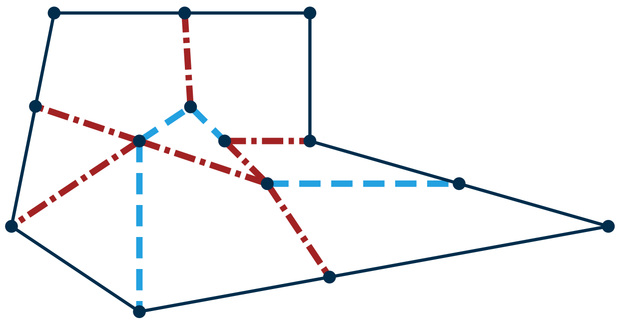
Computational Geometry

Computational Origami – Foldability

Thomas Bläsius

Flat-Foldable Crease Patterns



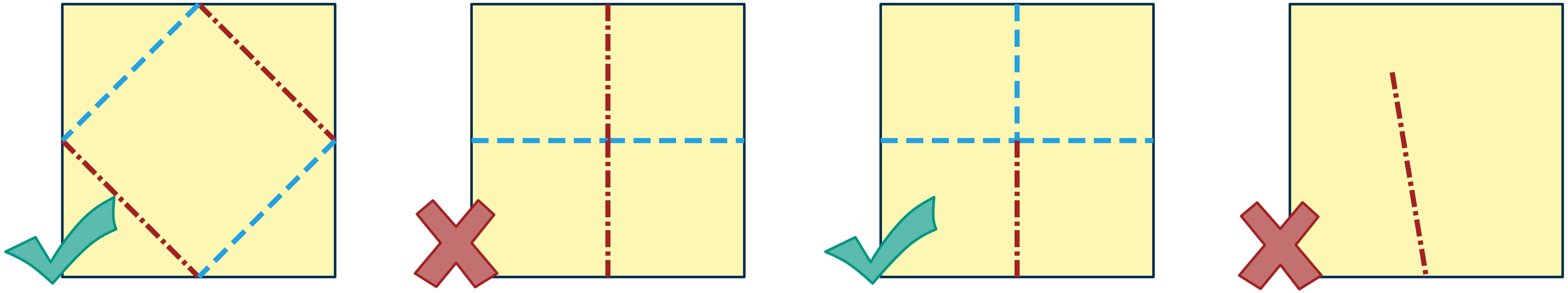
Given
- a geometric graph
- each inner edge is labeled either *mountain* or *valley*

Interpretation
- boundary of the outer face is a piece of paper (in the following usually a square)
- the geometric graph is a **crease pattern**
- folding direction is given via the **mountain/valley-assignment**

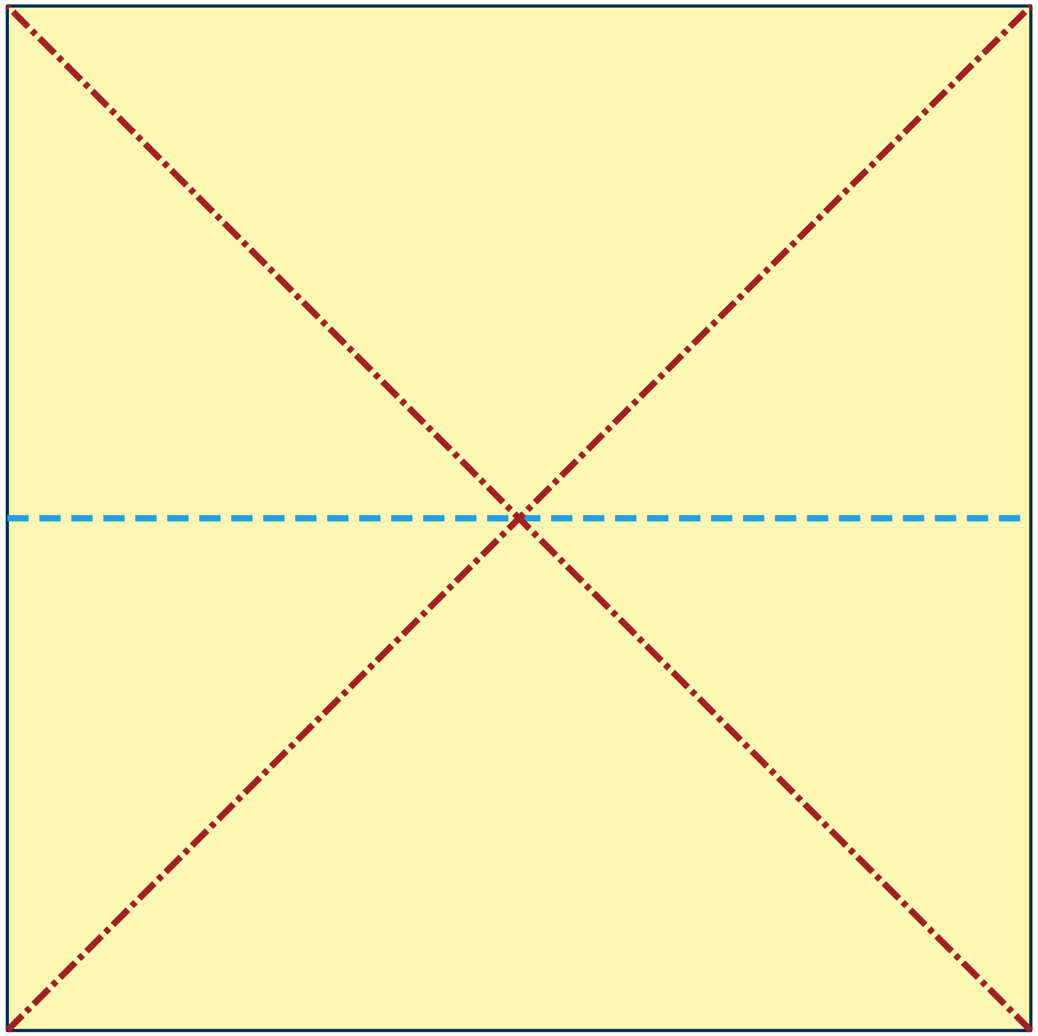
} **mountain/valley pattern**

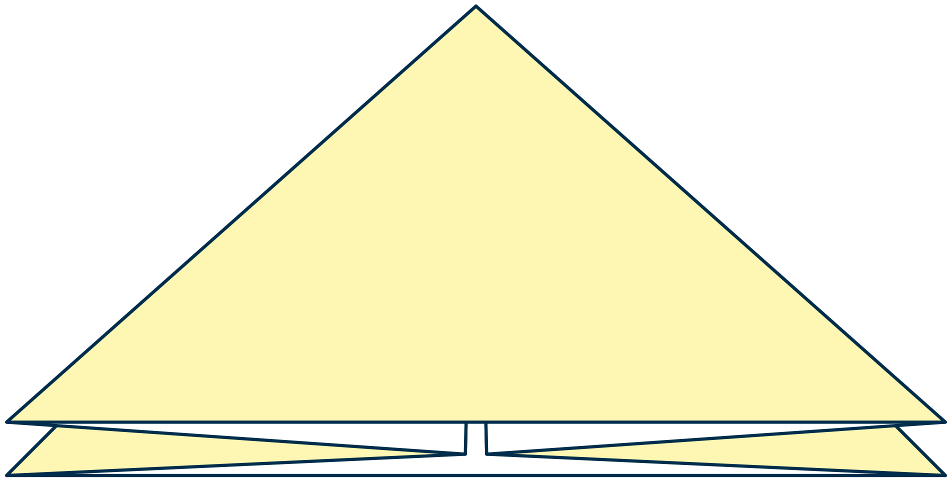
Problem: is the mountain/valley pattern flat foldable?

Examples:



Is The Mountain/Valley Pattern Flat Foldable?





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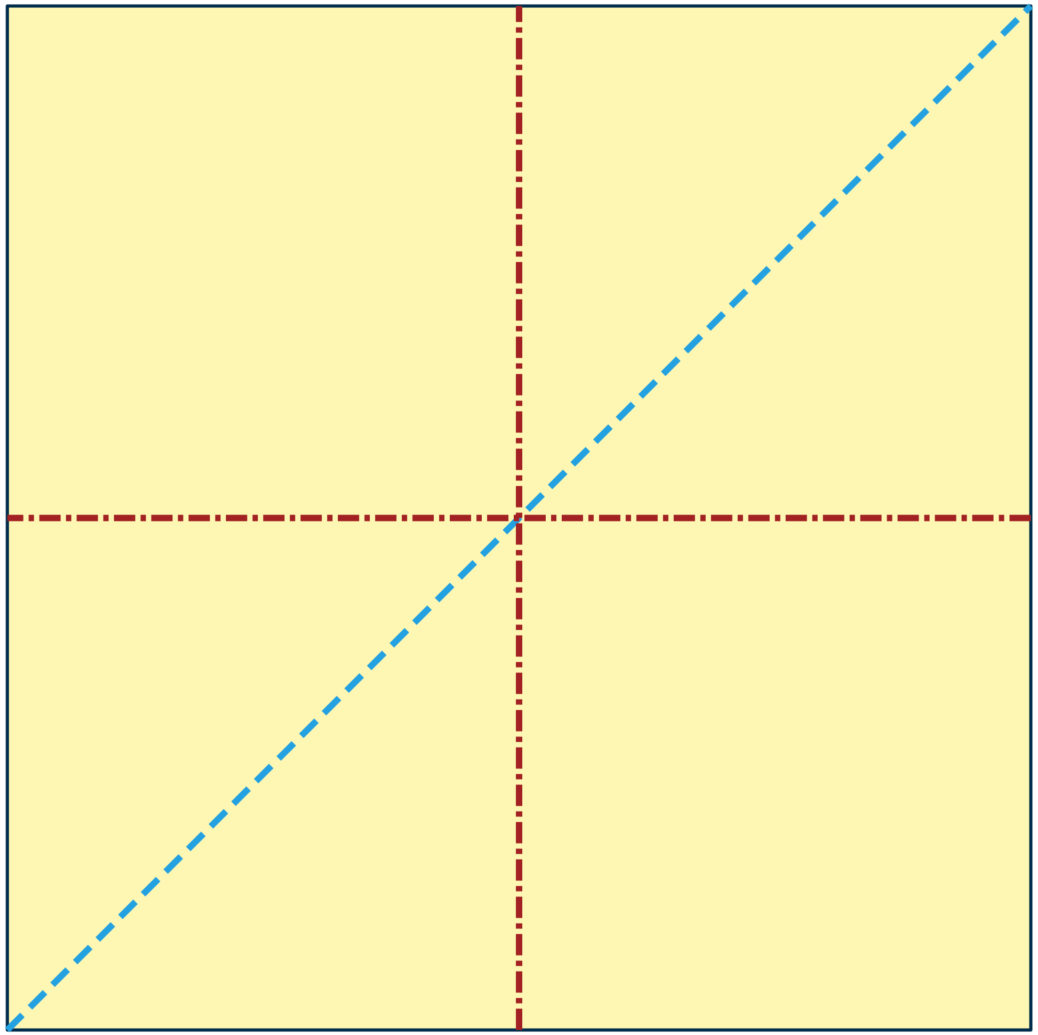

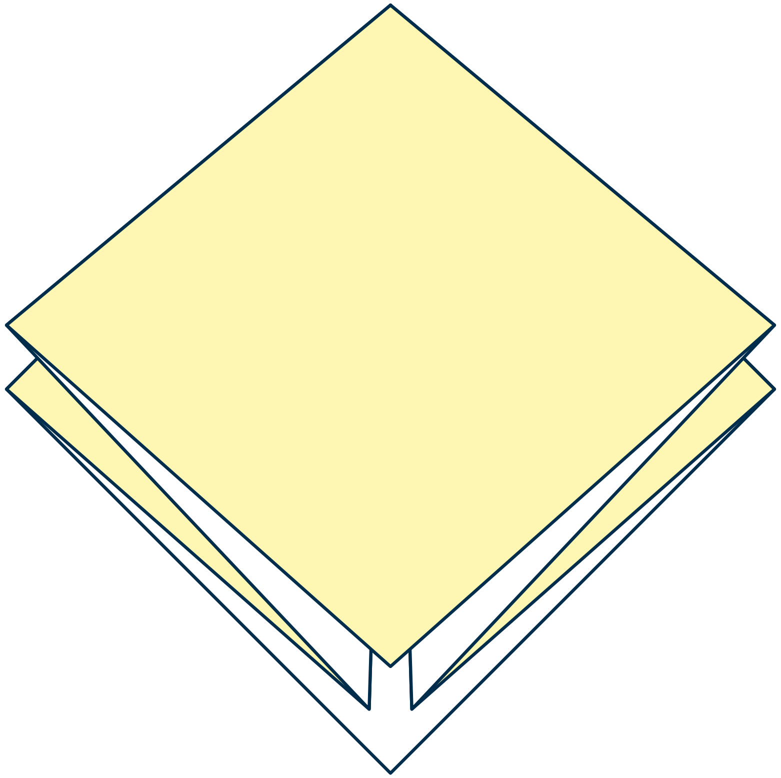

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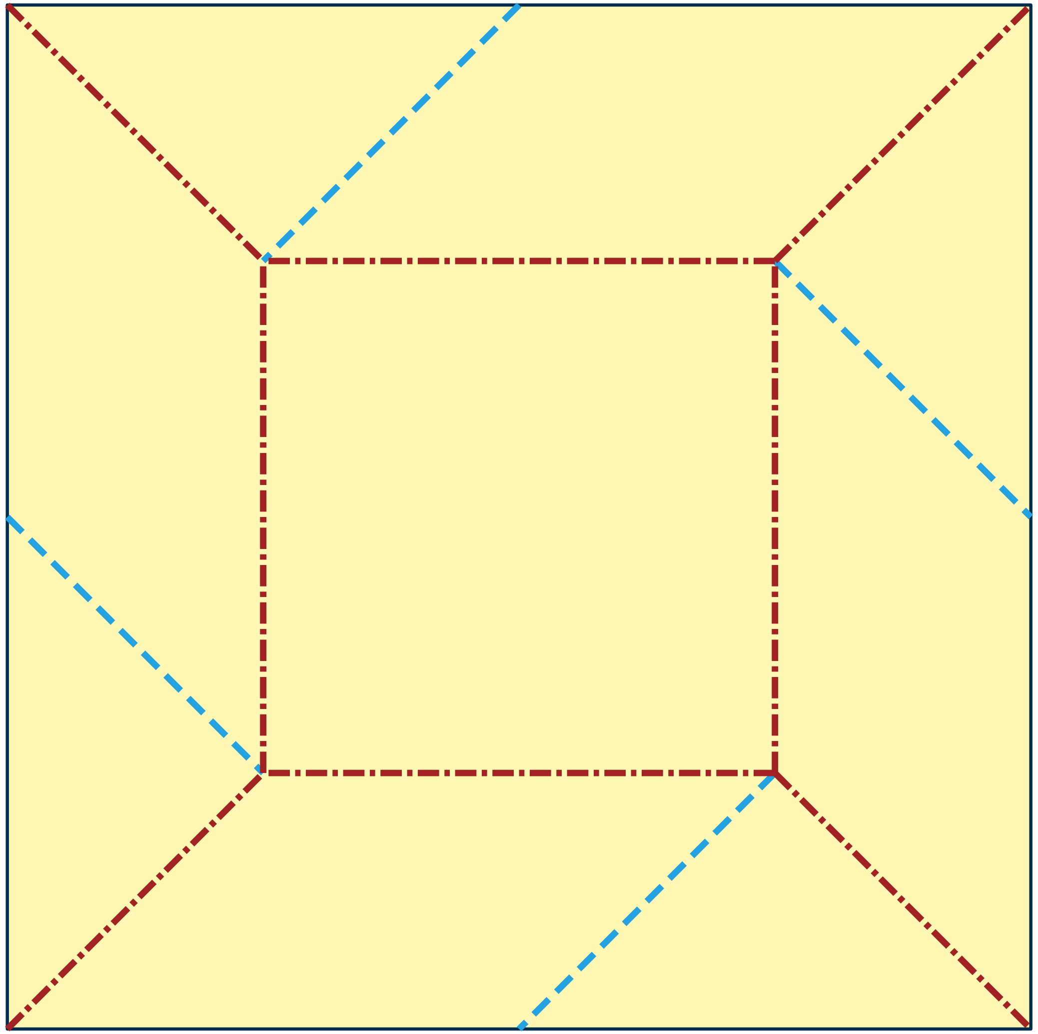

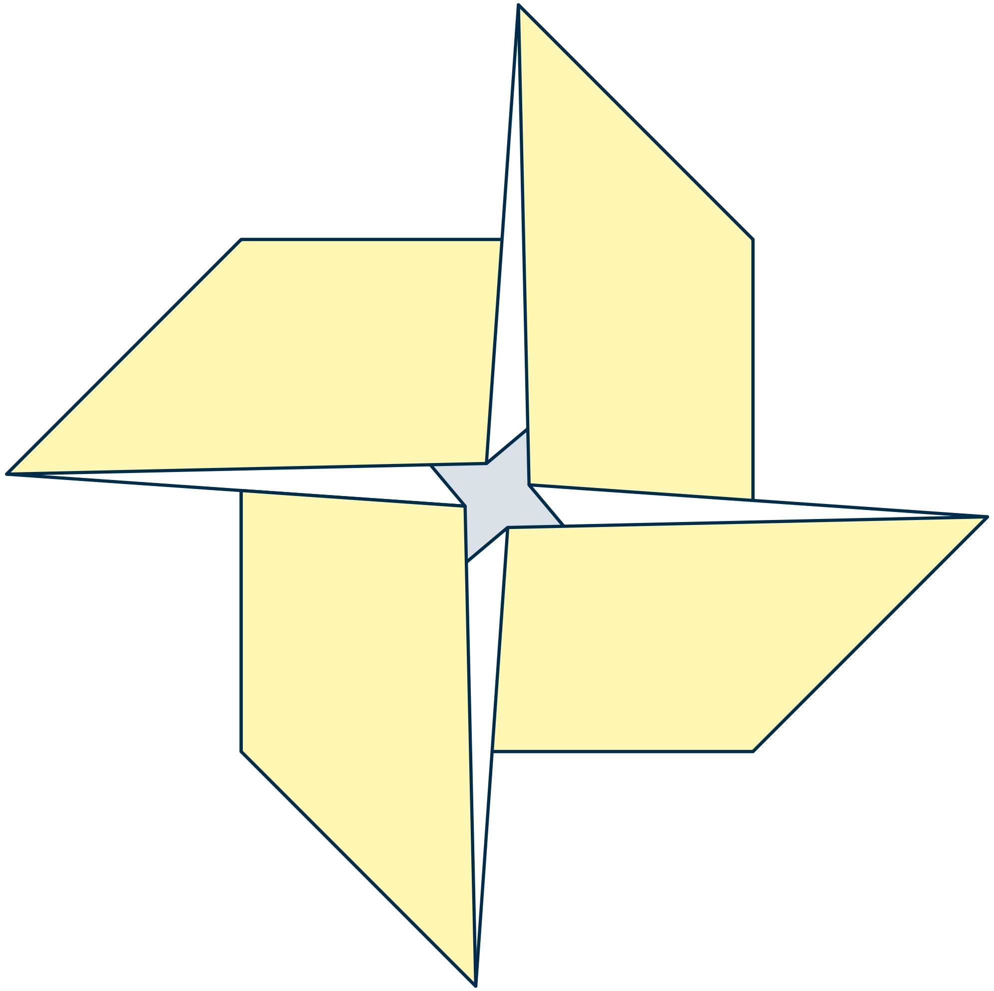

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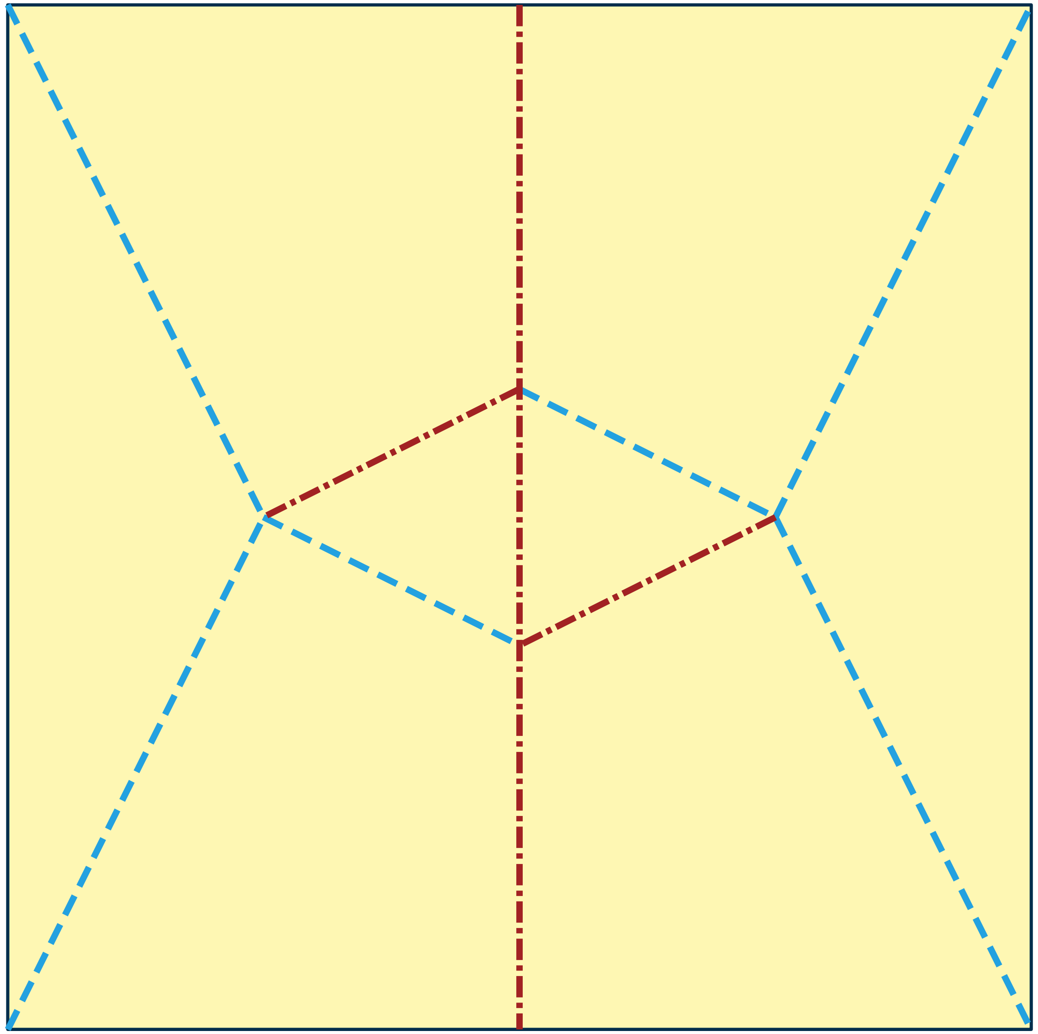

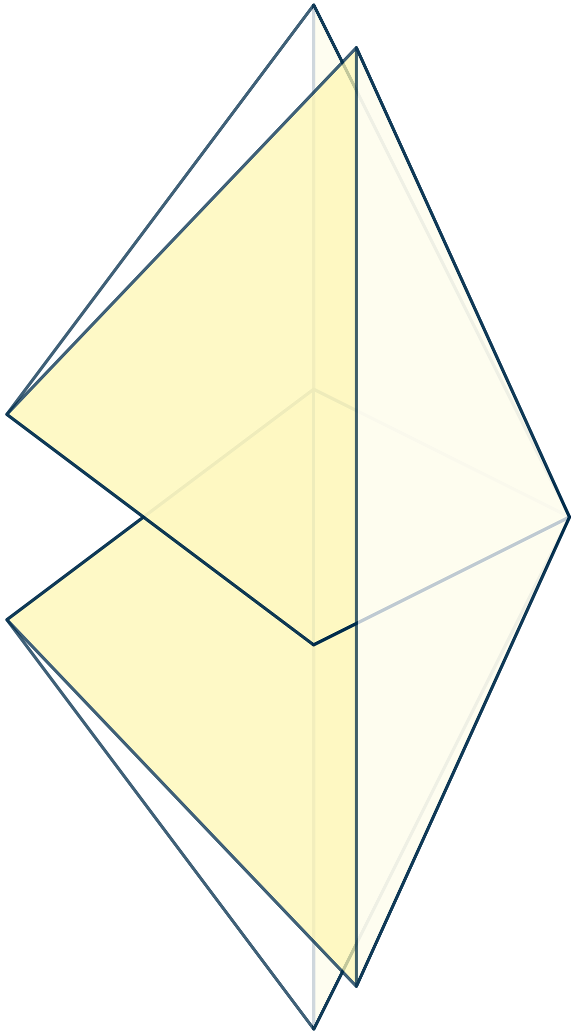

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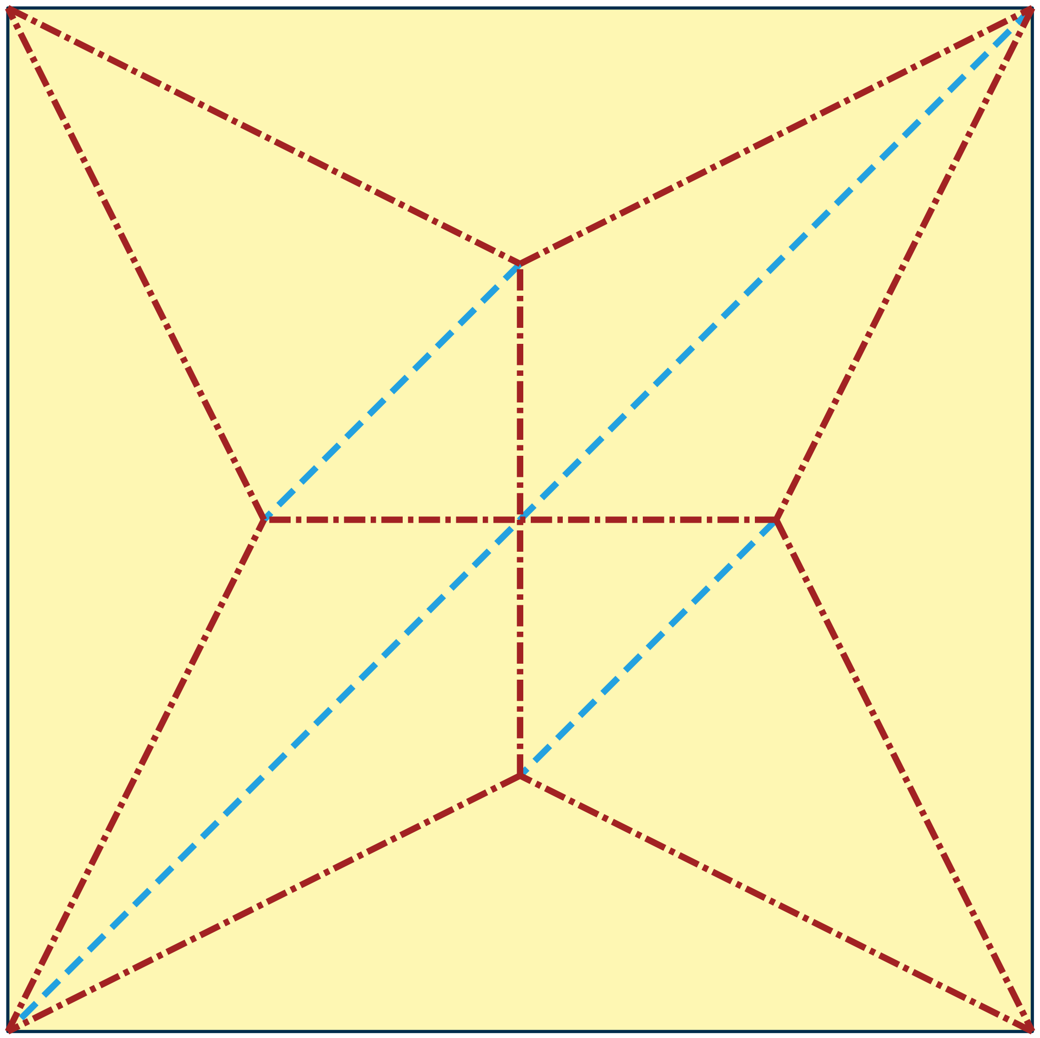
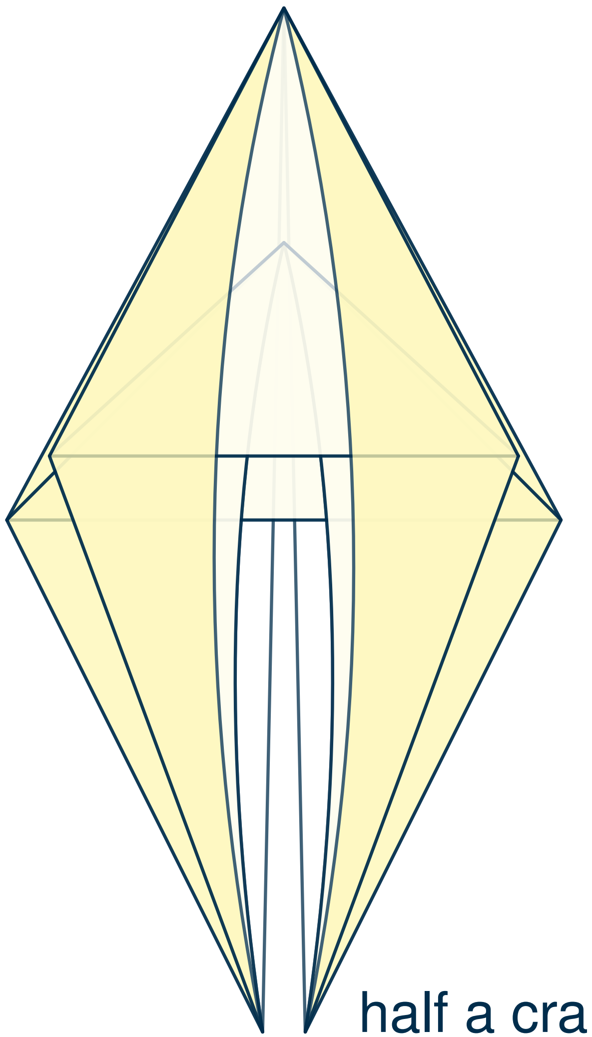




half a crane

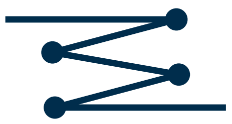
This Is All Way Too Complicated!

Two Problem Variants

- Given a mountain/valley pattern, is it flat foldable?
- Given a crease pattern, is there are mountain/valley assignment, such that it is flat foldable?

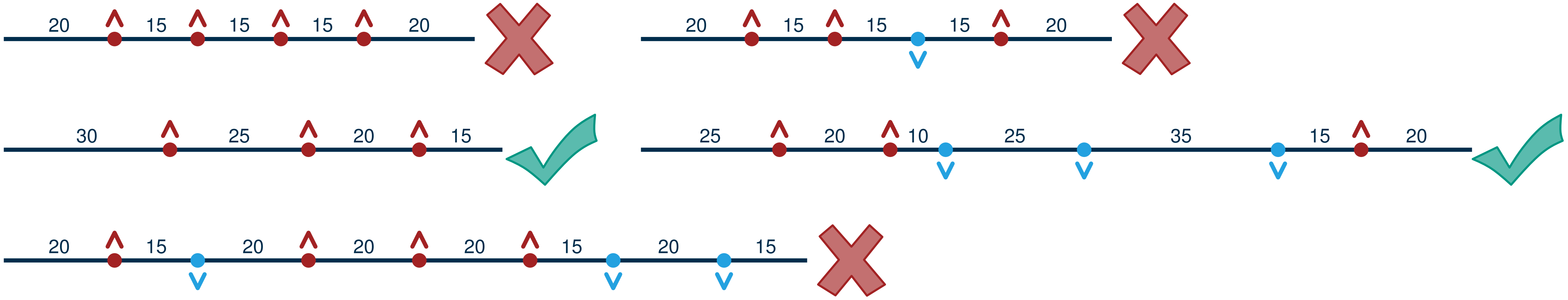
Simplifying A Bit: 1D-case $\rightarrow$ our “paper” is a line segment

1D Crease Pattern:



zig-zag always works

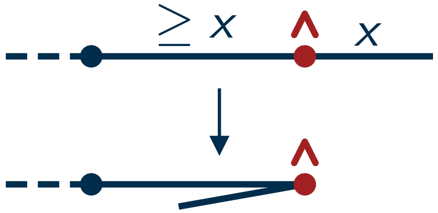
1D Mountain/Valley Pattern



Reduction Rules

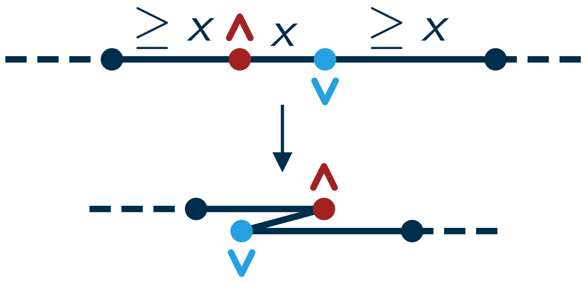
End-Fold

- condition: first/last piece smaller ($\leq$) than neighbor
- reduction: fold first/last vertex



Crimp

- condition: piece with mountain and valley vertex and larger neighbors
- reduction: fold both adjacent vertices



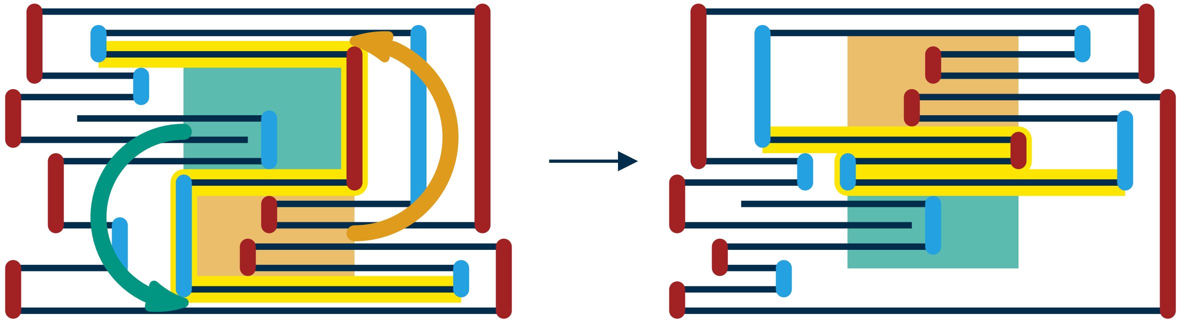
Lemma **(Safety first!)**

The reduction rules "end-fold" and "crimp" are safe.

(If a mountain/valley pattern is flat foldable, then it remains flat foldable after applying the reduction rules.)

Proof

- **End-Fold:** obvious
- **Crimp:** proof by picture

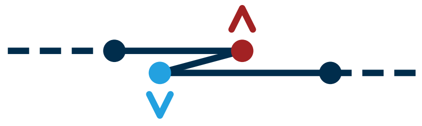


Reductions For The Win

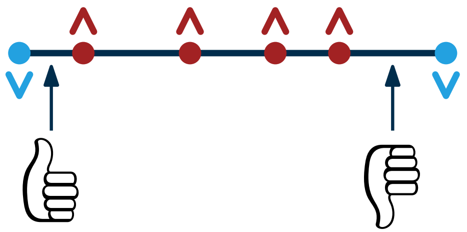
Theorem **(foldable ⇒ reduction)**
If the mountain/valley pattern is flat foldable, then there is an end-fold or a crimp.

Proof Plan

crimp:



- short pieces with one mountain and one valley vertex are good
- consider maximal sequence (wrt inclusion) with only mountain or only valley vertices
- left-short: first piece is shorter than second
- right-short: last piece shorter than second-to-last



Lemma **(foldable ⇒ left/right-short)**
flat foldable ⇒ each maximal sequence is left-short or right-short

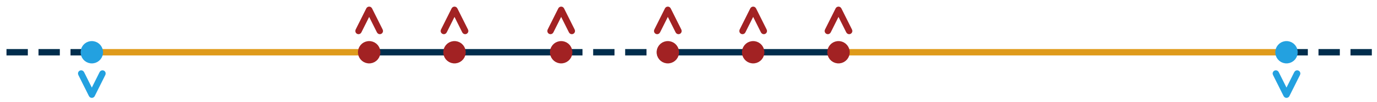
Lemma **(left/right-short ⇒ reduction)**
each maximal sequence is left-short or right-short ⇒ there is an end-fold or a crimp

Short Pieces Everywhere

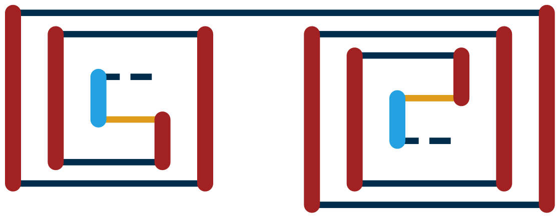
Lemma **(foldable ⇒ left/right-short)**
flat foldable ⇒ each maximal sequence is left-short or right-short

Proof

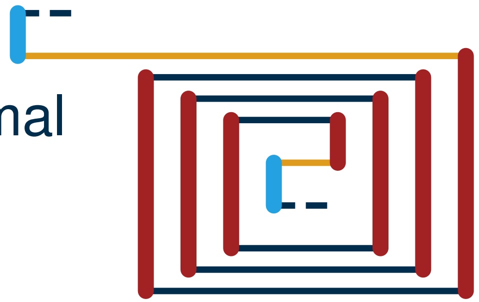
- consider maximal sequence that is neither left-short nor right-short



- we cannot escape from the spiral



- being one of left-short or right-short would be enough (for the maximal sequence in isolation)

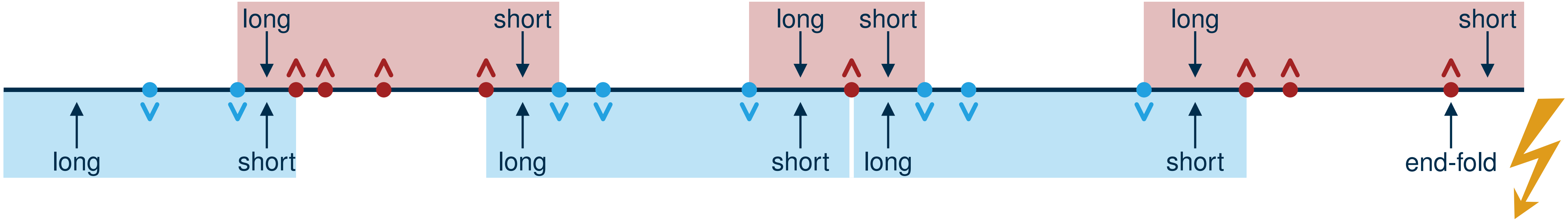


There Is Always One More Reduction

Lemma **(left/right-short ⇒ reduction)**
each maximal sequence is left-short or right-short ⇒ there is an end-fold or a crimp

Proof

- assume for contradiction: no end fold and no crimp
- no end-fold ⇒ first maximal sequence not left-short ⇒ right-short
- no crimp ⇒ second maximal sequence not left-short ⇒ right-short
- iterate argument ⇒ every maximal sequence is right-short
- but: this gives an end-fold for the last segment



Wrap-Up: 1D Origami

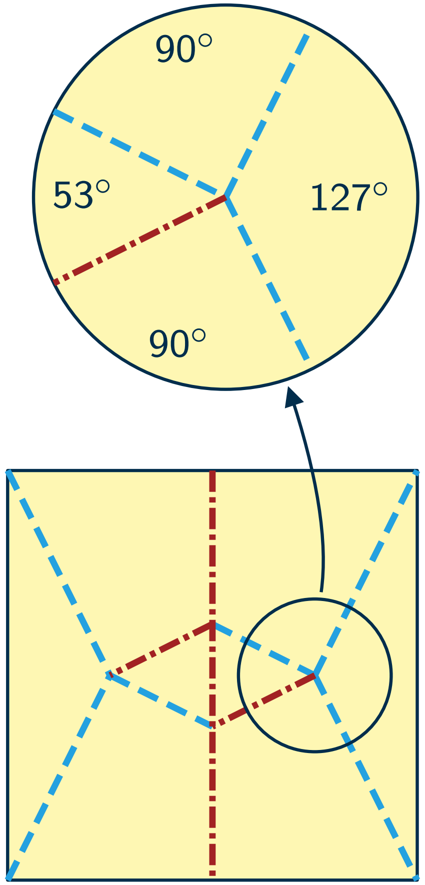
Lemma **(Safety first!)**
The reduction rules “end-fold” and “crimp” are safe.
(If a mountain/valley pattern is flat foldable, then it remains flat foldable after applying the reduction rules.)

Theorem **(foldable $\Rightarrow$ reduction)**
If the mountain/valley pattern is flat foldable, then there is an end-fold or a crimp.

Algorithm For Recognizing Flat Foldable 1D Mountain/Valley Patterns

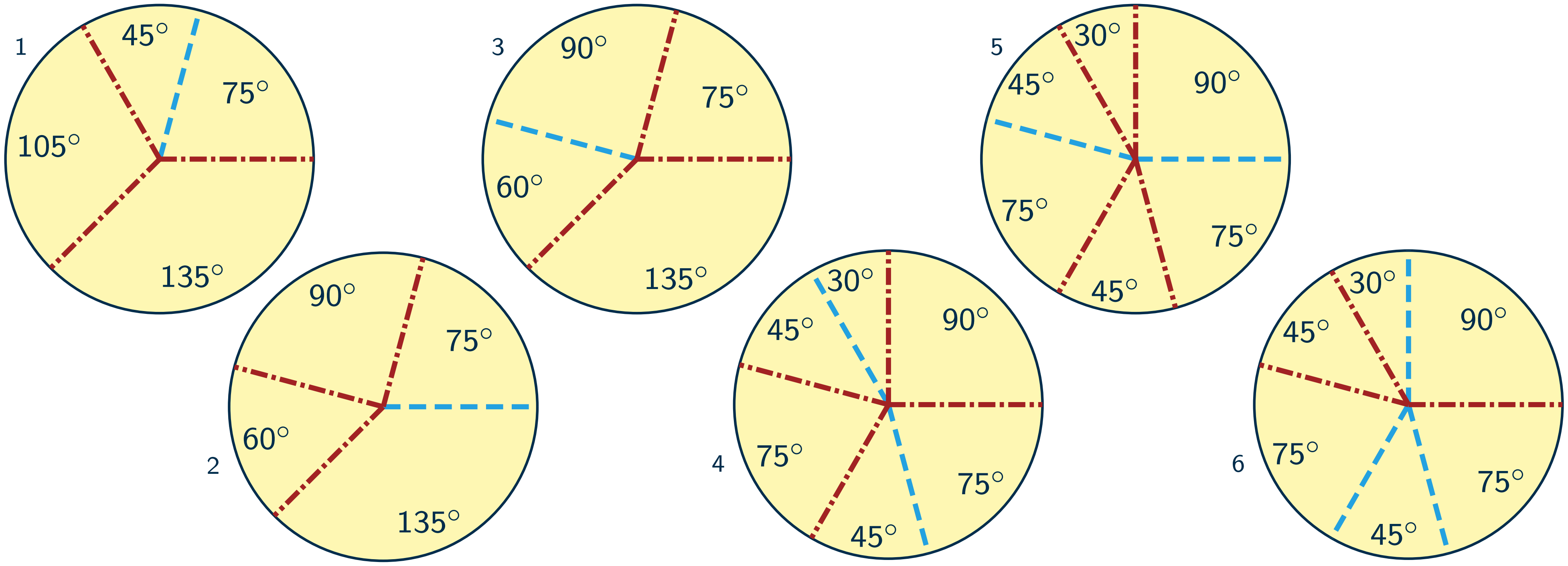
- while there is an end-fold or a crimp, apply an end-fold or a crimp
- result is a flat folding $\Rightarrow$ flat foldable
- result is not a flat folding $\Rightarrow$ not flat foldable
- running time: $O(n) \rightarrow$ exercise

2D Origami (With Only One Vertex)



Necessary Condition

- mountain/valley pattern foldable ⇒ each vertex locally foldable
- Which 1-vertex mountain/valley patterns are flat foldable?



2D Origami (With Only One Vertex)

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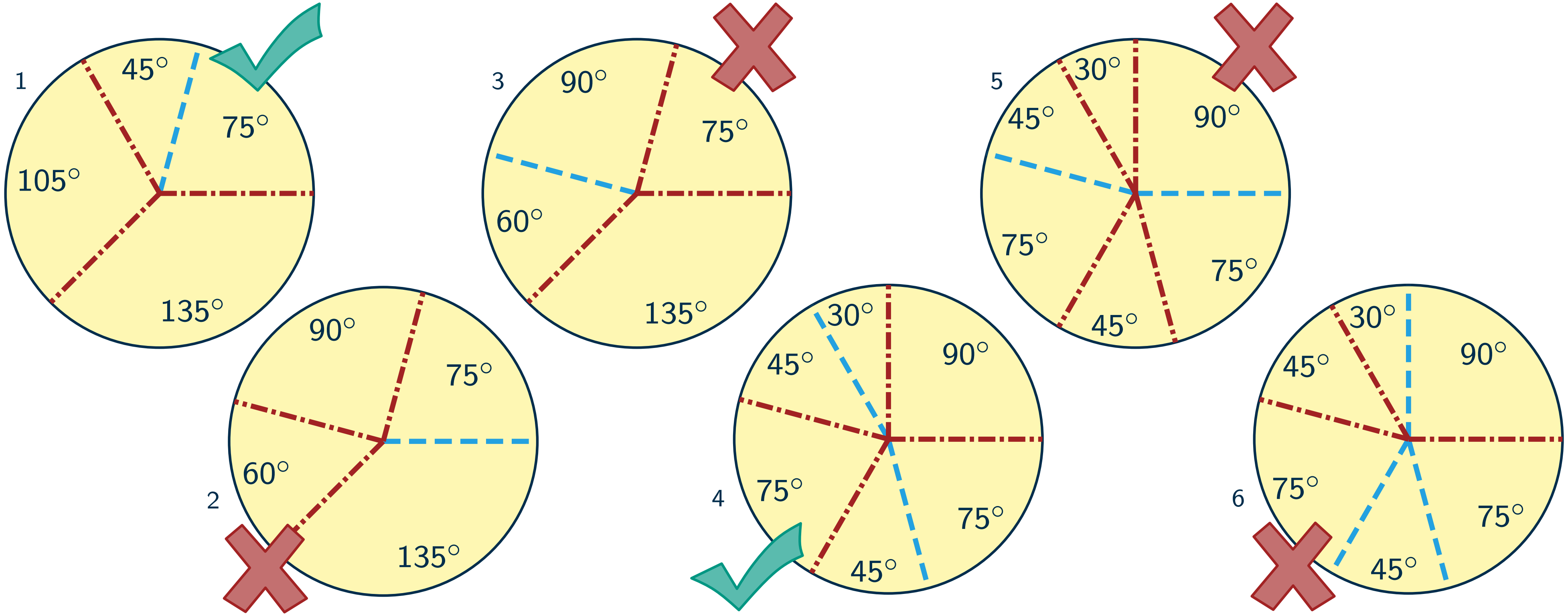
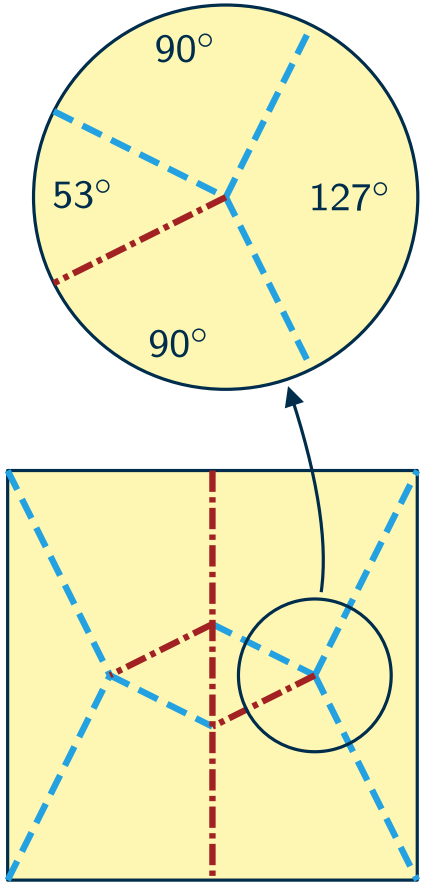




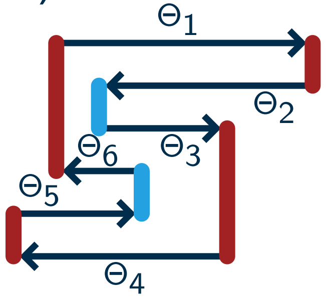
Crease Patterns With One Vertex

Observations

- similarity to 1D-case: sequence of angles $\Theta_1, \ldots, \Theta_n$
- now, our paper is a circle, instead of a line segment
- not every crease pattern is flat foldable (in 1D, zig-zag always works)

Necessary Condition For The Crease Pattern

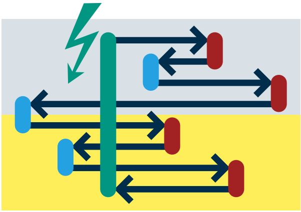
- orientations of the circular arcs alternate $\Rightarrow$ n is even
- $\Theta_1 + \Theta_3 + \cdots + \Theta_{n-1} = \Theta_2 + \Theta_4 + \cdots + \Theta_n = 180^\circ$



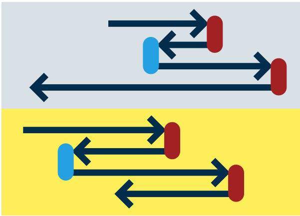
Theorem
The crease pattern is flat-foldable if and only if the sum of even and odd angles is 180° each.

Proof

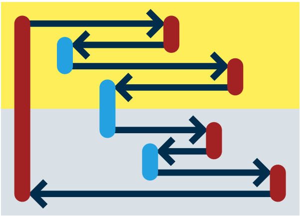
1. try zig-zag



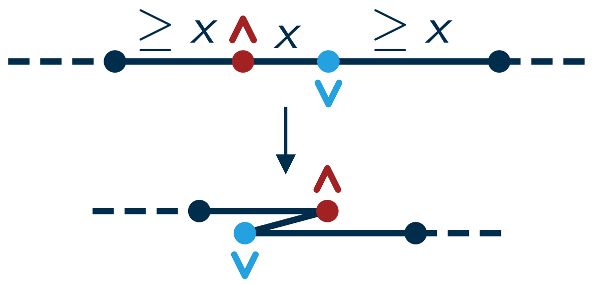
2. cut at left-most point



3. swap order
4. reconnect



Mountain/Valley Patterns With Only One Vertex



Safe Reduction Rule
- define crimp as in the 1D-case
- same argument as before: reduction rule is safe
- end-folds obviously no longer exist

Flat Foldable $\Rightarrow$ Crimp
- is still true, except if there are only two angles of equal size (in which case we are done)

Theorem
If a mountain/valley pattern with one vertex is flat foldable, then there is a crimp and every sequence of crimps yields a flat folding.

Bonus Observation: flat foldable $\Rightarrow$ $\#\text{mountains} - \#\text{valleys} = \pm 2$

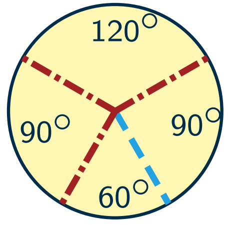
Wrap-Up

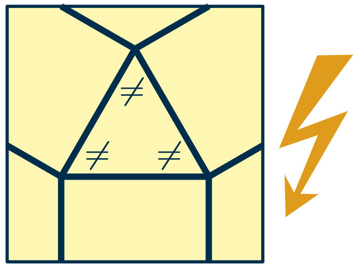
Seen Today

- 1D crease pattern: always flat foldable (zig-zag)
- 1D mountain/valley pattern: flat foldable $\Rightarrow$ safe reduction rule crimp or end-fold applicable
- 2D 1-vertex crease pattern: flat foldable $\Leftrightarrow \Theta_1 + \Theta_3 + \cdots + \Theta_{n-1} = \Theta_2 + \Theta_4 + \cdots + \Theta_n = 180^\circ$
- 2D 1-vertex mountain/valley pattern: flat foldable $\Rightarrow$ safe reduction rule crimp applicable

What Else Is There?

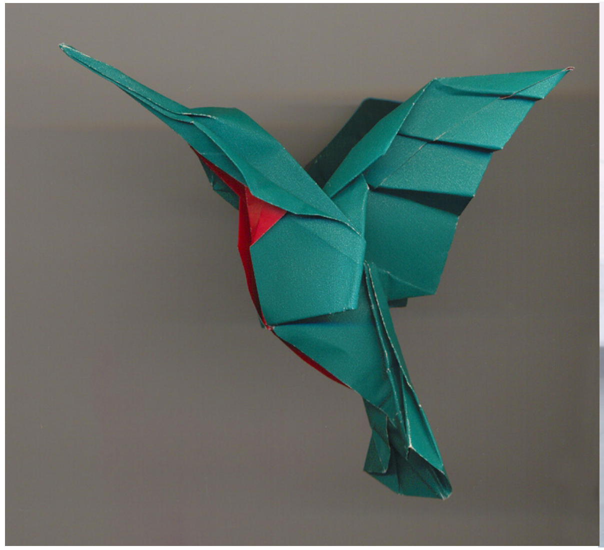
- check if a crease pattern is flat foldable: NP-hard
- check if mountain/valley-pattern is flat foldable: NP-hard
- local foldability: $O(n)$
  (find mountain/valley assignment, such that each vertex alone is flat foldable)

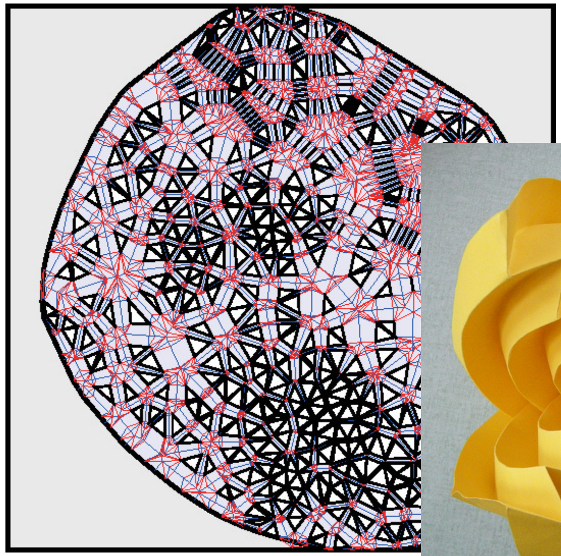


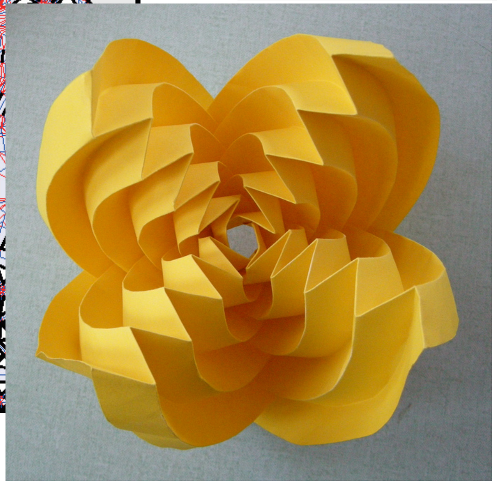


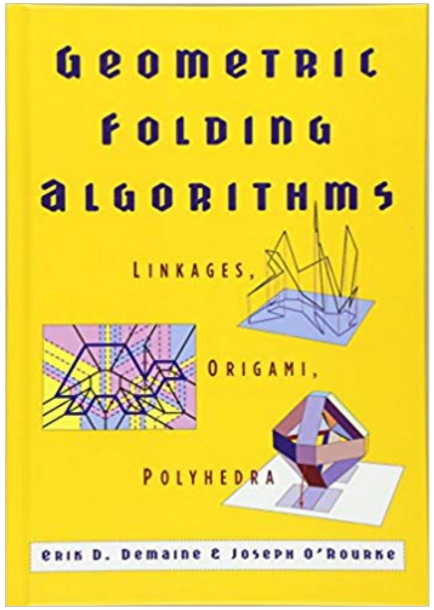
What Else Is There?

Origami everything is foldable (for different definitions of “everything”)

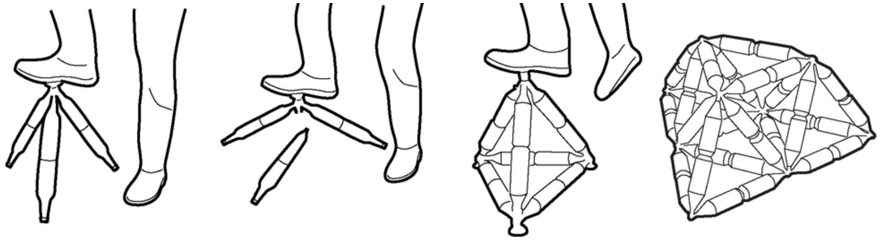






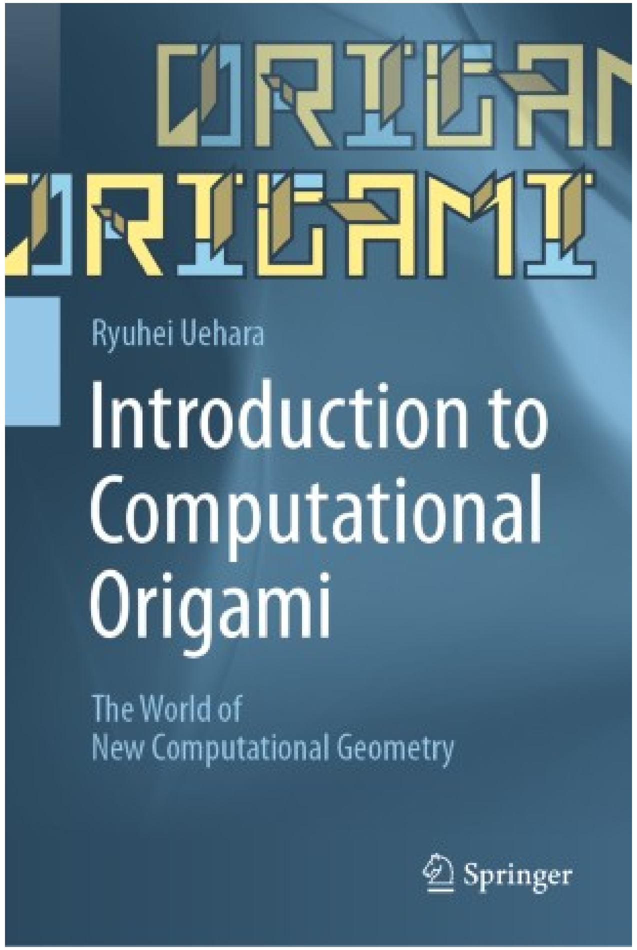


1D-Linkages



Fold-and-Cut → Thursday
bring scissors!

More Books



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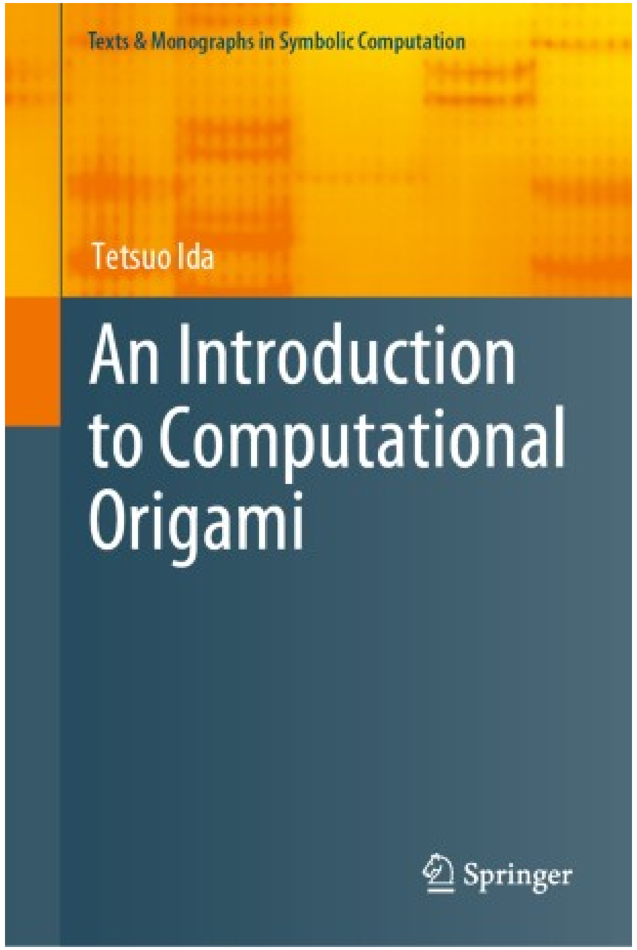

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AI Generated Origami