Computational Geometry

Orthogonal Range Queries: Fractional Cascading

Thomas Bläsius

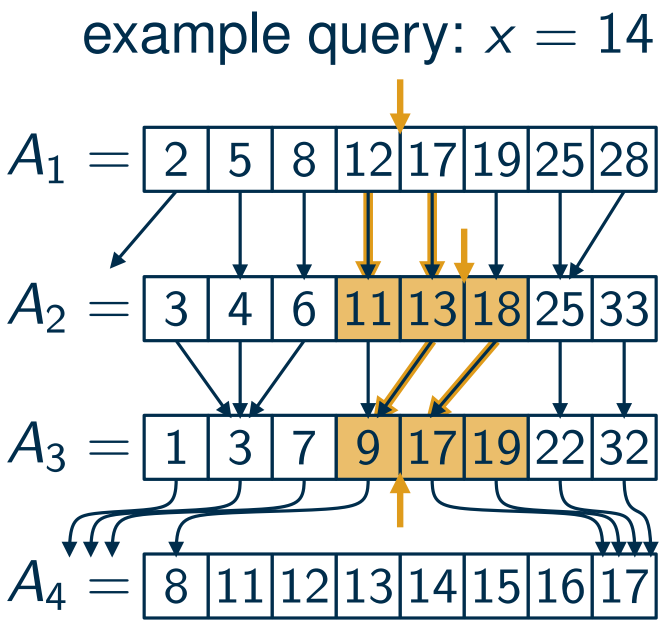
Searching In Many Arrays

Situation

- consider ℓ sorted arrays $A_1, \dots, A_\ell$ with $\leq n$ elements each
- find the position of x in all arrays
- obvious solution: $O(\ell \log n)$
- last lecture: $O(\ell + \log n)$ if $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_\ell$

Is $\ell + \log n$ Possible In General?

- hope: search x in A_1, find x in $A_2, \dots, A_\ell$ via pointers
- problem: position of x in A_i may not help to find position in A_{i+1}



Observation

- $A_i \supseteq A_{i+1} \Rightarrow$ position in A_i determines position in A_{i+1}
- A_i contains many elements from $A_{i+1} \Rightarrow$ position in A_i roughly determines position in A_{i+1}
- idea: insert some elements from A_{i+1} into A_i

Fractional Cascading

Shared Elements

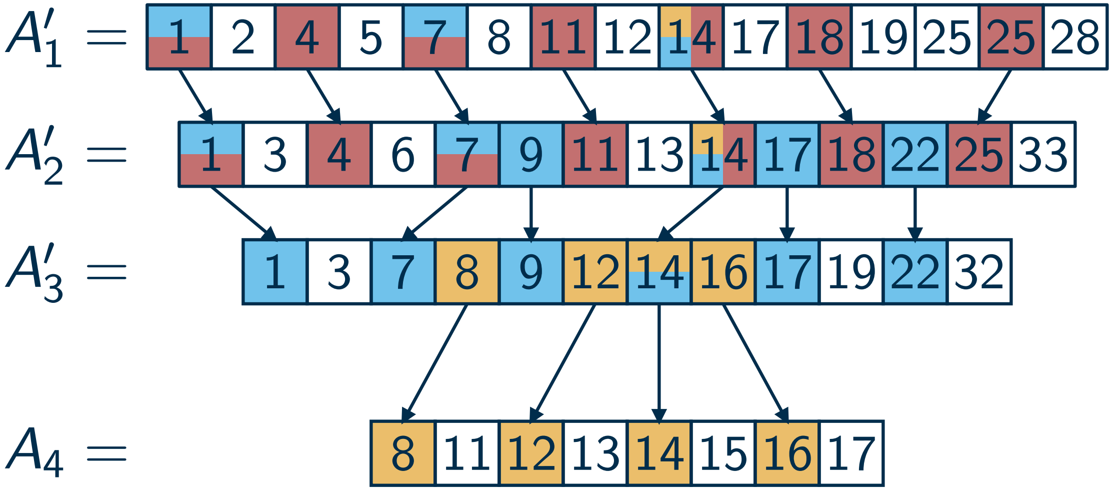
- new array A'_3: insert every other element from A_4 into A_3
- store pointers to copies
- pointers from $A'_3 \setminus A_4$ to prev / next element from A_4 $\Rightarrow$ position in A'_3 gives position in A_4 (± 1)
- pointers from elements in A_4 to prev / next in $A'_3 \setminus A_4 \Rightarrow$ position in A'_3 gives position in A_3
- cascade the process for all previous A_i

$A_1 = [2, 5, 8, 12, 17, 19, 25, 28]$

$A_2 = [3, 4, 6, 11, 13, 18, 25, 33]$

$A_3 = [1, 3, 7, 9, 17, 19, 22, 32]$

$A_4 = [8, 11, 12, 13, 14, 15, 16, 17]$



Fractional Cascading – Running Time

Cost For The Search

- one search in $A'_1 \to O(\log(|A'_1|))$
- $O(1)$ for every subsequent array $\to O(\ell)$

$\}$ total: $O(\ell + \log(|A'_1|))$

How Large is A'_1?

(assumption: $|A_i| = n$ for all i)

- $|A'_{\ell-1}| = (\frac{1}{2} + 1)n$
- $|A'_{\ell-2}| = (\frac{1}{4} + \frac{1}{2} + 1)n$
- $|A'_{\ell-3}| = (\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1)n$
- $|A'_1| \leq 2n$

$\Rightarrow$ search takes $O(\ell + \log n)$ time

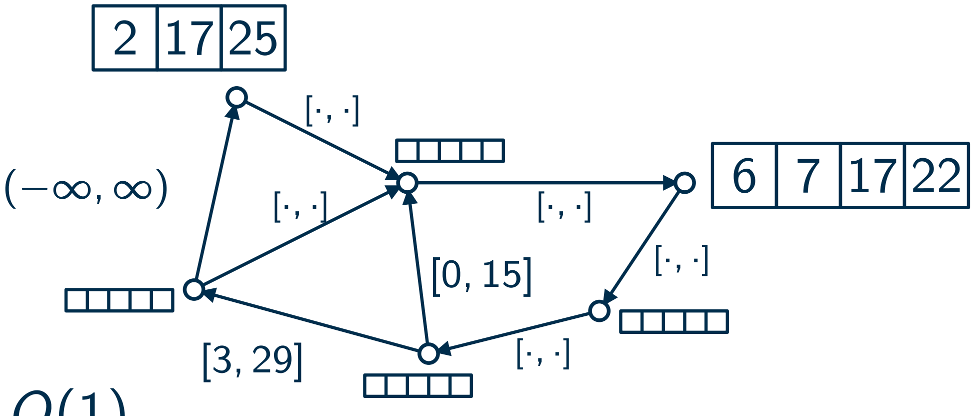
Memory Consumption

- only a constant factor overhead
- also true if not all arrays have the same size

Precomputation Time

- linear in the input

General Fractional Cascading



Now With A Directed Graph $G = (V, E)$

- sorted array A_v for every vertex v
- an interval I_e for every edge e
- for every number x and $u \in V$: $|\{uv \in E \mid x \in I_{uv}\}| \in O(1)$

How is this a generalization?

A Game Between Alice And Bob

- precomputes a data structure
- answers the question
- answers the question

- choose a number x and $u \in V$
- asks where x lies in A_u
- choose edge uv with $x \in I_{uv}$
- asks where x lies in A_v

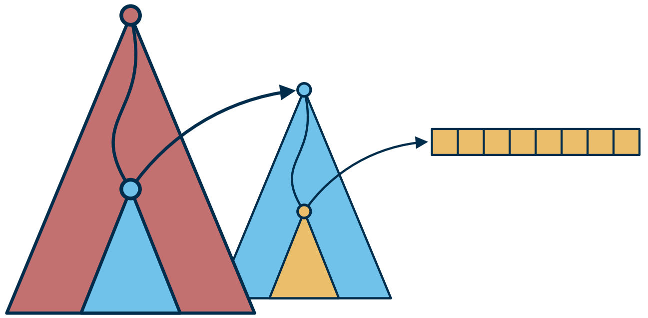
Similar Guarantee To The Path Setting (without proof)

(s = total array size)

- precomputation: $O(s)$ time and $O(s)$ space
- query: $O(\log s)$ for the first, then $O(1)$

Back To The Range Queries

query: $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$
direction: x y z



Query In 3D Range Tree (Simple Variant)

- walk down the x-tree $\rightarrow O(\log n)$
- walk down in $O(\log n)$ y-trees $\rightarrow O(\log n \log n)$
- search in $O(\log n \log n)$ z-arrays $\rightarrow O(\log n \log n \log n)$

Last Lecture: Do z-Search Earlier

- walk down the x-tree $\rightarrow O(\log n)$ time
- search in z-arrays in roots of $O(\log n)$ y-trees $\rightarrow O(\log n \log n)$
- walk down in $O(\log n)$ y-trees (and follow z-array pointers) $\rightarrow O(\log n \log n)$

Idea: Do The z-Search Even Earlier

- search z-array in root of x-tree $\rightarrow O(\log n)$
- walk down the x-tree $\rightarrow O(\log n)$ time
- walk down in $O(\log n)$ y-trees $\rightarrow O(\log n \log n)$

Observation

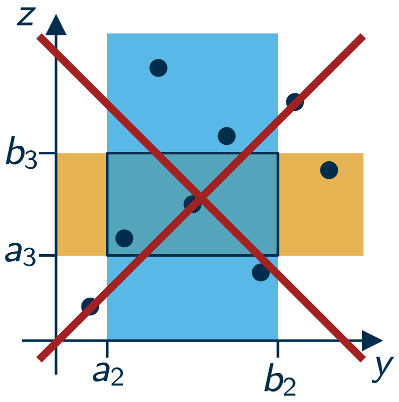
- getting rid of $\log n$ seems easy
- getting rid of $\log n$ seems hard
- goal: 2D DS with query time $O(\log n)$

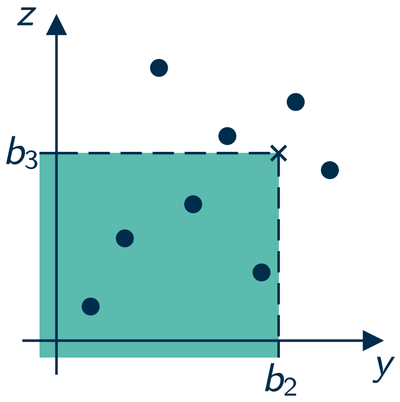
One-Sided 2D Range Queries

goal: 2D DS with query time $O(\log n)$

One-Sided Queries: Half The Sides, Half The Trouble

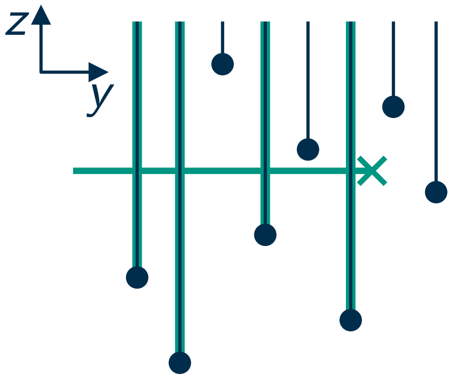
- goal: answer queries of the form $(-\infty, b_2] \times (-\infty, b_3]$ (instead of $[a_2, b_2] \times [a_3, b_3]$)





Alternative Perspective

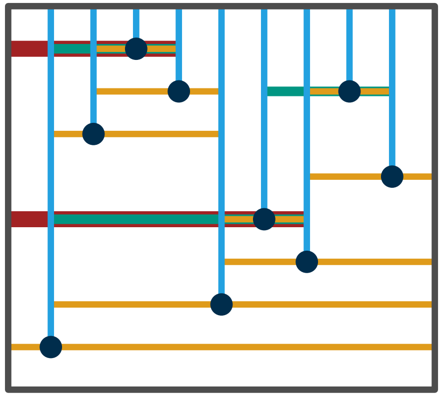
- shoot a ray from each point upwards
- ray from $\langle b_2, b_3 \rangle$ to the left
- intersecting rays yield desired points



Find All Intersecting Rays

- collect the intersecting rays from left to right
- we basically walk from cell to cell
- each cells knows its right neighbors sorted by $z \Rightarrow O(k \log n)$

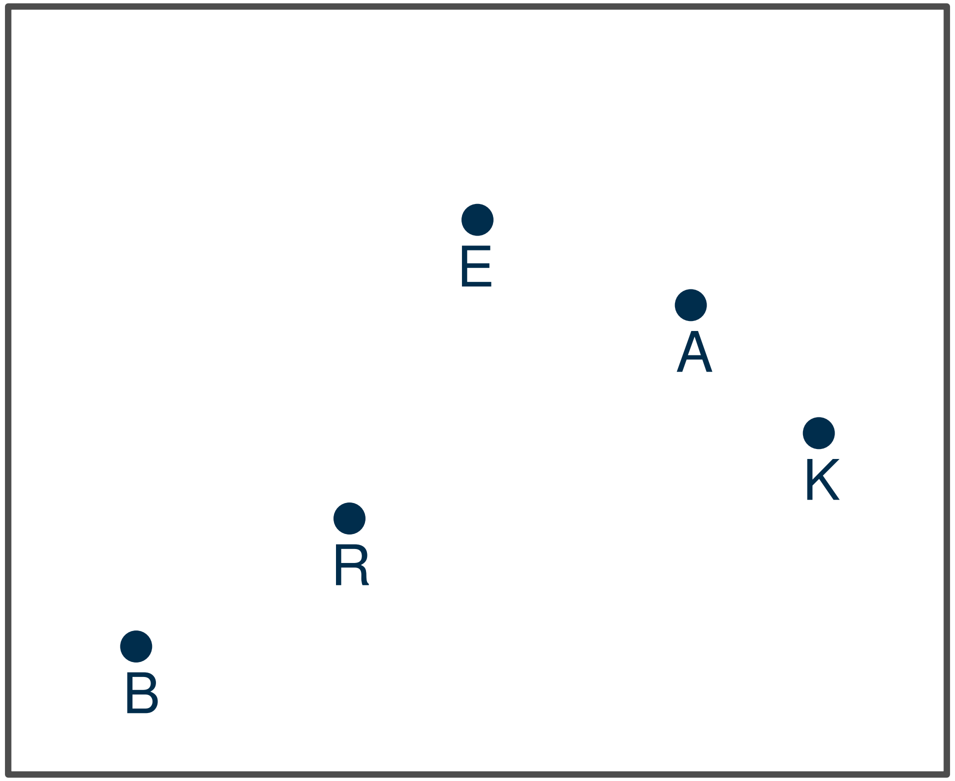
Can we do $\log n + k$?



Fractional Cascading!

Count The Cells

How many cells do we get (with and without fractional cascading)?



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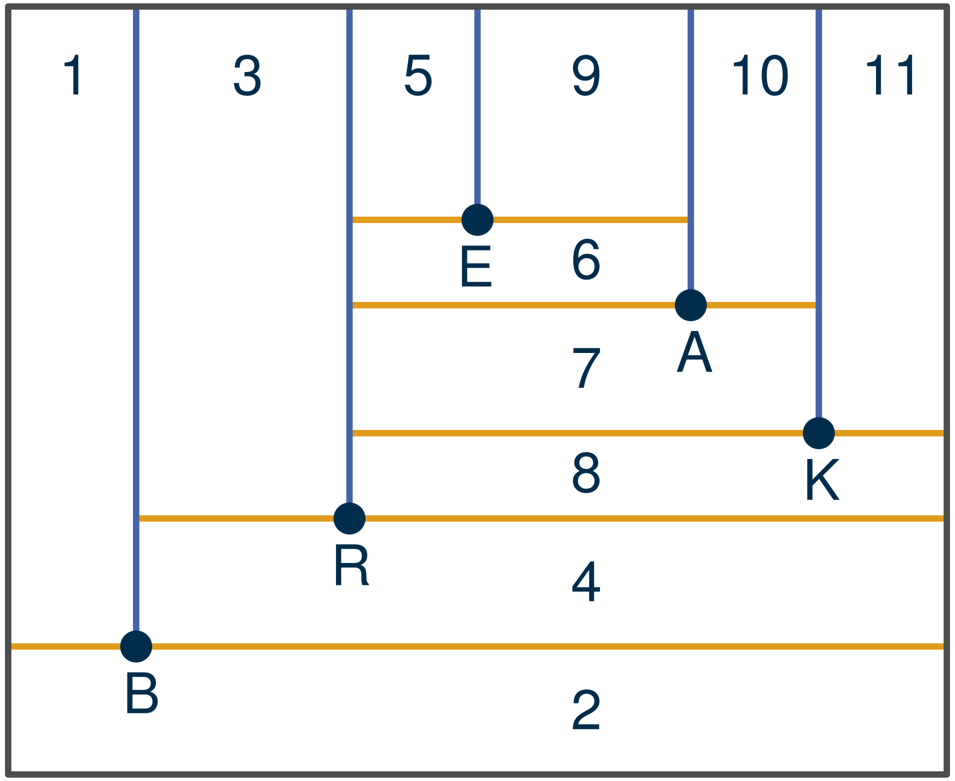

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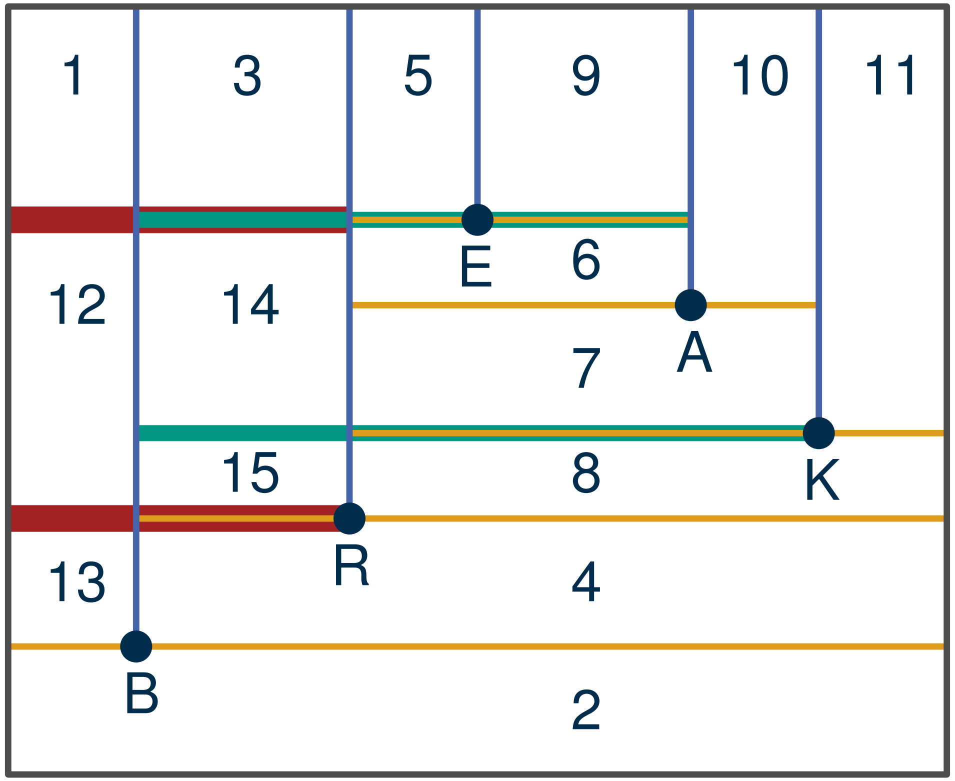
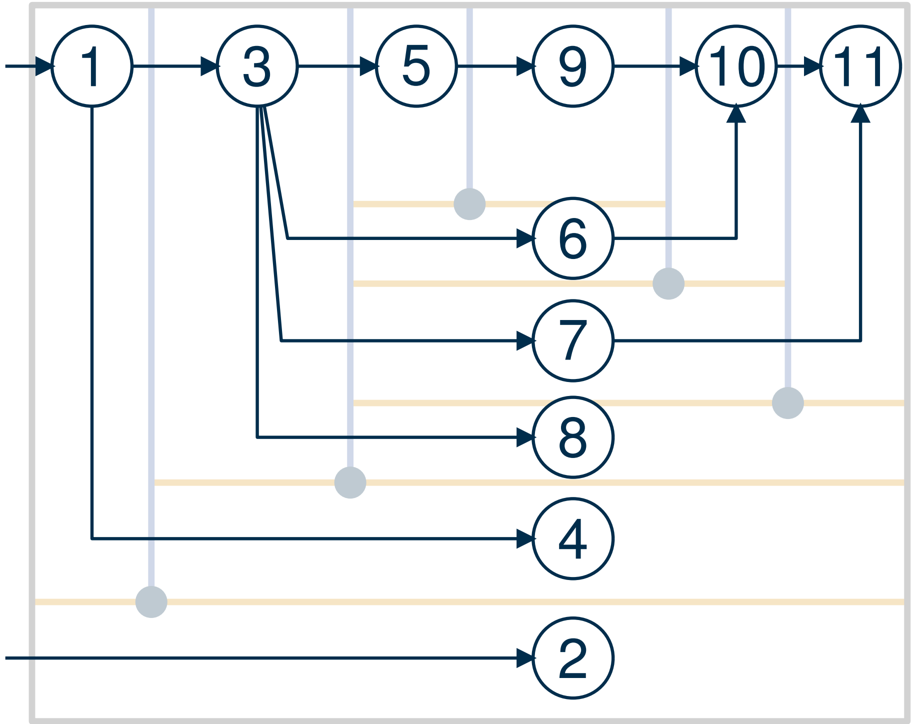
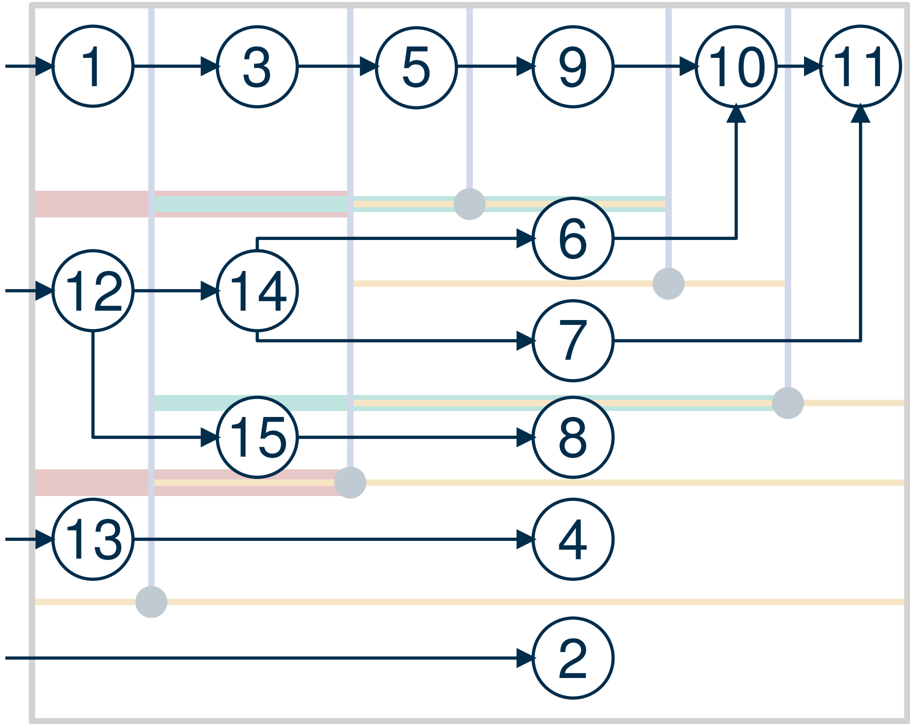


General Framework vs. Specific Situation

Useful Way Of Thinking

- mental shortcut: multiple searches for the same number $\rightarrow$ fractional cascading probably helps
- specific situation: problem-specific argument often easier than pressing it into the framework





One-Sided 3D Range Queries

Seen So Far: queries $(-\infty, b_2] \times (-\infty, b_3]$ can be answered in $O(\log n + k)$ **(DS1)**

one search in z direction

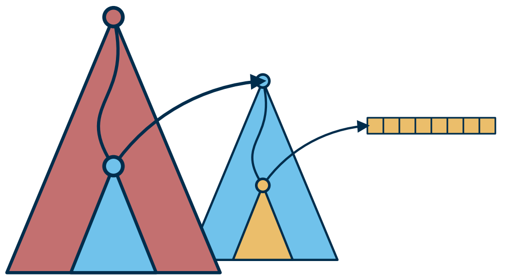
output size

New Goal: answer queries of the form $[a_1, b_1] \times (-\infty, b_2] \times (-\infty, b_3]$

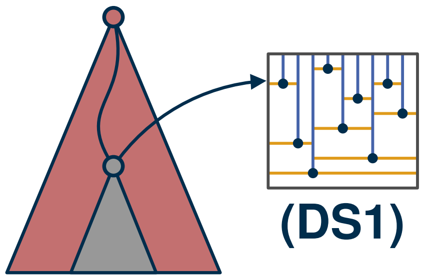
We Already Know How To Do This ...

- binary search tree for x-direction
- every node stores **(DS1)** for the corresponding points

Last Lecture



Now



Don't We Have To Search In $O(\log n)$ Many (DS1)?

- yes, but ... **Fractional Cascading!**
- search once in z-direction in the root of the x-tree
- follow pointers for the z-positions while walking down the x-tree
- save the first search in **(DS1)** $\Rightarrow$ total running time $O(\log n + k)$

One-Sided $\to$ Two-Sided

Lemma **(DS2)**
For n points in $\mathbb{R}^3$, we can answer queries of the form $[a_1, b_1] \times (-\infty, b_2] \times (-\infty, b_3]$ in $O(\log n + k)$ time after $O(n \log n)$ preprocessing with $O(n \log n)$ memory.

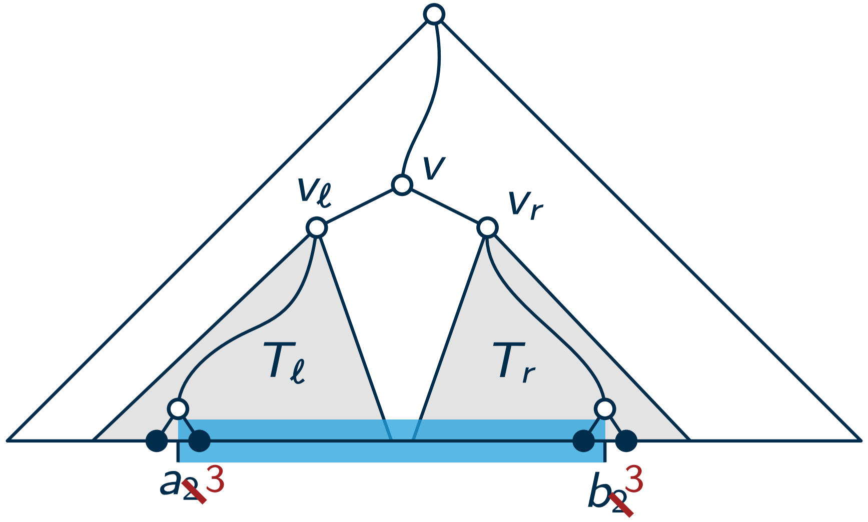
Plan
- use **(DS2)** as black box
- y-inverted variant $\to$ $[a_2, \infty)$ queries
- query $[a_2, \infty)$ and $(-\infty, b_2]$ to get $[a_2, b_2]$

Why can't we just use the intersection of two queries?

$$\begin{array}{ccccccc} [a_1, b_1] & \times & [a_2, b_2] & \times & [a_3, b_3] & = \\ [a_1, b_1] & \times & (-\infty, b_2] & \times & (-\infty, b_3] & \cap \\ [a_1, b_1] & \times & [a_2, \infty) & \times & [a_3, \infty) & \end{array}$$

Theorem **(DS4)**
For n point in $\mathbb{R}^3$, we can answer queries of the form $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ in $O(\log n + k)$ time after $O(n \log^3 n)$ preprocessing with $O(n \log^3 n)$ memory.

Two-Sided Query In ~~y~~ z-Direction



Simplified Perspective: Ignore x And ~~z~~ y-Direction

- **(DS~~2~~3)** allows $[a_{\cancel{2}3}, \infty)$ and $(-\infty, b_{\cancel{2}3}]$ queries
- goal: build data structure, that allows $[a_{\cancel{2}3}, b_{\cancel{2}3}]$ queries

Binary Search Tree In ~~y~~ z-Direction

- search for $a_{\cancel{2}3}$ and $b_{\cancel{2}3}$ splits at v to v_ℓ and v_r
- queries in instances of **(DS~~2~~3)**: $[a_{\cancel{2}3}, \infty)$ on points in T_ℓ and $(-\infty, b_{\cancel{2}3}]$ on points in T_r
- running time: $O(\log n)$ for search in ~~y~~ z-tree plus $O(\log n + k)$ for two queries in **(DS~~2~~3)**
- memory: $O(\log n) \cdot$ (memory for **DS~~2~~3**)

~~Lemma~~ Theorem **(DS~~3~~4)**
For n point in $\mathbb{R}^3$, we can answer queries of the form $[a_1, b_1] \times [a_2, b_2] \times (\cancel{-\infty}\ a_3, b_3]$ in $O(\log n + k)$ time after $O(n \log^{\cancel{2}3} n)$ preprocessing with $O(n \log^{\cancel{2}3} n)$ memory.

The Big Picture

(DS4)

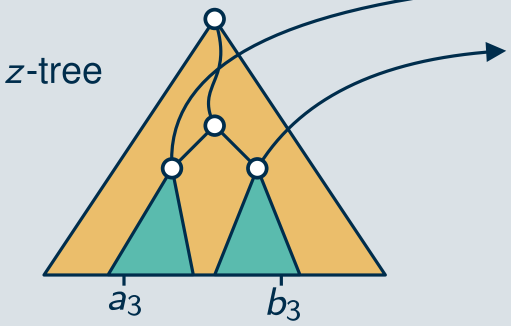
$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

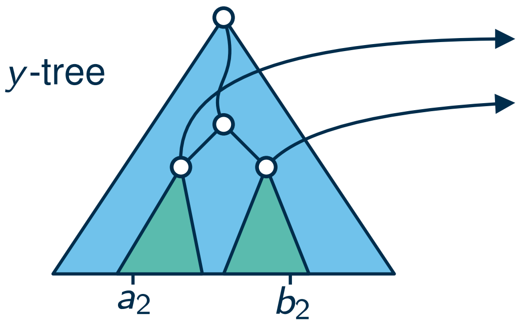
precomp: $O(n \log^3 n)$

memory: $O(n \log^3 n)$

query: $O(\log n + k)$

- search for a_3, b_3 in z-tree
- split → two **(DS3)** queries





(DS3)

$[a_1, b_1] \times [a_2, b_2] \times (-\infty, b_3]$

precomp: $O(n \log^2 n)$

memory: $O(n \log^2 n)$

query: $O(\log n + k)$

- search for a_2, b_2 in y-tree
- splits → two **(DS2)** queries

(DS2)

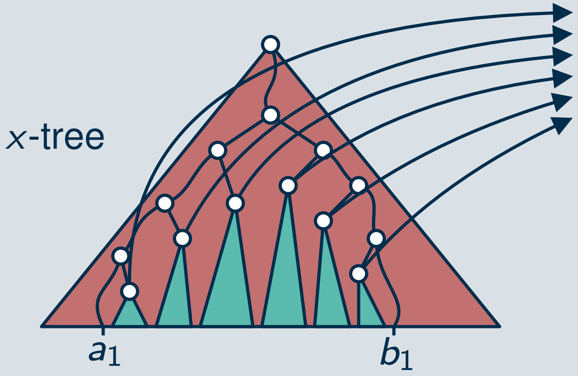
$[a_1, b_1] \times (-\infty, b_2] \times (-\infty, b_3]$

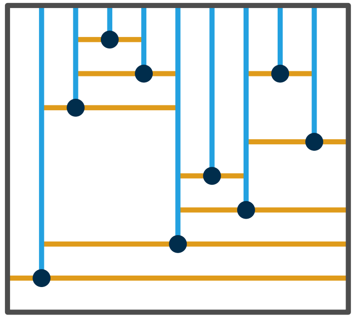
precomp: $O(n \log n)$

memory: $O(n \log n)$

query: $O(\log n + k)$

- search for z in the root
- walk down x-tree
- follow z-pointers
- each x-subtree: **(DS1)** query (with initial z-pos)





(DS1)

$(-\infty, b_2] \times (-\infty, b_3]$

query: $O(k)$

precomp: $O(n)$ (if points are sorted)

memory: $O(n)$

- initial z-position known
- walk in y-direction
- decide for cell by z-query
- output intersected rays

Wrap-Up

What Have We Learned Today?

- fractional cascading: search only once and then follow pointers
- clever geometric solution for simplified queries $(-\infty, b_2] \times (-\infty, b_3]$
- transformation from $(-\infty, b]$ to $[a, b]$

Theorem **(DS4)**
For n point in $\mathbb{R}^3$, we can answer queries of the form $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ in $O(\log n + k)$ time after $O(n \log^3 n)$ preprocessing with $O(n \log^3 n)$ memory.

What Else Is There?

- many applications of fractional cascading
- dynamic range queries: inserting and deleting points
- $O(\log n \cdot (\log n / \log \log n)^{d-3} + k)$ queries with $O(n \cdot (\log n / \log \log n)^{d-3})$ memory
- even better results with some bit-hacking in the word RAM model