathbb{R}$
- it is actually $\exists\mathbb{R}$-complete
- we believe: recognizing unit disk graphs is strictly harder than every problem in NP

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Wrap-Up

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- negation gadget (if you have a place where you need negative literals but can only get positive literals)
- crossing gadget (if you reduce from a non-planar SAT variant)

Literature

- **Algorithmic Aspects of Proportional Symbol Maps** (2009)
  Sergio Cabello, Herman Haverkort, Marc van Kreveld, Bettina Speckmann
  https://doi.org/10.1007/s00453-009-9281-8

- **Unit disk graph recognition is NP-hard** (1998)
  Heinz Breu, David G. Kirkpatrick
  https://doi.org/10.1016/S0925-7721(97)00014-X

- **Optimal Binary Space Partitions in the Plane** (2010)
  Mark de Berg, Amirali Khosravi
  https://doi.org/10.1007/978-3-642-14031-0_25
  (NP-hardness for planar monotone 3-SAT)

- **Sphere and Dot Product Representations of Graphs** (2012)
  Ross J. Kang, Tobias Müller
  https://doi.org/10.1007/s00454-012-9394-8
  ($\exists\mathbb{R}$-hardness for recognizing unit disk graphs)