Computational Geometry

Linear Programs & Half-Plane Intersection

Thomas Bläsius

Developing A Mold

Casting (Manufacturing Technique)

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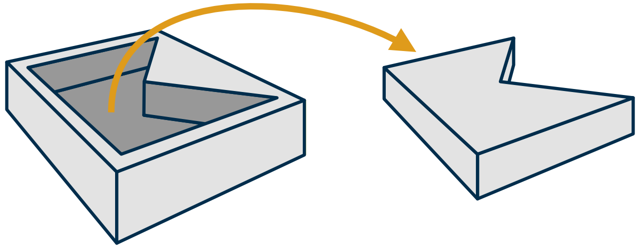

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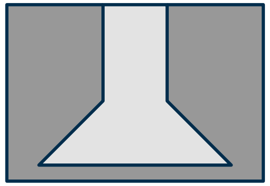

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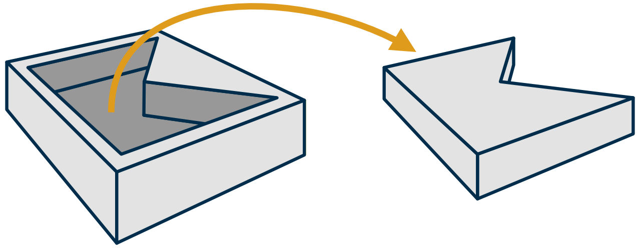

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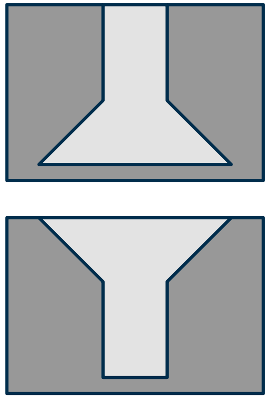

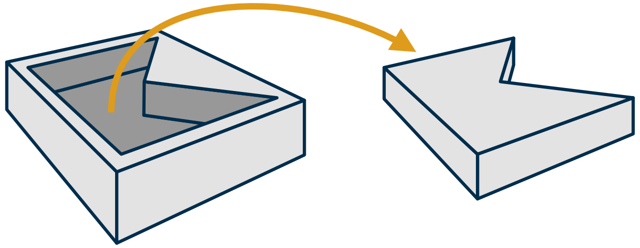

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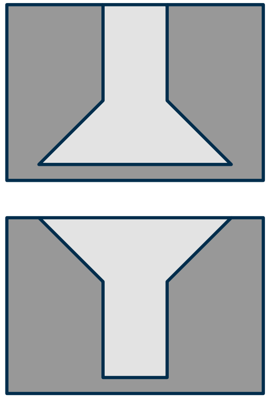

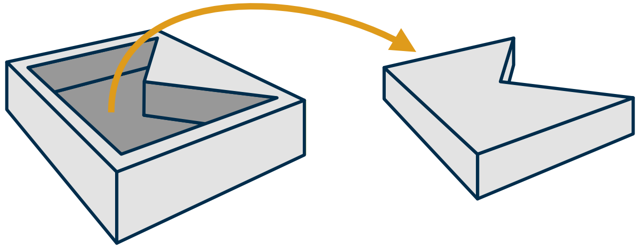

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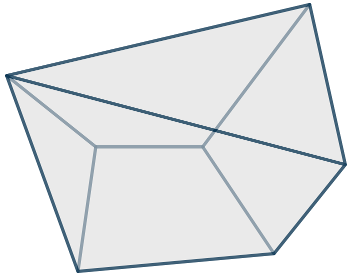

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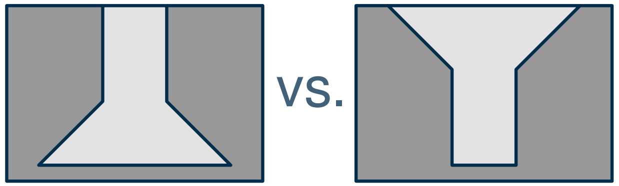

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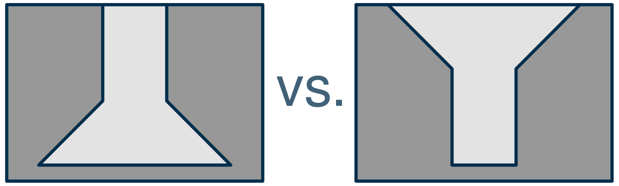

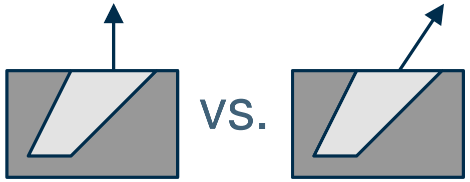

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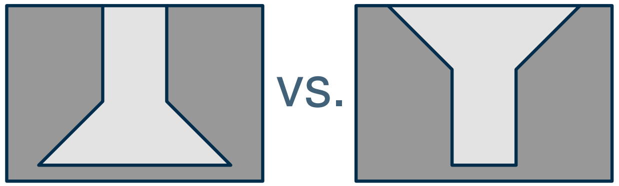

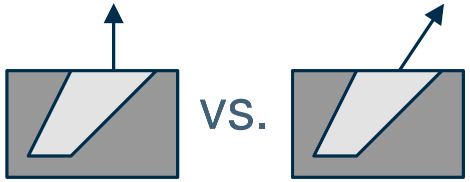

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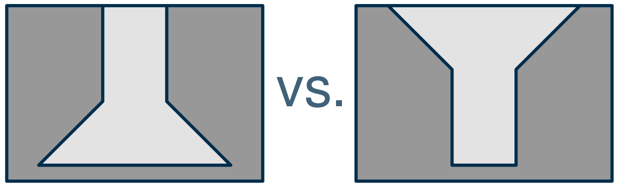

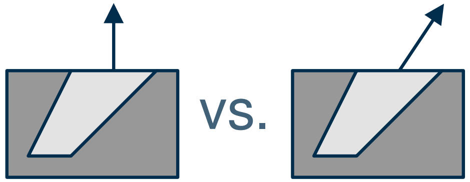

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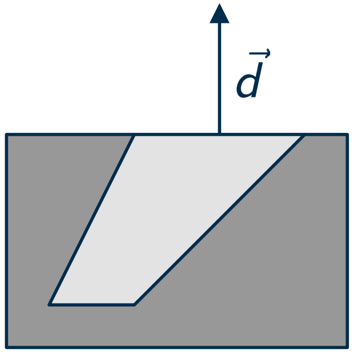

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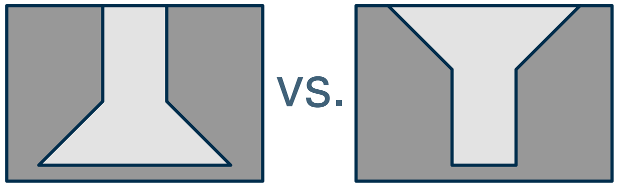

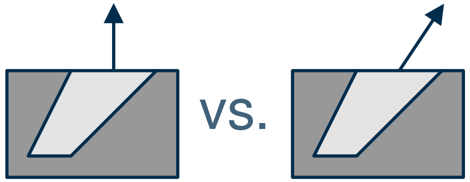

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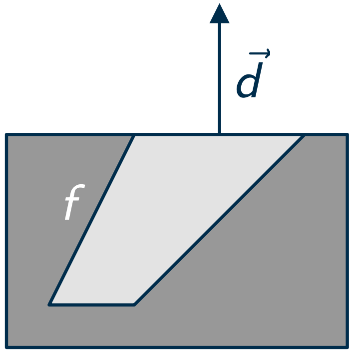

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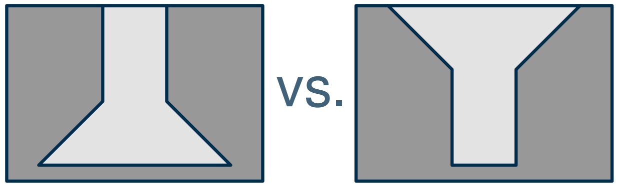

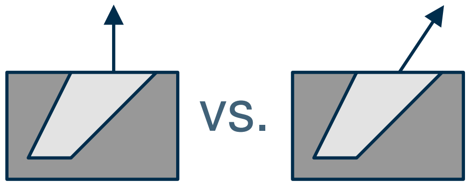

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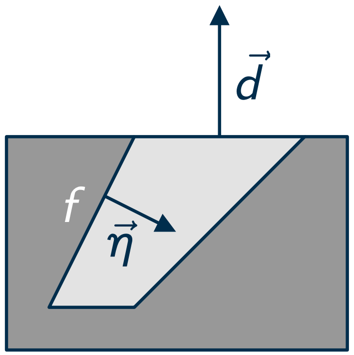

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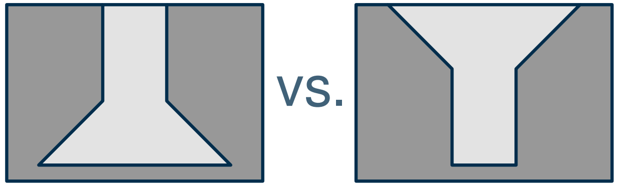

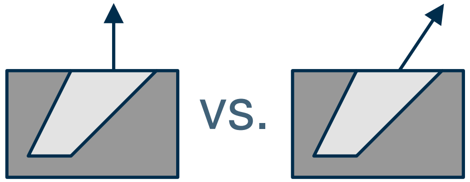

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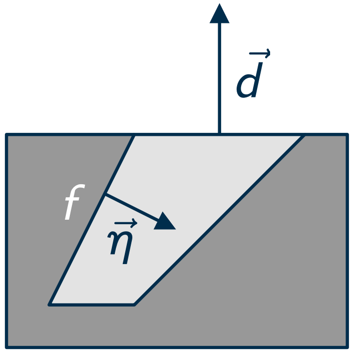

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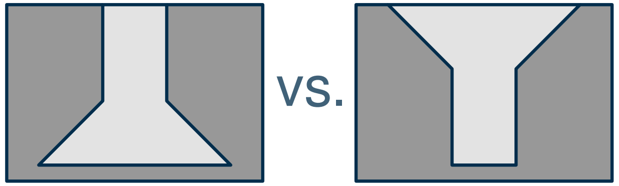

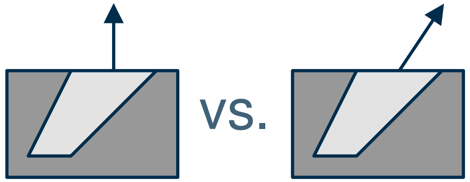

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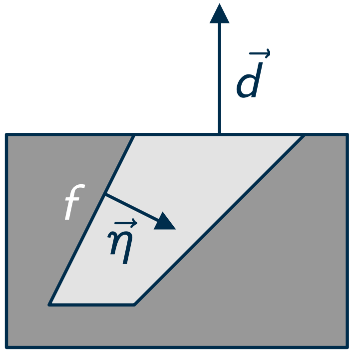

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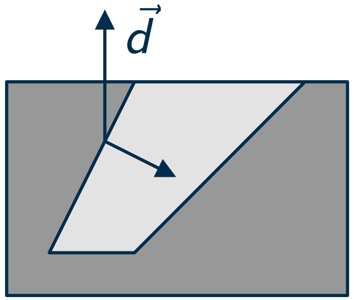

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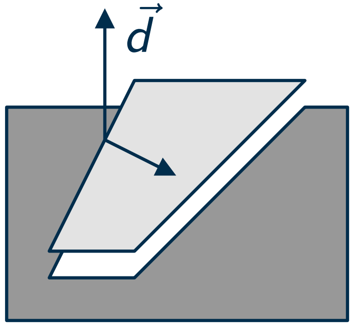

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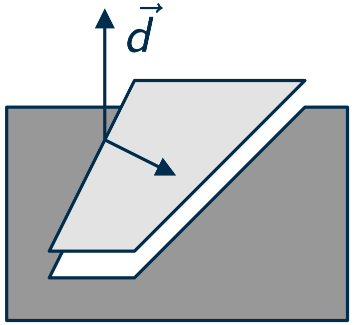

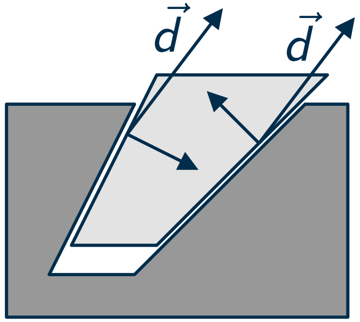

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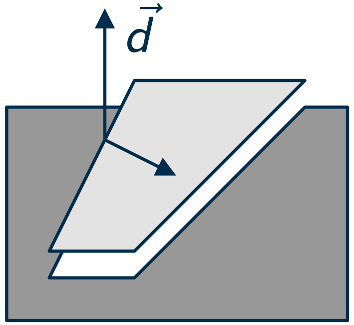

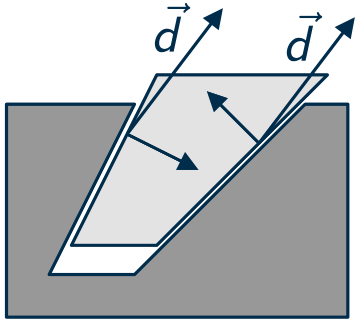

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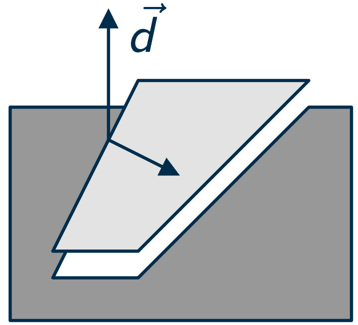

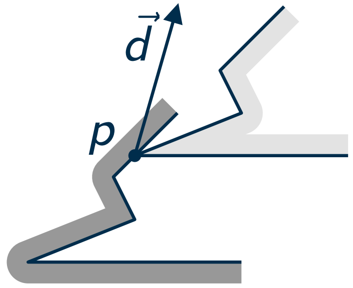

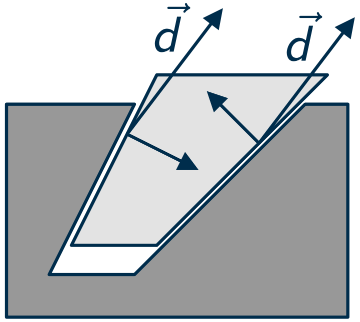

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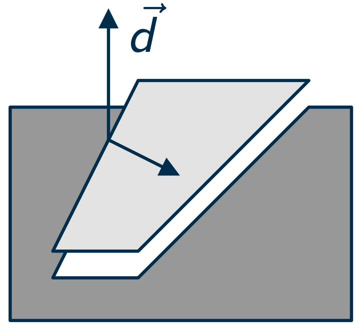

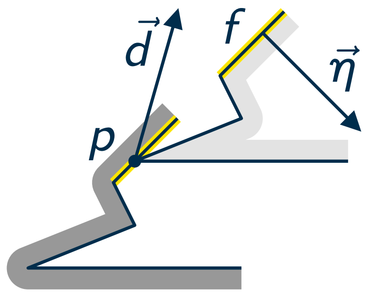

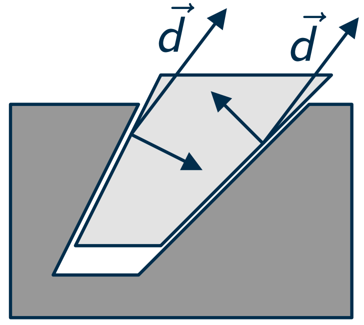

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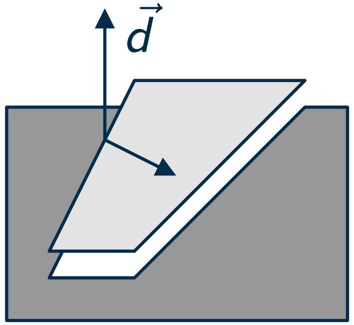

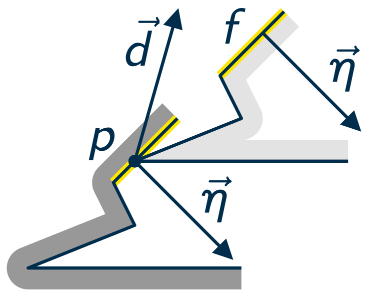

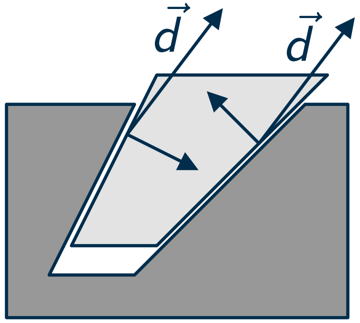

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Goal

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- this is a linear program (LP)

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Why?

Reminder (Dot Product): angle between $\vec{d}$ and $\vec{\eta} \leq 90^\circ \Leftrightarrow \vec{d} \cdot \vec{\eta} \geq 0$

Restating The Problem (For A Fixed Upper Face)

- find d_x and d_y
- such that for every regular face we have: $\eta_x \cdot d_x + \eta_y \cdot d_y + \eta_z \geq 0$
- this is a linear program (LP)

Is the inequality really linear?

Linear Programs

Linear Programs

General Form Of An LP

- variables $x_1, \ldots, x_d$
- an objective function
- $a_{i,j}$, b_i, c_i are constants
- n constraints

$$
\begin{aligned}
\text{maximize} \quad & c_1x_1 + c_2x_2 + \cdots + c_dx_d \\
\text{such that} \quad & a_{1,1}x_1 + \cdots + a_{1,d}x_d \leq b_1 \\
& a_{2,1}x_1 + \cdots + a_{2,d}x_d \leq b_2 \\
& \qquad \vdots \\
& a_{n,1}x_1 + \cdots + a_{n,d}x_d \leq b_n
\end{aligned}
$$

Linear Programs

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Our Specific LP

Linear Programs

General Form Of An LP

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Our Specific LP

- variables $d_x, d_y \to$ dimension 2
- one constraint for each face: $\eta_x \cdot d_x + \eta_y \cdot d_y + \eta_z \geq 0$

Linear Programs

General Form Of An LP

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Algorithm For The Mold Creation Problem

Linear Programs

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Algorithm For The Mold Creation Problem

- choose each of the n faces once as upper face
- for every upper face, solve a 2-dimensional LP with $n - 1$ constraints

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Goal In The Following: efficient algorithm to solve a 2-dimensional LP

Find An Optimal Solution

maximize: $x_1 + x_2$

such that:

$$
\begin{aligned}
x_1 &\geq 0 && (B)\\
x_2 &\geq 0 && (R)\\
x_2 - x_1 &\leq 1 && (E)\\
x_1 + 6x_2 &\leq 15 && (A)\\
4x_1 - x_2 &\leq 10 && (K)
\end{aligned}
$$

2D LPs

Example

$$
\begin{aligned}
\text{maximize:} \quad & x_1 + x_2 \\
\text{such that:} \quad & x_1 \geq 0 \\
& x_2 \geq 0 \\
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Geometric Interpretation

2D LPs

Example

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$x_2 - x_1 \leq 1$

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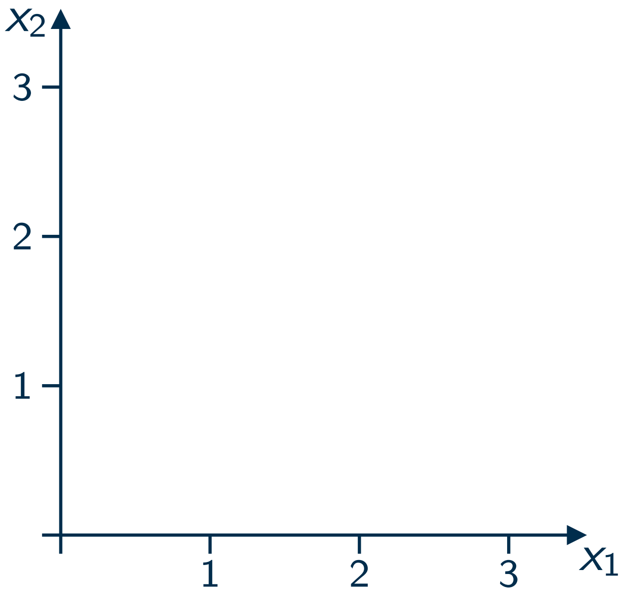
Geometric Interpretation



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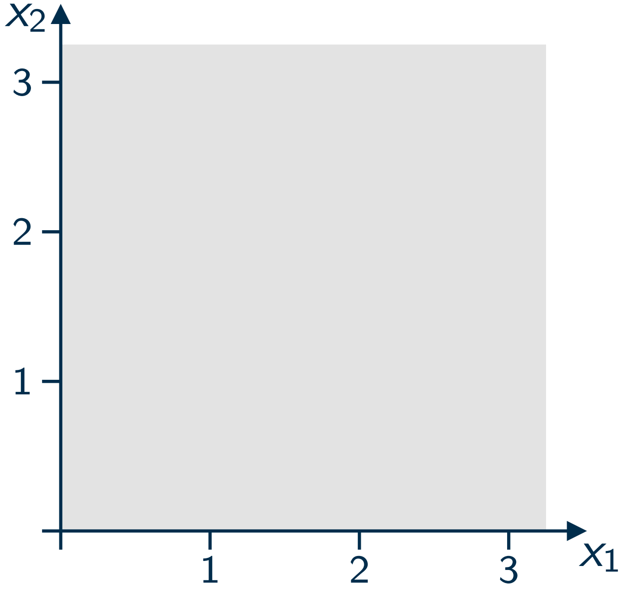

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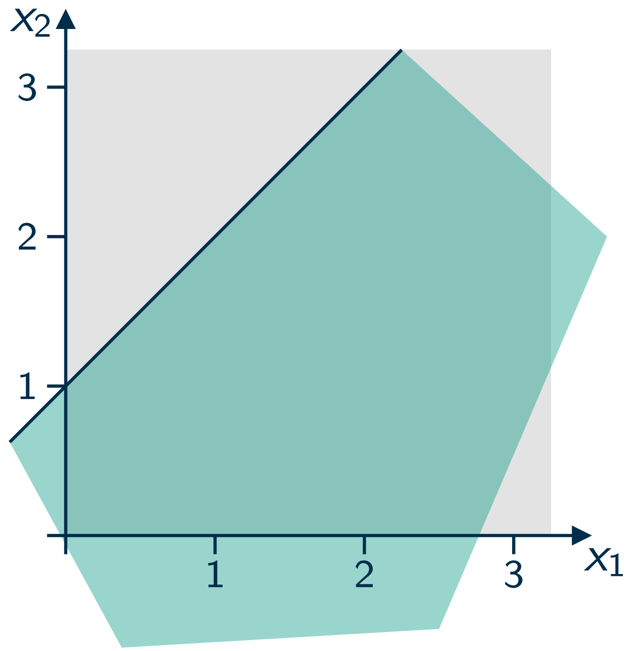

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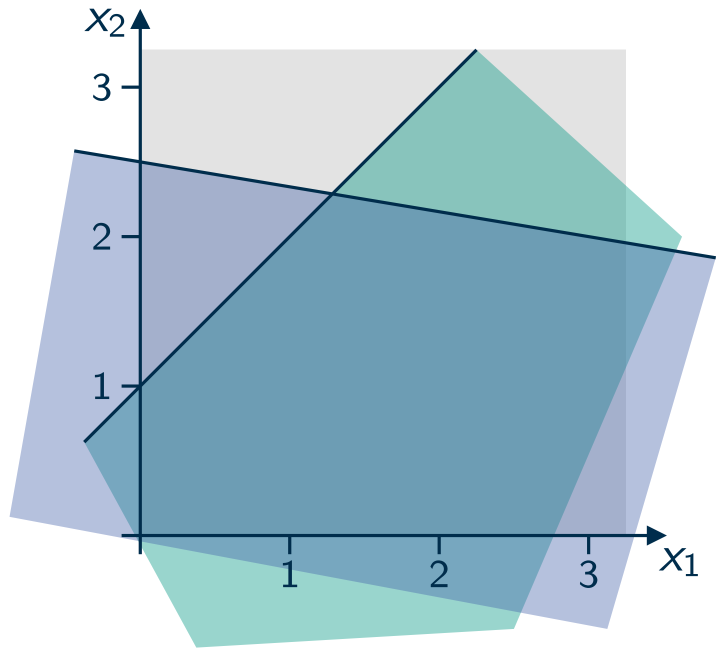

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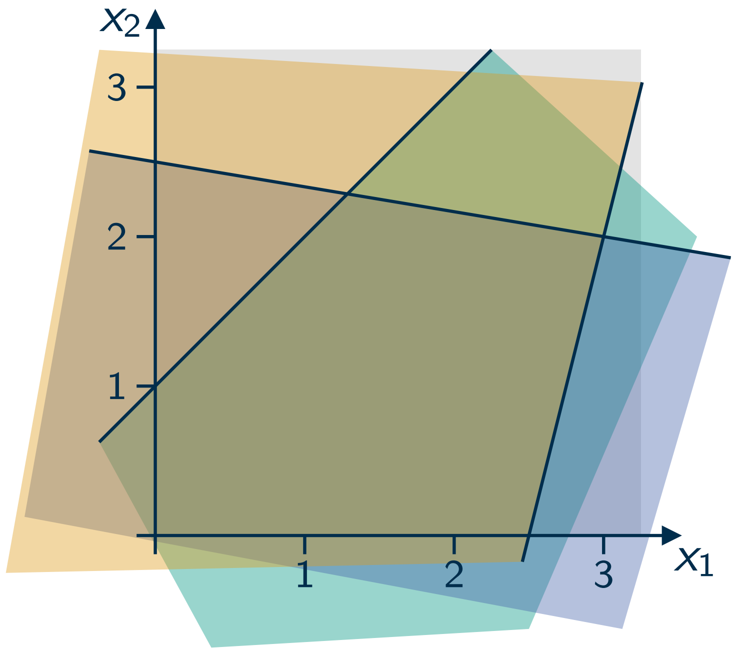

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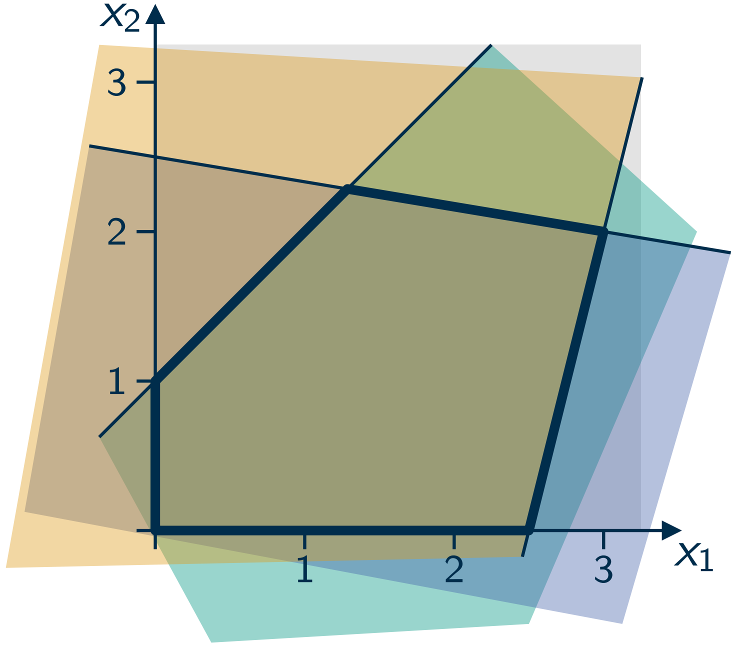

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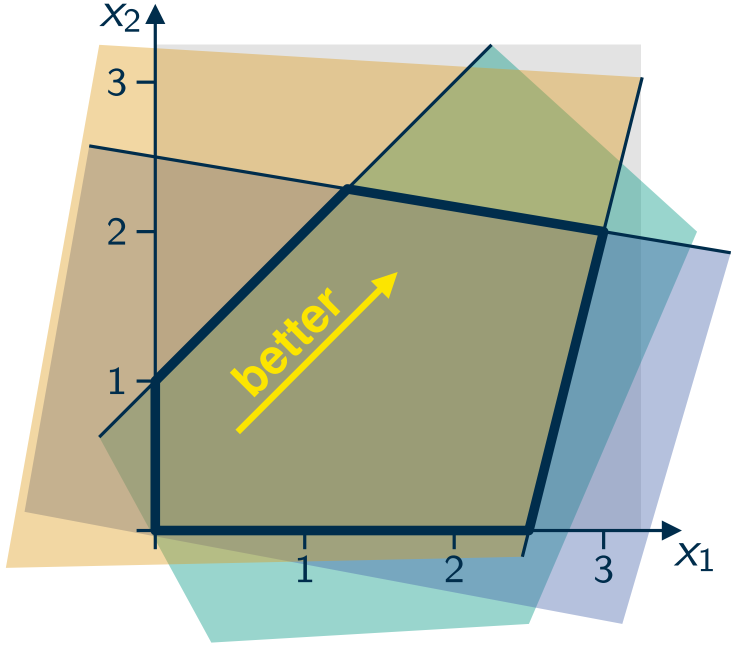

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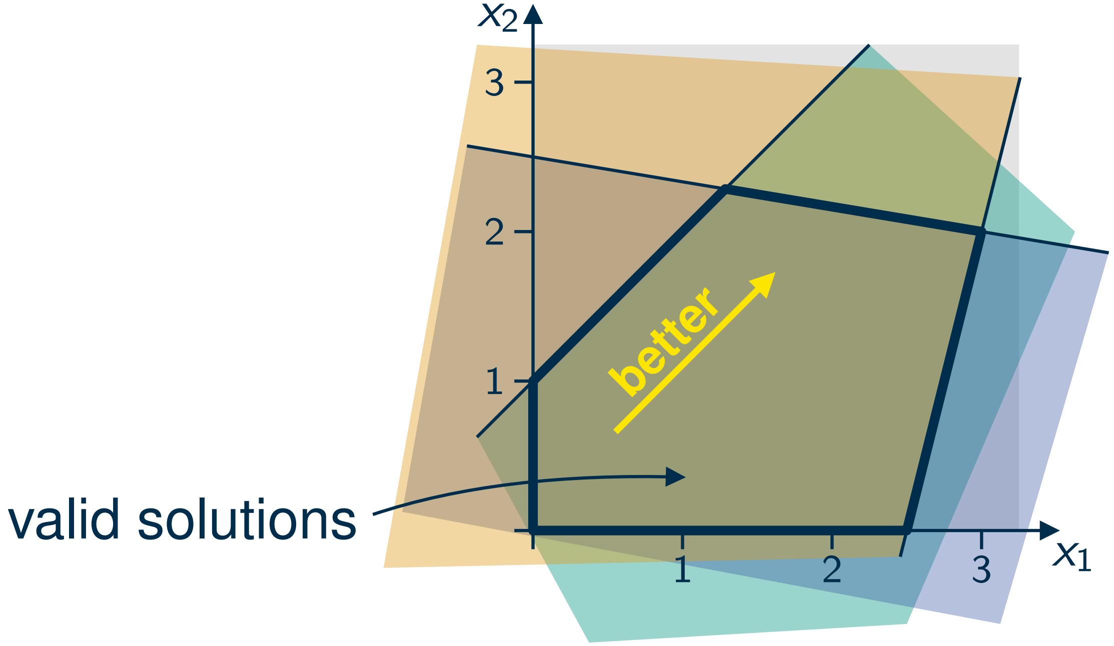

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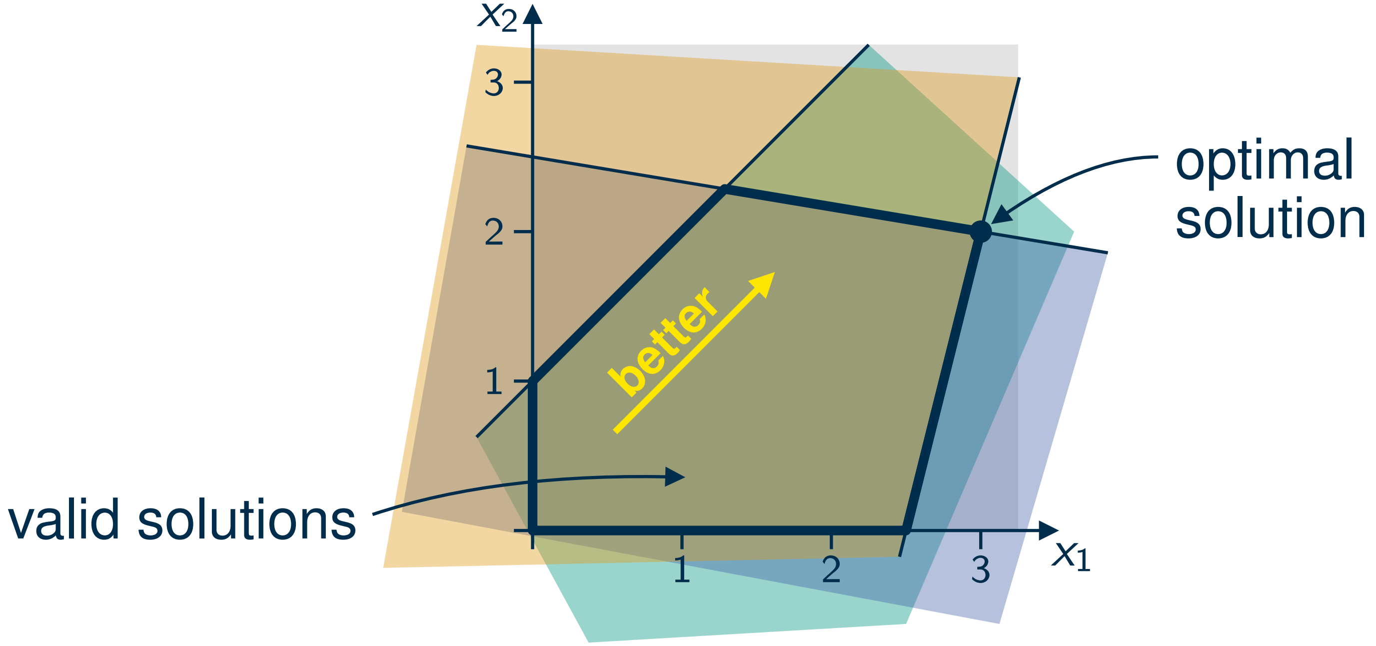

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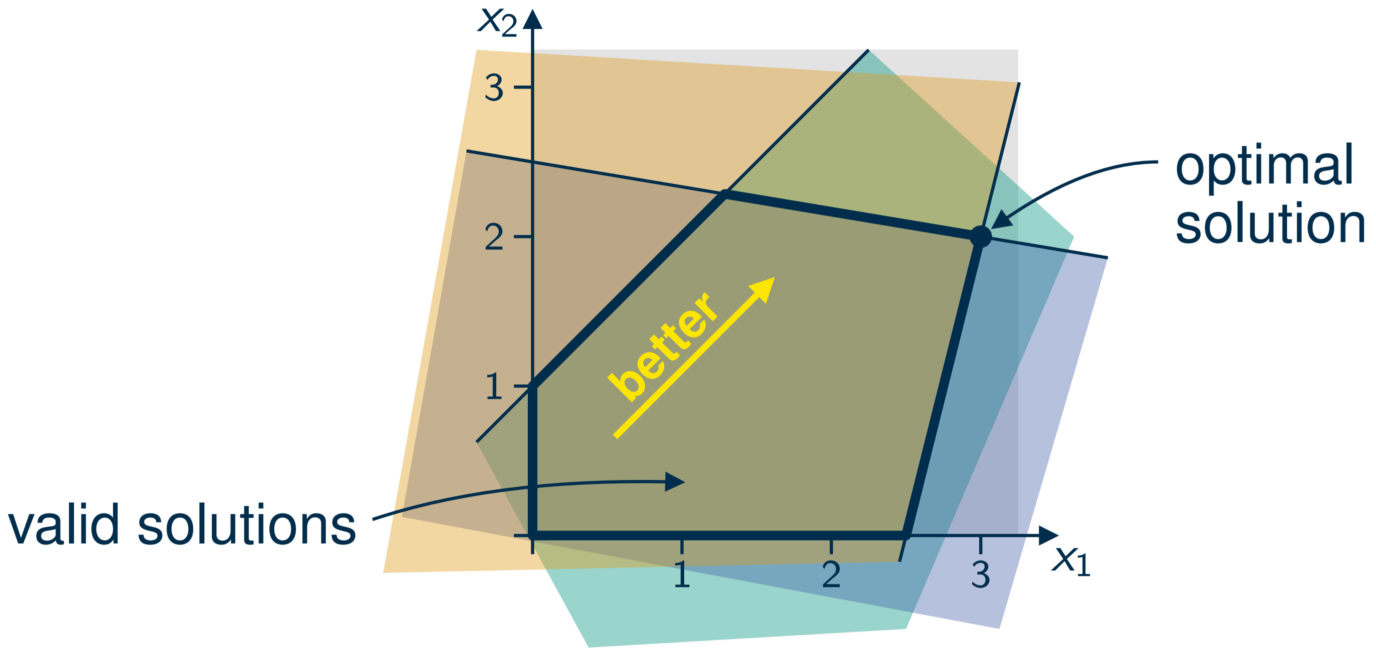

Properties Of The LP

- **infeasible:** no valid solution

2D LPs

Example

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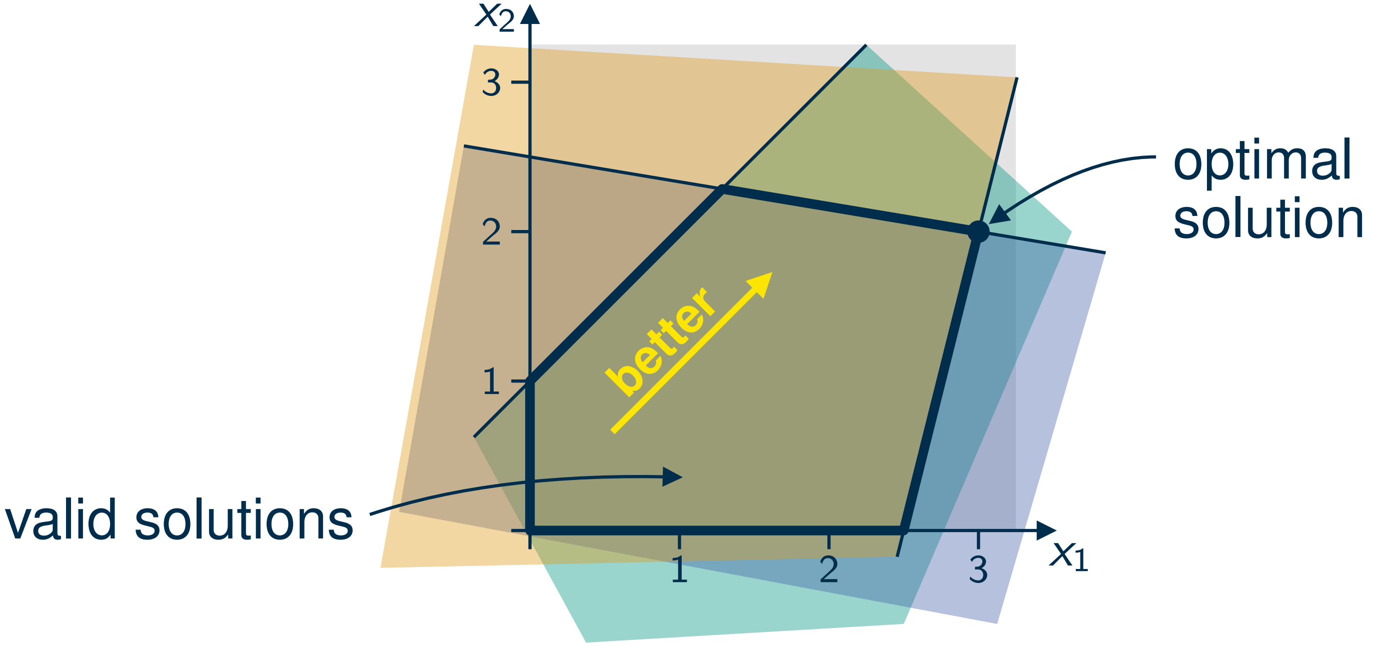

Properties Of The LP

- **infeasible:** no valid solution
- **unbounded:** there are solutions with arbitrarily large objective

2D LPs

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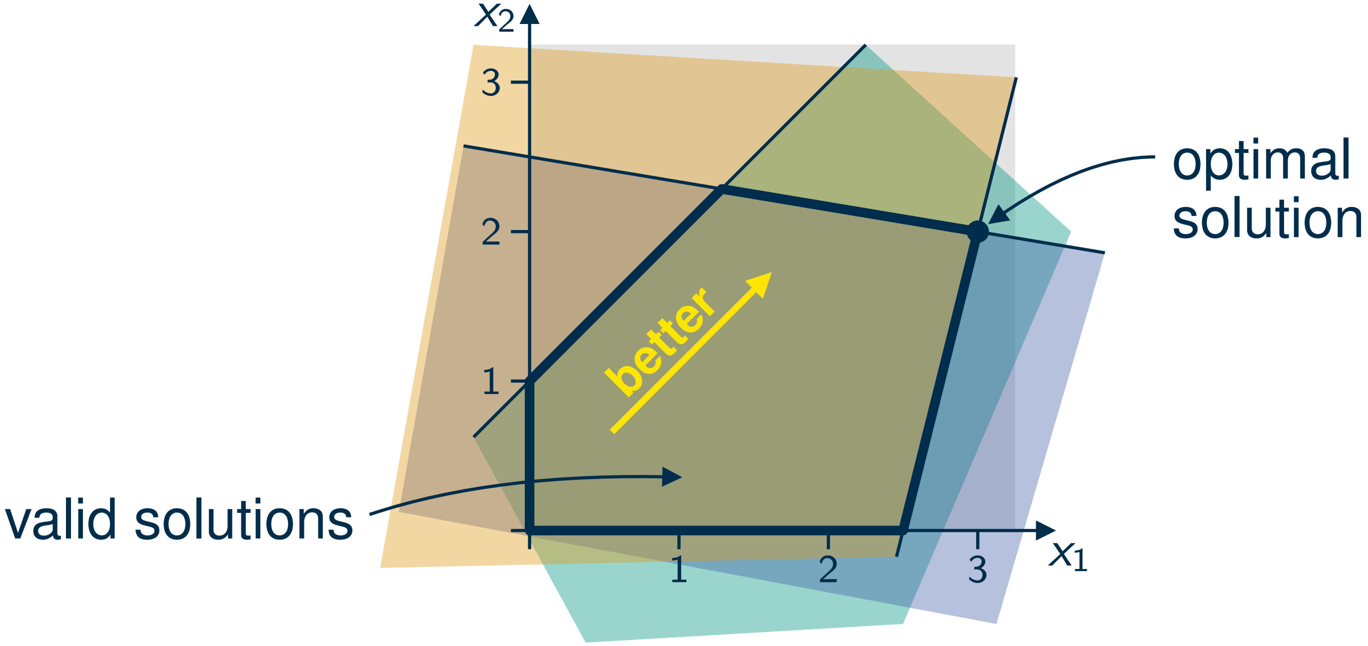

Properties Of The LP

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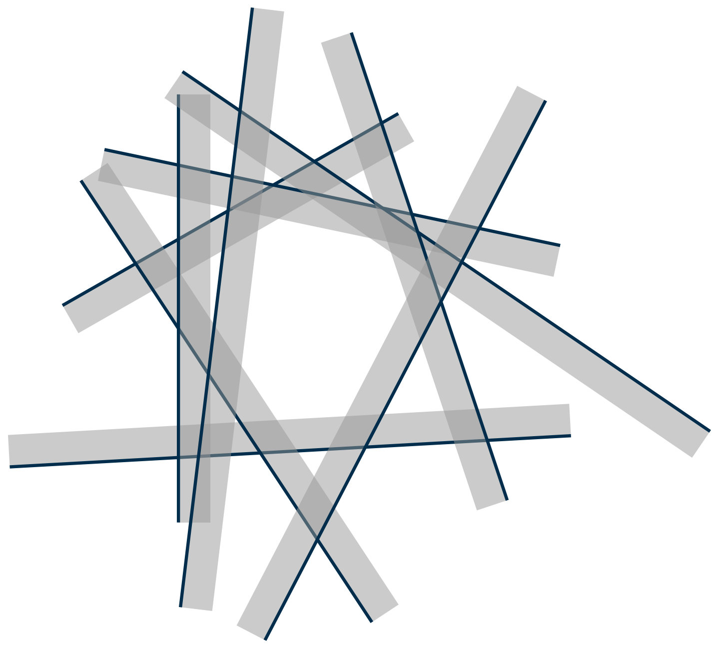
Problem: Half-Plane Intersection
Given n half planes, compute their intersection.

Half-Plane Intersection

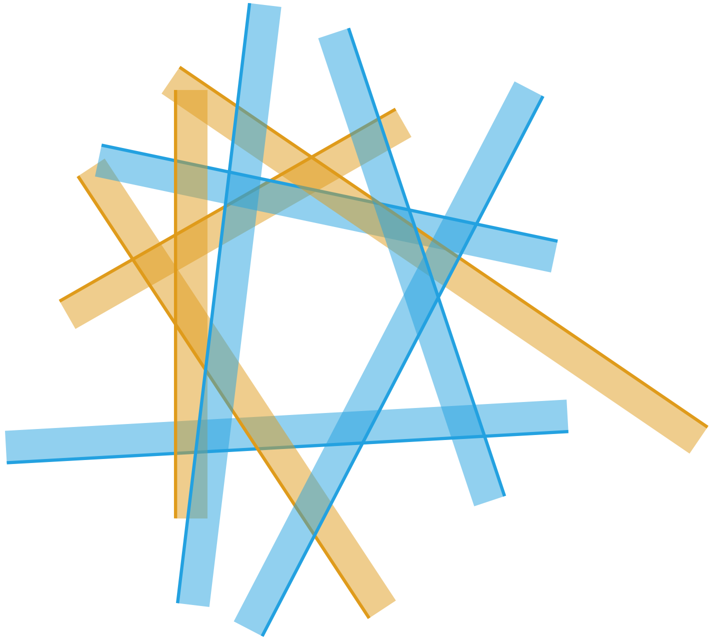
Plan: Divide And Conquer



Half-Plane Intersection

Plan: Divide And Conquer

- split half planes into two groups of roughly equal size



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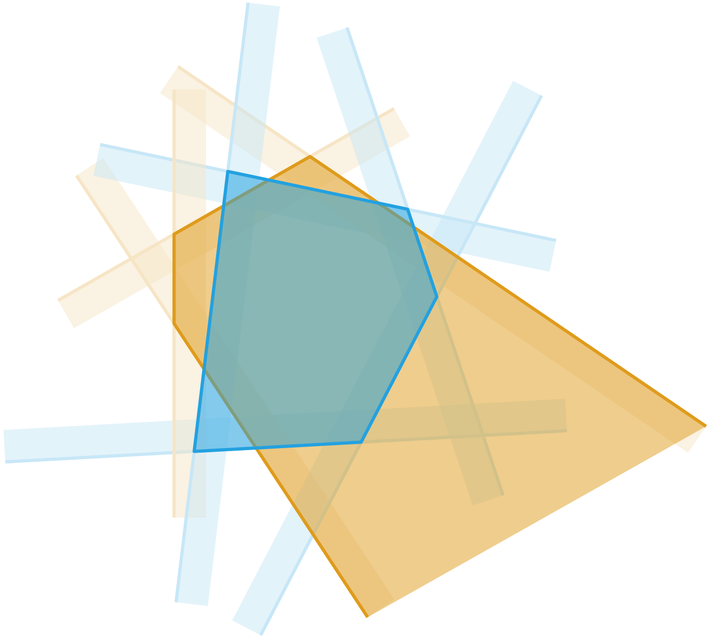
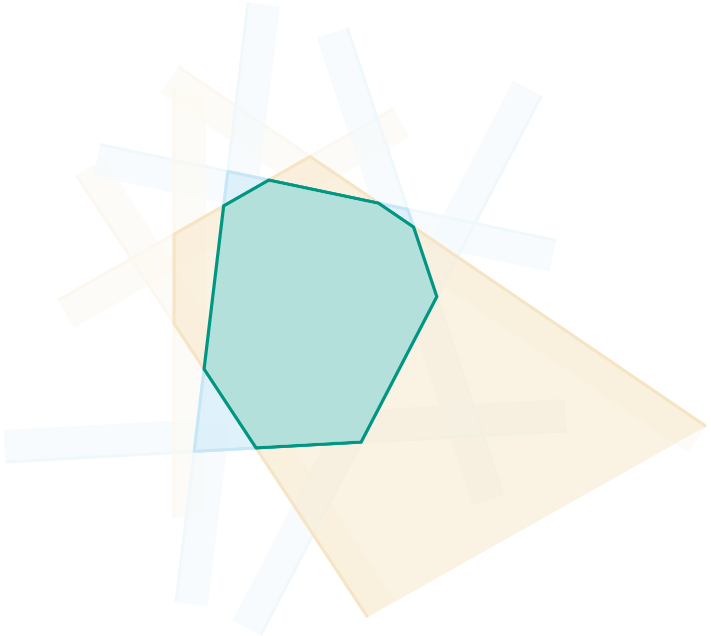


Plan: Divide And Conquer

- split half planes into two groups of roughly equal size
- compute intersection for each group

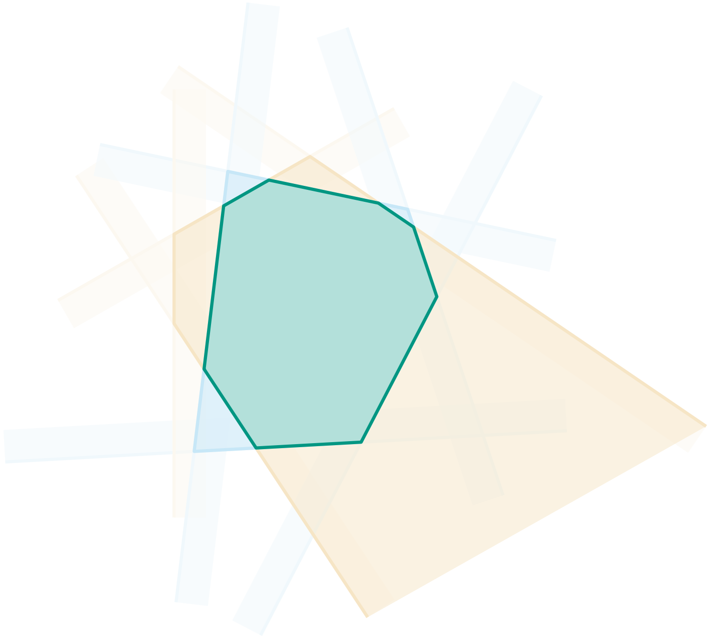
Half-Plane Intersection



Plan: Divide And Conquer

- split half planes into two groups of roughly equal size
- compute intersection for each group
- compute intersection of the two resulting regions

Half-Plane Intersection



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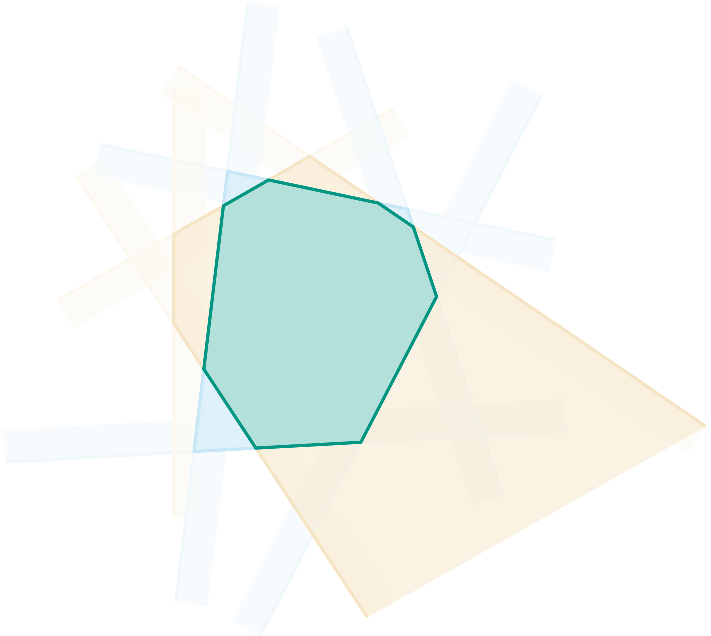
Intersecting The Two Results

Half-Plane Intersection



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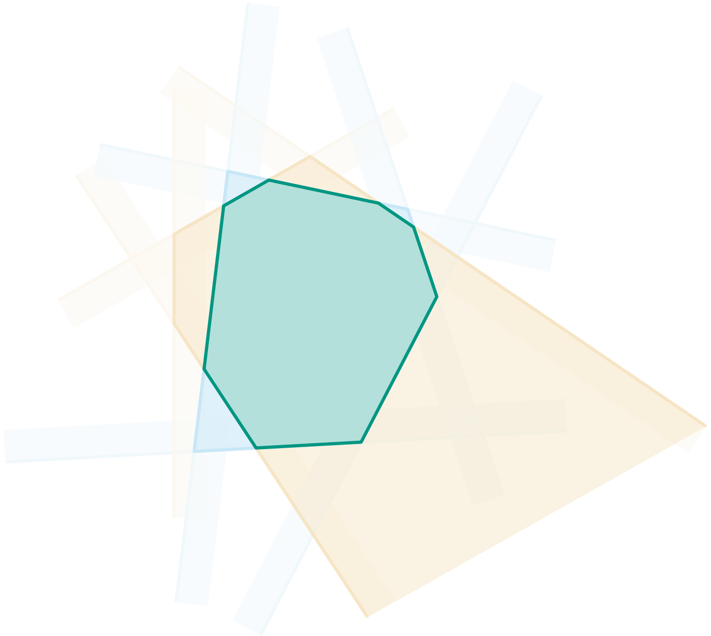
Intersecting The Two Results

- more or less the intersection of two convex polygons

Half-Plane Intersection



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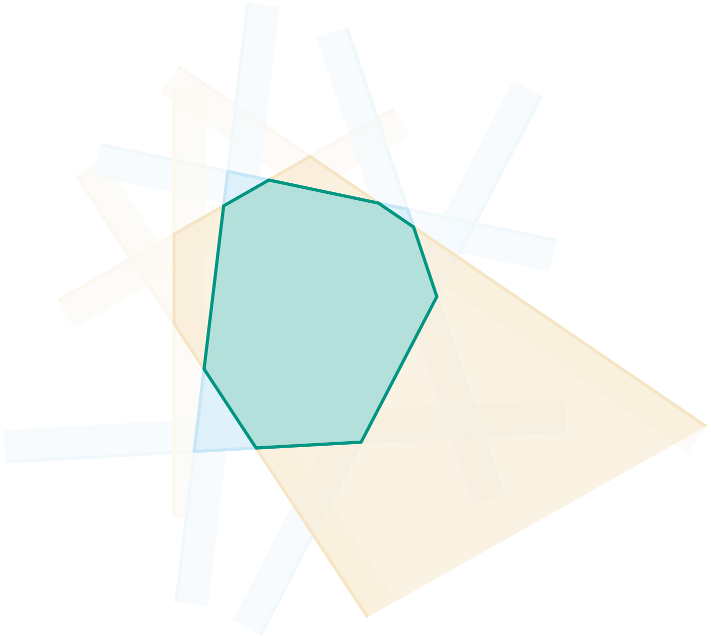

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- sweep-line algorithm can be adjusted accordingly $\rightarrow$ running time $O(n \log n)$

this is a hint for one exercise on the current sheet

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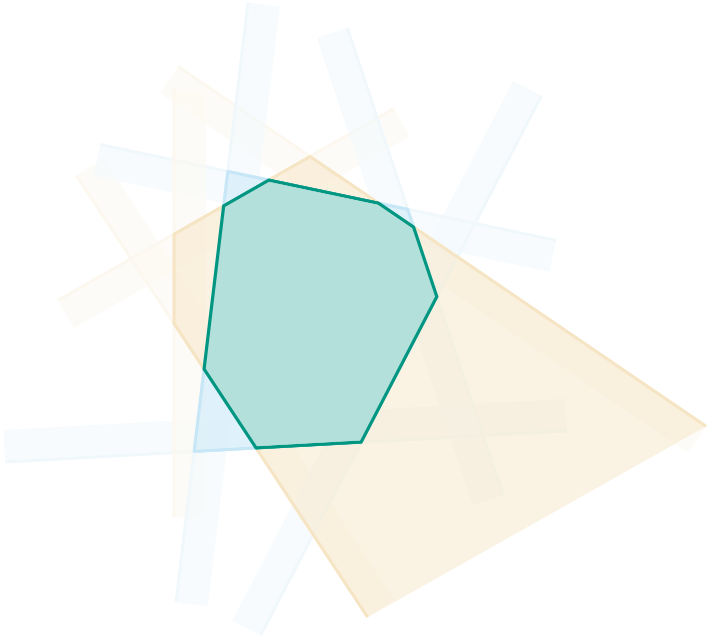

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- using convexity $\rightarrow$ running time $O(n)$

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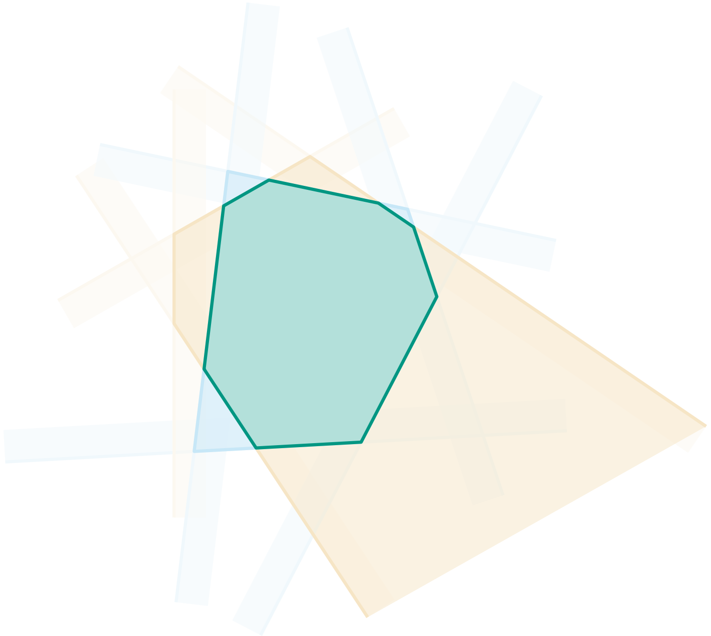

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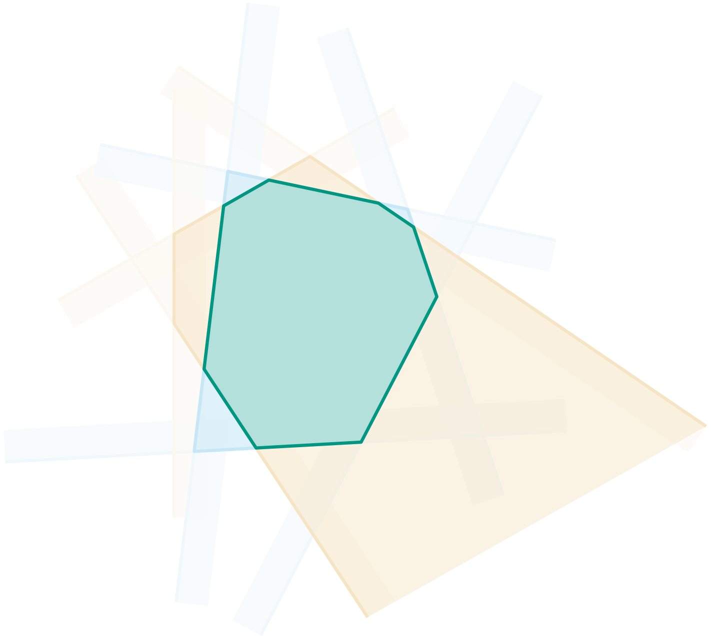
Total Time:

Half-Plane Intersection



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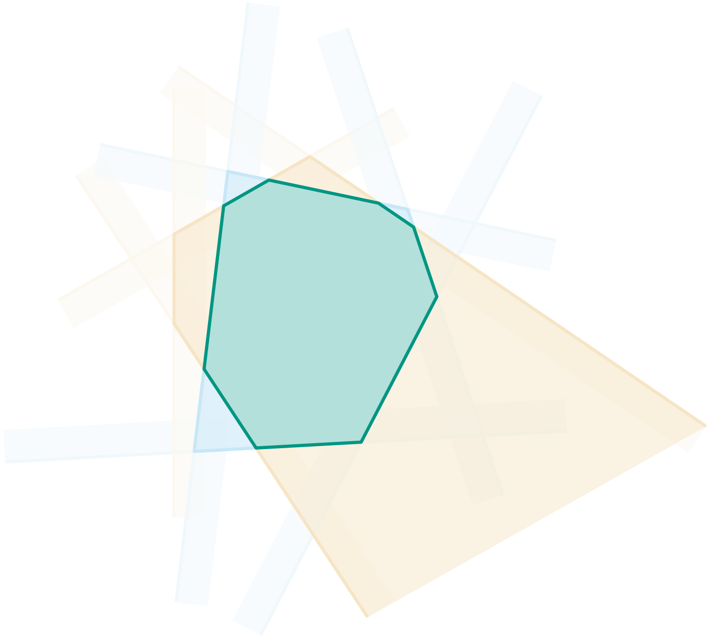
Total Time: $T(n) = O(n) + 2T(n/2)$

Half-Plane Intersection



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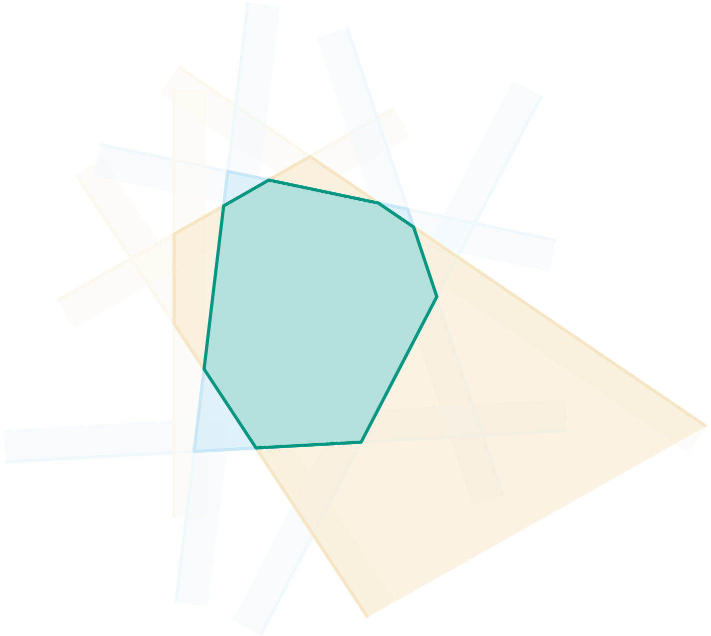

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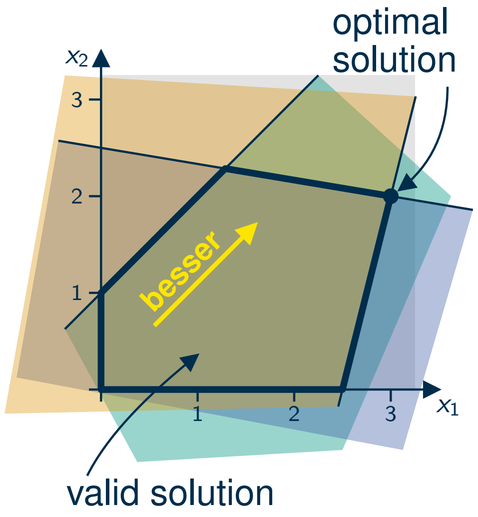
Total Time: $T(n) = O(n) + 2T(n/2)$
$\Rightarrow O(n \log n)$

Can This Be Improved?

- closely related to convex hull (via duality)
- lower bound: $\Omega(n \log n)$

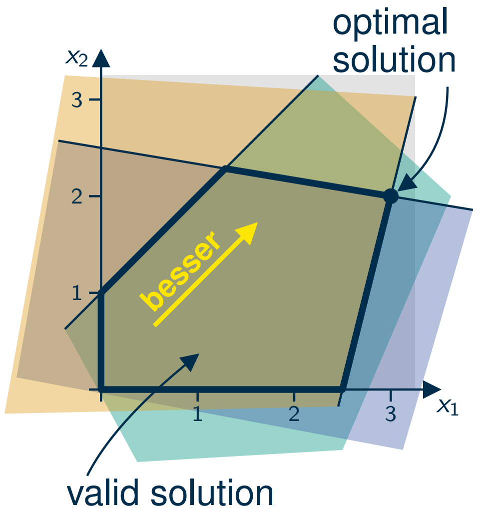
Incremental Algorithm For 2D LPs



Observation

- we do not actually need to compute the valid region explicitly
- it is sufficient to compute a valid point that maximizes the objective

Incremental Algorithm For 2D LPs



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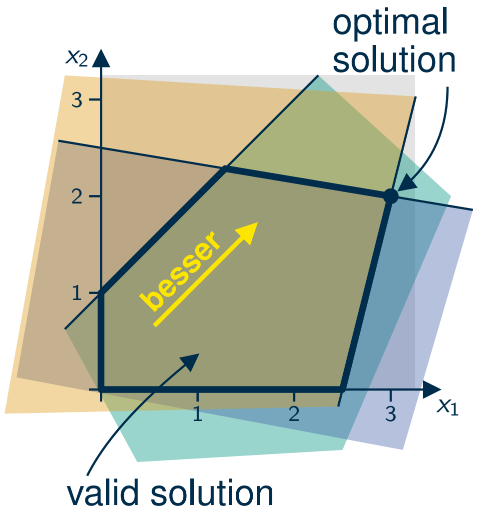
Incremental Approach

- let $h_1, \dots, h_n$ he the half planes (constraints)
- let $C_i = h_1 \cap h_2 \cap \dots \cap h_i$ (feasible region with respect to $h_1, \dots, h_i$)

Incremental Algorithm For 2D LPs



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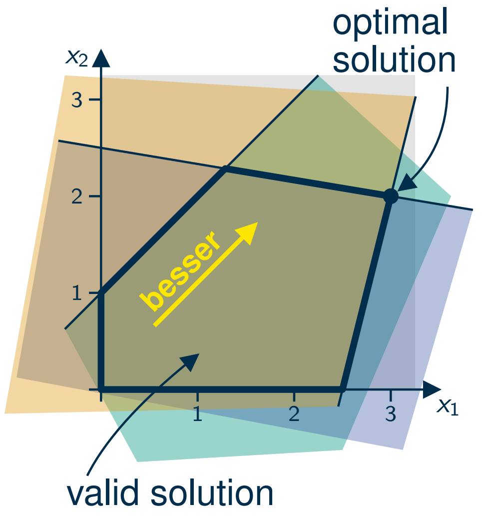

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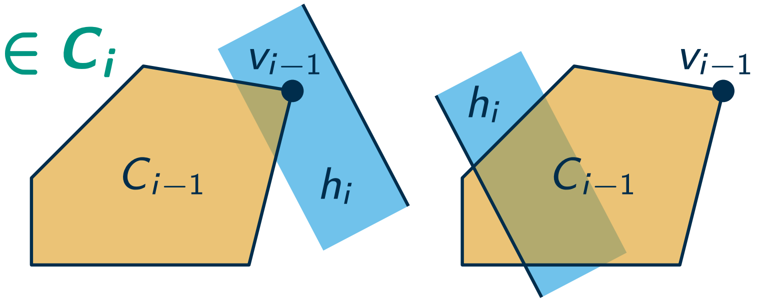

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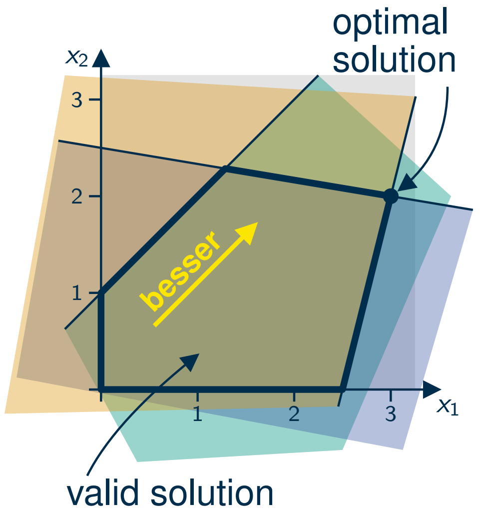

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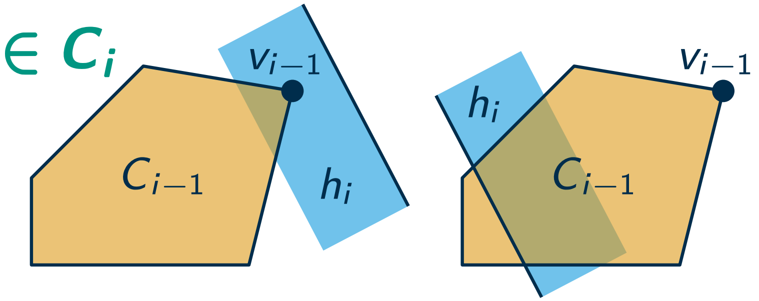

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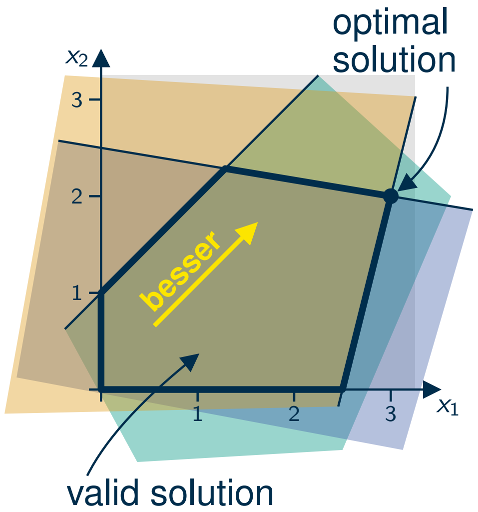

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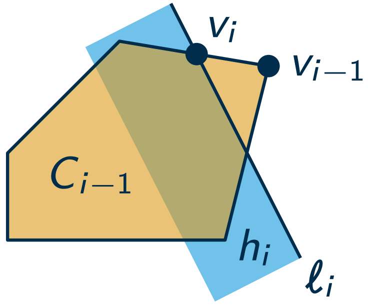

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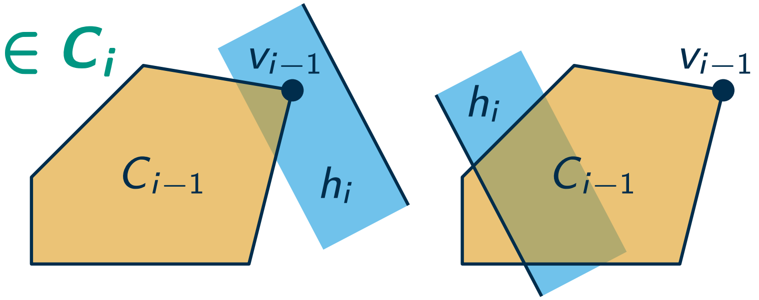

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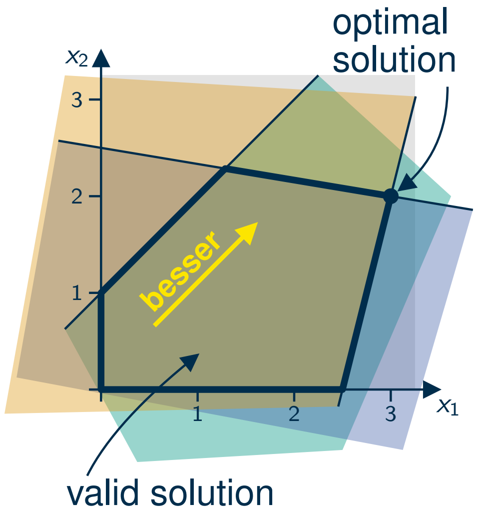

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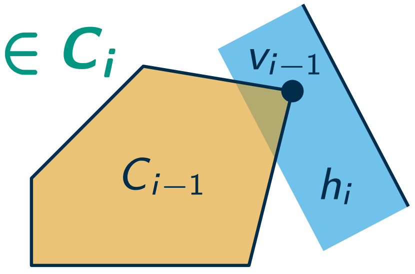

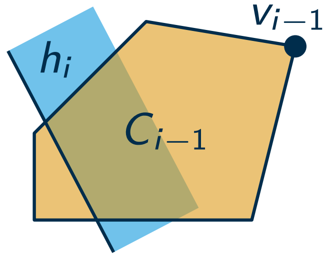

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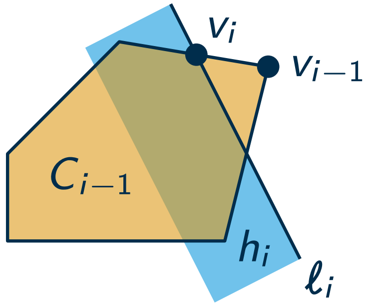

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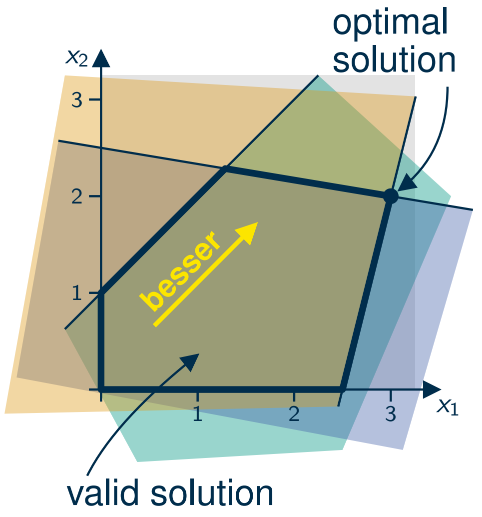
Why?

Incremental Algorithm For 2D LPs



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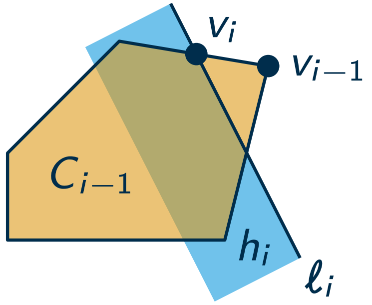

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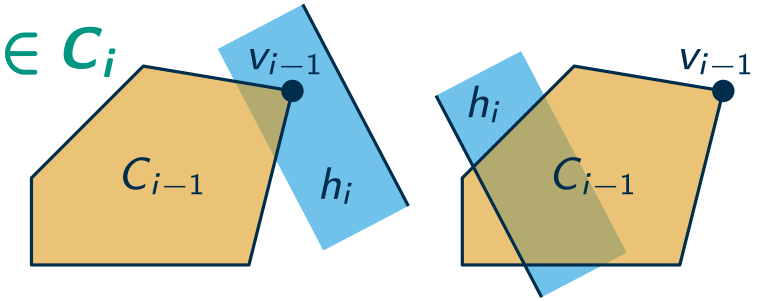

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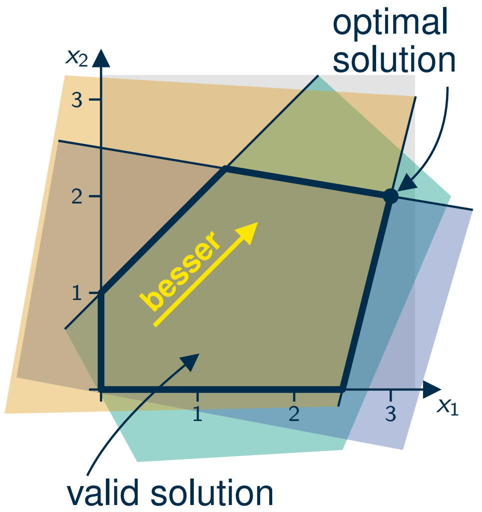

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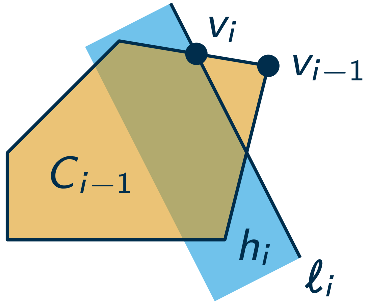

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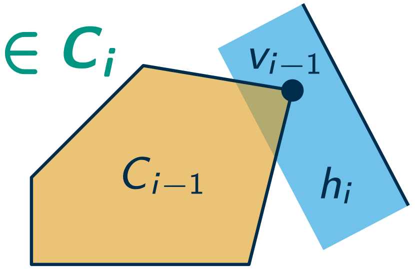

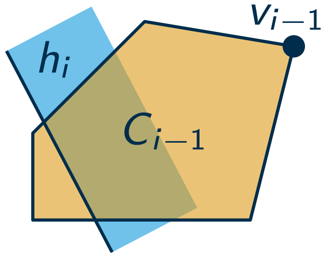

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- v_i lies on the line ℓ_i bounding h_i
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- can be solved in $O(i)$ time

How?

Incremental Algorithm For 2D LPs

Open Questions

- How do we start?
- Why do we care about an $O(n^2)$ algorithm?

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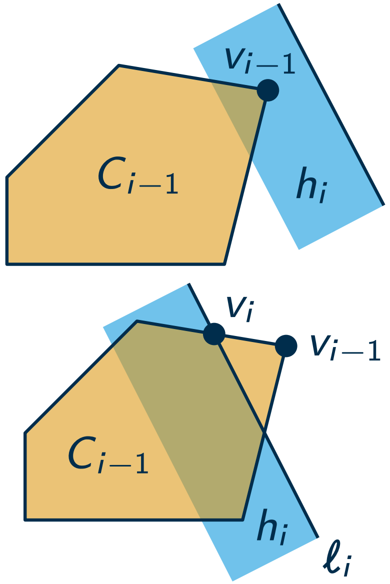

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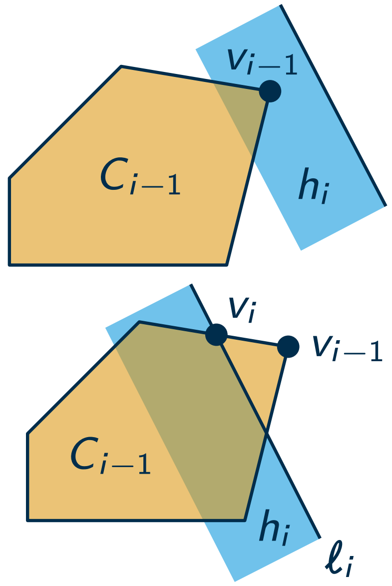

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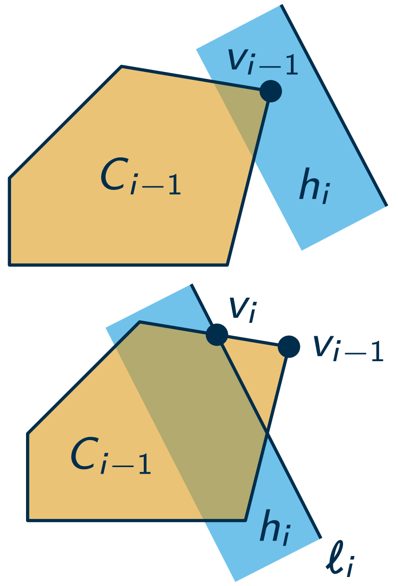

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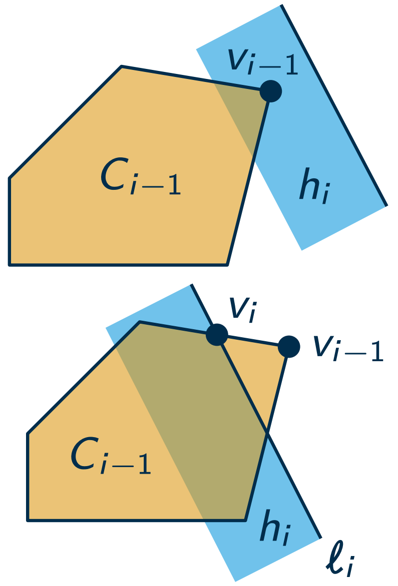

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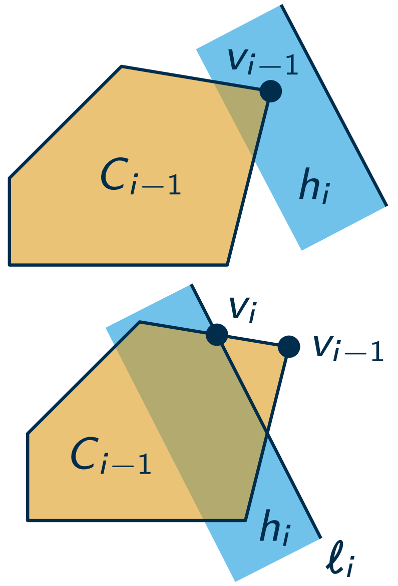

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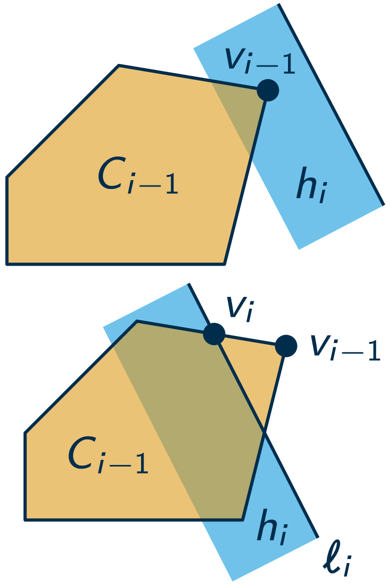

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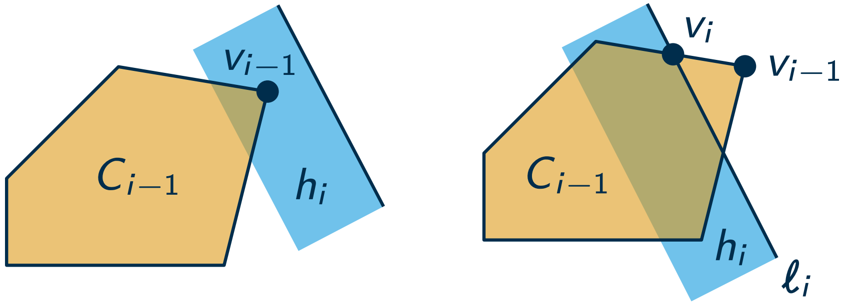

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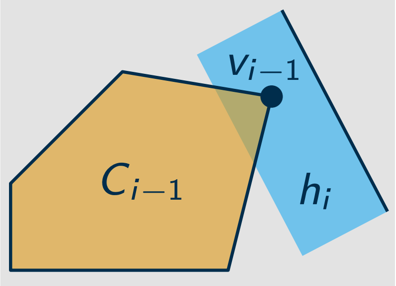

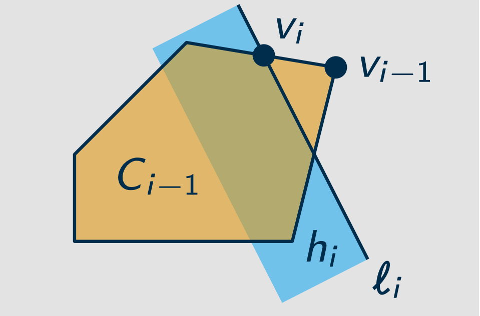

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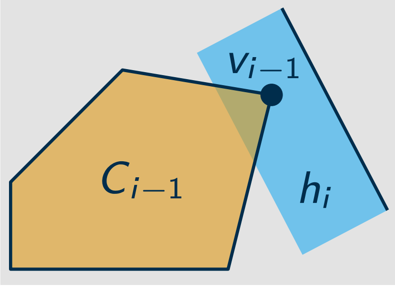

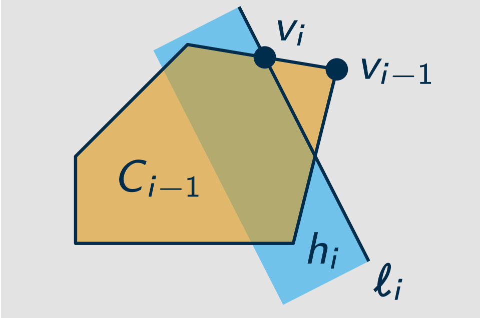

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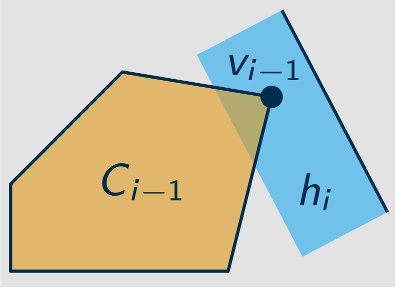

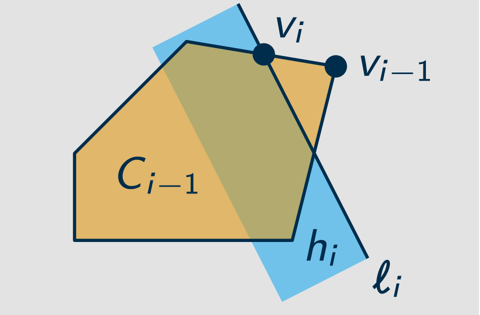

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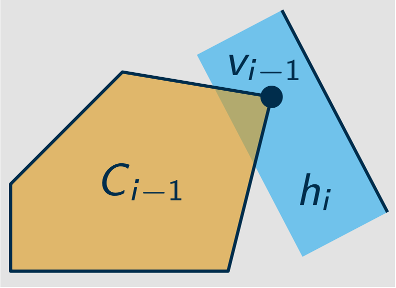

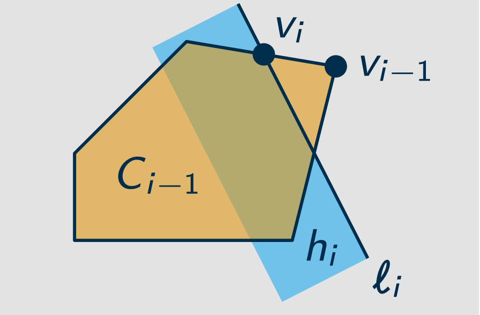

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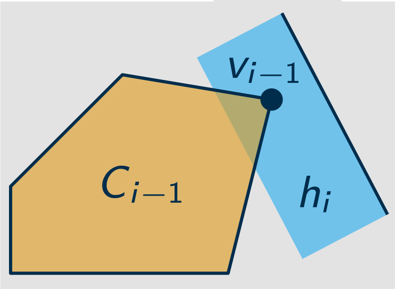

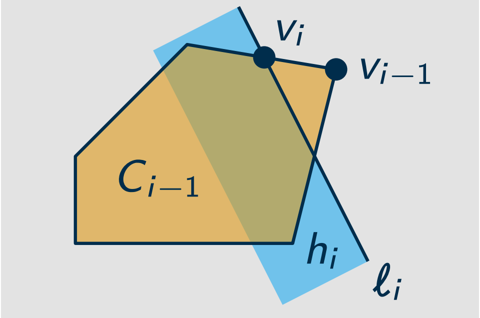

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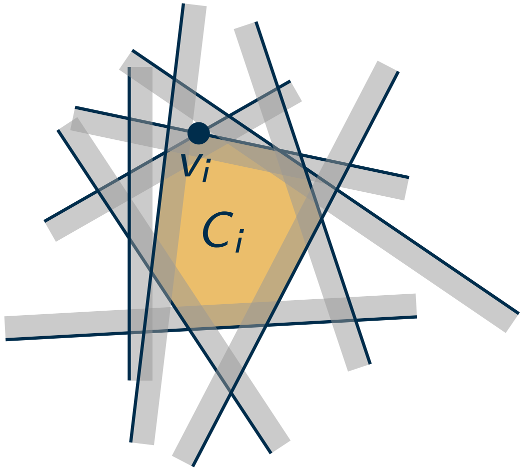

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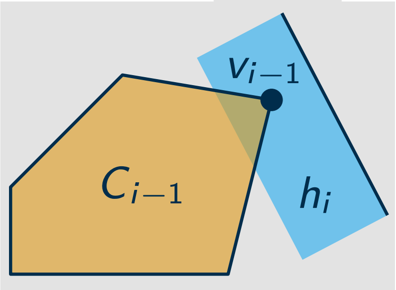

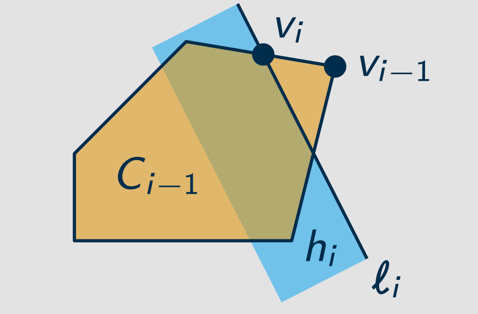

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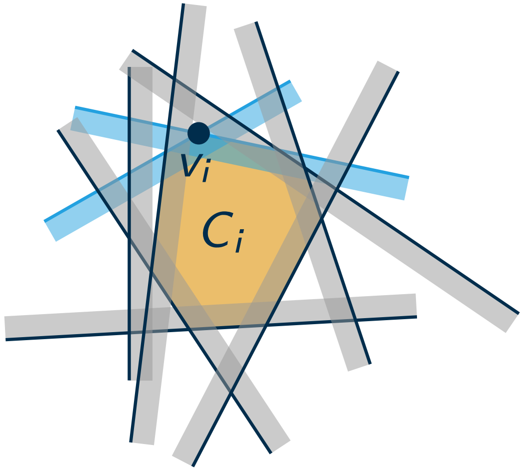

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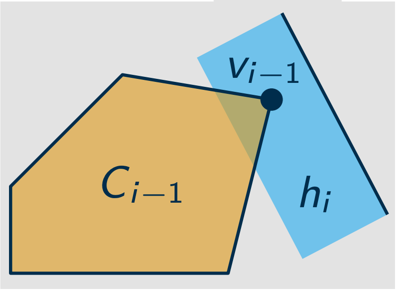

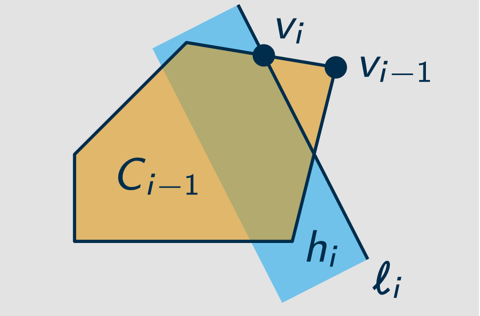

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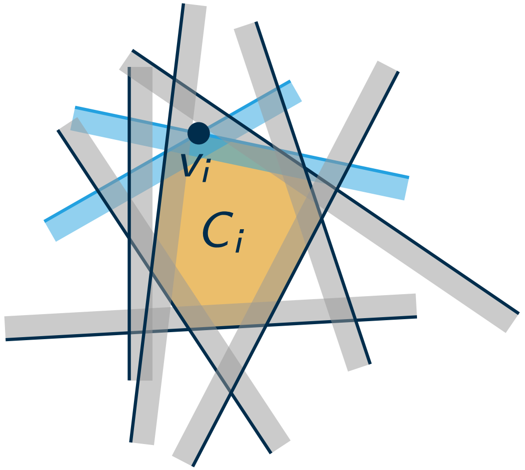

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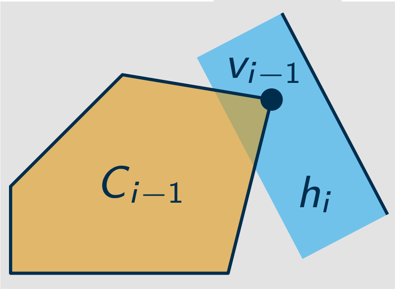

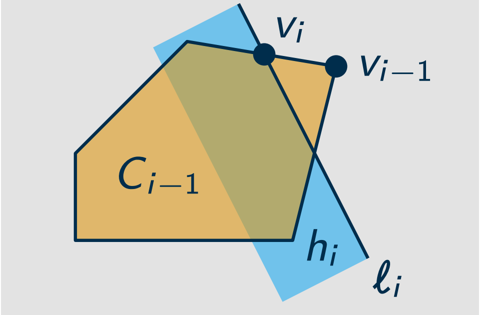

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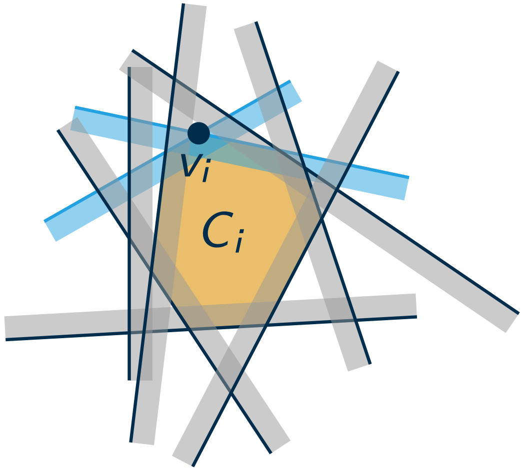

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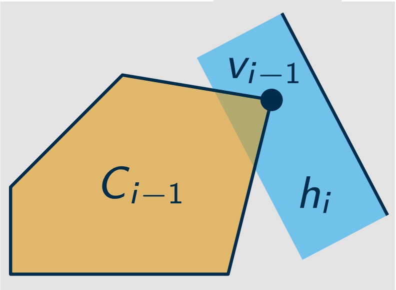

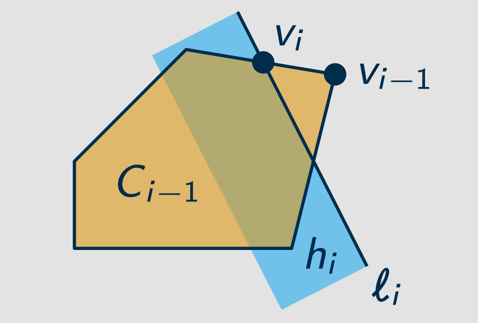

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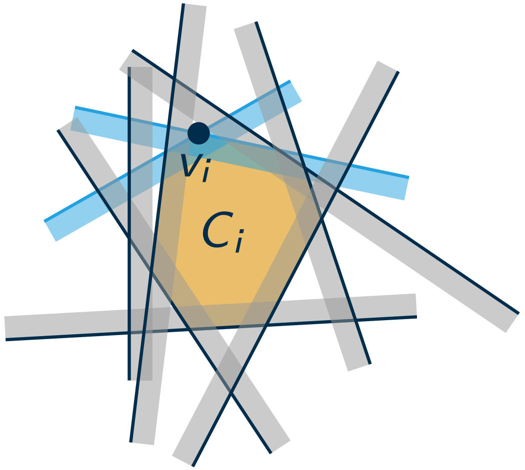

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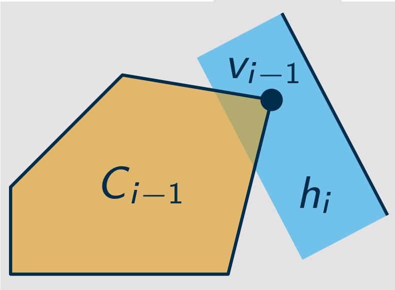

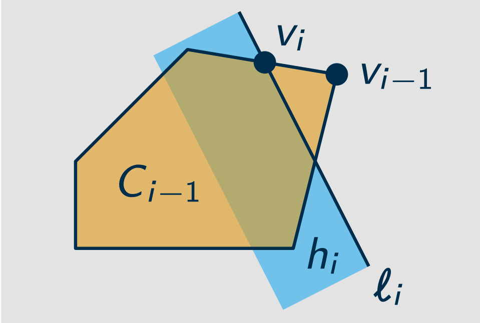

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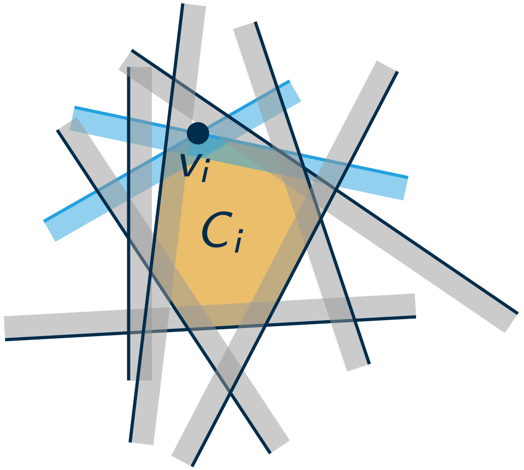

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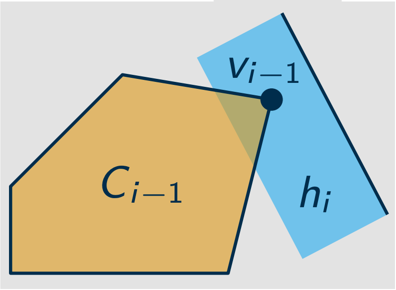

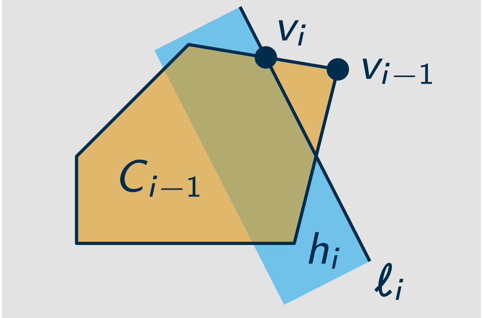

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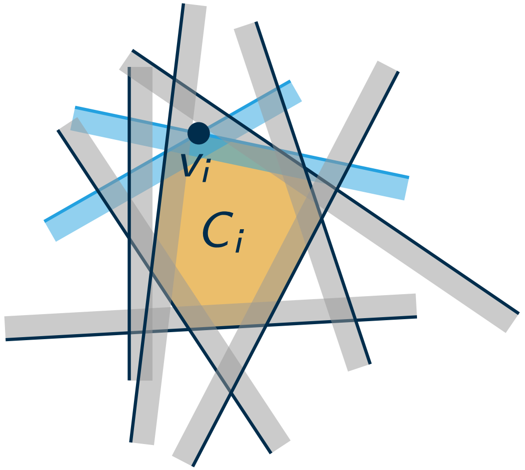

$$\Rightarrow \mathbb{E}[X_i] \in O(1) \Rightarrow \mathbb{E}[X] \in O(n)$$

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What Have We Learned Today?

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What Else Is There?

- the randomized algorithm also works for higher dimensions
  - running time: still $O(n)$ in expectation, if d constant
  - grows super-exponentially in d
- this type of randomization (and analysis) works for other geometric problems
  - prominent example: $O(n \log n)$ algo for convex hull in 3D