Computational Geometry

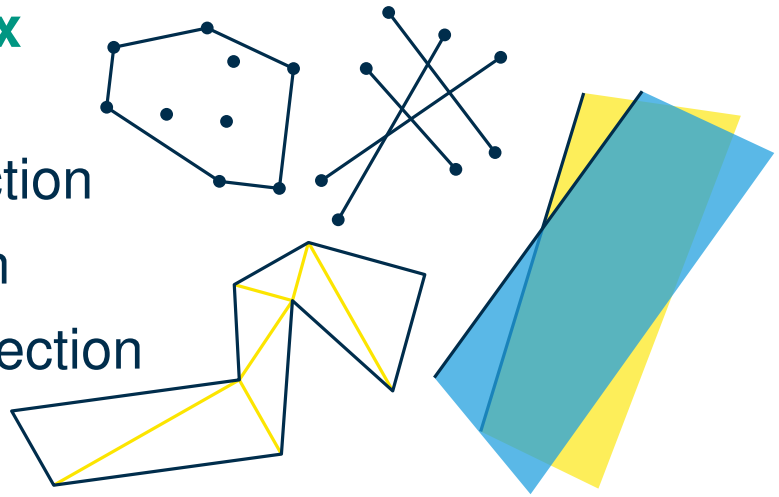
Voronoi Diagram: Fortunes Algorithm

Thomas Bläsius

Overview

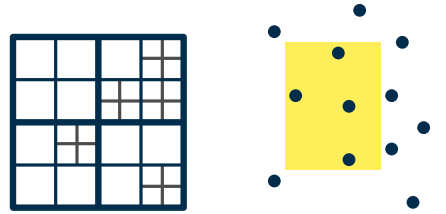
Basic Toolbox

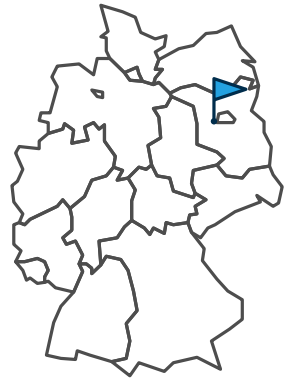
- convex hull
- line intersection
- triangulation
- plane intersection



Geometric Data Structures

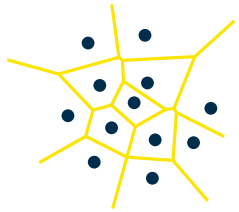
- orthogonal range searching
- space partitioning
- point location

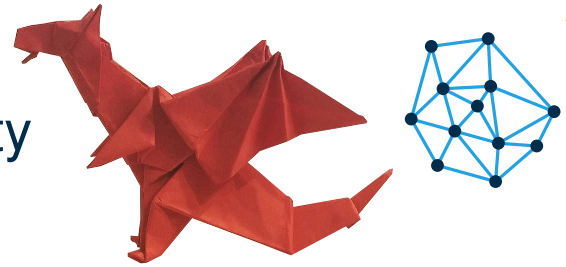




Advanced Toolbox

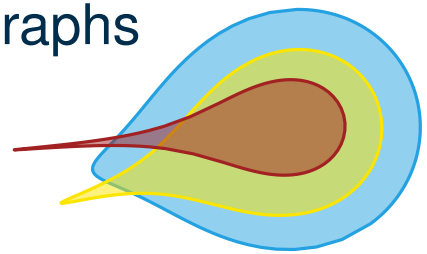
- Voronoi diagrams
- Delaunay triangulations
- origami
- complexity

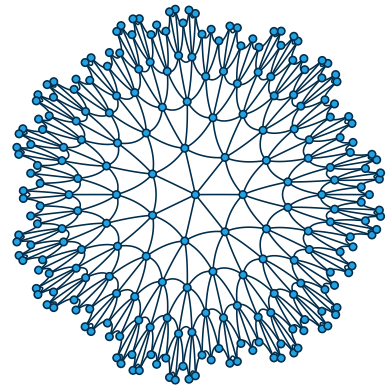




Related Topics

- What is geometry?
- hyperbolic geometry
- geometric graphs

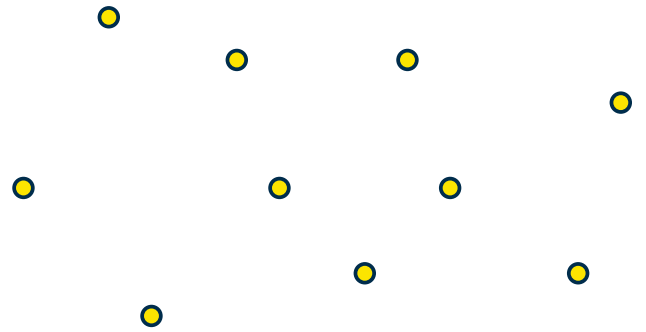




Who Uses Which Mailbox?

Situation

- given: set S of sites
- Which points are closest to which site?
  (assuming Euclidean distances)



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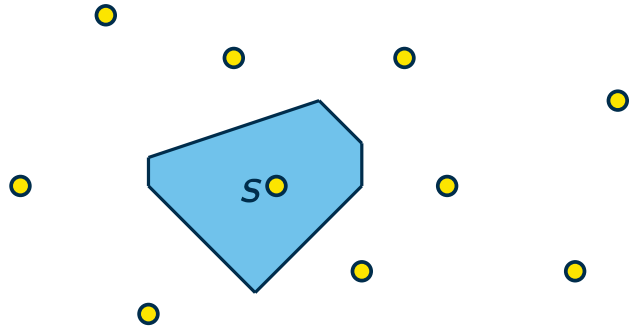

Voronoi-Diagramm – Some Definitions

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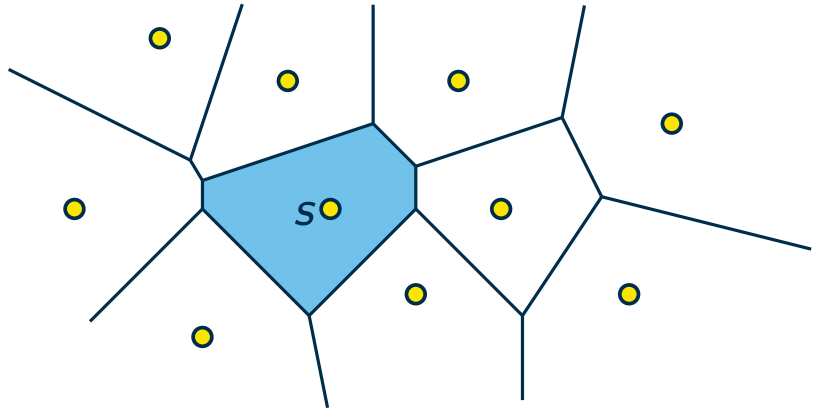

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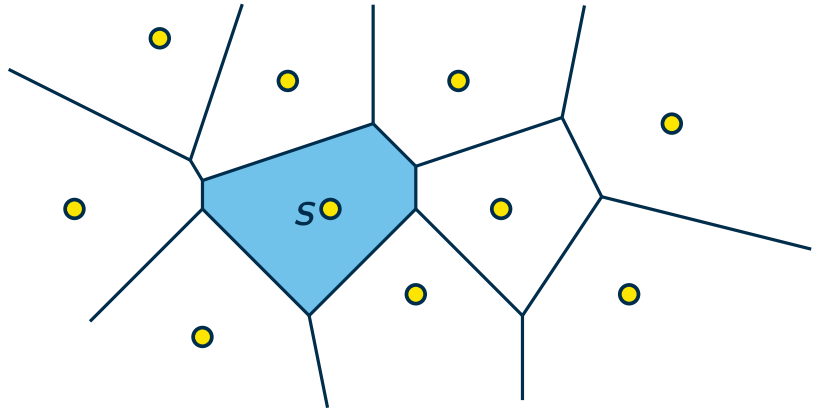

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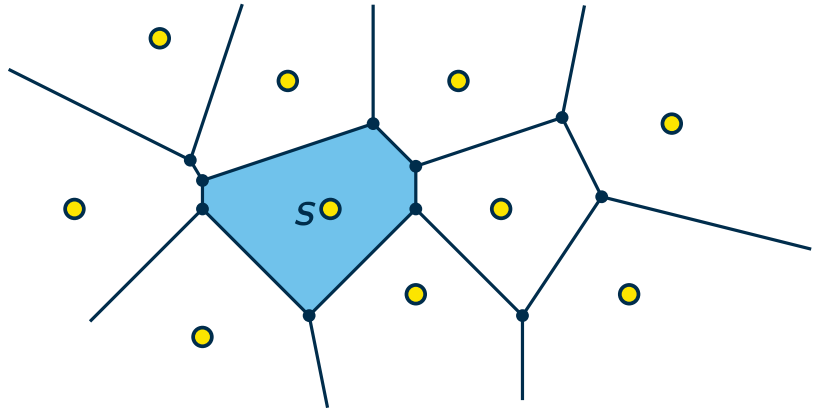

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- boundary between cells are called **edges**
- endpoints of these edges are **vertices**

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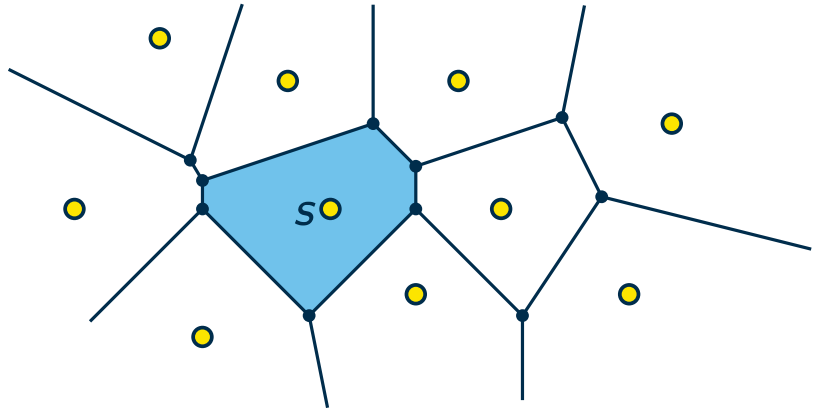

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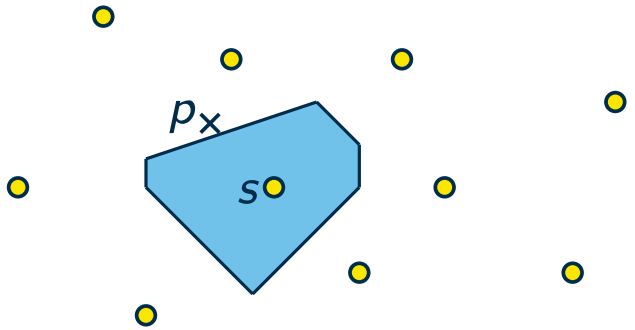
Goal For Today: compute $\mathrm{Vor}(S)$

Basic Properties



Points Belonging To The Cell Of s

- Why does p not lie in the cell of s?

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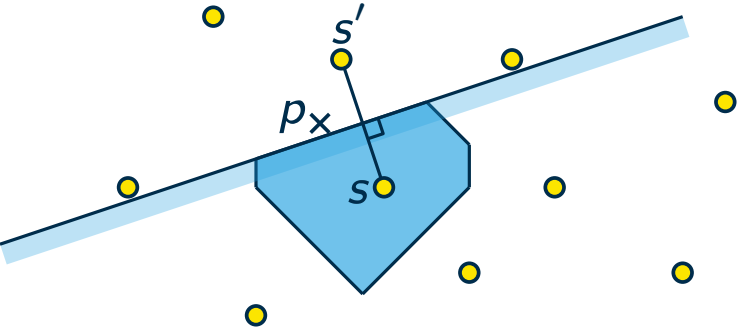

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- Why does p not lie in the cell of s?
- s' is closer $p \Leftrightarrow s \perp s'$ separates s from p

  perpendicular bisector of s and s'

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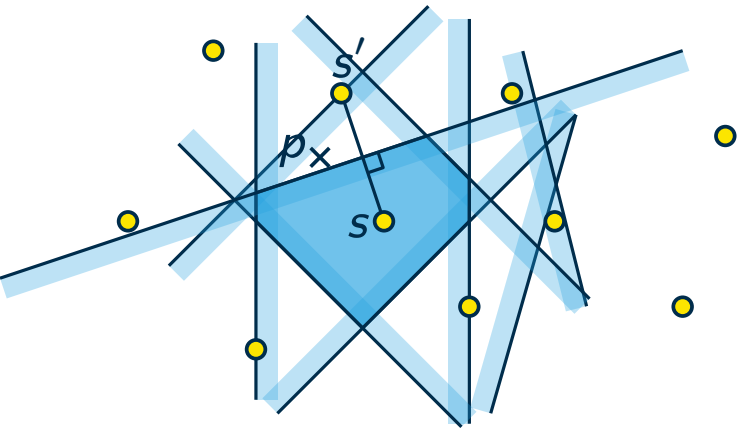

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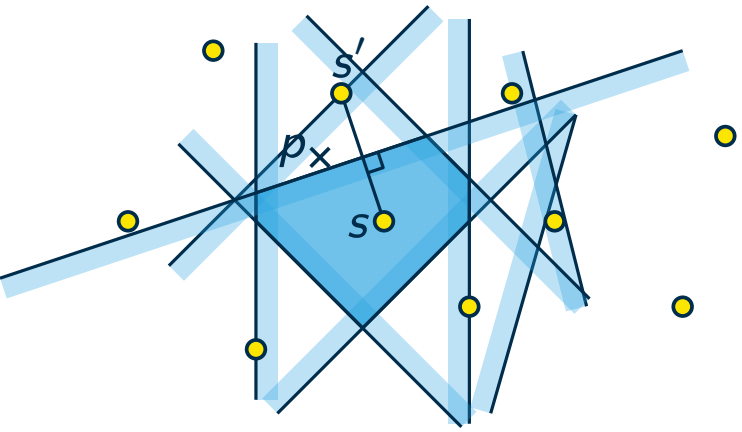

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How slow exactly?

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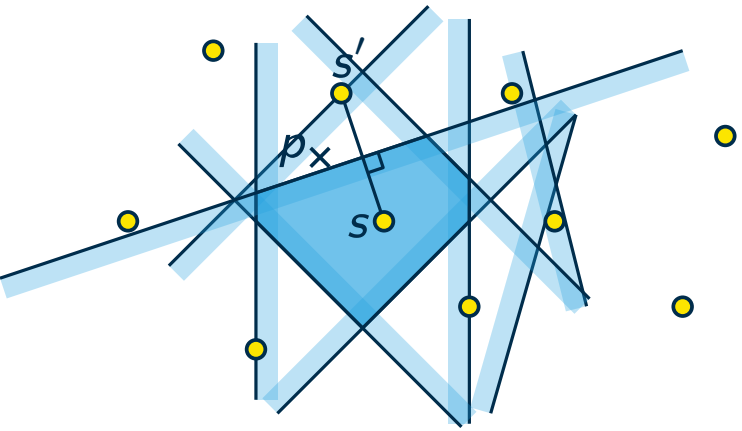

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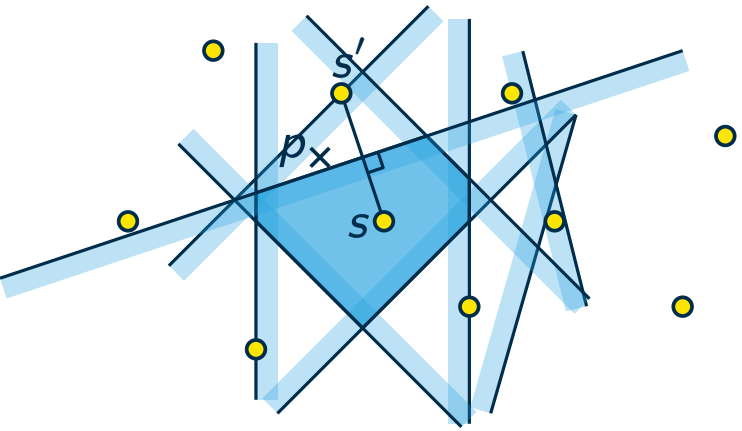

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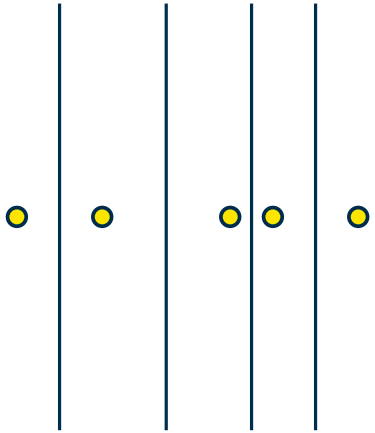
(viewing $\mathrm{Vor}(S)$ as the union of edges, i.e., the geometric “graph”)



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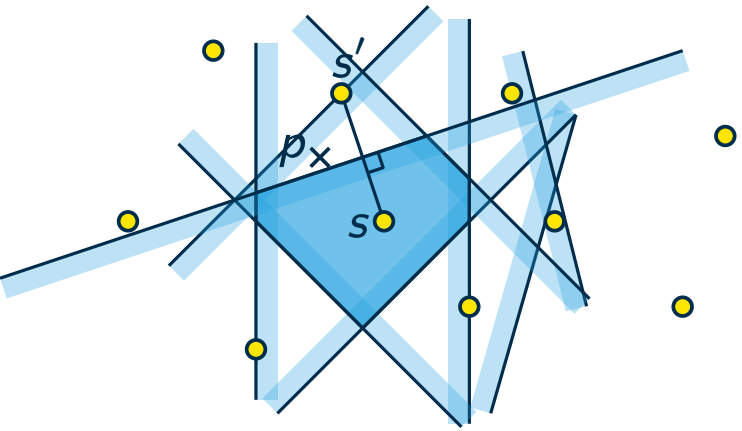

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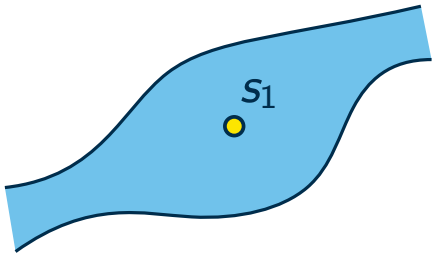

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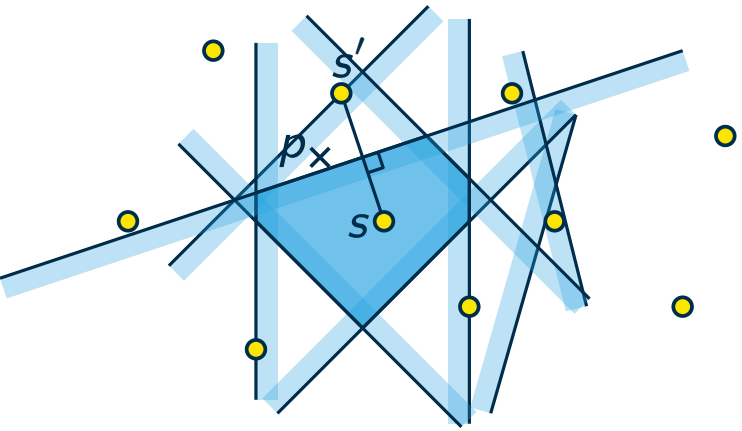

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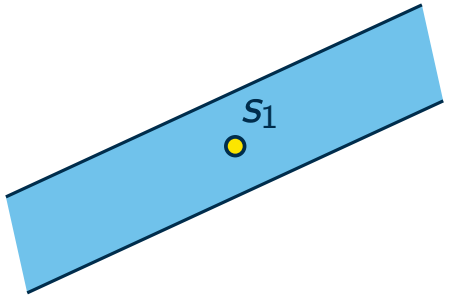

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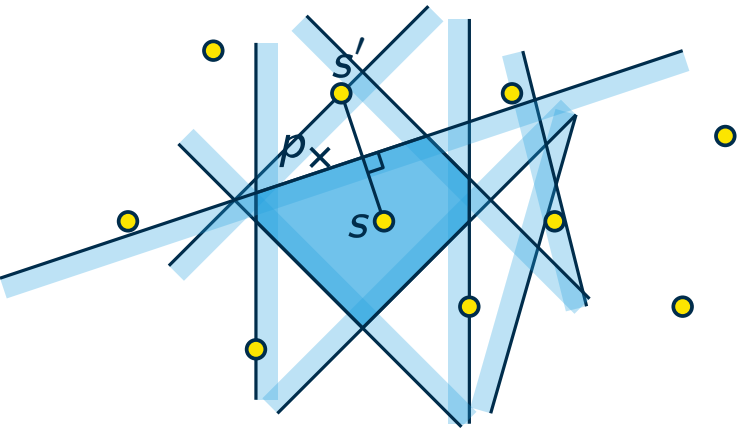

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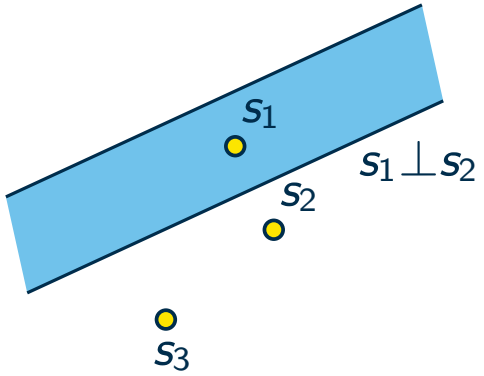

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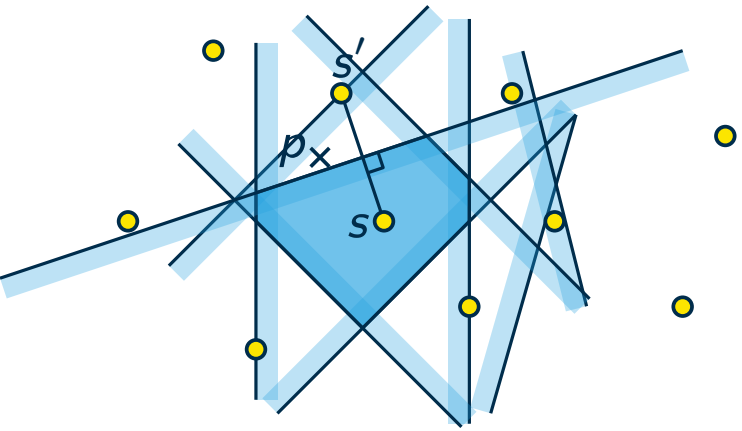

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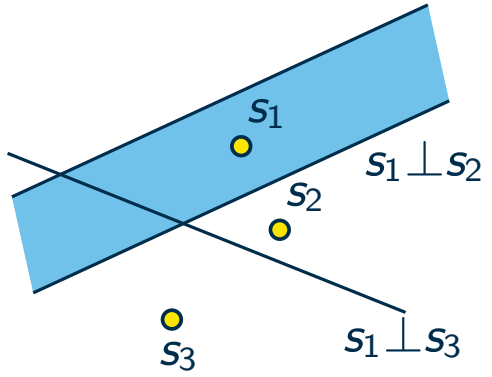
  - $\Rightarrow s_1 \perp s_2$ intersects $s_1 \perp s_3$



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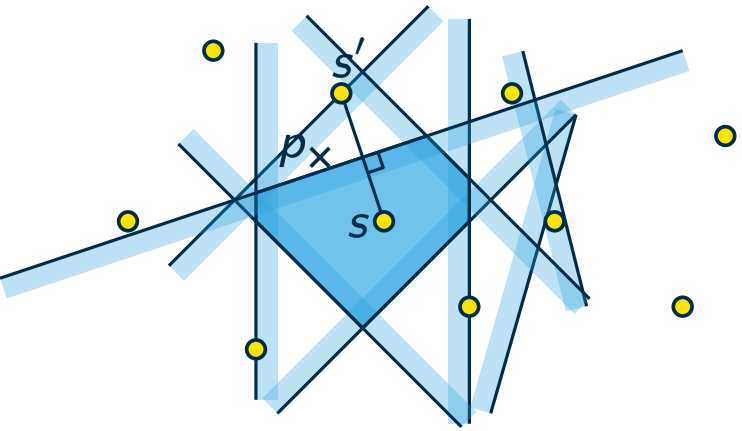

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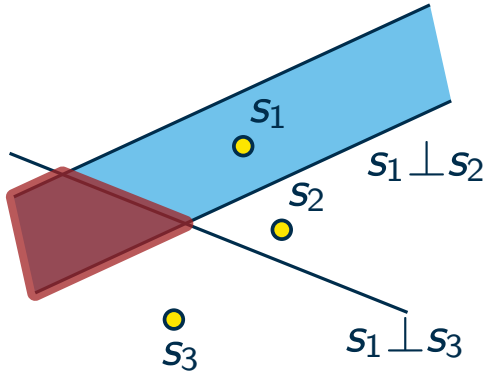
  - $\Rightarrow$ $s_1 \perp s_2$ intersects $s_1 \perp s_3$ $\Rightarrow$ contradiction



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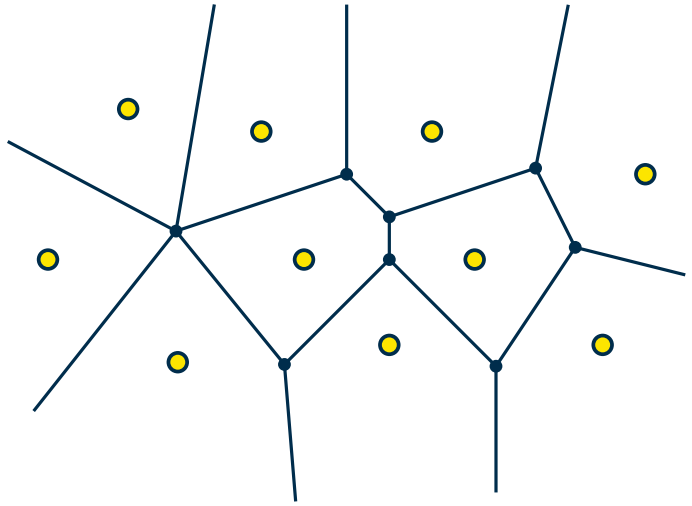

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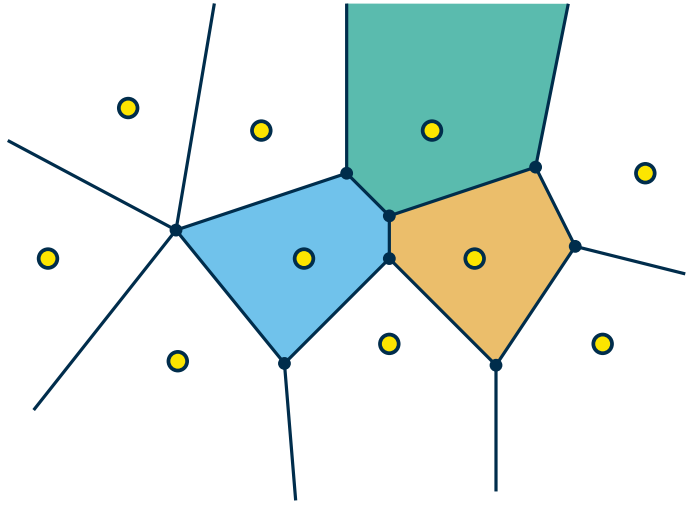

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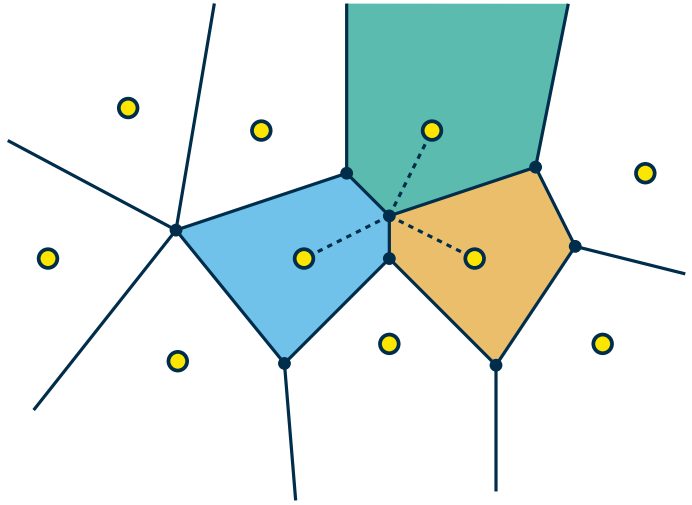

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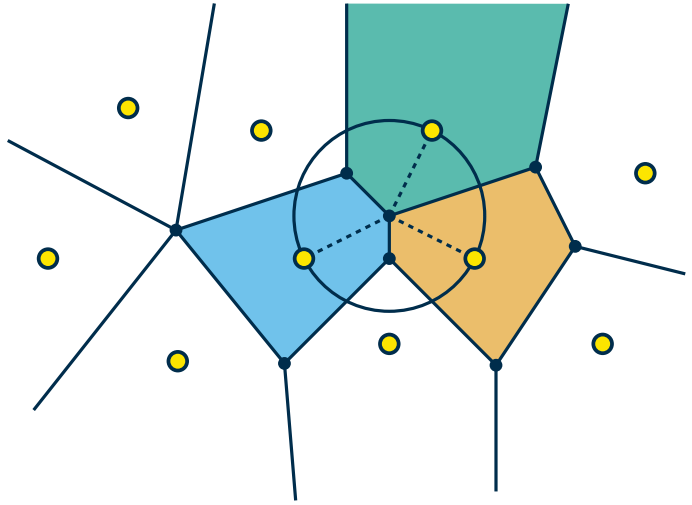
- at each a vertex v: ≥ 3 cells meet
- v has equal distance to these sites $\Rightarrow$ circle with center v



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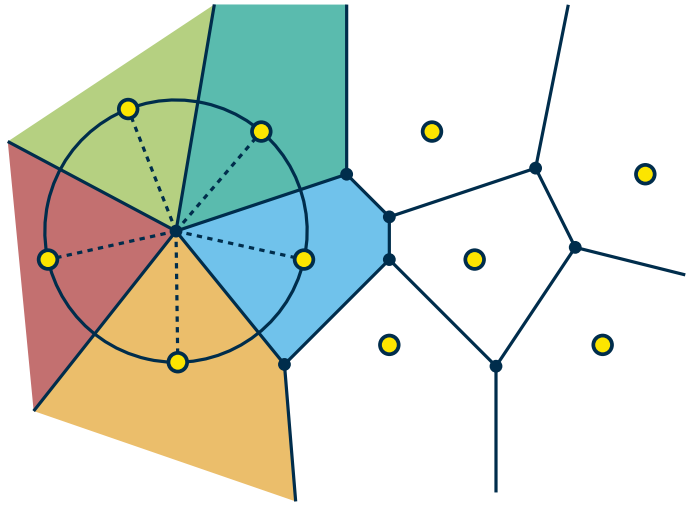

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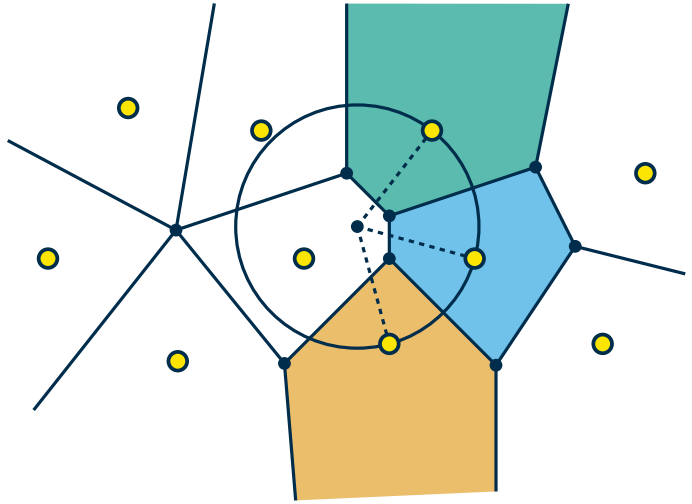
- not every circle through ≥ 3 sites has vertex as center



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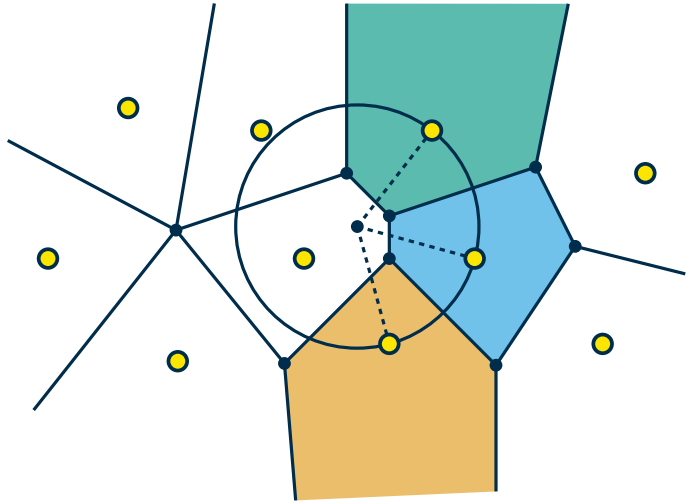

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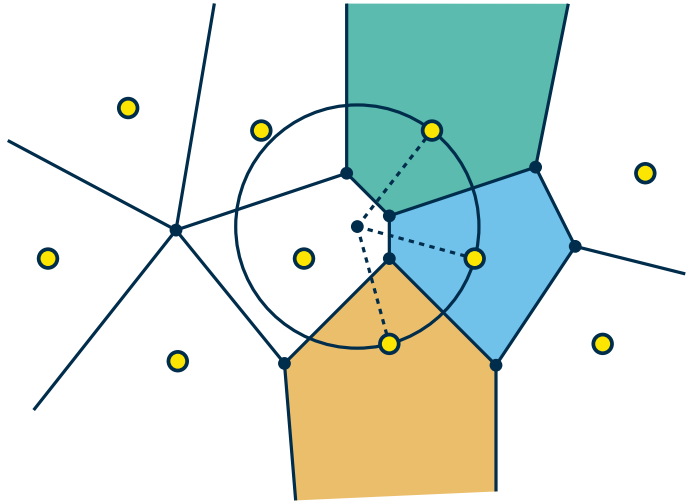
- not every circle through ≥ 3 sites has vertex as center
- the circle cannot have any sites in its interior
- $C_S(p) =$ largest circle around p, with no site from S in its interior



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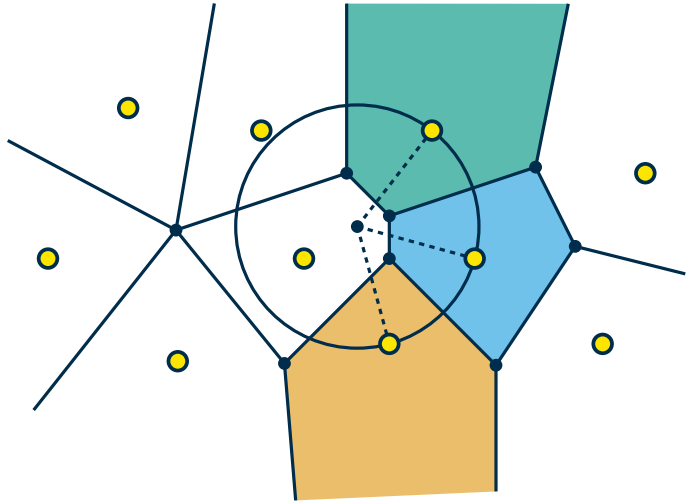
Which Points Are Vertices Of Vor(S)?

- at each a vertex v: ≥ 3 cells meet
- v has equal distance to these sites $\Rightarrow$ circle with center v
- not every circle through ≥ 3 sites has vertex as center
- the circle cannot have any sites in its interior
- $C_S(p) =$ largest circle around p, with no site from S in its interior



Theorem **(vertices of Vor(S))**
The point $p \in \mathbb{R}^2$ is a vertex of $\mathrm{Vor}(S)$ if and only if at least three sites lie on the circle $C_S(p)$.

Basic Properties (2)

Theorem **(proof: exercise)**
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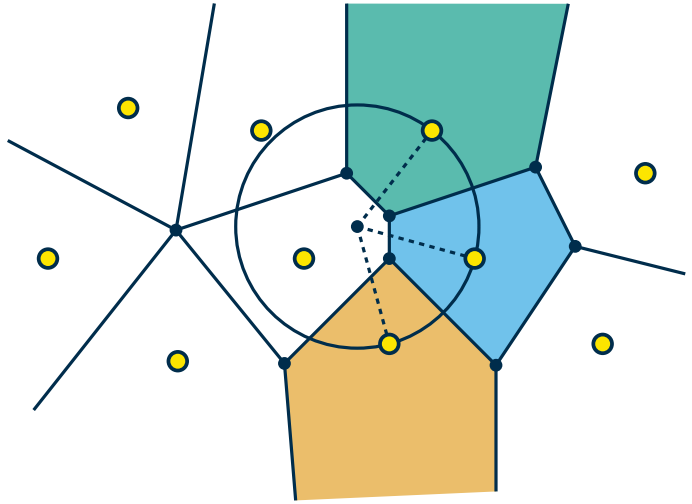


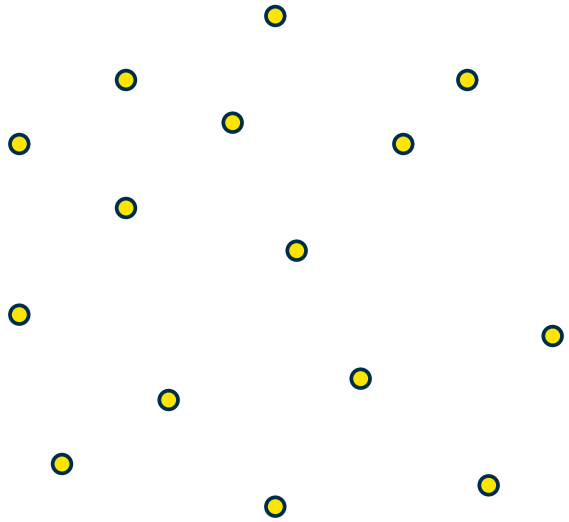
Theorem **(vertices of Vor(S))**
The point $p \in \mathbb{R}^2$ is a vertex of $\mathrm{Vor}(S)$ if and only if at least three sites lie on the circle $C_S(p)$.

Which similar statement holds for edges?

Sweep-Line Algo

General Approach: Sweep Line

- move sweep line from top to bottom
- at every point in time: know $\mathrm{Vor}(S)$ above the sweep line



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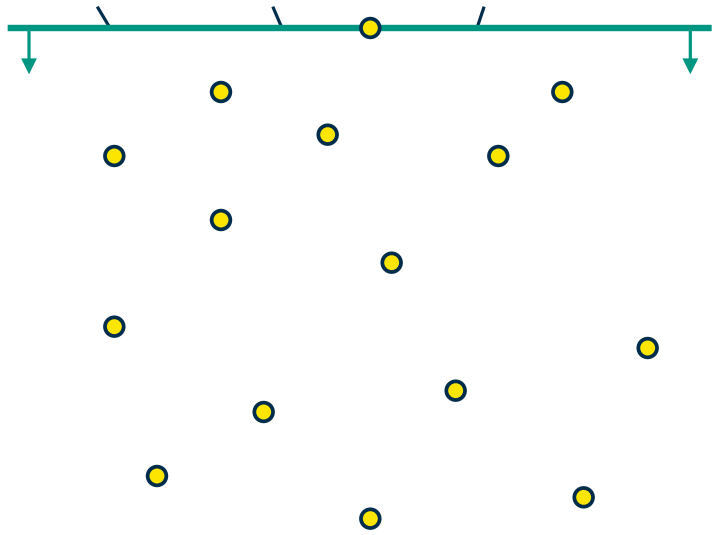

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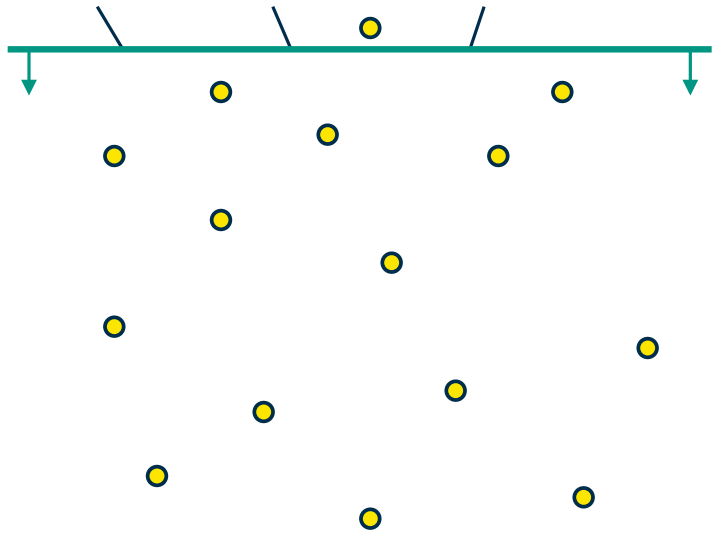

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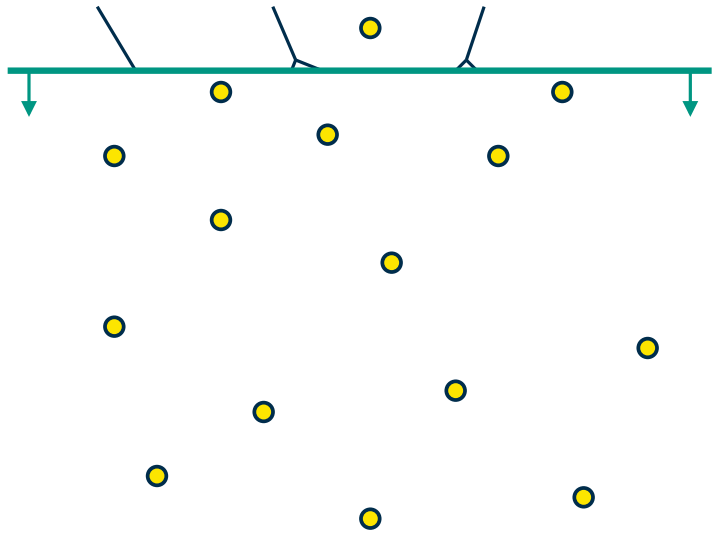

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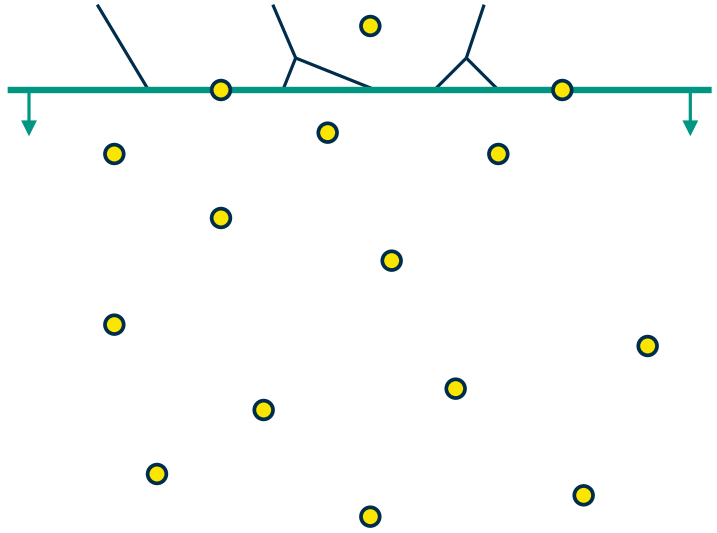

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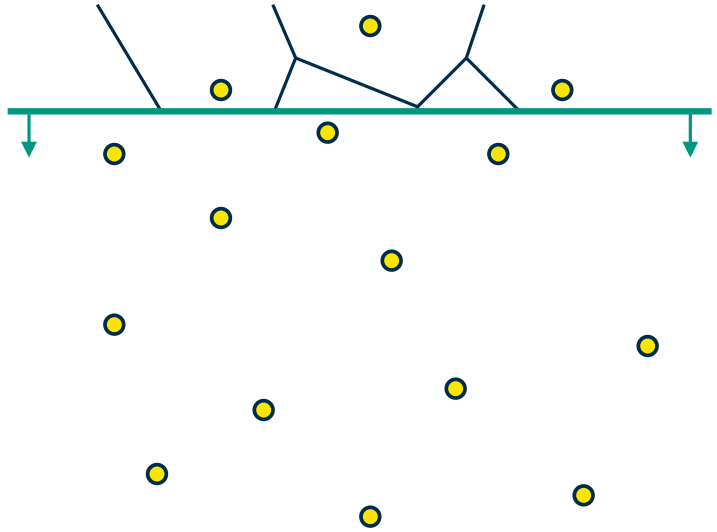

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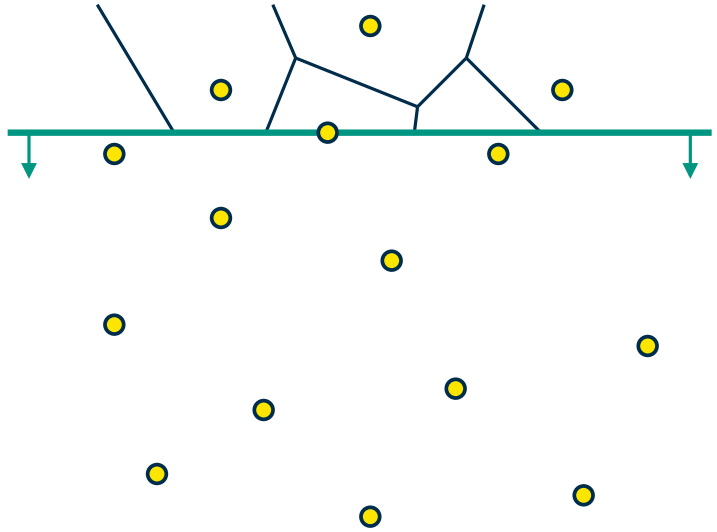

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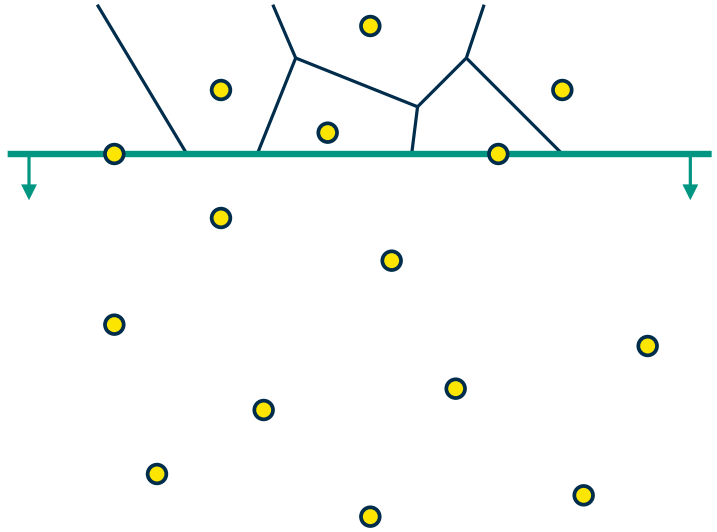

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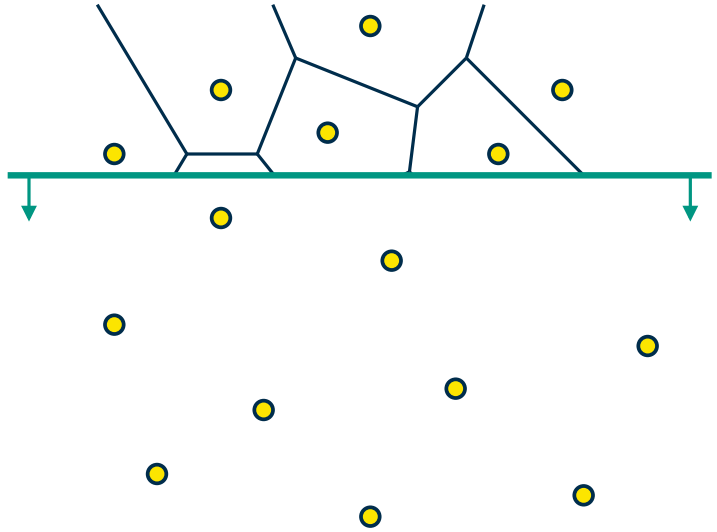

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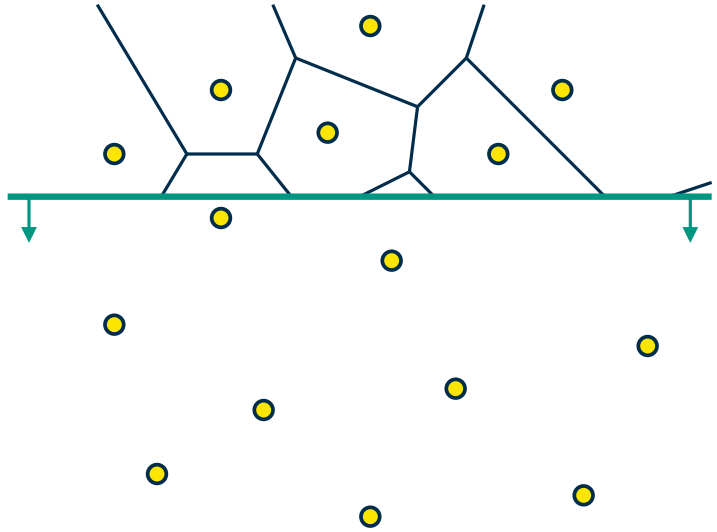

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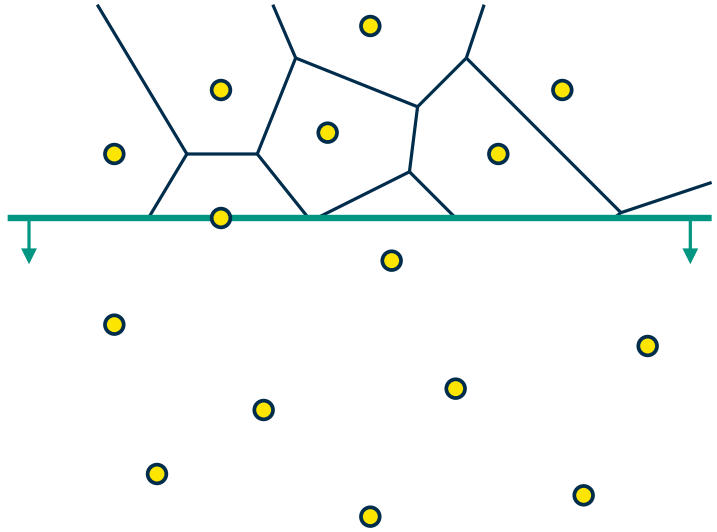

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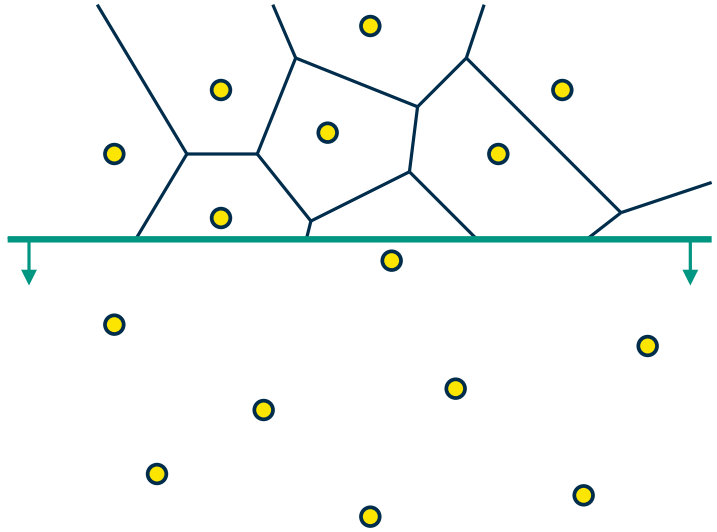

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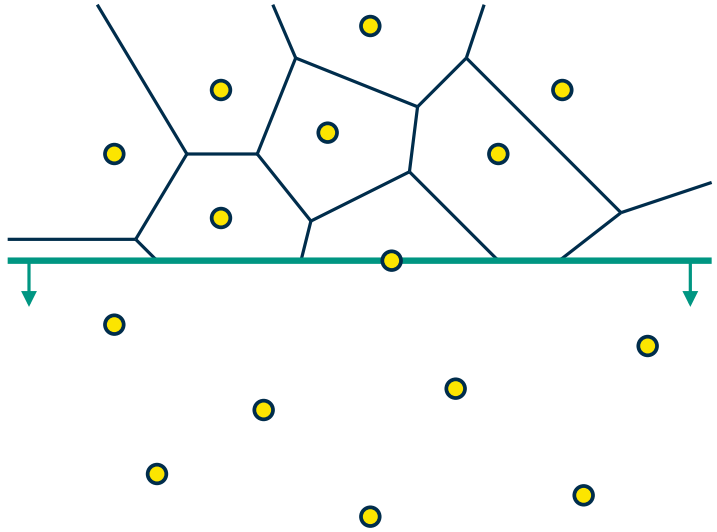

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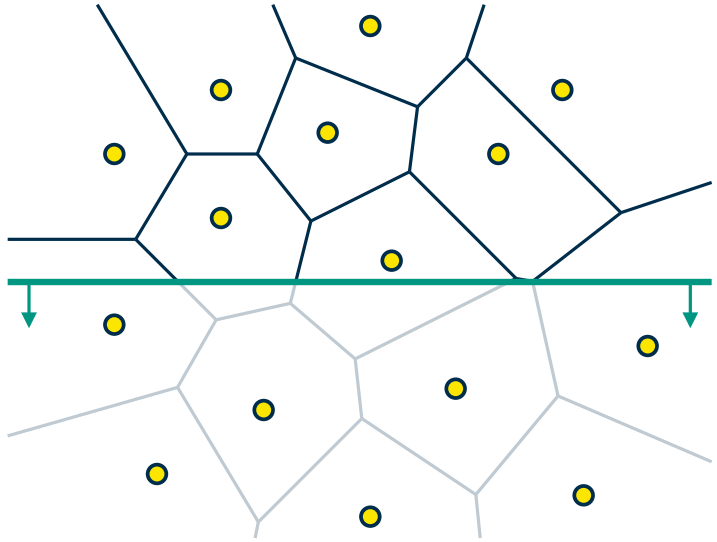

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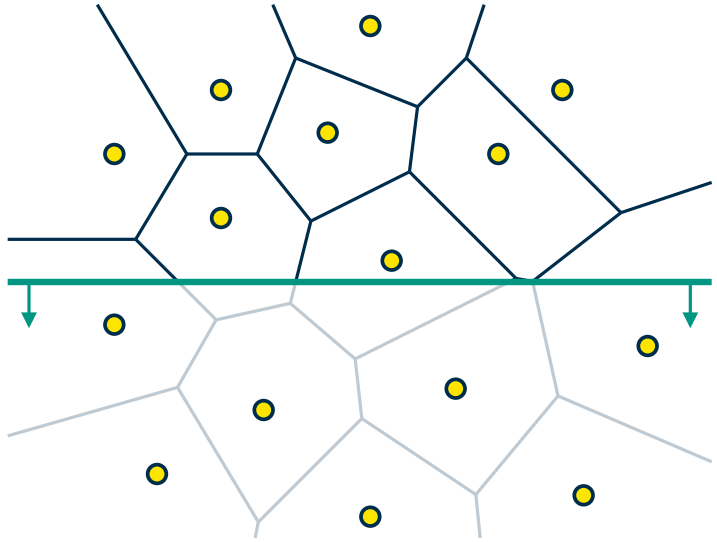

Problem

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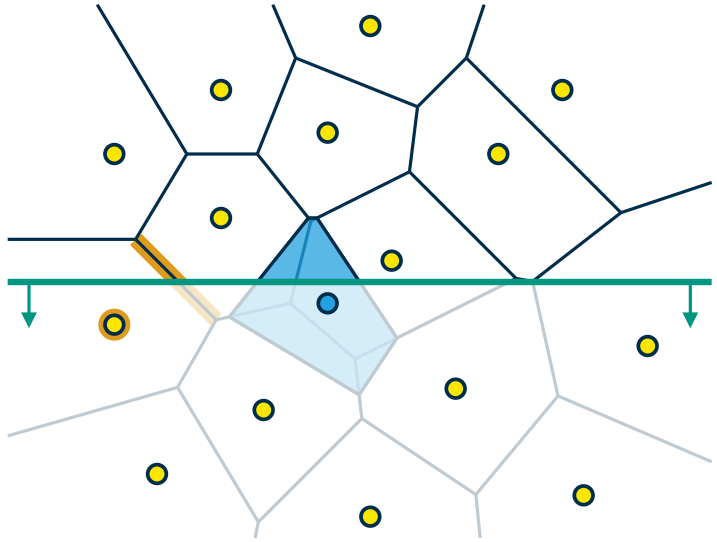
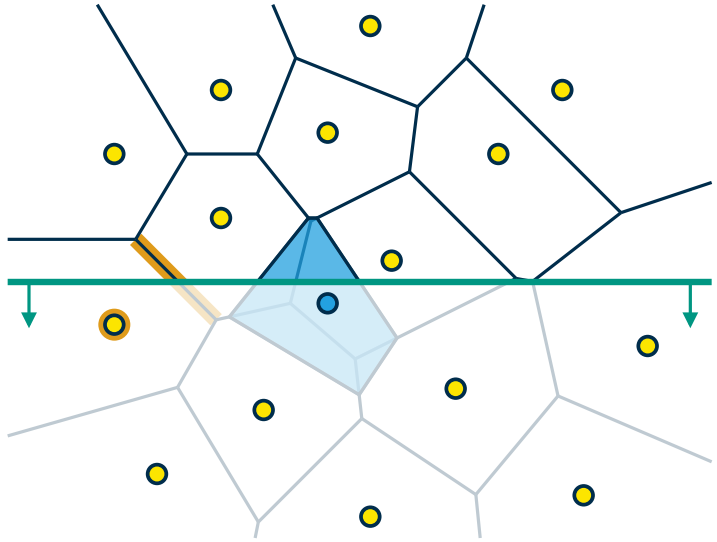


Problem

- parts above the sweep line depend on sites below the sweep line

Sweep-Line Algo



General Approach: Sweep Line

- move sweep line from top to bottom
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Problem

- parts above the sweep line depend on sites below the sweep line
- that (hopefully) happens only slightly above the sweep line!?!

Sweep-Line Algo

General Approach: Sweep Line

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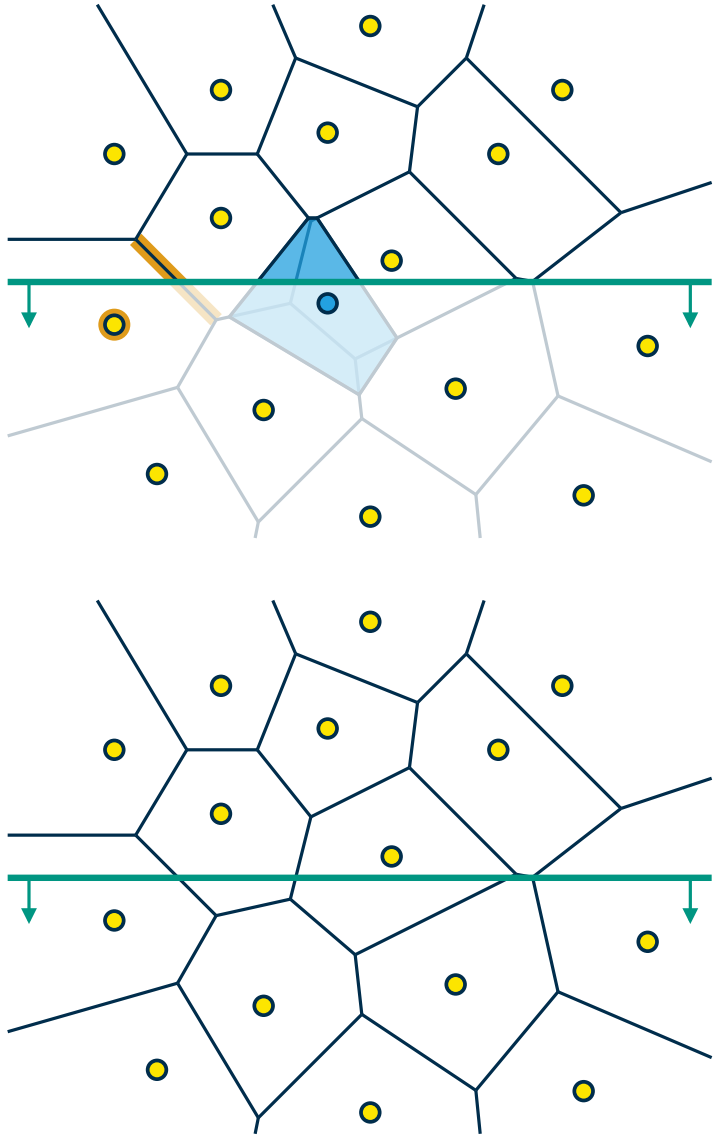
Adjusted Sweep-Line Approach

- region of seen sites $\neq$ region in which $\mathrm{Vor}(S)$ is computed



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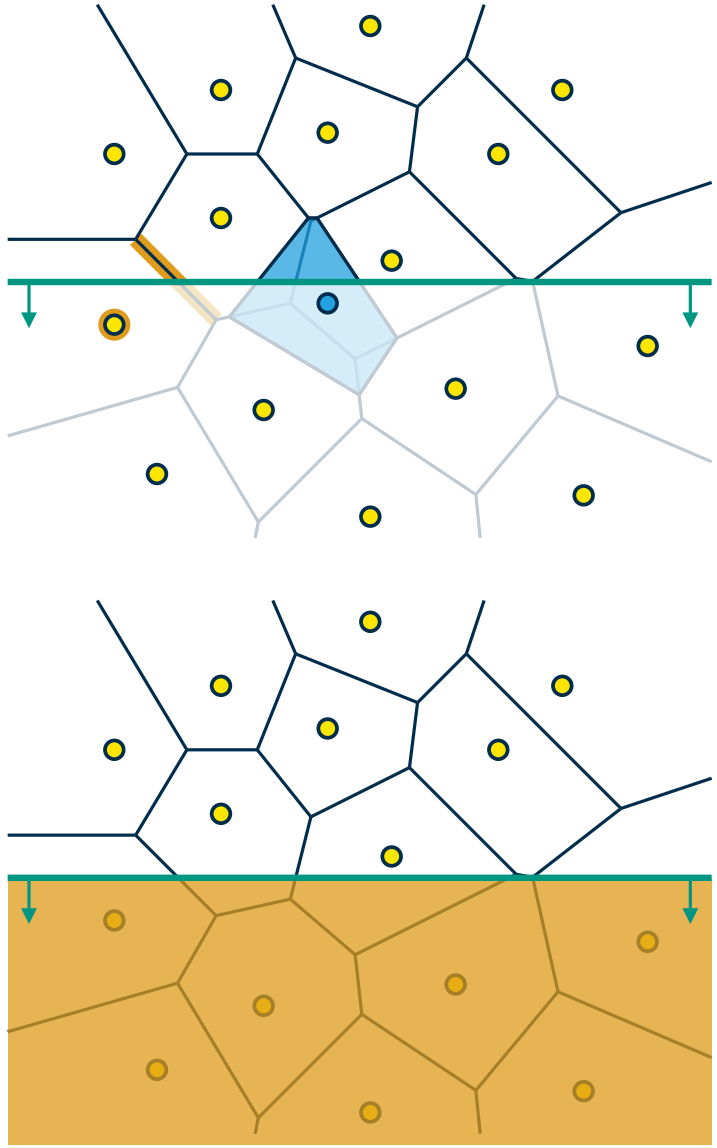
Adjusted Sweep-Line Approach

- region of seen sites $\neq$ region in which $\mathrm{Vor}(S)$ is computed
- below the sweep line: unknown area



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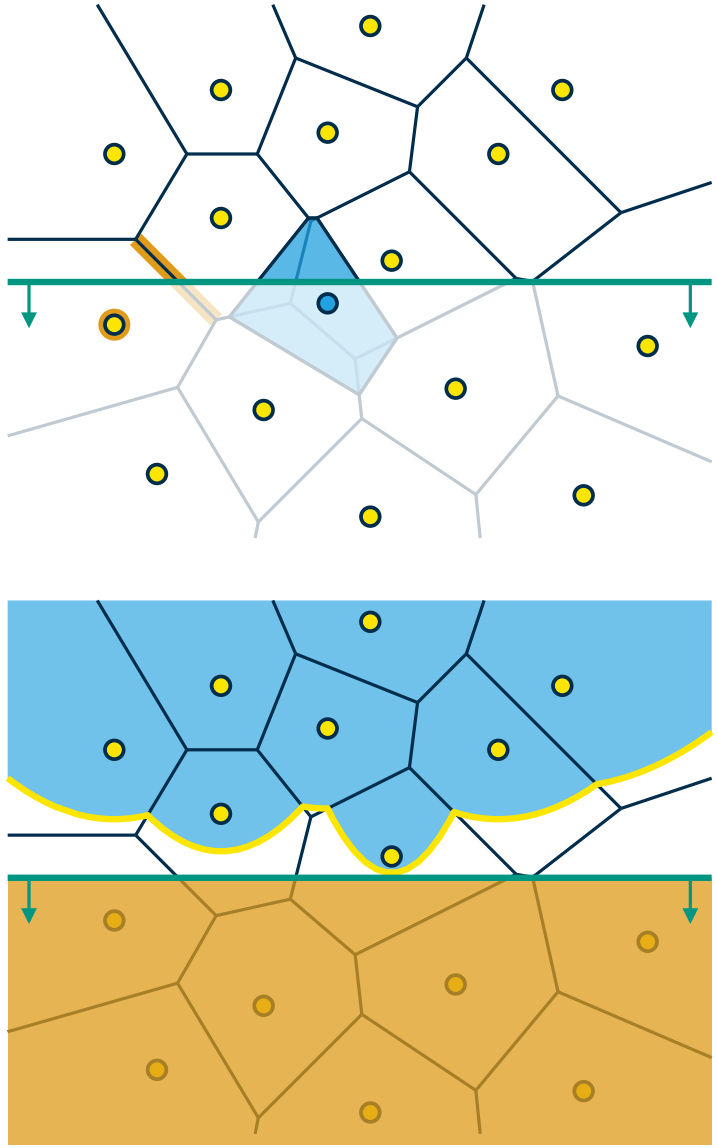
- below the sweep line: unknown area
- some region above the sweep line: known area



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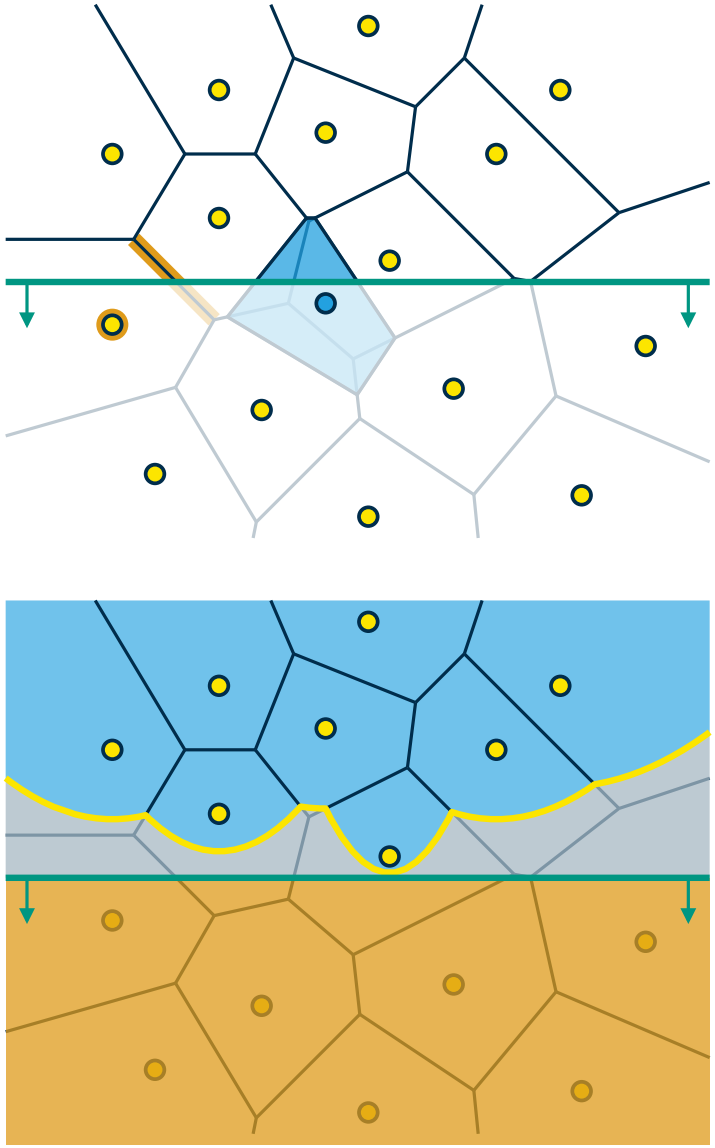
- below the sweep line: unknown area
- some region above the sweep line: known area
- some gray area in between



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Where Does The Known Area End?

- consider a point $p \in \mathbb{R}^2$ above ℓ

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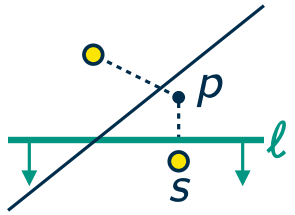

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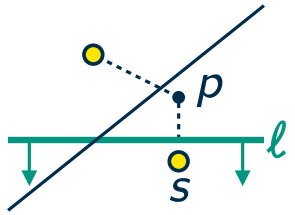

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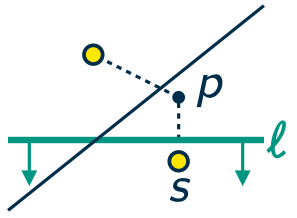

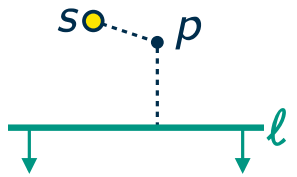

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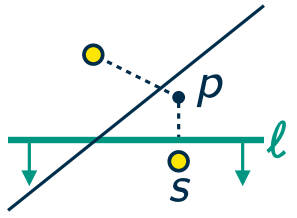

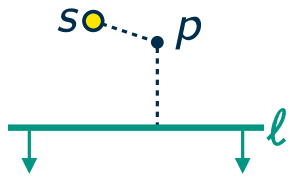

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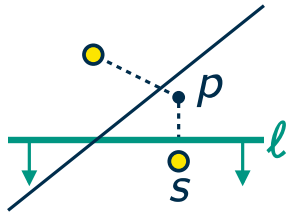

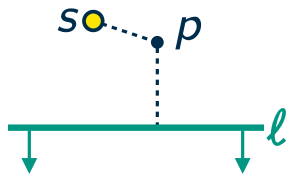

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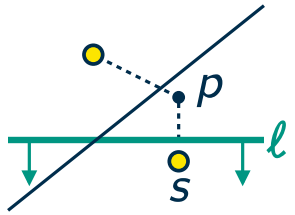

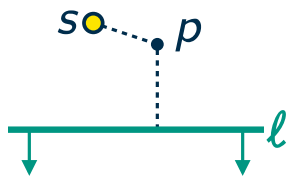

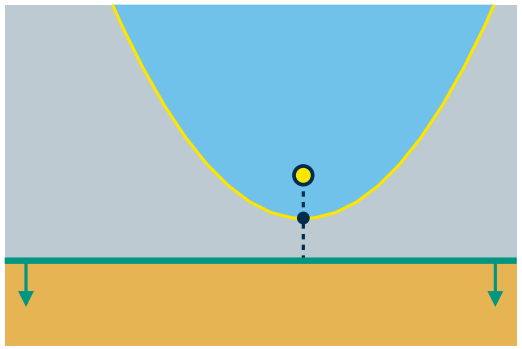

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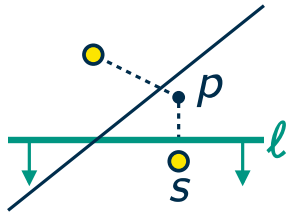

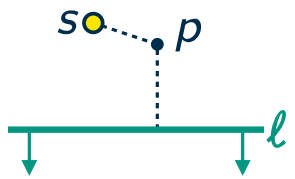

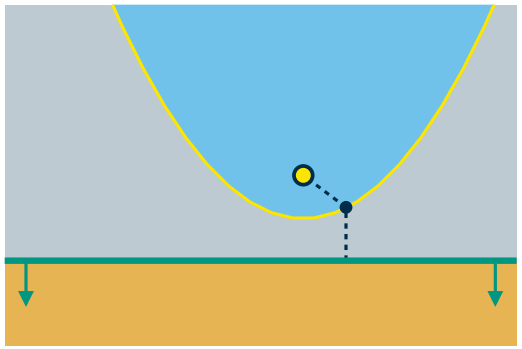

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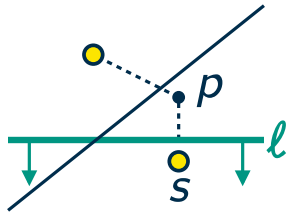

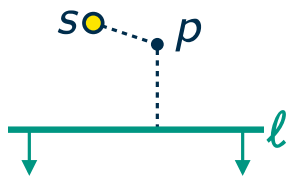

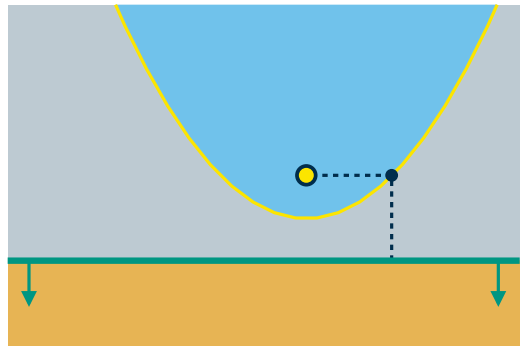

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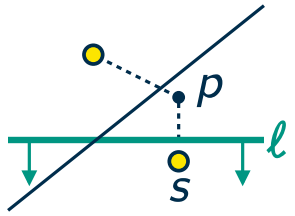

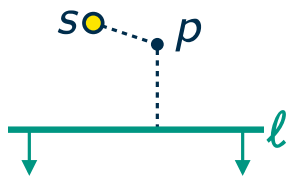

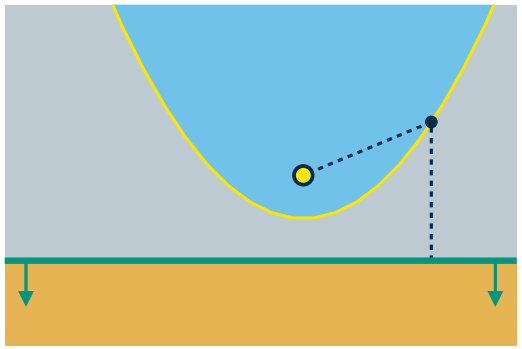

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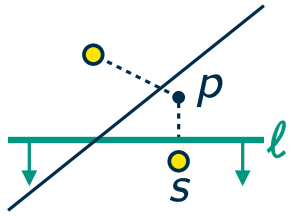

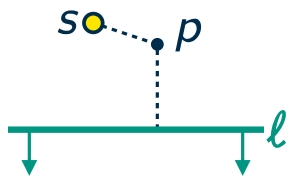

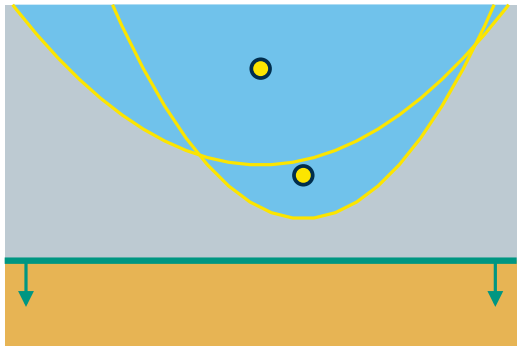

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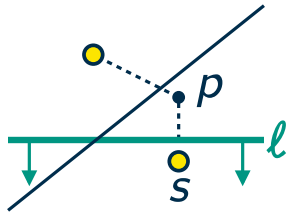

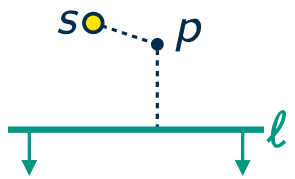

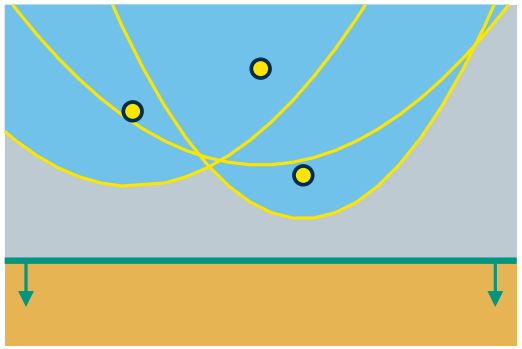

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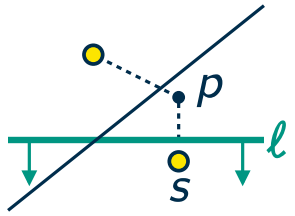

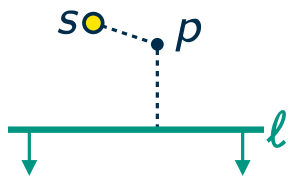

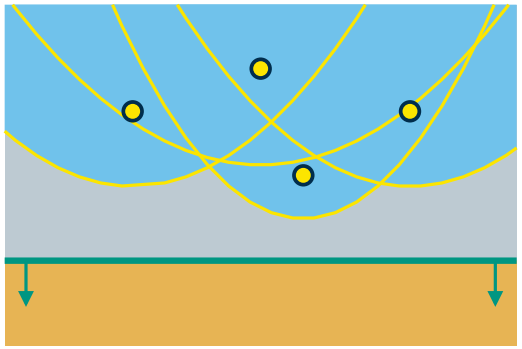

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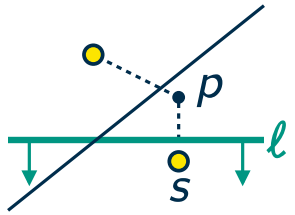

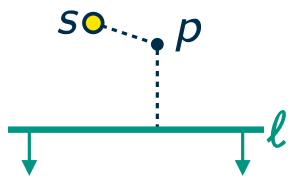

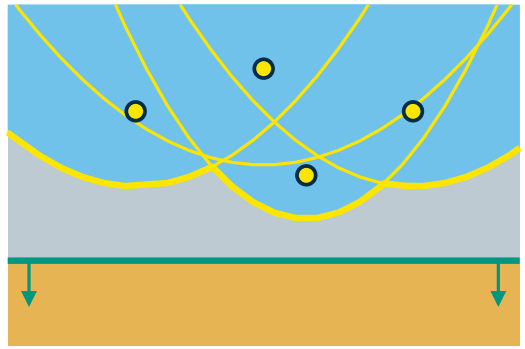

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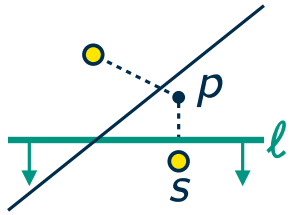
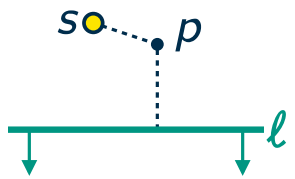
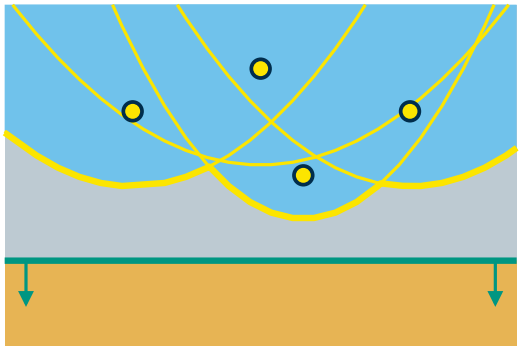
- each piece of the curve belongs to one site







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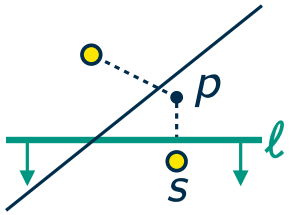

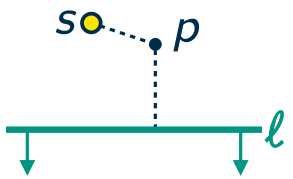

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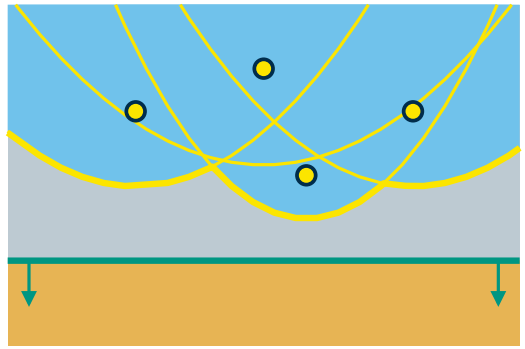
- each piece of the curve belongs to one site
- next slide: each piece is a parabola



Shape Of The Beach Line

Relevant Coordinates

- $s = (s_x, s_y)$
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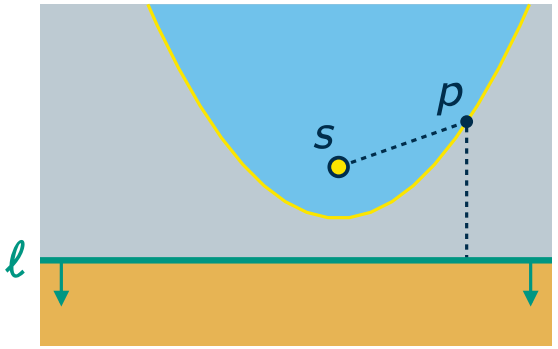
- y-Koordinate von ℓ: ℓ_y



Shape Of The Beach Line

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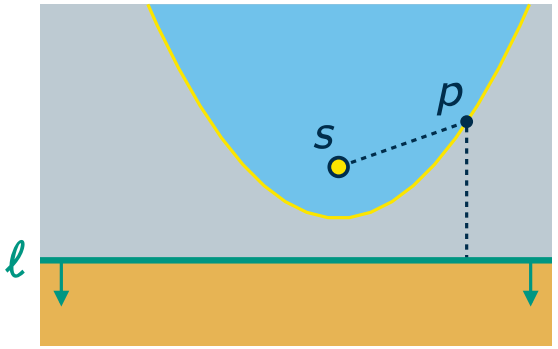

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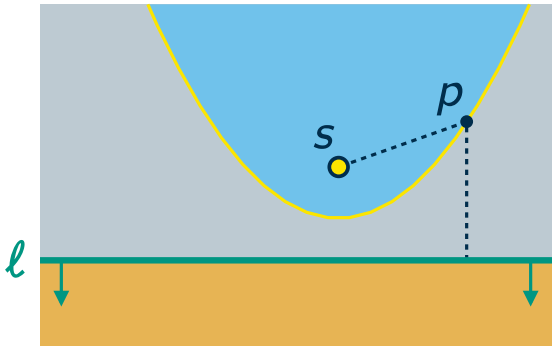

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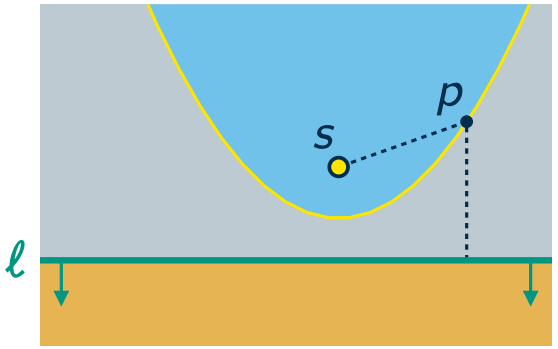

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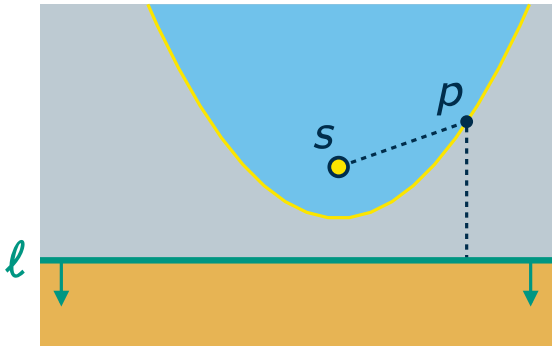

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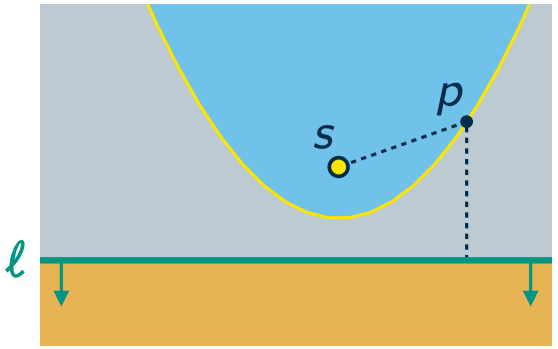

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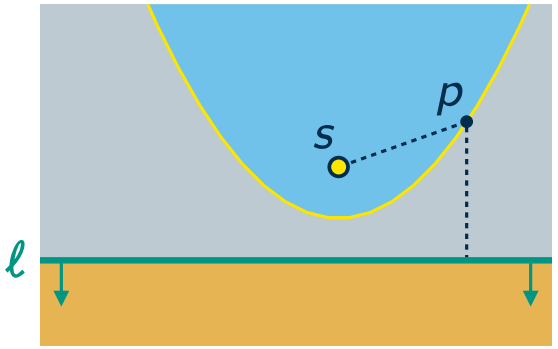

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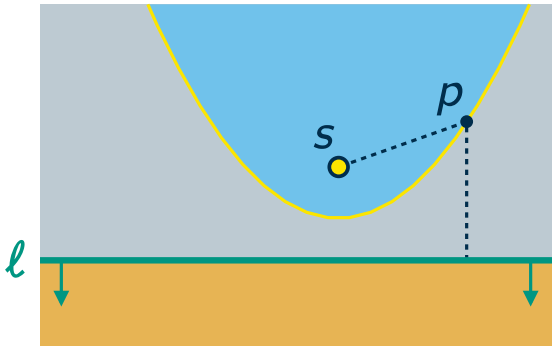

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translation in x-direction

parabola

$$\frac{(p_x - s_x)^2}{2s_y - 2\ell_y} + \frac{s_y + \ell_y}{2} = p_y$$

translation in y-direction

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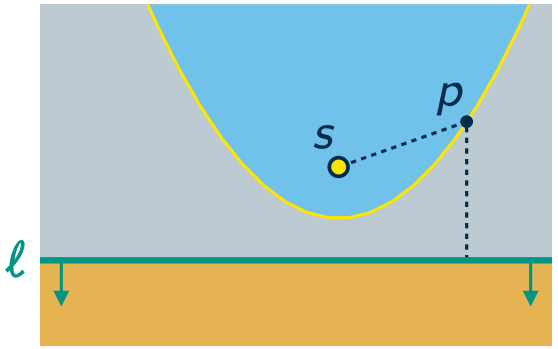

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$$(p_x - s_x)^2 + p_y^2 - 2p_y s_y + s_y^2 = p_y^2 - 2p_y \ell_y + \ell_y^2$$

$$(p_x - s_x)^2 + s_y^2 - \ell_y^2 = 2p_y s_y - 2p_y \ell_y$$

$$\frac{(p_x - s_x)^2}{2s_y - 2\ell_y} + \frac{s_y^2 - \ell_y^2}{2s_y - 2\ell_y} = p_y$$

translation in x-direction

parabola

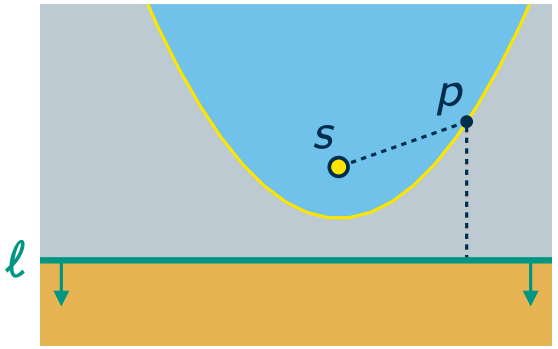
$$\frac{(p_x - s_x)^2}{2s_y - 2\ell_y} + \frac{s_y + \ell_y}{2} = p_y$$

translation in y-direction

- parabola with vertex at the center between s and ℓ
- larger $\text{dist}(s,\ell)$ yields a wider parabola

Relevant Coordinates

- $s = (s_x, s_y)$
- $p = (p_x, p_y)$
- y-Koordinate von ℓ: ℓ_y



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Sweep-Line Algo

- sweep-line status: state of the beach line
- events: points in time, when BL changes

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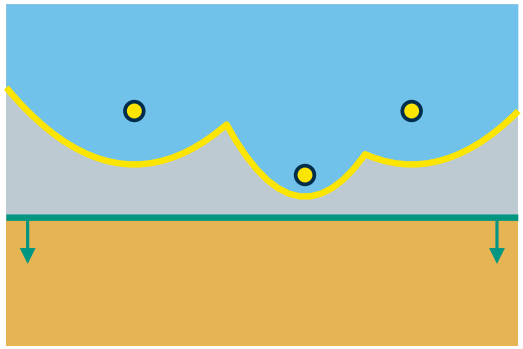

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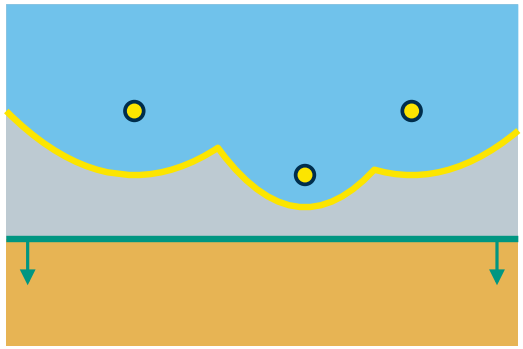

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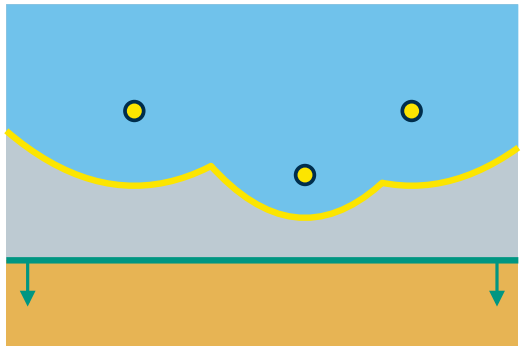

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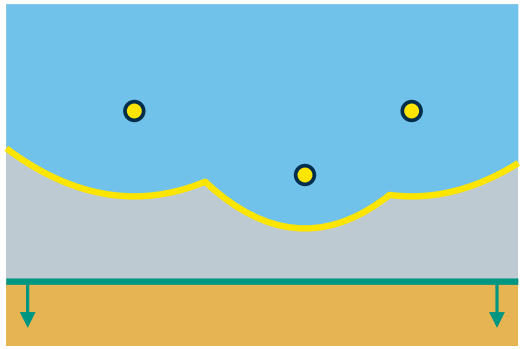

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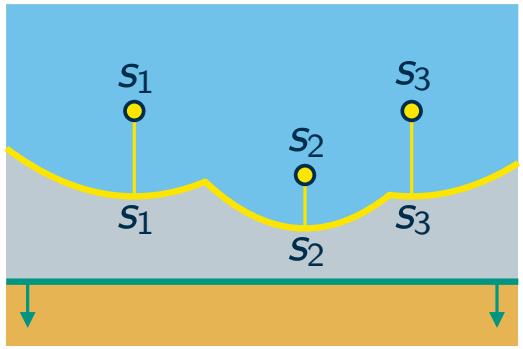

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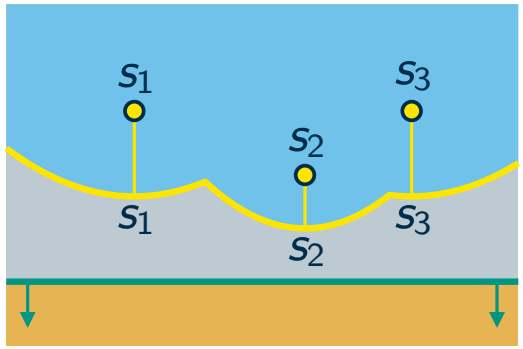

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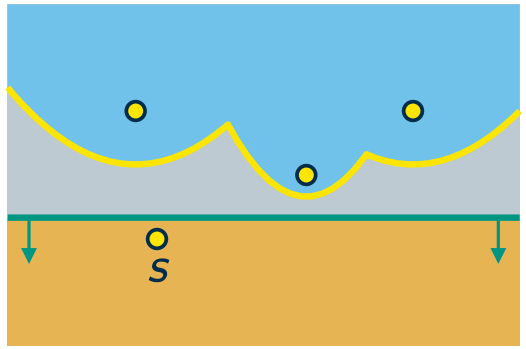

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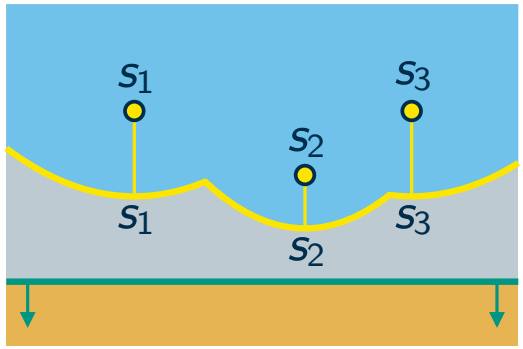

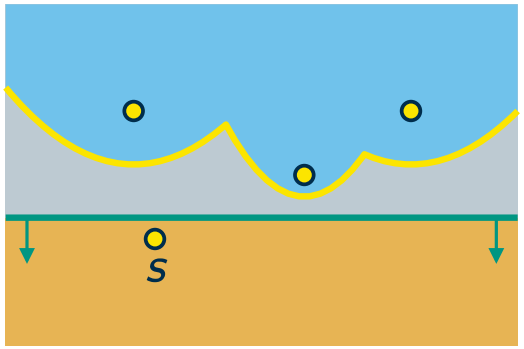

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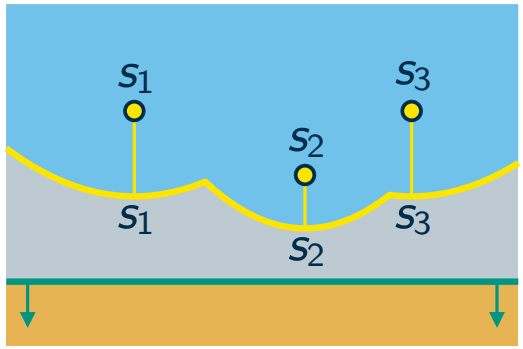

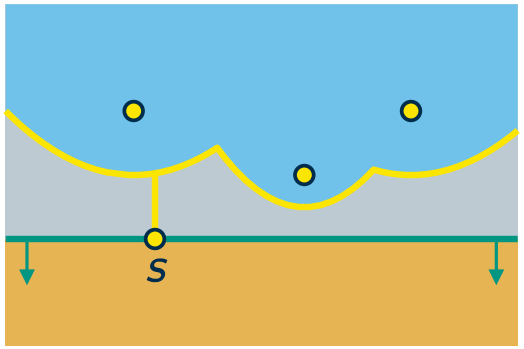

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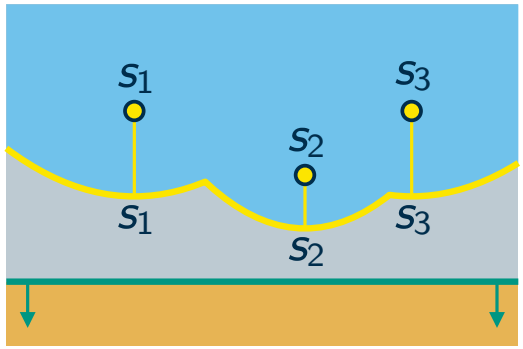

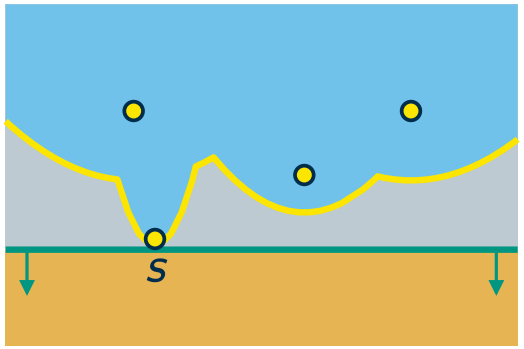

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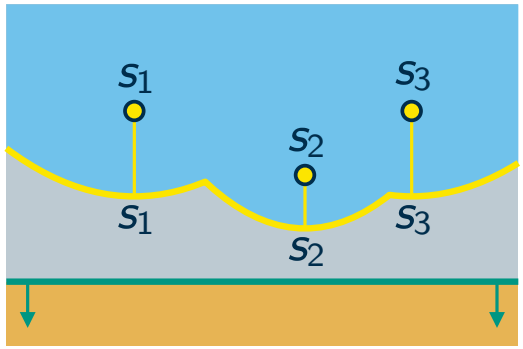

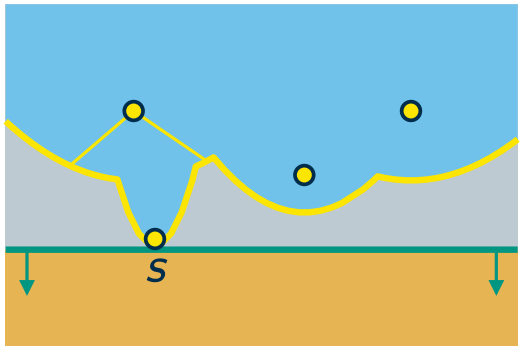

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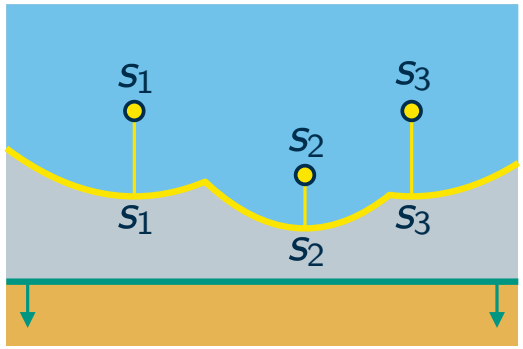

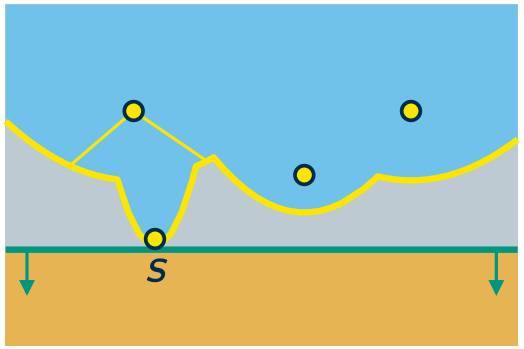

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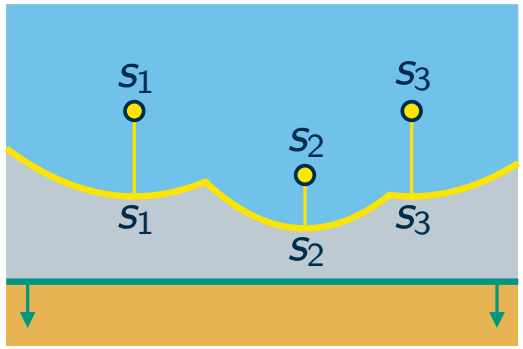

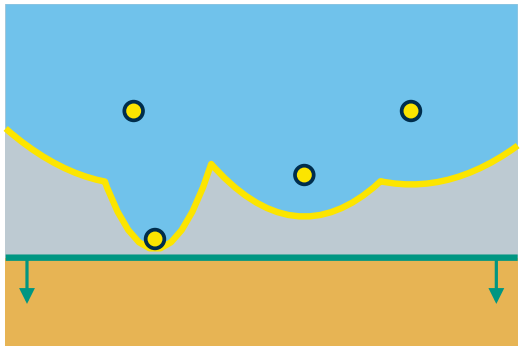

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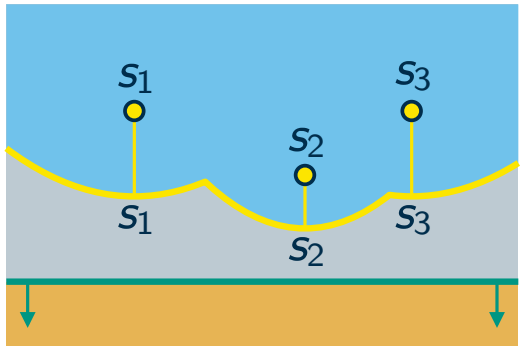

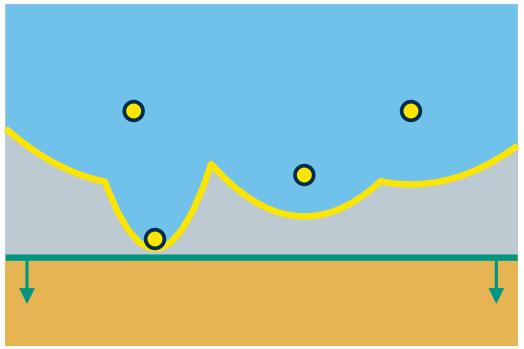

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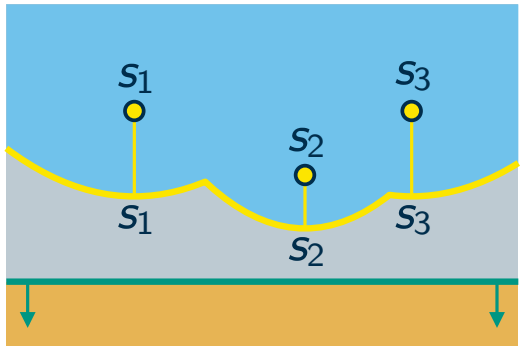

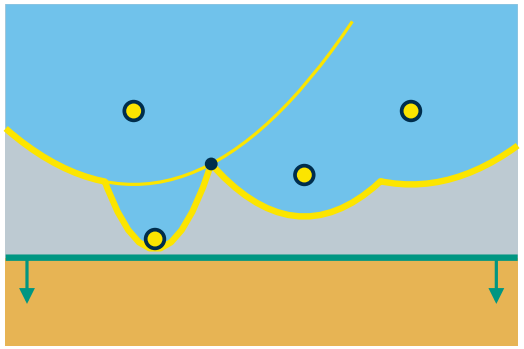

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- new parabola is inserted into the BL
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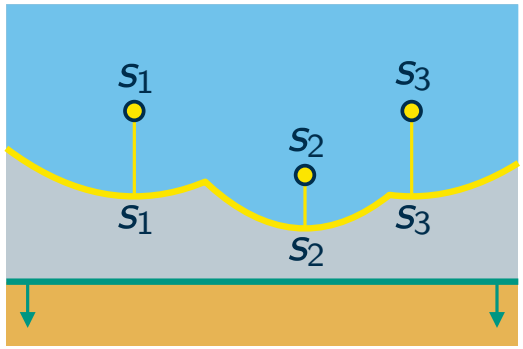

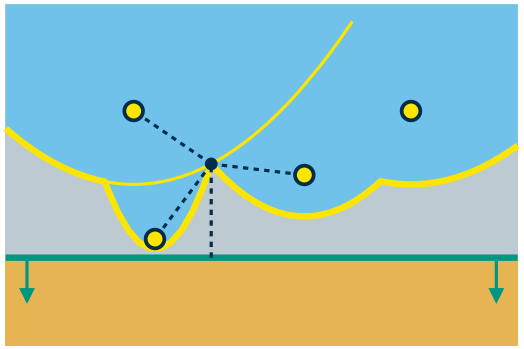

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- How do we know when this happens?
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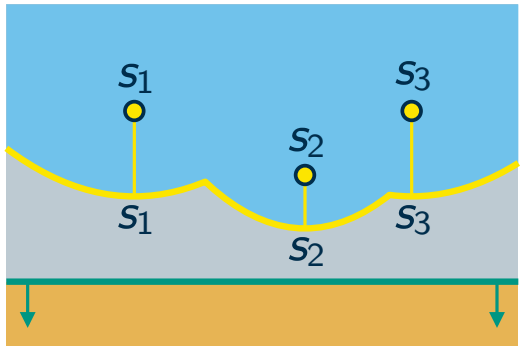

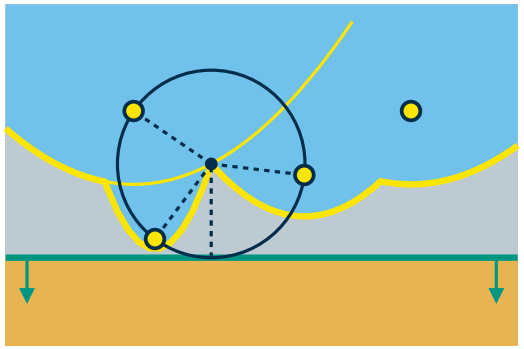

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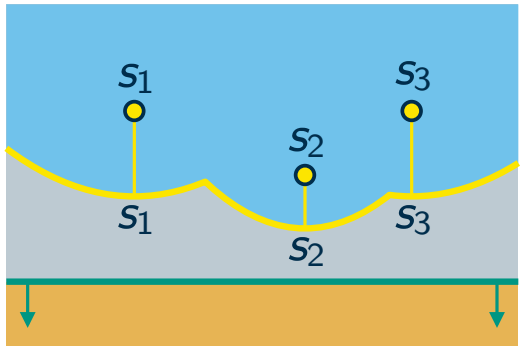

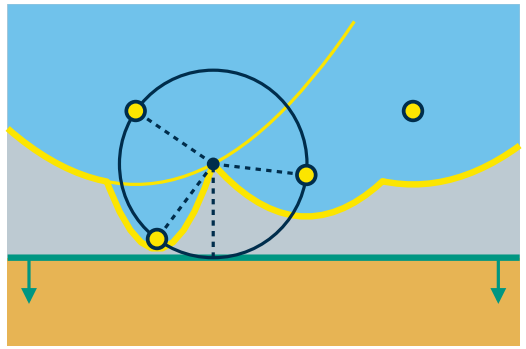

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- parabolas of three points intersect
- intersection has same distance to the three sites and to the sweep line
- no other site is closer
- we call this a circle event
- position defined by the sites corresponding to three consecutive parabola pieces

Why not?

What Do We Already Know?

What Do We Already Know?

Event Points: Beach Line Changes

- site event $\rightarrow$ insert new parabola piece
- circle event $\rightarrow$ remove parabola piece

(without proof: these are the only situations, in which the beach line changes)

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Finding Circle Events

- every beach line change: new triples of consecutive parabola pieces $\rightarrow$ maybe new circle event

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Voronoi Diagram

- circle event $\rightarrow$ new vertex of the Voronoi diagram
- every vertex is found this way

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- How do we manage the beach line?

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- How do we manage the beach line?
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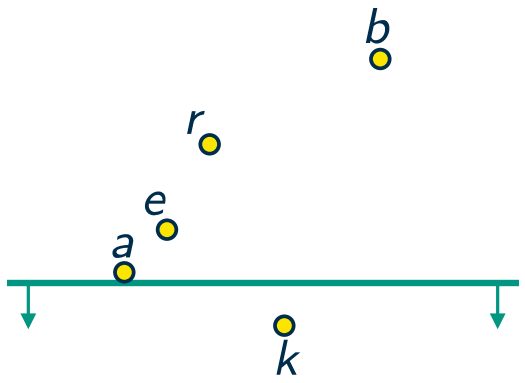
Special Cases

- sites with same y-coordinate
- vertices with degree > 3
- what else?

How Many Events Are In The Queue?



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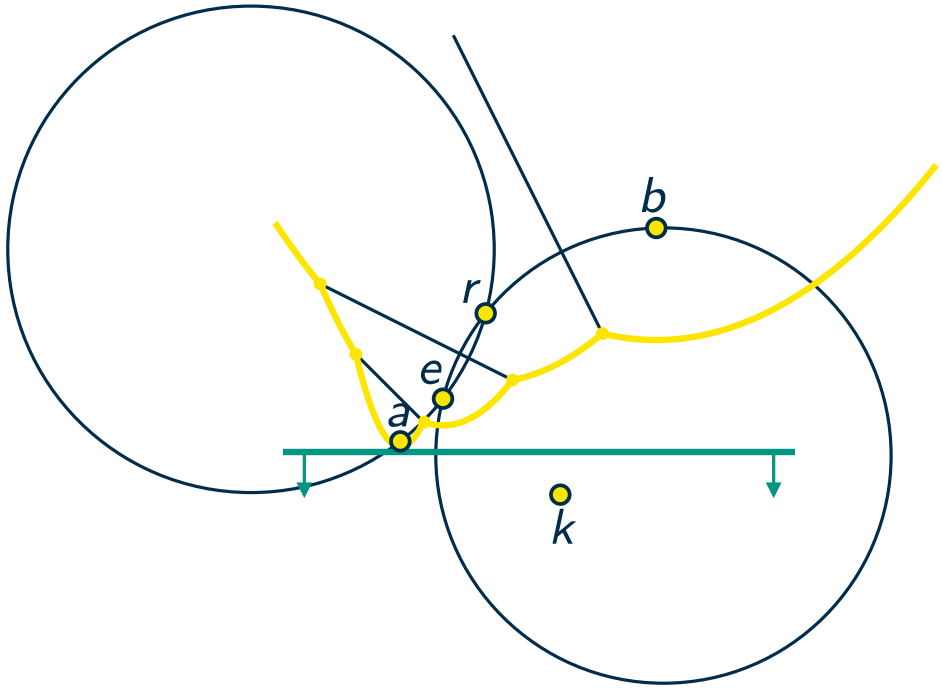
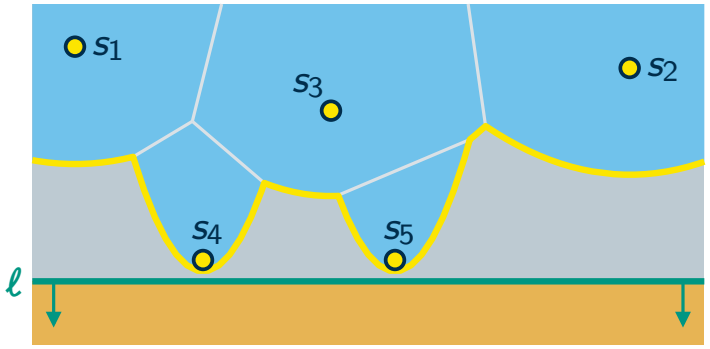


Answer: 3 (one site and two circle events)

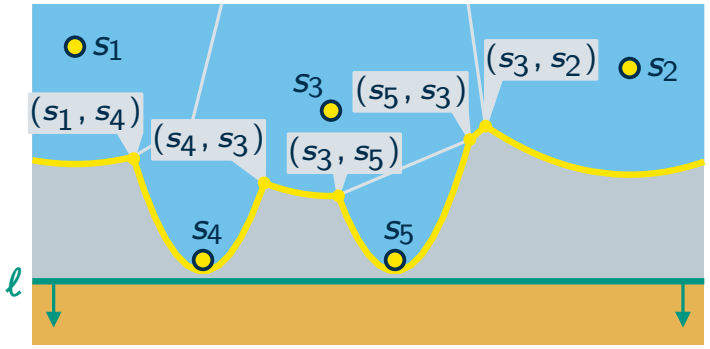
Managing The Beach Line



Observation

- parabolas can be split into multiple pieces
- knowing the site does not uniquely identify the parabola piece

Managing The Beach Line



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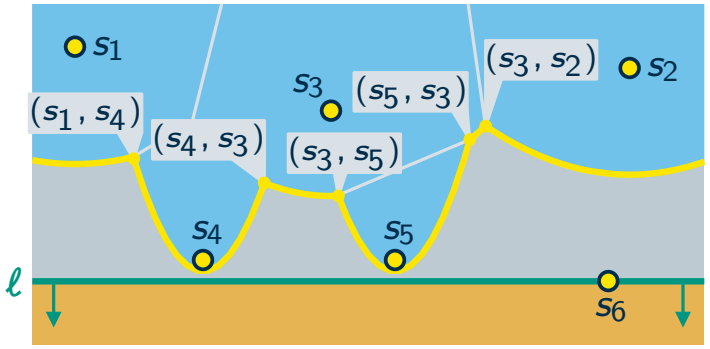
- instead: store a representation of the intersection points that is independent of ℓ's exact position

Managing The Beach Line



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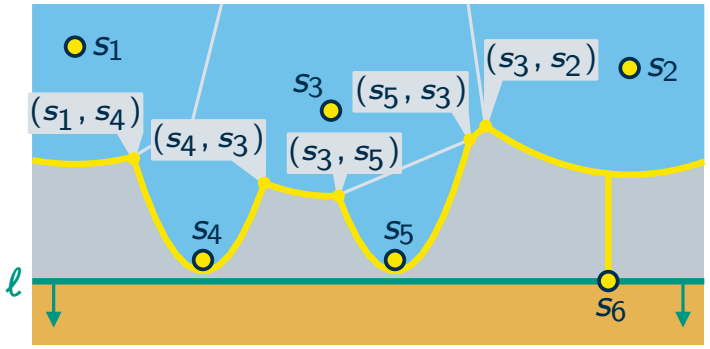
Site Event

Managing The Beach Line



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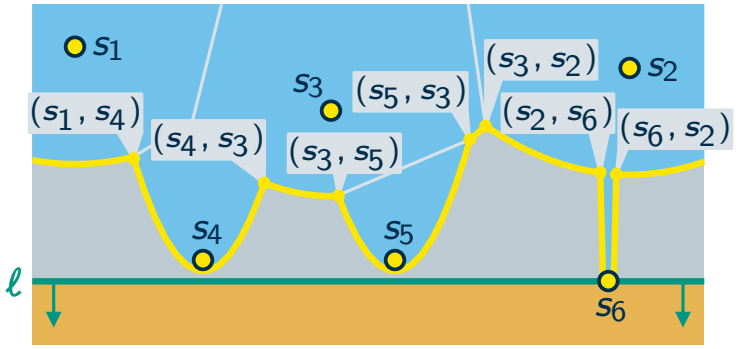
Site Event

- find x-coordinate of the new site in the set of intersection points

Managing The Beach Line



Observation

- parabolas can be split into multiple pieces
- knowing the site does not uniquely identify the parabola piece
- instead: store a representation of the intersection points that is independent of ℓ's exact position

Site Event

- find x-coordinate of the new site in the set of intersection points
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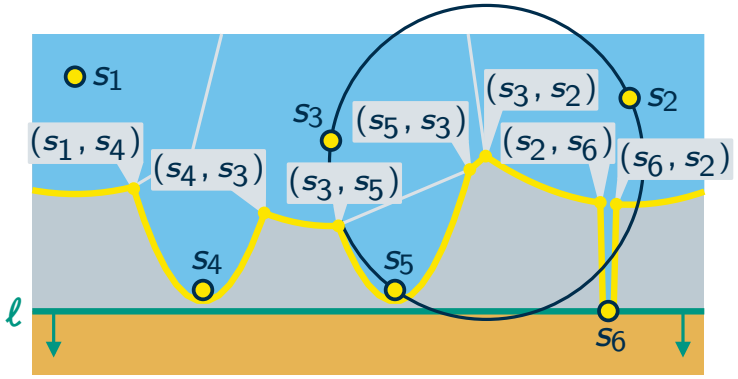
Managing The Beach Line



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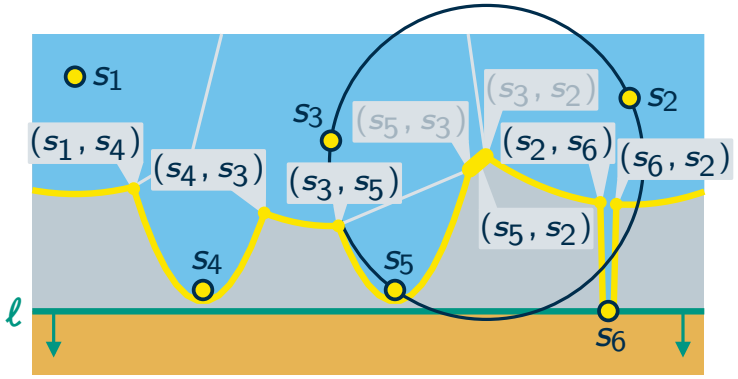

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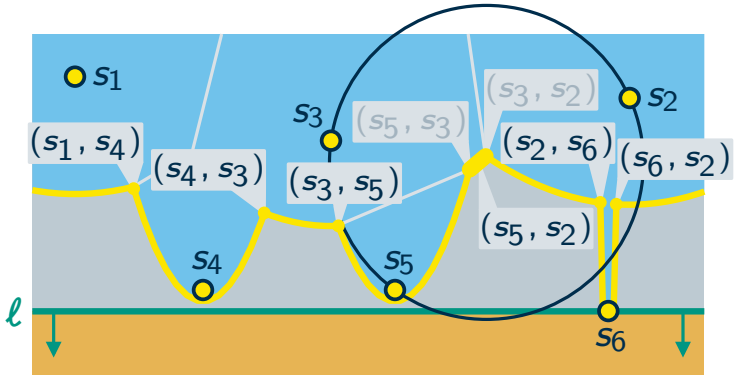
Managing The Beach Line



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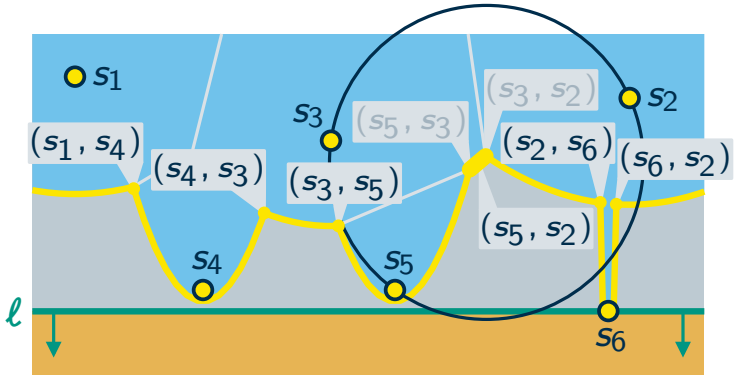
Managing The Beach Line



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Data Structure: search (for x), insert, and delete $\rightarrow$ search tree ($O(\log n)$ per operation)

Managing The Events

Observation

- we know all site events from the start
- circle events: defined by adjacent intersection points on the beach line

Managing The Events

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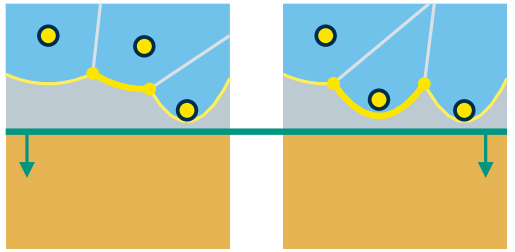

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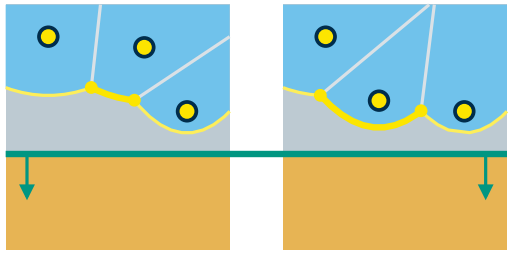

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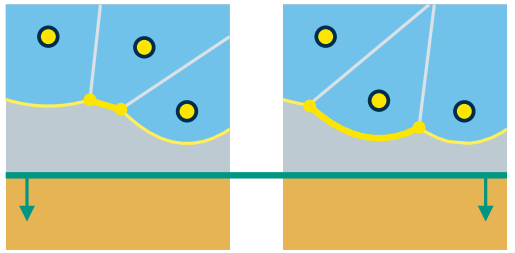

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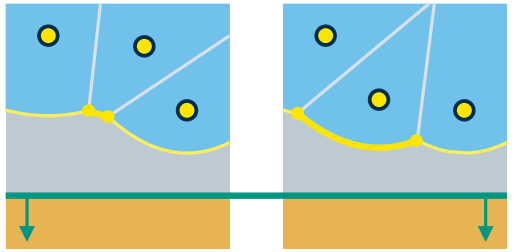

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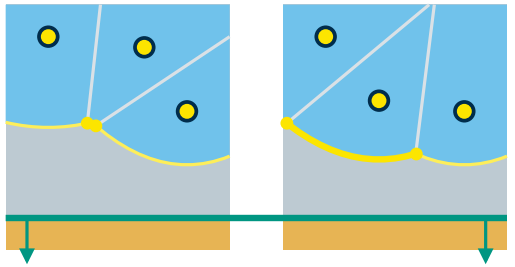

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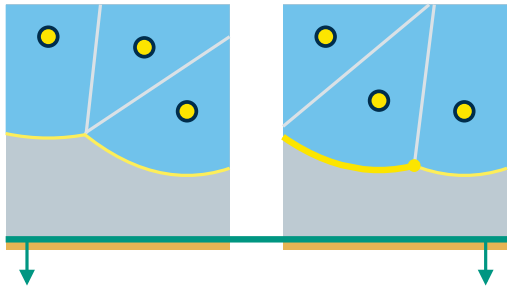

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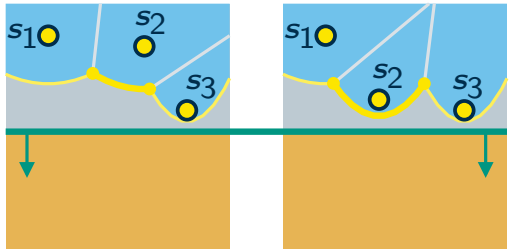

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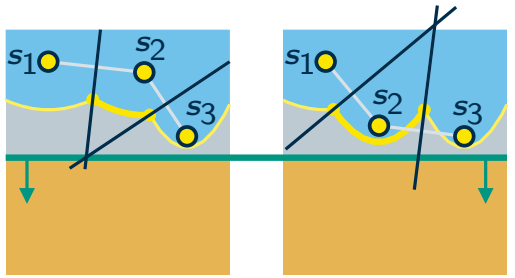

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Managing The Events

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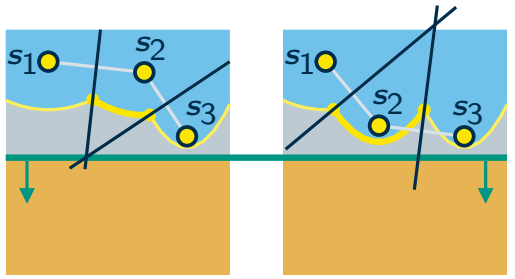

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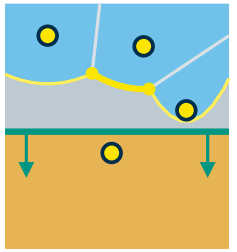

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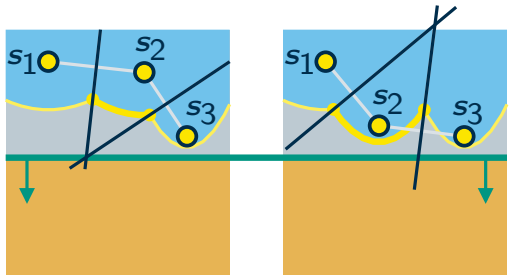

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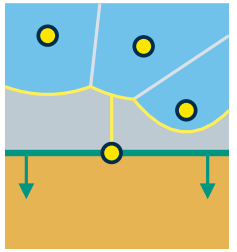

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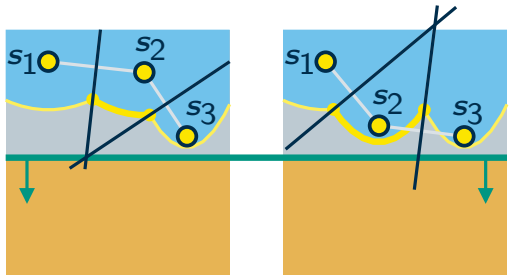

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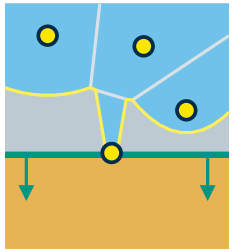

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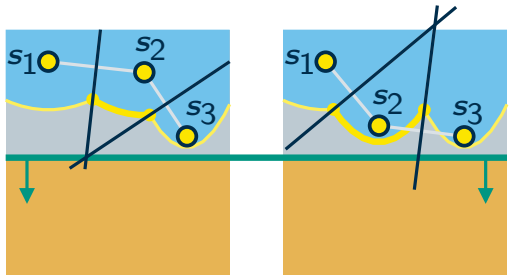

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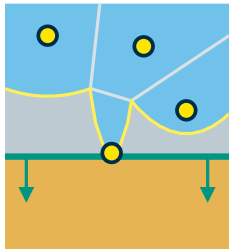

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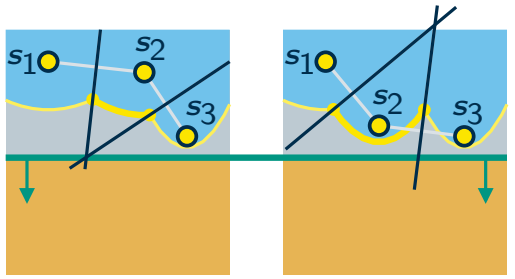

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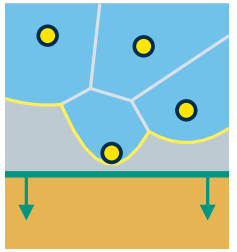

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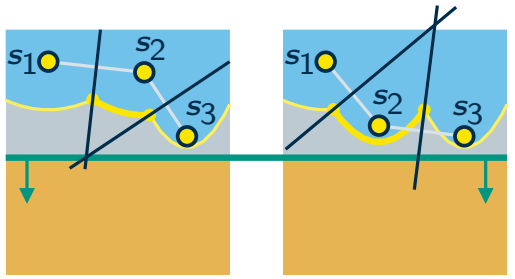

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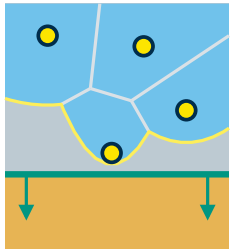

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- at site event: remove circle event of split parabola (if it exists)

Managing The Events

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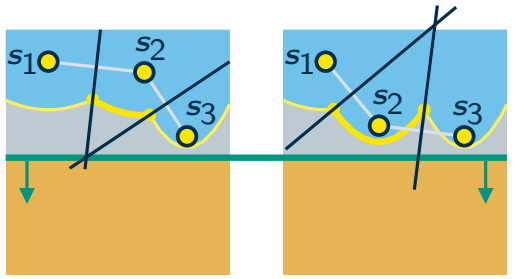

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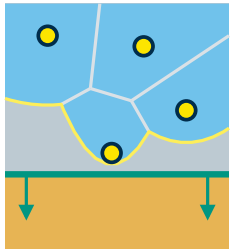

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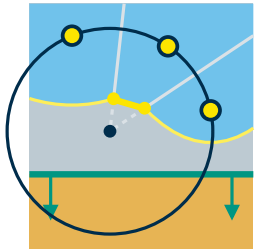
- no problem: use search tree for the event queue $\rightarrow O(\log n)$ per operation

Constructing The Voronoi Diagram

Already Seen

- point p is vertex $\Leftrightarrow$ p is the center for a circle event
- p lies on an edge $\Leftrightarrow$ p is at some point the intersection of parabola pieces



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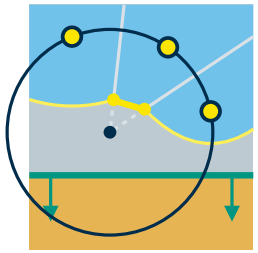

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- compute doubly-connected edge list of $\mathrm{Vor}(S)$ during the sweep (with special handling for infinite edges)

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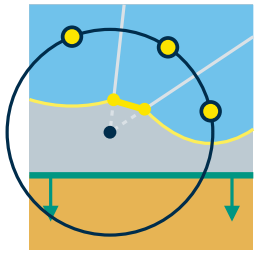

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- compute doubly-connected edge list of $\mathrm{Vor}(S)$ during the sweep (with special handling for infinite edges)
- site event: new parabola $\rightarrow$ new intersections $\rightarrow$ new half edges

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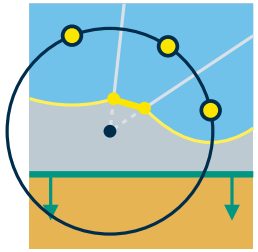

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- circle event: new vertex $\rightarrow$ attach corresponding half edges to vertex
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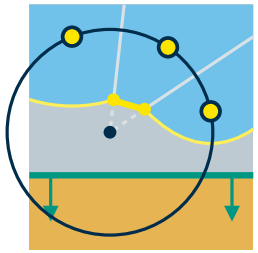

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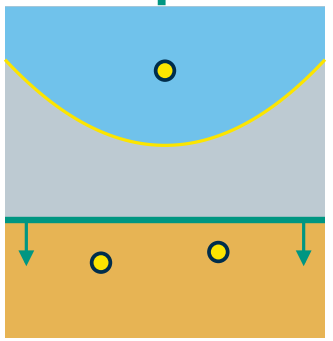

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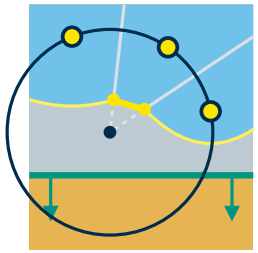

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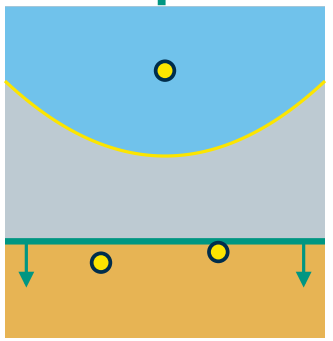

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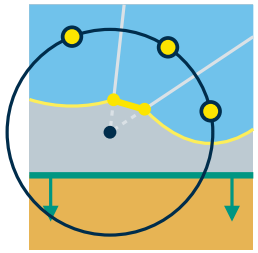

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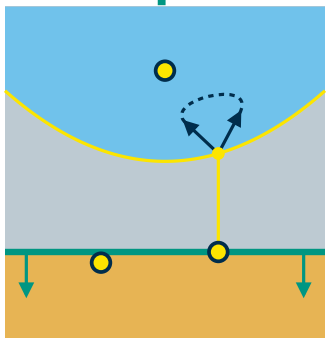

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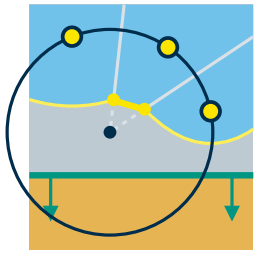

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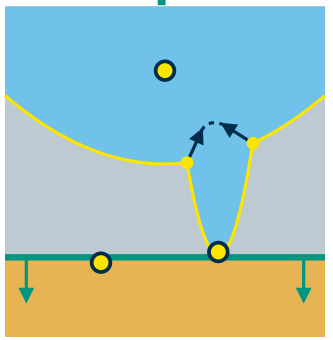
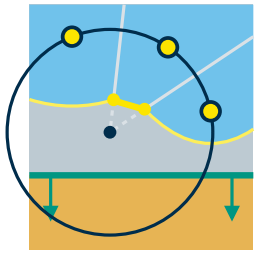


Constructing The Voronoi Diagram

Already Seen

- point p is vertex $\Leftrightarrow$ p is the center for a circle event
- p lies on an edge $\Leftrightarrow$ p is at some point the intersection of parabola pieces



Plan

- compute doubly-connected edge list of $\mathrm{Vor}(S)$ during the sweep (with special handling for infinite edges)
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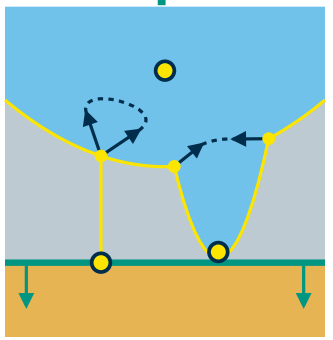

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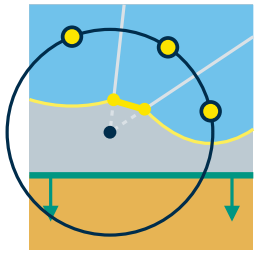

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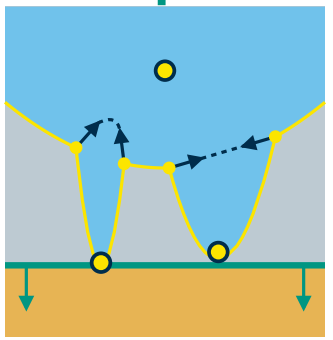

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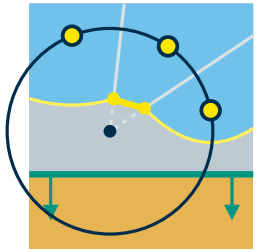

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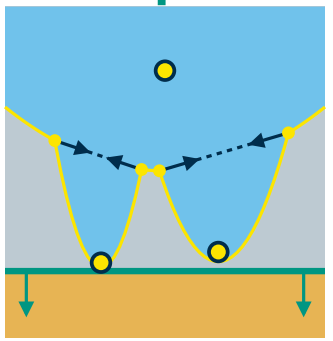

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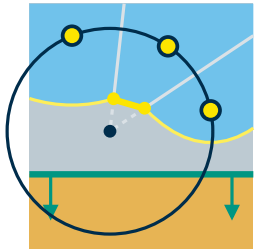

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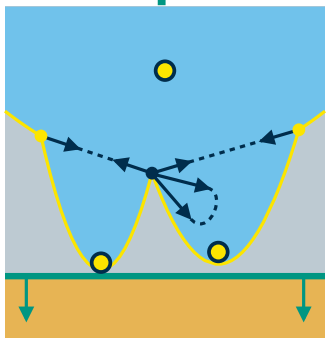

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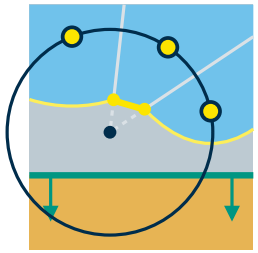

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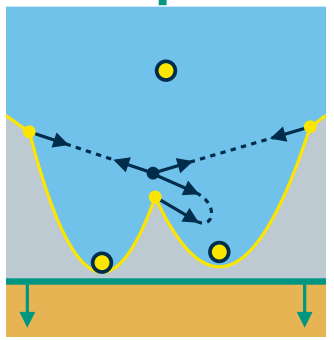

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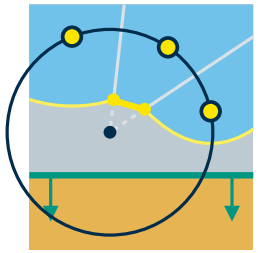

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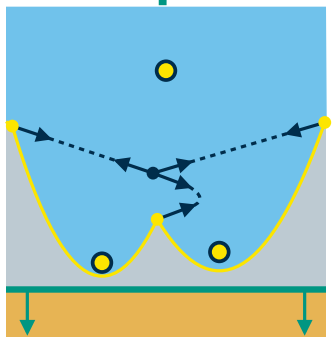

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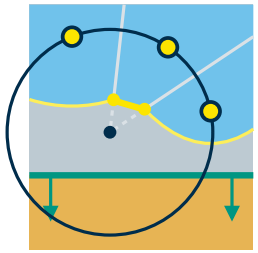

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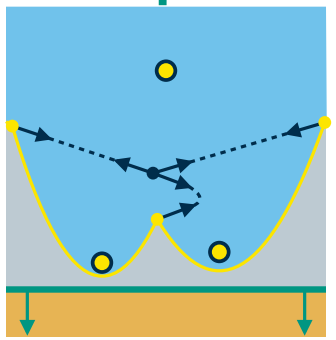

Infinite Edges

Constructing The Voronoi Diagram

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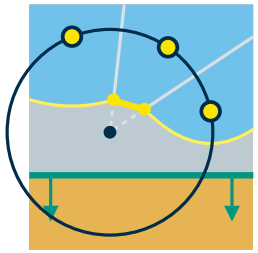

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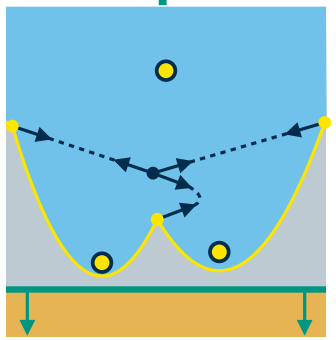
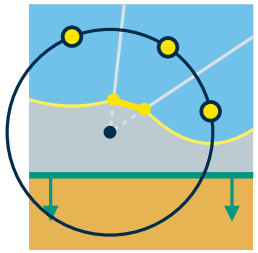


Infinite Edges

- some half edges do not end, as their parabola pieces never disappear

Constructing The Voronoi Diagram

Already Seen



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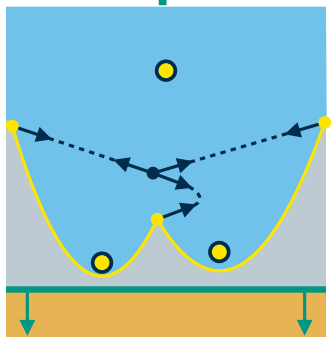
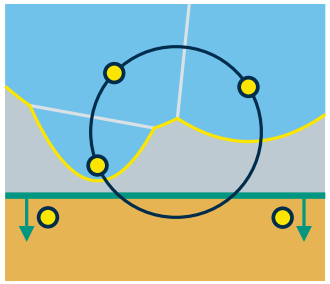


Infinite Edges

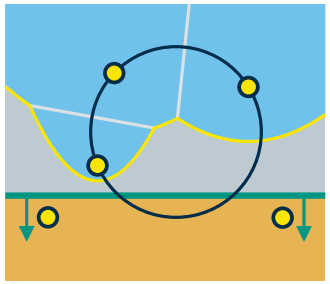
- some half edges do not end, as their parabola pieces never disappear
- empty queue $\rightarrow$ some final clean up remaining intersections in beach line
  (depending on how your doubly-connected edge list deals with infinite edges)

Special Cases

Events With Equal y-Coordinate



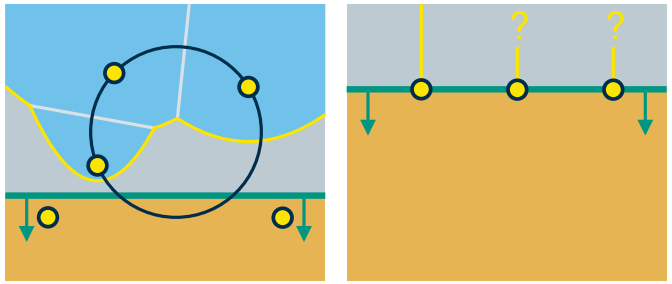
Special Cases



Events With Equal y-Coordinate

- handle in arbitrary order

Special Cases



Events With Equal *y*-Coordinate

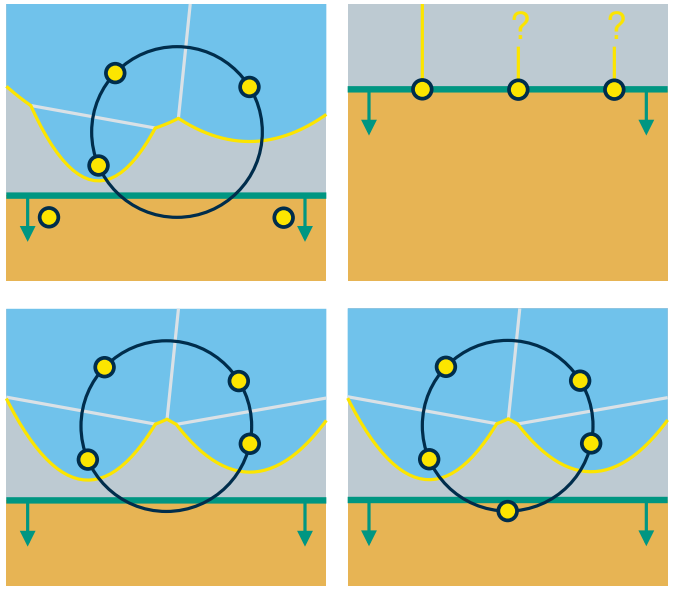
- handle in arbitrary order
- exception: initial site events (no parabola above exists)

Special Cases

Events With Equal y-Coordinate

- handle in arbitrary order
- exception: initial site events (no parabola above exists)

Equal x- and y-Coordinate



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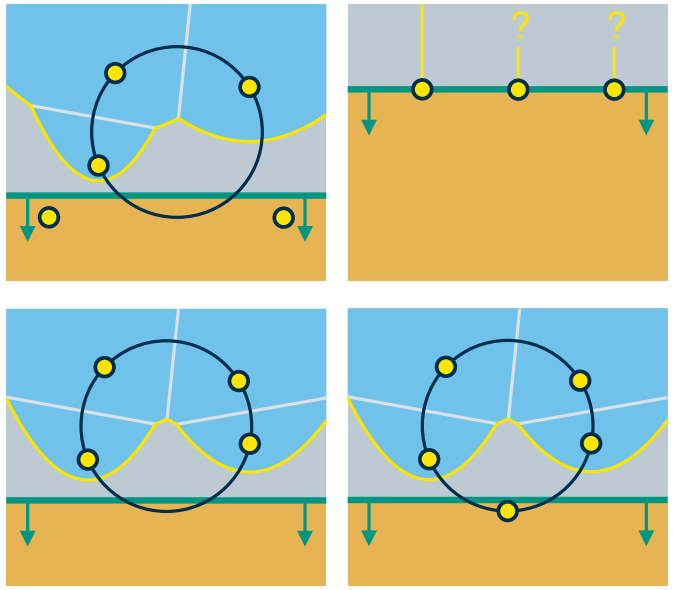

Events With Equal y-Coordinate

- handle in arbitrary order
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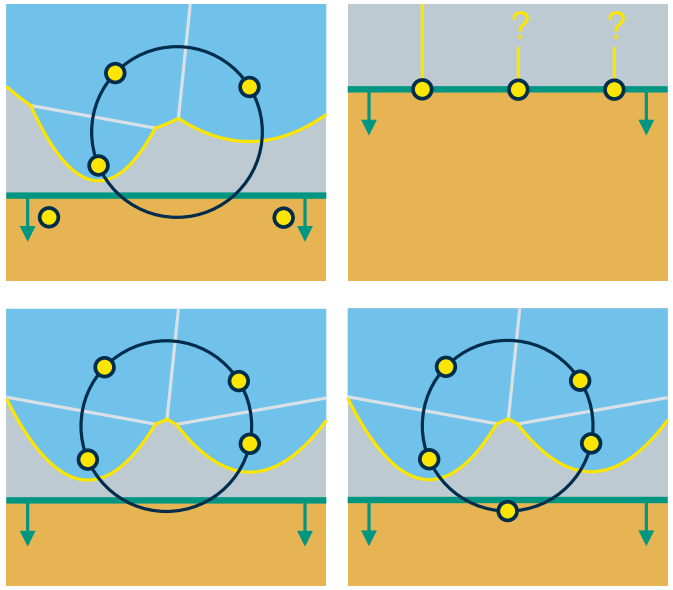
Equal x- and y-Coordinate

- vertex in $\mathrm{Vor}(S)$ with degree > 3

Special Cases



Events With Equal y-Coordinate

- handle in arbitrary order
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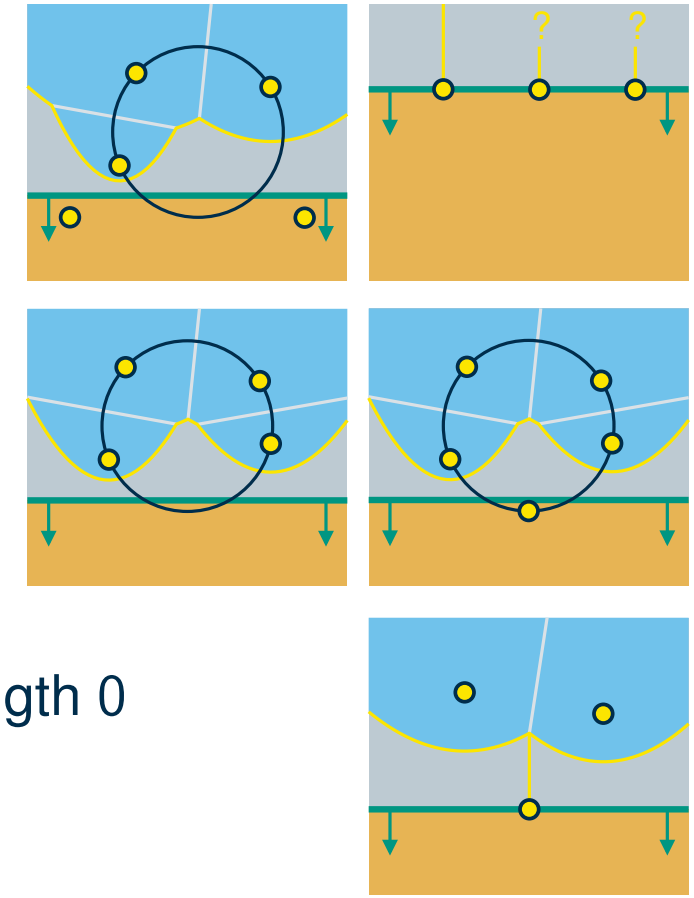
Equal x- and y-Coordinate

- vertex in $\mathrm{Vor}(S)$ with degree > 3
- special handling $\rightarrow$ vertex with degree > 3
- no special handling $\rightarrow$ multiple degree-3 vertices with edges of length 0

Special Cases



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New Parabola Starts At Intersection

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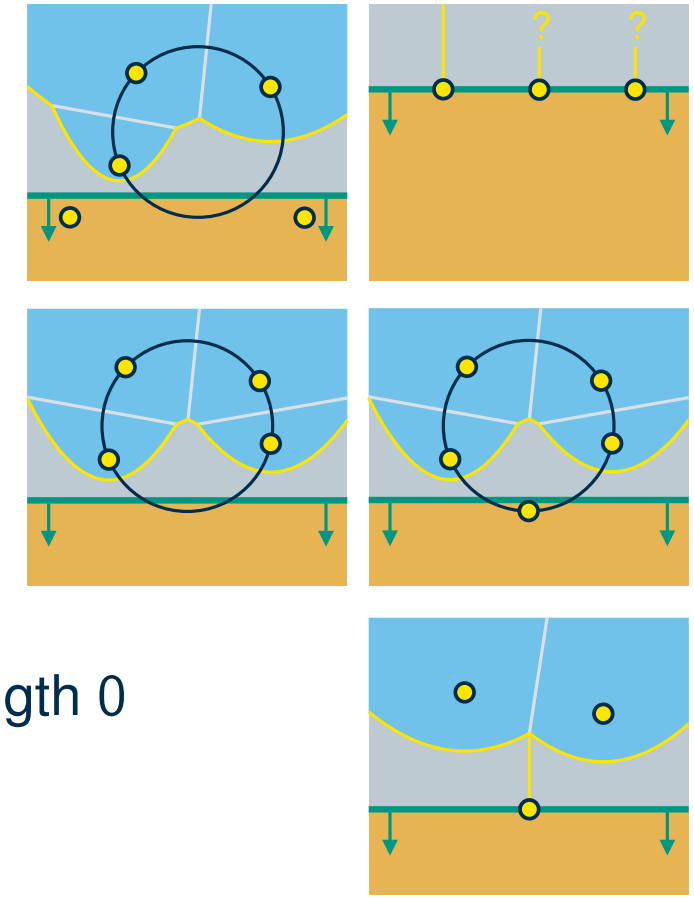

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New Parabola Starts At Intersection

- a vertex should be created

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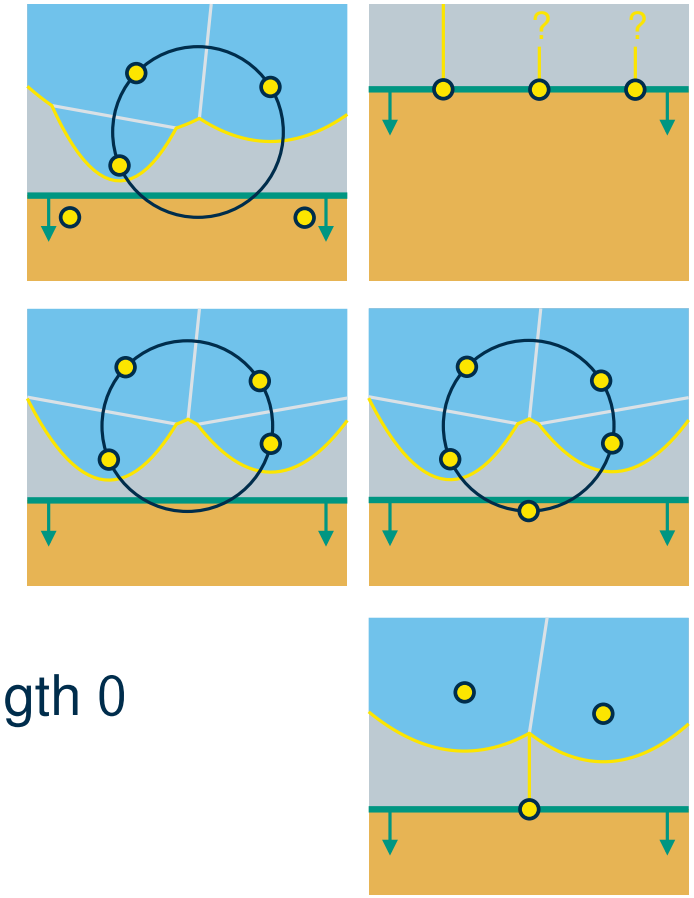

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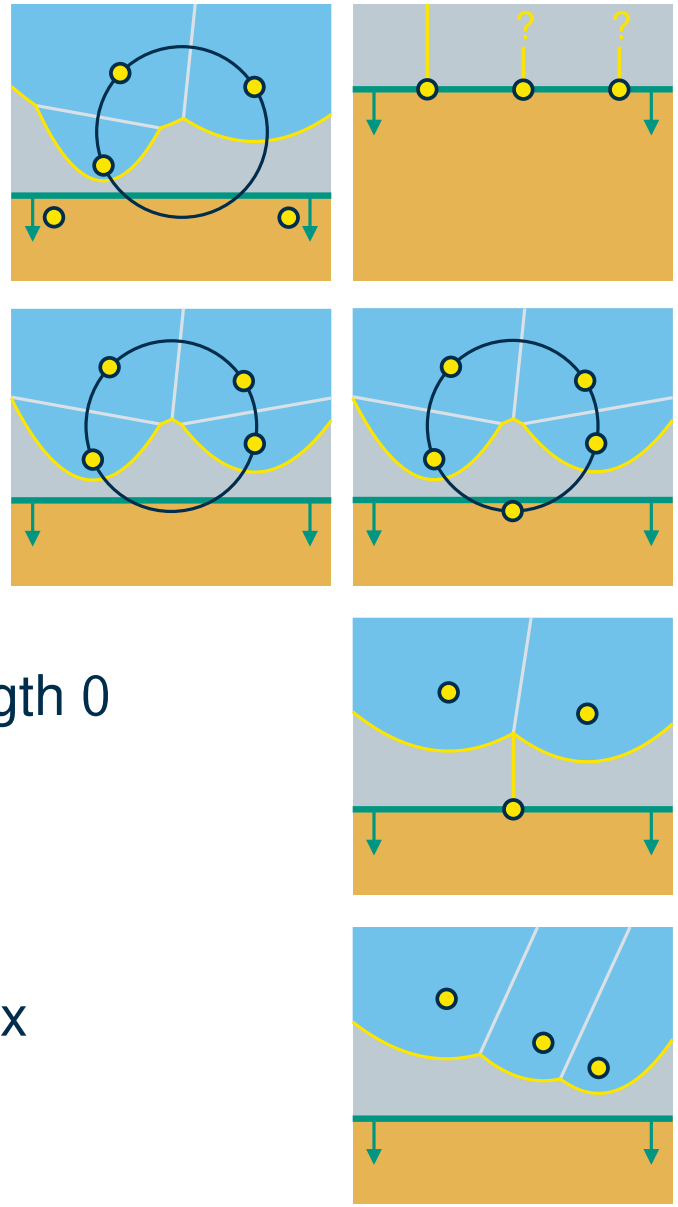
New Parabola Starts At Intersection

- a vertex should be created
- special handling: create vertex and half edges
- no special handling: split one parabola $\rightarrow$ circle event $\rightarrow$ new vertex

Special Cases



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Collinear Sites With Consecutive Parabola Pieces

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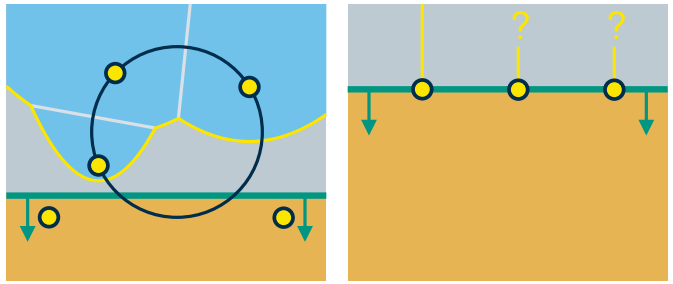

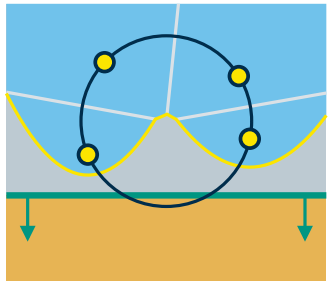

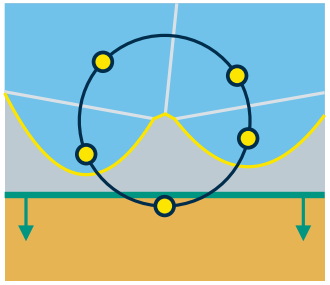

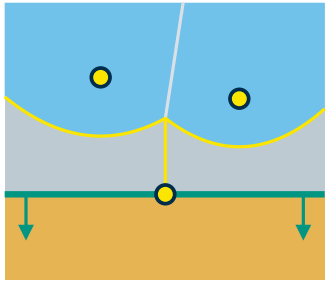

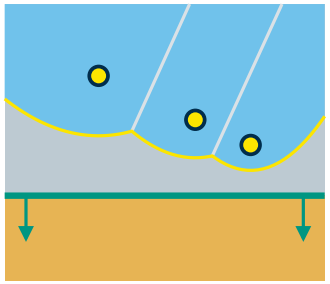

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Collinear Sites With Consecutive Parabola Pieces

- careful, when computing the circle

Overview

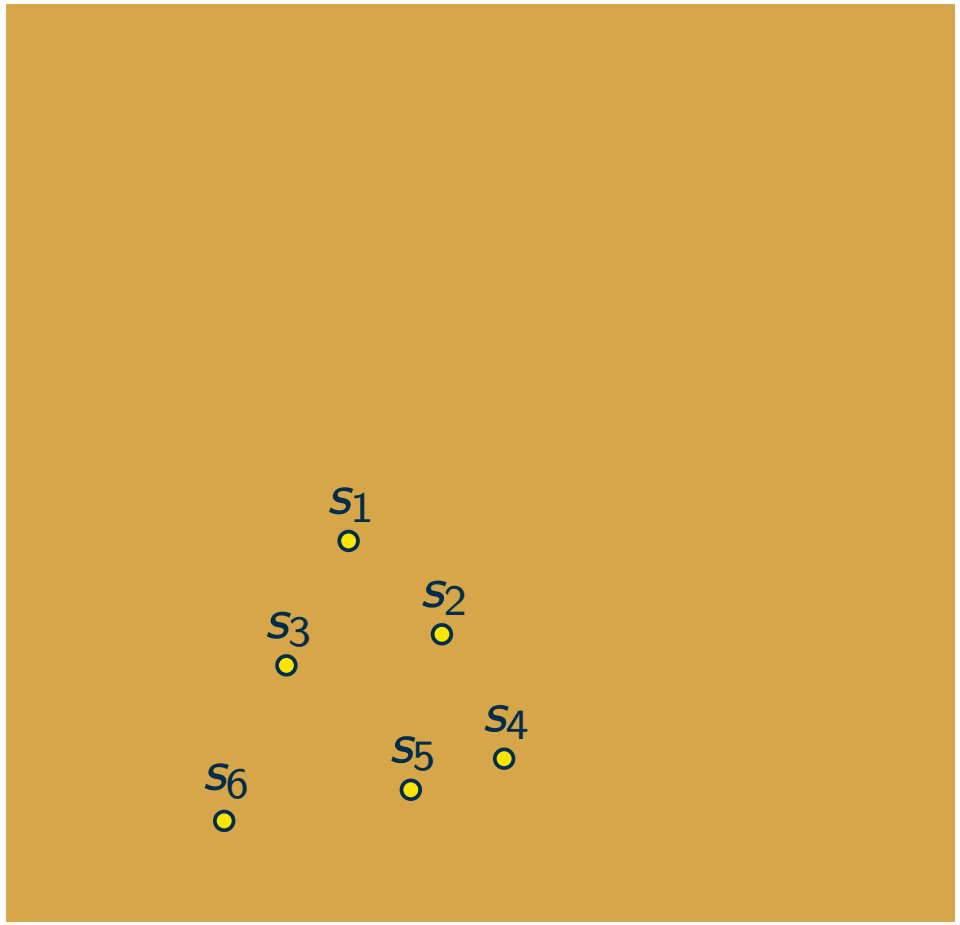
Site Event (Site s)

- find s_x in the beach line
- insert two new intersections
- handle false alarm for intersected parabola piece
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



beach line:

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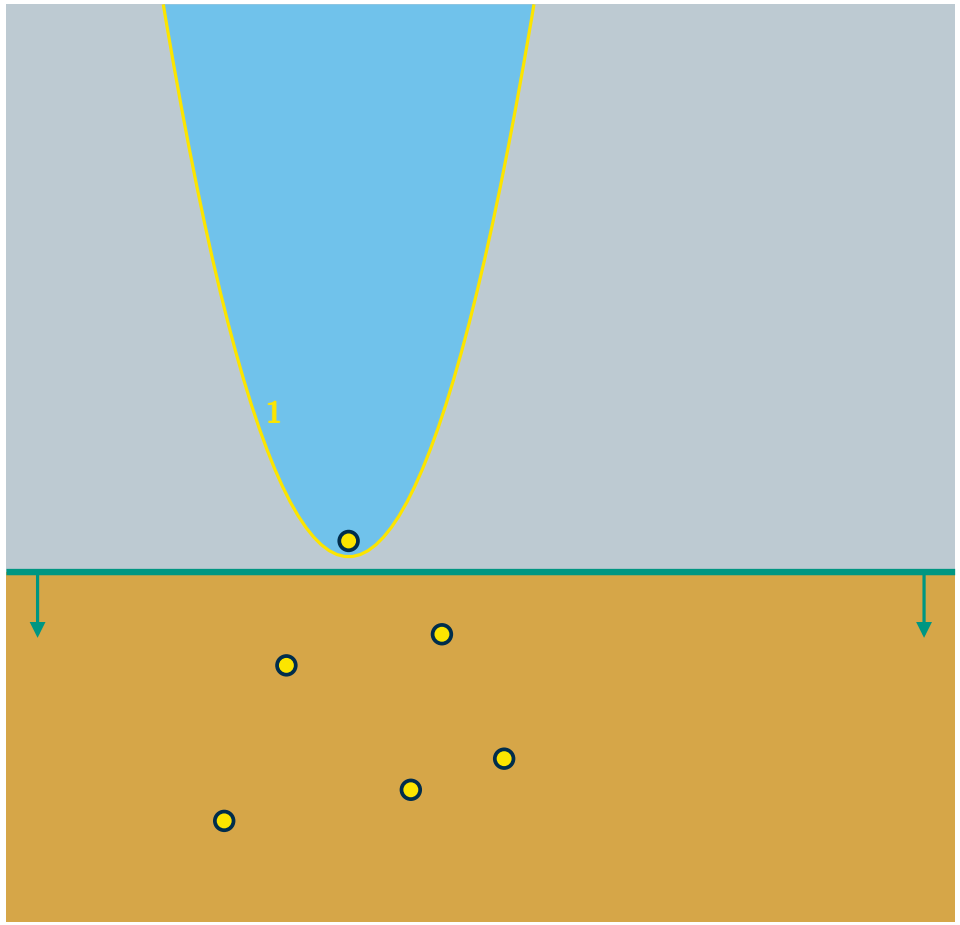

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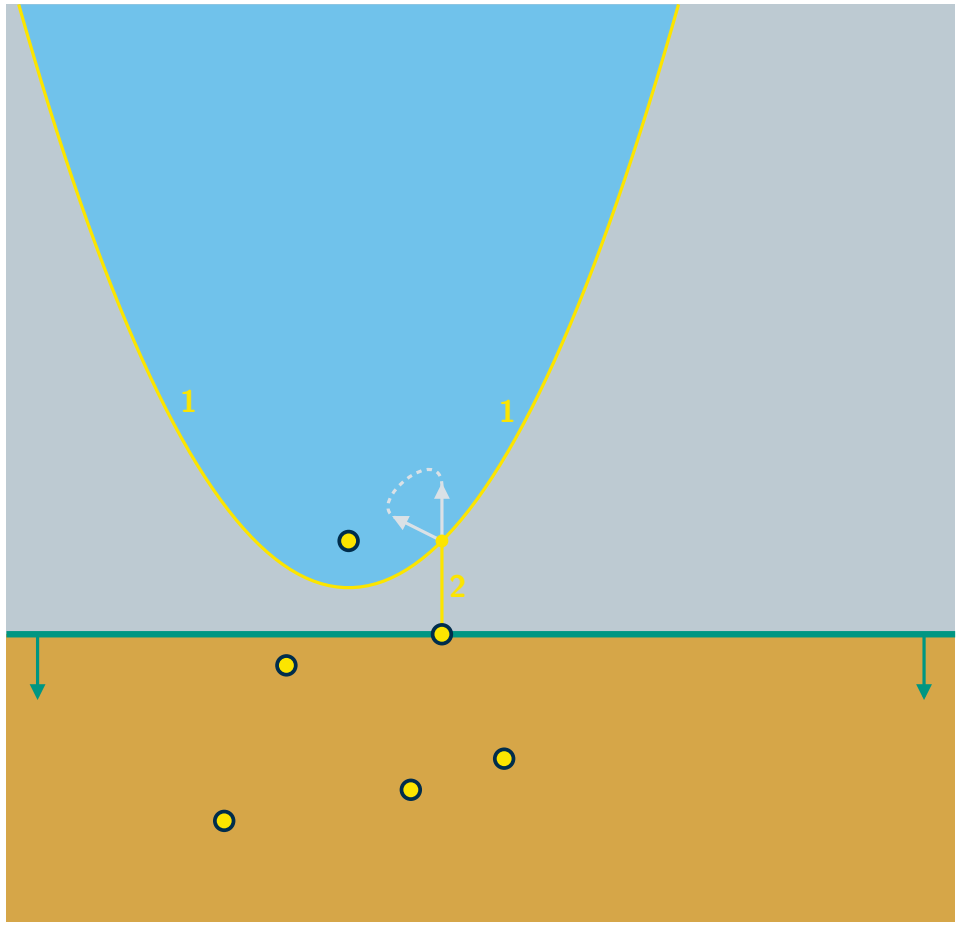

beach line: (s_1, s_2) (s_2, s_1)

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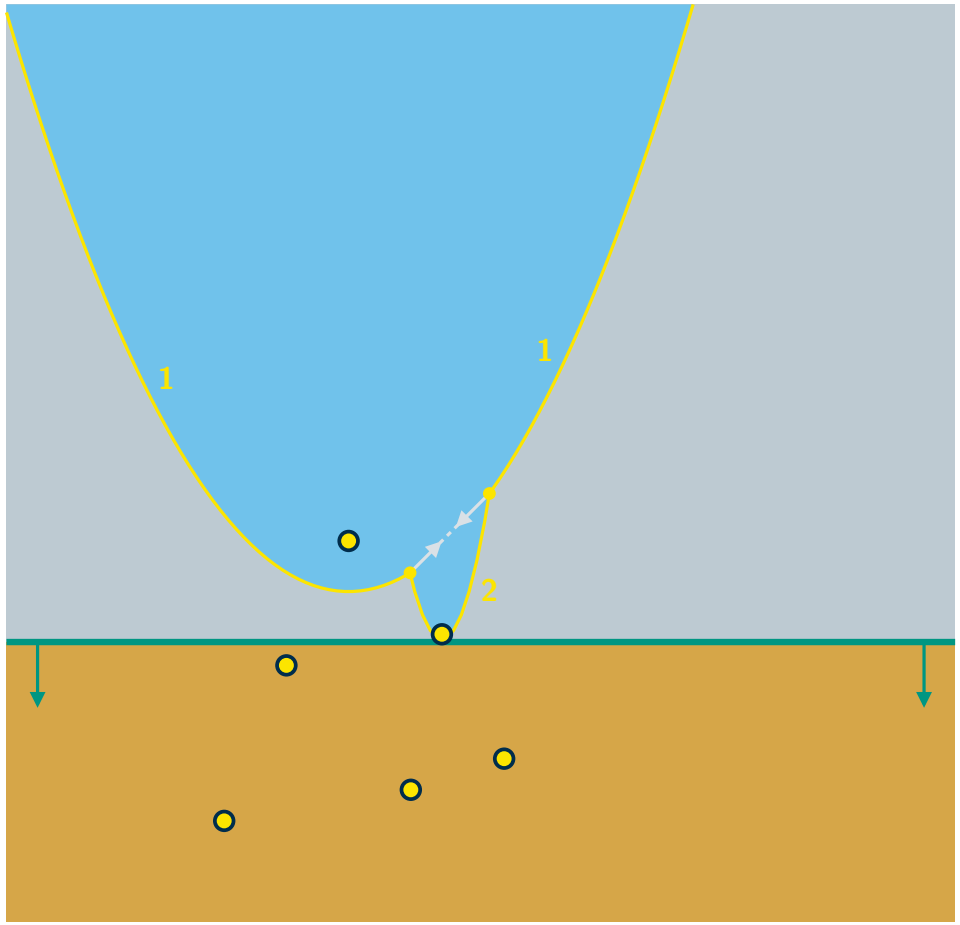


beach line: (s_1, s_2) (s_2, s_1)

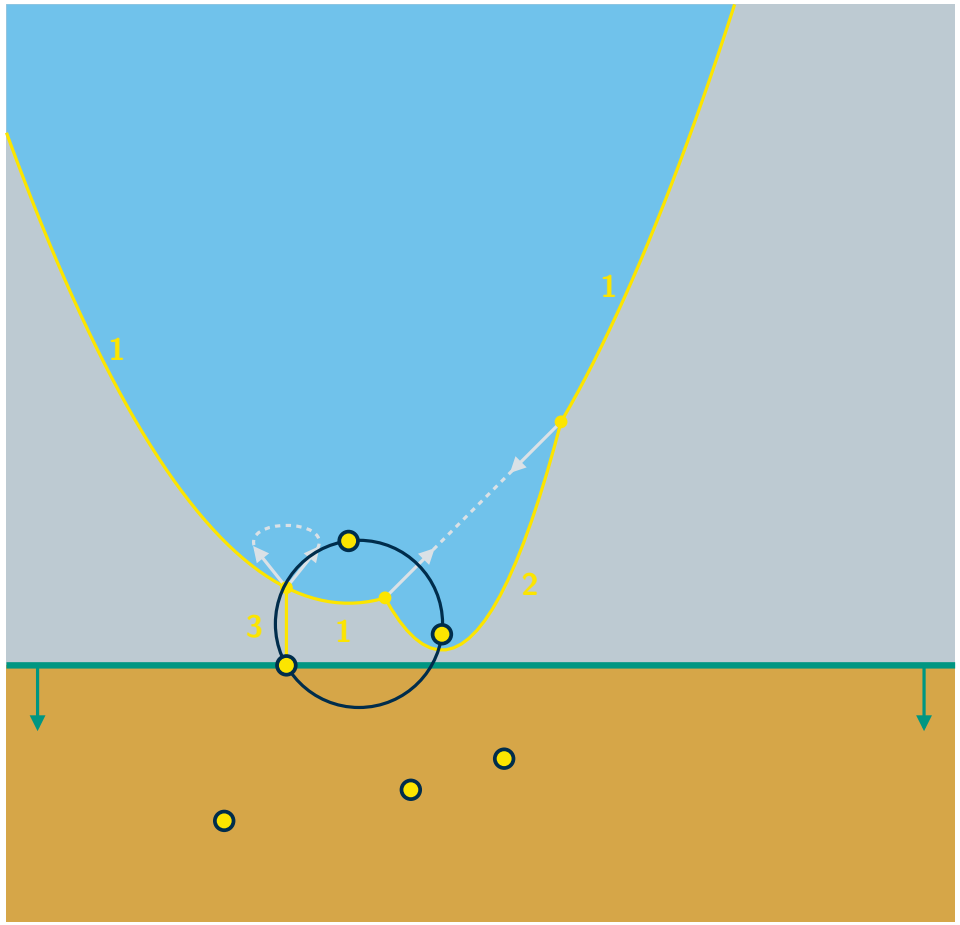
Site Event (Site s)

- find s_x in the beach line
- insert two new intersections
- handle false alarm for intersected parabola piece
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



beach line: (s_1, s_3) (s_3, s_1) (s_1, s_2) (s_2, s_1)

Site Event (Site s)

- find s_x in the beach line
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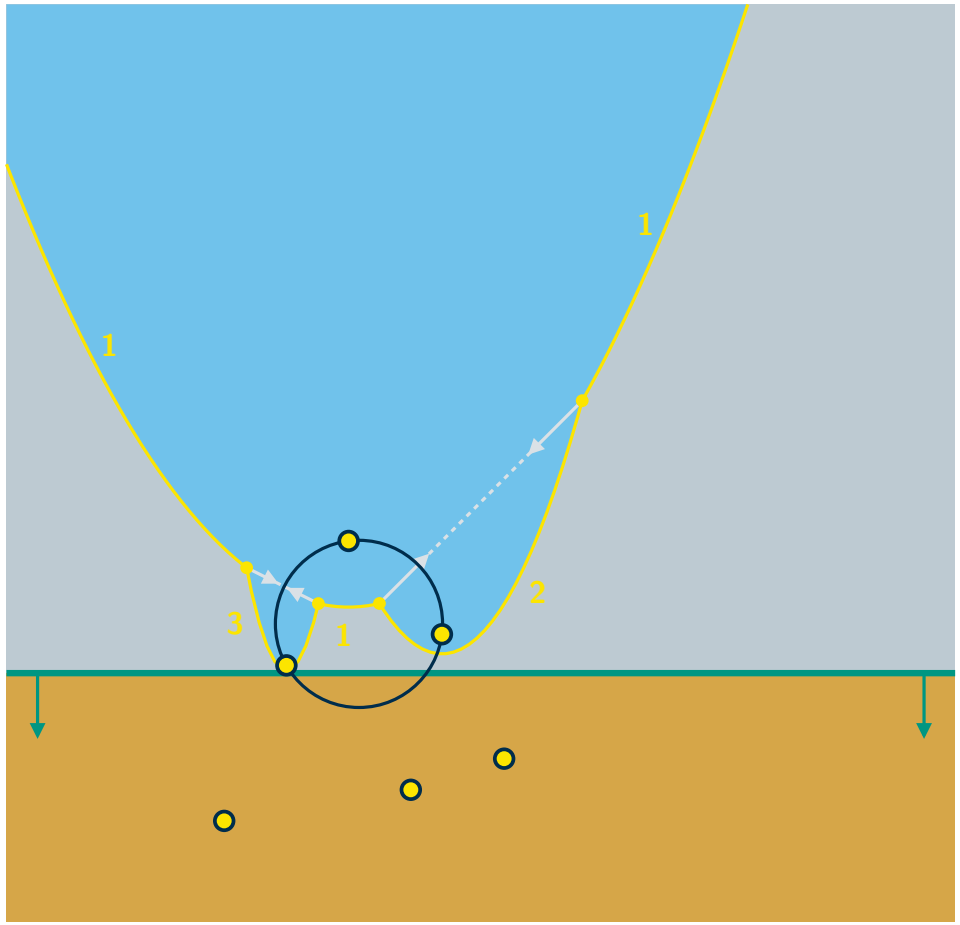
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



beach line: (s_1, s_3) (s_3, s_1) (s_1, s_2) (s_2, s_1)

Site Event (Site s)

- find s_x in the beach line
- insert two new intersections
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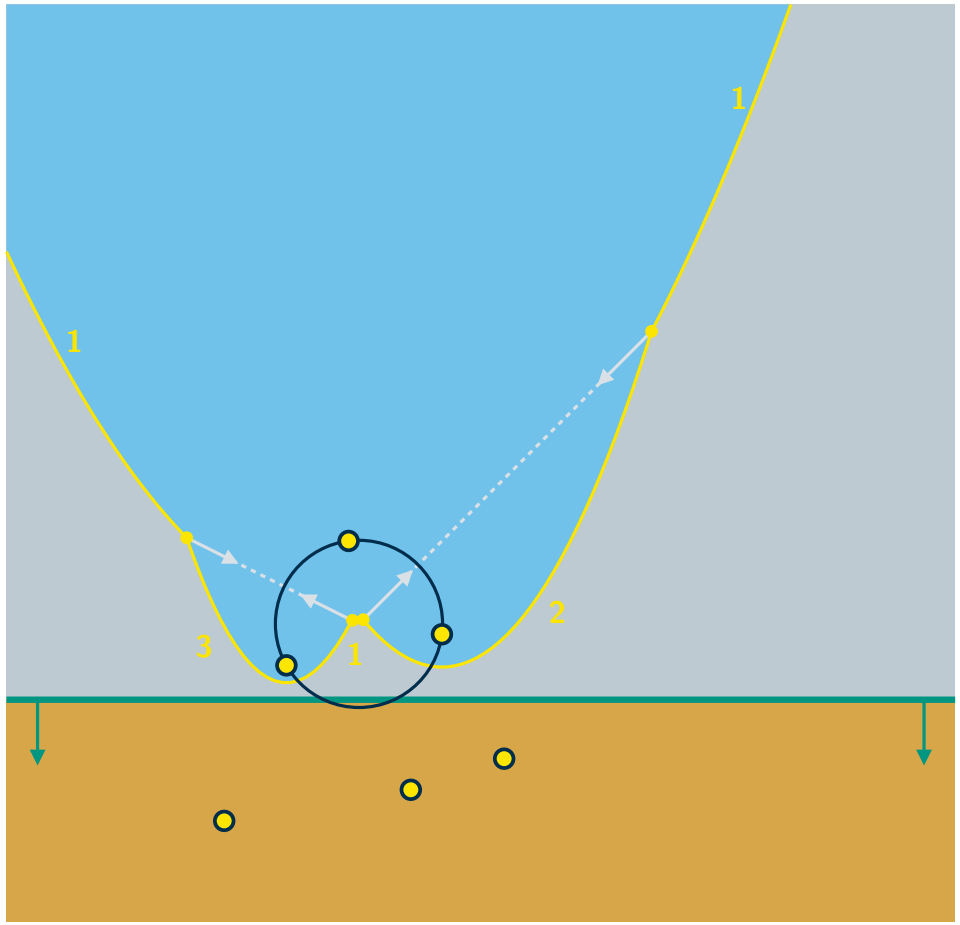
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



beach line: (s_1, s_3) (s_3, s_1) (s_1, s_2) (s_2, s_1)

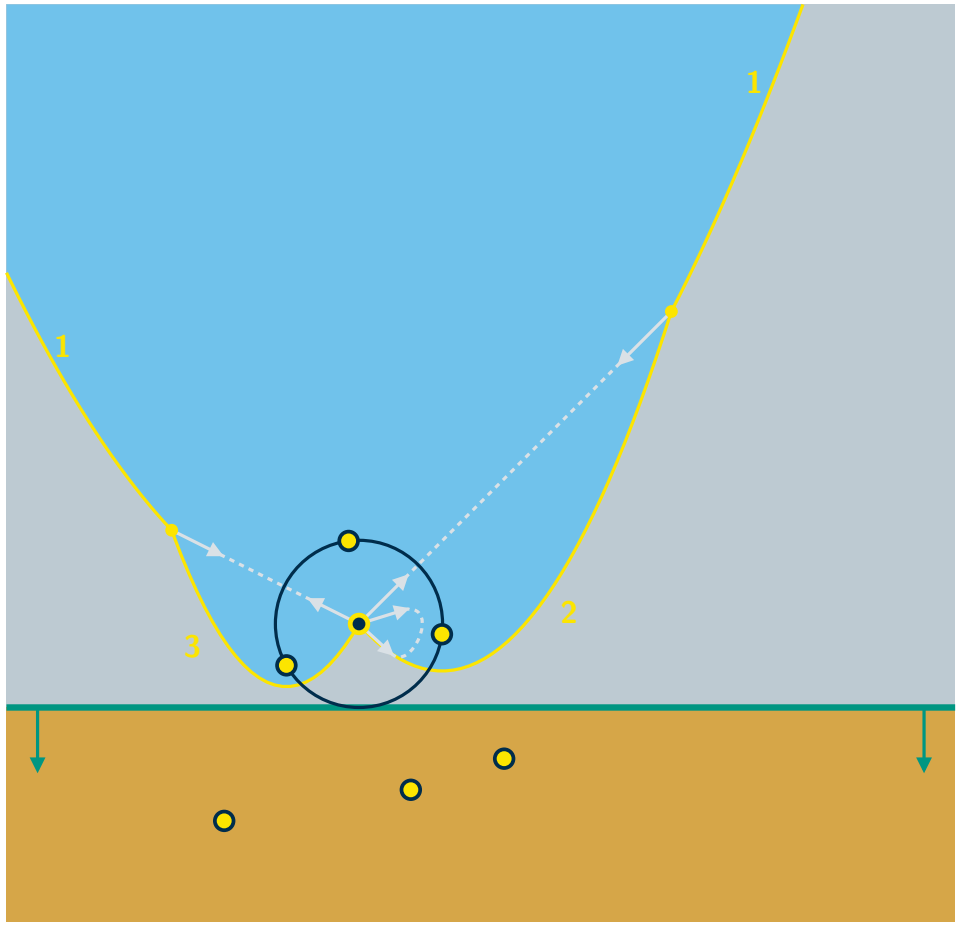
Site Event (Site s)

- find s_x in the beach line
- insert two new intersections
- handle false alarm for intersected parabola piece
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



beach line: (s_1, s_3) (s_3, s_2) (s_2, s_1)

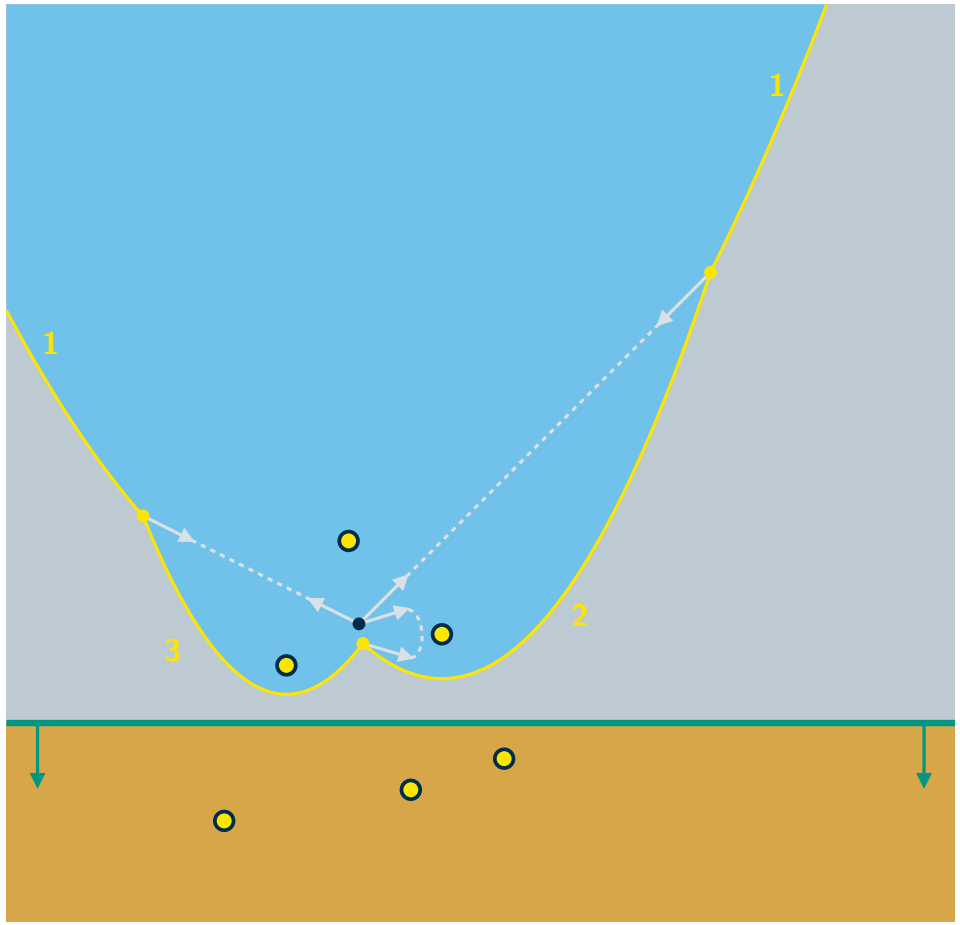
Site Event (Site s)

- find s_x in the beach line
- insert two new intersections
- handle false alarm for intersected parabola piece
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections (s_i, s_j), (s_j, s_k))

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



beach line: (s_1, s_3) (s_3, s_2) (s_2, s_1)

Site Event (Site s)

- find s_x in the beach line
- insert two new intersections
- handle false alarm for intersected parabola piece
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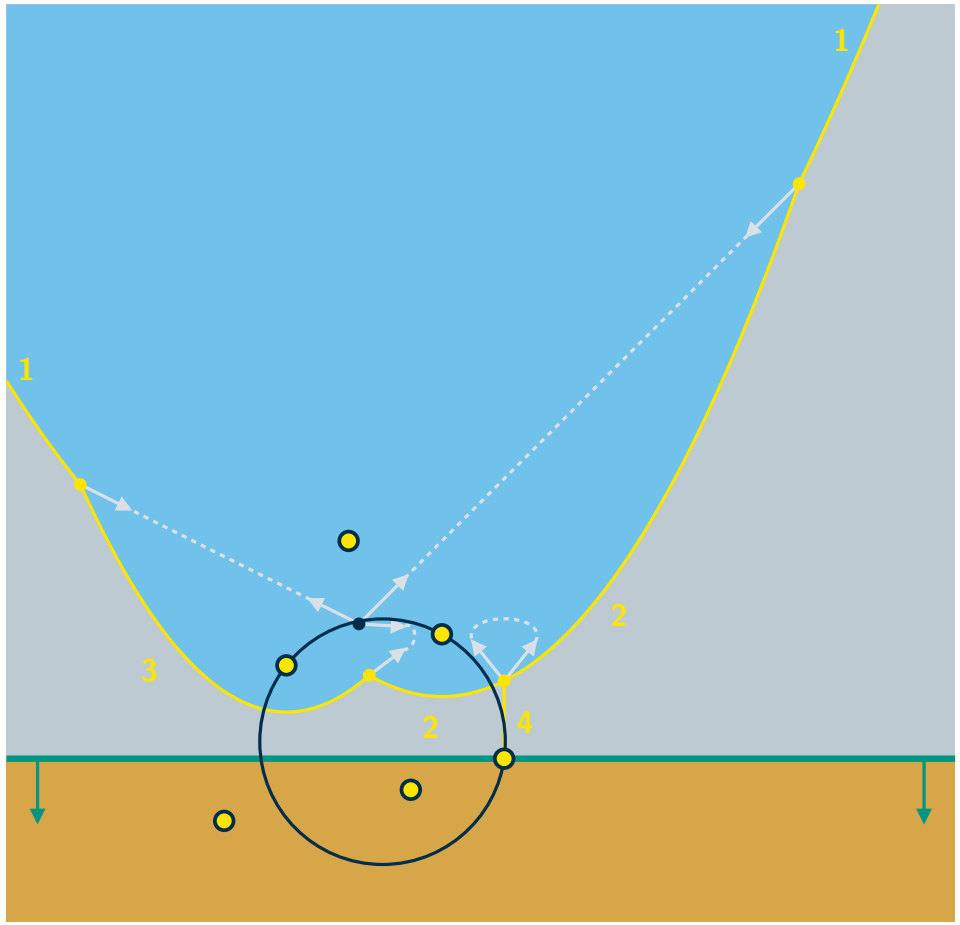
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



beach line: (s_1, s_3) (s_3, s_2) (s_2, s_4) (s_4, s_2) (s_2, s_1)

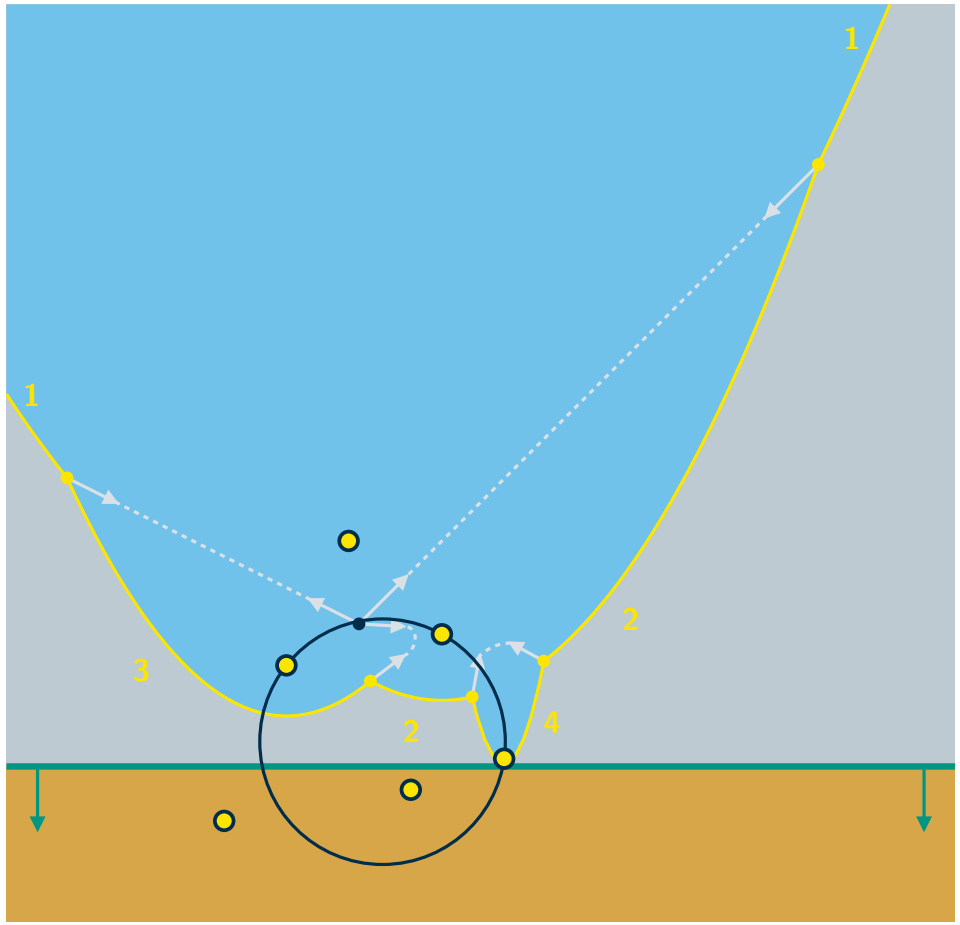
Site Event (Site s)

- find s_x in the beach line
- insert two new intersections
- handle false alarm for intersected parabola piece
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections (s_i, s_j), (s_j, s_k))

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



beach line: (s_1, s_3) (s_3, s_2) (s_2, s_4) (s_4, s_2) (s_2, s_1)

Site Event (Site s)

- find s_x in the beach line
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- handle false alarm for intersected parabola piece
- test new intersections for circle events
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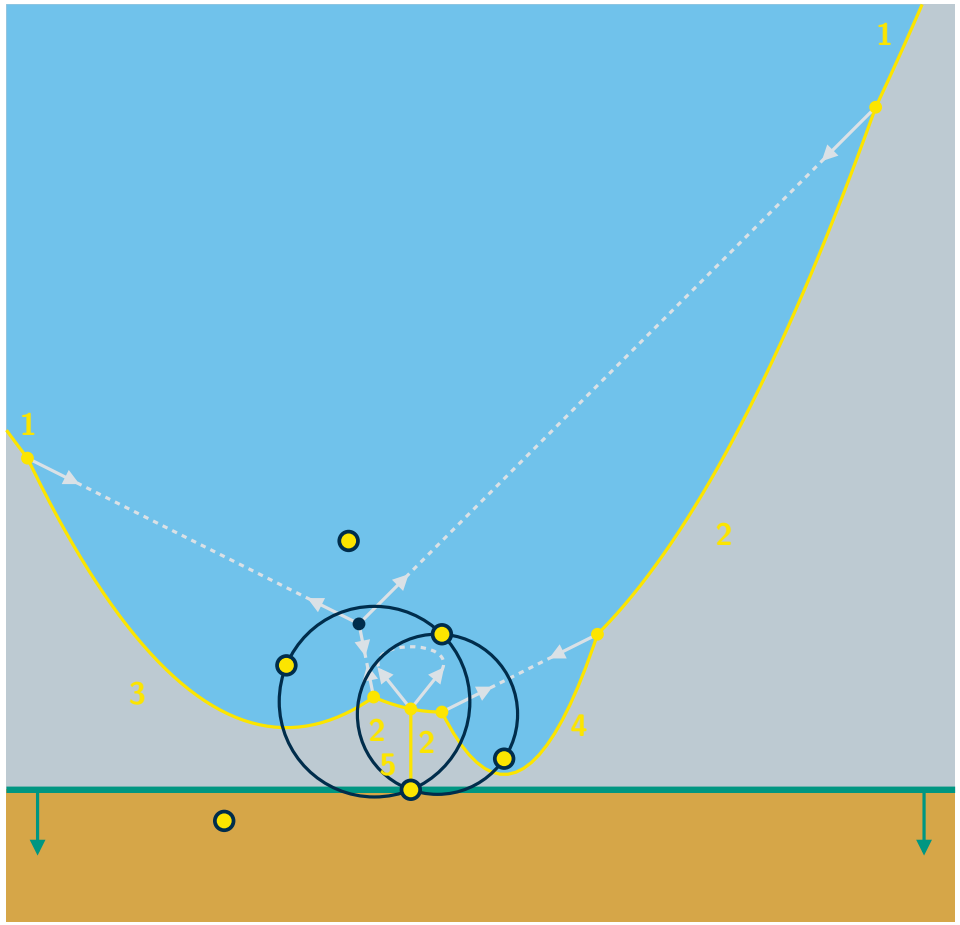
Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



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- find s_x in the beach line
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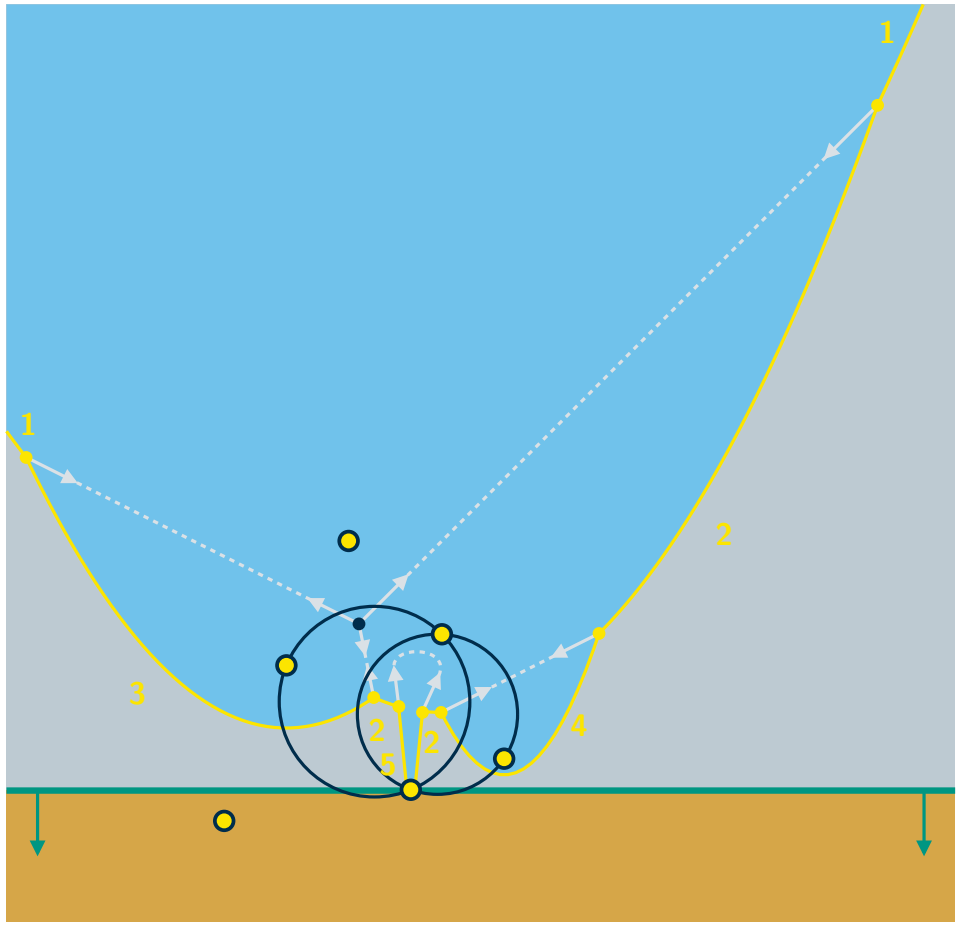
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

beach line: (s_1, s_3) (s_3, s_2) (s_2, s_5) (s_5, s_2) (s_2, s_4) (s_4, s_2) (s_2, s_1)

Overview



Site Event (Site s)

- find s_x in the beach line
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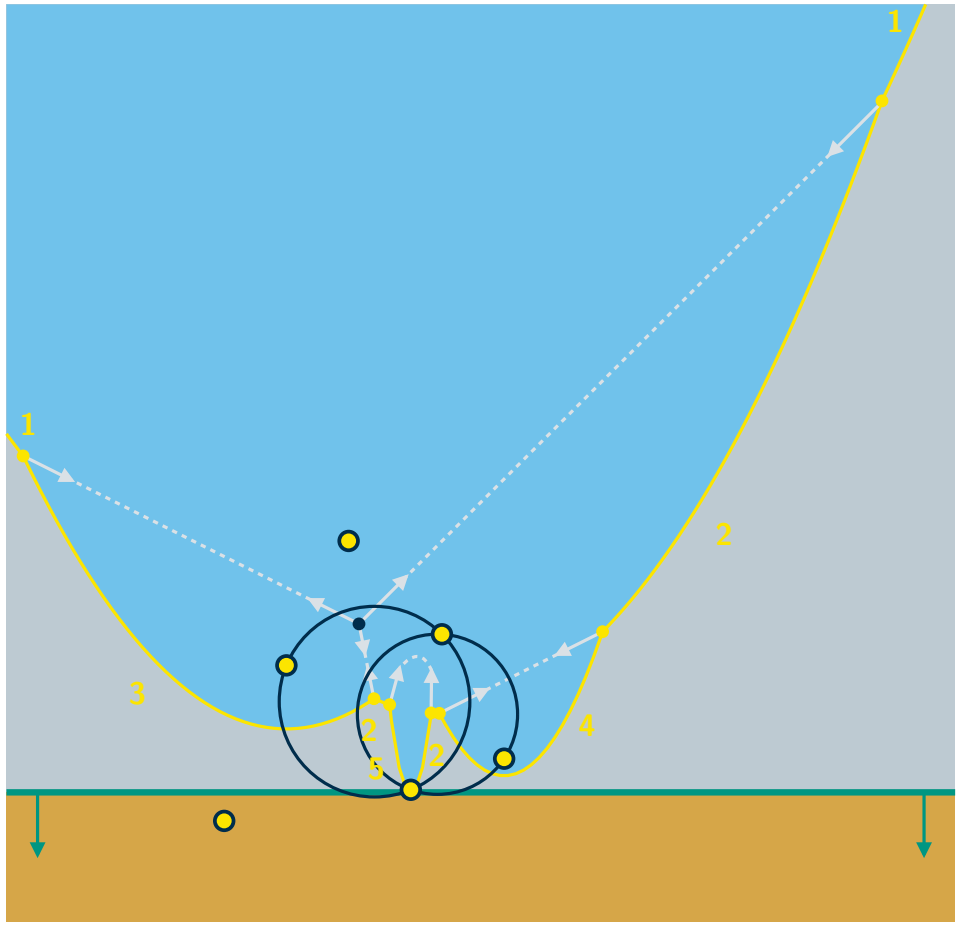
- handle false alarm for intersected parabola piece
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

beach line: (s_1, s_3) (s_3, s_2) (s_2, s_5) (s_5, s_2) (s_2, s_4) (s_4, s_2) (s_2, s_1)

Overview



Site Event (Site s)

- find s_x in the beach line
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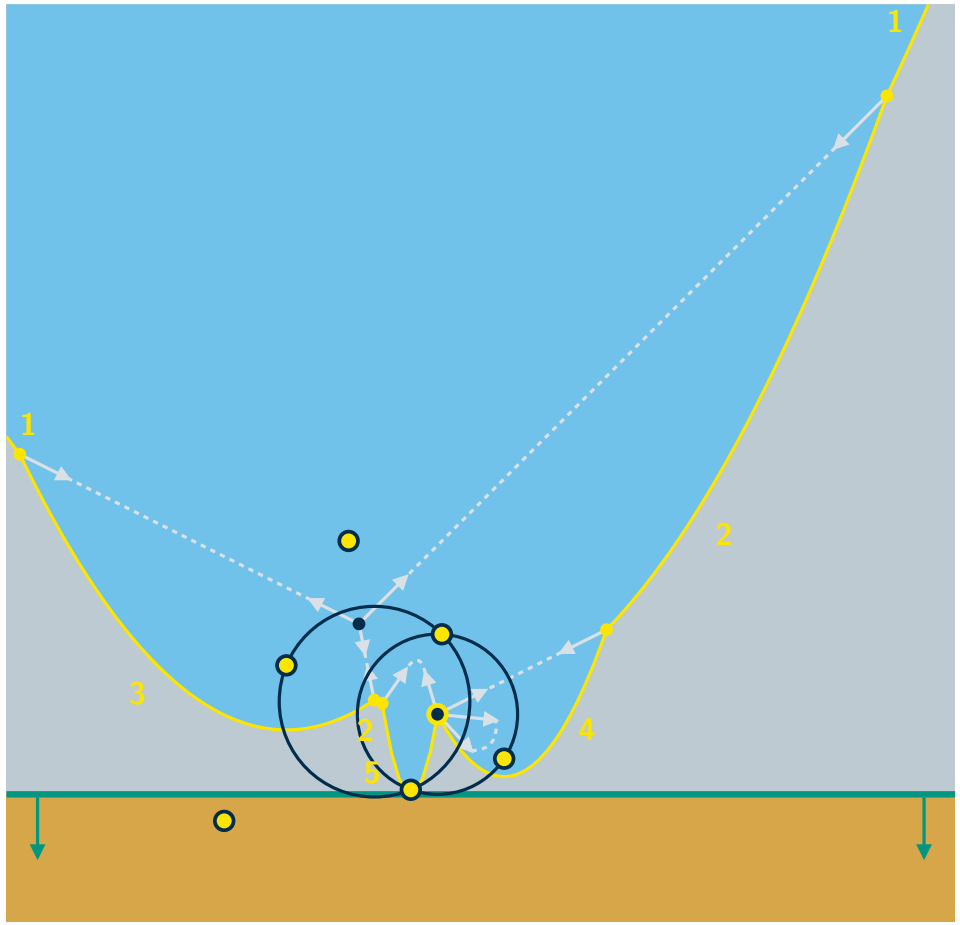
- handle false alarm for intersected parabola piece
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

beach line: (s_1, s_3) (s_3, s_2) (s_2, s_5) (s_5, s_2) (s_2, s_4) (s_4, s_2) (s_2, s_1)

Overview



Site Event (Site s)

- find s_x in the beach line
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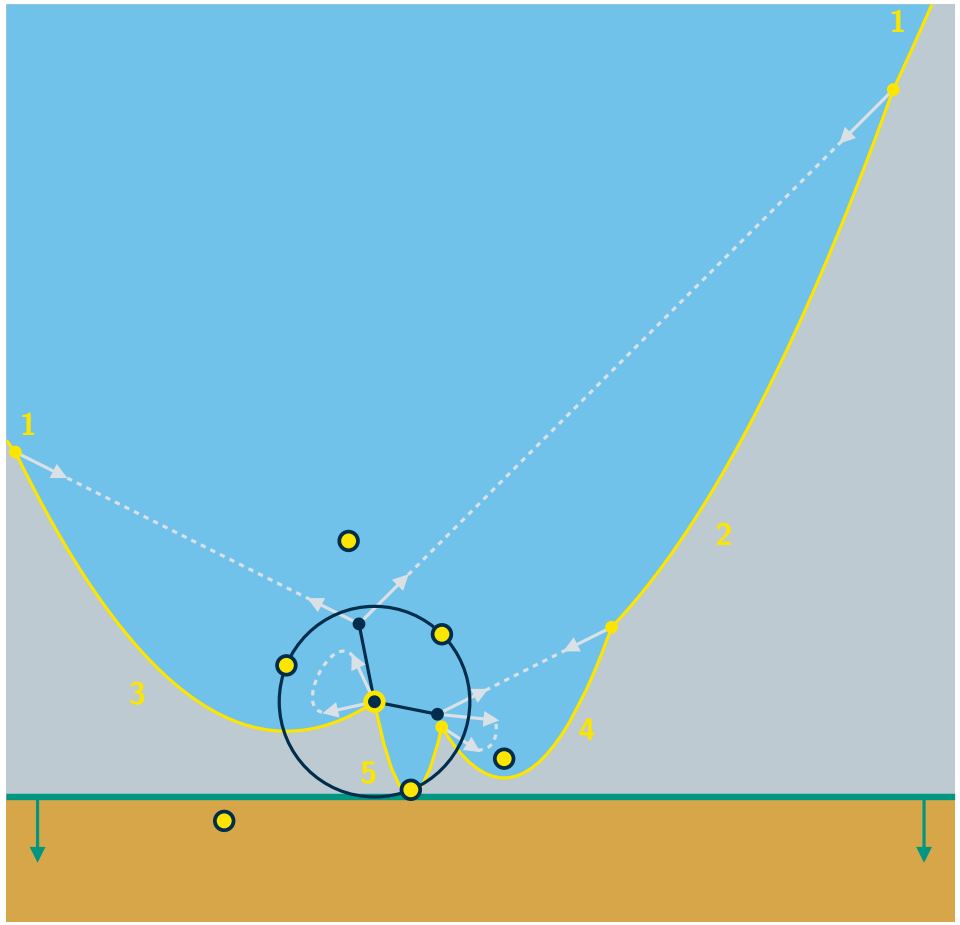
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

beach line: (s_1, s_3) (s_3, s_2) (s_2, s_5) (s_5, s_4) (s_4, s_2) (s_2, s_1)

Overview



beach line: (s_1, s_3) (s_3, s_5) (s_5, s_4) (s_4, s_2) (s_2, s_1)

Site Event (Site s)

- find s_x in the beach line
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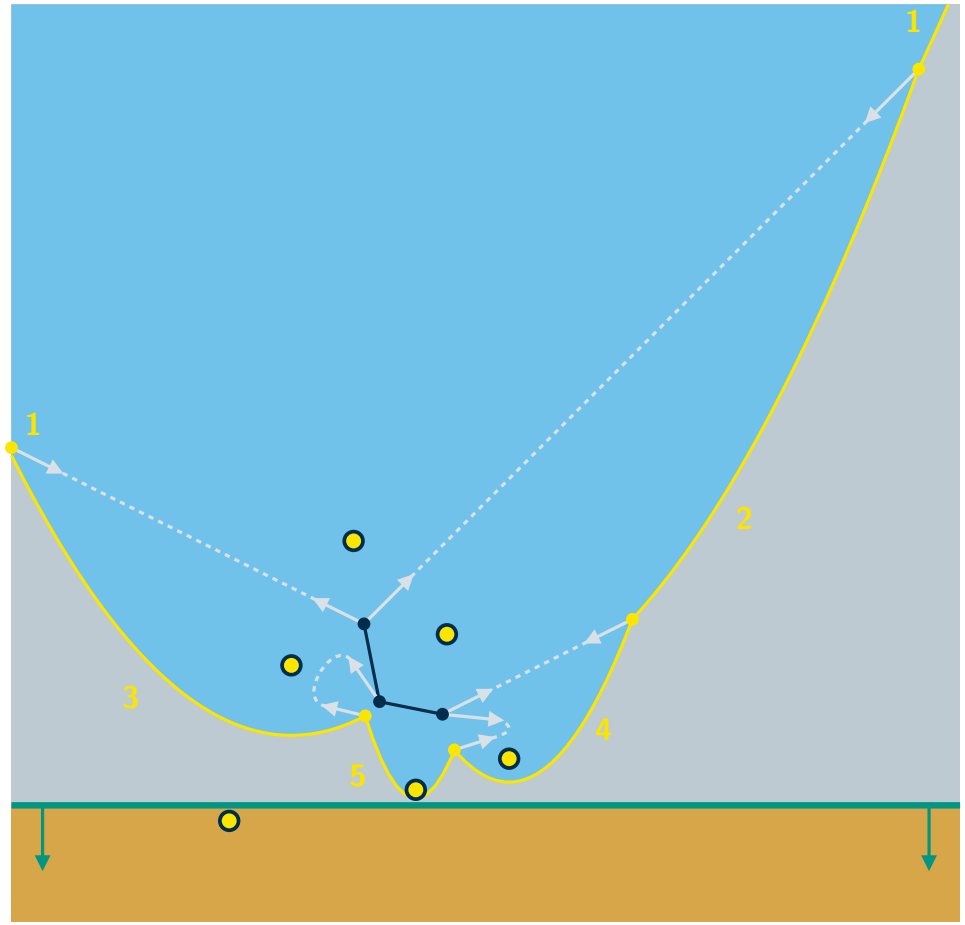
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



beach line: (s_1, s_3) (s_3, s_5) (s_5, s_4) (s_4, s_2) (s_2, s_1)

Site Event (Site s)

- find s_x in the beach line
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- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
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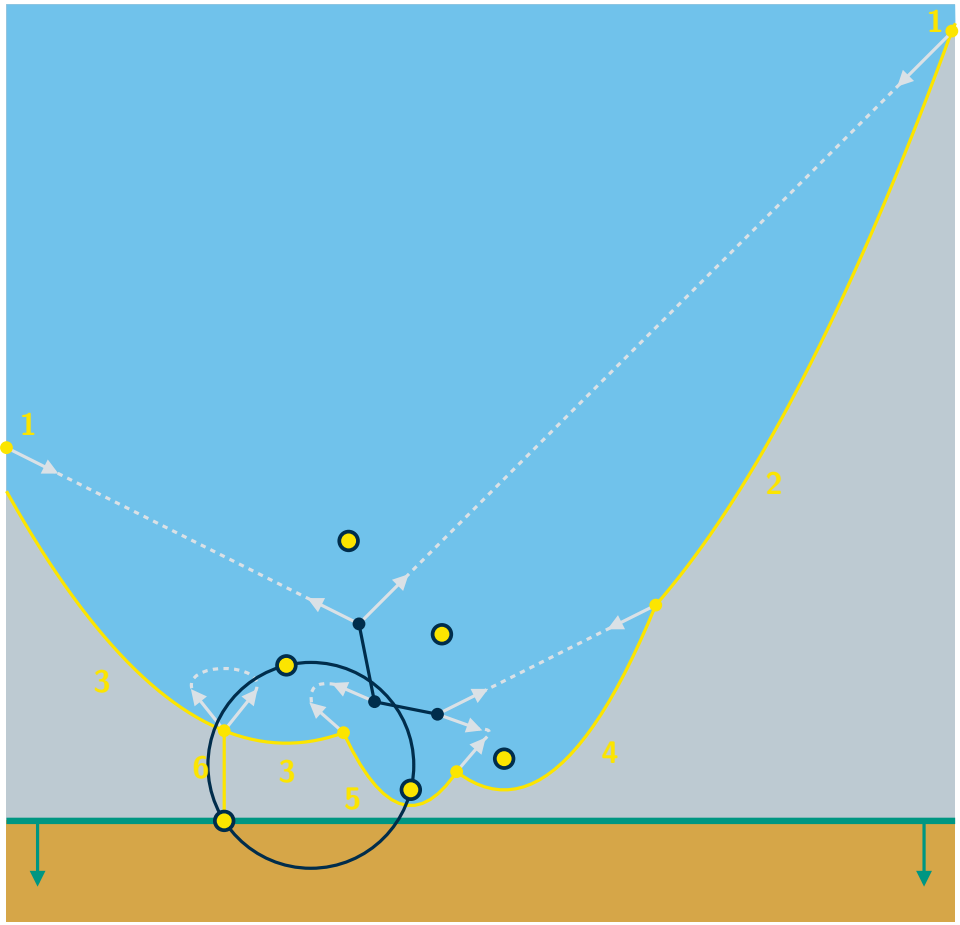
Overview



Site Event (Site s)

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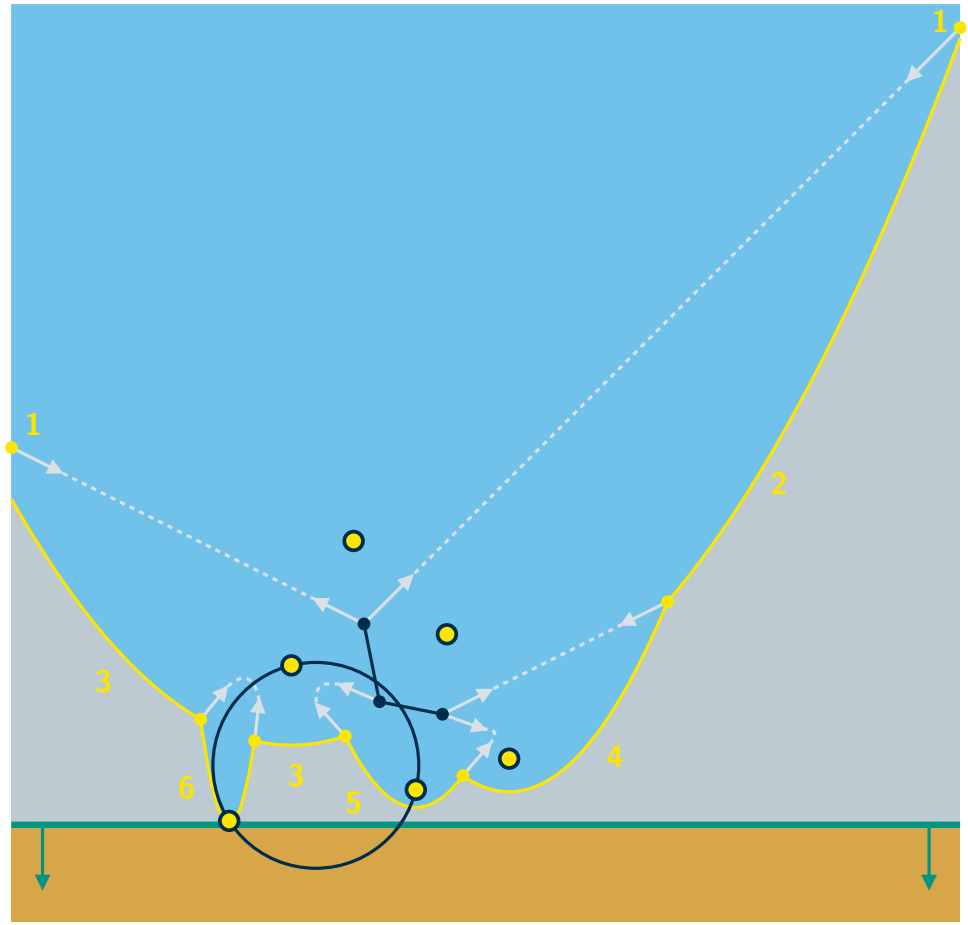
Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

beach line: (s_1, s_3) (s_3, s_6) (s_6, s_3) (s_3, s_5) (s_5, s_4) (s_4, s_2) (s_2, s_1)

Overview



Site Event (Site s)

- find s_x in the beach line
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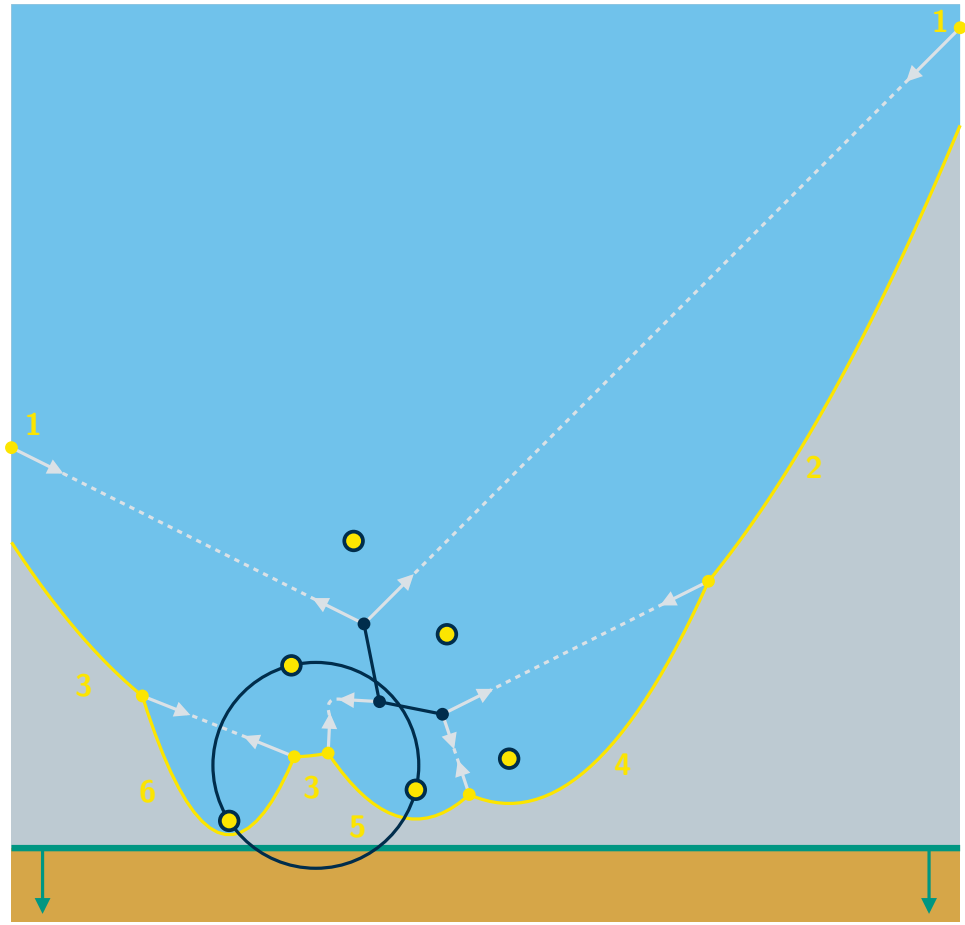
- handle false alarm for intersected parabola piece
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

beach line: (s_1, s_3) (s_3, s_6) (s_6, s_3) (s_3, s_5) (s_5, s_4) (s_4, s_2) (s_2, s_1)

Overview



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- test new intersections for circle events
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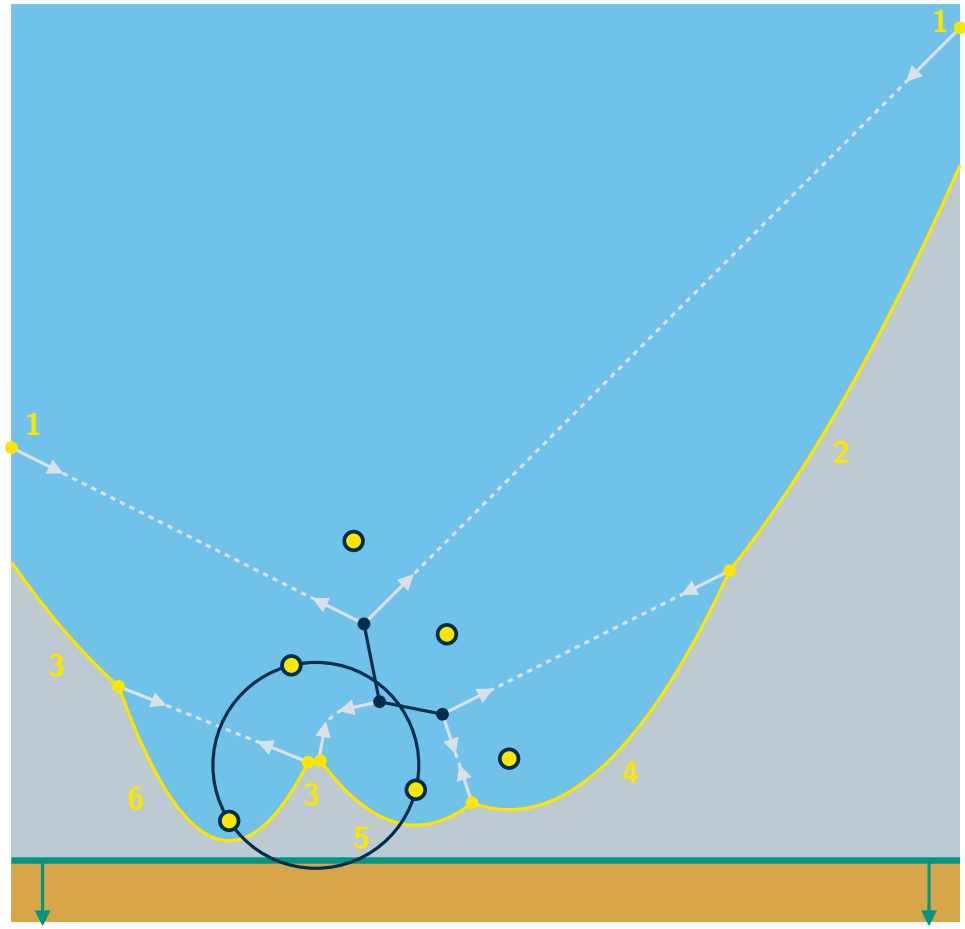
Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

beach line: (s_1, s_3) (s_3, s_6) (s_6, s_3) (s_3, s_5) (s_5, s_4) (s_4, s_2) (s_2, s_1)

Overview



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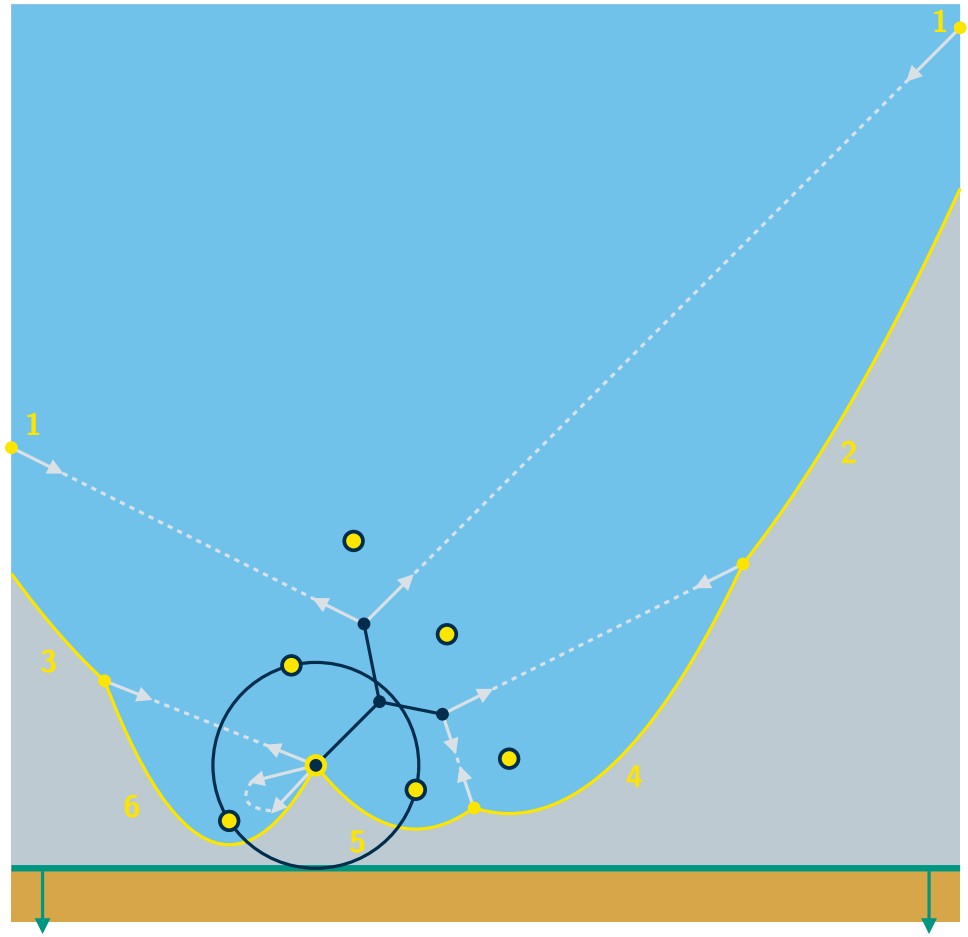
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

beach line: (s_1, s_3) (s_3, s_6) (s_6, s_3) (s_3, s_5) (s_5, s_4) (s_4, s_2) (s_2, s_1)

Overview



beach line: (s_1, s_3) (s_3, s_6) (s_6, s_5) (s_5, s_4) (s_4, s_2) (s_2, s_1)

Site Event (Site s)

- find s_x in the beach line
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- test new intersections for circle events
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Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
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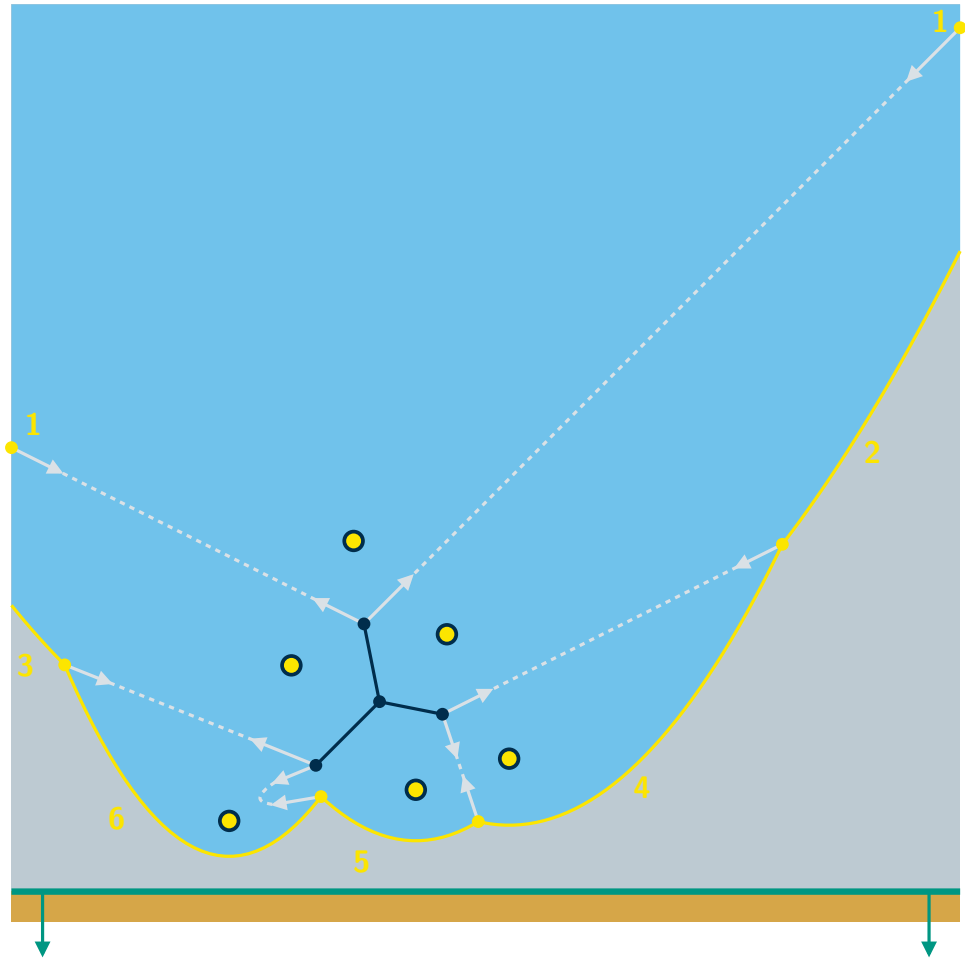
Overview



beach line: (s_1, s_3) (s_3, s_6) (s_6, s_5) (s_5, s_4) (s_4, s_2) (s_2, s_1)

Site Event (Site s)

- find s_x in the beach line
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Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
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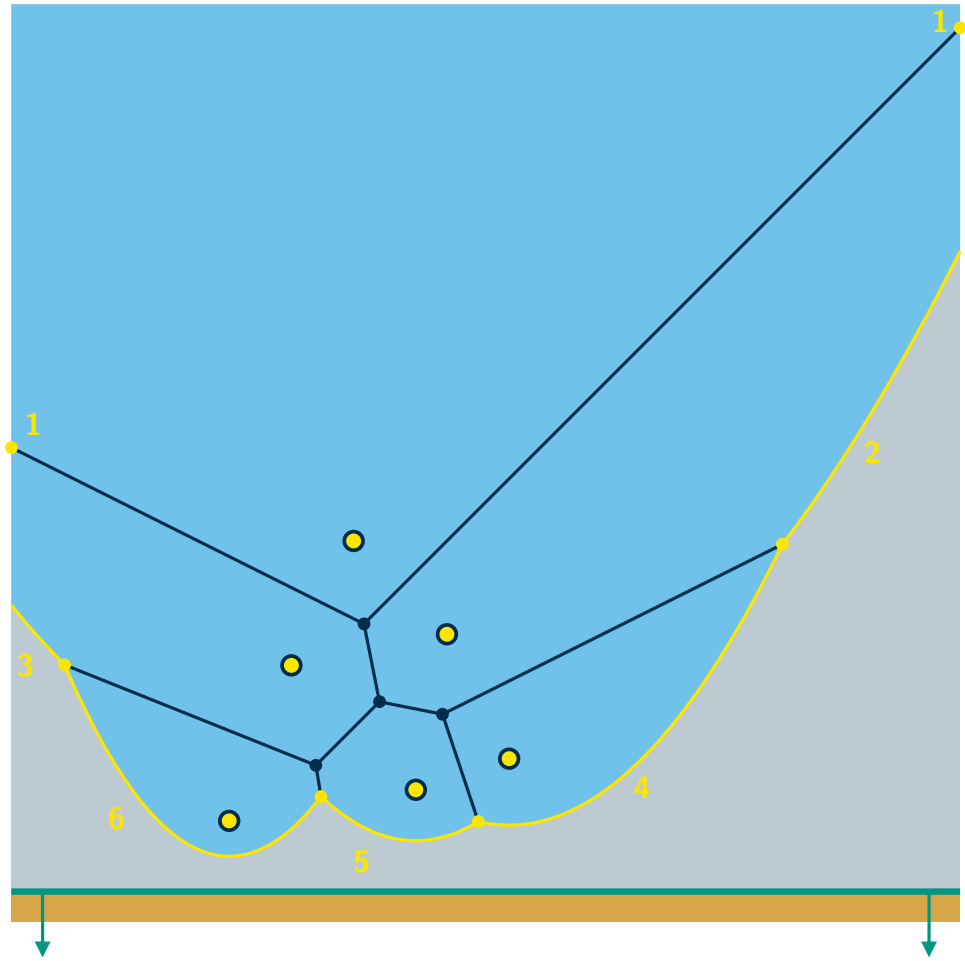
Overview



beach line: (s_1, s_3) (s_3, s_6) (s_6, s_5) (s_5, s_4) (s_4, s_2) (s_2, s_1)

Site Event (Site s)

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- test new intersections for circle events
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Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
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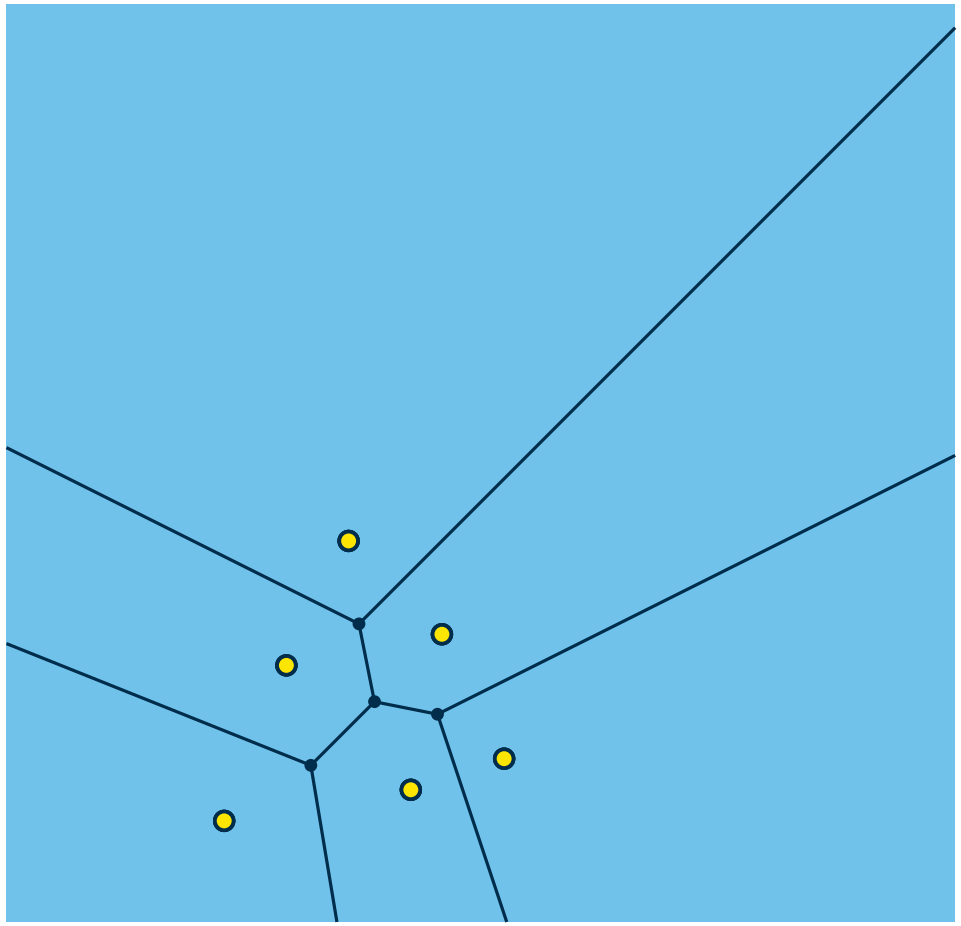
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Overview



beach line:

Site Event (Site s)

- find s_x in the beach line
- insert two new intersections
- handle false alarm for intersected parabola piece
- test new intersections for circle events
- new pair of half edges for the intersections

Circle Event (For Intersections $(s_i, s_j), (s_j, s_k)$)

- remove (s_i, s_j) and (s_j, s_k)
- insert (s_i, s_k)
- test (s_i, s_k) with BL-neighbors for circle events
- create new vertex v
- attach half edge of (s_i, s_j) and (s_j, s_k) to v
- new pair of half edges: one at v, one at (s_i, s_k)

Wrap-Up

Theorem
The Voronoi diagram of n sites can be computed in $O(n \log n)$ time using $O(n)$ space.

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  - sites that are more complicated than just points (e.g., line segments)
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- visualization: `http://www.raymondhill.net/voronoi/rhill-voronoi.html`