Computational Geometry

Height Interpolation & Delaunay Triangulation

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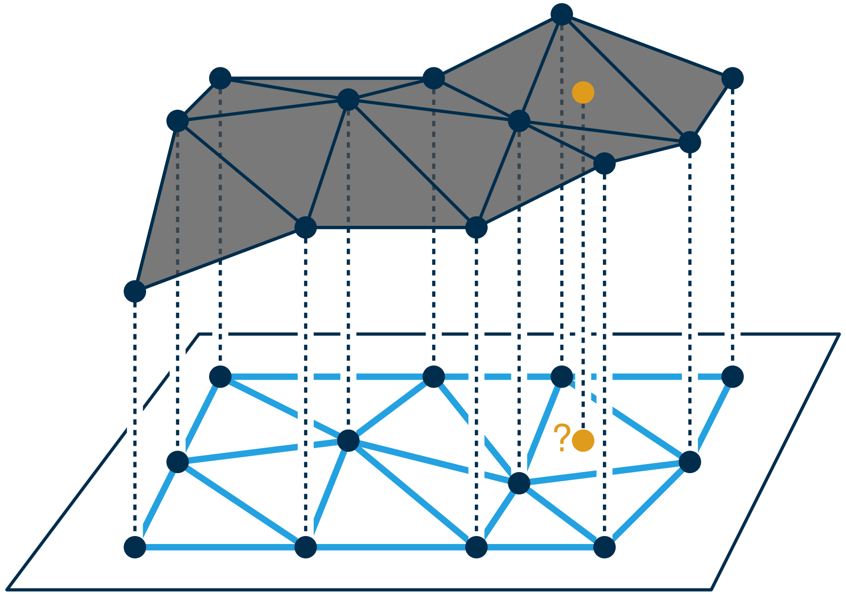
Height Interpolation

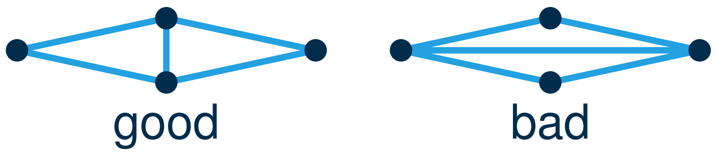


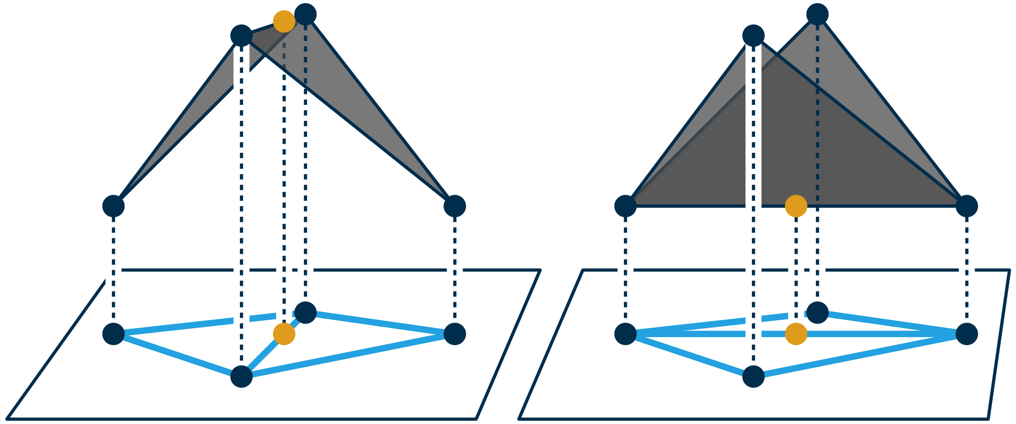
Sample Points Of Measuring The Height Of A Terrain

- What is the altitude of a point that was not measured?
- triangulate the points in the plane
- $\rightarrow$ also triangulates the measured 3d points
- deduce the altitude based on the triangulation

What Makes A Good Triangulation?

- different triangulations yield different results
- goal: avoid thin triangles





Good And Bad Triangulations

Angle Vector

- consider triangulation $\mathcal{T}$ of a point set with m triangles
- interior angles of triangles sorted increasingly: $\alpha_1, \dots, \alpha_{3m}$
- angle vector: $\alpha(\mathcal{T}) = (\alpha_1, \dots, \alpha_{3m})$
- order the angle vectors lexicographically

  $(\alpha(\mathcal{T}) > \alpha(\mathcal{T}') \Leftrightarrow \exists i \in \{1, \dots, 3m\} \colon a_i > a'_i$ und $\forall j < i \colon a_j = a'_j)$

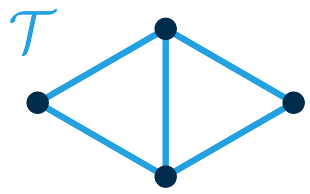
Optimal Triangulation

- triangulation is optimal, if it is maximal (wrt this order)

  ($\mathcal{T}$ optimal $\Leftrightarrow \mathcal{T} \geq \mathcal{T}'$ for all triangulations $\mathcal{T}'$)

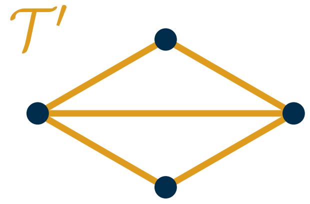
Questions For Today

- How can we recognize an optimal triangulation?
- Can we compute an optimal triangulation?
- Is the optimal triangulation unique?



$\alpha(\mathcal{T}) = (60°, 60°, 60°, 60°, 60°, 60°)$

$\alpha(\mathcal{T}') = (30°, 30°, 30°, 30°, 120°, 120°)$



Forbidden Edges

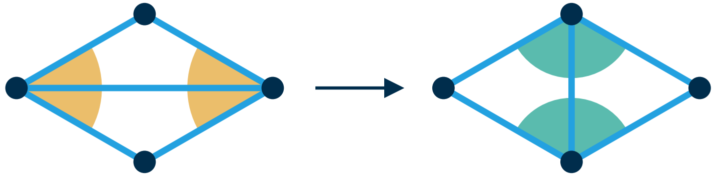
Idea

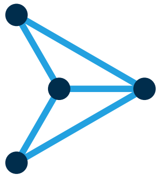
- iteratively improve triangulation by local improvements
- yields a local optimum
- hope: it is also a global optimum

Does this terminate?

Edge Flip

- remove an (inner) edge
- insert a new diagonal



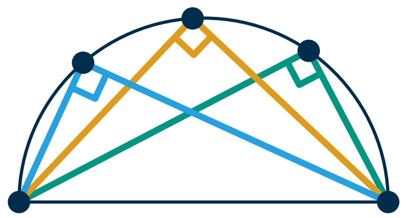


Observations

- is not feasible for every edge
- we are interested in the four angles at the flipped edge
- we call an edge **forbidden** if flipping increases the smallest of those angles
- let e in $\mathcal{T}$ be a forbidden edge and let $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$; then: $\alpha(\mathcal{T}') > \alpha(\mathcal{T})$

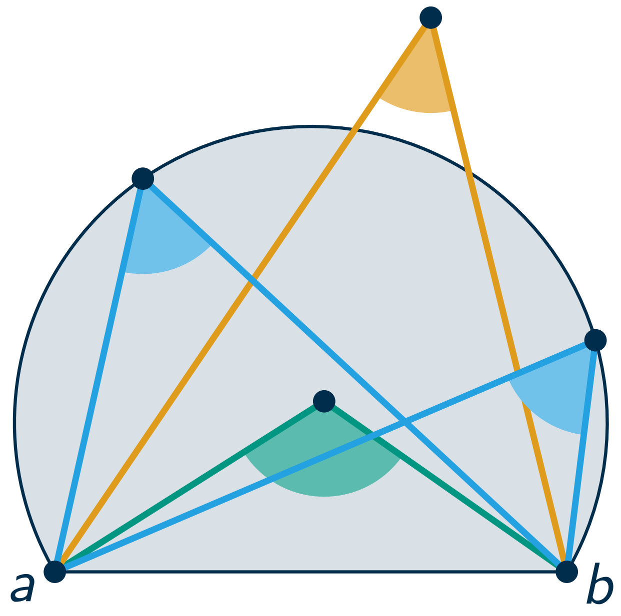
Thales’s Theorem

Thales’s Theorem
The line segments from a point on a circle to the endpoints of a diameter form a right angle.



Generalization

- fix an arbitrary chord of the circle (endpoints: a and b)
- also fix one of the two circular segment
- the line segments from every point on the circular arc to a and b form the same angle
- for points in the interior, the angle is bigger
- for points in the exterior, the angle is smaller
  (on the same side of ab)



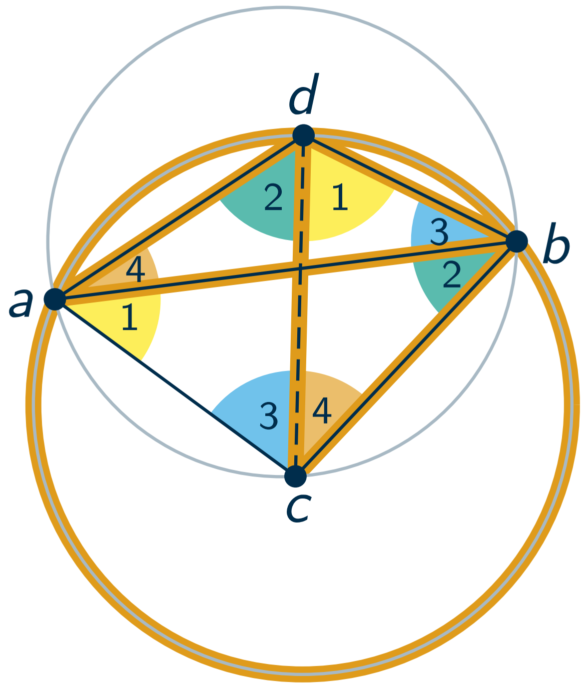
Which Edges Are Forbidden?

Lemma
Let $\triangle abc$ and $\triangle abd$ be two triangles such that a, b, c, d are in convex position. The circumcircle of $\triangle abc$ contains d if and only if ab is forbidden.

Proof: d in circumcircle of $\triangle abc \Rightarrow ab$ forbidden

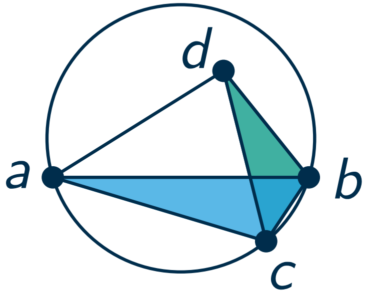
- note: the circumcircle of $\triangle abd$ then contains c
- consider bijection between angles at a and b, and “opposite” angles at c and d of the flipped edge
- claim: each angle at ab is smaller than corresponding angle at cd
- follows from Thales’s theorem, when looking at the proper triangles

$\Rightarrow$ minimum angle at cd is bigger $\Rightarrow$ ab is forbidden



Other Direction

- similar argument by looking at the minimum angle



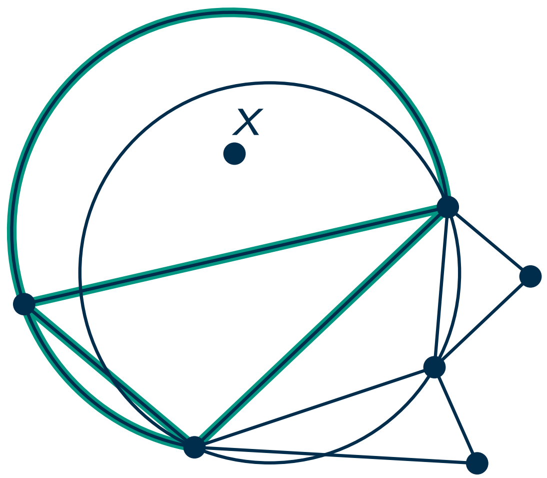
Forbidden Edges And Empty Circles

Theorem
A triangulation has a forbidden edge if and only if the circumcircle of a face contains a vertex.

Proof

- lemma: forbidden edge $\Rightarrow$ circumcircle contains vertex
- todo: circumcircle contains vertex $\Rightarrow$ forbidden edge
- consider the circumcircle of a triangle containing x
- lemma: x is adjacent $\Rightarrow$ forbidden edge
- problem: non-adjacent vertex x
  - consider triangle closer to x
  - circumcircle still contains x
  - iterate $\rightarrow$ at some point we should have x as neighbor
- cleaner argument: consider extreme situation, show contradiction

Lemma
Let $\triangle abc$ and $\triangle abd$ be two triangles such that a, b, c, d are in convex position. The circumcircle of $\triangle abc$ contains d if and only if ab is forbidden.



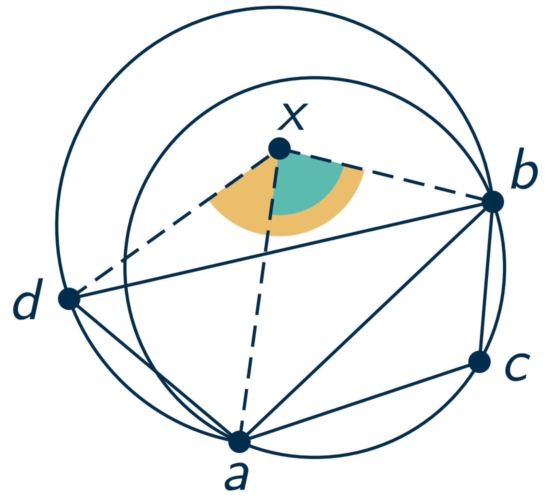
Forbidden Edges And Empty Circles

Theorem
A triangulation has a forbidden edge if and only if the circumcircle of a face contains a vertex.

Proof

- lemma: forbidden edge $\Rightarrow$ circumcircle contains vertex
- todo: circumcircle contains vertex $\Rightarrow$ forbidden edge
- consider the circumcircle of $\triangle abc$ containing x such that $\angle axb$ is maximal (without loss of generality: x lies next to ab)
- consider the other triangle $\triangle abd$ with side ab
- circumcircle of $\triangle abd$ also contains x
- $\angle dxb$ (or $\angle axd$) is bigger than $\angle axb$ $\Rightarrow$ contradiction

Lemma
Let $\triangle abc$ and $\triangle abd$ be two triangles such that a, b, c, d are in convex position. The circumcircle of $\triangle abc$ contains d if and only if ab is forbidden.



Locally Optimal Triangulations

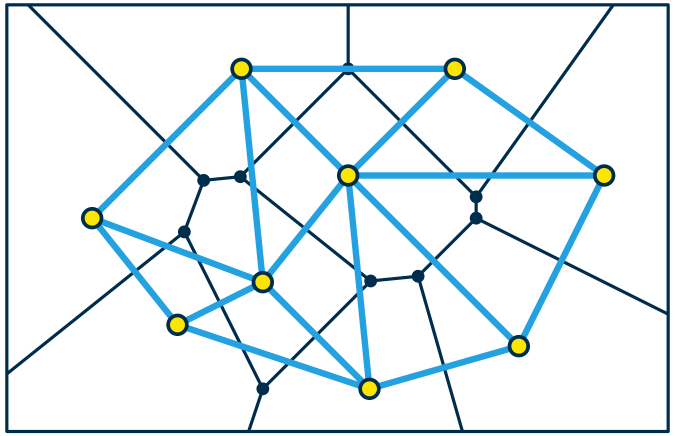
Equivalent:

- no forbidden edges
- locally optimal
- the circumcircle of each triangle is empty

Implications For Locally Optimal Triangulations

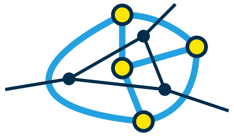
- for every triangle, the center of the circumcircle is a Voronoi vertex
- the triangle edges correspond to the Voronoi edges incident to the Voronoi vertex

  $\Rightarrow$ the triangulation is the dual graph of the Voronoi diagram



Conversely: Consider The Dual Of The Voronoi Diagram

- forms a triangulation (assumption: no four vertices lie on the same circle)
- circumcircle of each face is empty $\Rightarrow$ locally optimal triangulation



Don’t we need to show that there are no edge crossings?

Delaunay Triangulation

The dual graph of the Voronoi diagram is called **Delaunay triangulation**.

Just Seen

- the Delaunay triangulation is locally maximal
- if $\mathcal{T}$ is locally maximal, then $\mathcal{T}$ is the Delaunay triangulation

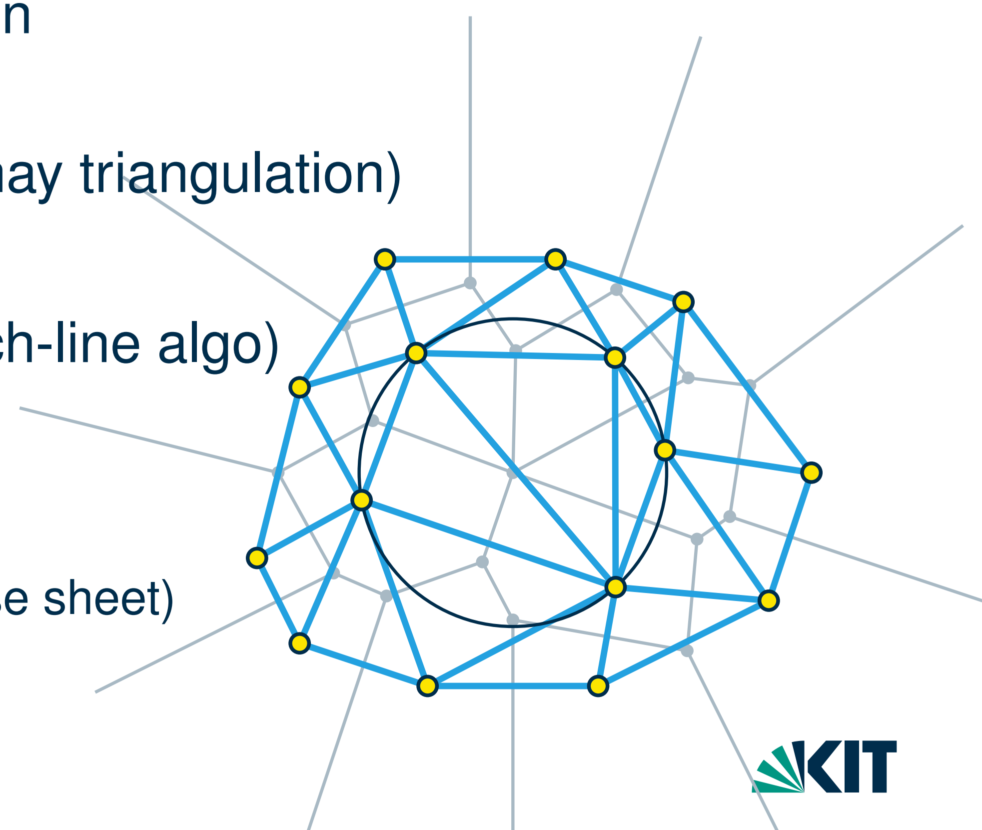
Implications

- the locally maximal triangulation is unique (it is the Delaunay triangulation)
- the Delaunay triangulation is globally maximal
- it can be computed in $O(n \log n)$ time (e.g., using the beach-line algo)

Multiple Points On A Circle

- Voronoi vertex has higher degree
- dual is not a triangulation
- multiple triangulations possible

(more on this on the exercise sheet)



Wrap-Up

Seen Today

- the Delaunay triangulation has nice properties
  - lexicographically maximal angle vector
  - dual of the Voronoi diagram
- proof technique: local optimum unique $\Rightarrow$ must be the global optimum

What Else Is There?

- the Delaunay triangulation has other nice properties (e.g., the MST is part of the triangulation)
- different optimization criteria: minimizing the total edge length is NP-hard

Exam Scheduling

Monday	Tuesday	Wednesday	Thursday	Friday	
28				1	
4				8	
11				15	August
18				22	
25				29	
1				5	
8				12	
15				19	September
22				26	
29				3	
6				10	
13				17	October
20				24	
27				31	

Help Us Improve

Evaluation

- survey open until 18:00
- link also in discord
- exercise sessions are evaluated separately

https://onlineumfrage.kit.edu/evasys/online.php?p=G4MUD