Computational Geometry

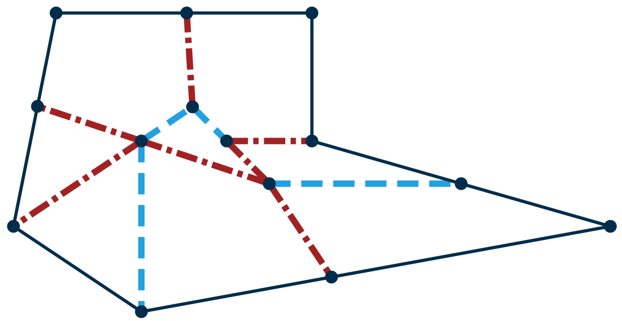
Computational Origami – Foldability

Thomas Bläsius

Flat-Foldable Crease Patterns

Given

- a geometric graph
- each inner edge is labeled either *mountain* or *valley*



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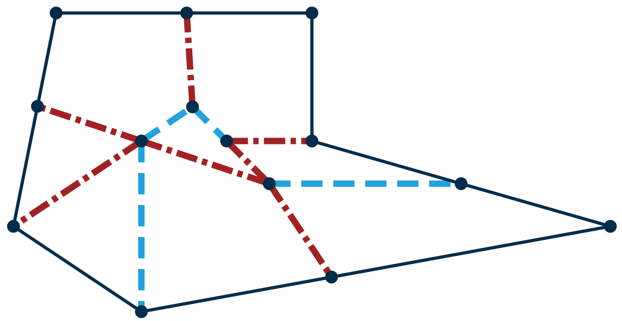

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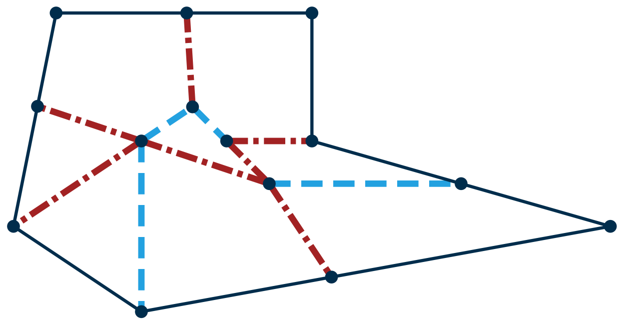

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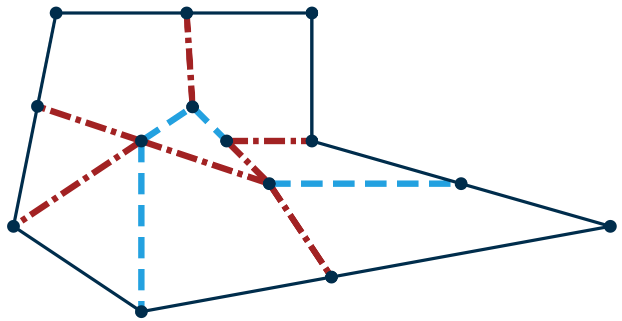

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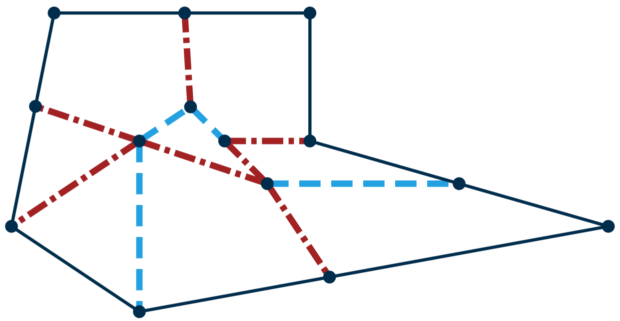

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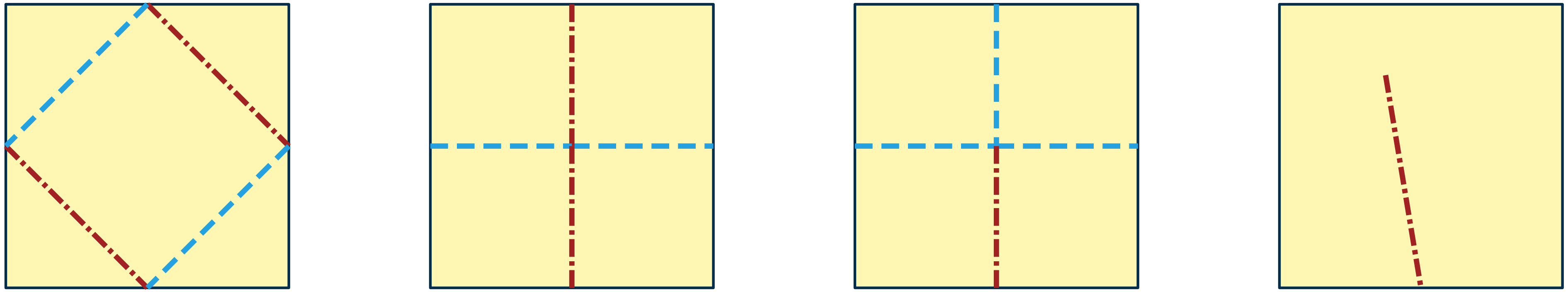

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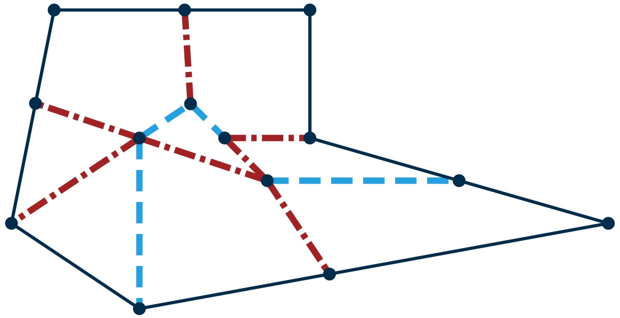

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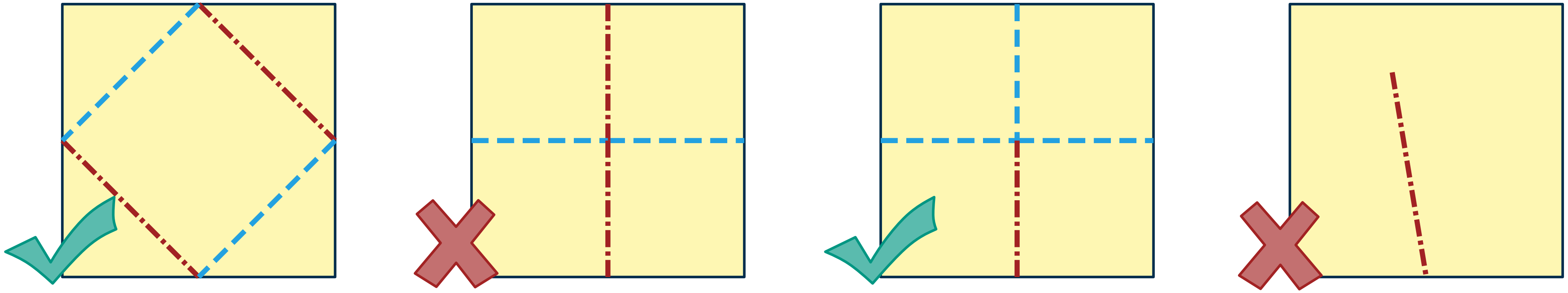

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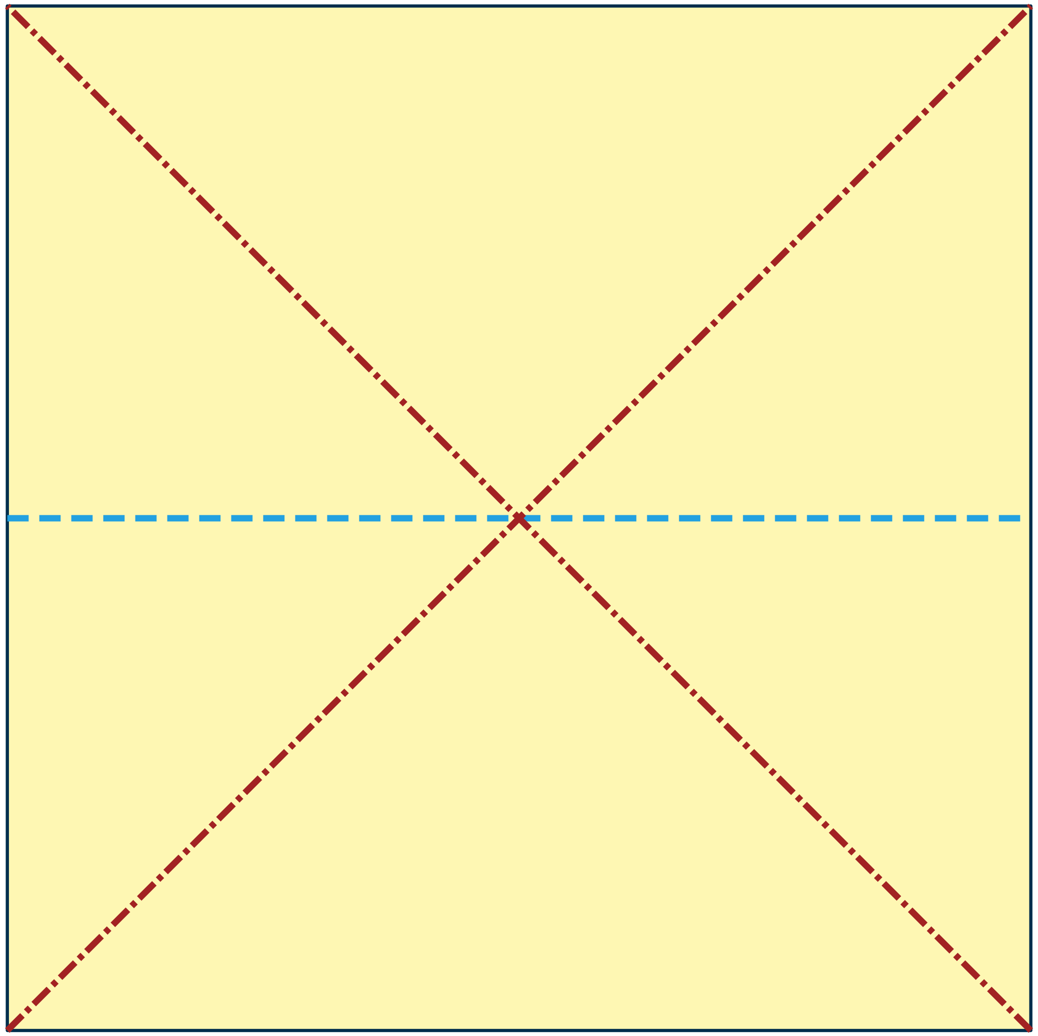

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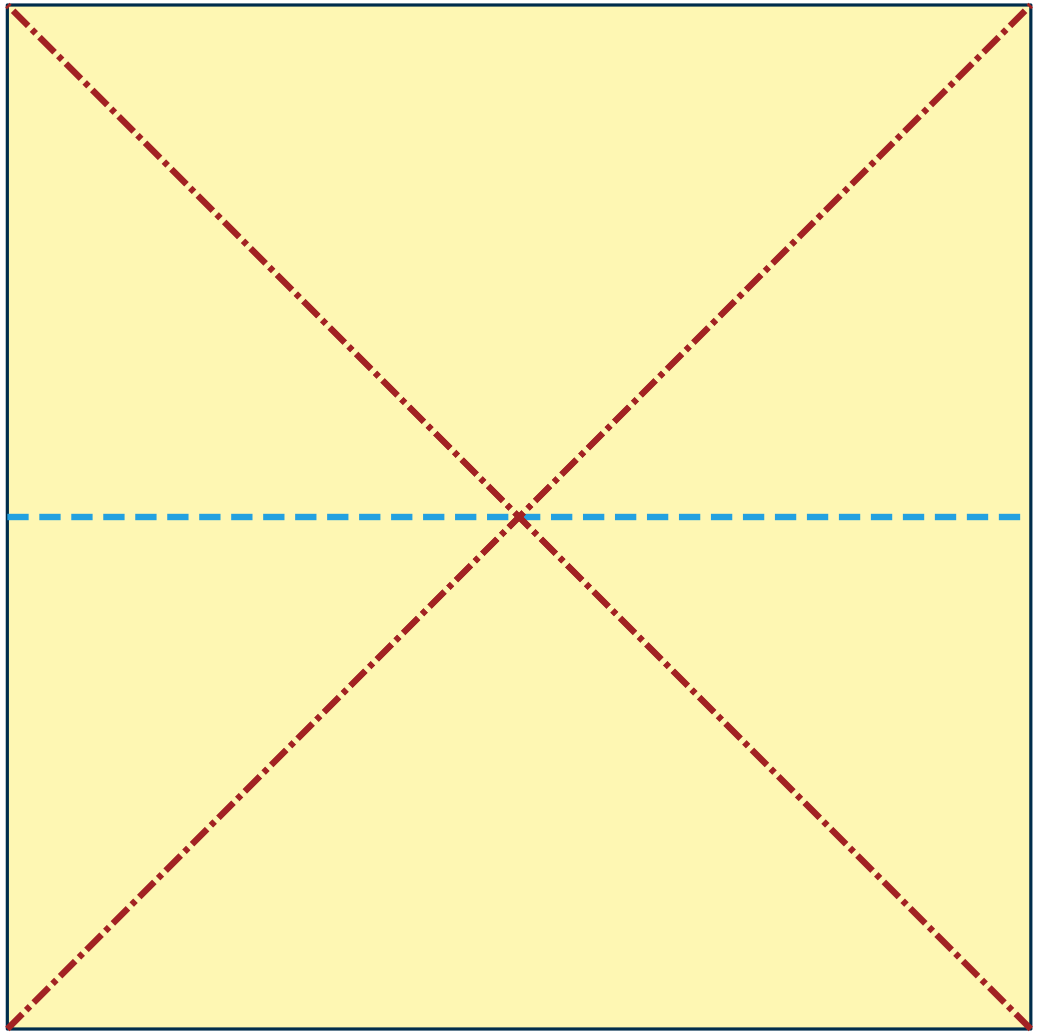

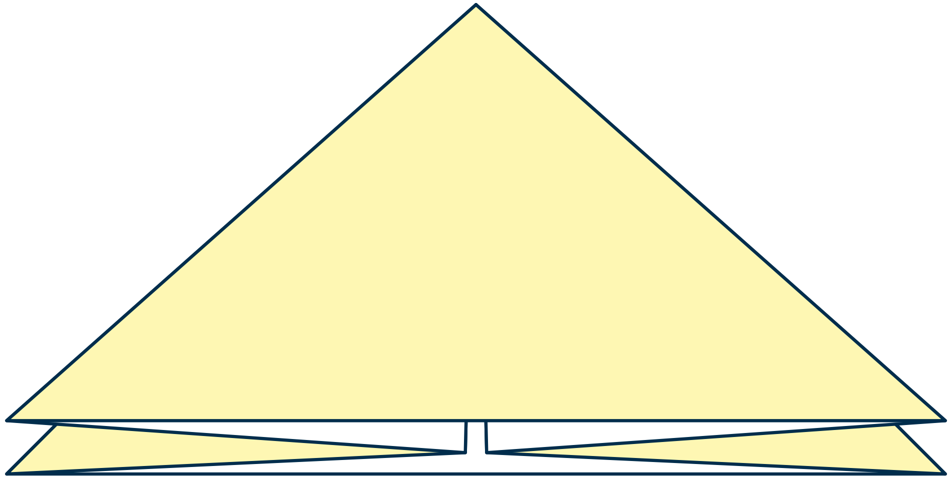

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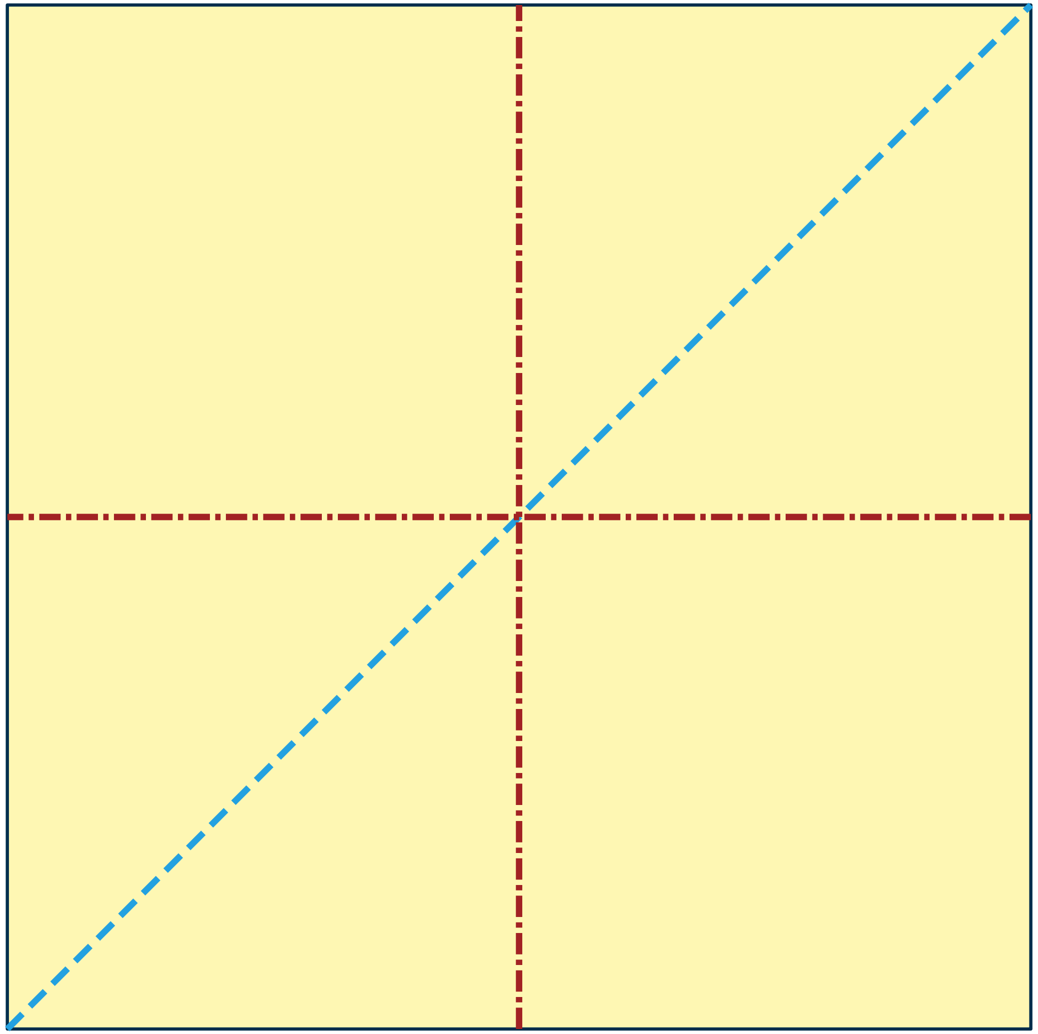

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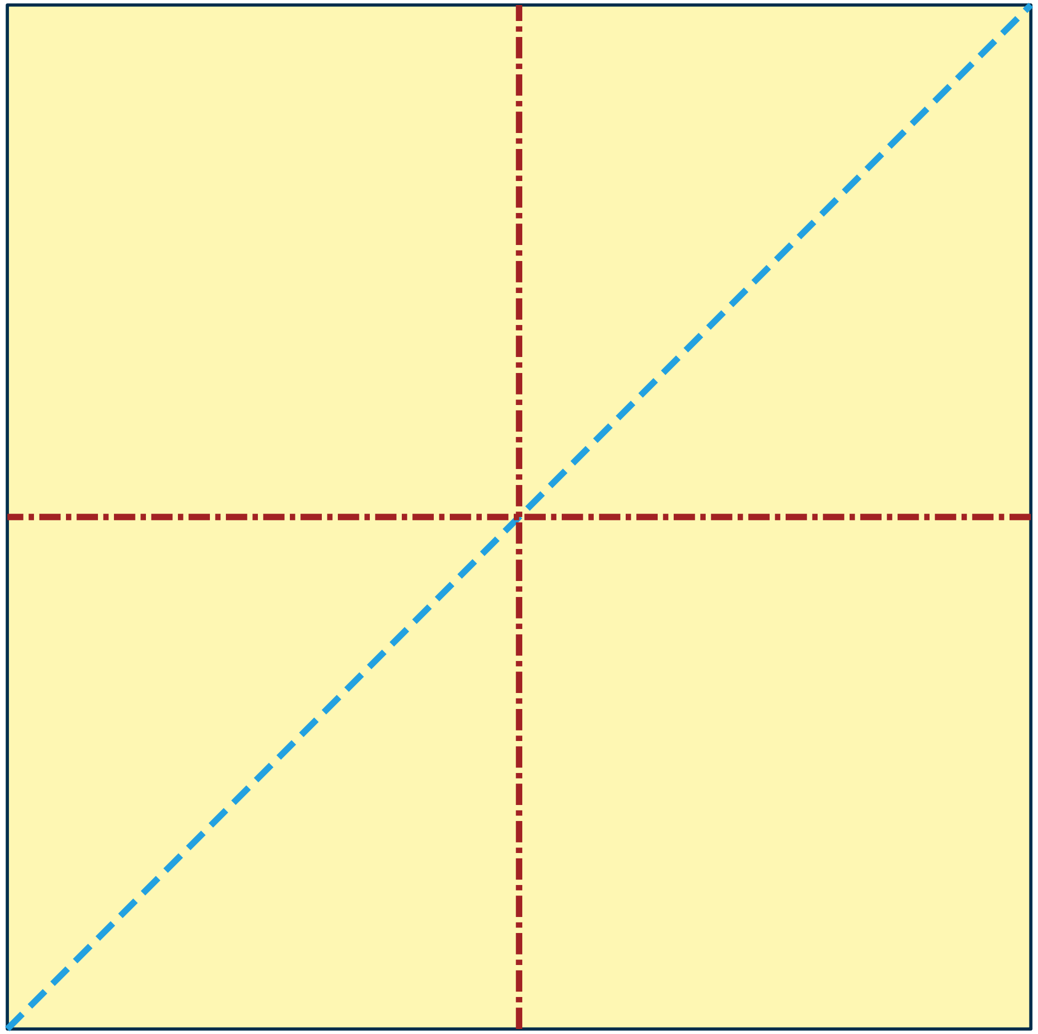

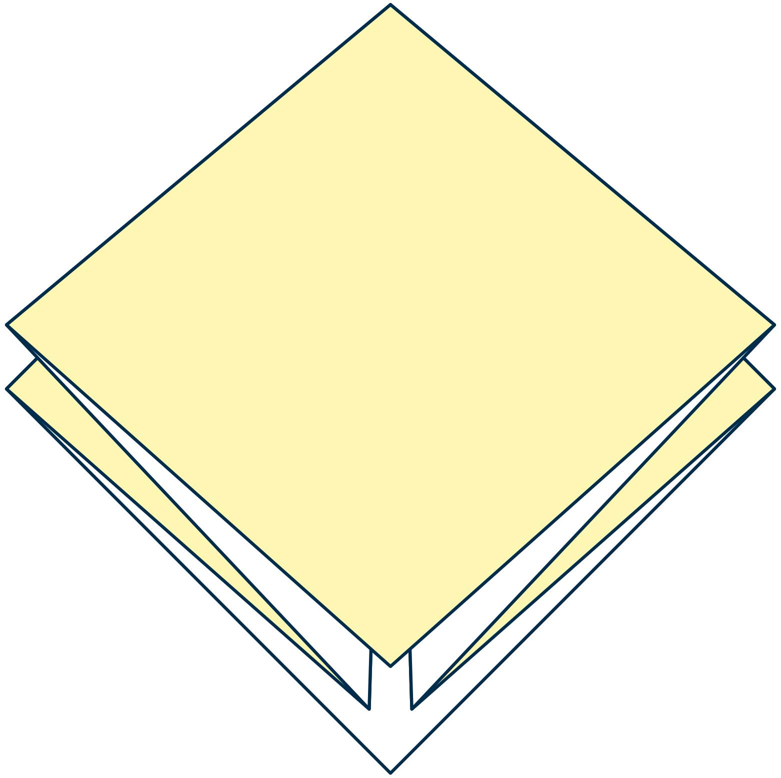

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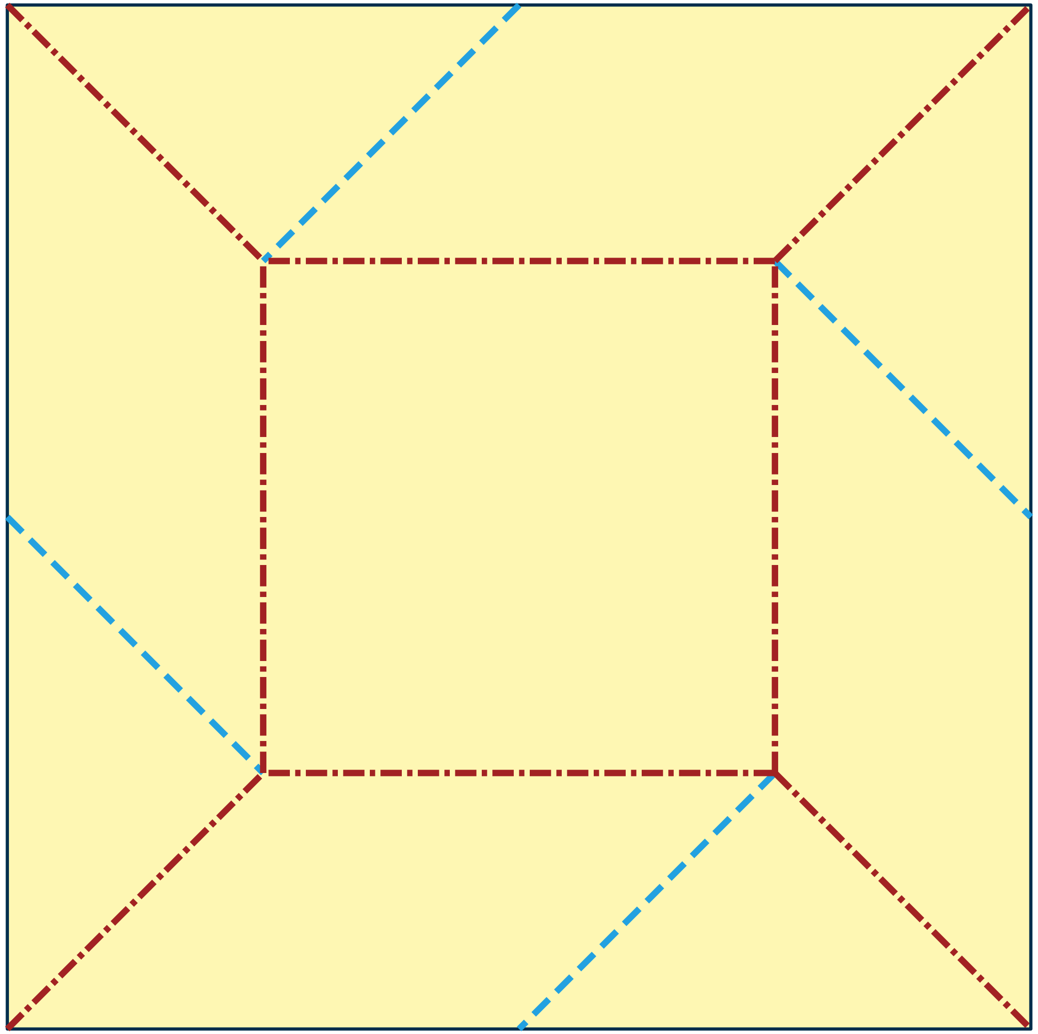

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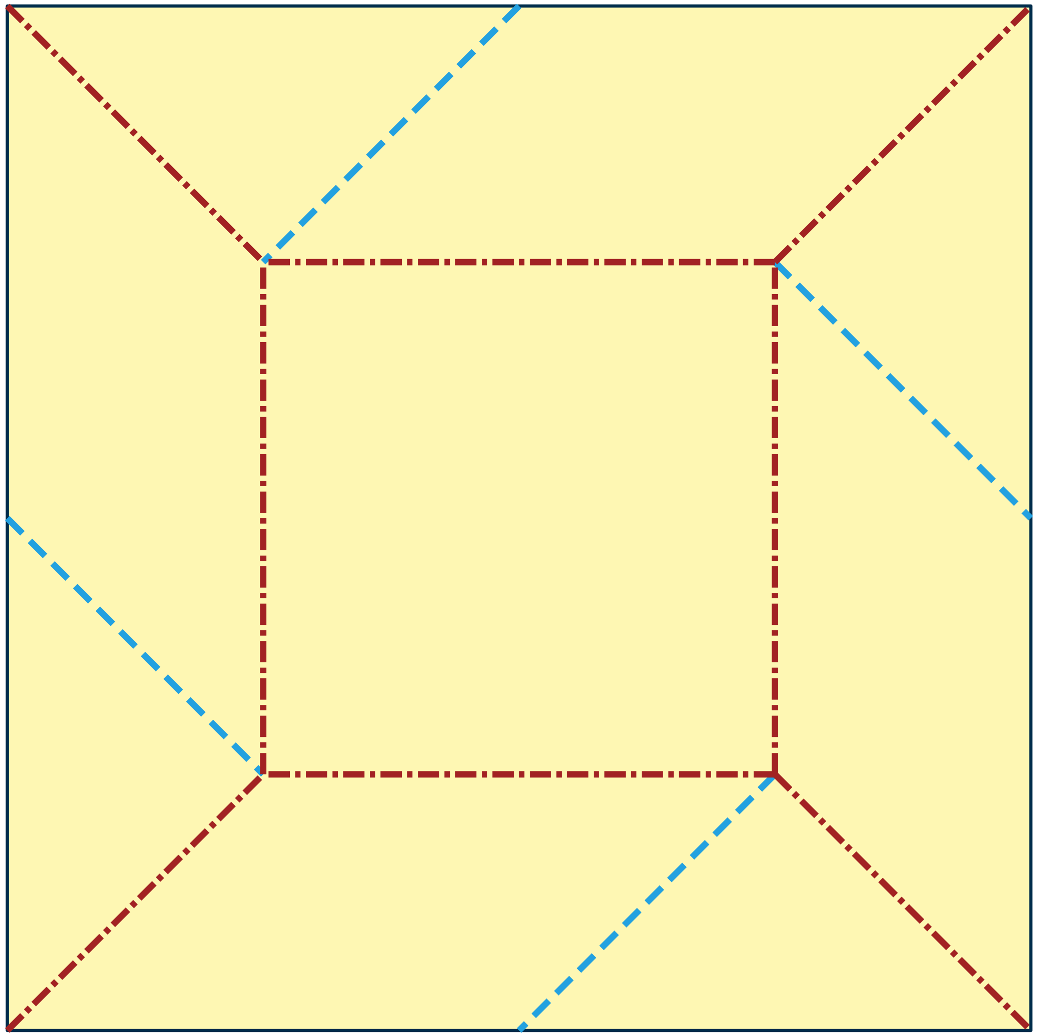

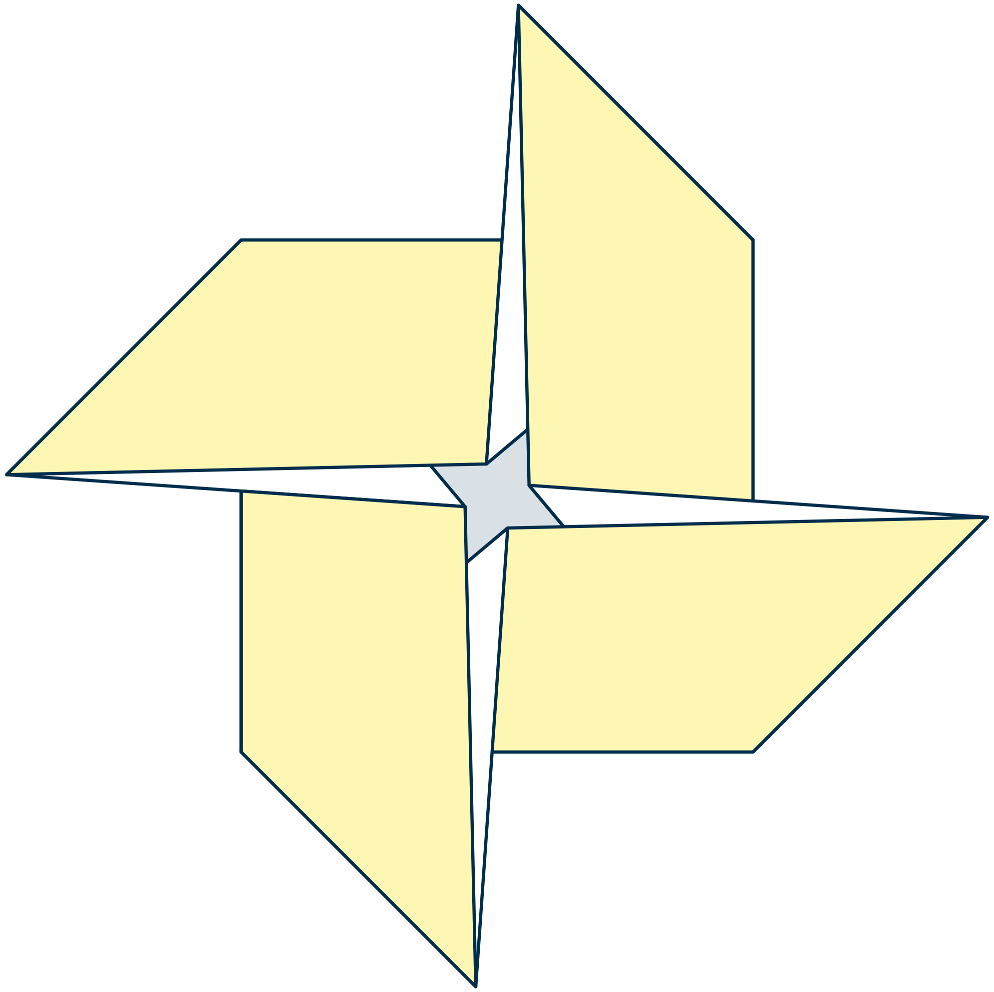

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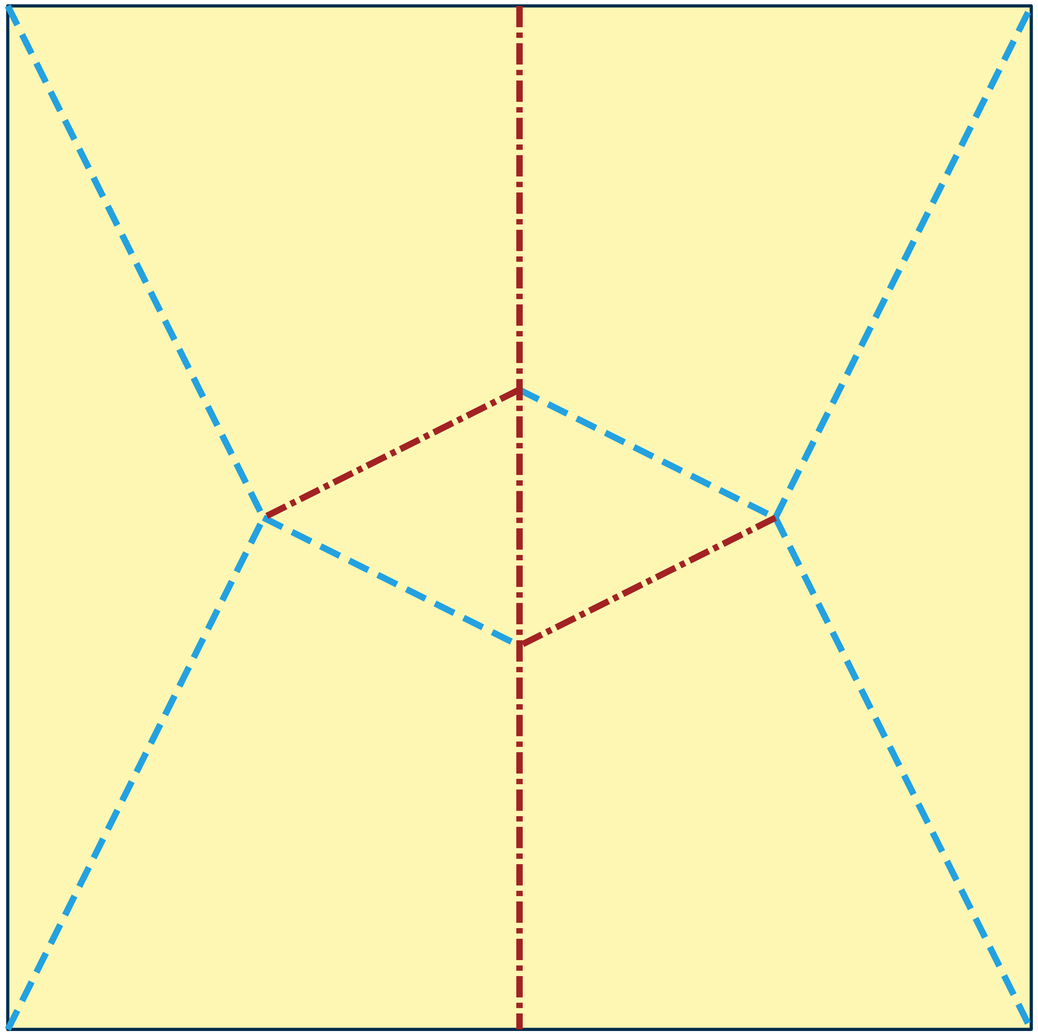

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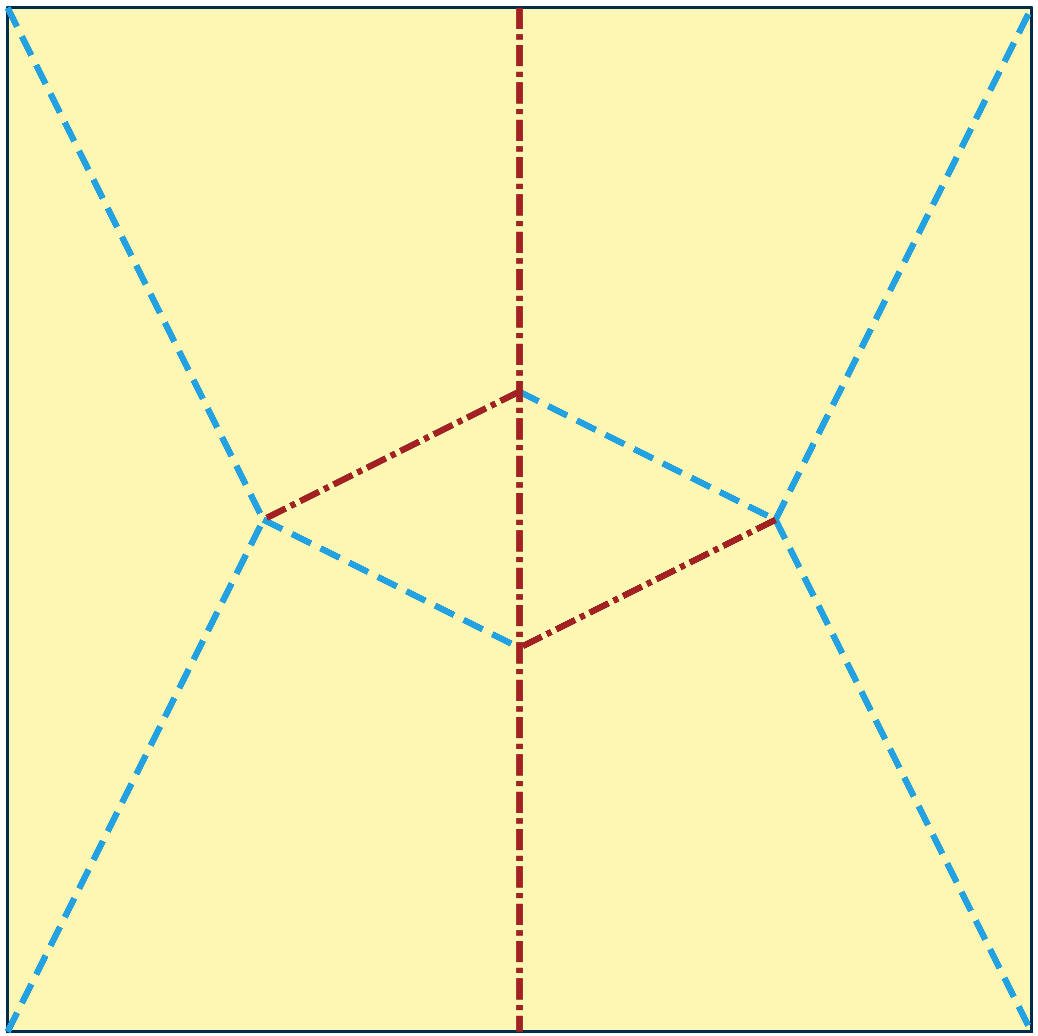

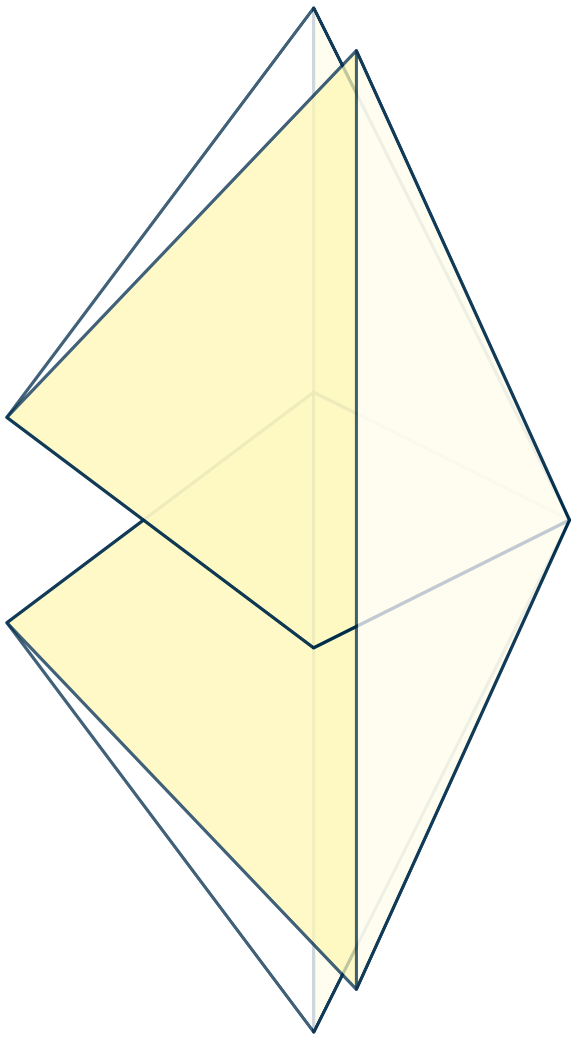

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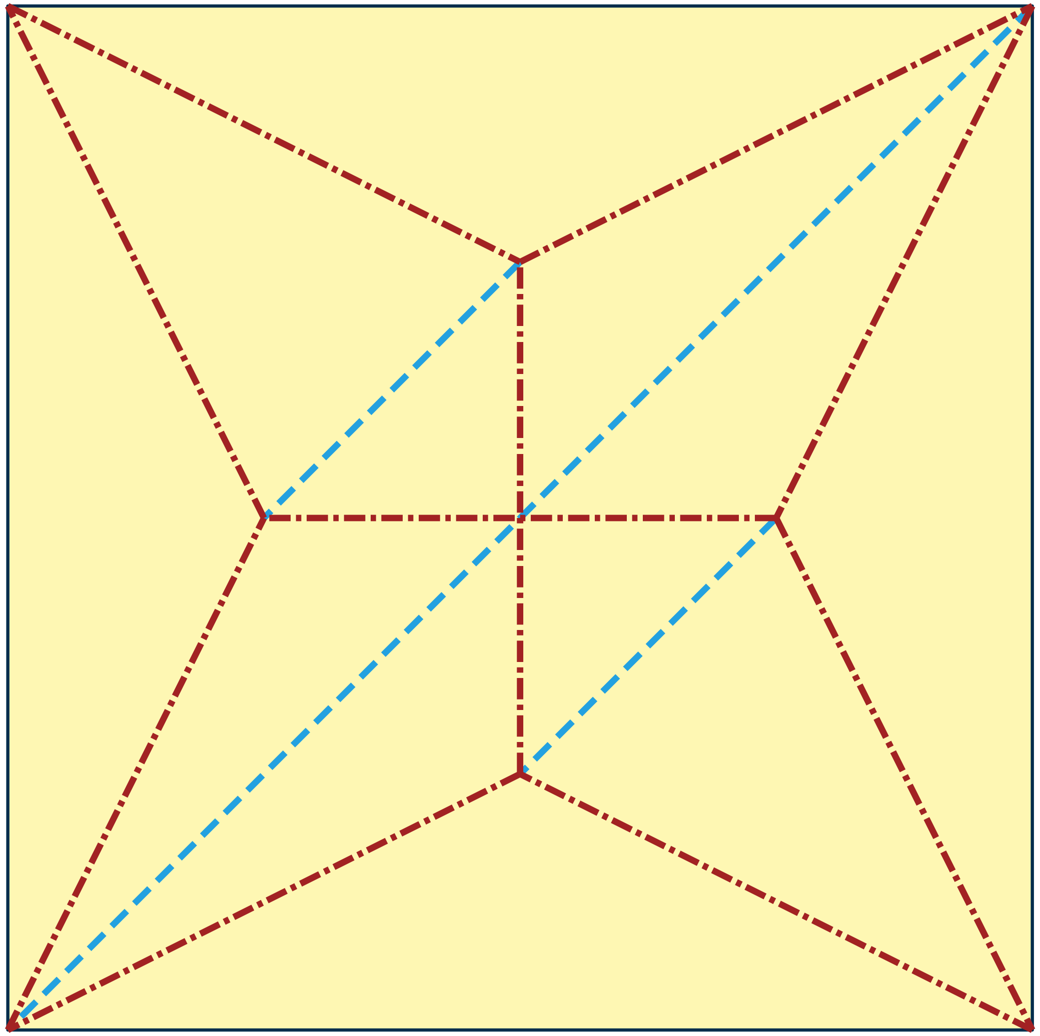

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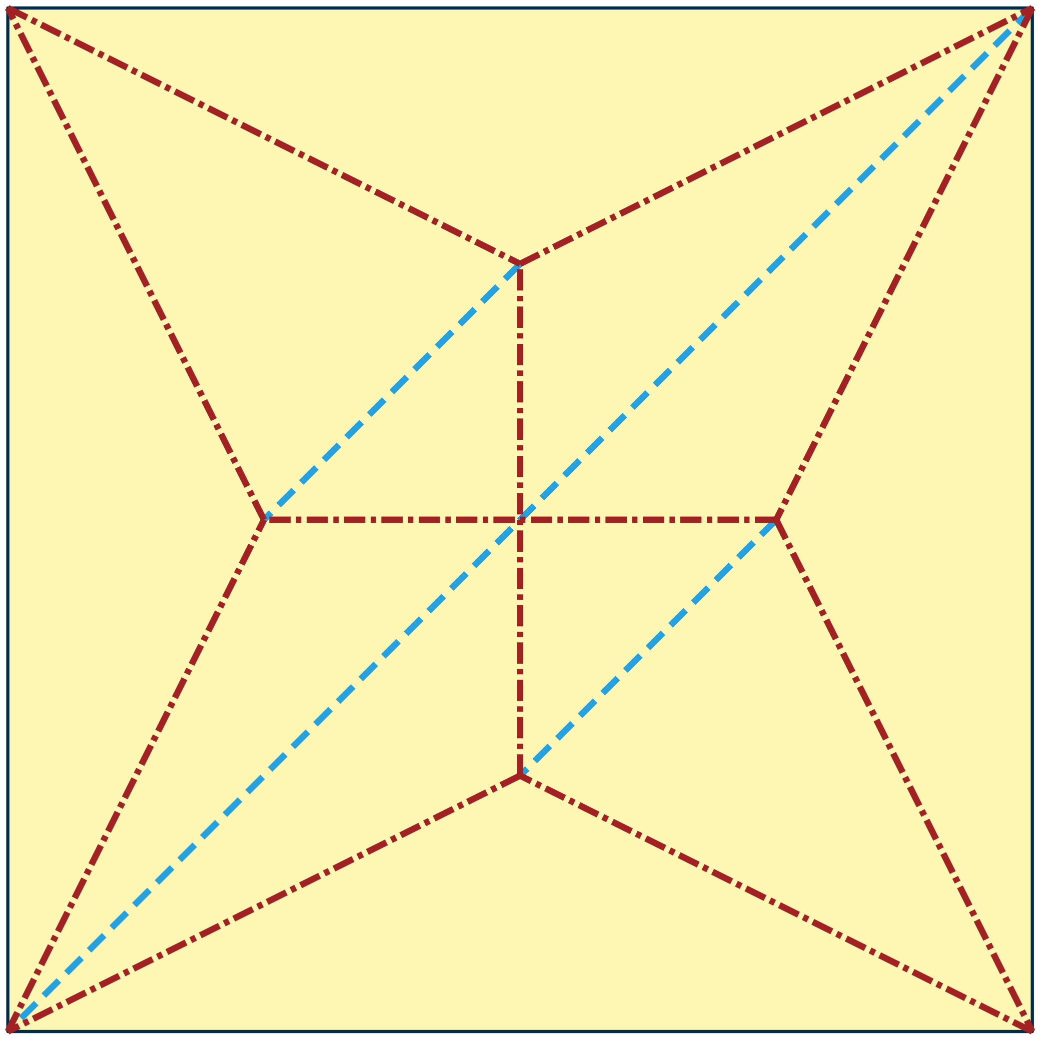

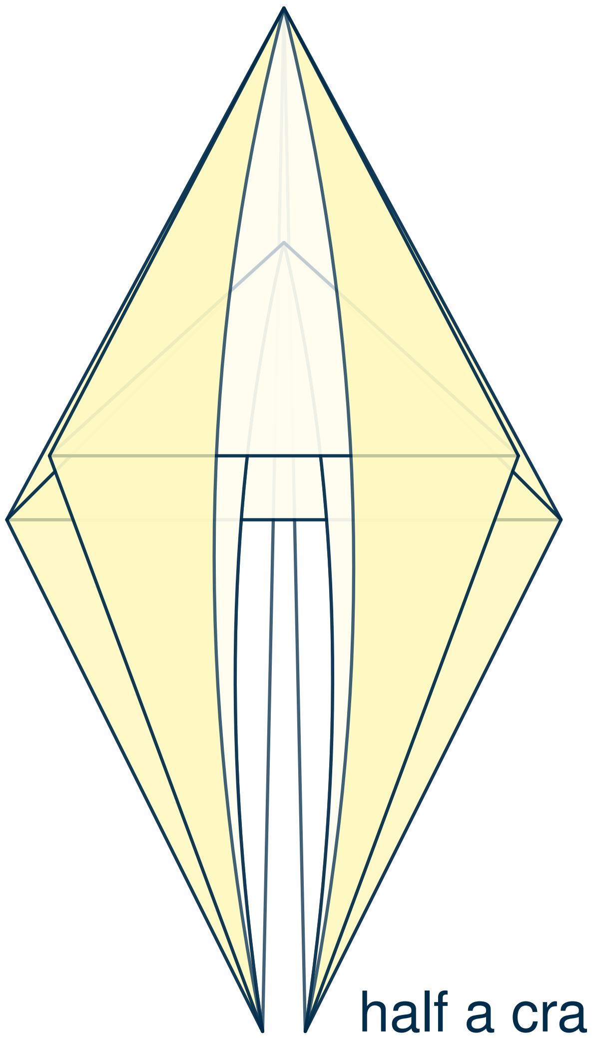

half a crane

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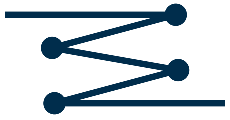

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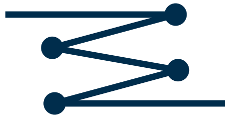

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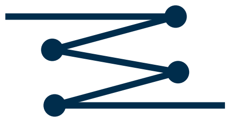

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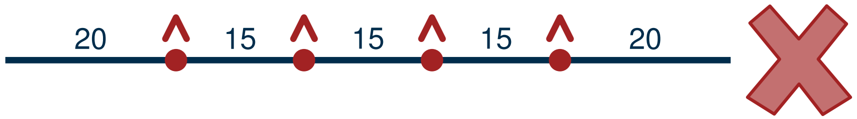

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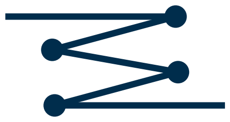

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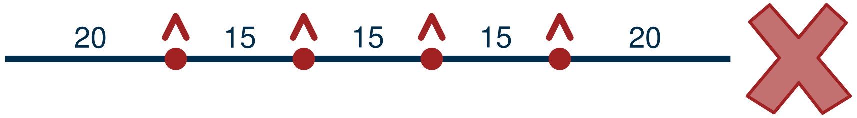

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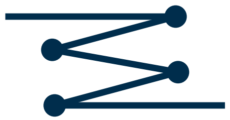

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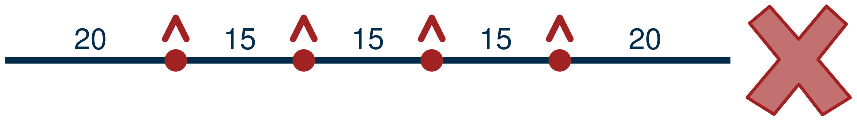

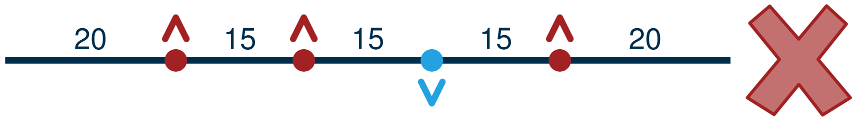

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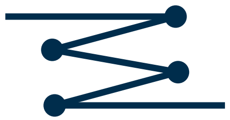

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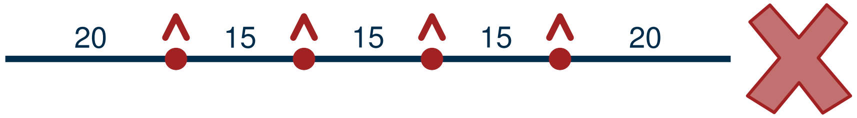

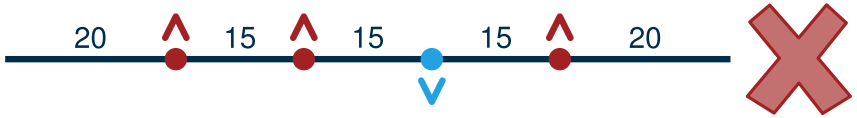

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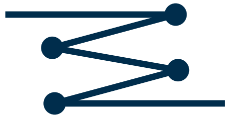

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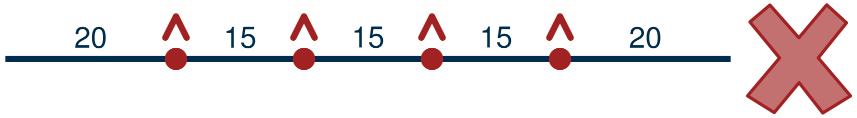

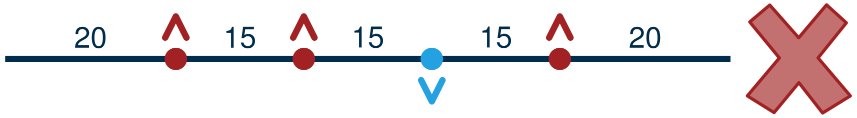

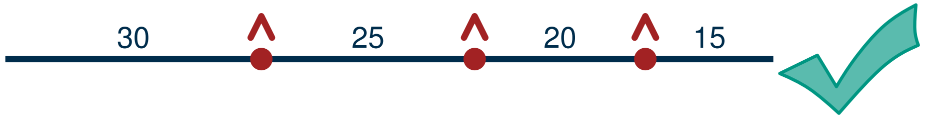

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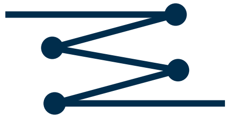

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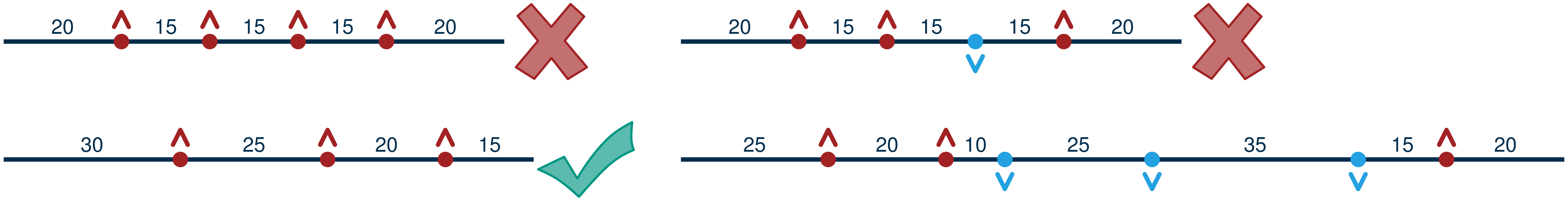

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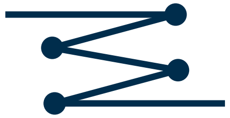

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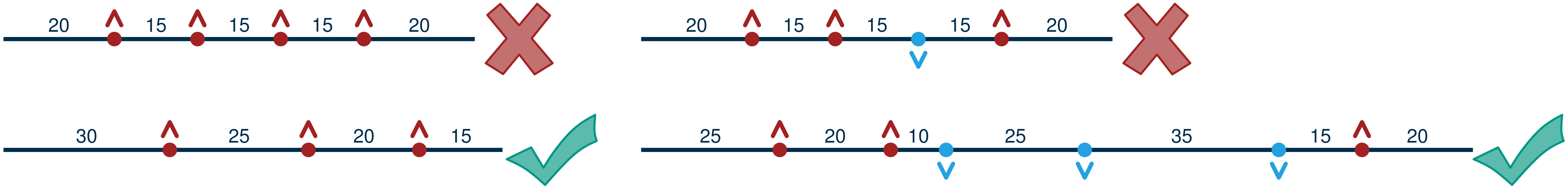

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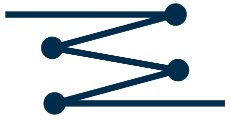

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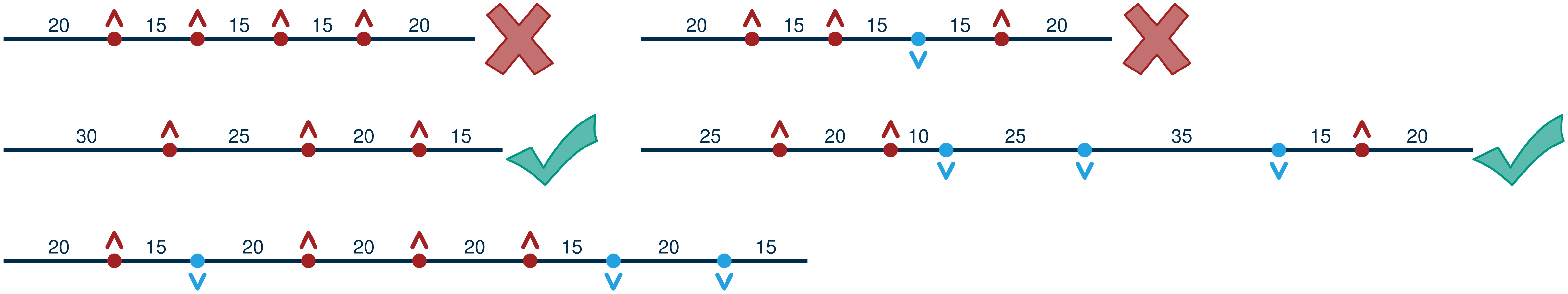

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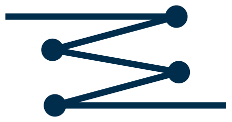

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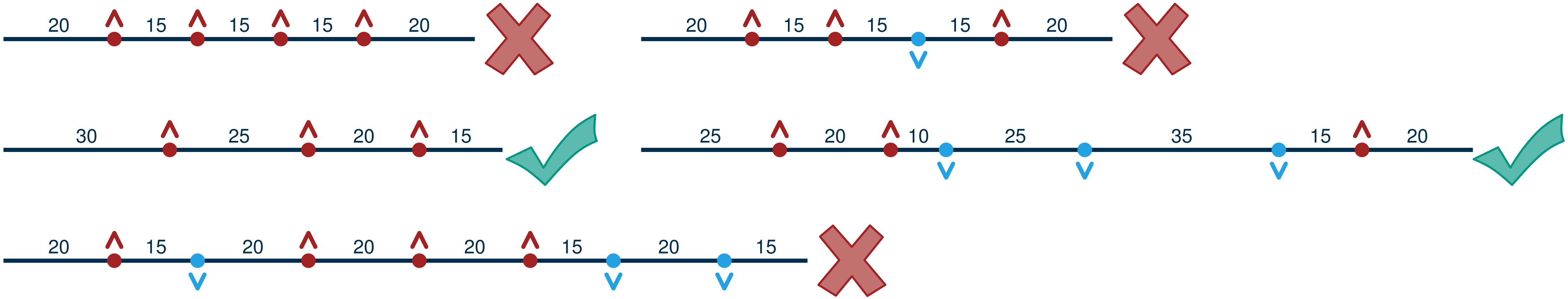

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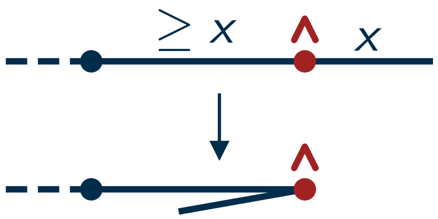

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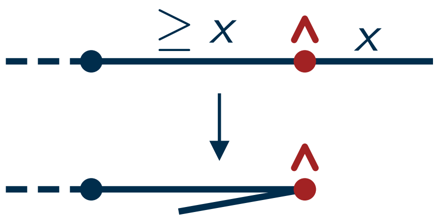

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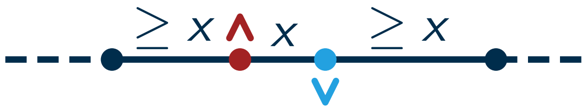
- condition: piece with mountain and valley vertex and larger neighbors



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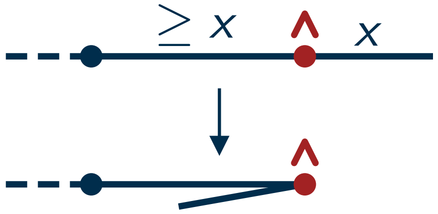

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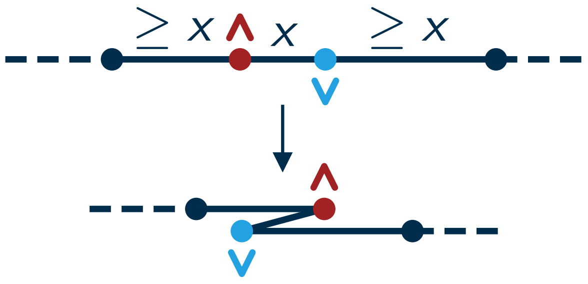

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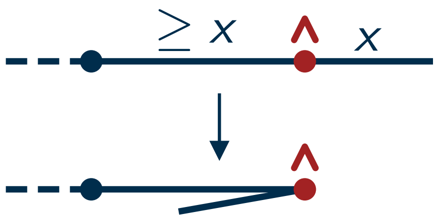

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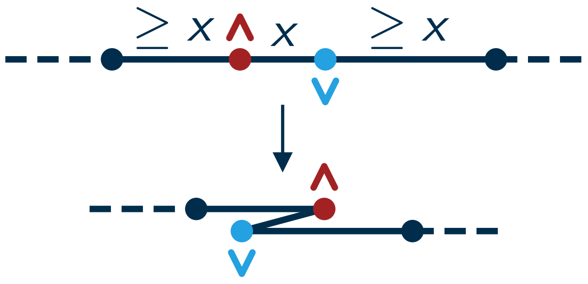

Lemma **(Safety first!)**

The reduction rules “end-fold” and “crimp” are safe.

(If a mountain/valley pattern is flat foldable, then it remains flat foldable after applying the reduction rules.)

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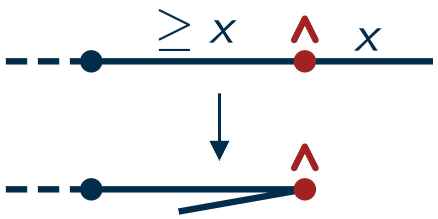

Crimp

- condition: piece with mountain and valley vertex and larger neighbors
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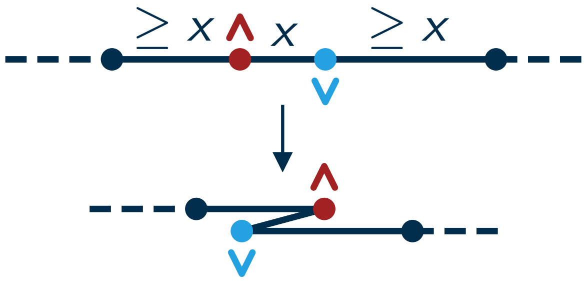


Lemma **(Safety first!)**

The reduction rules "end-fold" and "crimp" are safe.

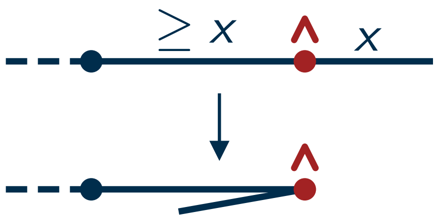
(If a mountain/valley pattern is flat foldable, then it remains flat foldable after applying the reduction rules.)

Proof

- **End-Fold:** obvious

Reduction Rules

End-Fold



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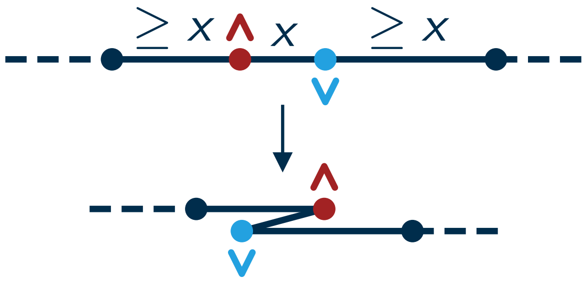

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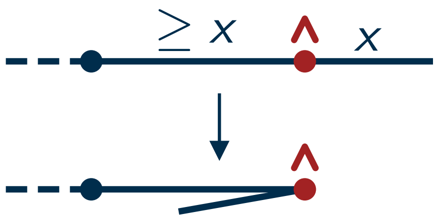

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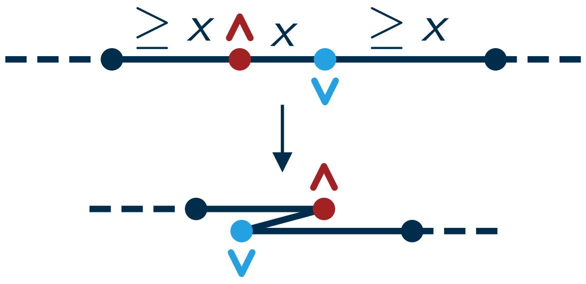

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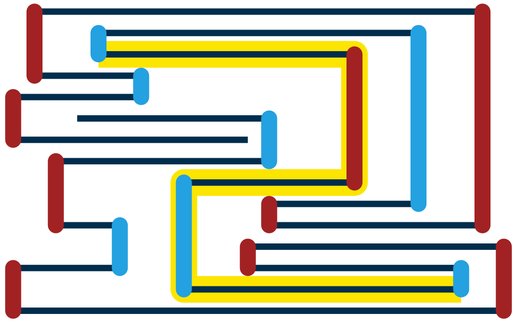

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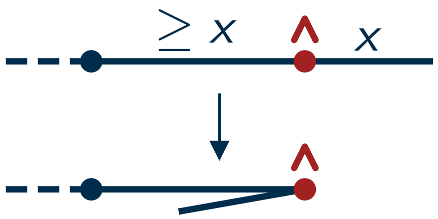

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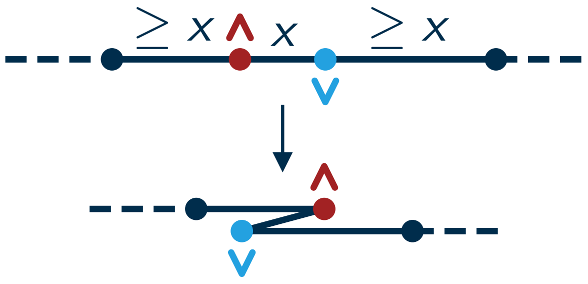

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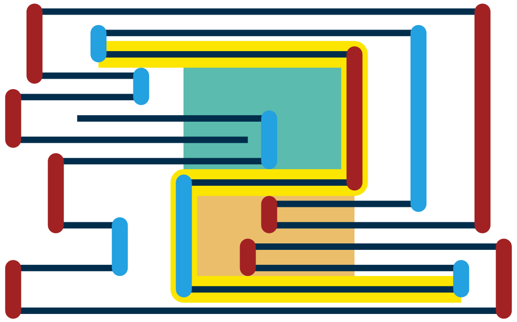

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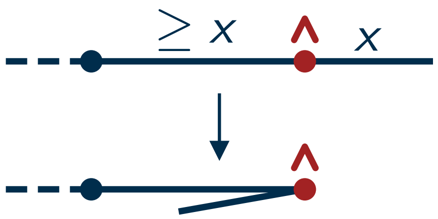

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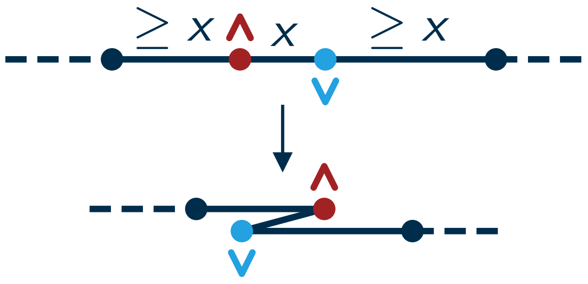

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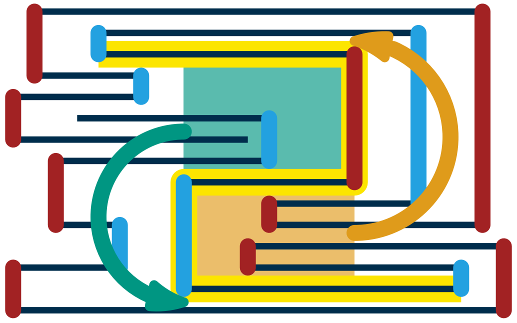

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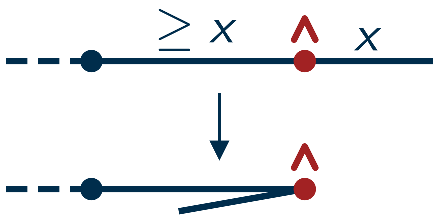

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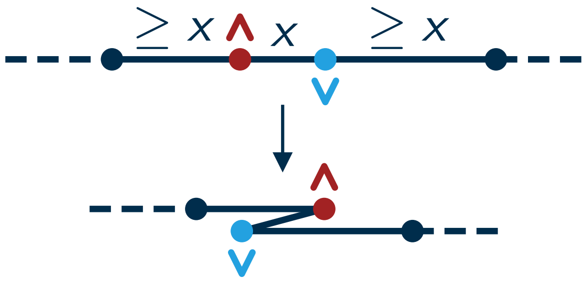

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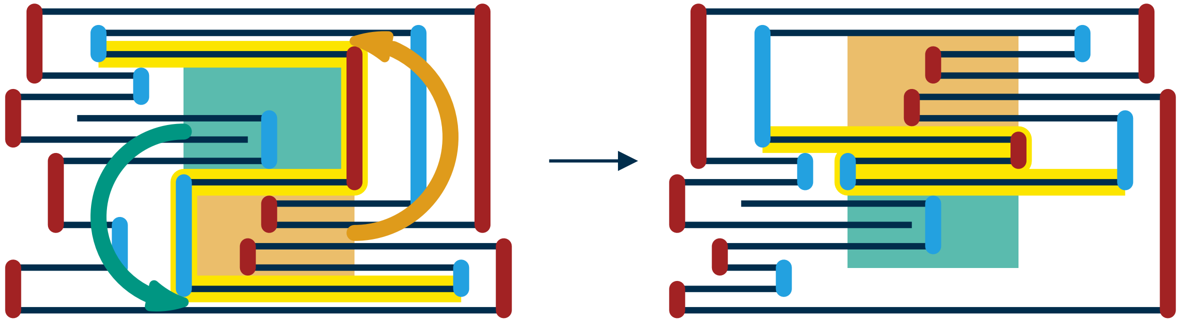

Reductions For The Win

Theorem **(foldable $\Rightarrow$ reduction)**

If the mountain/valley pattern is flat foldable, then there is an end-fold or a crimp.

Reductions For The Win

Theorem **(foldable ⇒ reduction)**
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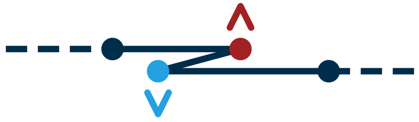
Proof Plan

- short pieces with one mountain and one valley vertex are good

crimp: 

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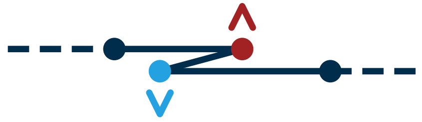
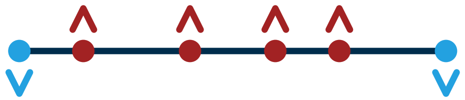
crimp: 

- short pieces with one mountain and one valley vertex are good
- consider maximal sequence (wrt inclusion) with only mountain or only valley vertices



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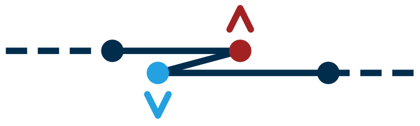

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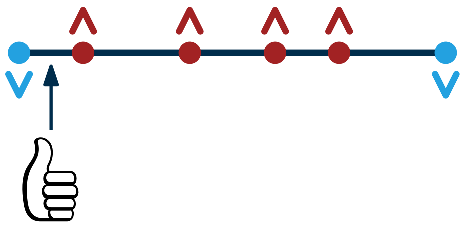
- left-short: first piece is shorter than second



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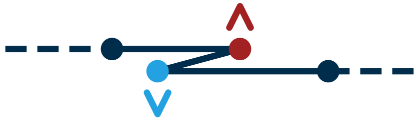

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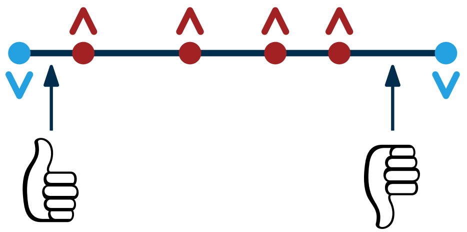

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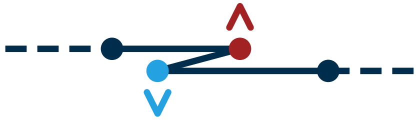

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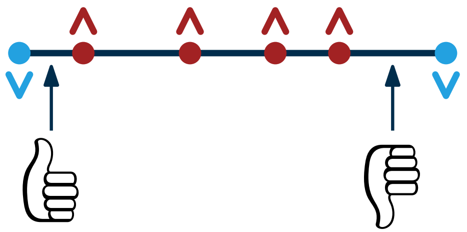

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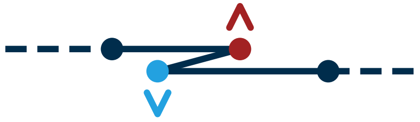
crimp: 

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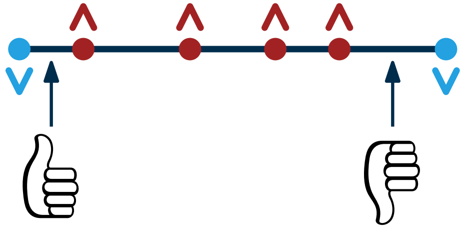

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Short Pieces Everywhere

Lemma **(foldable $\Rightarrow$ left/right-short)**

flat foldable $\Rightarrow$ each maximal sequence is left-short or right-short

Proof

Short Pieces Everywhere

Lemma **(foldable ⇒ left/right-short)**
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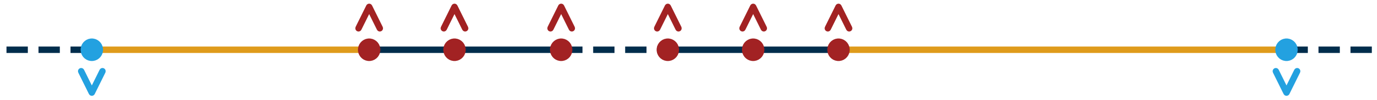
Proof

- consider maximal sequence that is neither left-short nor right-short



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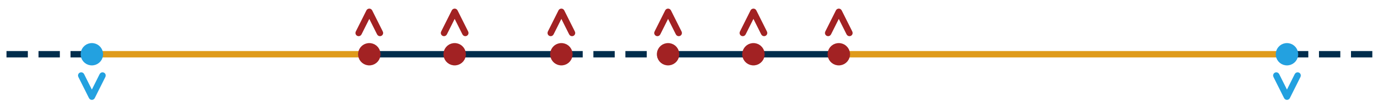
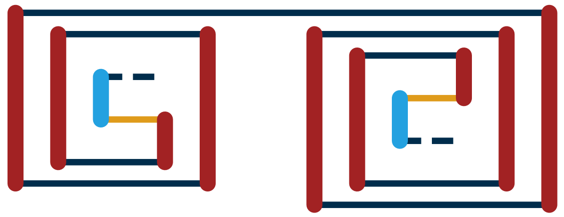


- we cannot escape from the spiral



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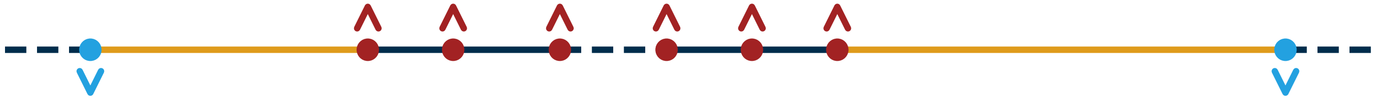

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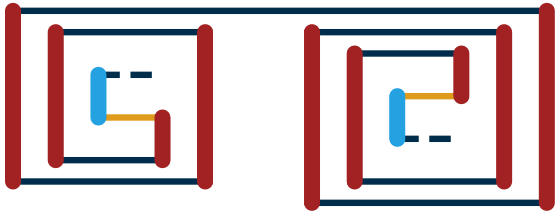

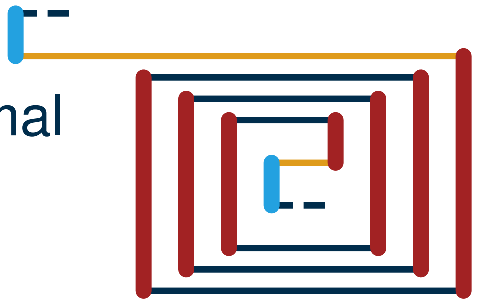

- being one of left-short or right-short would be enough (for the maximal sequence in isolation)

There Is Always One More Reduction

Lemma **(left/right-short $\Rightarrow$ reduction)**
each maximal sequence is left-short or right-short $\Rightarrow$ there is an end-fold or a crimp

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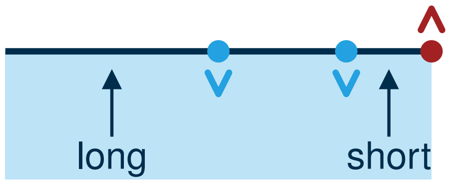
- no end-fold $\Rightarrow$ first maximal sequence not left-short $\Rightarrow$ right-short



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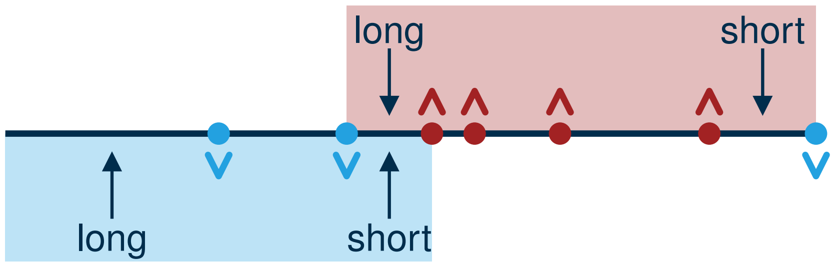

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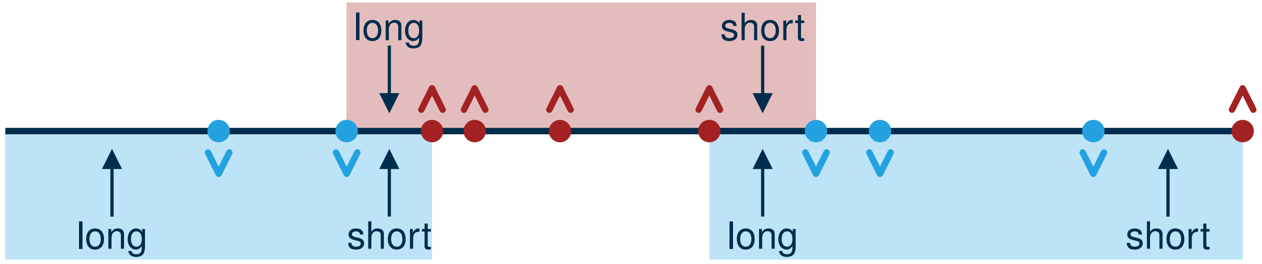
- iterate argument ⇒ every maximal sequence is right-short



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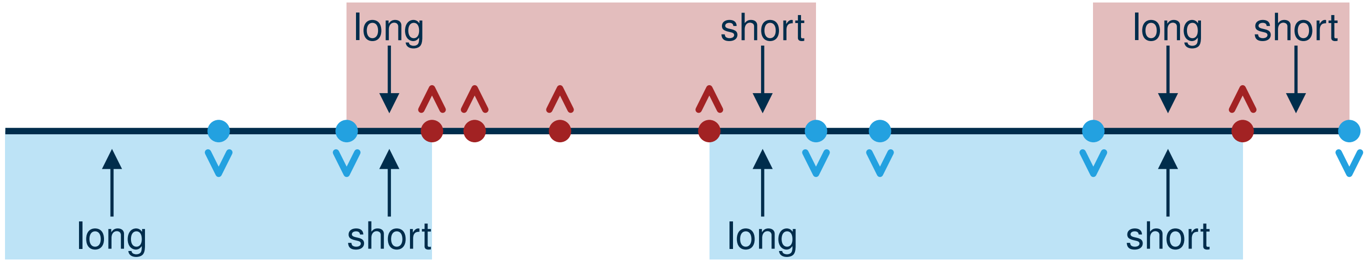

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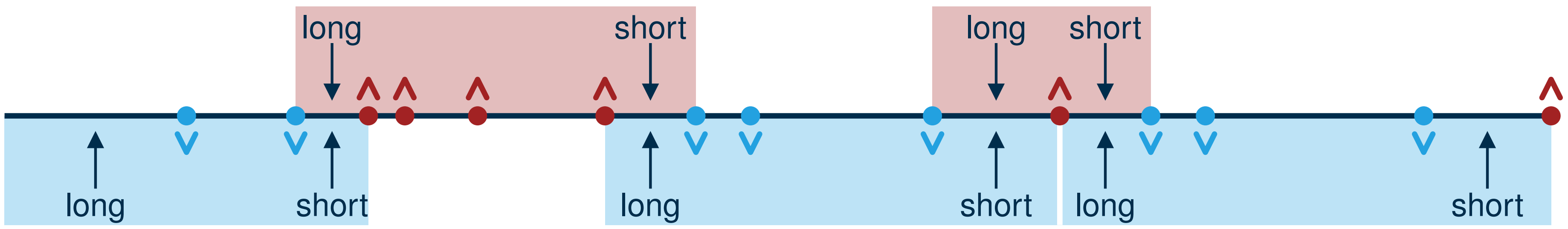

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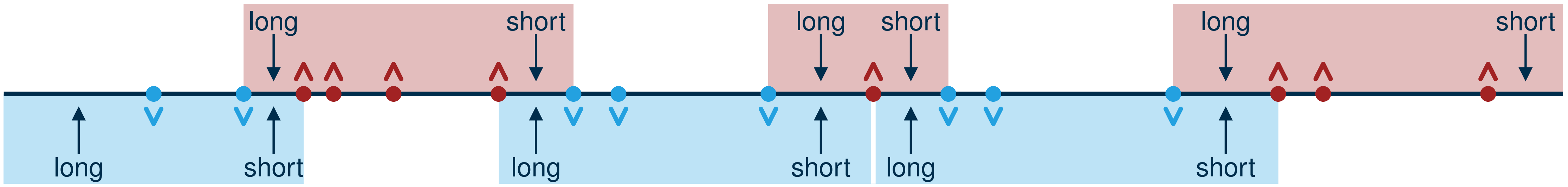

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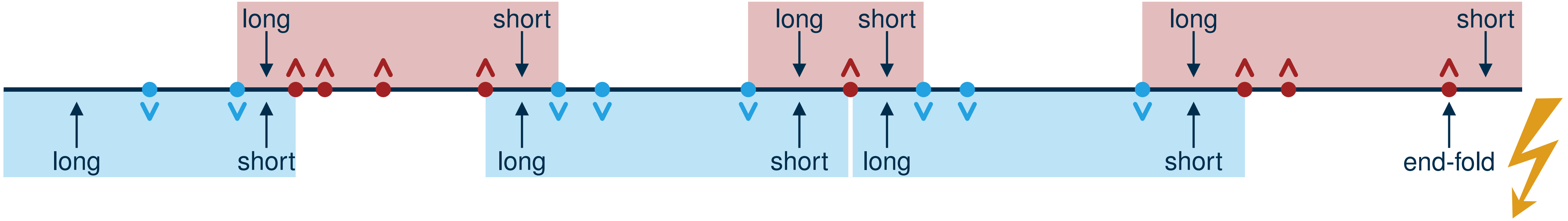
- but: this gives an end-fold for the last segment



Wrap-Up: 1D Origami

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- while there is an end-fold or a crimp, apply an end-fold or a crimp
- result is a flat folding $\Rightarrow$ flat foldable
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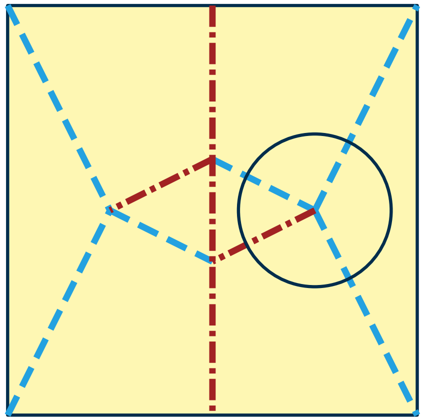
2D Origami (With Only One Vertex)

Necessary Condition

- mountain/valley pattern foldable $\Rightarrow$ each vertex locally foldable



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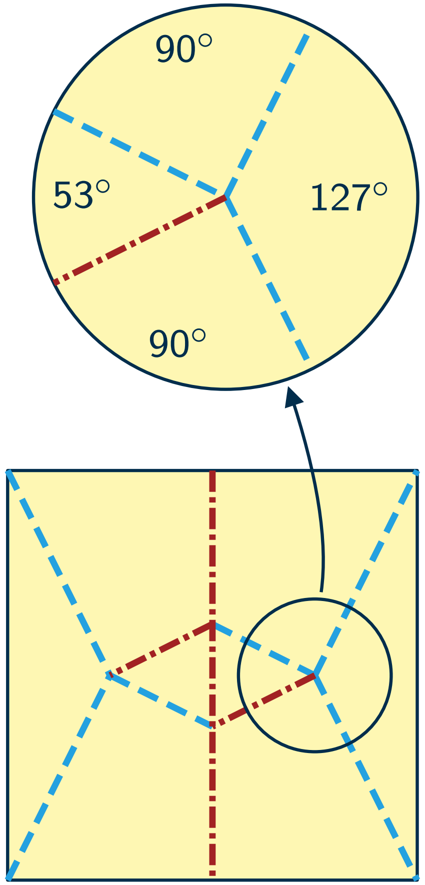

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- Which 1-vertex mountain/valley patterns are flat foldable?

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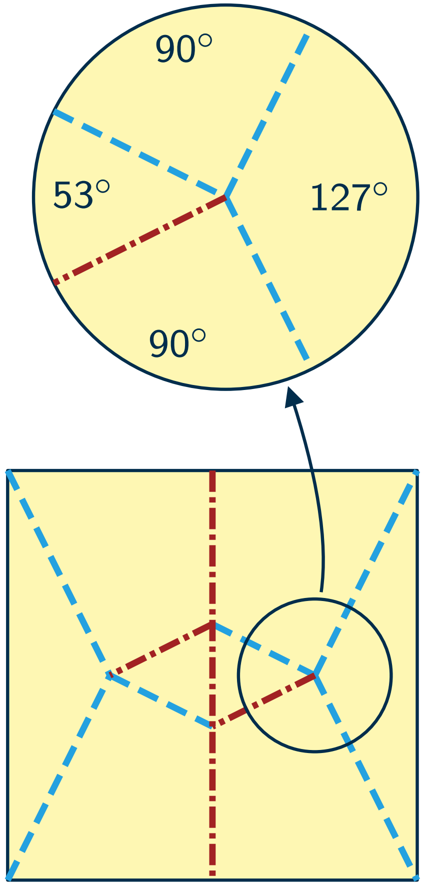

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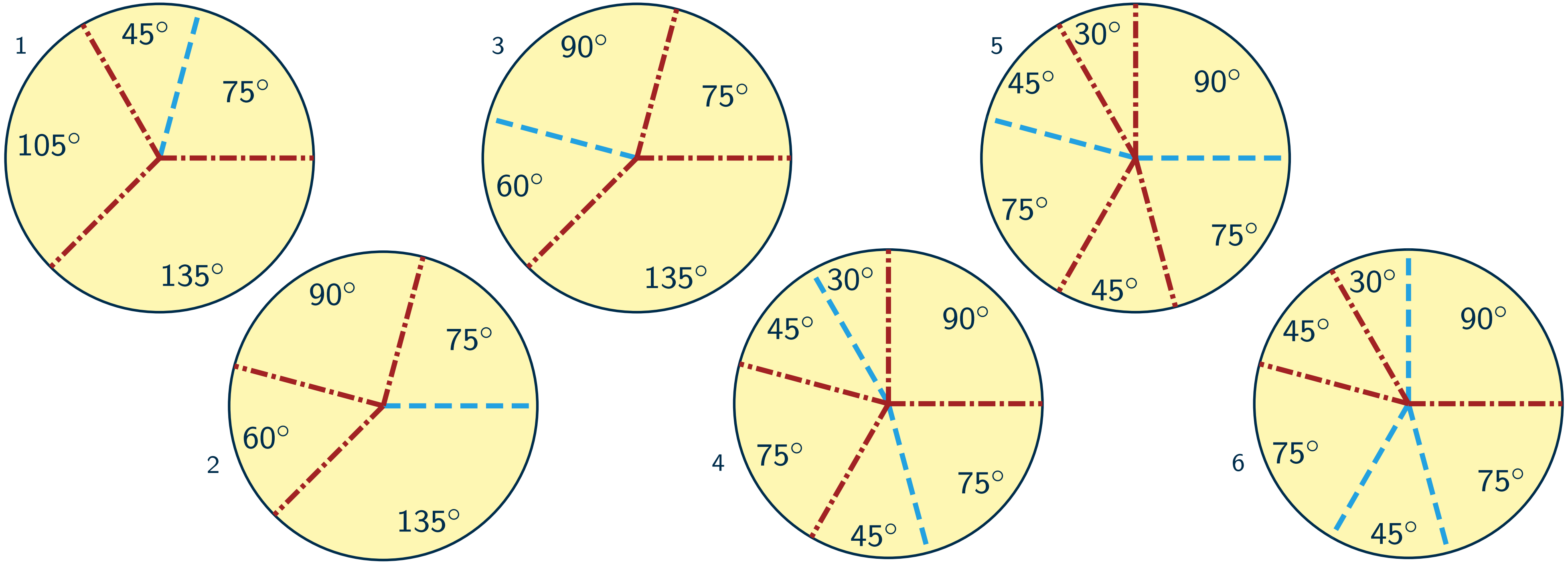

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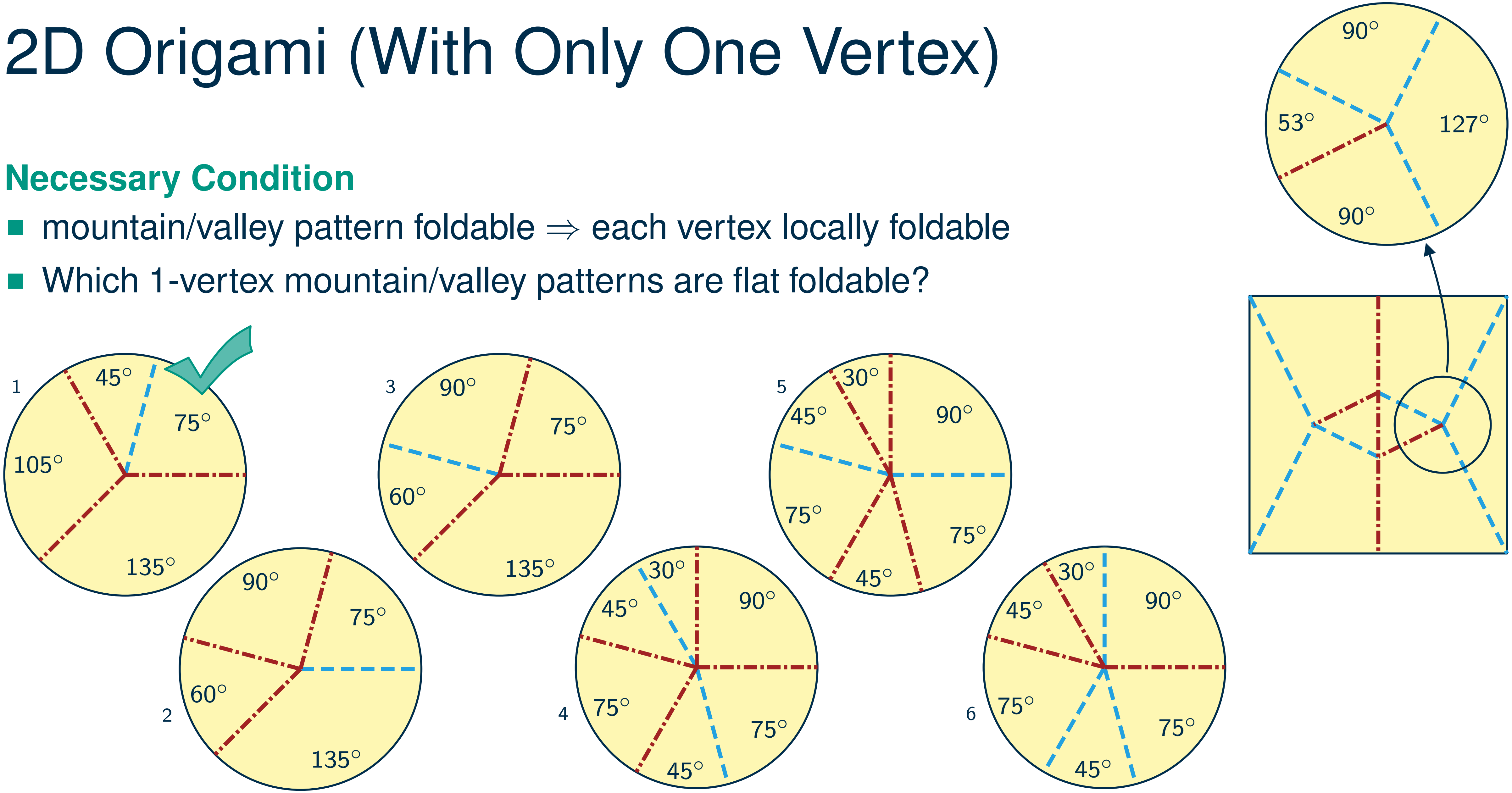

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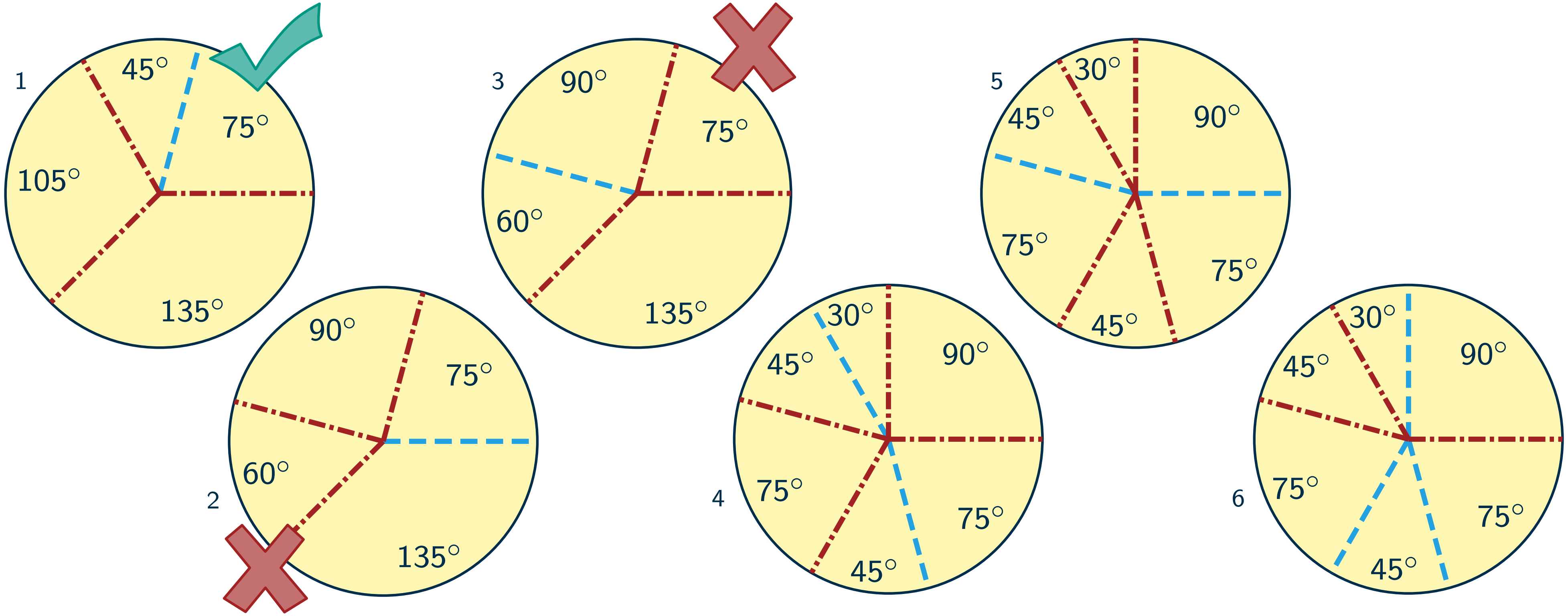
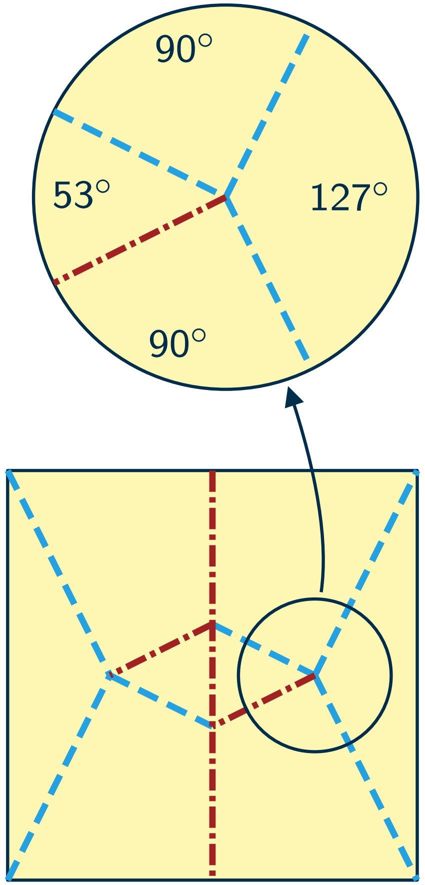
- mountain/valley pattern foldable $\Rightarrow$ each vertex locally foldable
- Which 1-vertex mountain/valley patterns are flat foldable?





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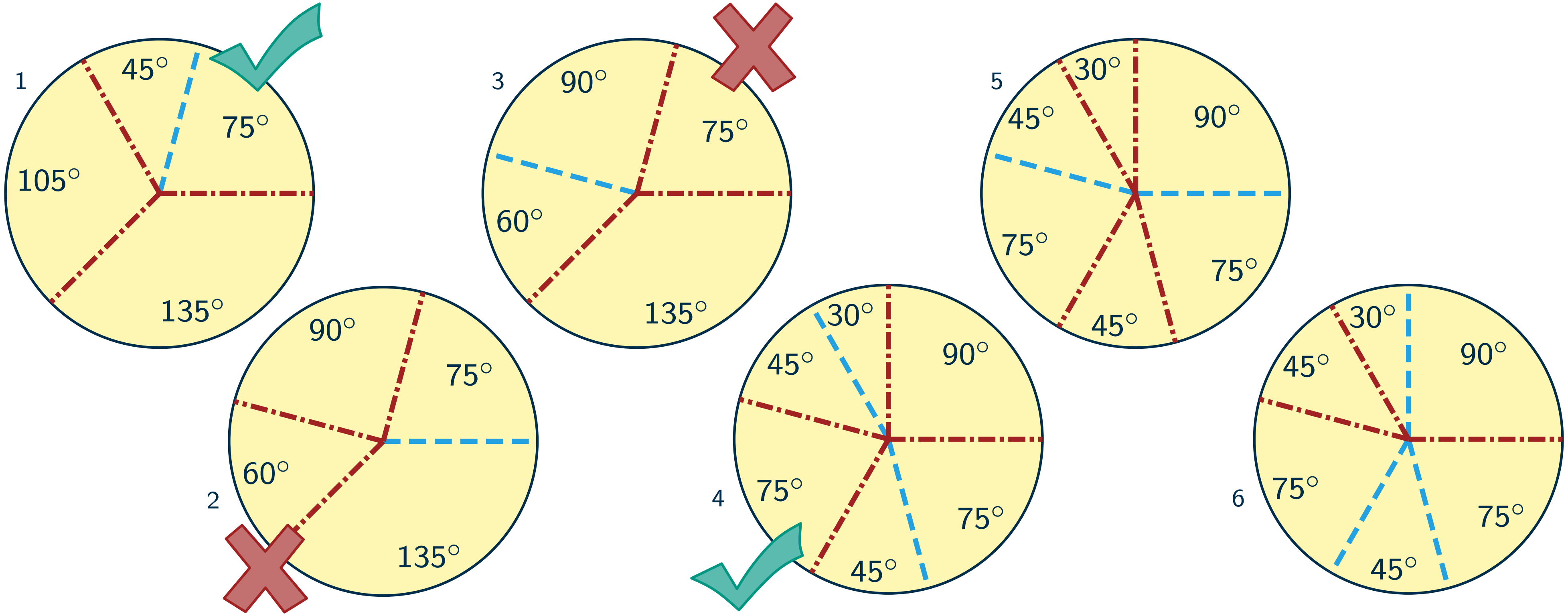

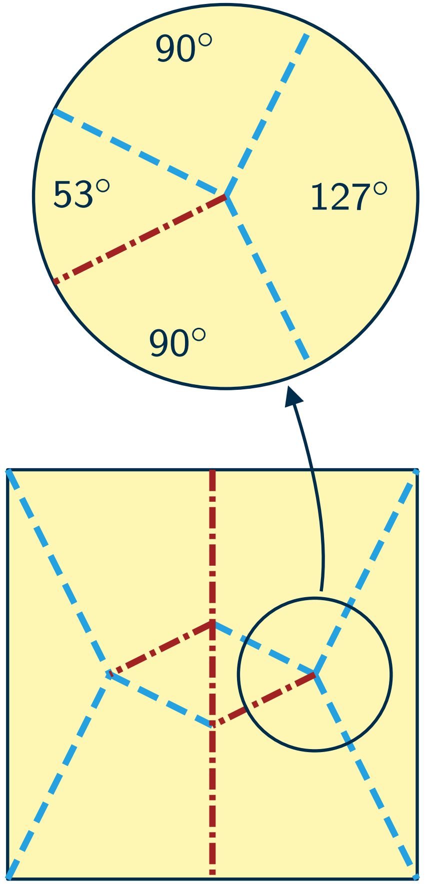

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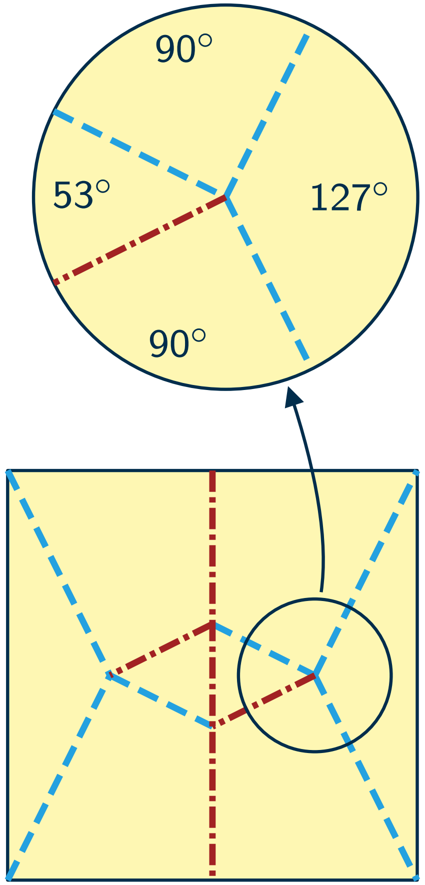

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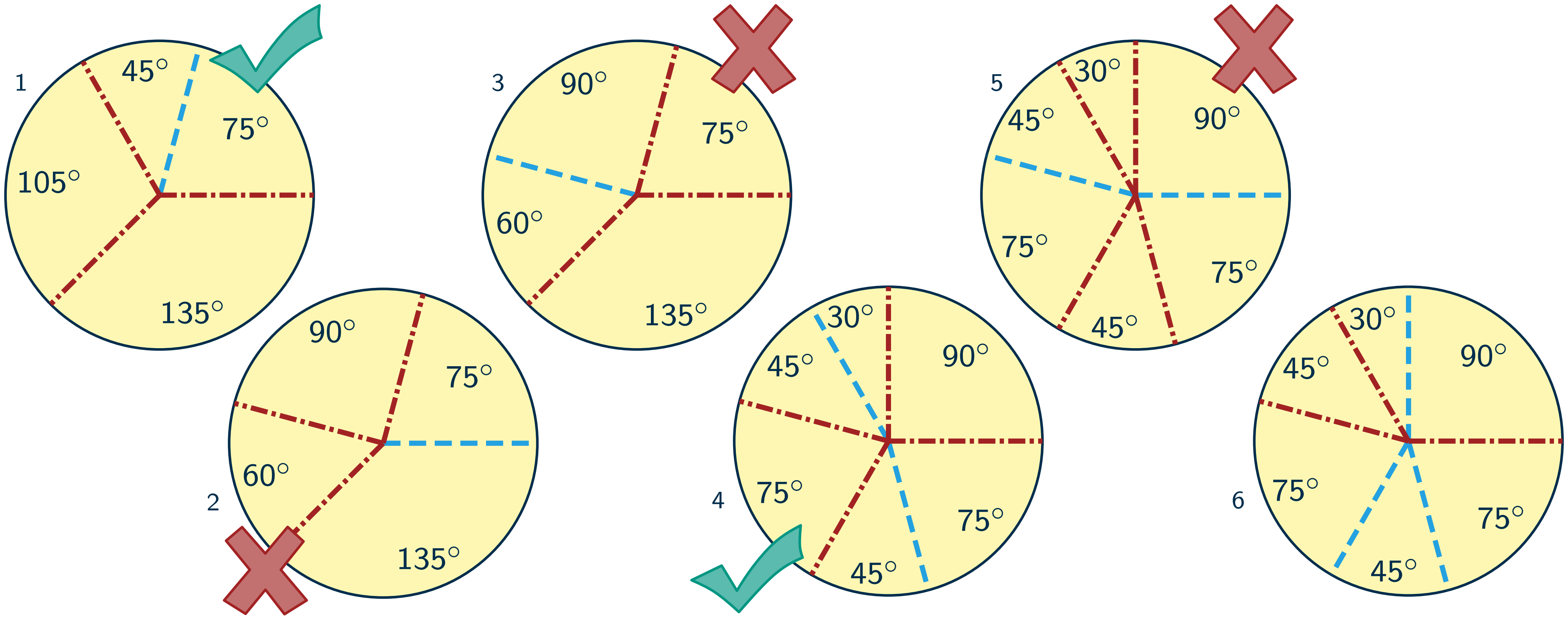

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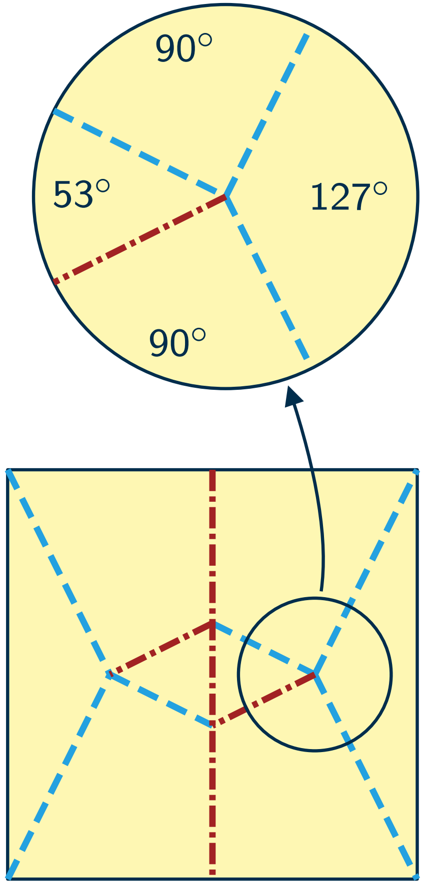

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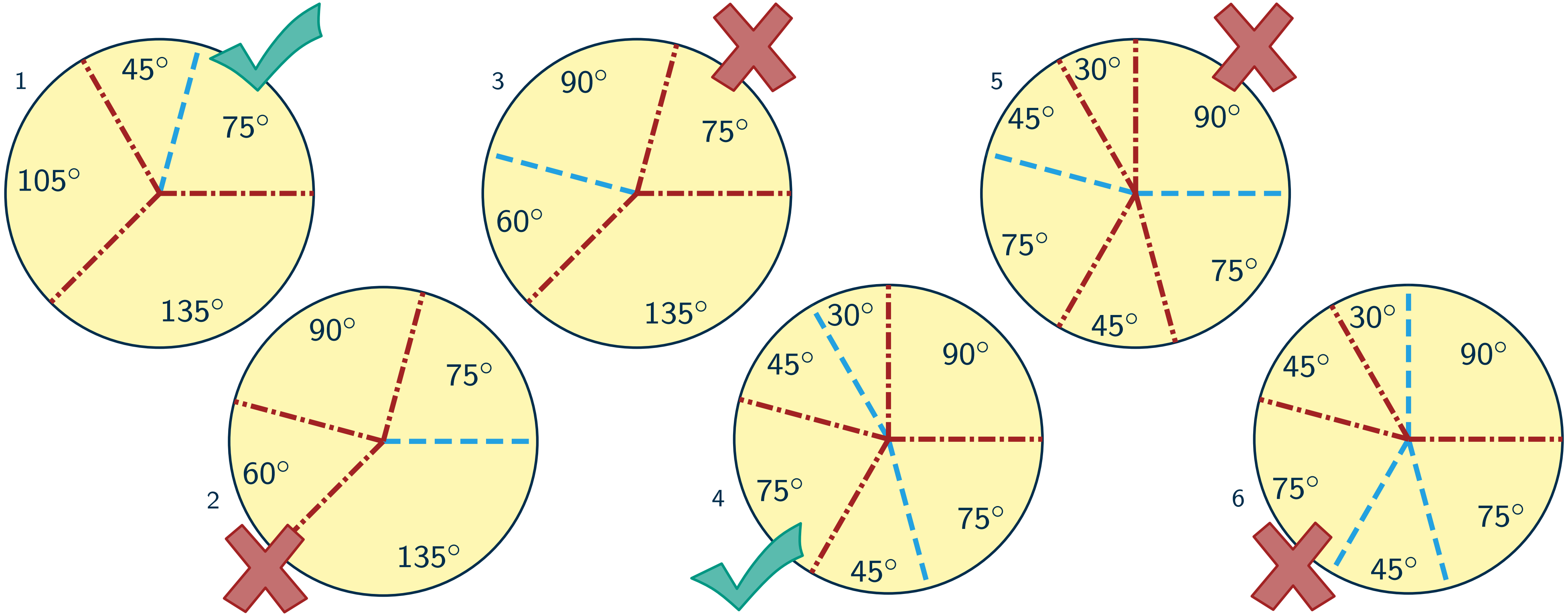

Crease Patterns With One Vertex

Observations

- similarity to 1D-case: sequence of angles $\Theta_1, \ldots, \Theta_n$
- now, our paper is a circle, instead of a line segment
- not every crease pattern is flat foldable (in 1D, zig-zag always works)

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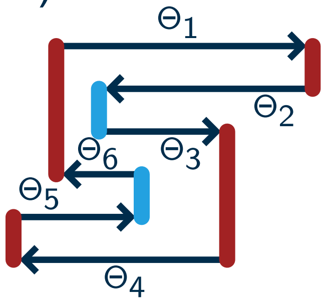

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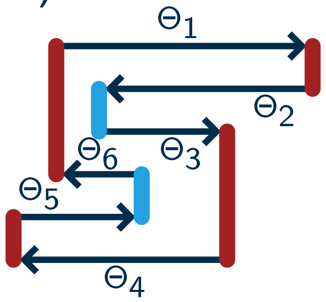

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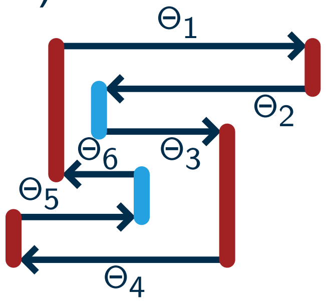

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- orientations of the circular arcs alternate $\Rightarrow$ n is even
- $\Theta_1 + \Theta_3 + \cdots + \Theta_{n-1} = \Theta_2 + \Theta_4 + \cdots + \Theta_n = 180°$

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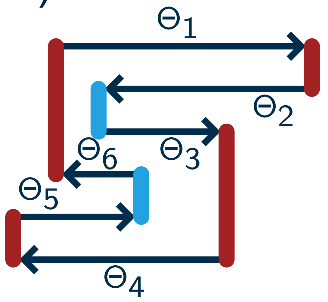

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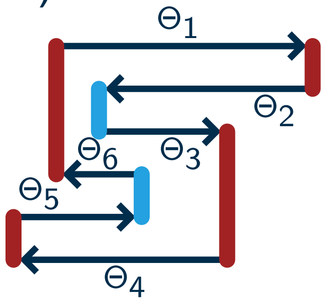

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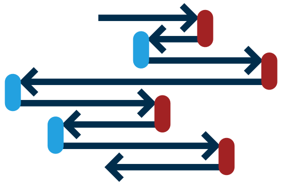

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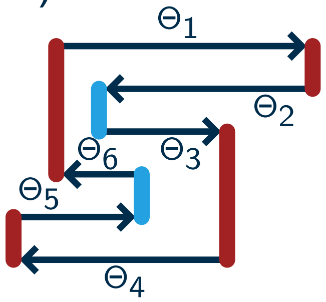

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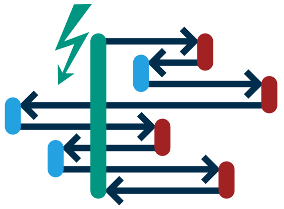

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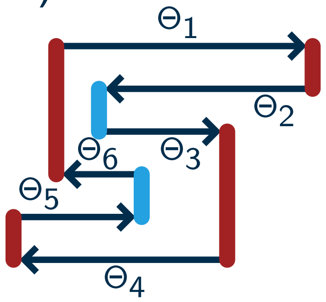

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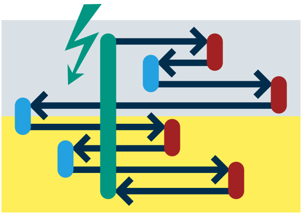
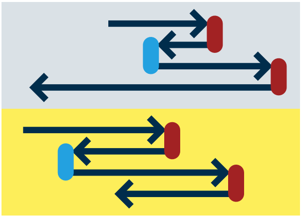
1. try zig-zag 

2. cut at left-most point 

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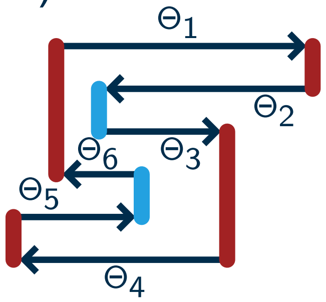

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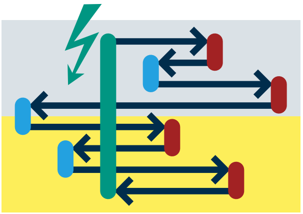

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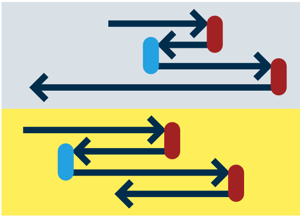

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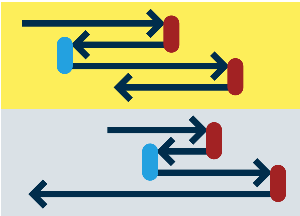

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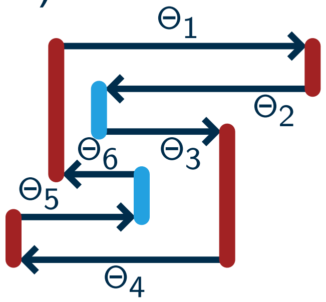

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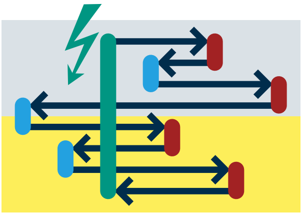

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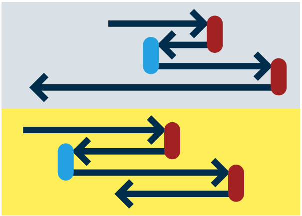

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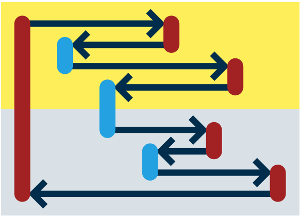

Mountain/Valley Patterns With Only One Vertex

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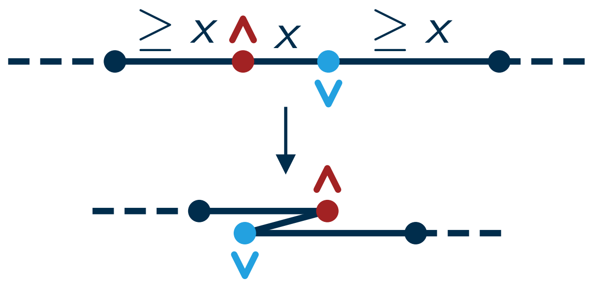

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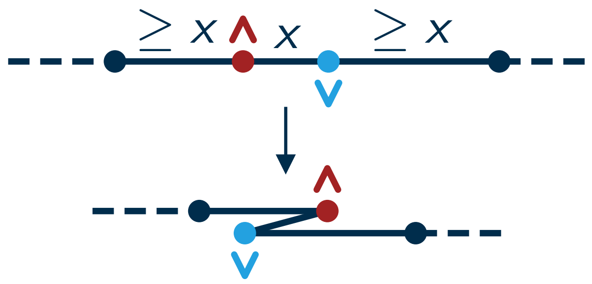

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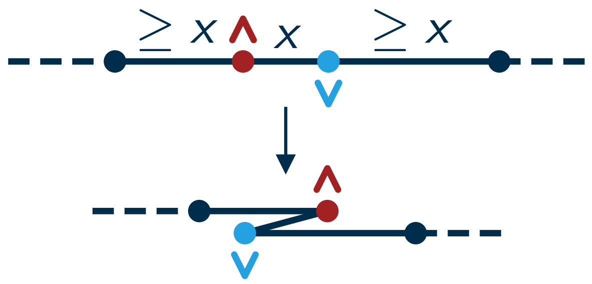

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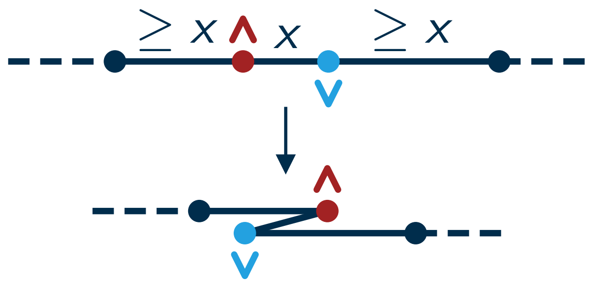

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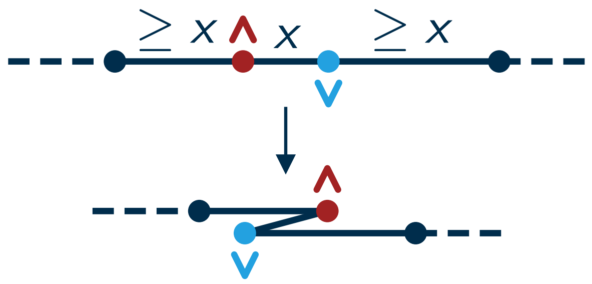

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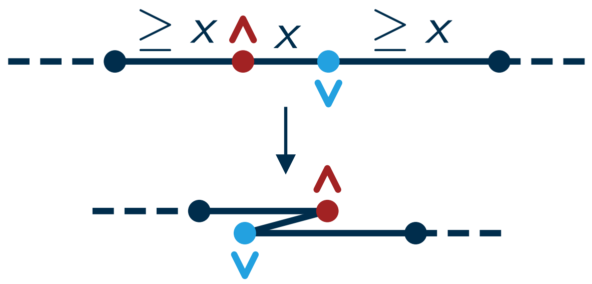

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Bonus Observation: flat foldable $\Rightarrow$ $\#$mountains $-$ $\#$valleys $= \pm 2$

Wrap-Up

Seen Today

- 1D crease pattern: always flat foldable (zig-zag)
- 1D mountain/valley pattern: flat foldable $\Rightarrow$ safe reduction rule crimp or end-fold applicable

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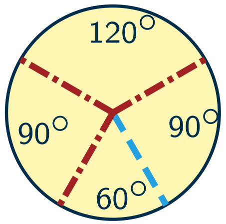
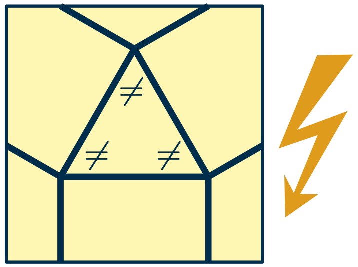
What Else Is There?

- check if a crease pattern is flat foldable: NP-hard
- check if mountain/valley-pattern is flat foldable: NP-hard
- local foldability: $O(n)$
  (find mountain/valley assignment, such that each vertex alone is flat foldable)





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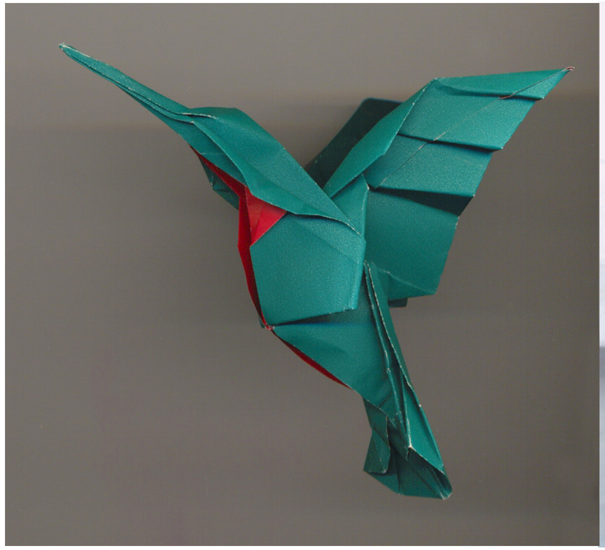

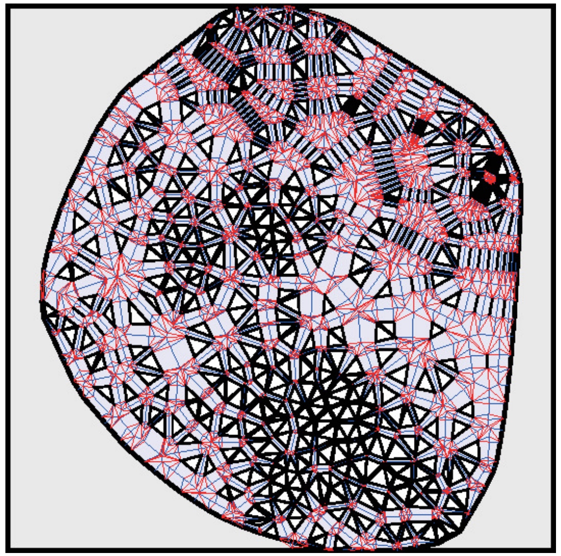

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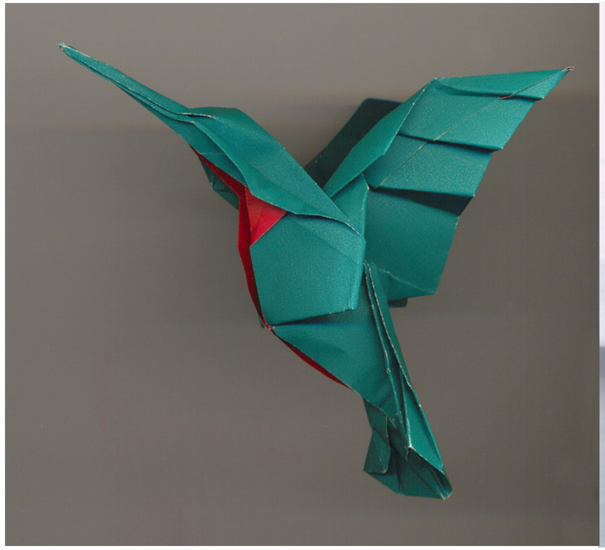

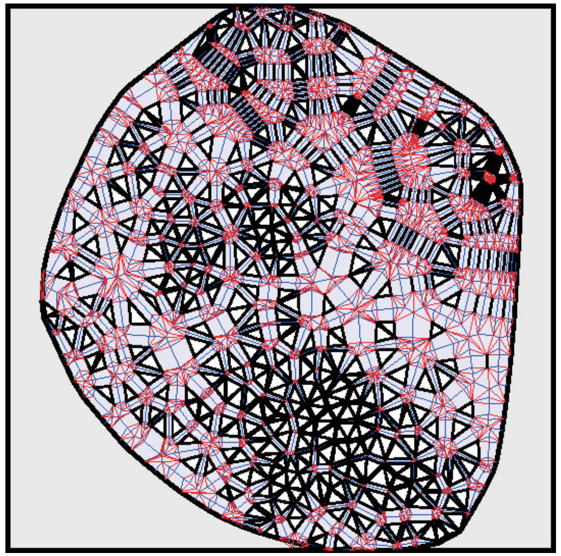

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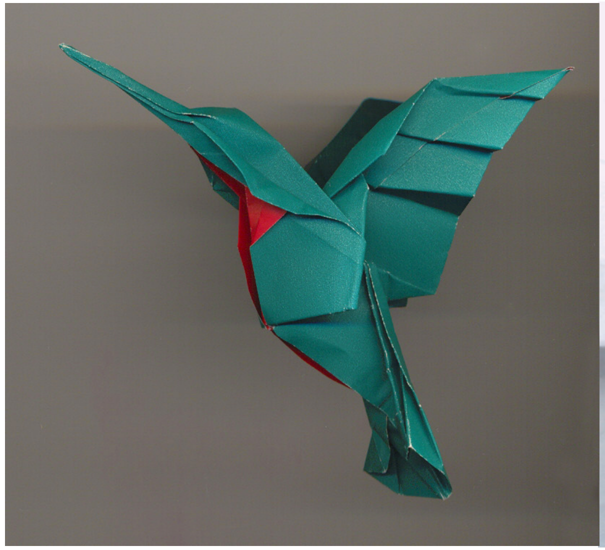

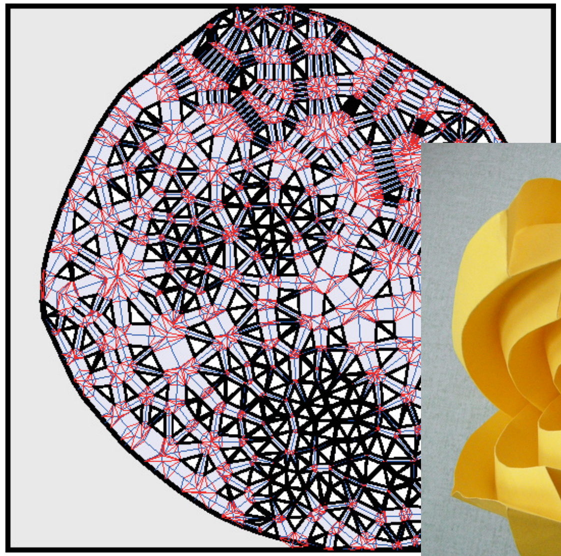

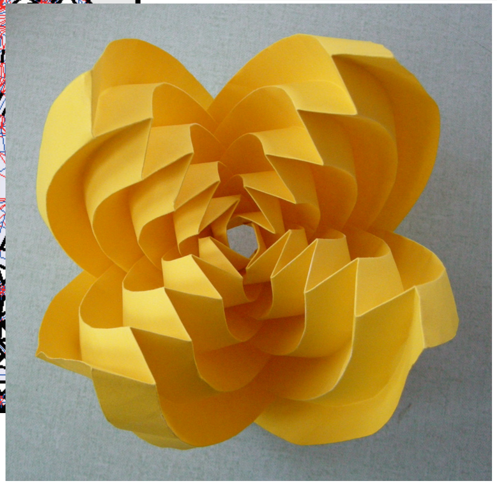

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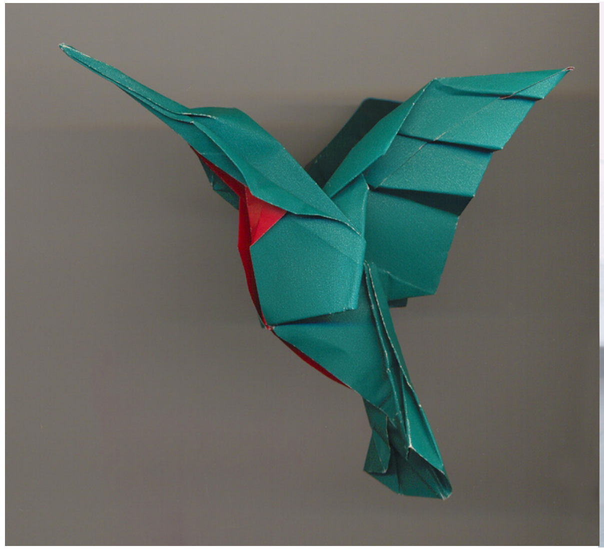

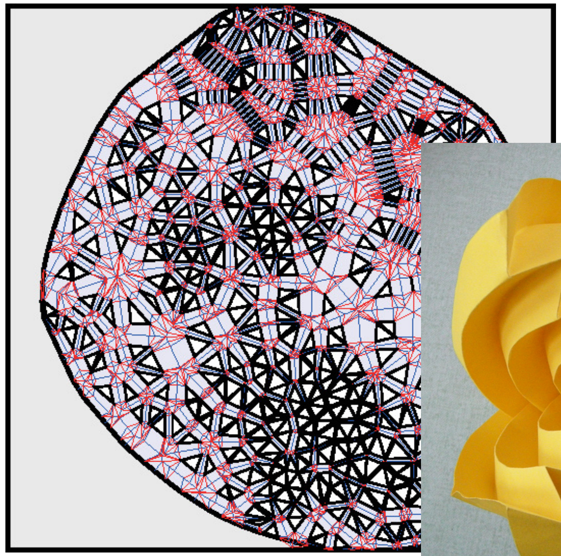

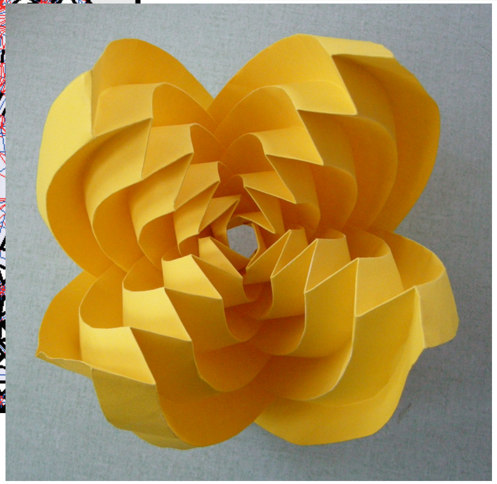

1D-Linkages

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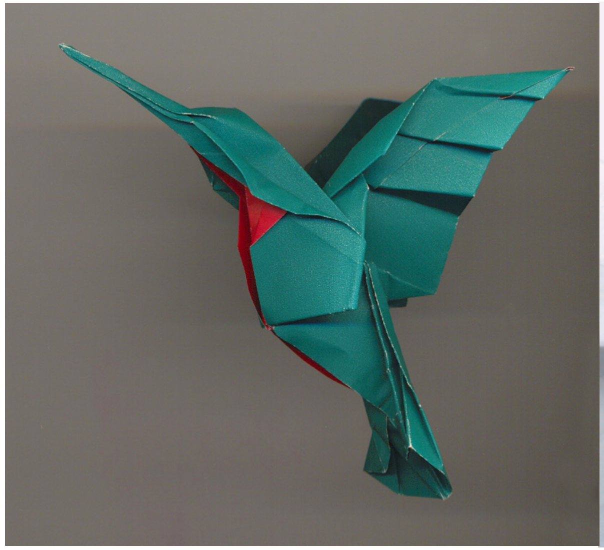
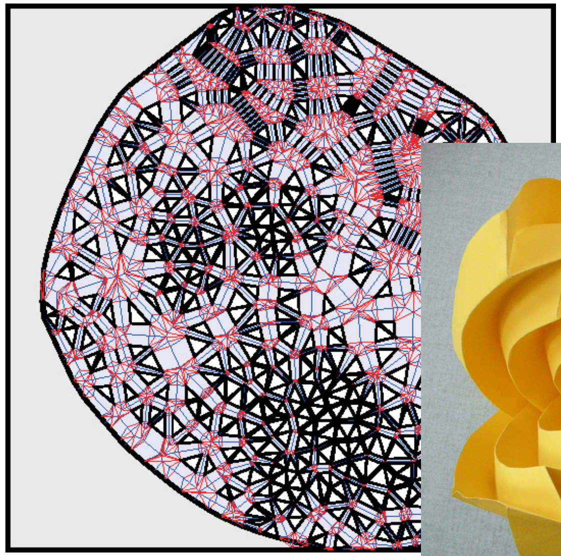
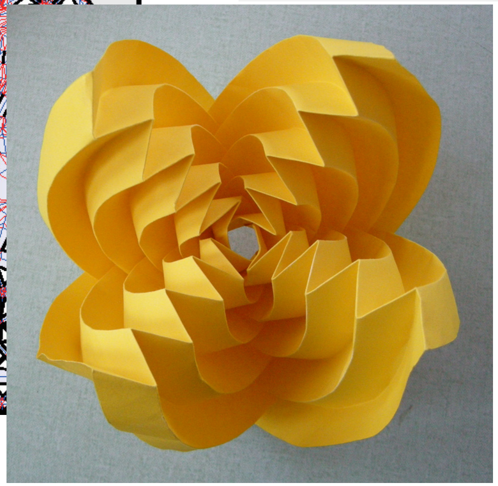
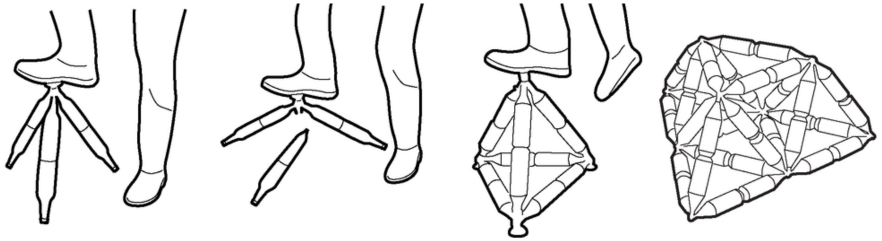






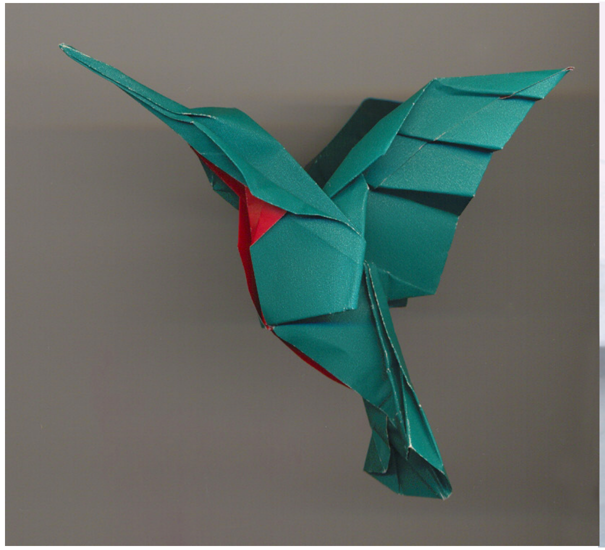
1D-Linkages

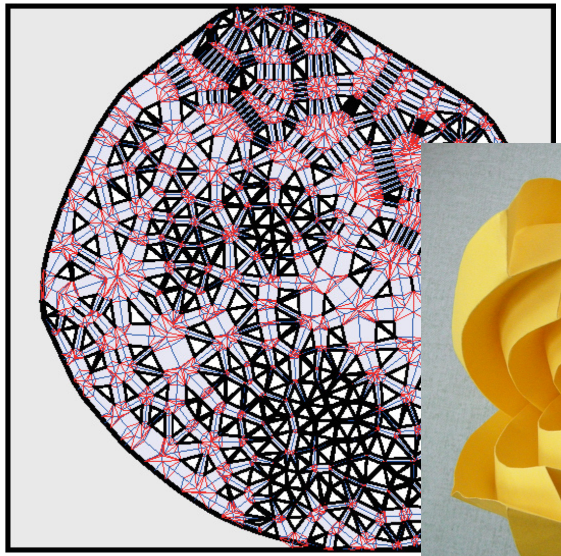


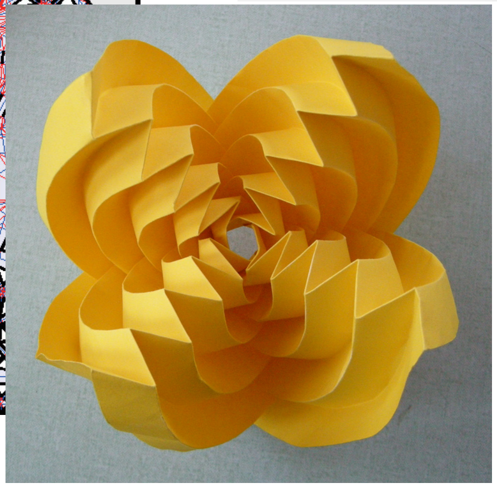
What Else Is There?

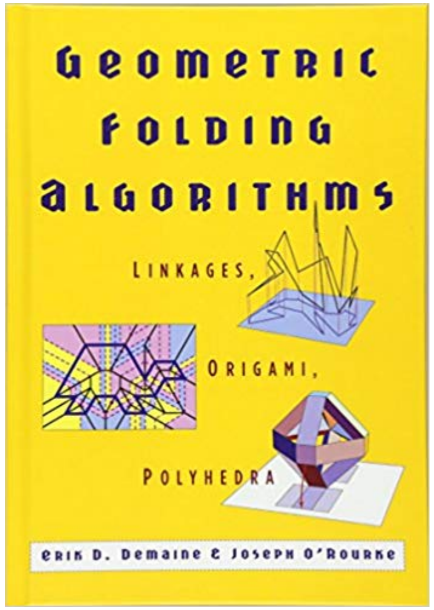
Origami

everything is foldable (for different definitions of “everything”)









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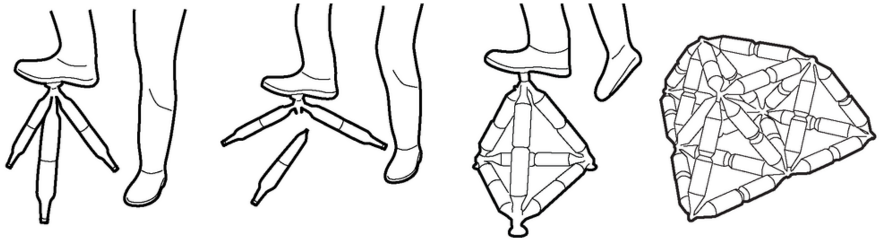

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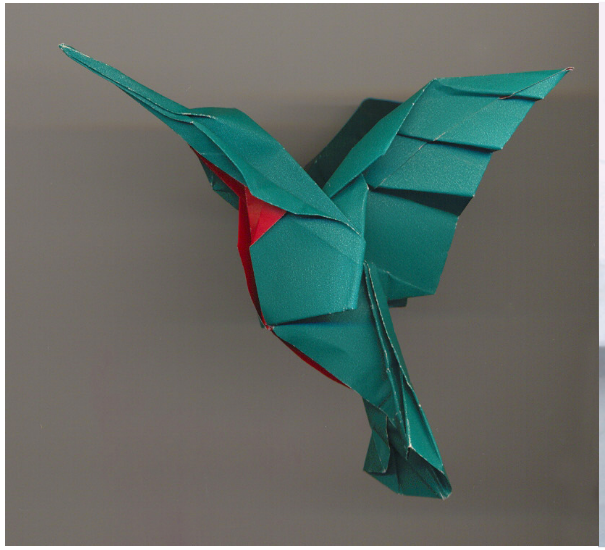
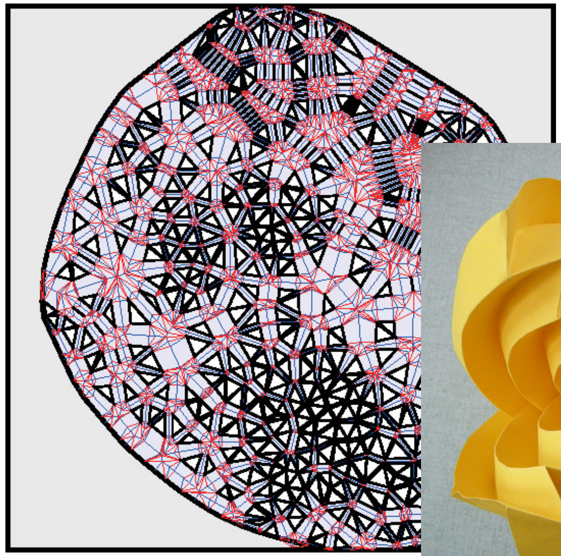
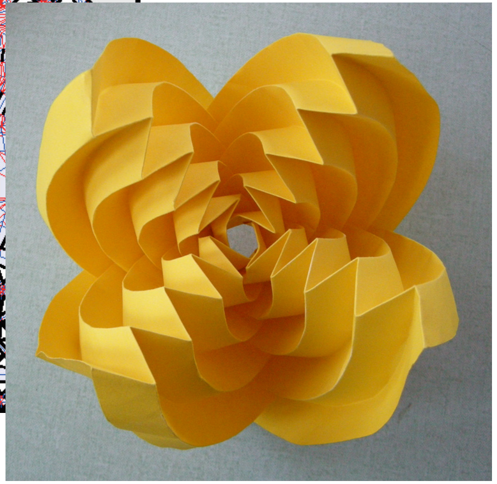
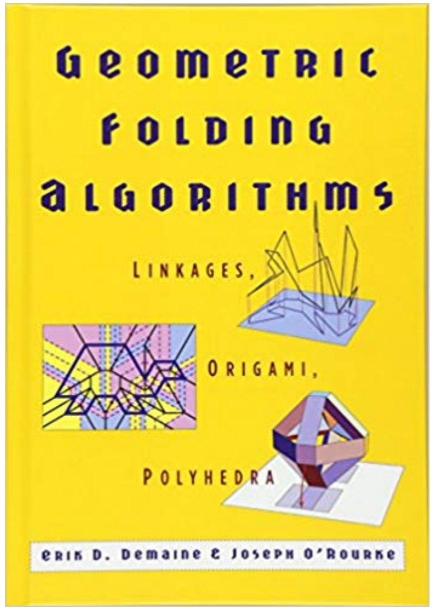
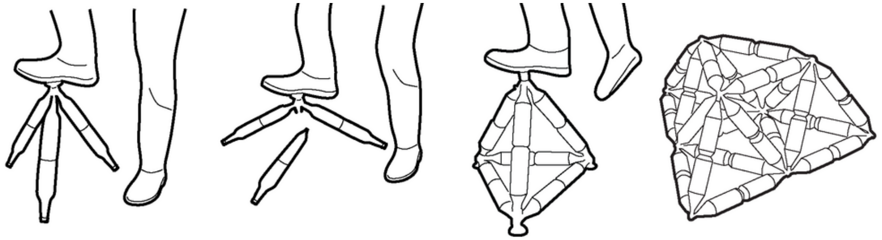








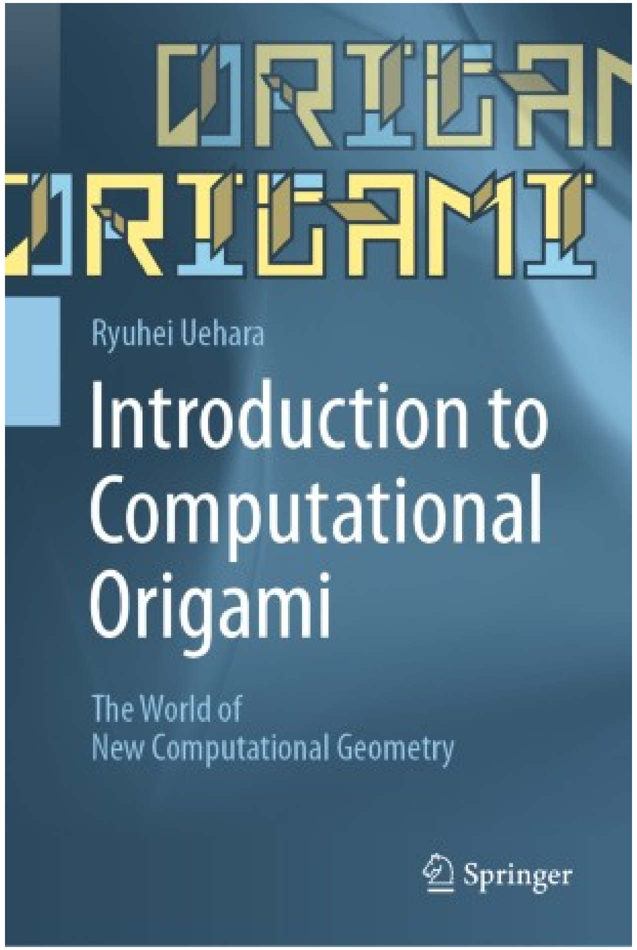
1D-Linkages



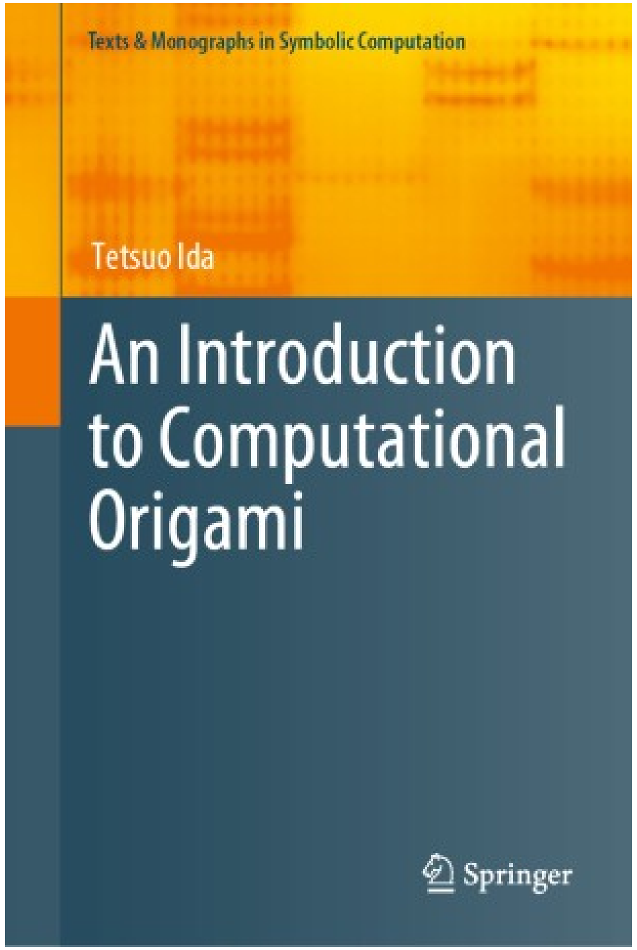
Fold-and-Cut → Thursday

bring scissors!

More Books



https://link.springer.com/book/10.1007/978-981-15-4470-5



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AI Generated Origami