Computational Geometry

Orthogonal Range Queries: Fractional Cascading

Thomas Bläsius

Searching In Many Arrays

Situation

- consider ℓ sorted arrays $A_1, \ldots, A_\ell$ with $\leq n$ elements each
- find the position of x in all arrays
- obvious solution: $O(\ell \log n)$

$A_1 =$

2	5	8	12	17	19	25	28

$A_2 =$

3	4	6	11	13	18	25	33

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Is $\ell + \log n$ Possible In General?

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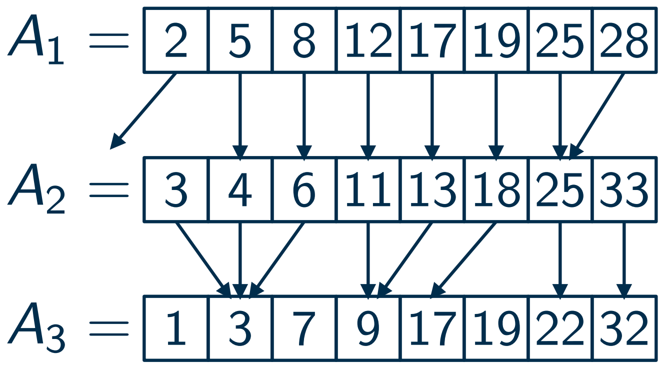

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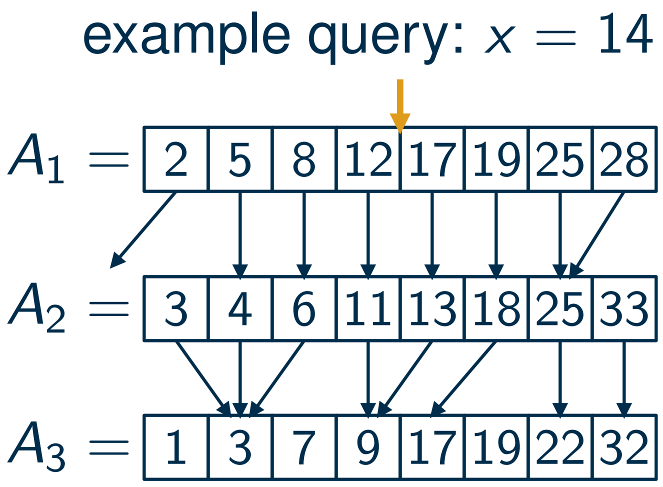

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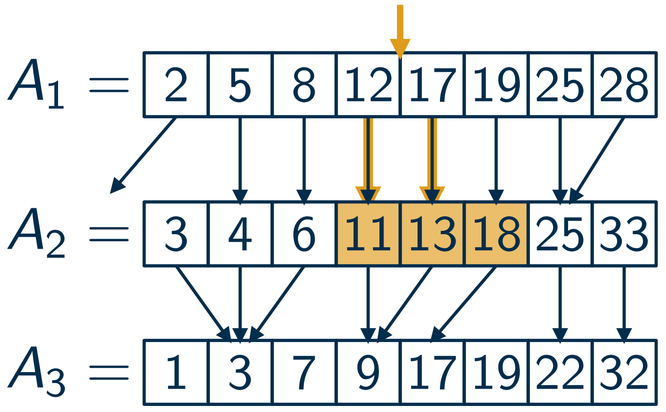
example query: $x = 14$



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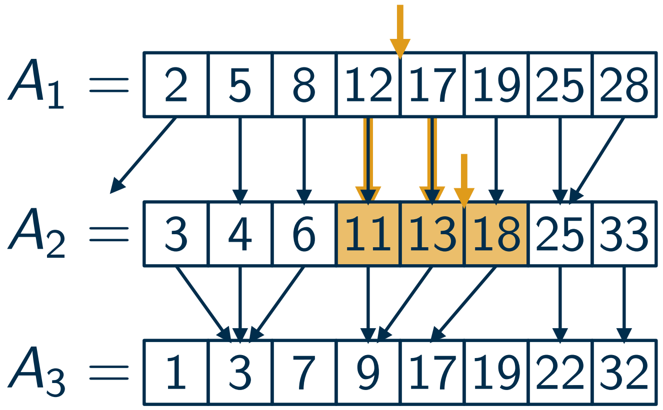

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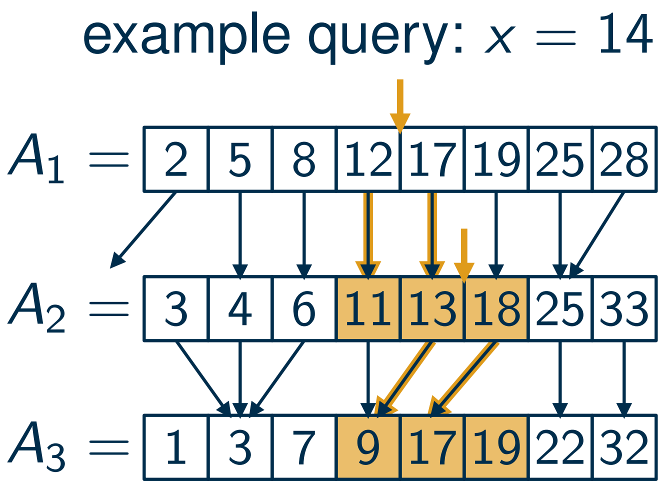

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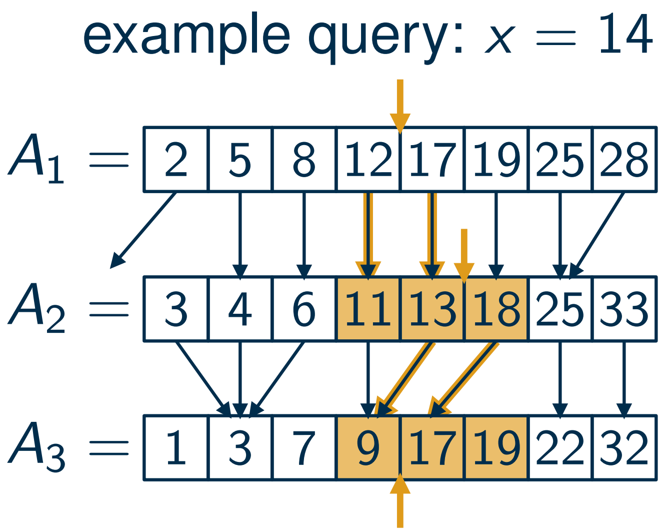

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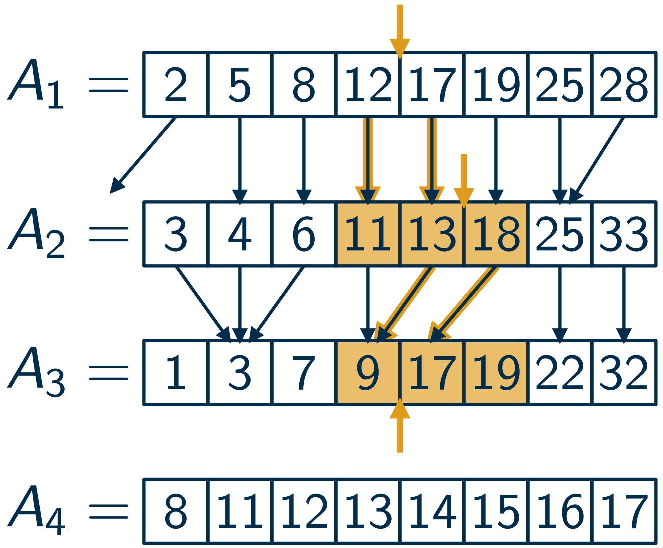

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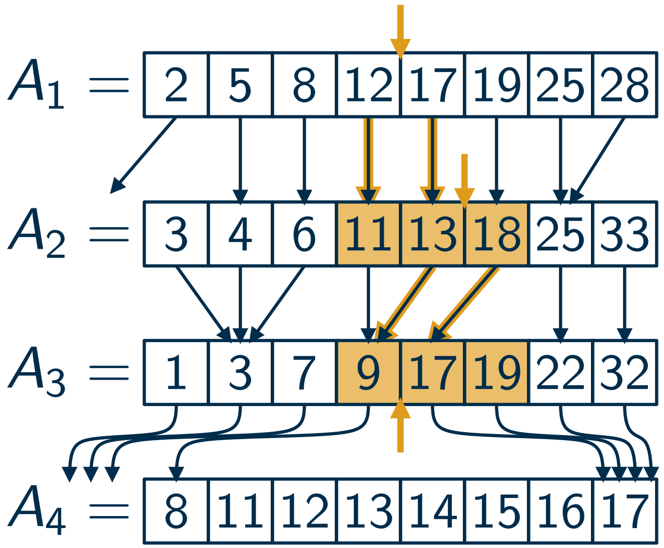

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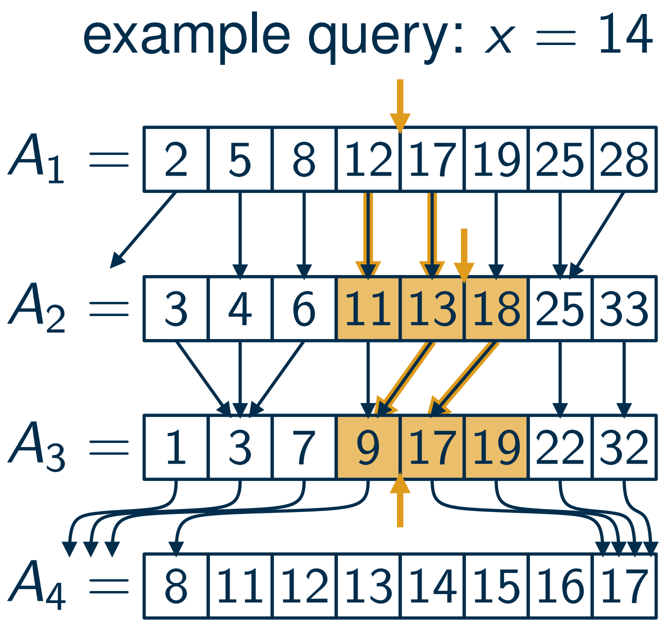
- problem: position of x in A_i may not help to find position in A_{i+1}



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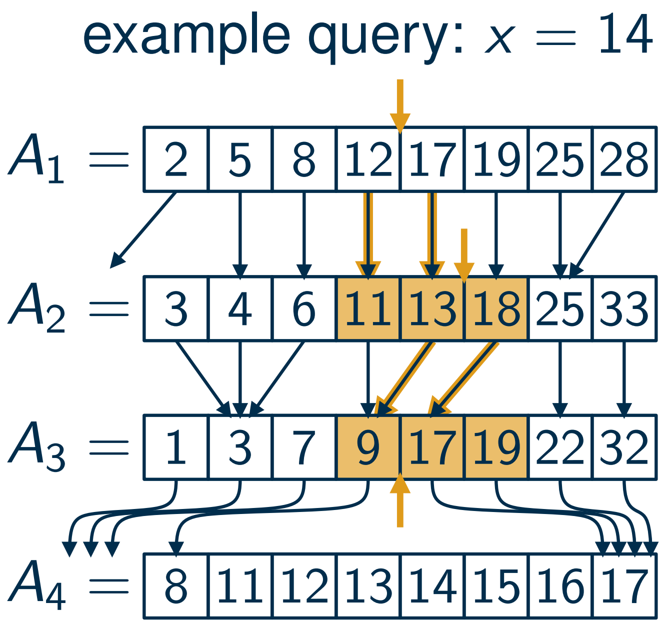
Observation

- $A_i \supseteq A_{i+1} \Rightarrow$ position in A_i determines position in A_{i+1}



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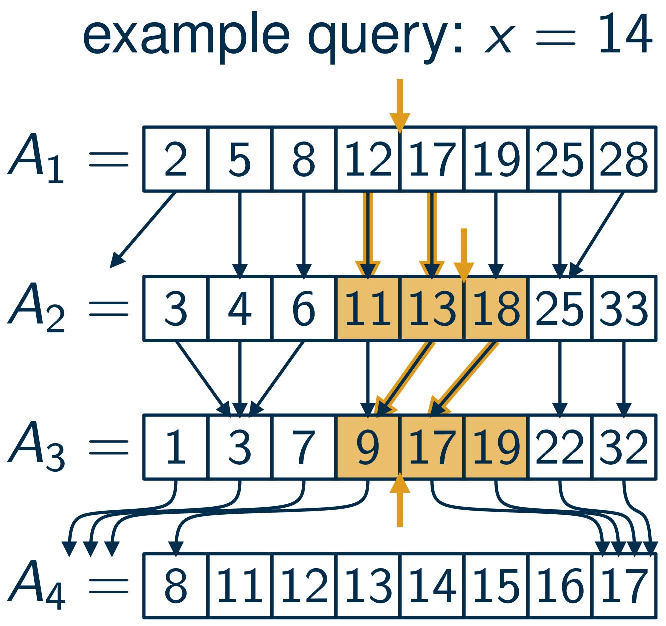

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Observation

- $A_i \supseteq A_{i+1} \Rightarrow$ position in A_i determines position in A_{i+1}
- A_i contains many elements from $A_{i+1} \Rightarrow$ position in A_i roughly determines position in A_{i+1}

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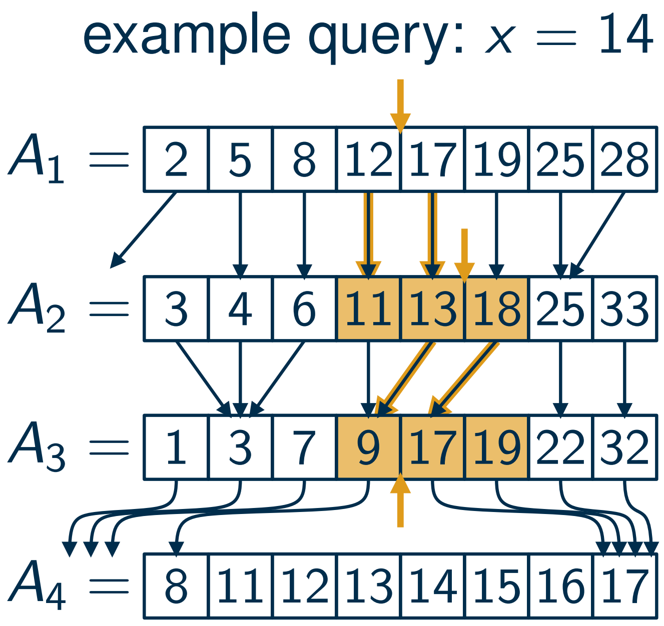

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- $A_i \supseteq A_{i+1} \Rightarrow$ position in A_i determines position in A_{i+1}
- A_i contains many elements from $A_{i+1} \Rightarrow$ position in A_i roughly determines position in A_{i+1}
- idea: insert some elements from A_{i+1} into A_i

Fractional Cascading

Shared Elements

- new array A'_3: insert every other element from A_4 into A_3

$A_3 =$

1	3	7	9	17	19	22	32

$A'_3 =$

1	3	7	8	9	12	14	16	17	19	22	32

$A_4 =$

8	11	12	13	14	15	16	17

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Shared Elements

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- store pointers to copies

$$A_3 = \boxed{1\,|\,3\,|\,7\,|\,9\,|\,17\,|\,19\,|\,22\,|\,32} \qquad A_3' = \boxed{1\,|\,3\,|\,7\,|\,8\,|\,9\,|\,12\,|\,14\,|\,16\,|\,17\,|\,19\,|\,22\,|\,32}$$

$$A_4 = \boxed{8\,|\,11\,|\,12\,|\,13\,|\,14\,|\,15\,|\,16\,|\,17} \qquad A_4 = \boxed{8\,|\,11\,|\,12\,|\,13\,|\,14\,|\,15\,|\,16\,|\,17}$$

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- new array A_3': insert every other element from A_4 into A_3
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- pointers from $A_3' \setminus A_4$ to prev / next element from A_4 $\Rightarrow$ position in A_3' gives position in A_4 (± 1)

How?

$A_3 =$ | 1 | 3 | 7 | 9 | 17 | 19 | 22 | 32 |

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$A_3' =$

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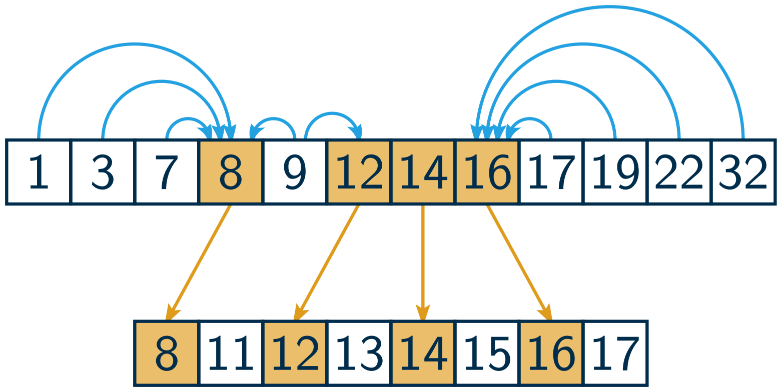

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How?

Why do we need that?

$A_3 = \boxed{1 \mid 3 \mid 7 \mid 9 \mid 17 \mid 19 \mid 22 \mid 32}$

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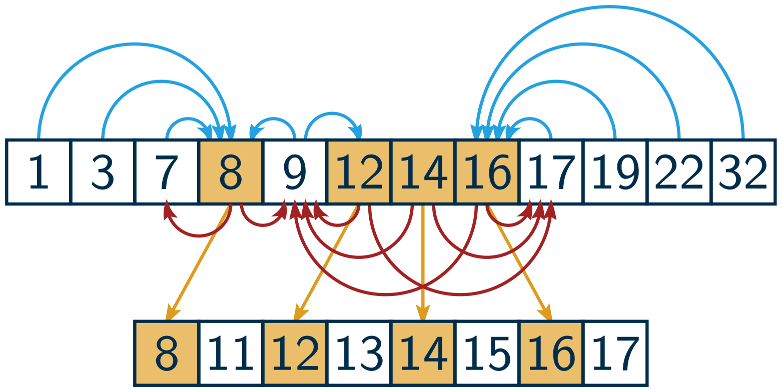

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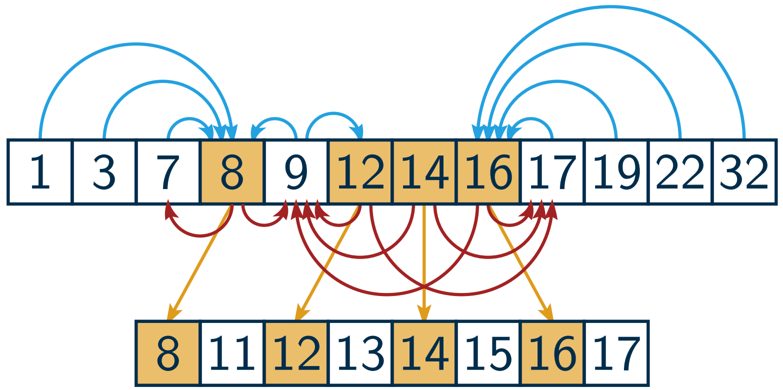

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- pointers from elements in A_4 to prev / next in $A'_3 \setminus A_4 \Rightarrow$ position in A'_3 gives position in A_3
- cascade the process for all previous A_i

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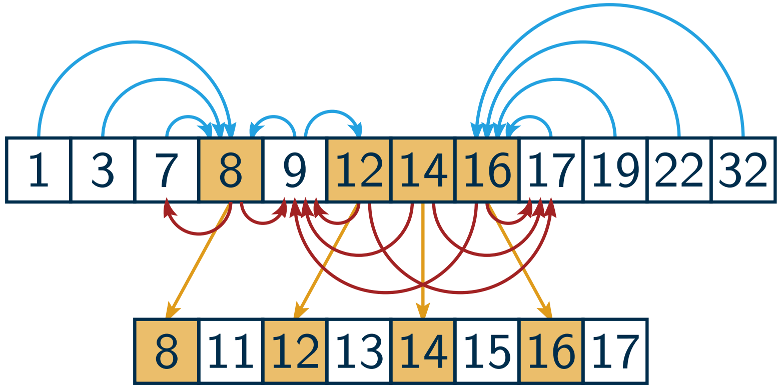

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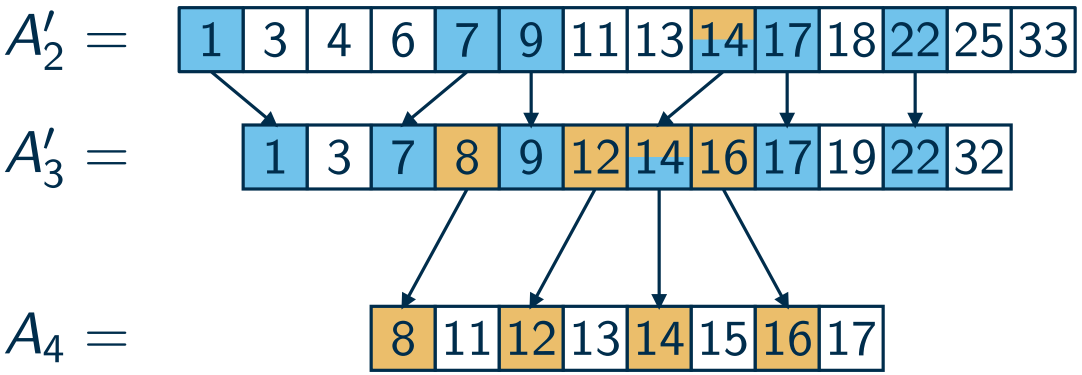

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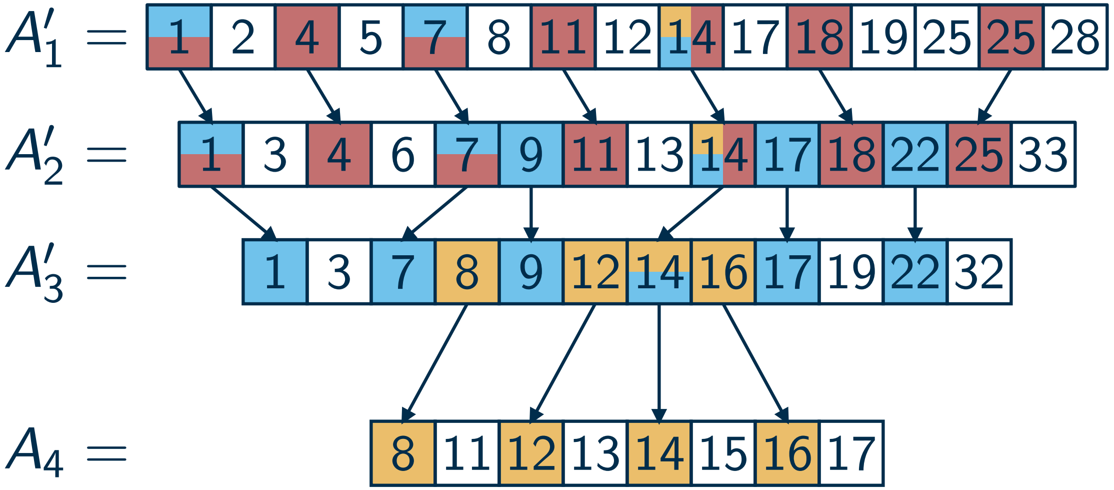

Fractional Cascading – Running Time

Cost For The Search

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Cost For The Search

$$\left.\begin{array}{l}\text{■ one search in } A'_1 \to O(\log(|A'_1|)) \\ \text{■ } O(1) \text{ for every subsequent array} \to O(\ell)\end{array}\right\} \text{total: } O(\ell + \log(|A'_1|))$$

Fractional Cascading – Running Time

Cost For The Search

- one search in $A'_1 \to O(\log(|A'_1|))$
- $O(1)$ for every subsequent array $\to O(\ell)$

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How Large is A'_1?

(assumption: $|A_i| = n$ for all i)

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- $|A'_{\ell-3}| = (\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1)n$
- $|A'_1| \leq 2n$

$\Rightarrow$ search takes $O(\ell + \log n)$ time

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Memory Consumption

- only a constant factor overhead
- also true if not all arrays have the same size

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How Large is A'_1?

(assumption: $|A_i| = n$ for all i)

- $|A'_{\ell-1}| = (\frac{1}{2} + 1)n$
- $|A'_{\ell-2}| = (\frac{1}{4} + \frac{1}{2} + 1)n$
- $|A'_{\ell-3}| = (\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1)n$
- $|A'_1| \leq 2n$

$\Rightarrow$ search takes $O(\ell + \log n)$ time

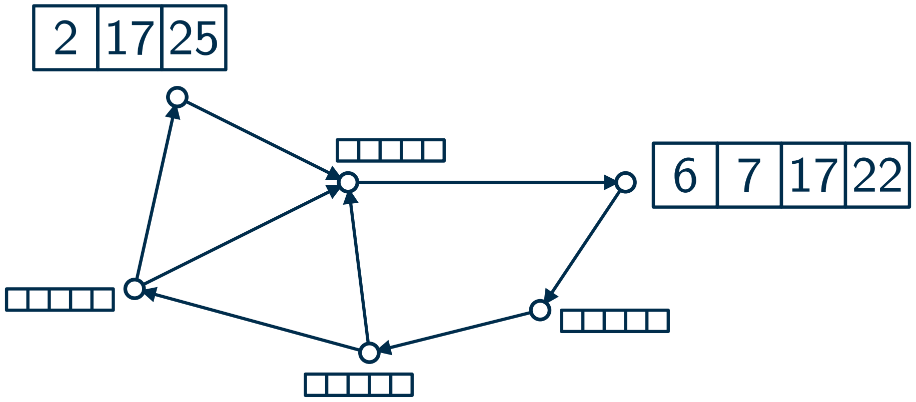
Memory Consumption

- only a constant factor overhead
- also true if not all arrays have the same size

Precomputation Time

- linear in the input

General Fractional Cascading



Now With A Directed Graph $G = (V, E)$

- sorted array A_v for every vertex v

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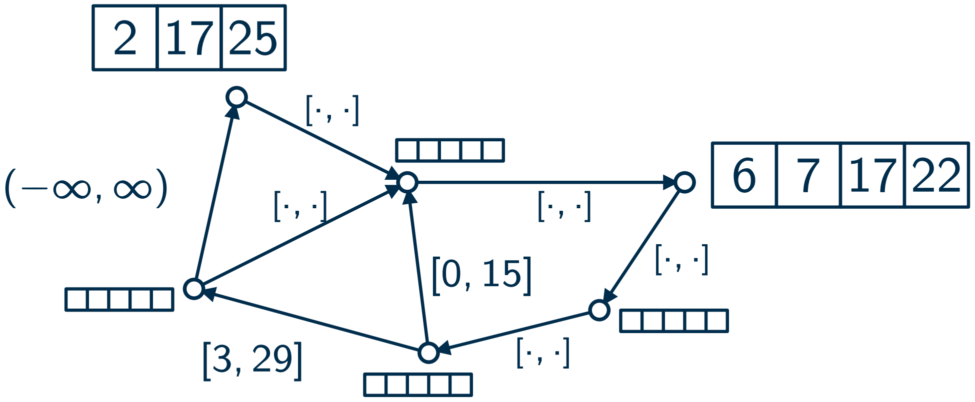

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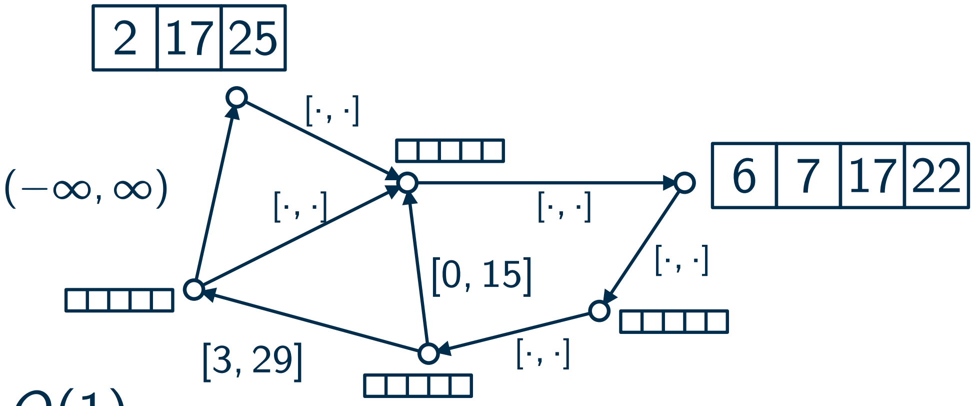

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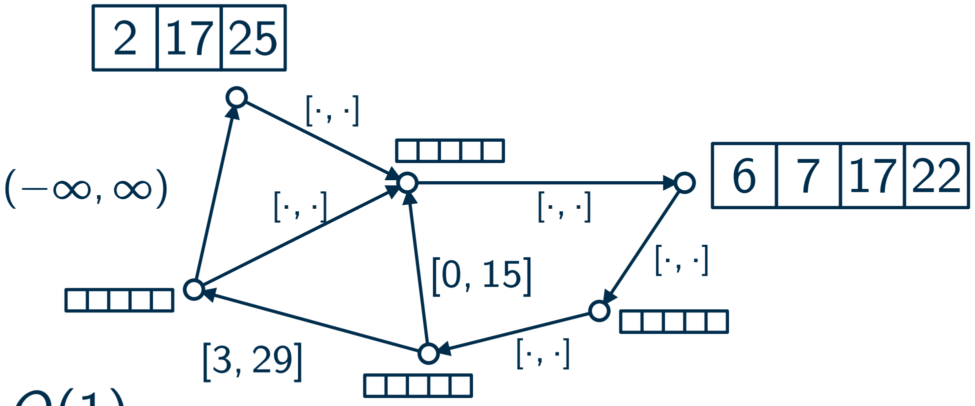

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A Game Between Alice And Bob

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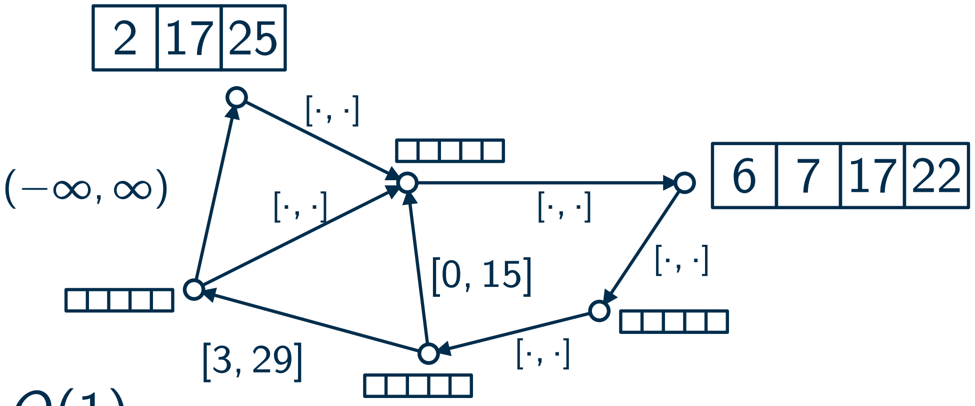

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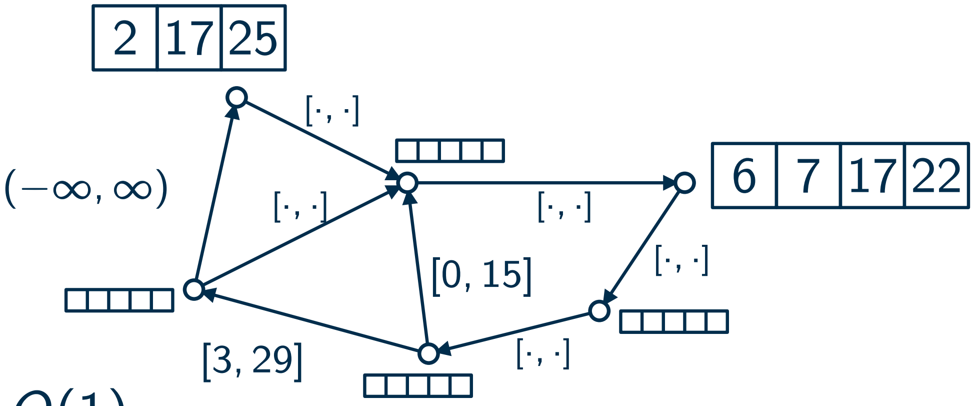

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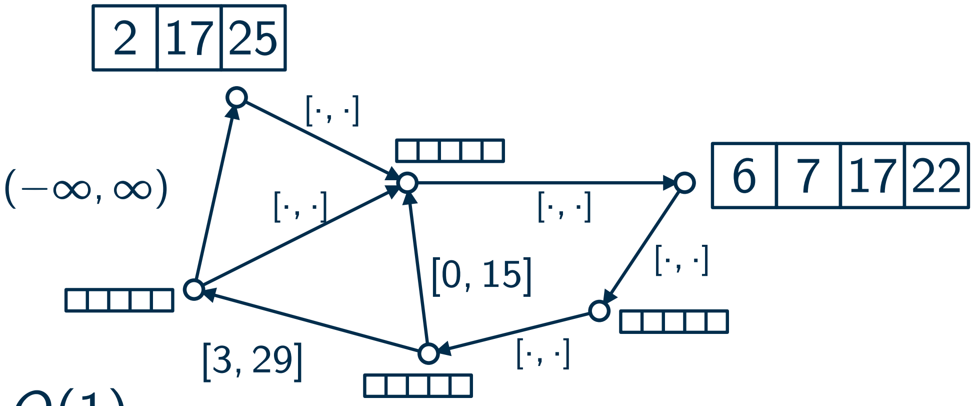

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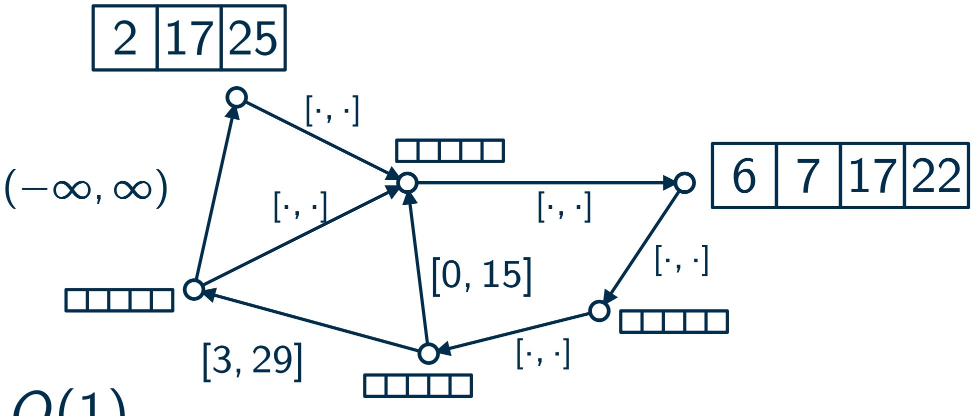

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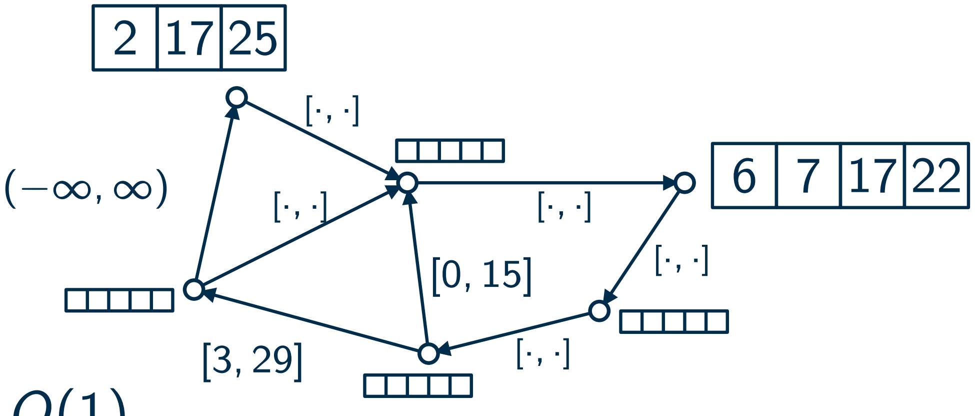

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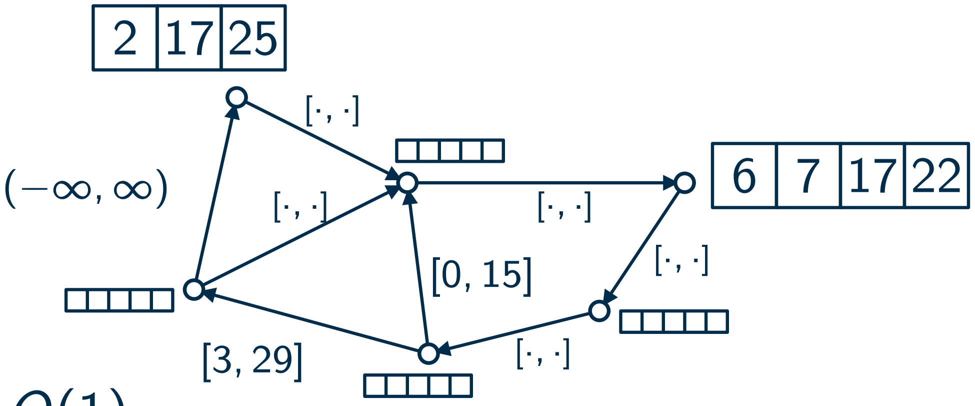

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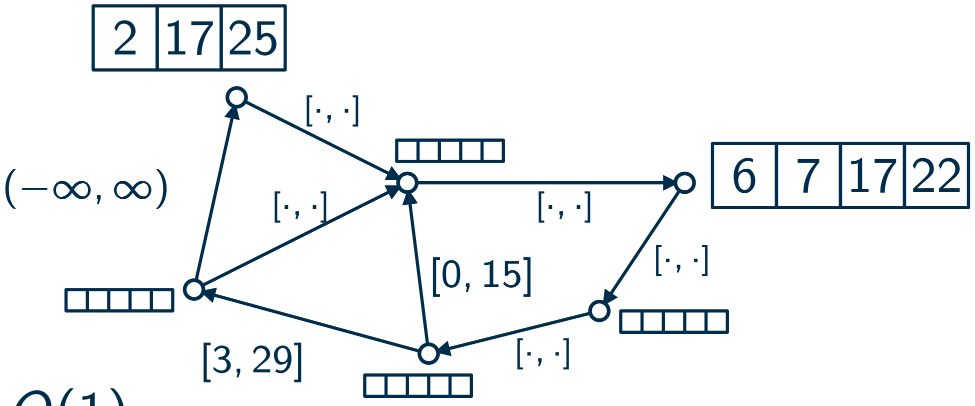

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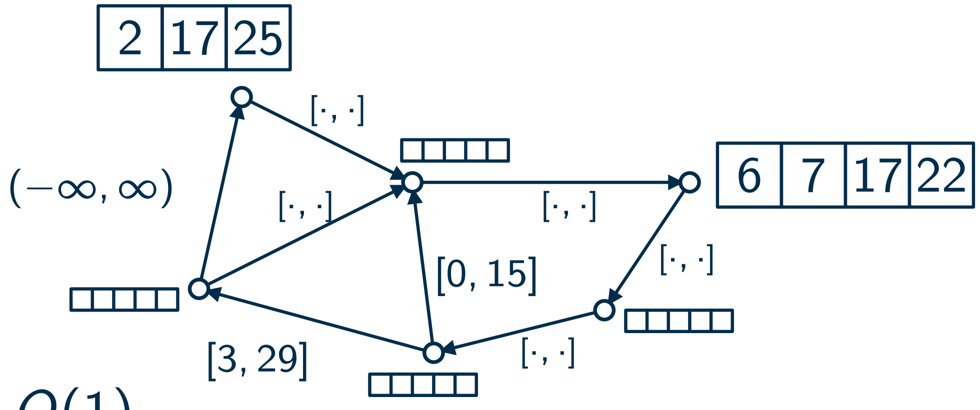

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Similar Guarantee To The Path Setting (without proof)

($s =$ Gesamtgröße der Arrays)

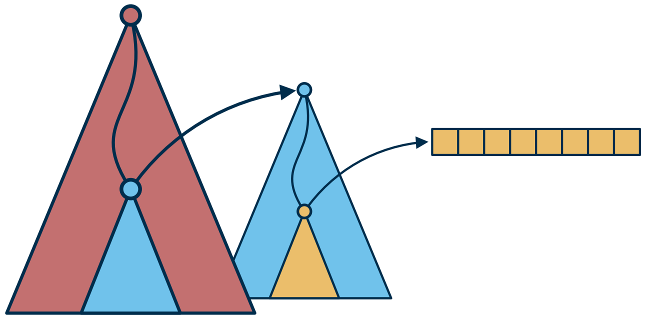
- precomputation: $O(s)$ time and $O(s)$ space
- query: $O(\log s)$ for the first, then $O(1)$

Back To The Range Queries

Back To The Range Queries

Query In 3D Range Tree (Simple Variant)

query: $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$
direction: x y z



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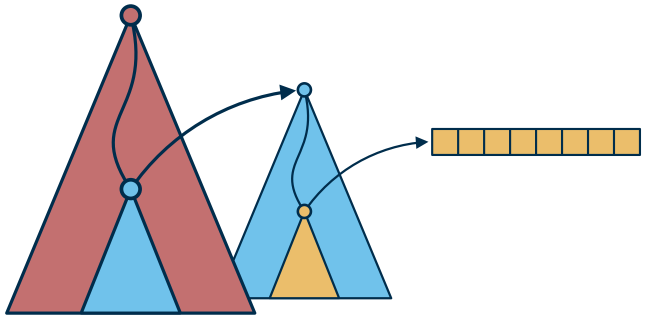

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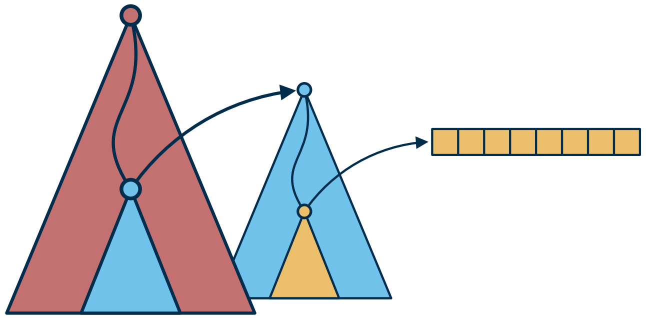

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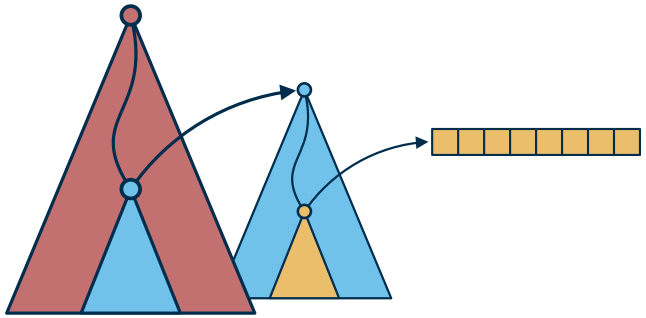

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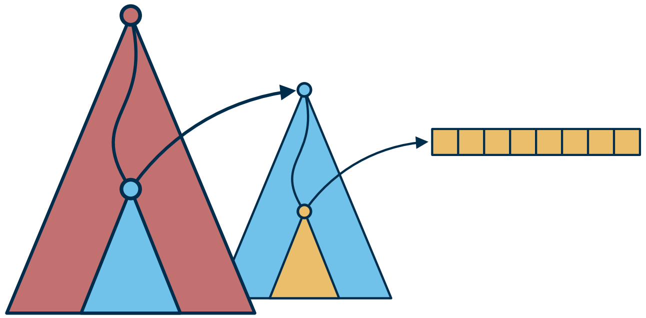

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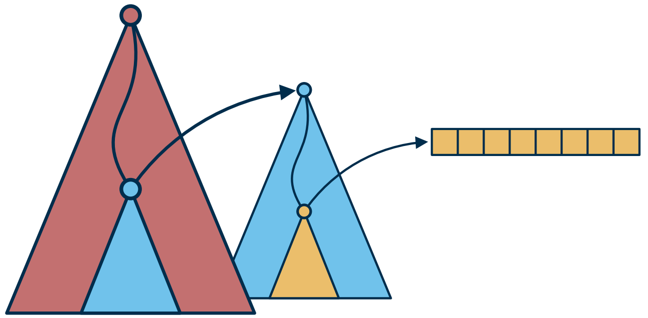

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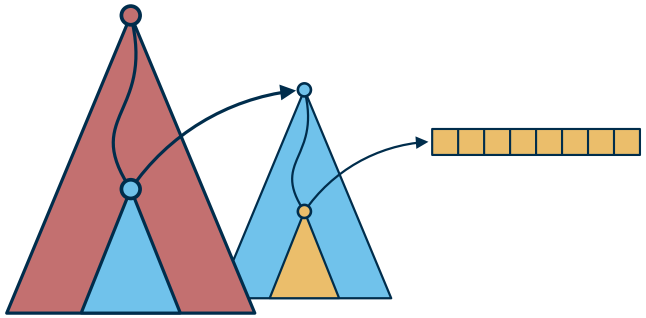

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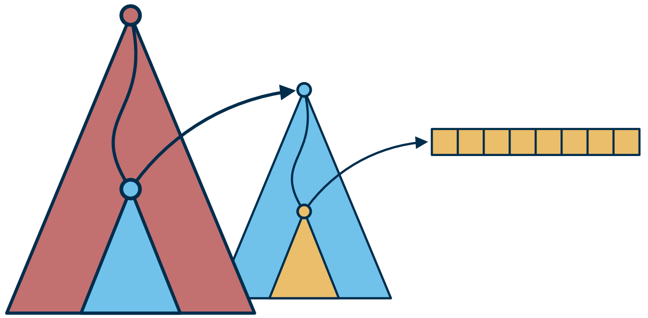

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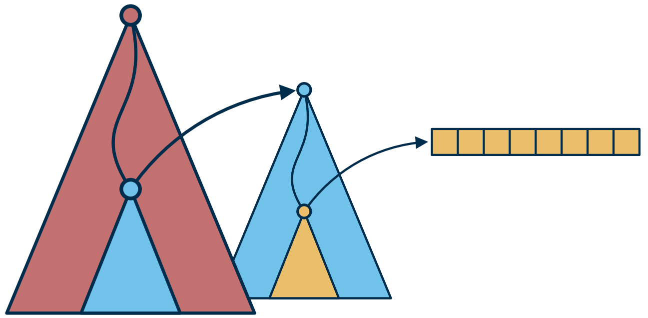

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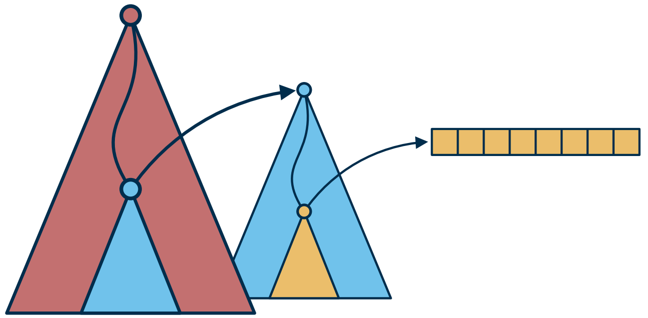

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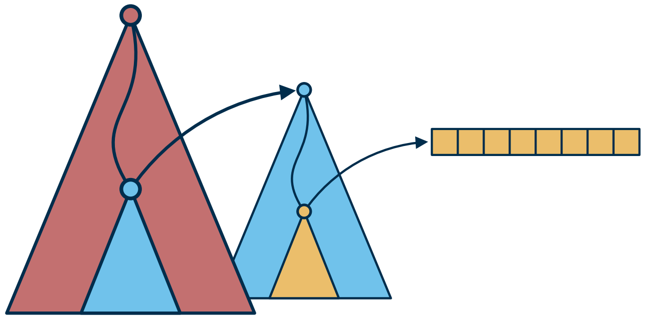

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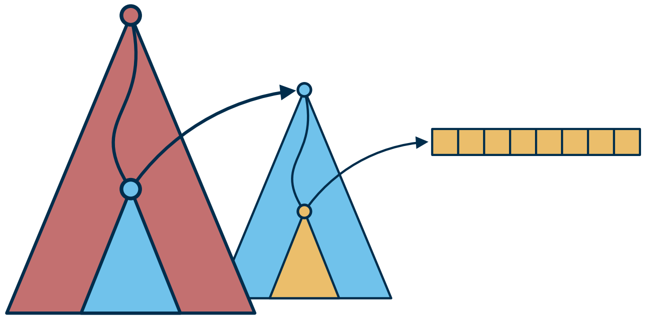

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One-Sided 2D Range Queries

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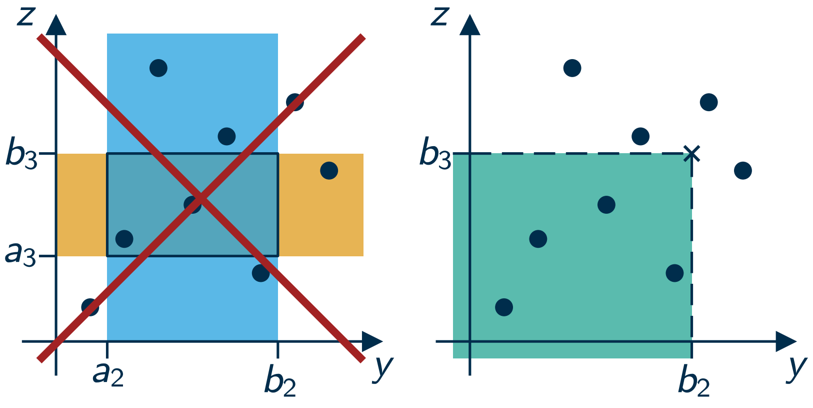
One-Sided Queries: Half The Sides, Half The Trouble

- goal: answer queries of the form $(-\infty, b_2] \times (-\infty, b_3]$ (instead of $[a_2, b_2] \times [a_3, b_3]$)



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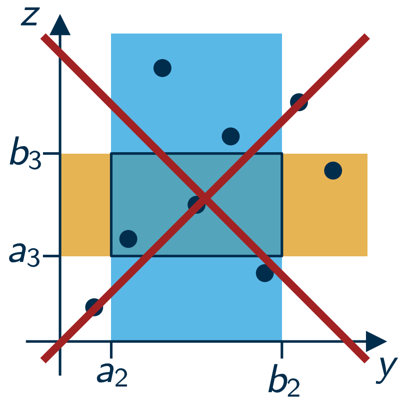
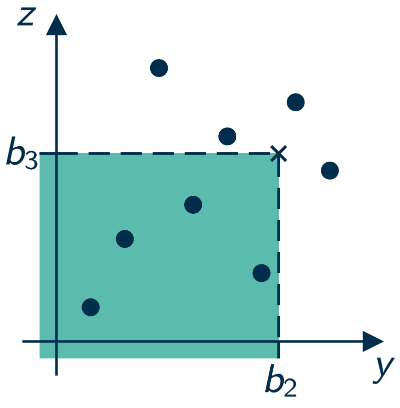
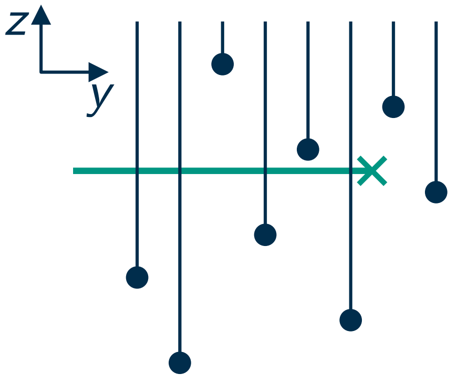




Alternative Perspective

- shoot a ray from each point upwards
- ray from $\langle b_2, b_3 \rangle$ to the left



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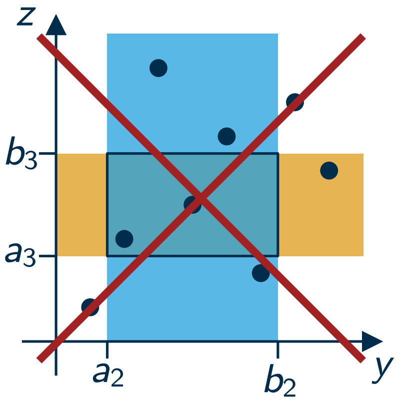

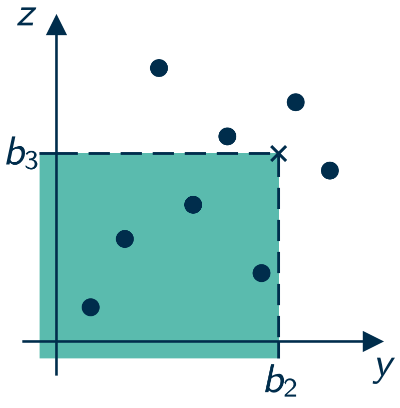

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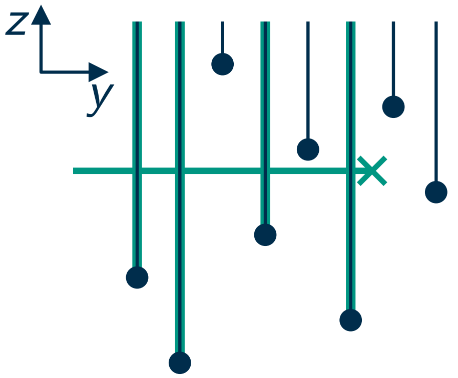

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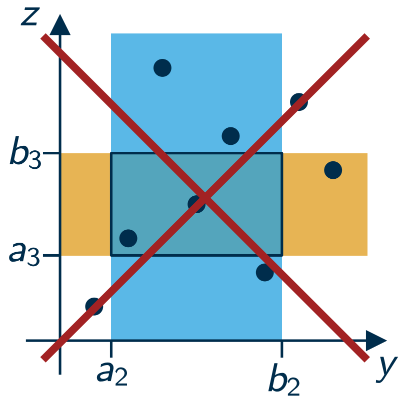

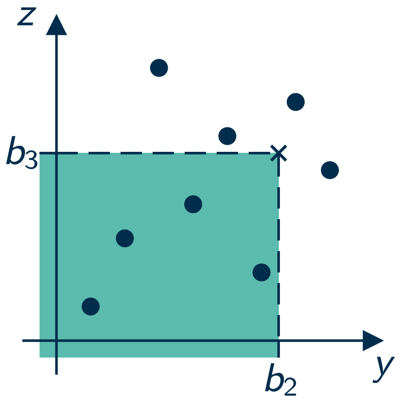

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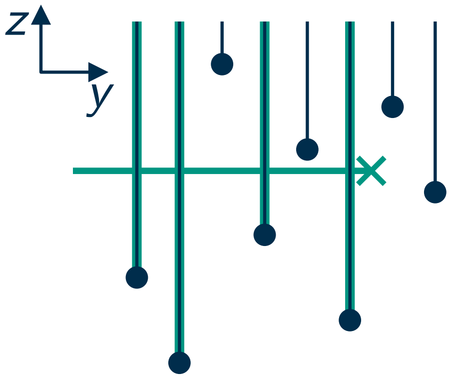

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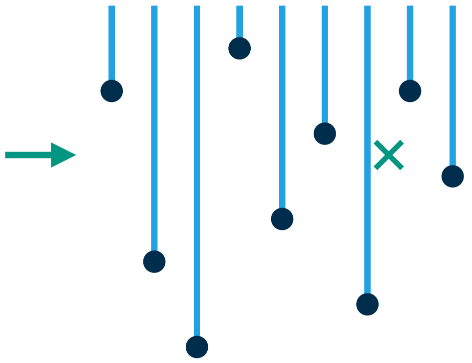

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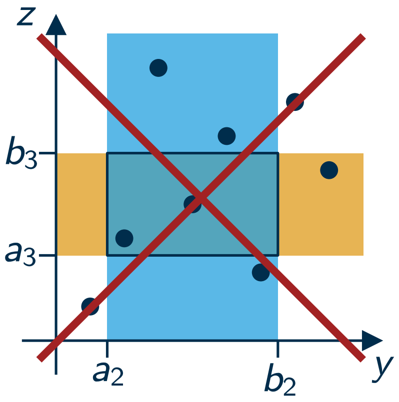

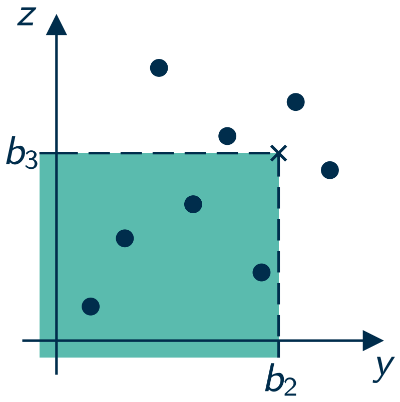

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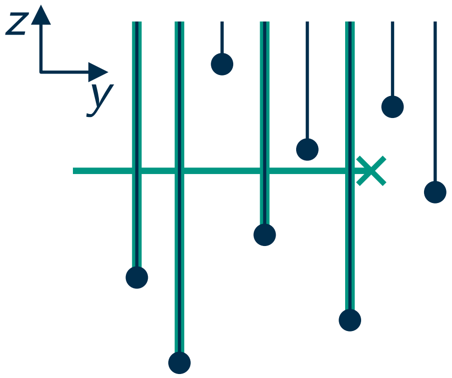

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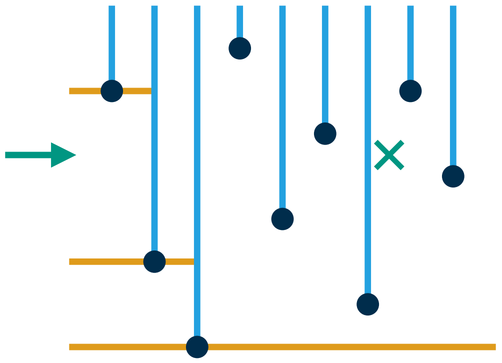

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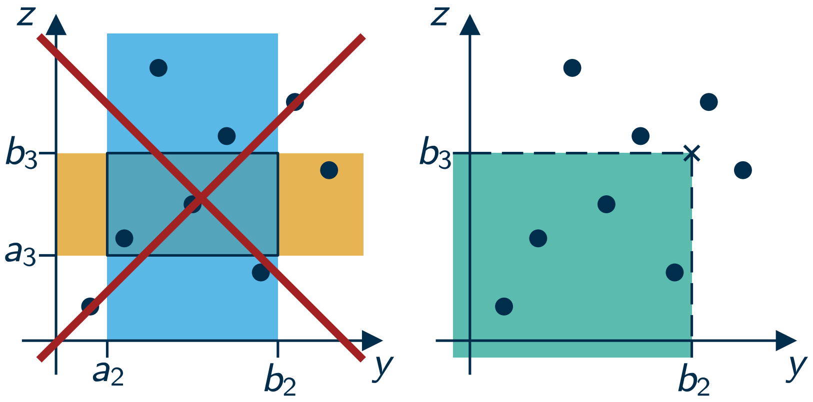

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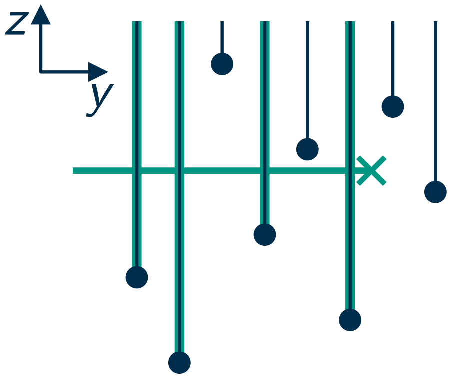

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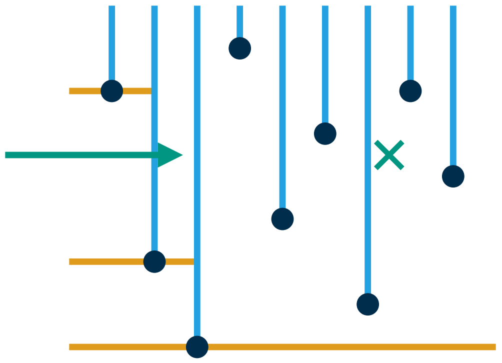

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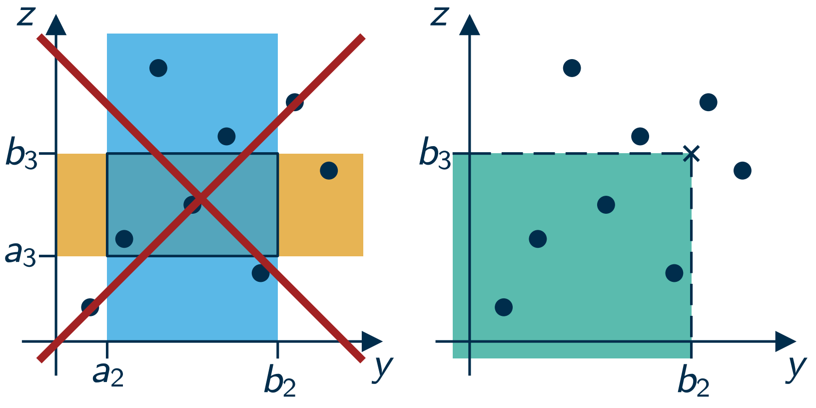

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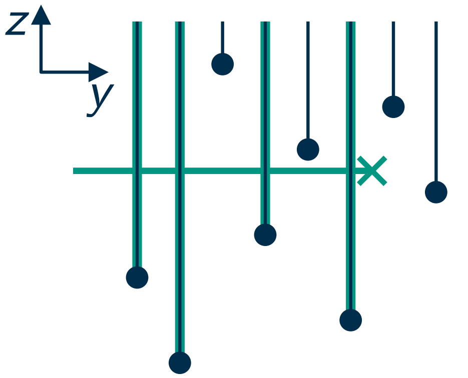

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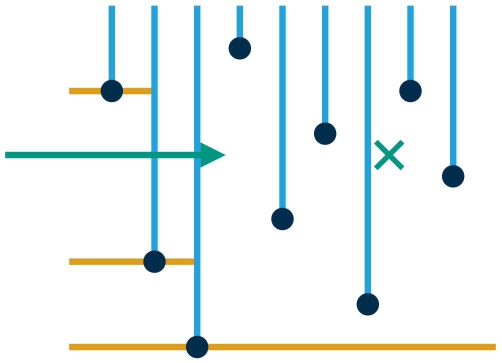

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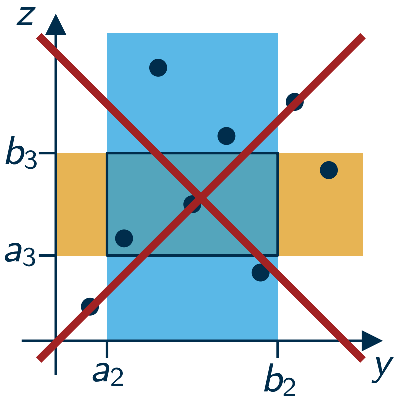

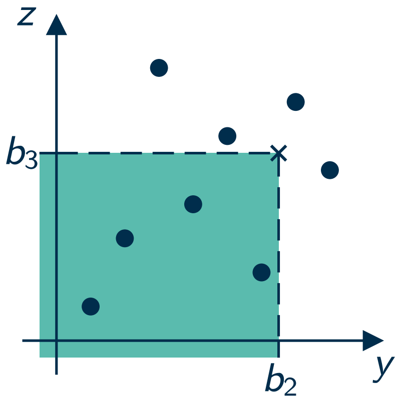

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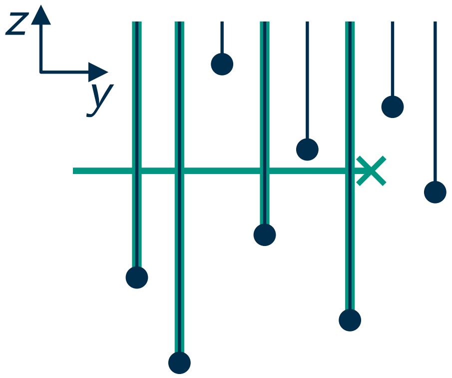

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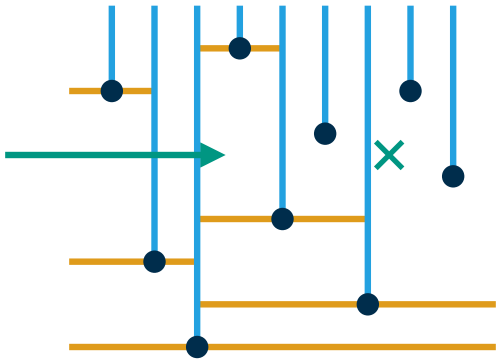

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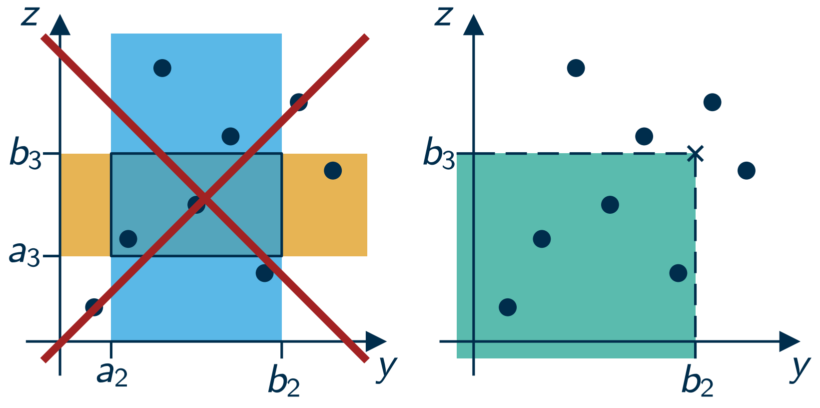

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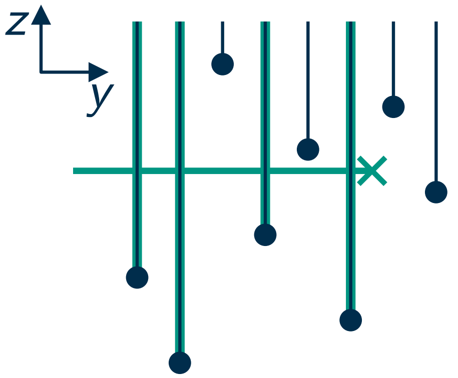

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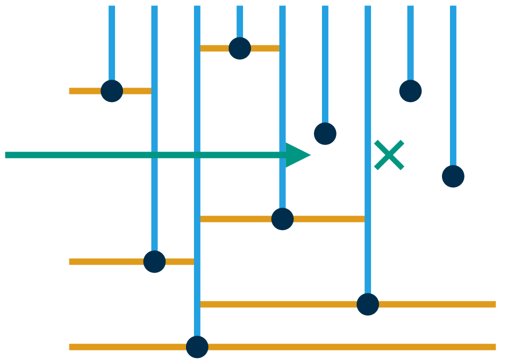

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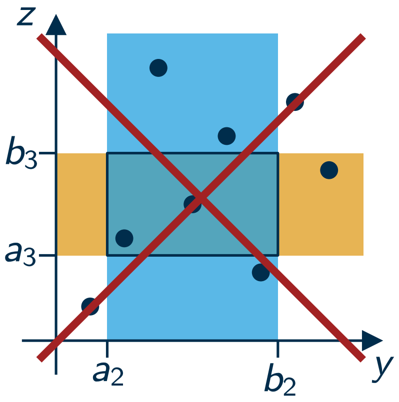

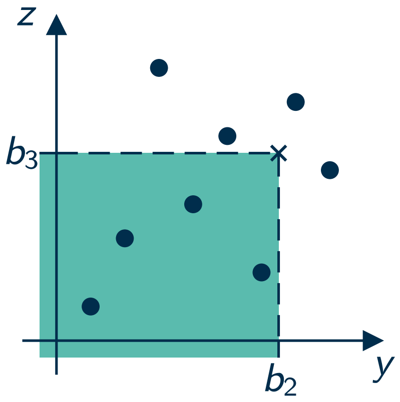

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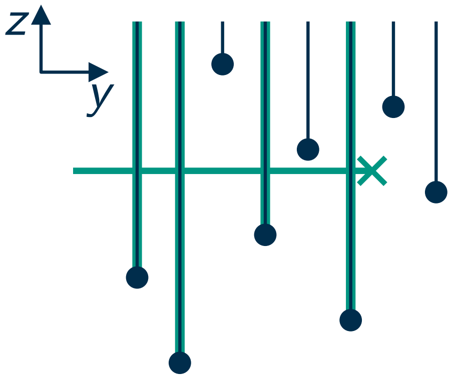

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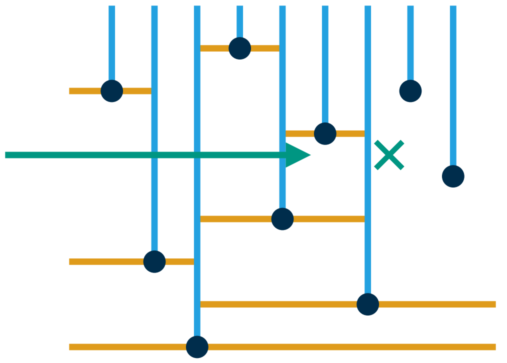

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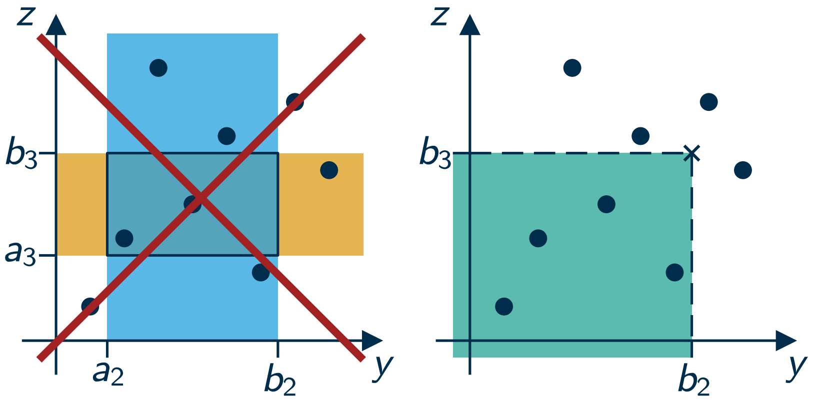

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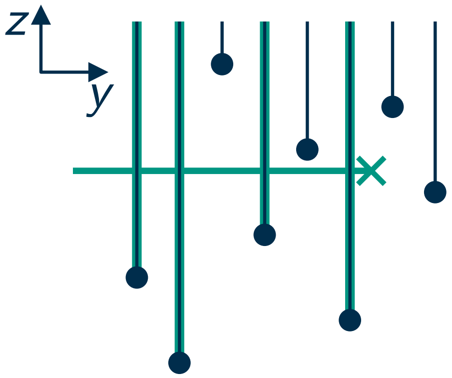

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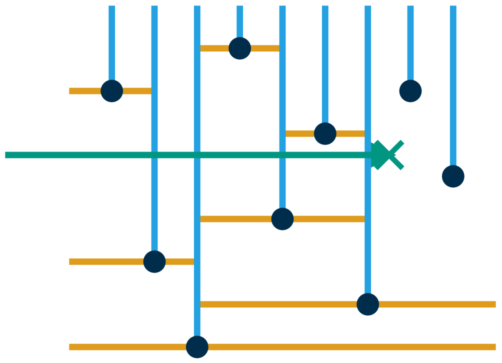

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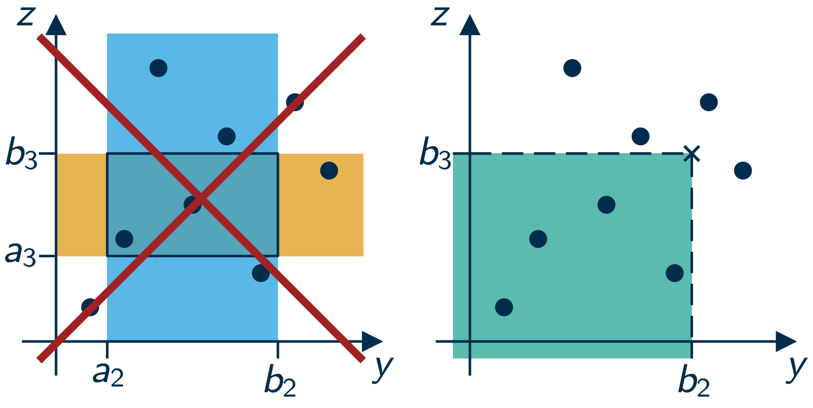

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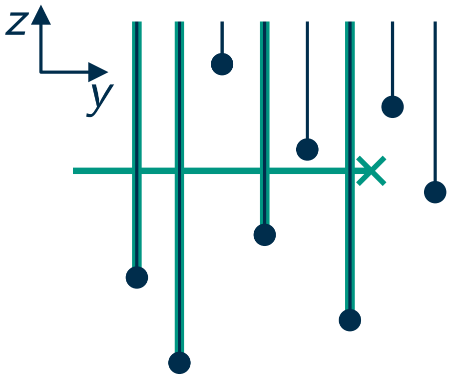

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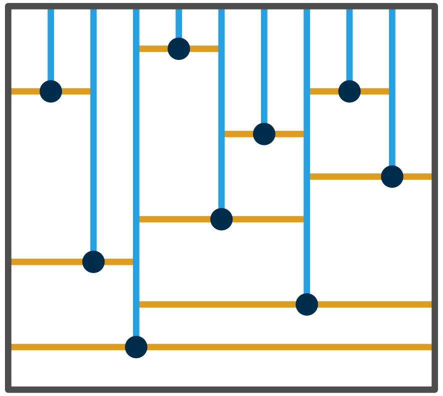

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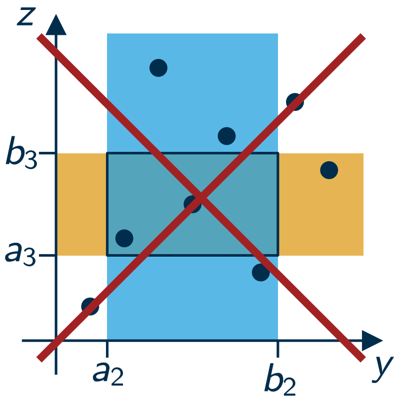

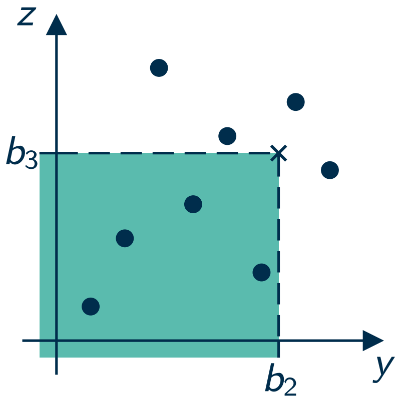

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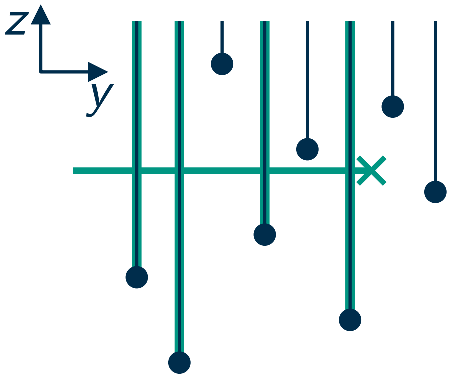

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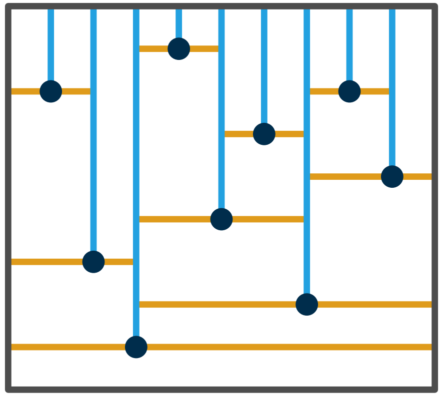

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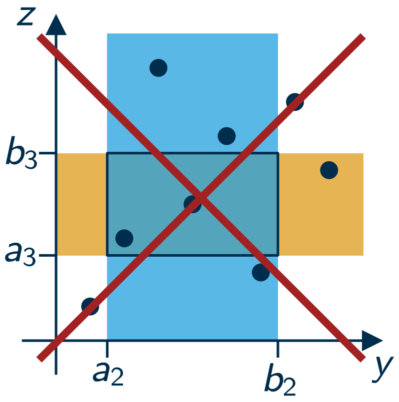

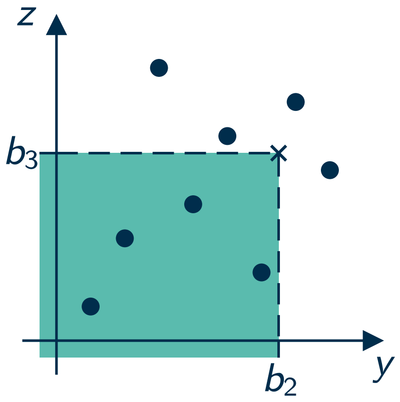

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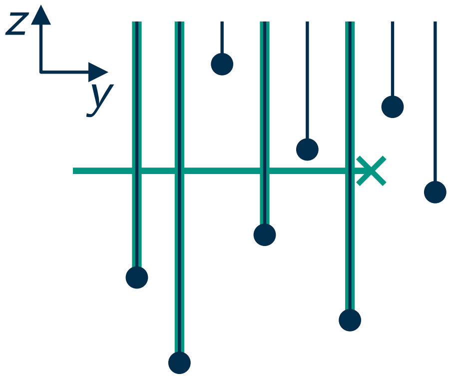

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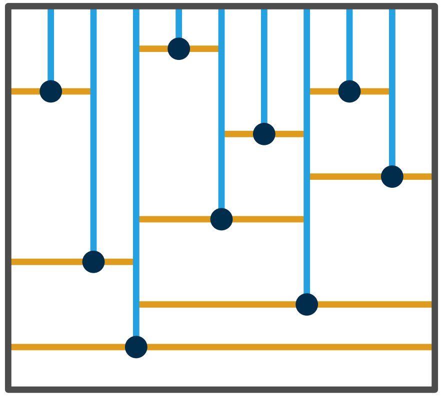

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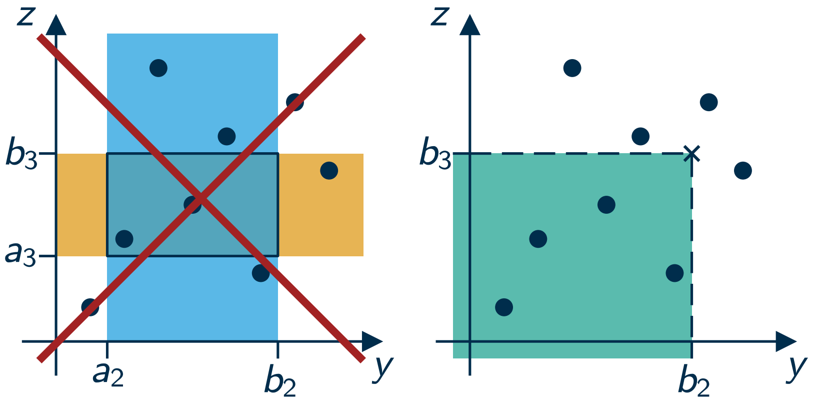

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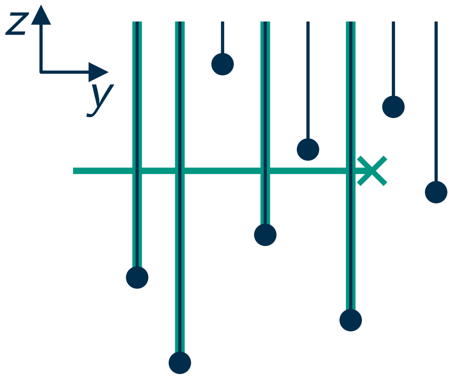

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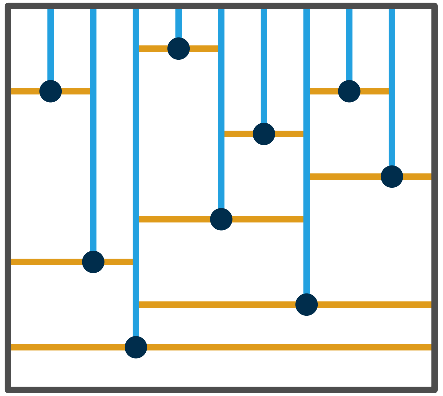

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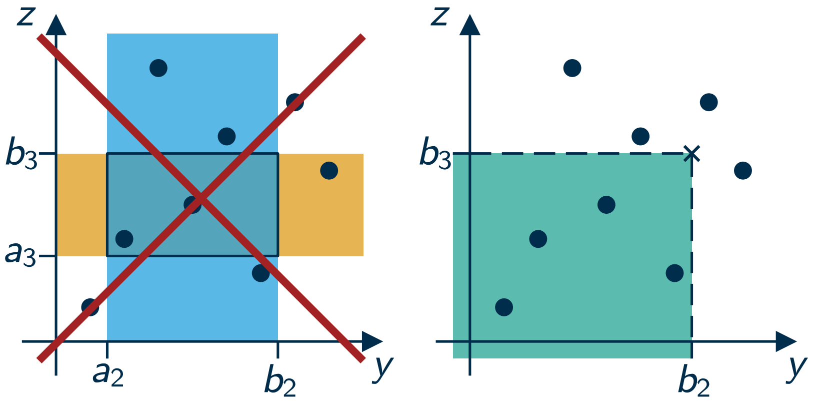

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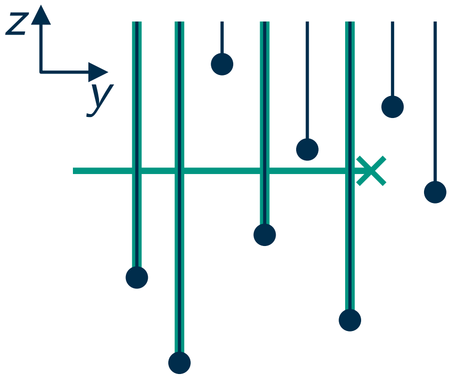

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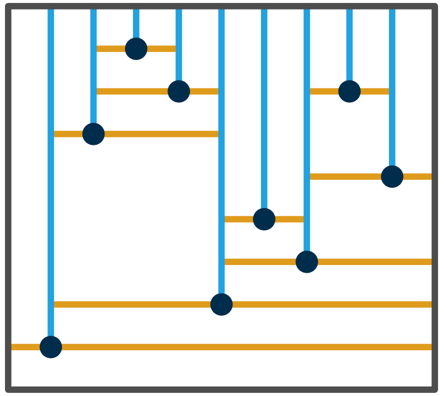

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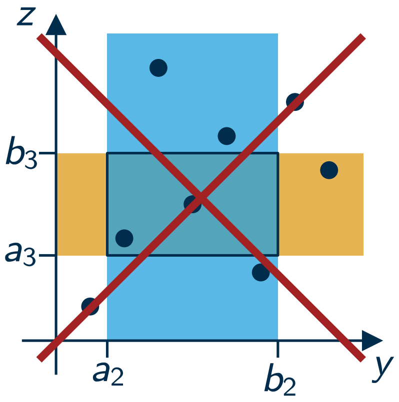

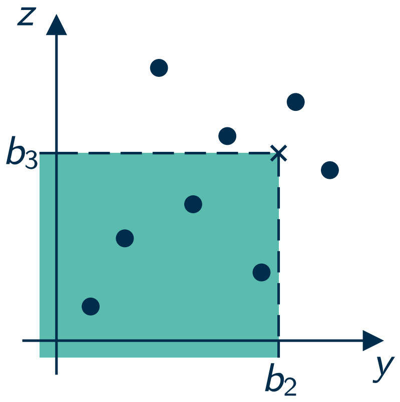

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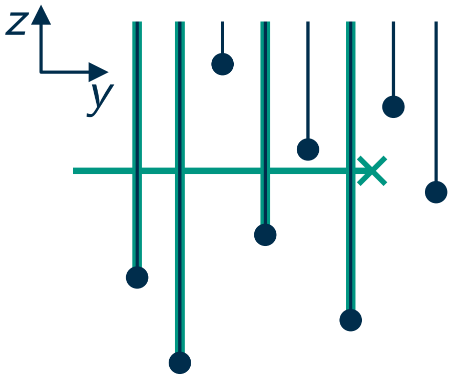

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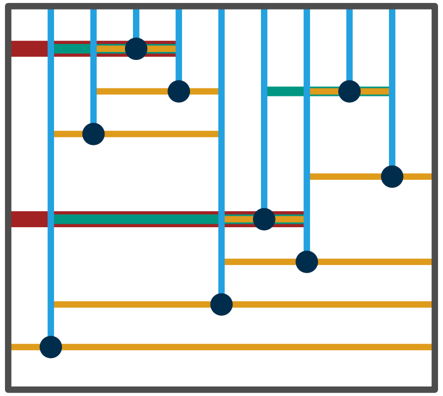

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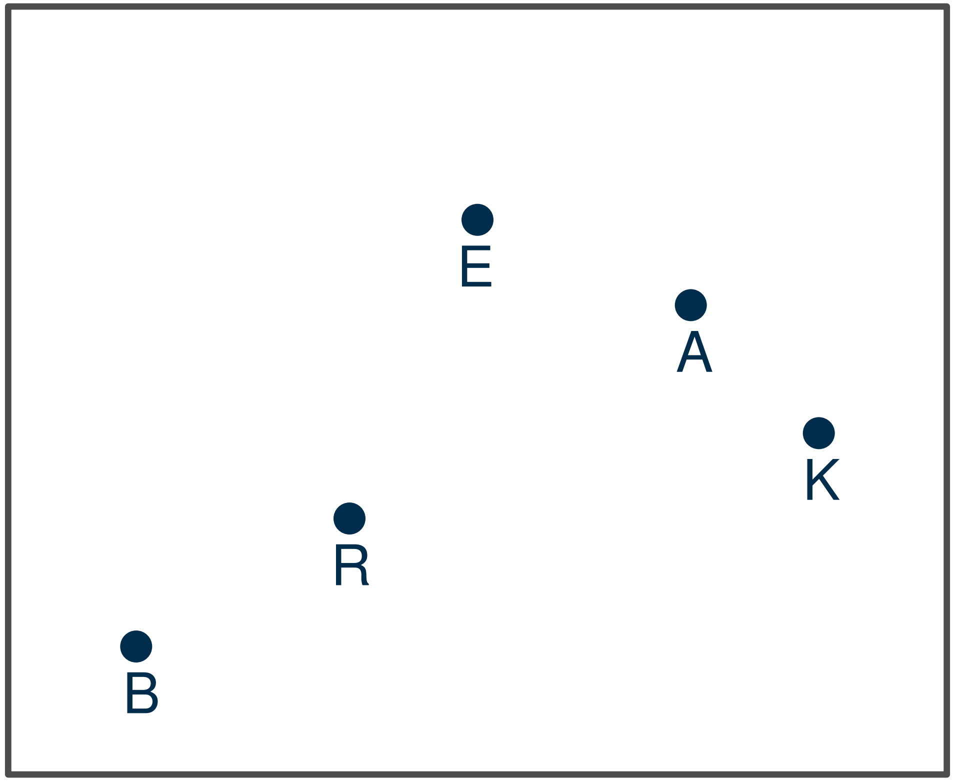

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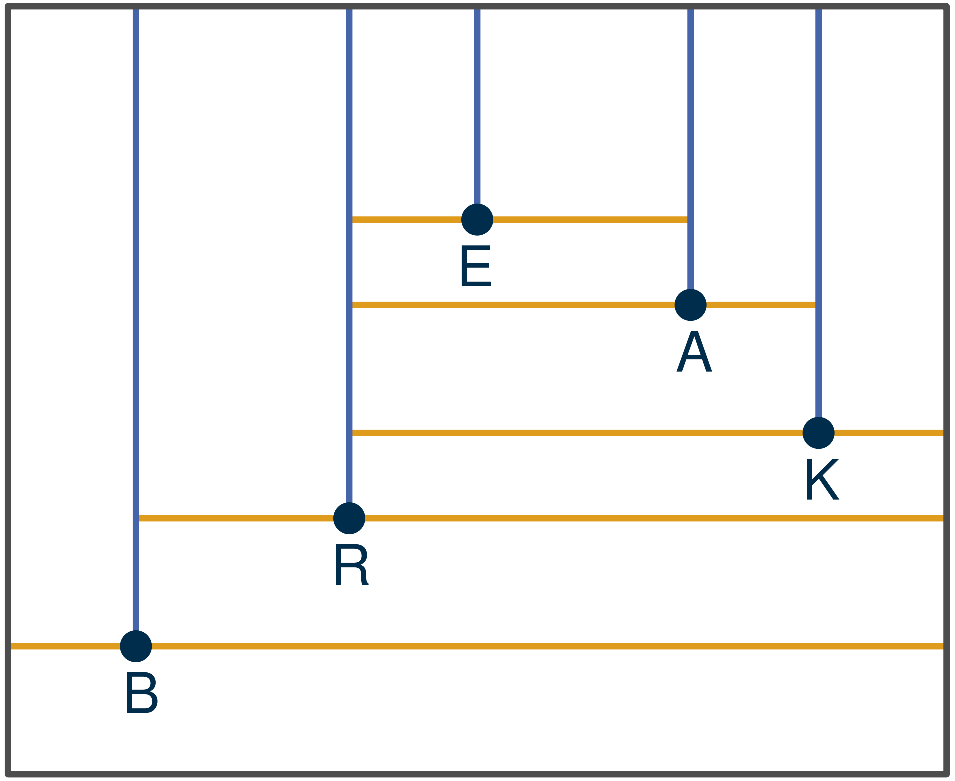

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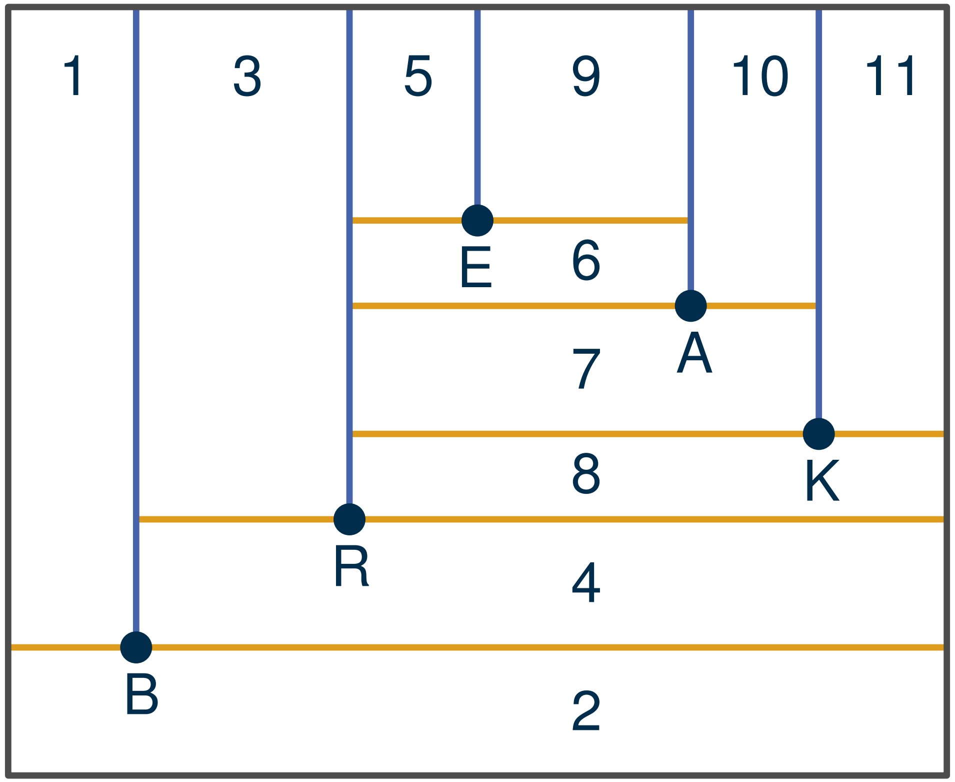

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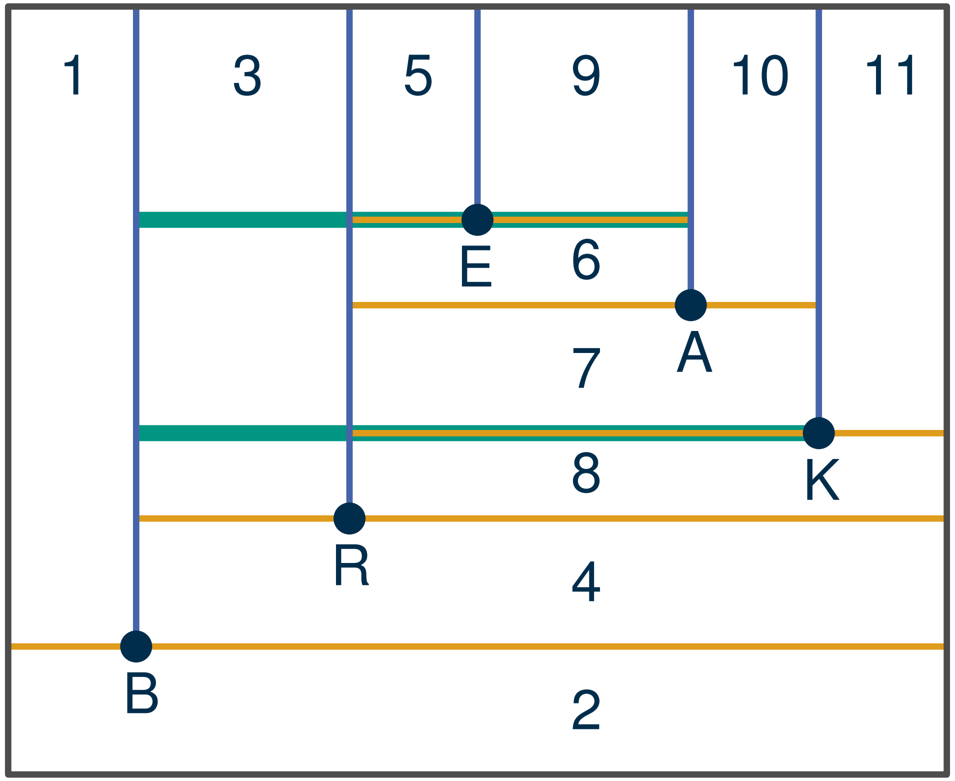

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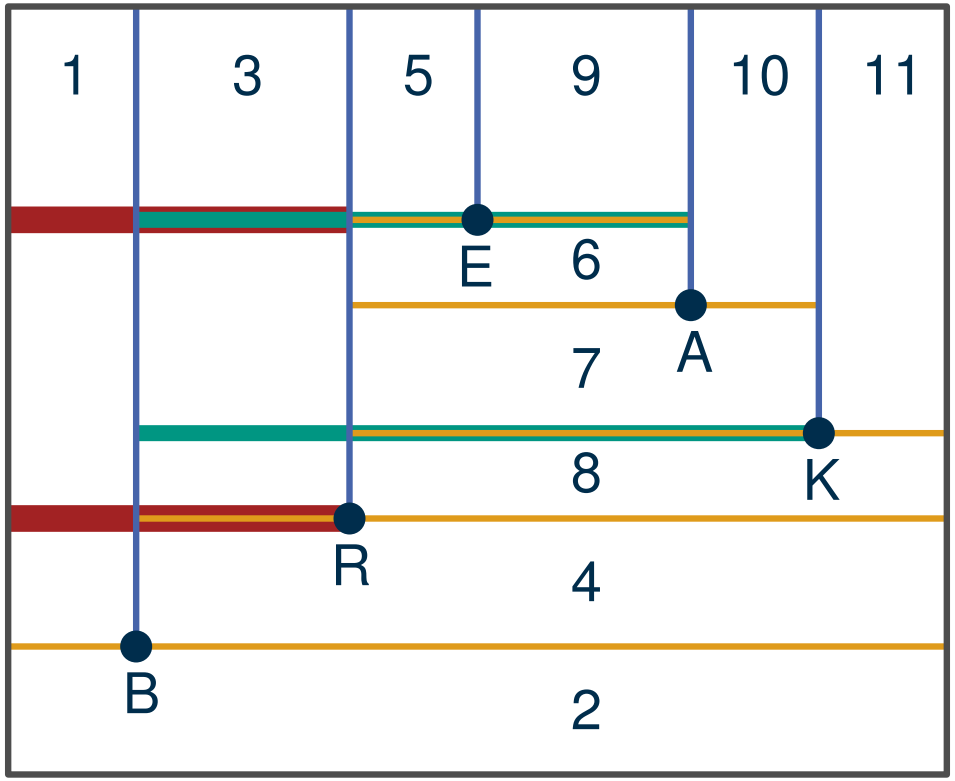

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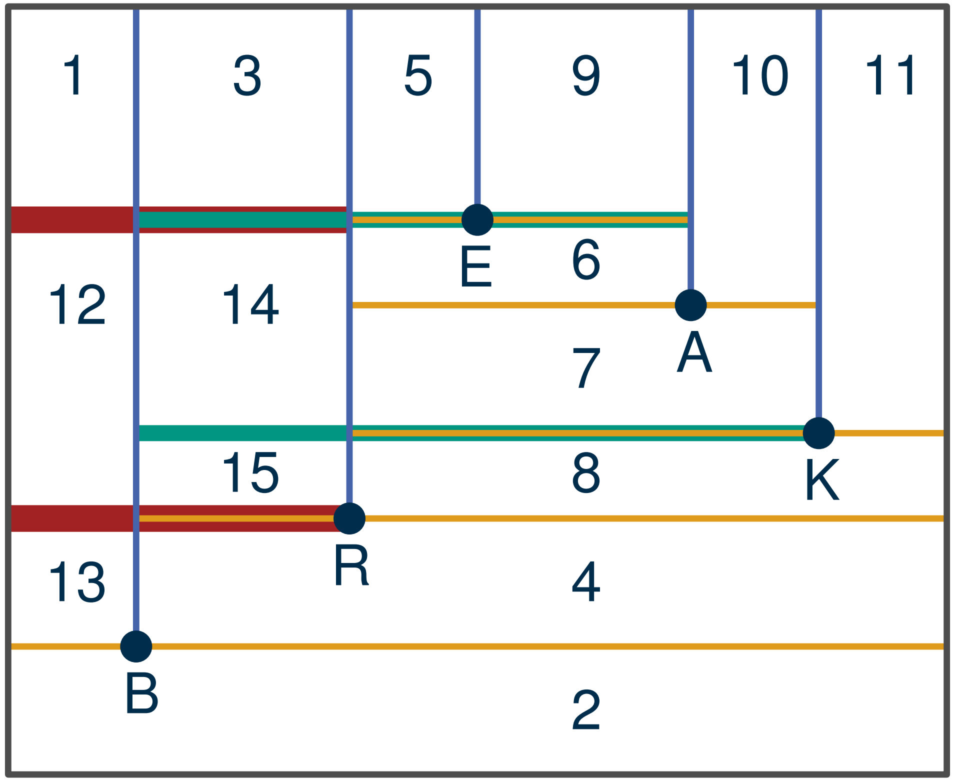

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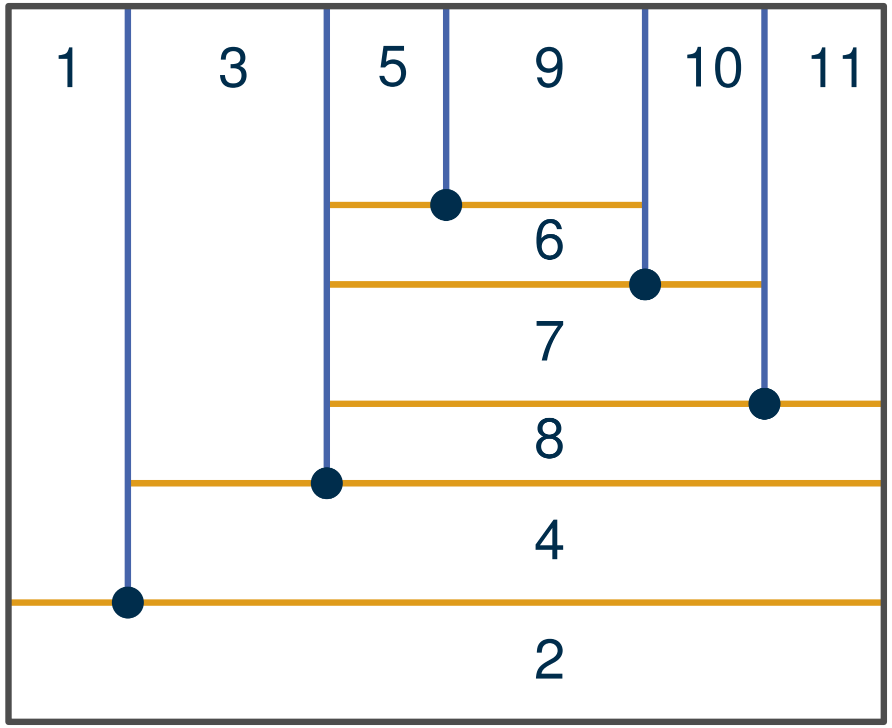
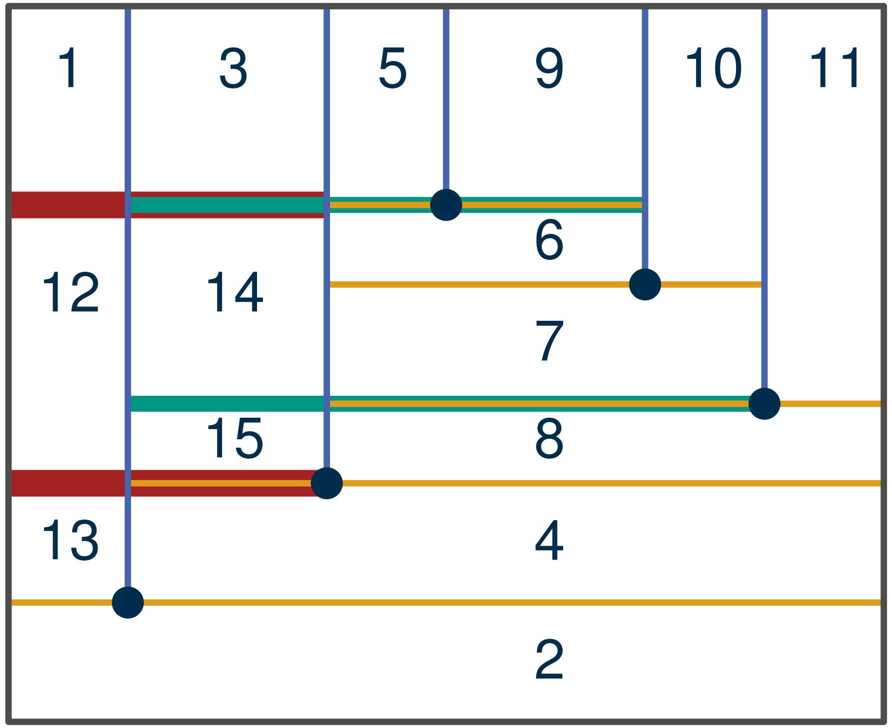
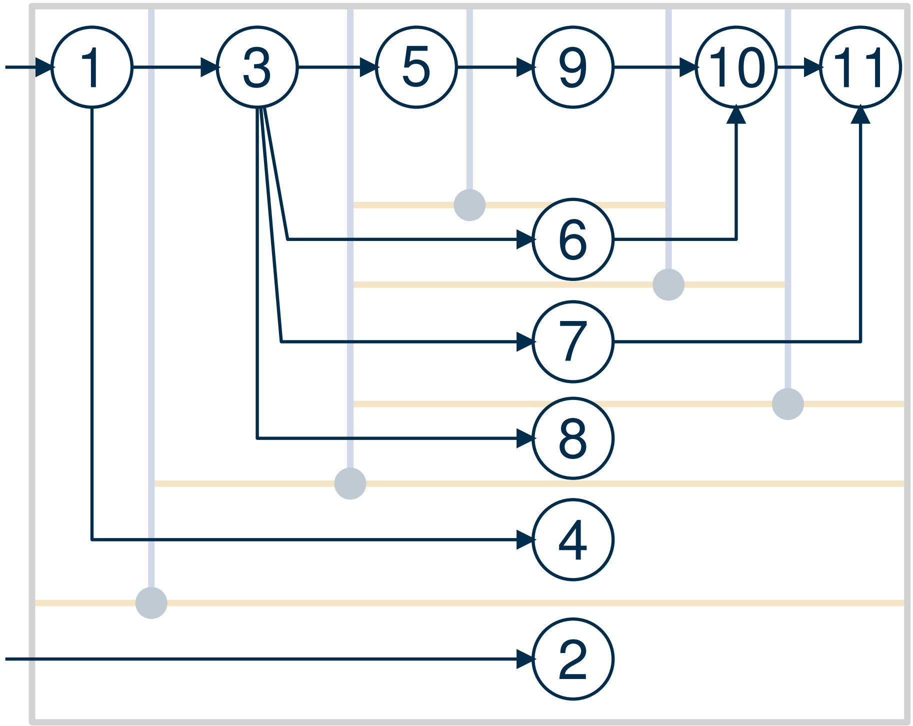


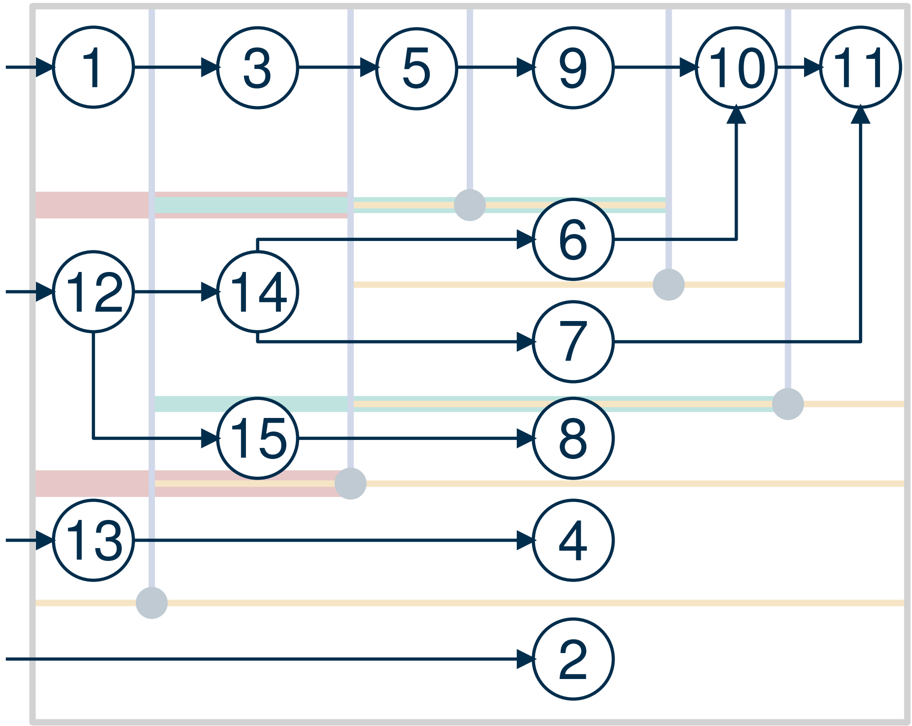


General Framework vs. Specific Situation

Useful Way Of Thinking

- mental shortcut: multiple searches for the same number $\rightarrow$ fractional cascading probably helps
- specific situation: problem-specific argument often easier than pressing it into the framework





One-Sided 3D Range Queries

one search in z direction

Seen So Far: queries $(-\infty, b_2] \times (-\infty, b_3]$ can be answered in $O(\log n + k)$ **(DS1)**

output size

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New Goal: answer queries of the form $[a_1, b_1] \times (-\infty, b_2] \times (-\infty, b_3]$

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- binary search tree for x-direction
- every node stores **(DS1)** for the corresponding points

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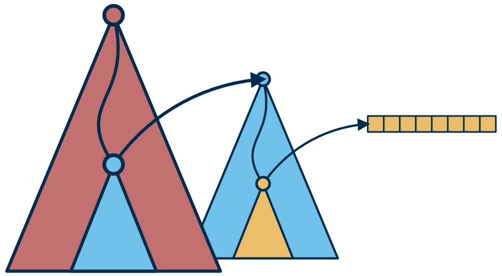
Last Lecture



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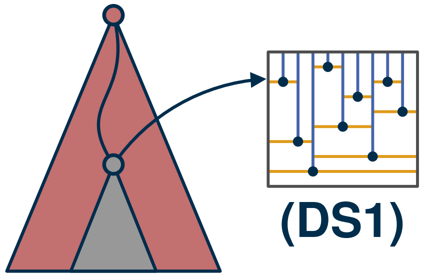

(DS1)

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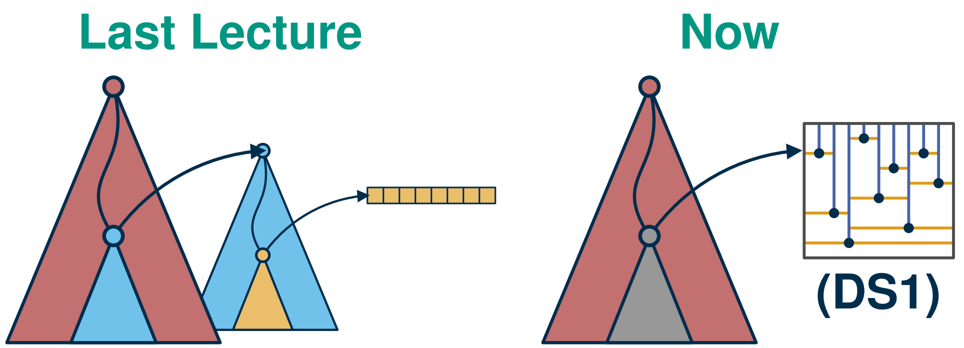

Don't We Have To Search In $O(\log n)$ Many (DS1)?

One-Sided 3D Range Queries

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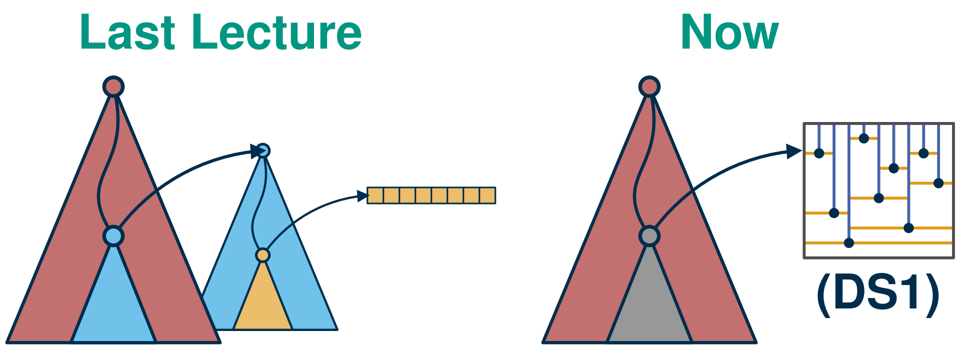

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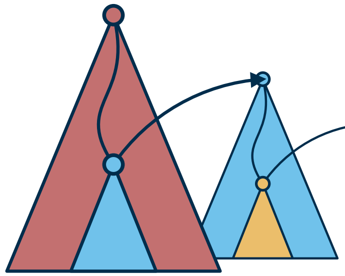

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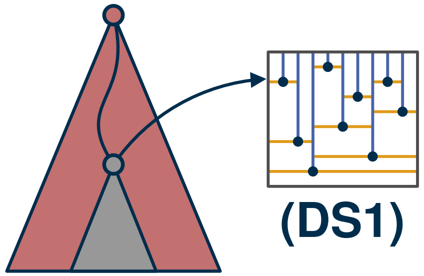

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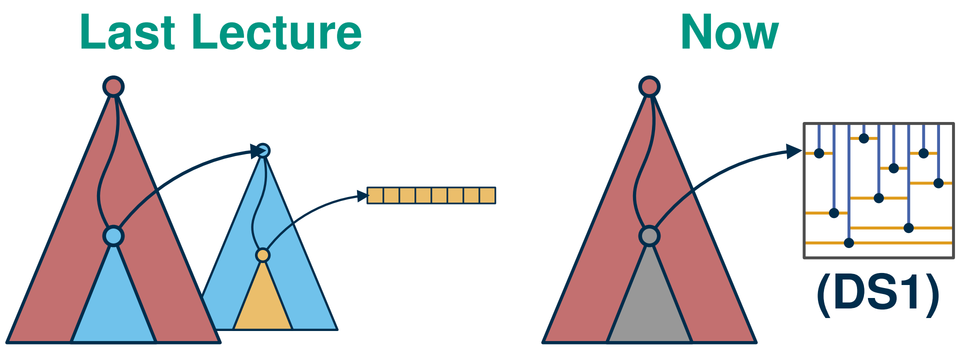

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- yes, but . . . **Fractional Cascading!**
- search once in z-direction in the root of the x-tree
- follow pointers for the z-positions while walking down the x-tree

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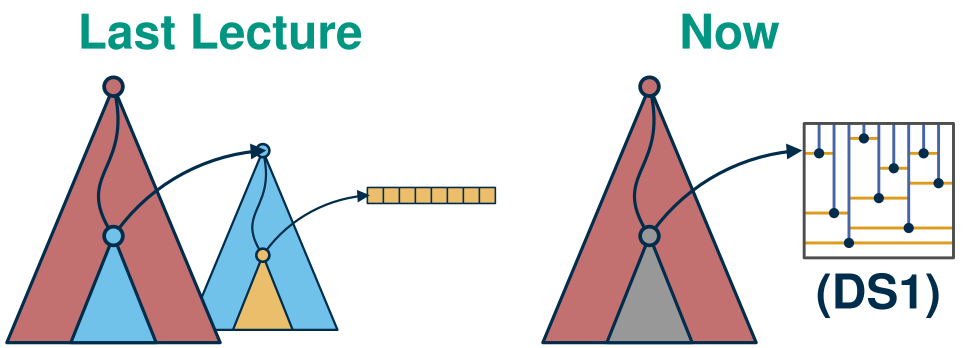

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- yes, but ... **Fractional Cascading!**
- search once in z-direction in the root of the x-tree
- follow pointers for the z-positions while walking down the x-tree
- save the first search in **(DS1)** $\Rightarrow$ total running time $O(\log n + k)$

One-Sided $\to$ Two-Sided

Lemma **(DS2)**
For n points in $\mathbb{R}^3$, we can answer queries of the form $[a_1, b_1] \times (-\infty, b_2] \times (-\infty, b_3]$ in $O(\log n + k)$ time after $O(n \log n)$ preprocessing with $O(n \log n)$ memory.

One-Sided $\rightarrow$ Two-Sided

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Plan

- use **(DS2)** as black box

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Plan

- use **(DS2)** as black box
- y-inverted variant $\rightarrow$ $[a_2, \infty)$ queries
- query $[a_2, \infty)$ and $(-\infty, b_2]$ to get $[a_2, b_2]$

One-Sided $\rightarrow$ Two-Sided

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Why can't we just use the intersection of two queries?

$$\begin{array}{ccccccc} [a_1, b_1] & \times & [a_2, b_2] & \times & [a_3, b_3] & = \\ [a_1, b_1] & \times & (-\infty, b_2] & \times & (-\infty, b_3] & \cap \\ [a_1, b_1] & \times & [a_2, \infty) & \times & [a_3, \infty) & \end{array}$$

One-Sided $\to$ Two-Sided

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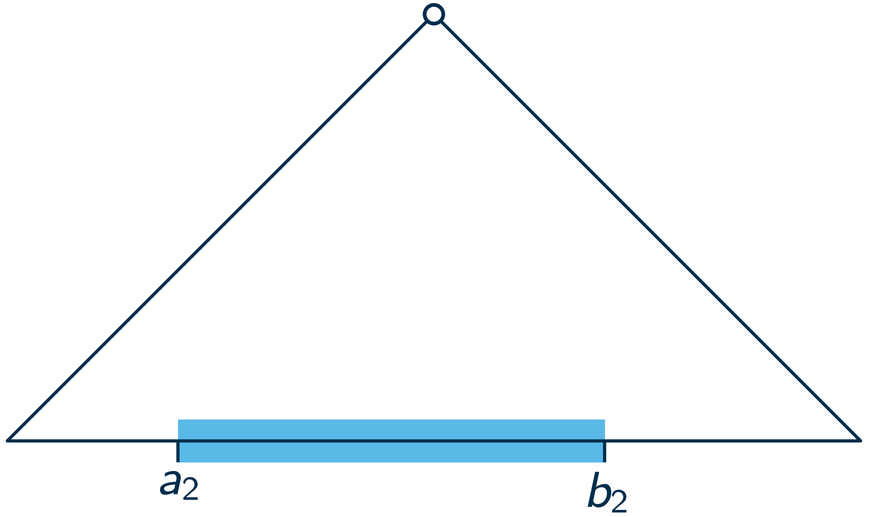
Theorem **(DS4)**
For n point in $\mathbb{R}^3$, we can answer queries of the form $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ in $O(\log n + k)$ time after $O(n \log^3 n)$ preprocessing with $O(n \log^3 n)$ memory.

Two-Sided Query In y-Direction

Simplified Perspective: Ignore x And z-Direction

- **(DS2)** allows $[a_2, \infty)$ and $(-\infty, b_2]$ queries
- goal: build data structure, that allows $[a_2, b_2]$ queries

Two-Sided Query In y-Direction



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Binary Search Tree In y-Direction

Two-Sided Query In y-Direction

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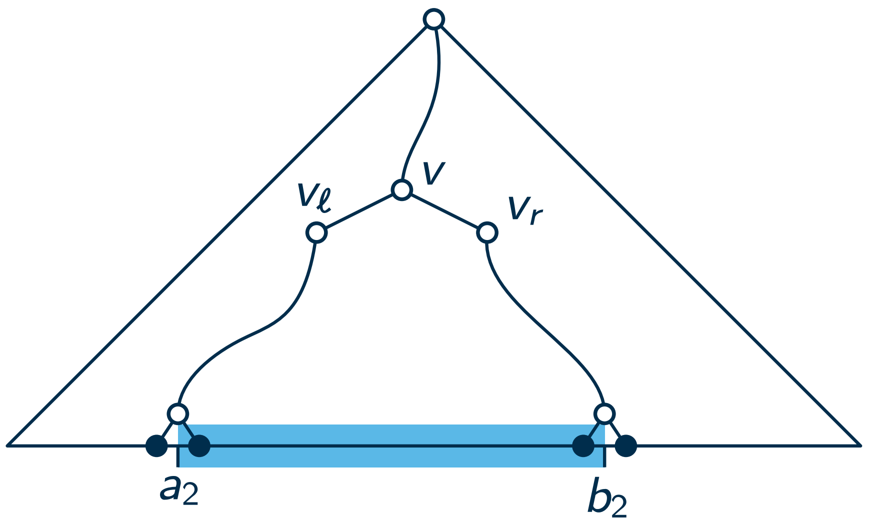
Binary Search Tree In y-Direction

- search for a_2 and b_2 splits at v to v_ℓ and v_r



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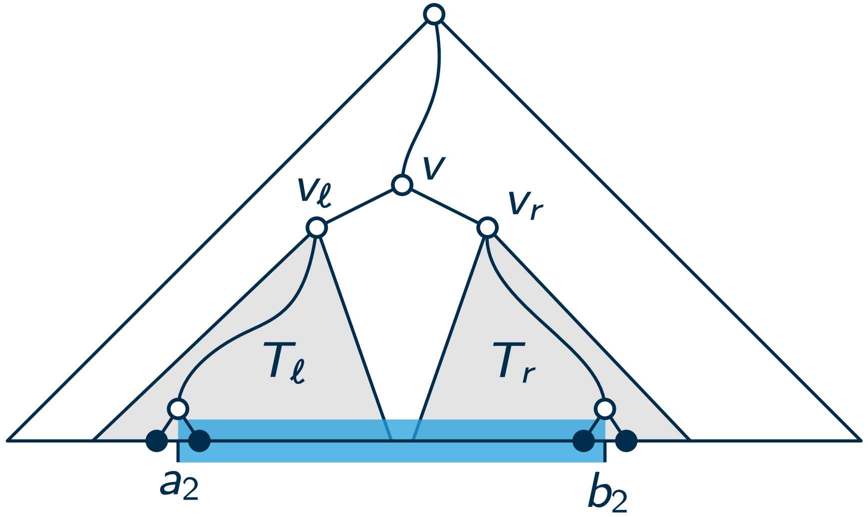

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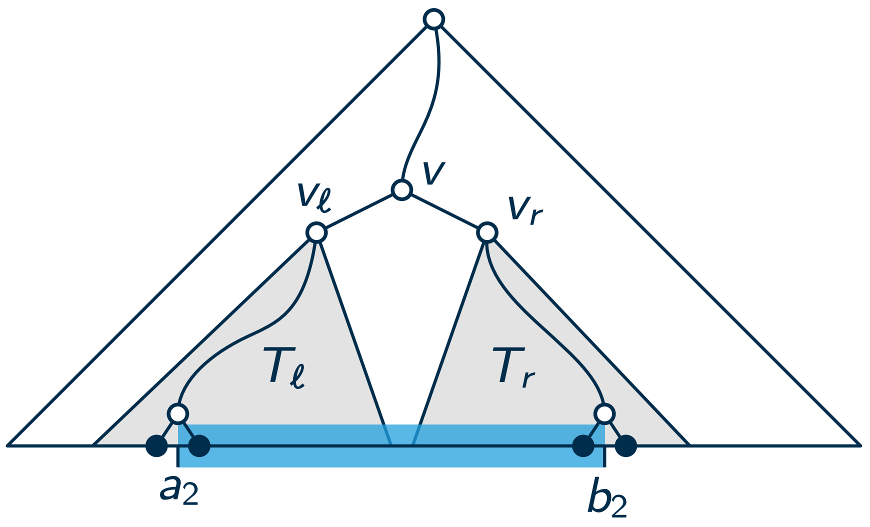
Binary Search Tree In y-Direction

- search for a_2 and b_2 splits at v to v_ℓ and v_r
- queries in instances of **(DS2)**: $[a_2, \infty)$ on points in T_ℓ and $(-\infty, b_2]$ on points in T_r

Two-Sided Query In y-Direction



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- running time: $O(\log n)$ for search in y-tree plus $O(\log n + k)$ for two queries in **(DS2)**

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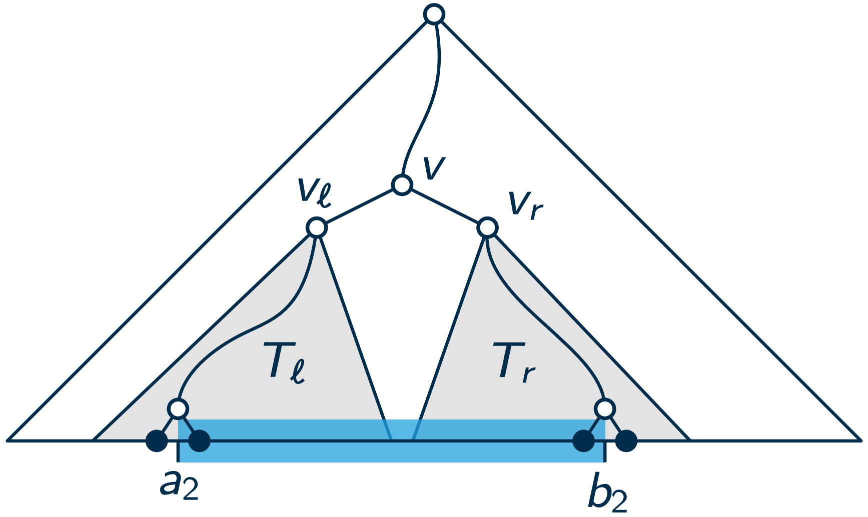

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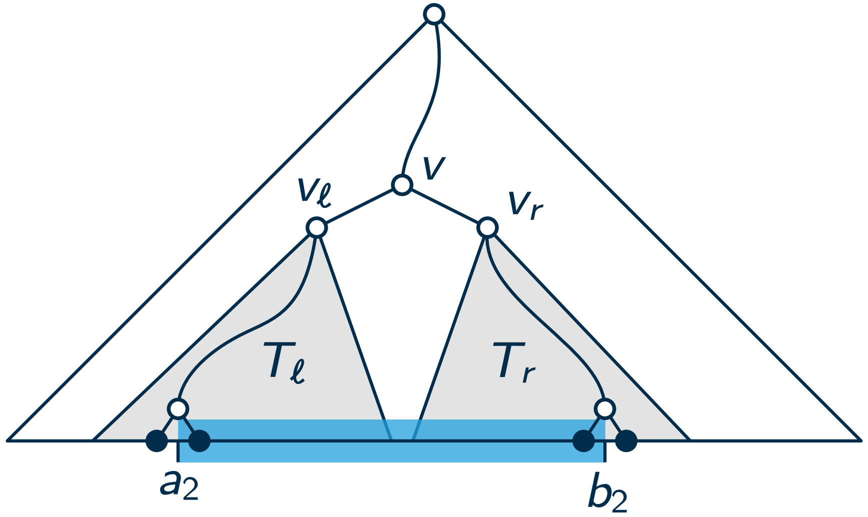

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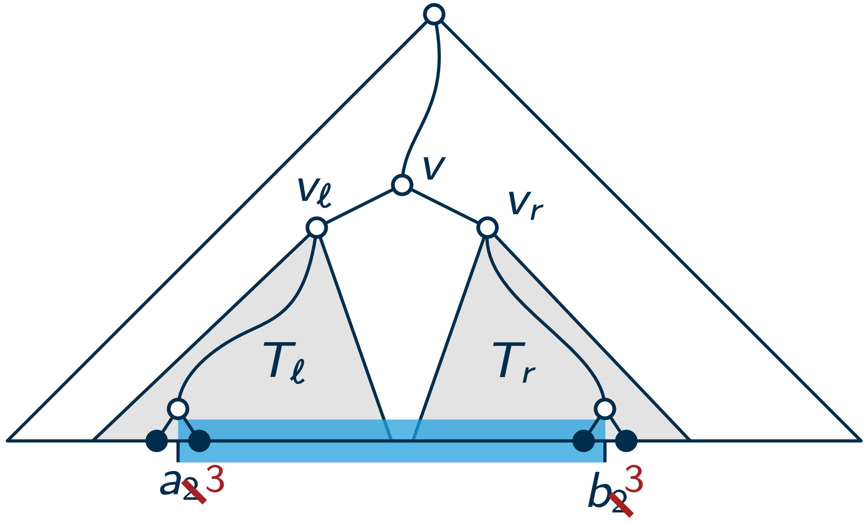
Lemma **(DS3)**
For n point in $\mathbb{R}^3$, we can answer queries of the form $[a_1, b_1] \times [a_2, b_2] \times (-\infty, b_3]$ in $O(\log n + k)$ time after $O(n \log^2 n)$ preprocessing with $O(n \log^2 n)$ memory.

Two-Sided Query In $\cancel{y}^{z}$-Direction



Simplified Perspective: Ignore x And $\cancel{z}^{y}$-Direction

- **(DS$\cancel{2}$3)** allows $[a_{\cancel{2}}^{3}, \infty)$ and $(-\infty, b_{\cancel{2}}^{3}]$ queries
- goal: build data structure, that allows $[a_{\cancel{2}}^{3}, b_{\cancel{2}}^{3}]$ queries

Binary Search Tree In $\cancel{y}^{z}$-Direction

- search for $a_{\cancel{2}}^{3}$ and $b_{\cancel{2}}^{3}$ splits at v to v_ℓ and v_r
- queries in instances of **(DS$\cancel{2}$3)**: $[a_{\cancel{2}}^{3}, \infty)$ on points in T_ℓ and $(-\infty, b_{\cancel{2}}^{3}]$ on points in T_r
- running time: $O(\log n)$ for search in $\cancel{y}^{z}$-tree plus $O(\log n + k)$ for two queries in **(DS$\cancel{2}$3)**
- memory: $O(\log n) \cdot$ (memory for **DS$\cancel{2}$3**)

~~Lemma~~ Theorem **(DS$\cancel{3}$4)**

For n point in $\mathbb{R}^3$, we can answer queries of the form $[a_1, b_1] \times [a_2, b_2] \times (\cancel{-\infty}^{a_3}, b_3]$ in $O(\log n + k)$ time after $O(n \log^{\cancel{2}3} n)$ preprocessing with $O(n \log^{\cancel{2}3} n)$ memory.

The Big Picture

(DS4)
$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$
precomp: $O(n \log^3 n)$
memory: $O(n \log^3 n)$
query: $O(\log n + k)$

The Big Picture

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$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

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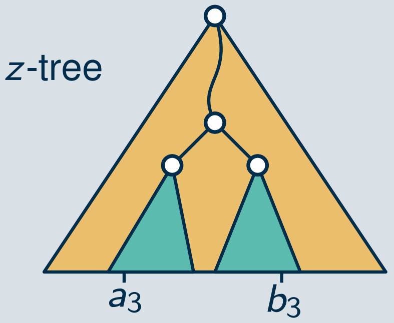
- search for a_3, b_3 in z-tree
- split $\rightarrow$ two **(DS3)** queries



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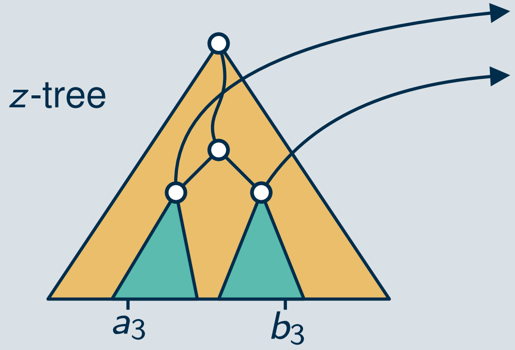


(DS3)
$[a_1, b_1] \times [a_2, b_2] \times (-\infty, b_3]$
precomp: $O(n \log^2 n)$
memory: $O(n \log^2 n)$
query: $O(\log n + k)$

The Big Picture

(DS4)

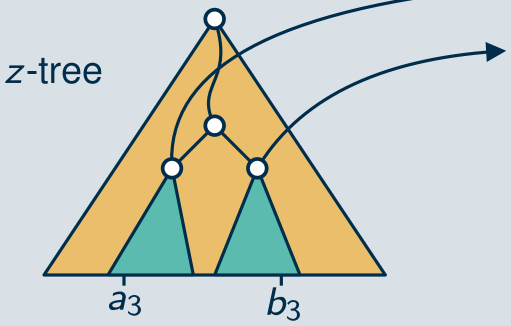
$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

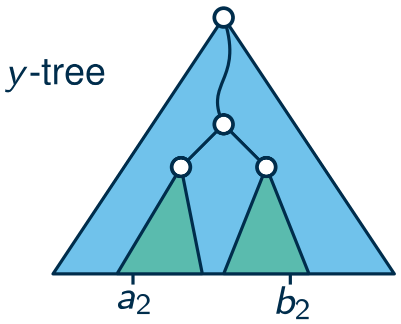
precomp: $O(n \log^3 n)$

memory: $O(n \log^3 n)$

query: $O(\log n + k)$

- search for a_3, b_3 in z-tree
- split $\rightarrow$ two **(DS3)** queries





(DS3)

$[a_1, b_1] \times [a_2, b_2] \times (-\infty, b_3]$

precomp: $O(n \log^2 n)$

memory: $O(n \log^2 n)$

query: $O(\log n + k)$

- search for a_2, b_2 in y-tree
- splits $\rightarrow$ two **(DS2)** queries

The Big Picture

(DS4)

$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

precomp: $O(n \log^3 n)$

memory: $O(n \log^3 n)$

query: $O(\log n + k)$

- search for a_3, b_3 in z-tree
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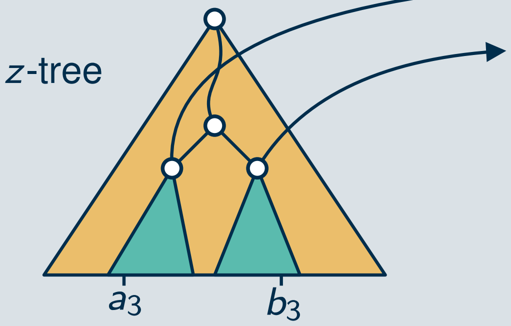

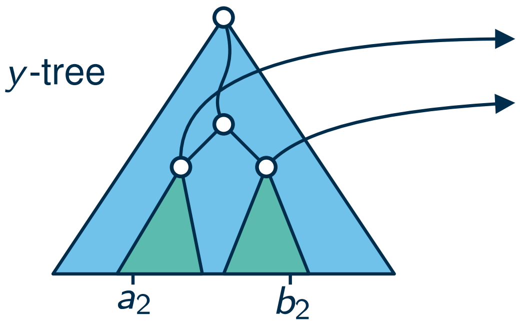

(DS2)

$[a_1, b_1] \times (-\infty, b_2] \times (-\infty, b_3]$

precomp: $O(n \log n)$

memory: $O(n \log n)$

query: $O(\log n + k)$

(DS3)

$[a_1, b_1] \times [a_2, b_2] \times (-\infty, b_3]$

precomp: $O(n \log^2 n)$

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- search for a_2, b_2 in y-tree
- splits $\rightarrow$ two **(DS2)** queries

The Big Picture

(DS4)

$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

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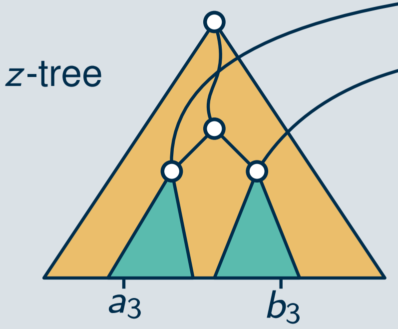

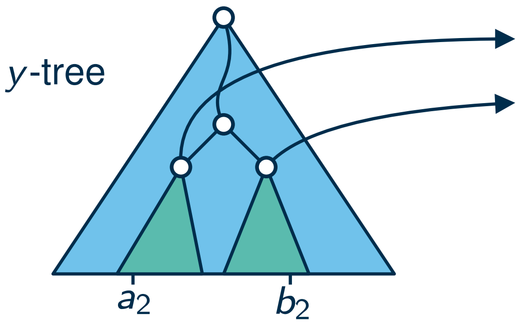

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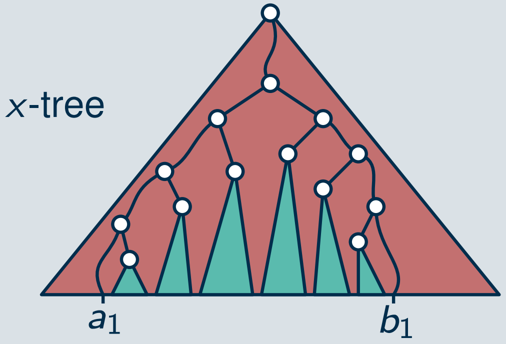

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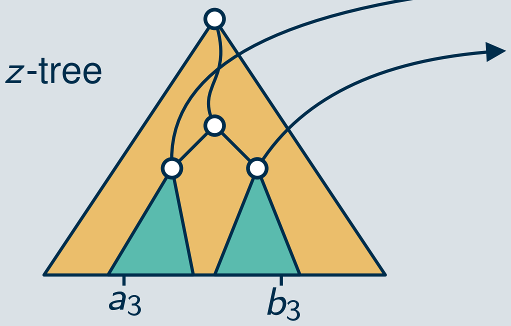

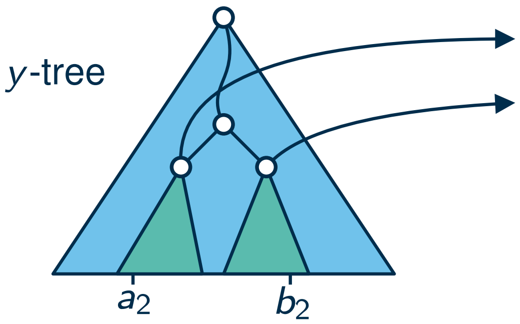

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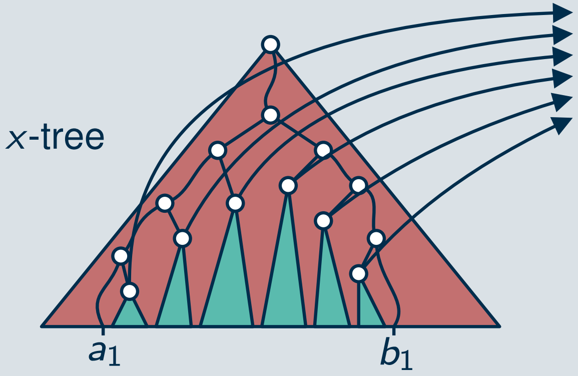

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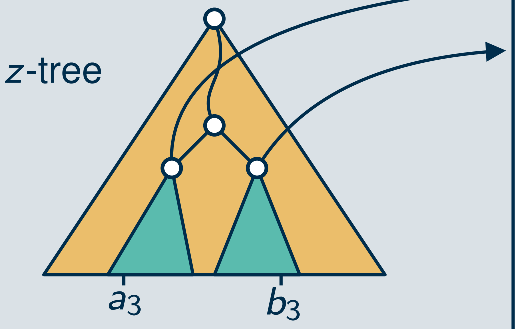

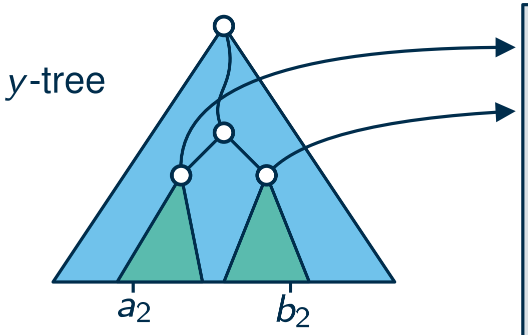

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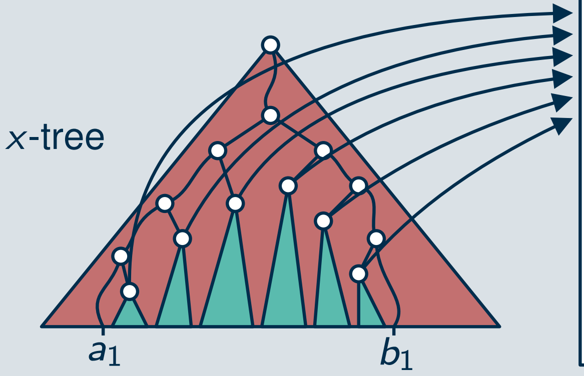

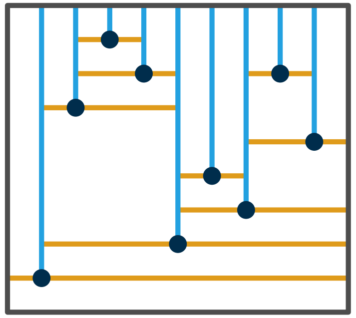

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$(-\infty, b_2] \times (-\infty, b_3]$

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- initial z-position known
- walk in y-direction
- decide for cell by z-query
- output intersected rays

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What Have We Learned Today?

- fractional cascading: search only once and then follow pointers

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For n point in $\mathbb{R}^3$, we can answer queries of the form $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ in $O(\log n + k)$ time after $O(n \log^3 n)$ preprocessing with $O(n \log^3 n)$ memory.

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- even better results with some bit-hacking in the word RAM model