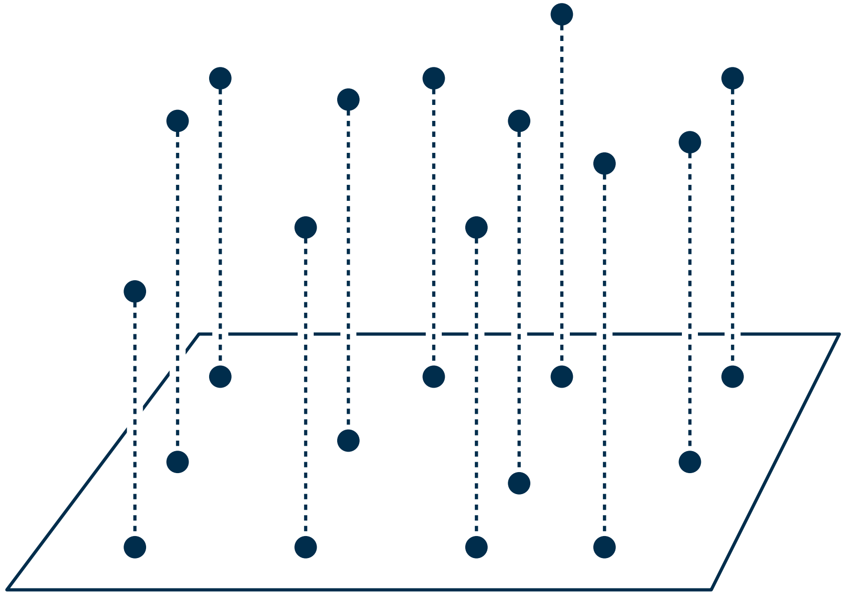
Computational Geometry

Height Interpolation & Delaunay Triangulation

Thomas Bläsius

Height Interpolation



Sample Points Of Measuring The Height Of A Terrain

Height Interpolation

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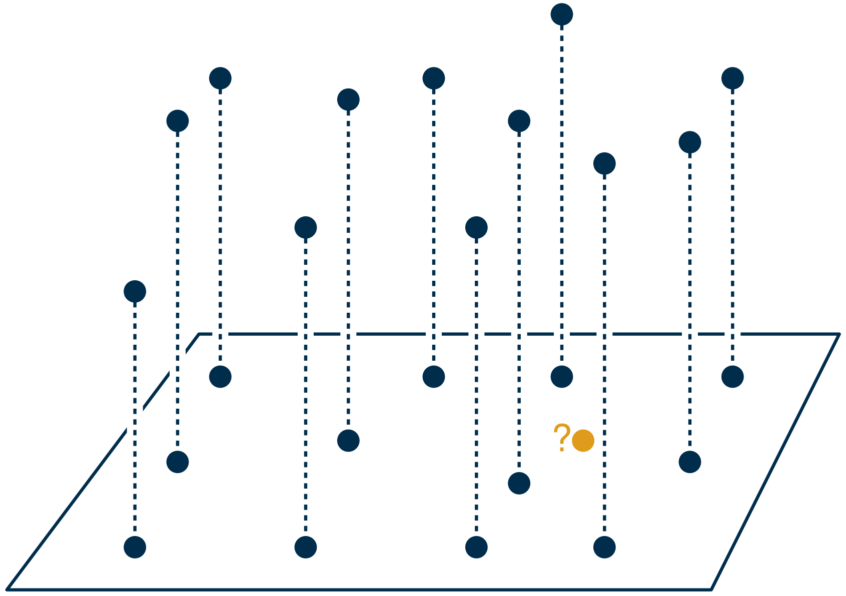
- What is the altitude of a point that was not measured?



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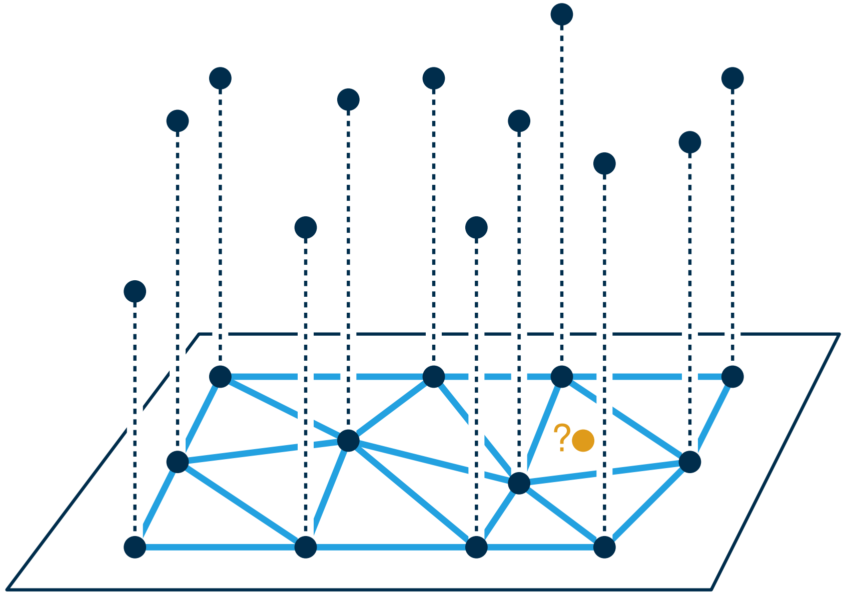
- What is the altitude of a point that was not measured?
- triangulate the points in the plane



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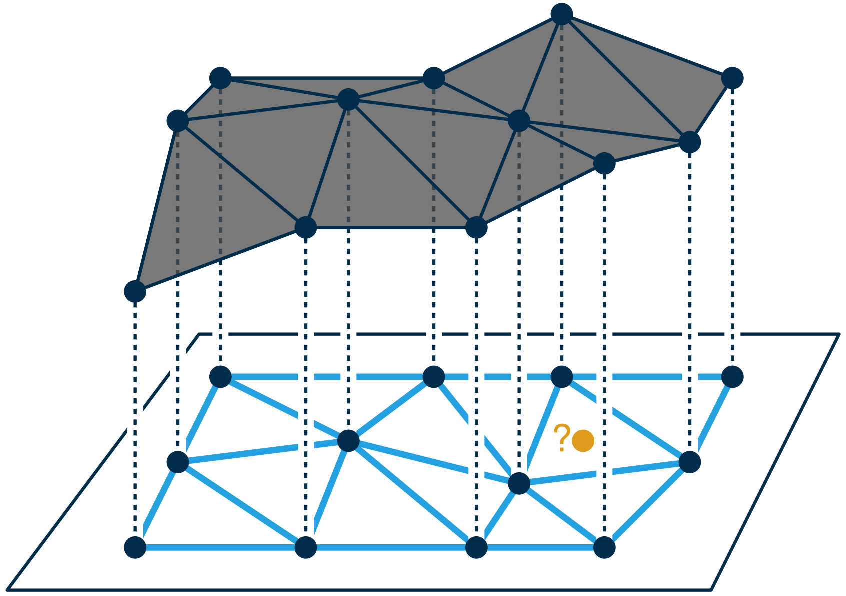

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- What is the altitude of a point that was not measured?
- triangulate the points in the plane
- $\rightarrow$ also triangulates the measured 3d points

Height Interpolation

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- What is the altitude of a point that was not measured?
- triangulate the points in the plane
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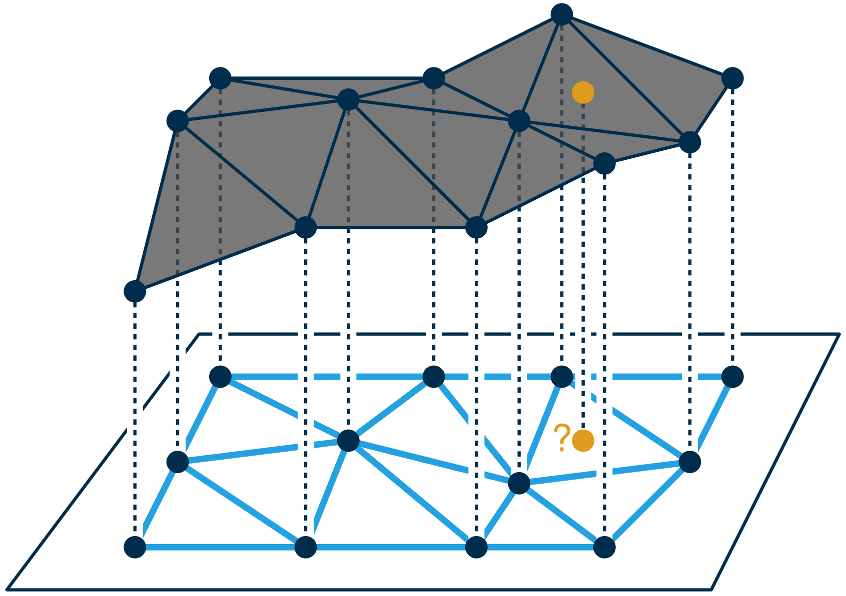
- deduce the altitude based on the triangulation



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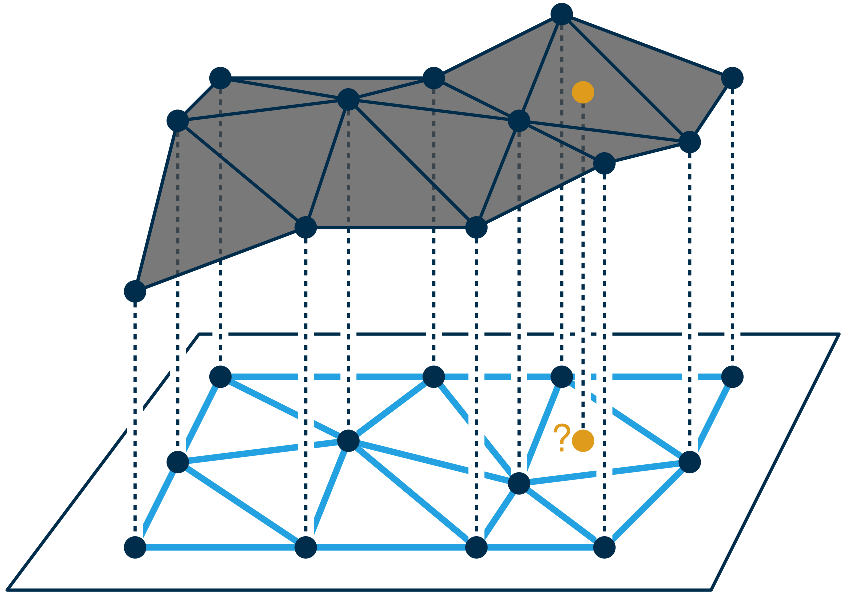
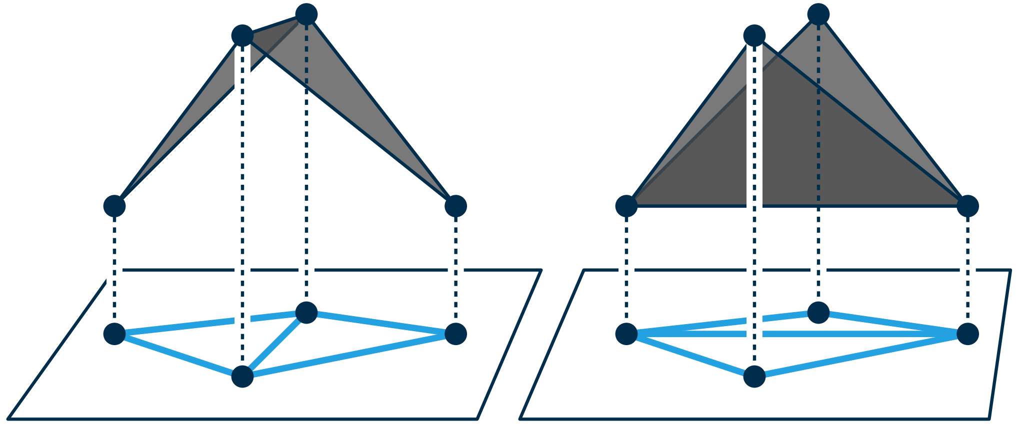


What Makes A Good Triangulation?



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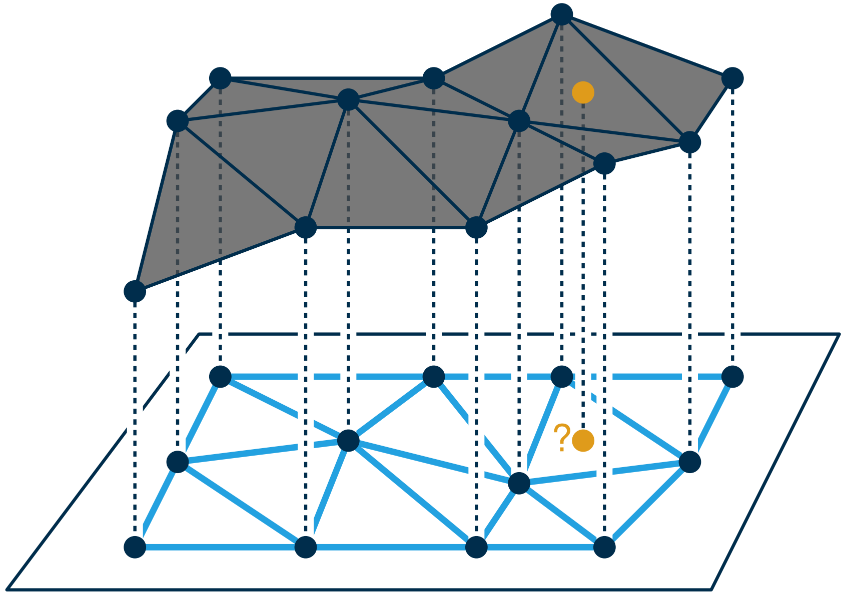
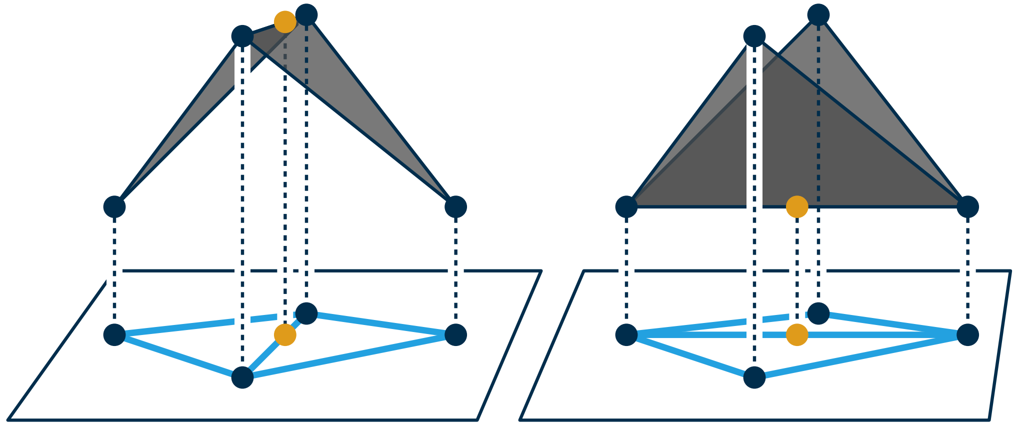


What Makes A Good Triangulation?

- different triangulations yield different results



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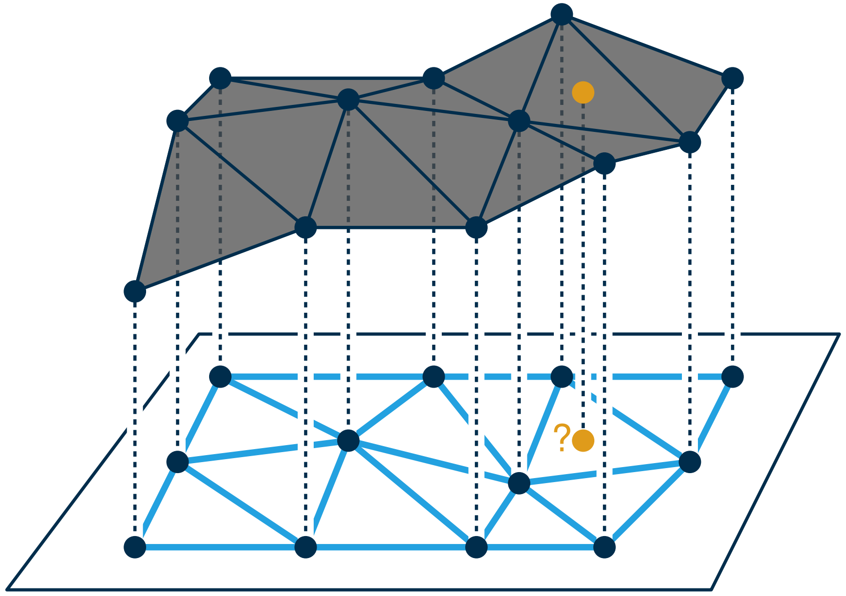
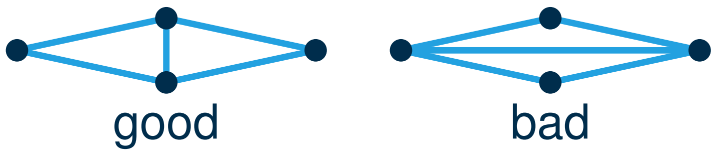
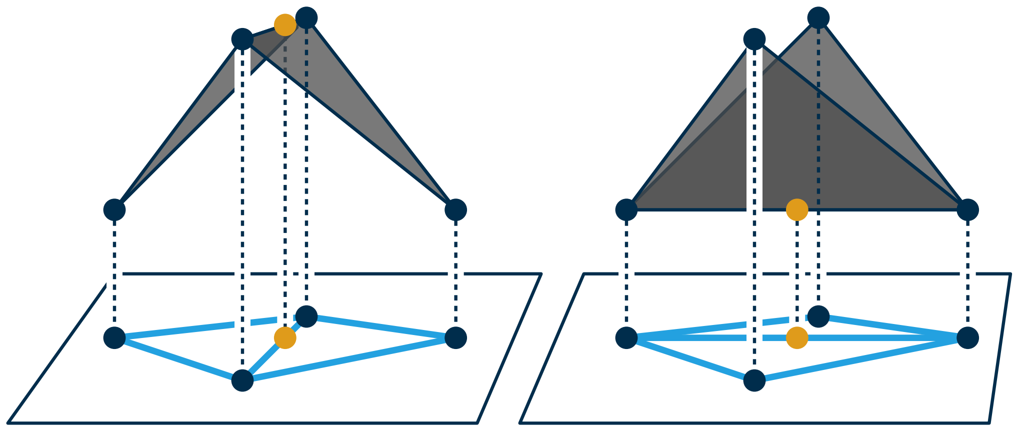


What Makes A Good Triangulation?

- different triangulations yield different results
- goal: avoid thin triangles

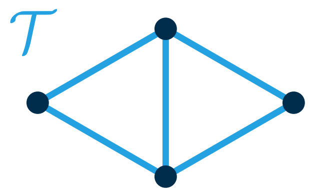




Good And Bad Triangulations

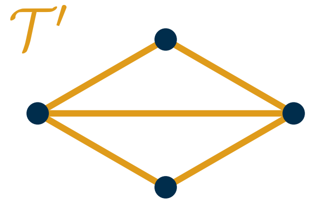
Angle Vector

- consider triangulation $\mathcal{T}$ of a point set with m triangles
- interior angles of triangles sorted increasingly: $\alpha_1, \ldots, \alpha_{3m}$
- angle vector: $\alpha(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$



$\alpha(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$

$\alpha(\mathcal{T}') = (30^\circ, 30^\circ, 30^\circ, 30^\circ, 120^\circ, 120^\circ)$



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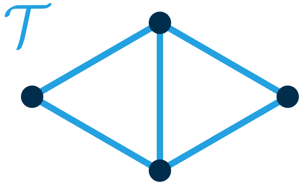
- order the angle vectors lexicographically

  $(\alpha(\mathcal{T}) > \alpha(\mathcal{T}') \Leftrightarrow \exists i \in \{1, \ldots, 3m\}\colon a_i > a'_i$ und $\forall j < i\colon a_j = a'_j)$

$\mathcal{T}$



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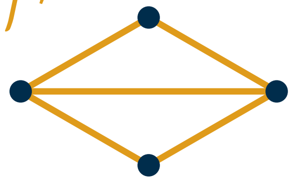

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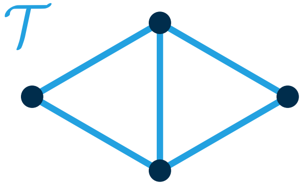
Optimal Triangulation

- triangulation is optimal, if it is maximal (wrt this order)
  ($\mathcal{T}$ optimal $\Leftrightarrow \mathcal{T} \geq \mathcal{T}'$ for all triangulations $\mathcal{T}'$)

$\mathcal{T}$



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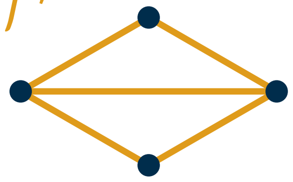

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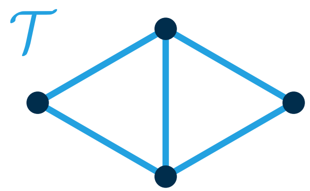

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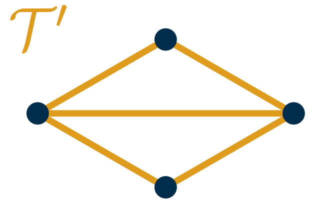

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Questions For Today

- How can we recognize an optimal triangulation?
- Can we compute an optimal triangulation?
- Is the optimal triangulation unique?

Forbidden Edges

Idea

- iteratively improve triangulation by local improvements
- yields a local optimum
- hope: it is also a global optimum

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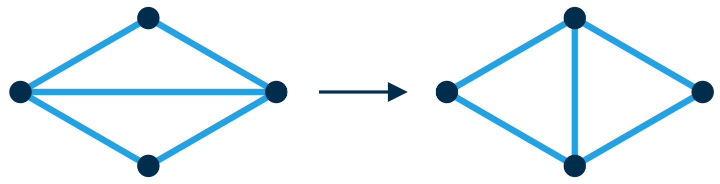
- remove an (inner) edge
- insert a new diagonal



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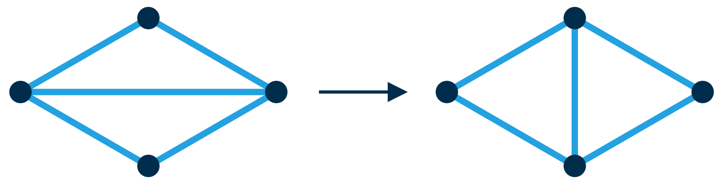
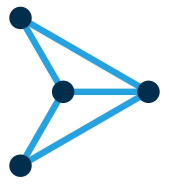


Observations

- is not feasible for every edge



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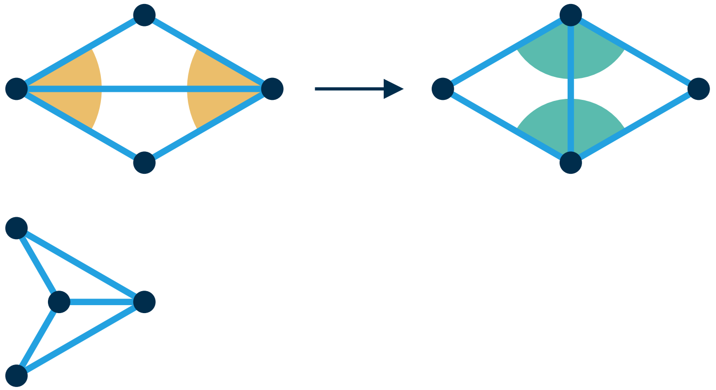

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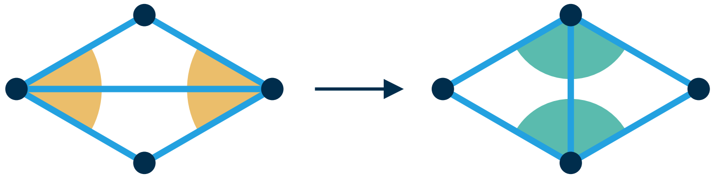

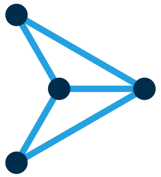

Observations

- is not feasible for every edge
- we are interested in the four angles at the flipped edge
- we call an edge **forbidden** if flipping increases the smallest of those angles
- let e in $\mathcal{T}$ be a forbidden edge and let $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$; then: $\alpha(\mathcal{T}') > \alpha(\mathcal{T})$

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- iteratively improve triangulation by local improvements
- yields a local optimum
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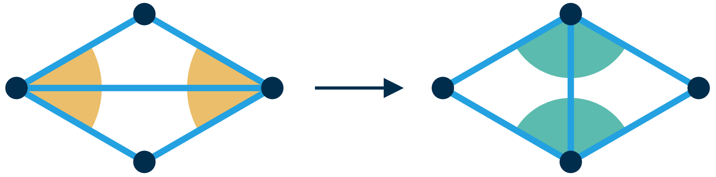
Does this terminate?

Edge Flip



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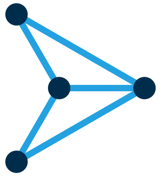

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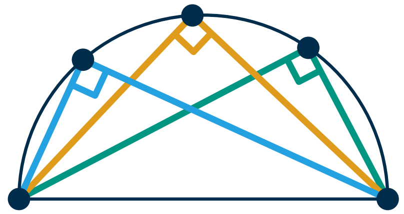
The line segments from a point on a circle to the endpoints of a diameter form a right angle.



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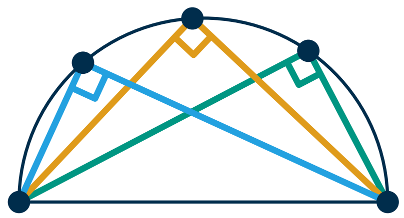
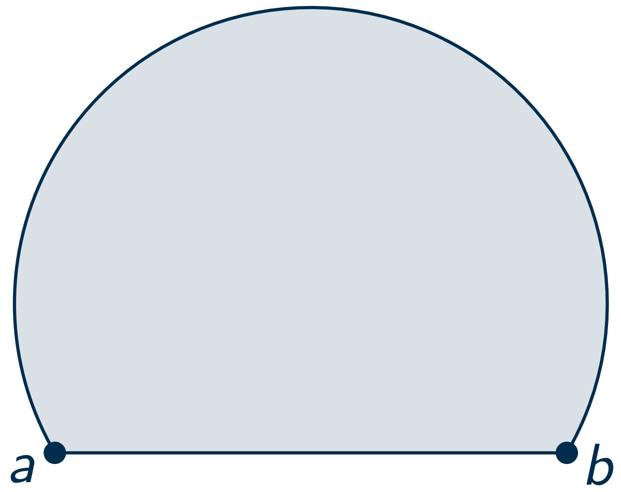


Generalization

- fix an arbitrary chord of the circle (endpoints: a and b)
- also fix one of the two circular segment



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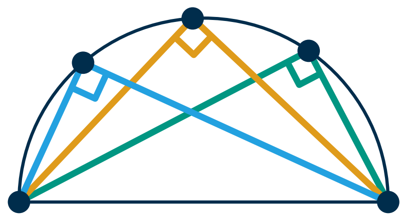

Generalization

- fix an arbitrary chord of the circle (endpoints: a and b)
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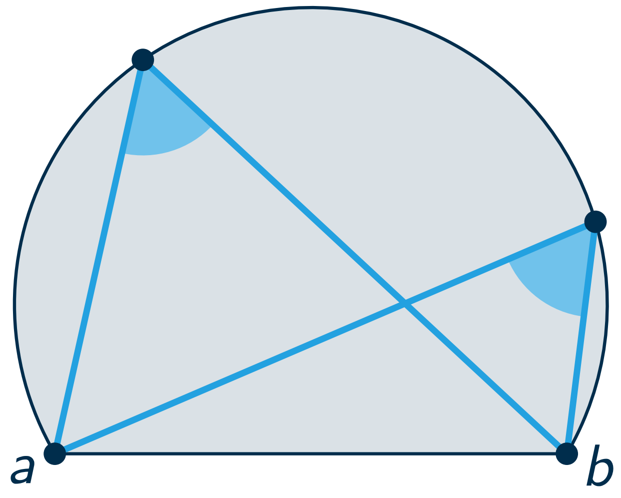
- the line segments from every point on the circular arc to a and b form the same angle



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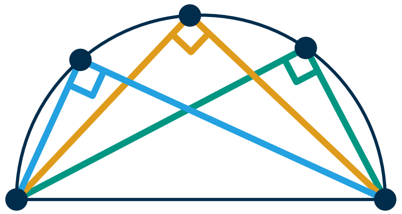

Generalization

- fix an arbitrary chord of the circle (endpoints: a and b)
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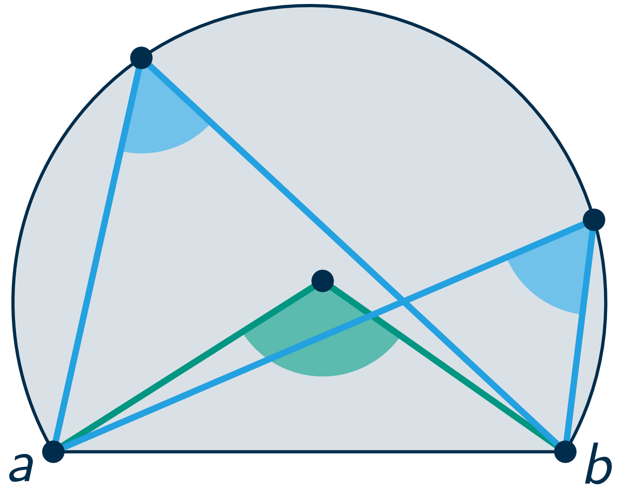
- the line segments from every point on the circular arc to a and b form the same angle
- for points in the interior, the angle is bigger



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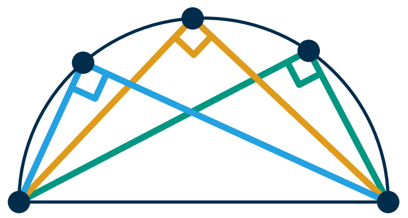
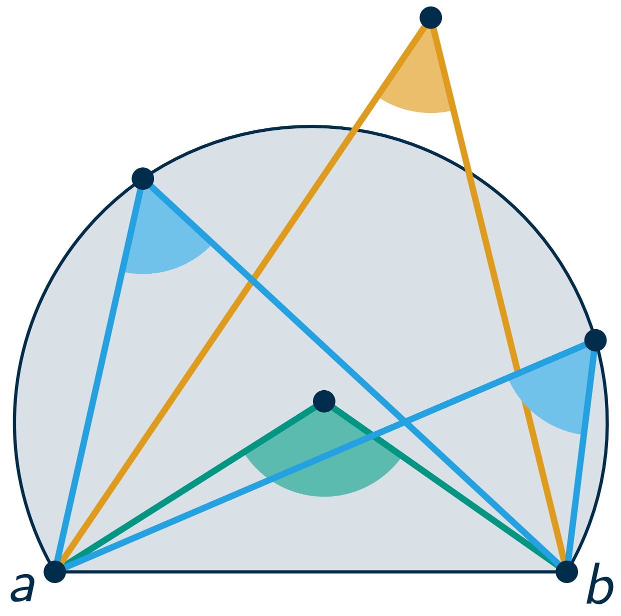


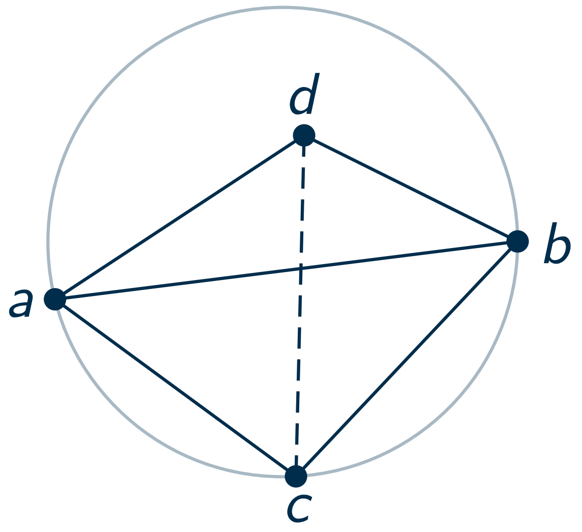
Generalization

- fix an arbitrary chord of the circle (endpoints: a and b)
- also fix one of the two circular segment
- the line segments from every point on the circular arc to a and b form the same angle
- for points in the interior, the angle is bigger
- for points in the exterior, the angle is smaller
  (on the same side of ab)



Which Edges Are Forbidden?

Lemma
Let $\triangle abc$ and $\triangle abd$ be two triangles such that a, b, c, d are in convex position. The circumcircle of $\triangle abc$ contains d if and only if ab is forbidden.



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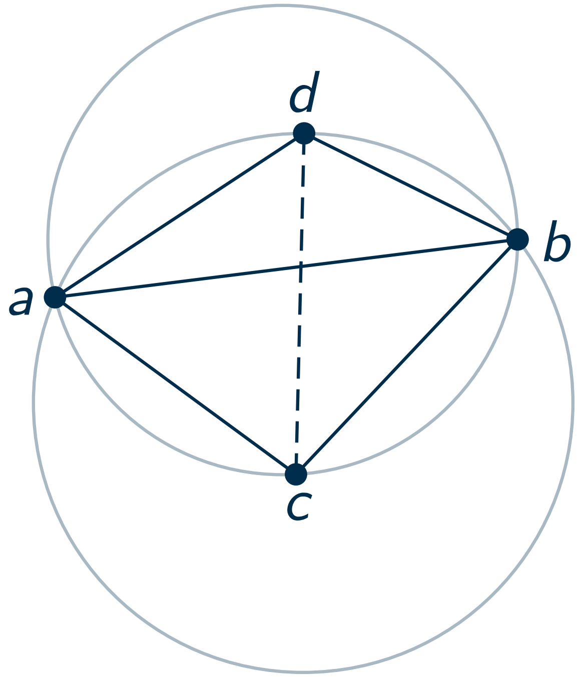
Proof: d in circumcircle of $\triangle abc \Rightarrow ab$ forbidden

- note: the circumcircle of $\triangle abd$ then contains c

Why?



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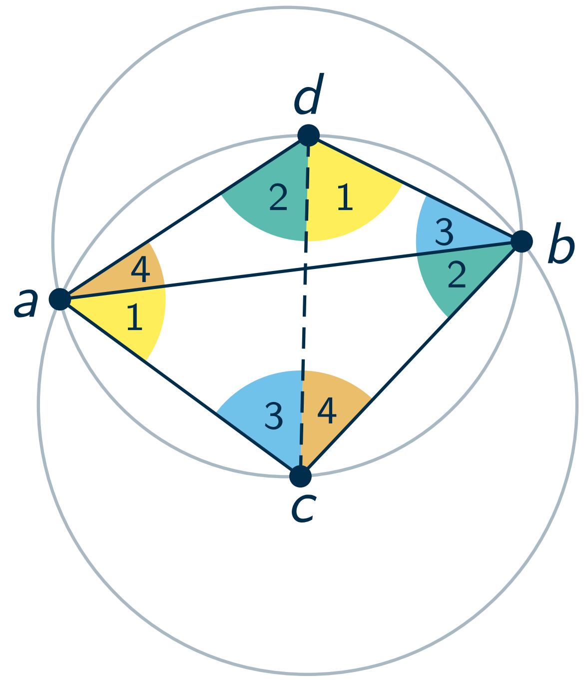

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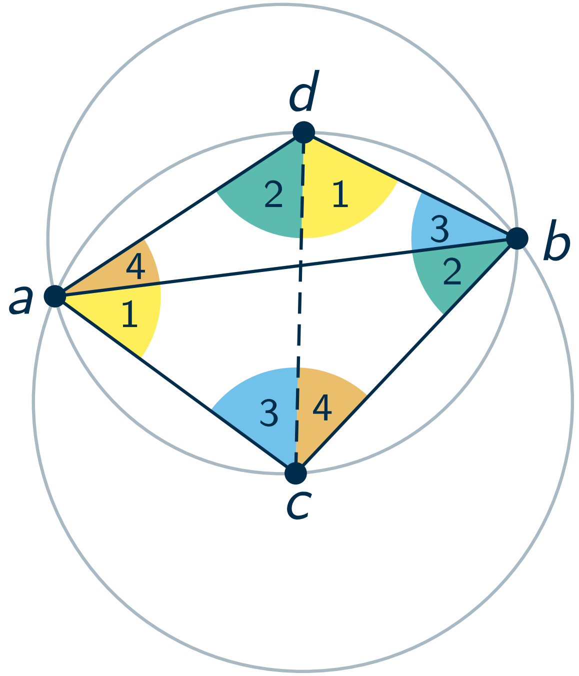
- consider bijection between angles at a and b, and "opposite" angles at c and d of the flipped edge
- claim: each angle at ab is smaller than corresponding angle at cd



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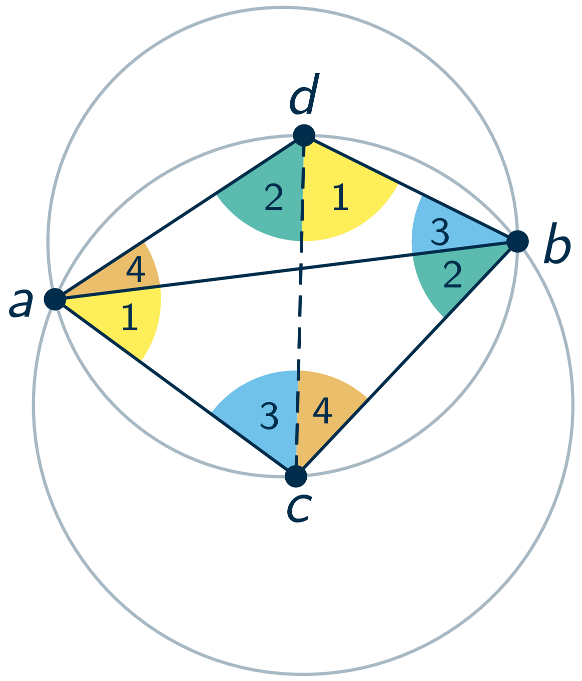

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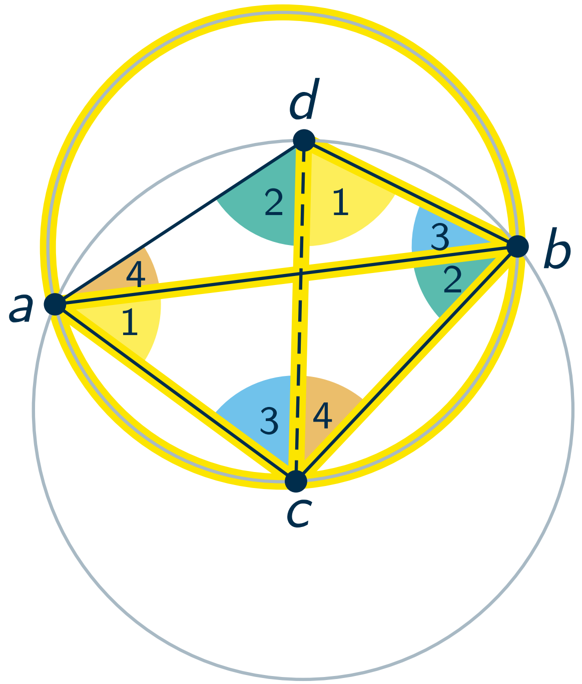

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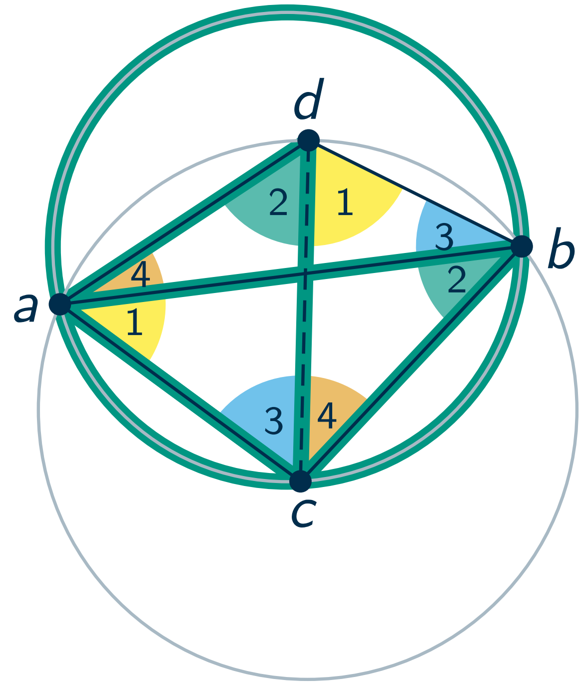

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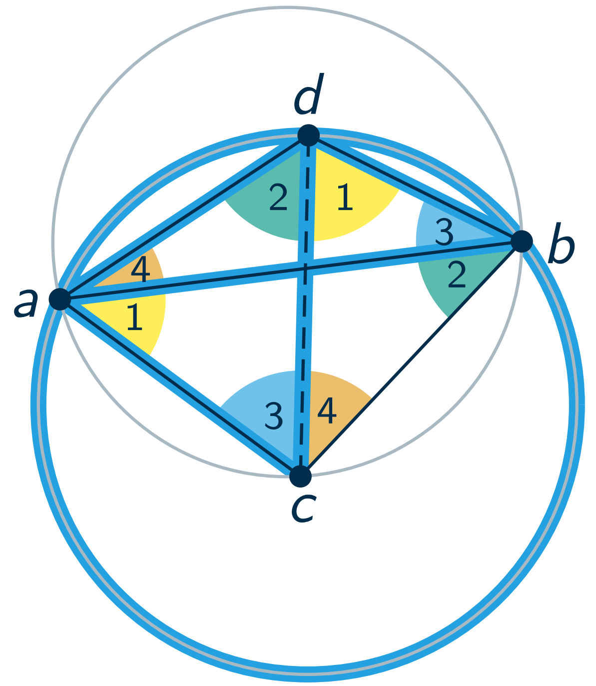

Which Edges Are Forbidden?

Lemma
Let $\triangle abc$ and $\triangle abd$ be two triangles such that a, b, c, d are in convex position. The circumcircle of $\triangle abc$ contains d if and only if ab is forbidden.

Proof: d in circumcircle of $\triangle abc \Rightarrow ab$ forbidden

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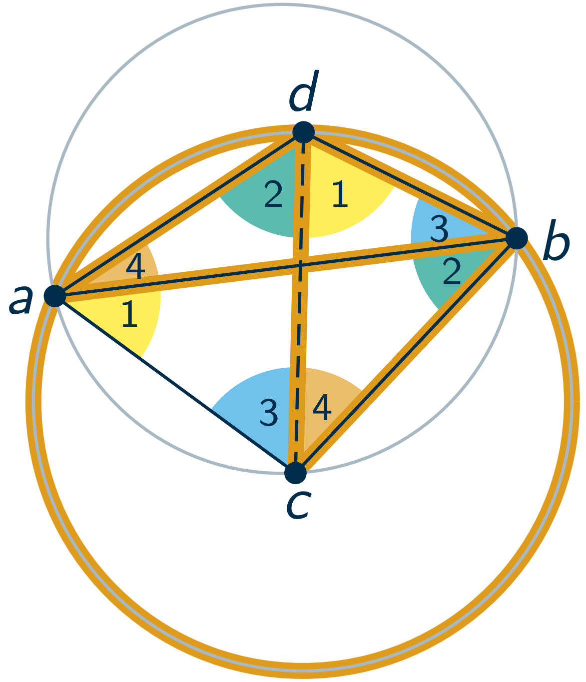

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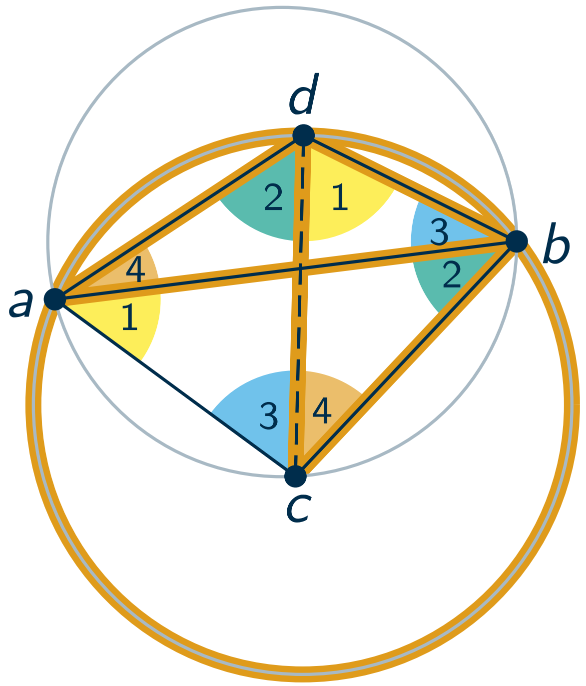

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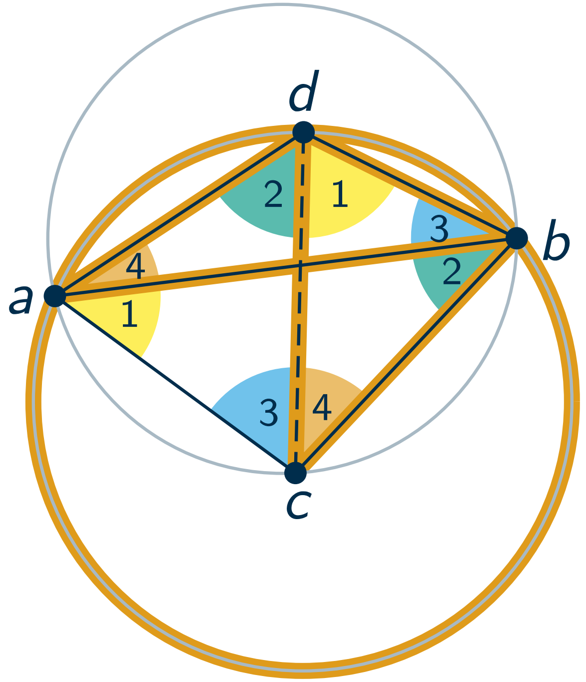
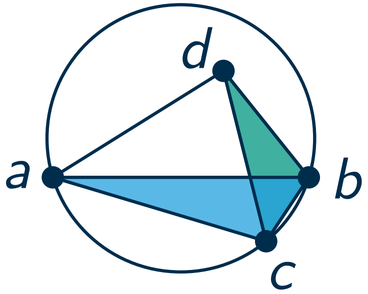


Other Direction

- similar argument by looking at the minimum angle



Forbidden Edges And Empty Circles

Theorem
A triangulation has a forbidden edge if and only if the circumcircle of a face contains a vertex.

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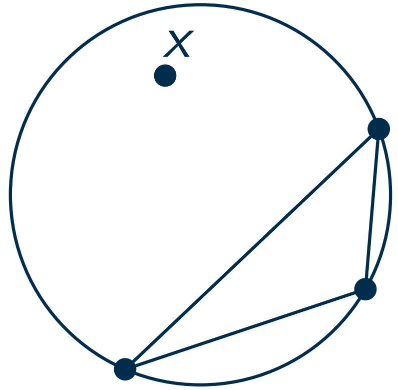

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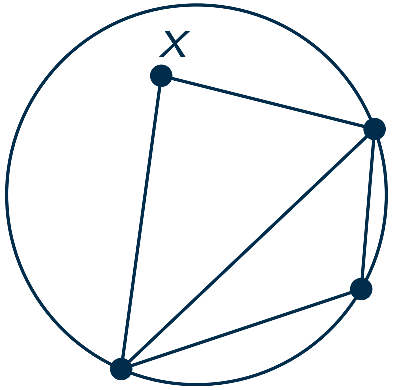

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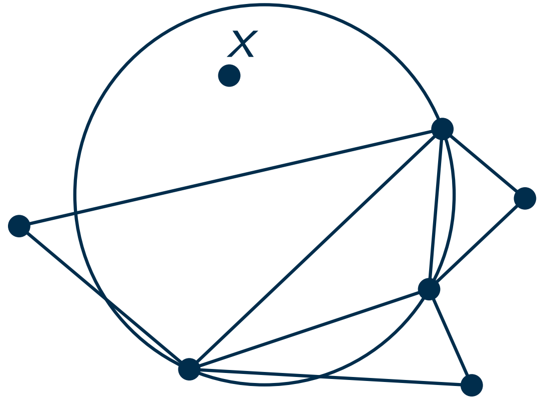

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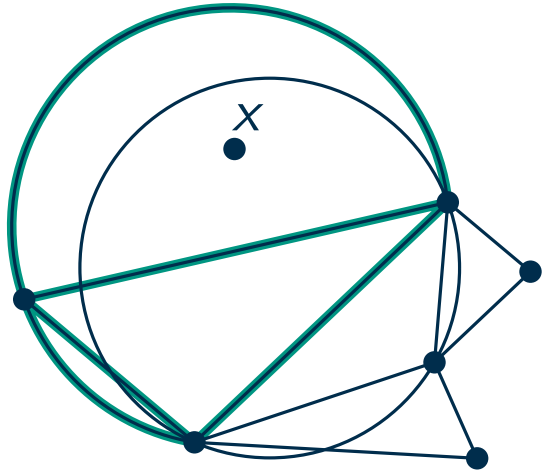

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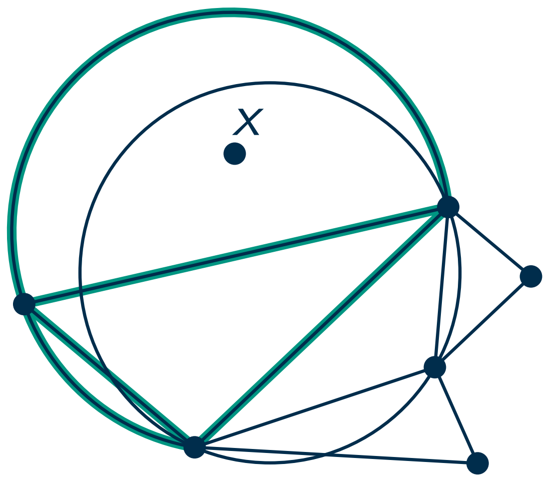
- cleaner argument: consider extreme situation, show contradiction



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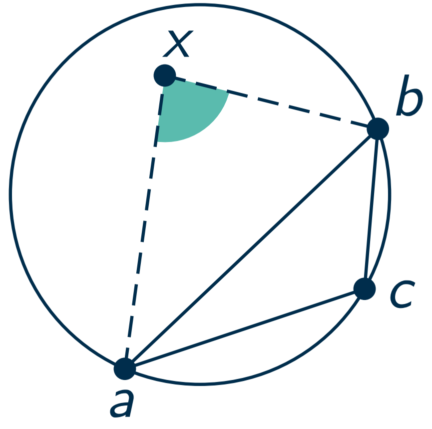

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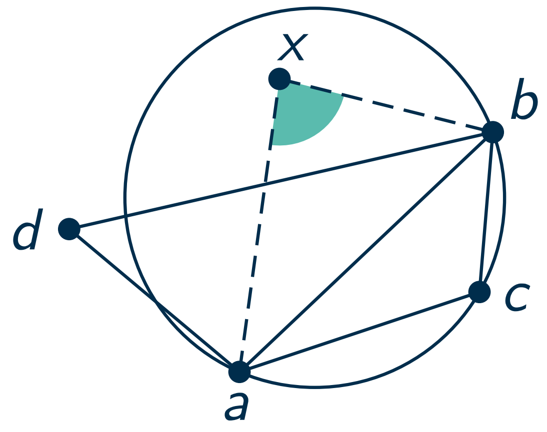

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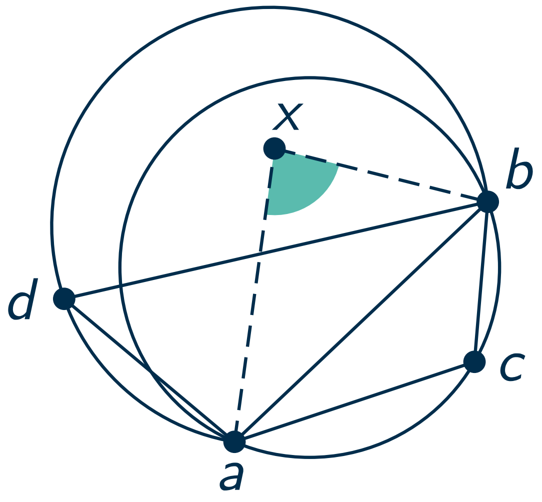

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- $\angle dxb$ is bigger than $\angle axb$ $\Rightarrow$ contradiction
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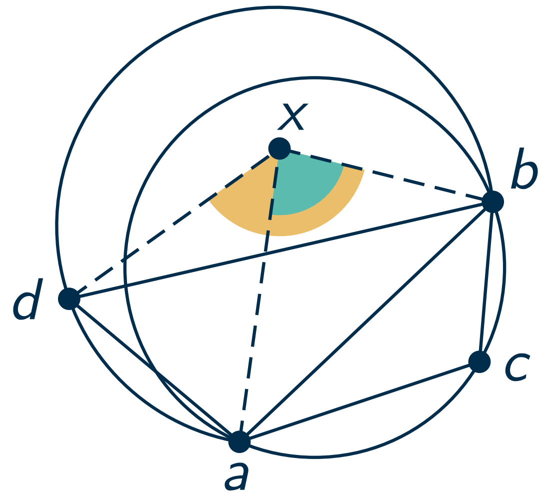

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Equivalent:

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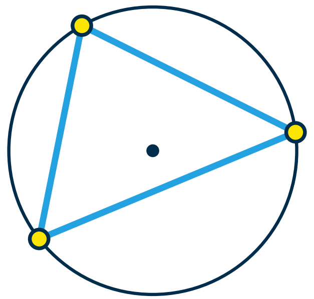

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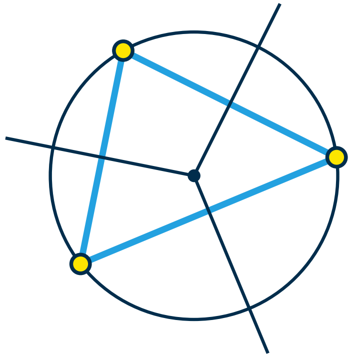

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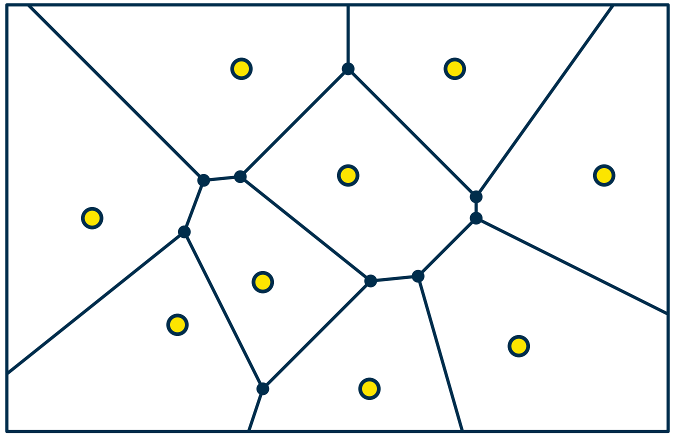

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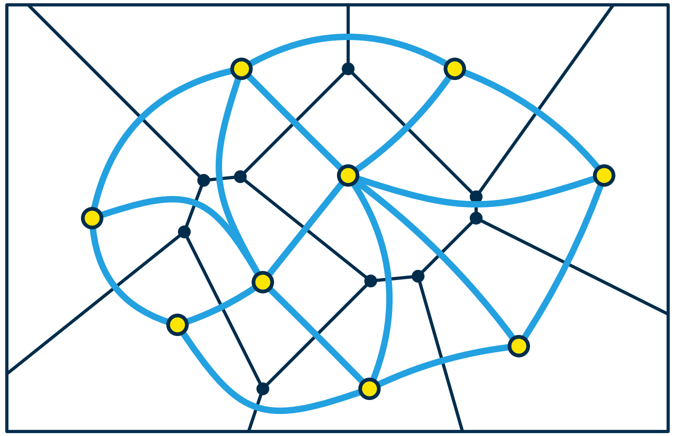

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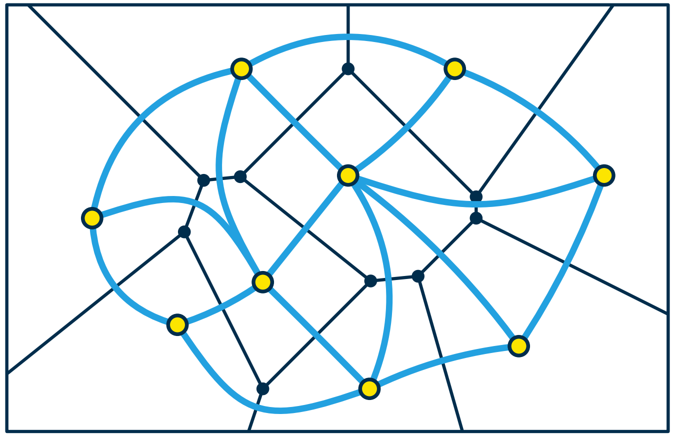

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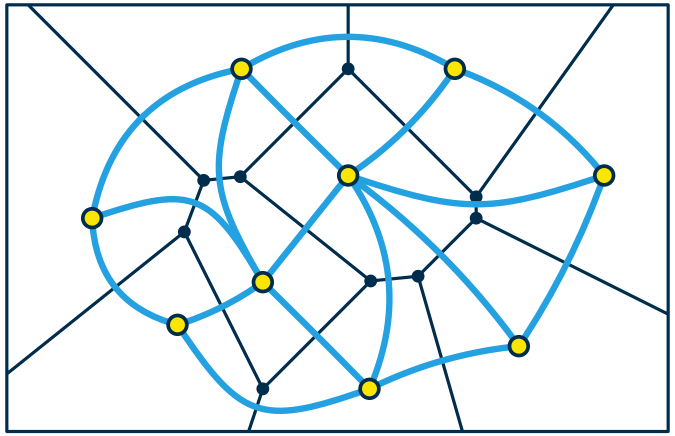

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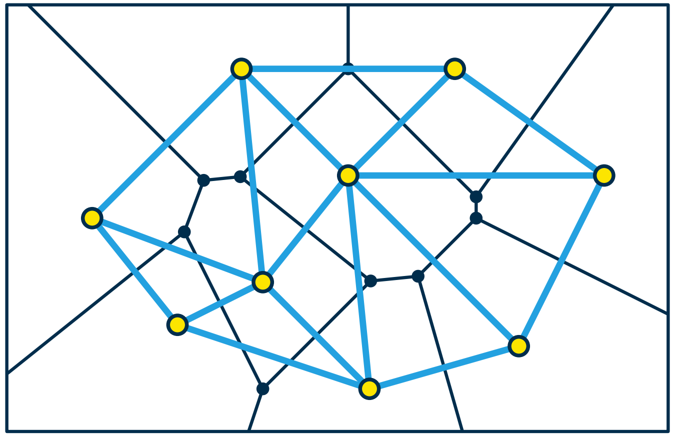

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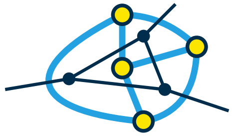

Don't we need to show that there are no edge crossings?

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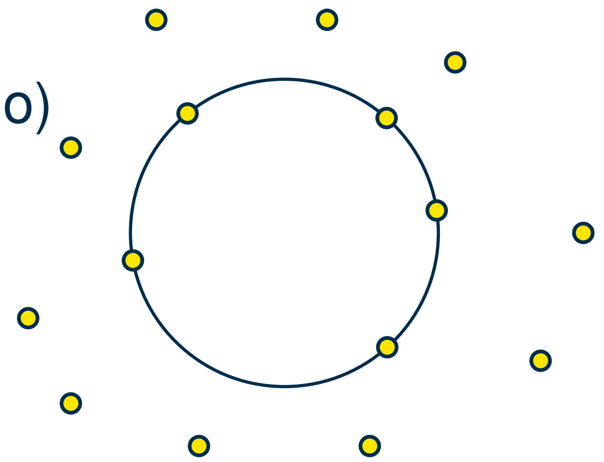

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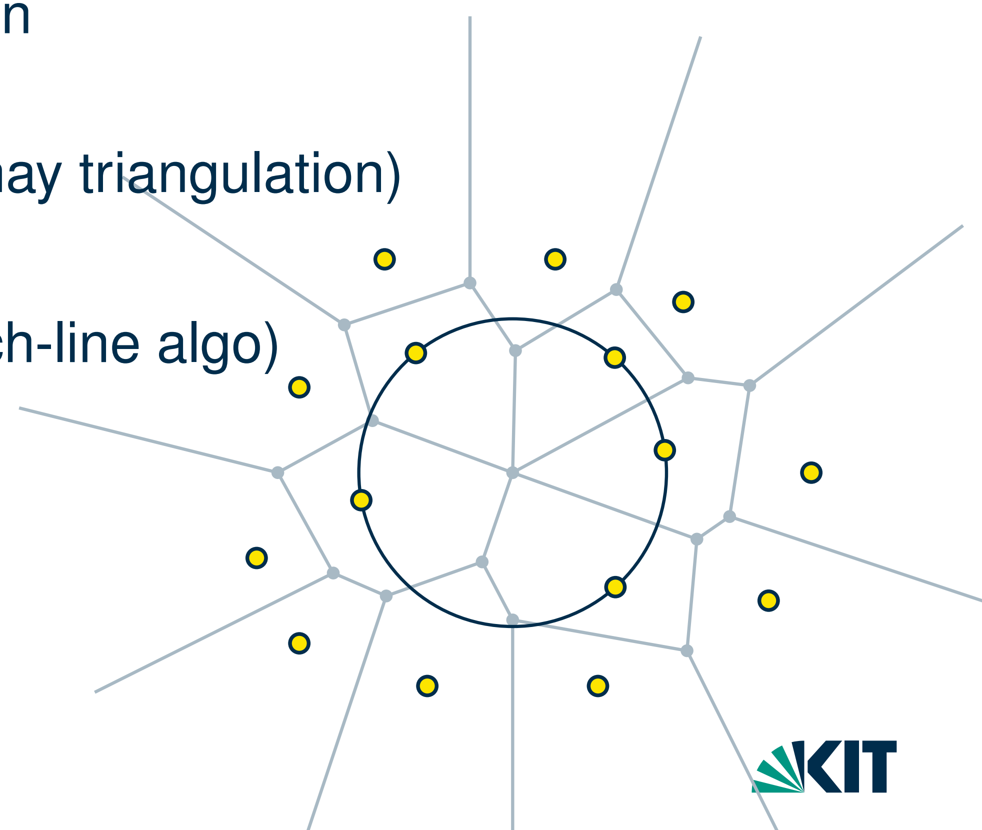

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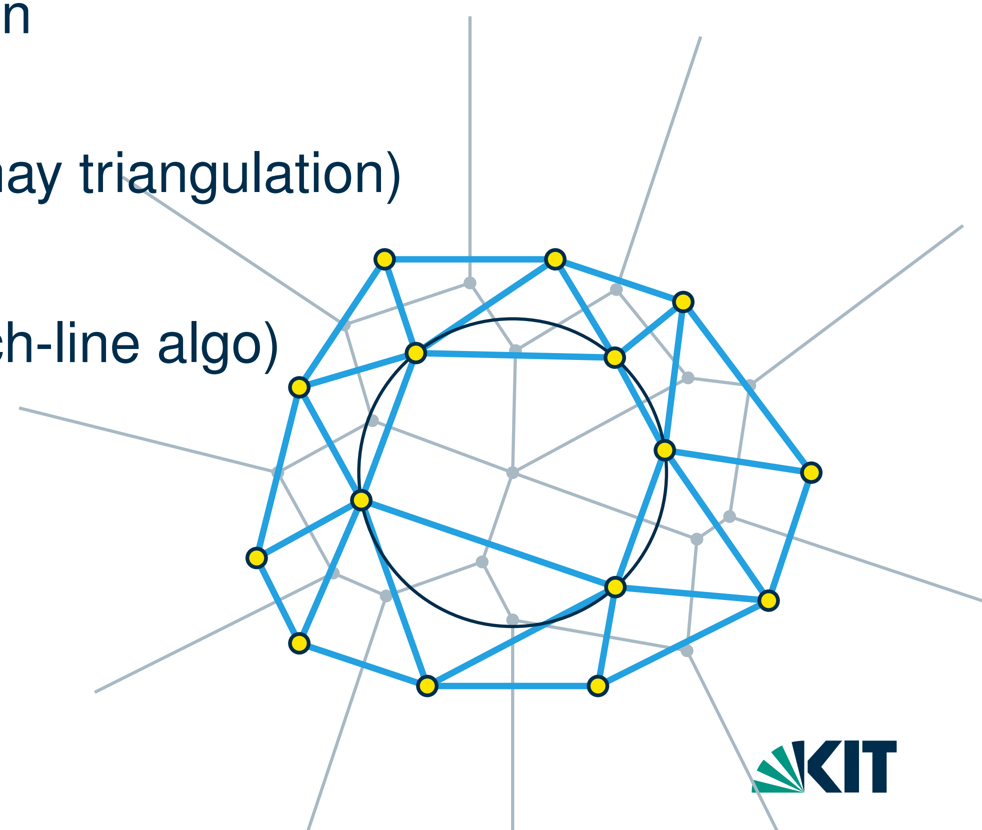

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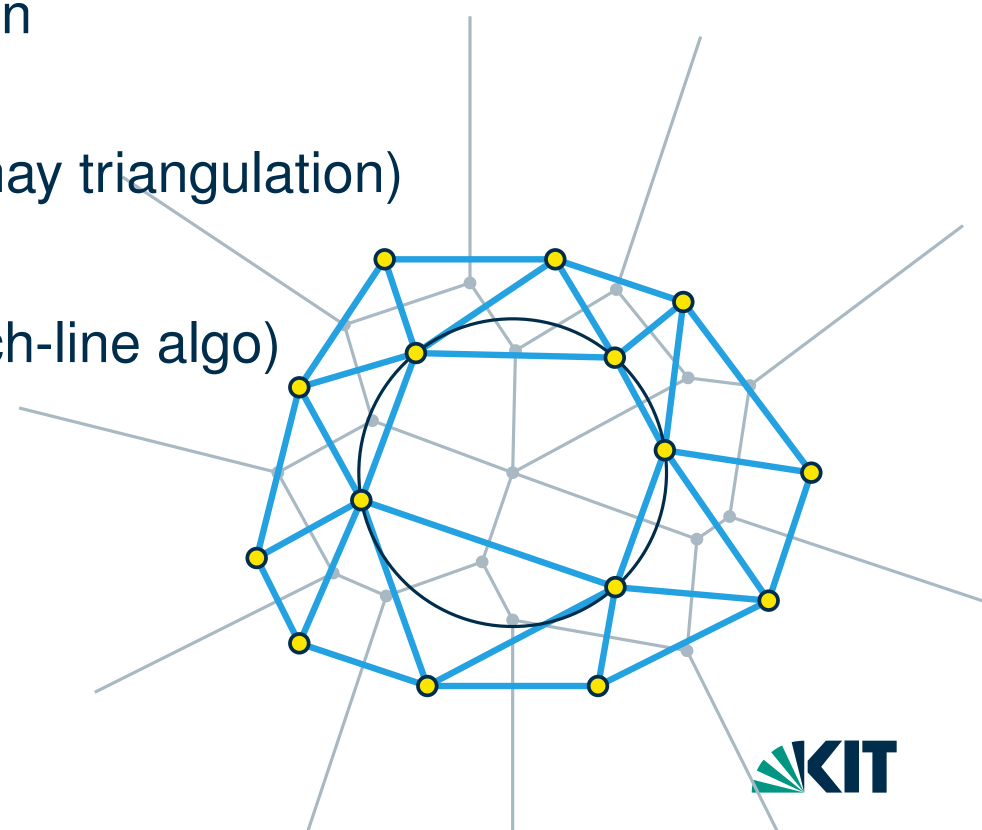

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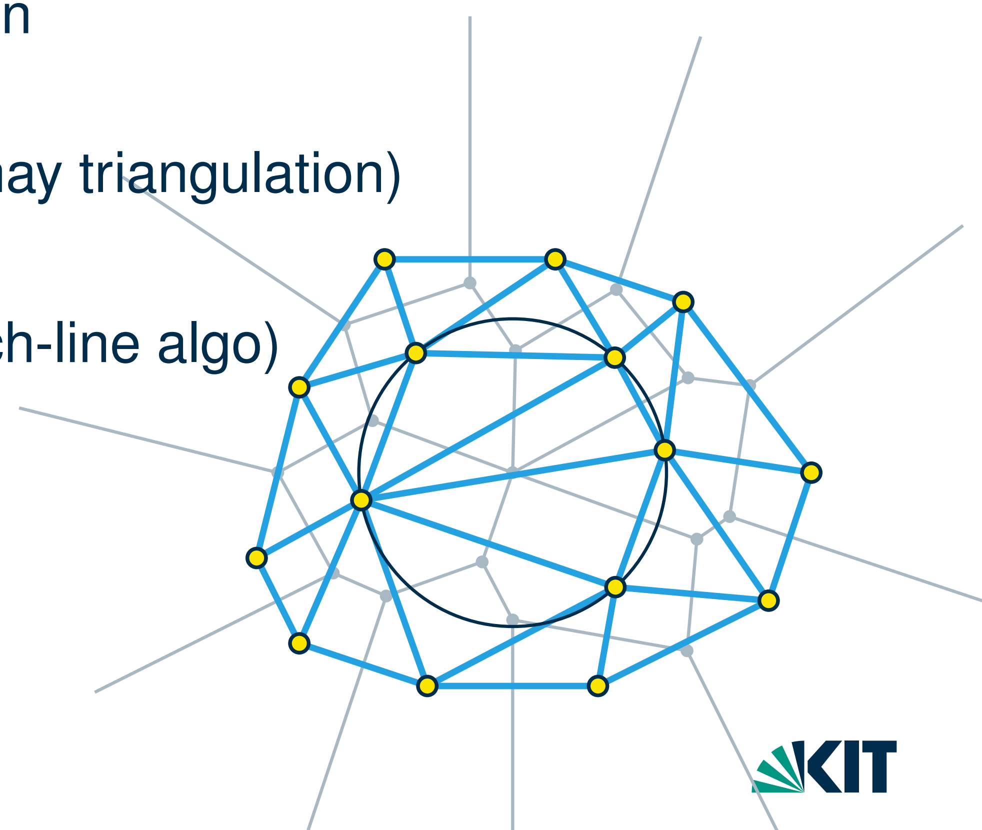

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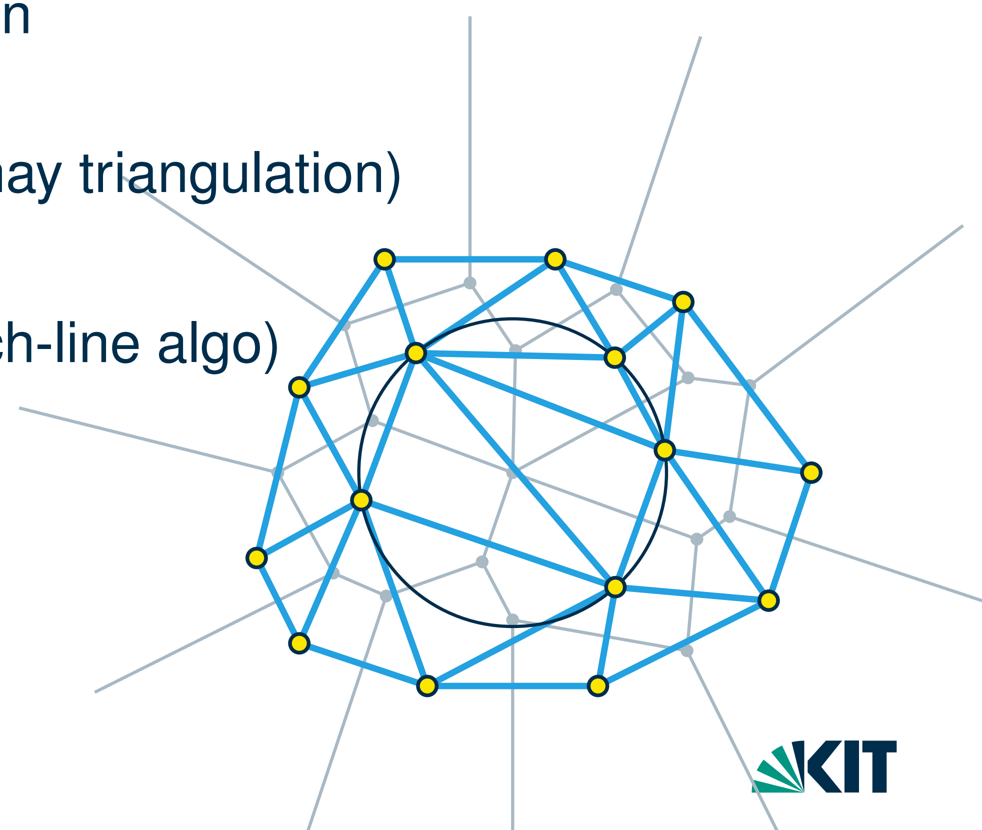

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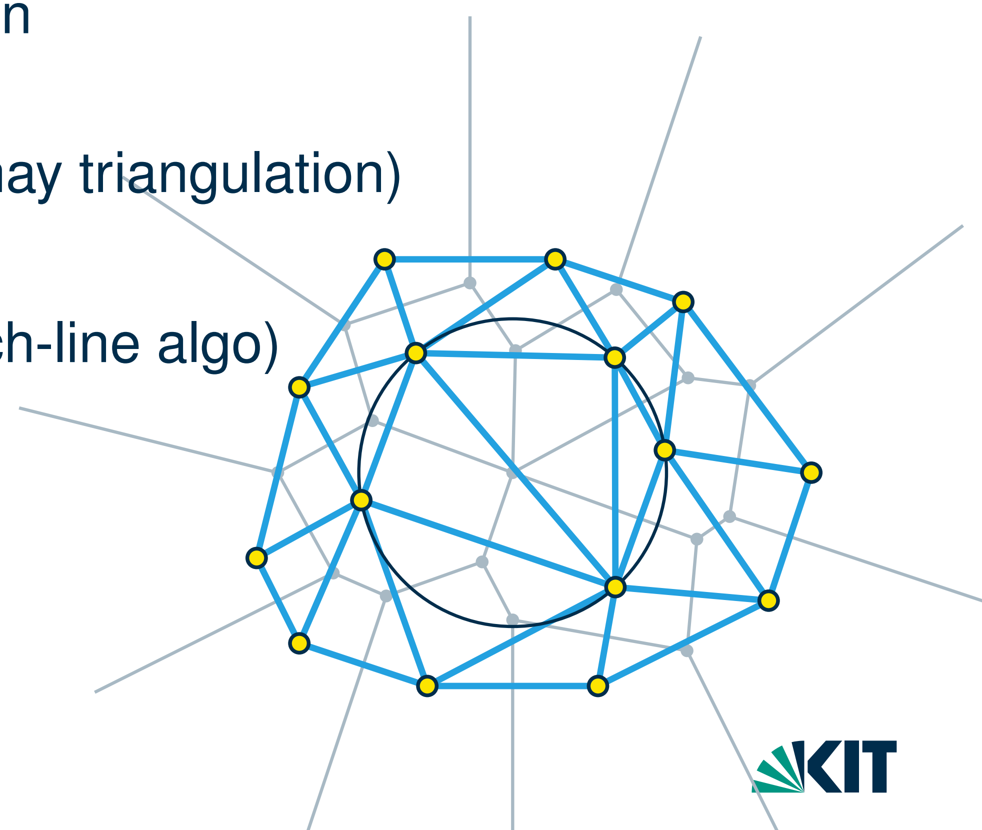

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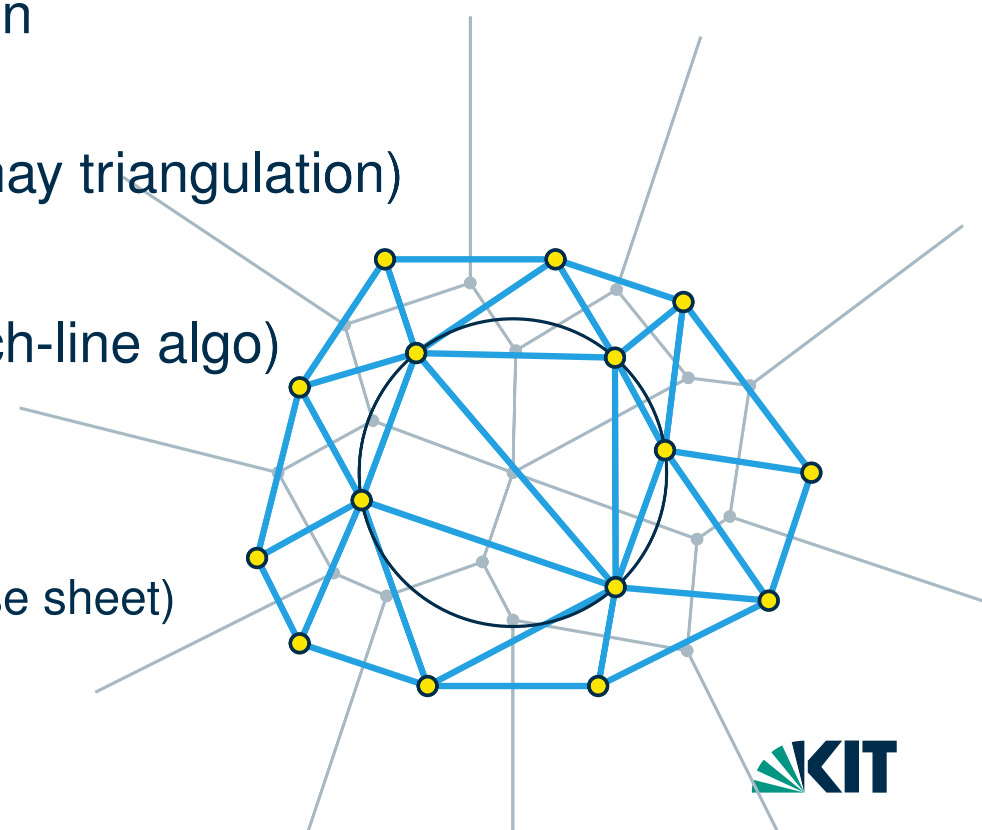
- multiple triangulations possible

(more on this on the exercise sheet)



Wrap-Up

Seen Today

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  - dual of the Voronoi diagram

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  - dual of the Voronoi diagram
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What Else Is There?

- the Delaunay triangulation has other nice properties (e.g., the MST is part of the triangulation)
- different optimization criteria: minimizing the total edge length is NP-hard

Exam Scheduling

Monday	Tuesday	Wednesday	Thursday	Friday	
28				1	
4				8	
11				15	August
18				22	
25				29	
1				5	
8				12	
15				19	September
22				26	
29				3	
6				10	
13				17	October
20				24	
27				31	

Help Us Improve

Evaluation

- survey open until 18:00
- link also in discord
- exercise sessions are evaluated separately

https://onlineumfrage.kit.edu/evasys/online.php?p=G4MUD