Computational Geometry

Polygon Triangulation

Thomas Bläsius

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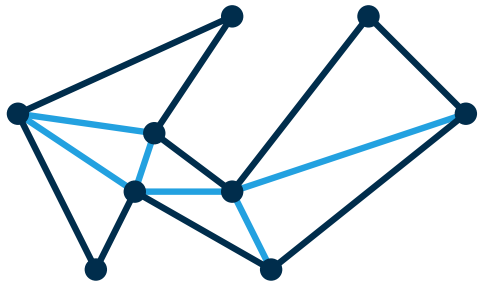

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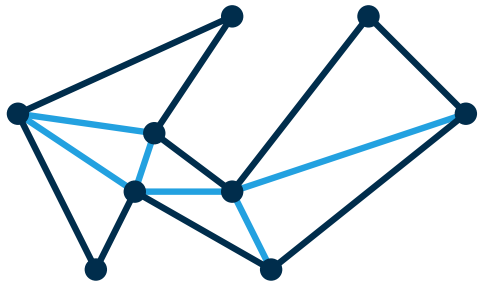

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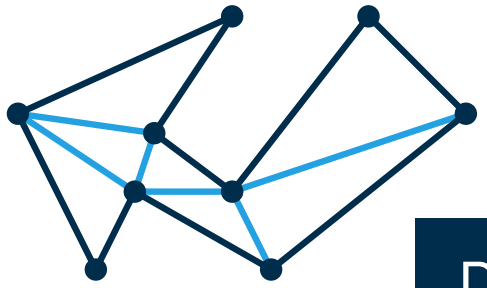

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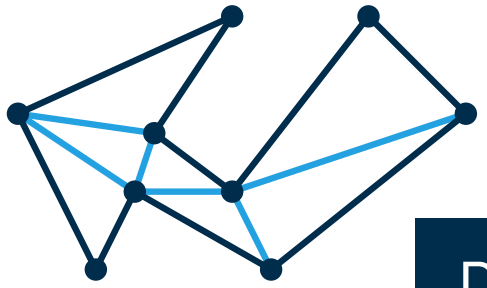

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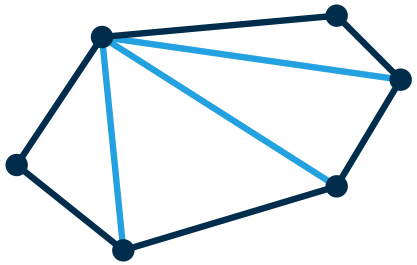

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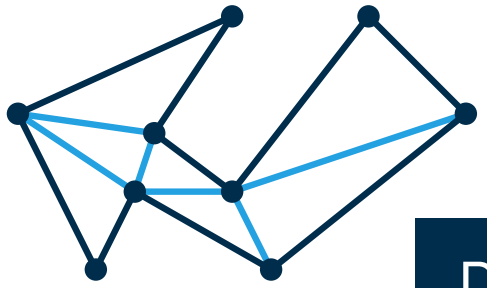

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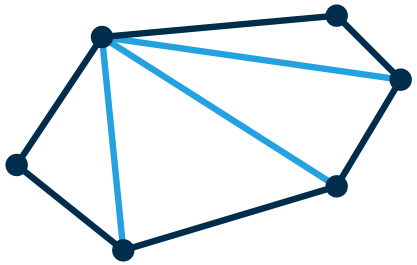

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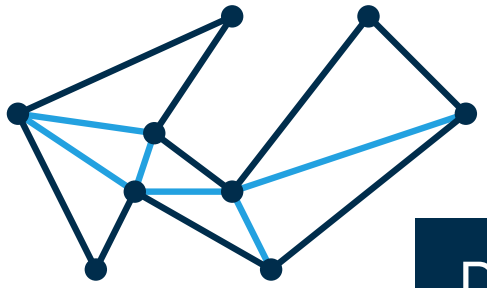

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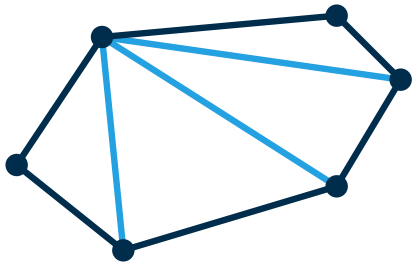

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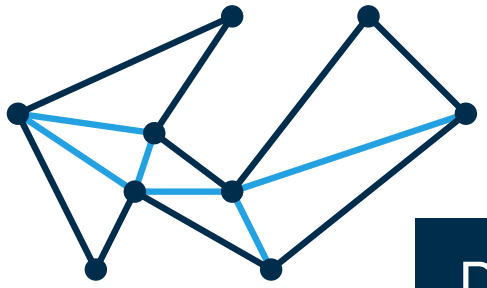

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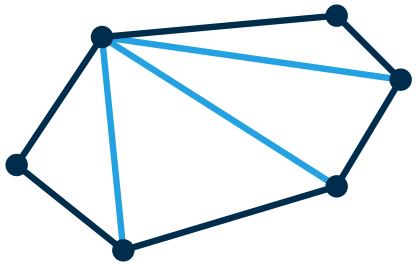

Our Plan

- find a weaker condition than convexity

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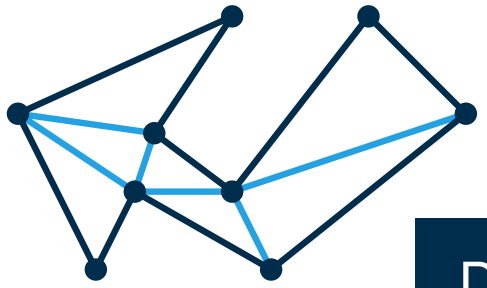

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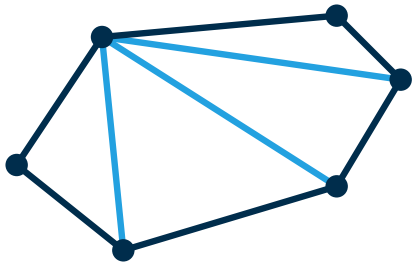


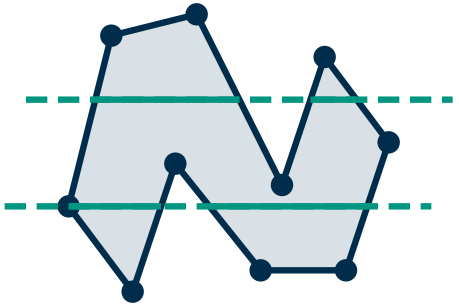
Our Plan

- find a weaker condition than convexity
- subdividing P into pieces with this property becomes easier
- triangulating the pieces becomes more difficult

y-Monotone Polygons

Definition
A polygon is ***y*-monotone** if the intersection with every horizontal line is connected.

not *y*-monotone



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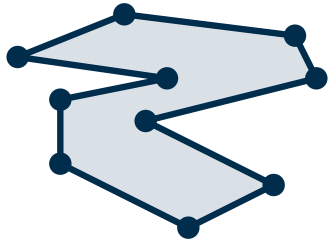

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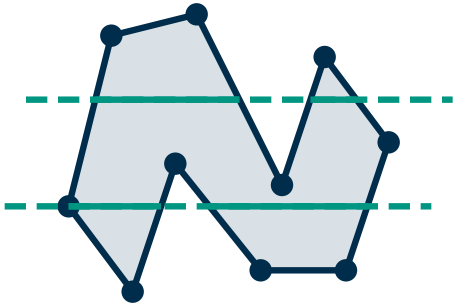

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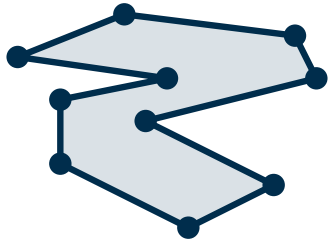

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- convex polygons are monotone in every direction

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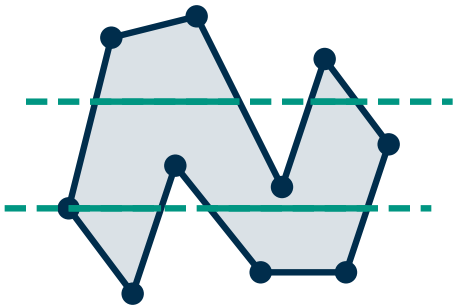

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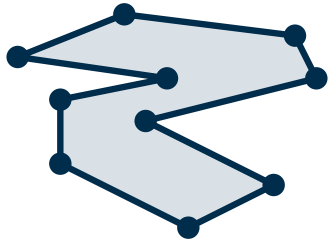

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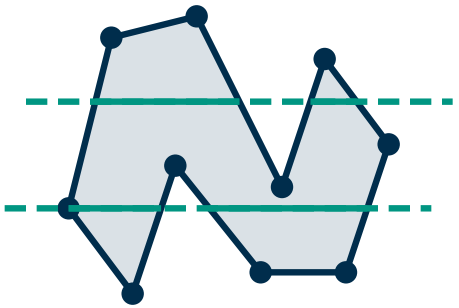

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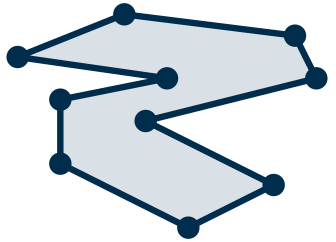

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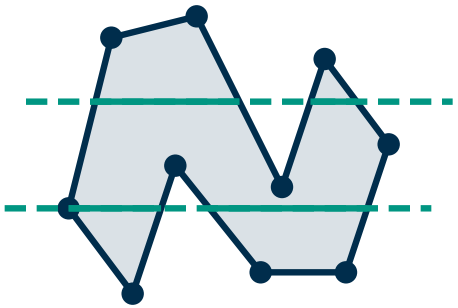

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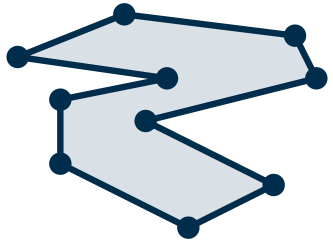

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- triangulate a y-monotone polygon in $O(n)$ time $\rightarrow$ exercise sheet

What Makes A Polygon Not y-Monotone?

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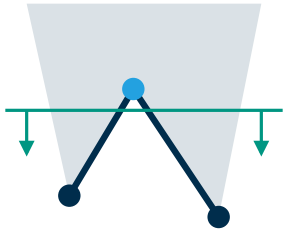
Split Vertex



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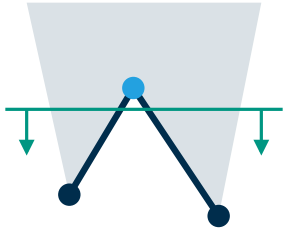

Merge Vertex

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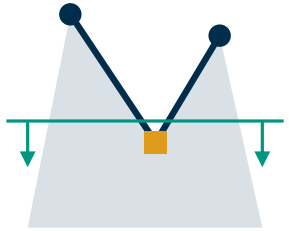

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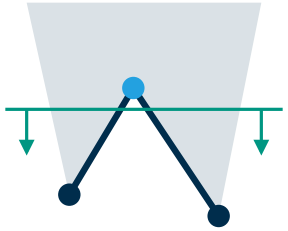

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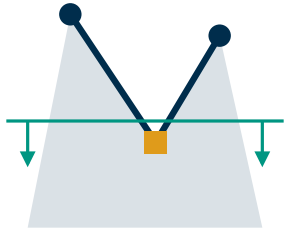

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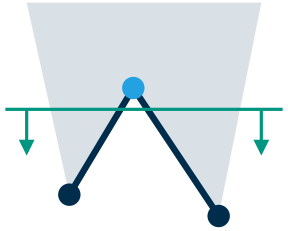

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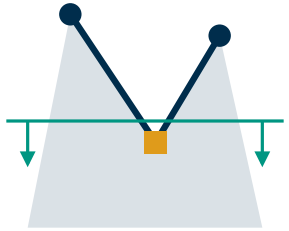
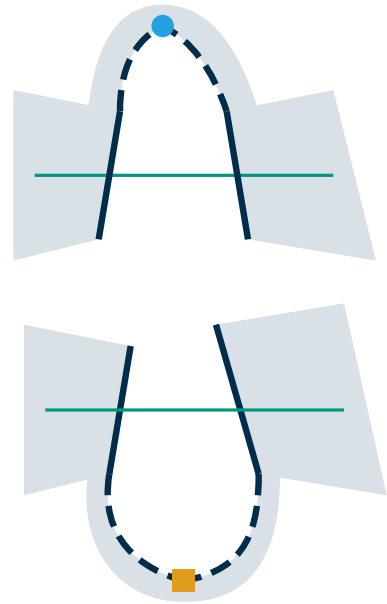


Observation

- a merge or split vertex exists $\Rightarrow$ the polygon is not y-monotone

Lemma **(y-monotonicity)**
A polygon is y-monotone if and only if it has no split or merge vertex.

proof by picture



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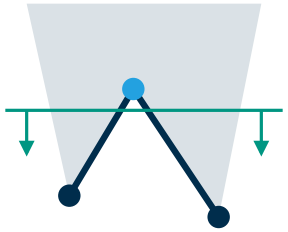

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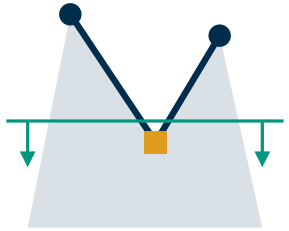

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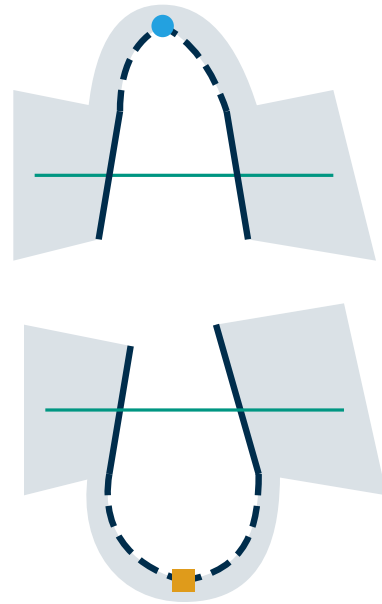

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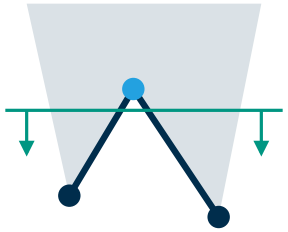

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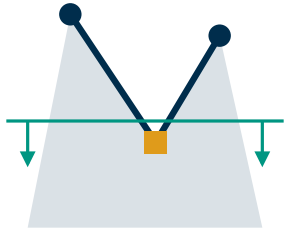

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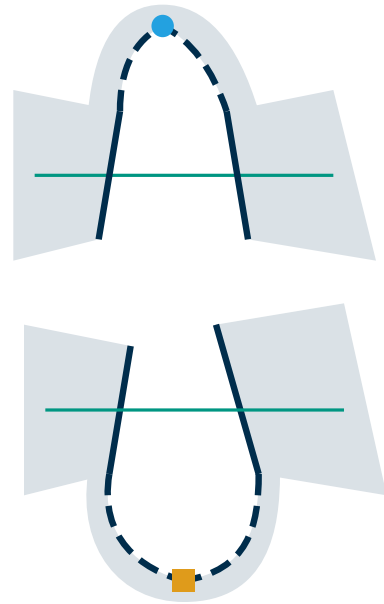

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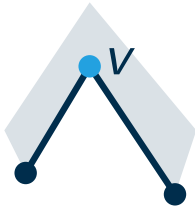
- eliminate all split and merge vertices by inserting diagonals
- upwards for split vertices and downwards for merge vertices

Eliminating Split Vertices

Idea For Split Vertex *v*

- idea: connect *v* to vertex *u* that is above *v* and close to *v*



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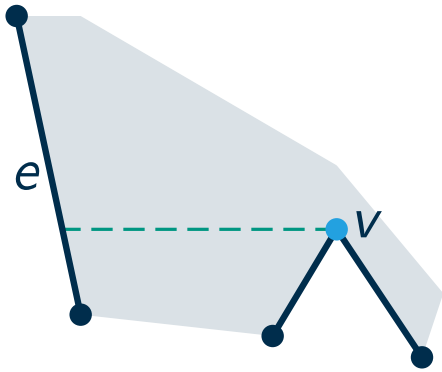

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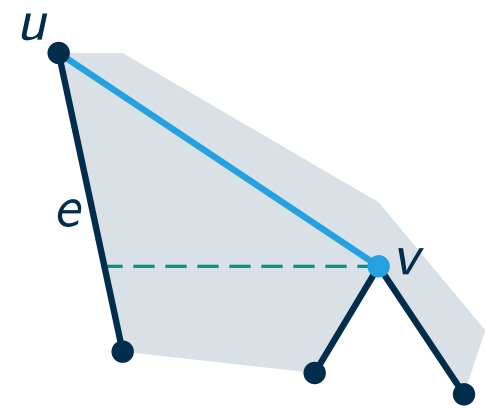

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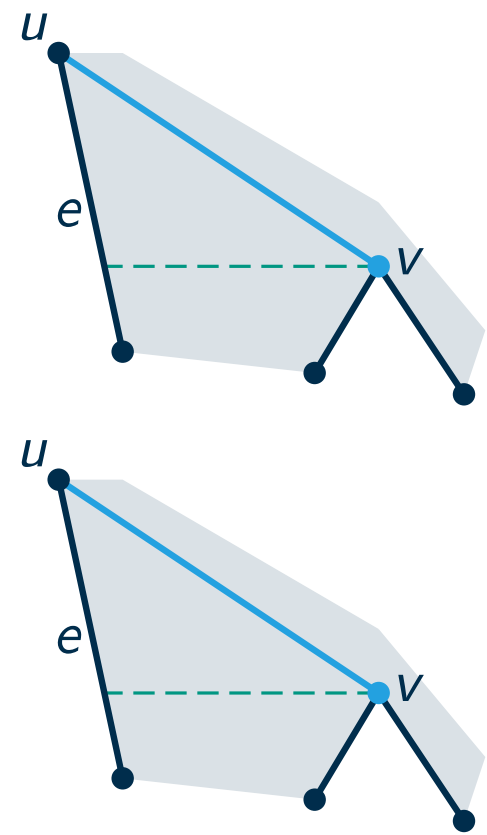

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Issue (And How To Fix It)

Eliminating Split Vertices

Idea For Split Vertex v

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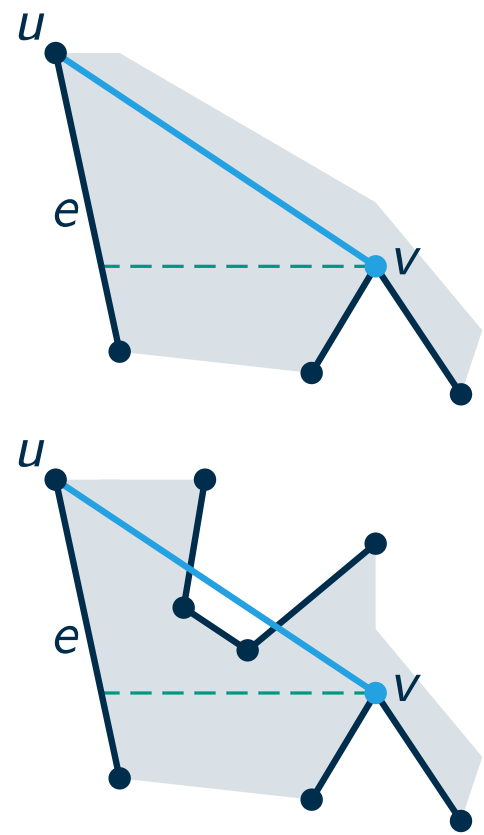
Issue (And How To Fix It)

- uv might intersect another edge of the polygon



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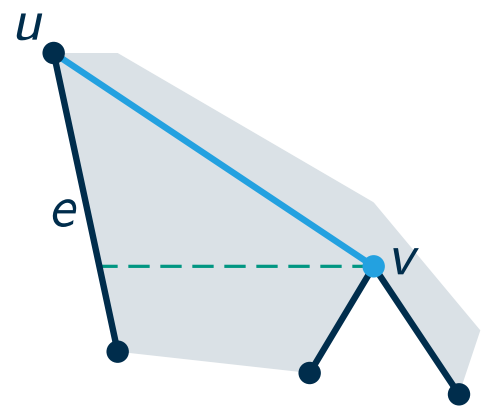

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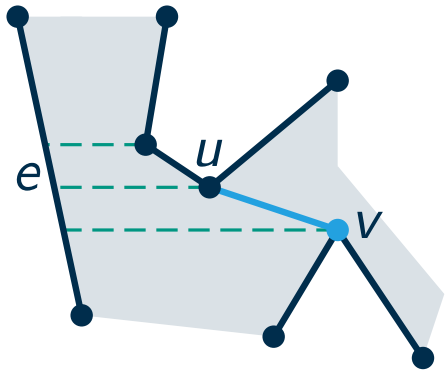

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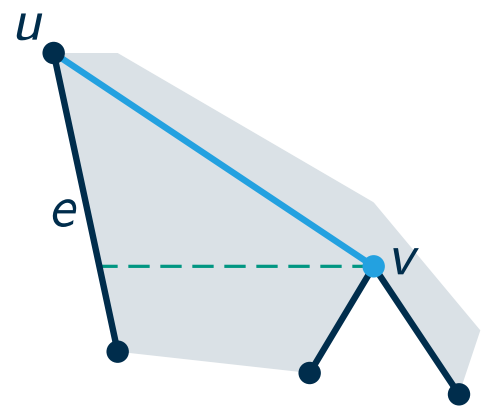
- *uv* might intersect another edge of the polygon
- fix: choose for *u* the lowest vertex above *v* such that *e* is to the left of *u*

Eliminating Split Vertices



Idea For Split Vertex *v*

- idea: connect *v* to vertex *u* that is above *v* and close to *v*
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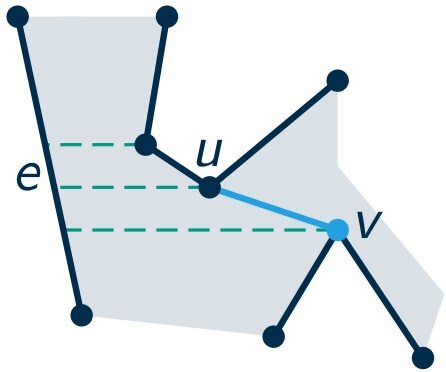
- choose *u* to be the upper vertex of *e*



Issue (And How To Fix It)

- *uv* might intersect another edge of the polygon
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- we call *u* the **helper** of *e* (note: it depends on *v*)

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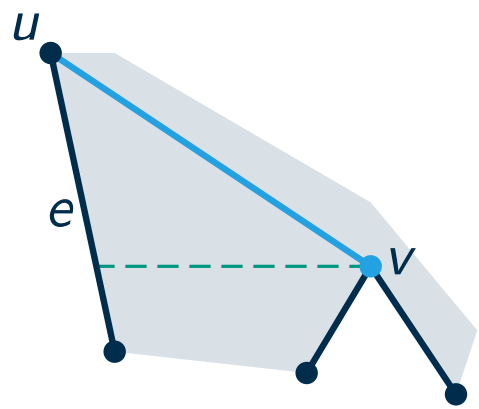

Idea For Split Vertex v

- idea: connect v to vertex u that is above v and close to v
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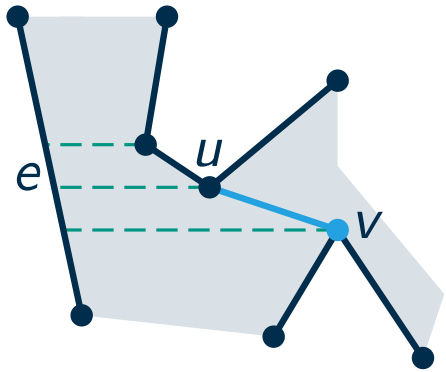
- choose u to be the upper vertex of e



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- uv might intersect another edge of the polygon
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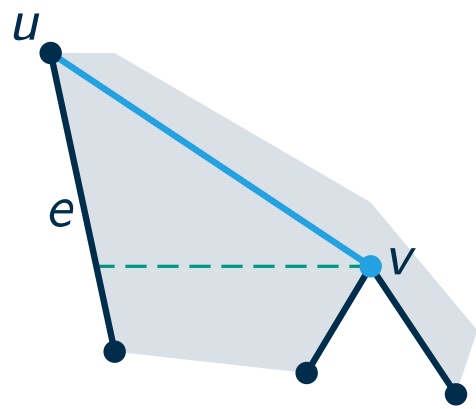
Lemma **(the helper is helpful)**
Let v be a split vertex, e the edge left of v, and u the helper of e (wrt v). Then uv does not intersect an edge of the polygon (except in u and v).

Eliminating Split Vertices



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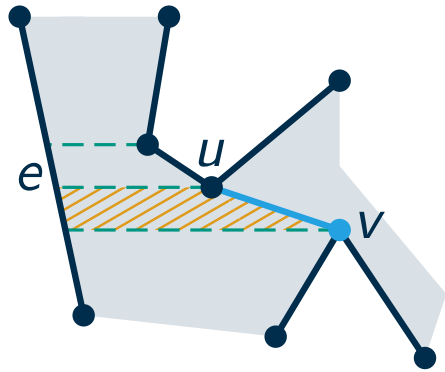

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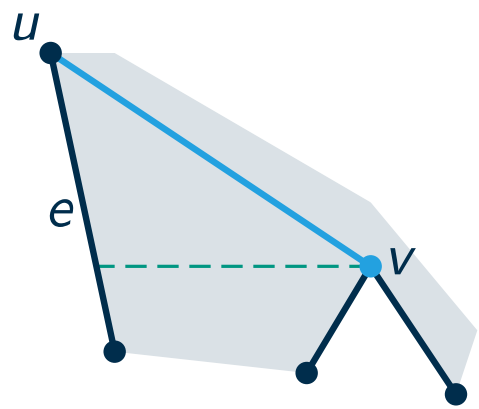
Proof

- the quadrilateral between *uv* and *e* contains no vertex

Why?

Eliminating Split Vertices



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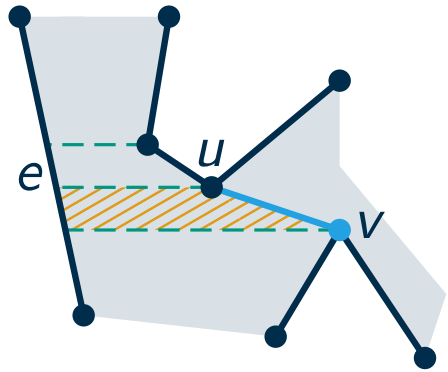

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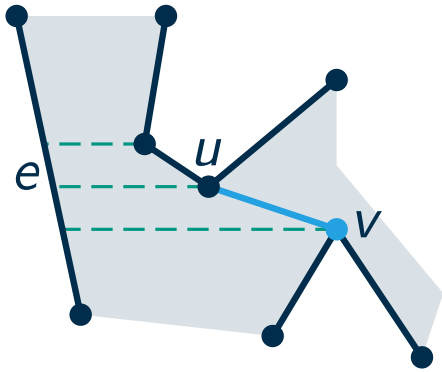

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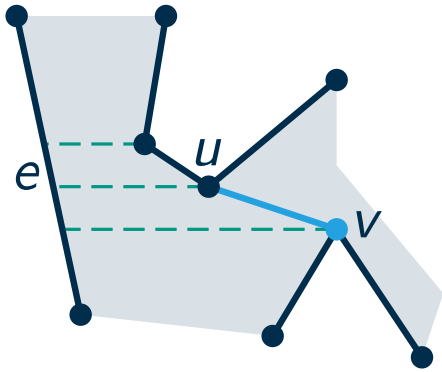

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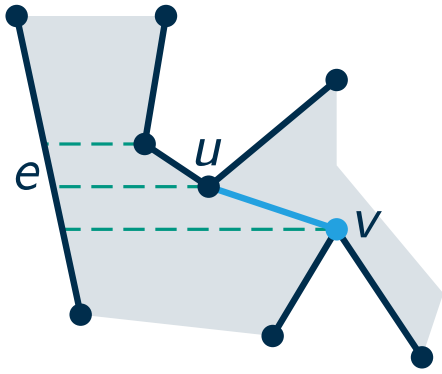

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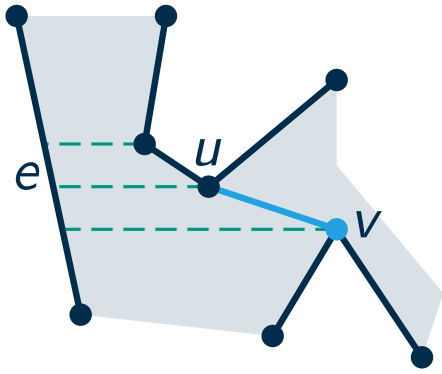

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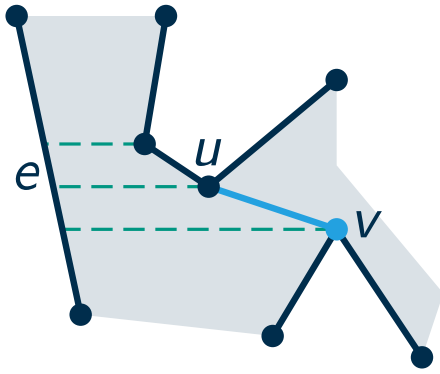

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Event Queue

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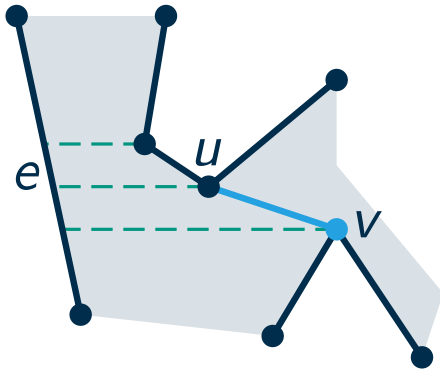

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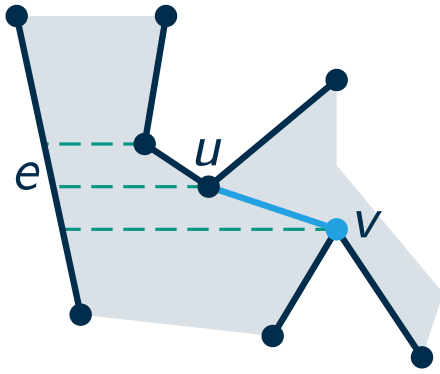
Eliminating Split Vertices



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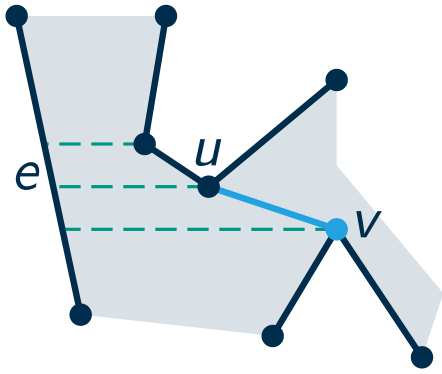
Eliminating Split Vertices



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Sweep Line Status

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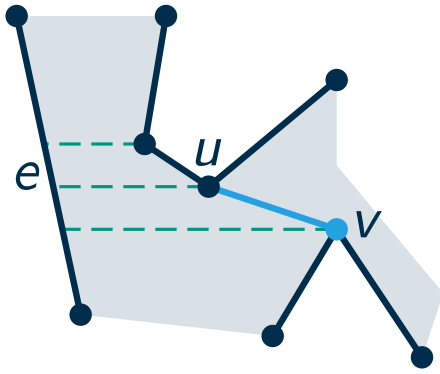
- edges that have the polygon to their right suffice

Eliminating Split Vertices



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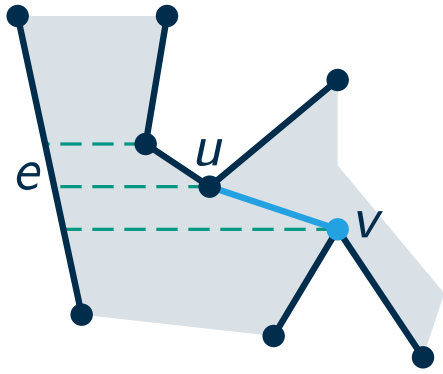
Eliminating Split Vertices



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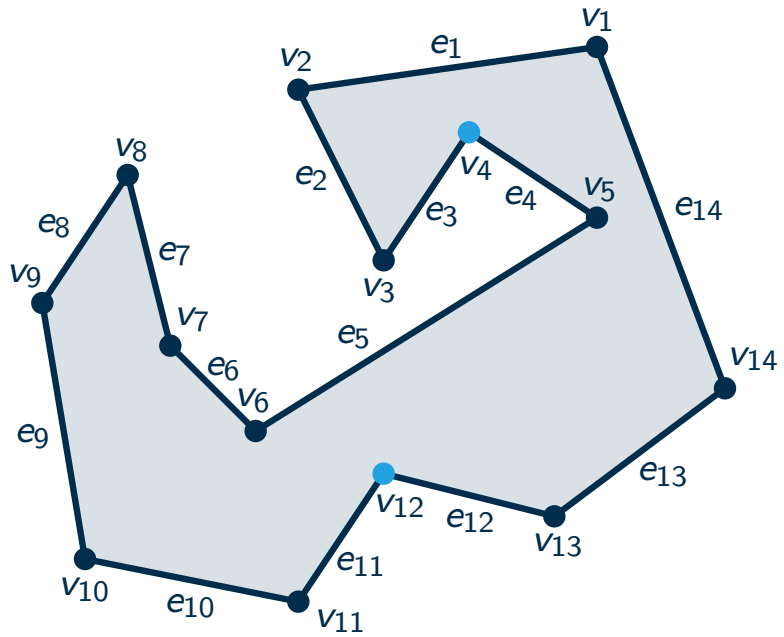


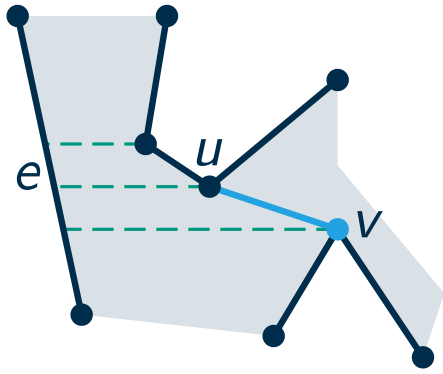
Queue

v_1 v_2 v_4 v_8 v_5 v_3 v_9 v_7 v_{14} v_6 v_{12} v_{13} v_{10} v_{11}

Status

Helper

Eliminating Split Vertices



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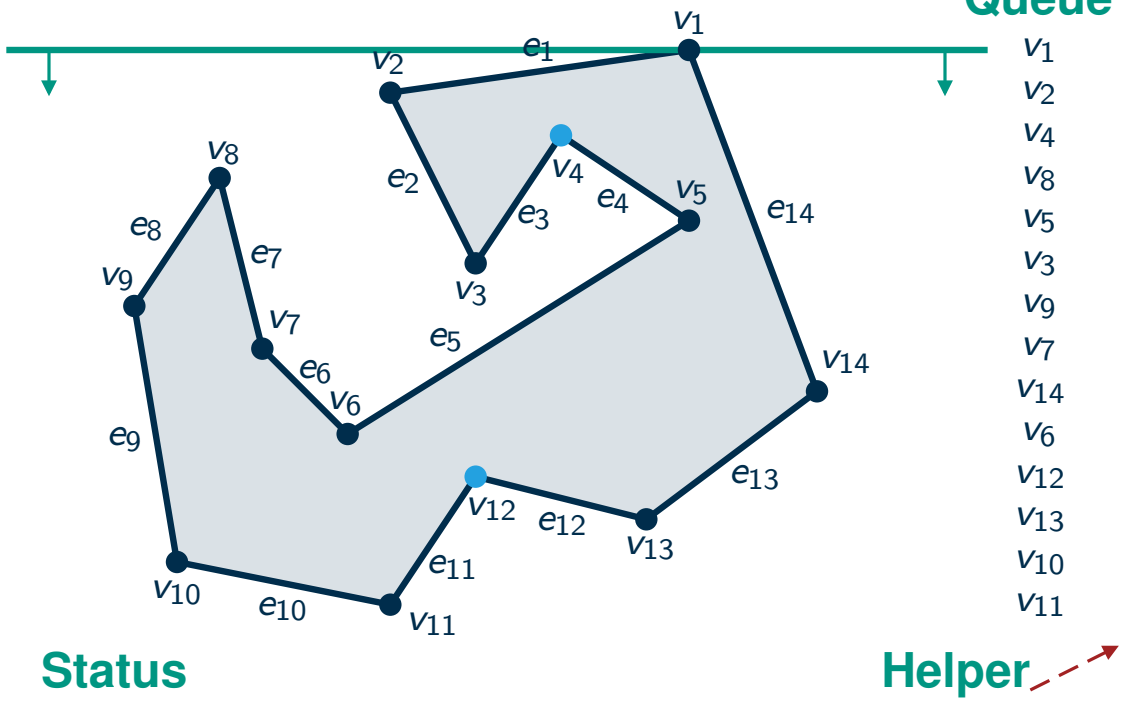

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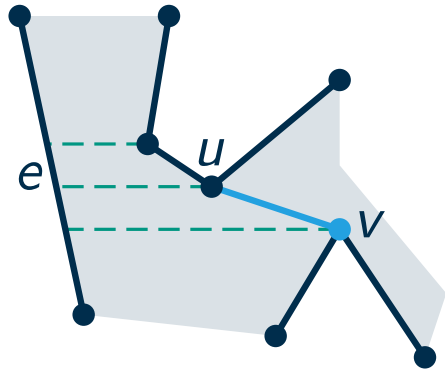

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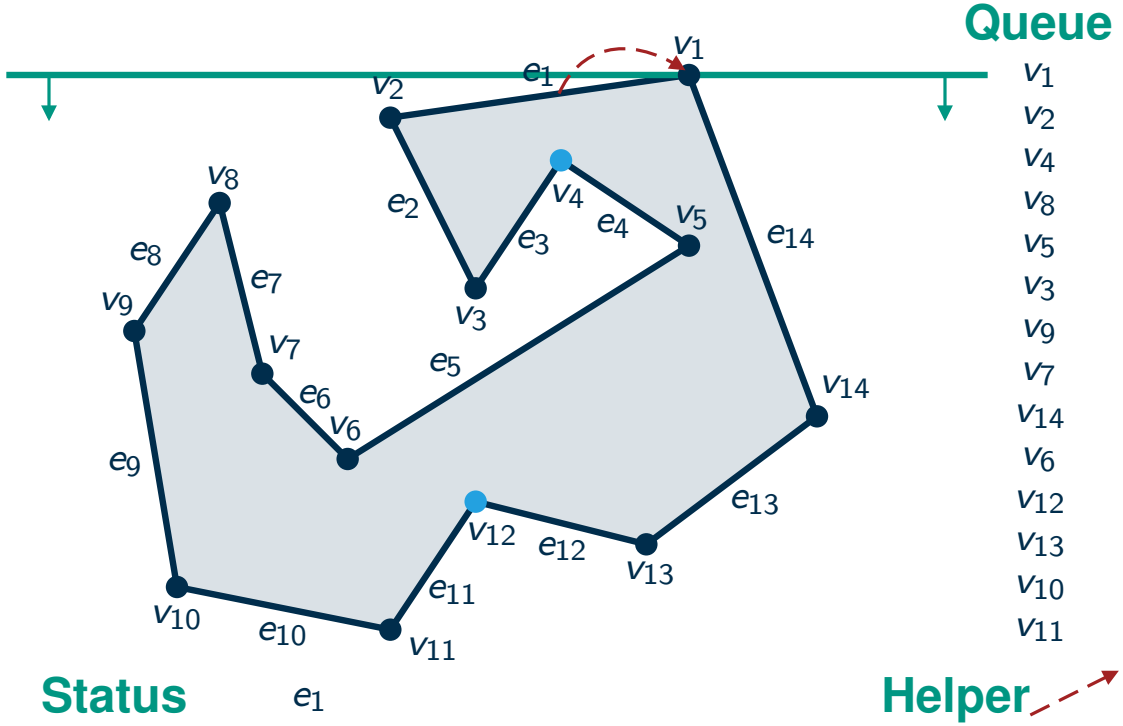

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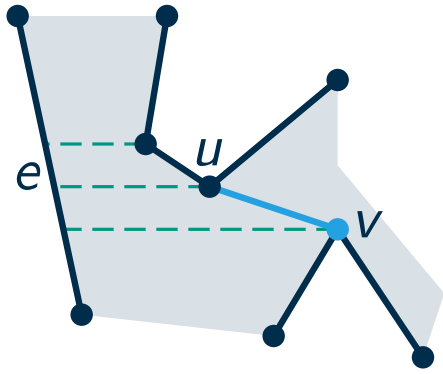

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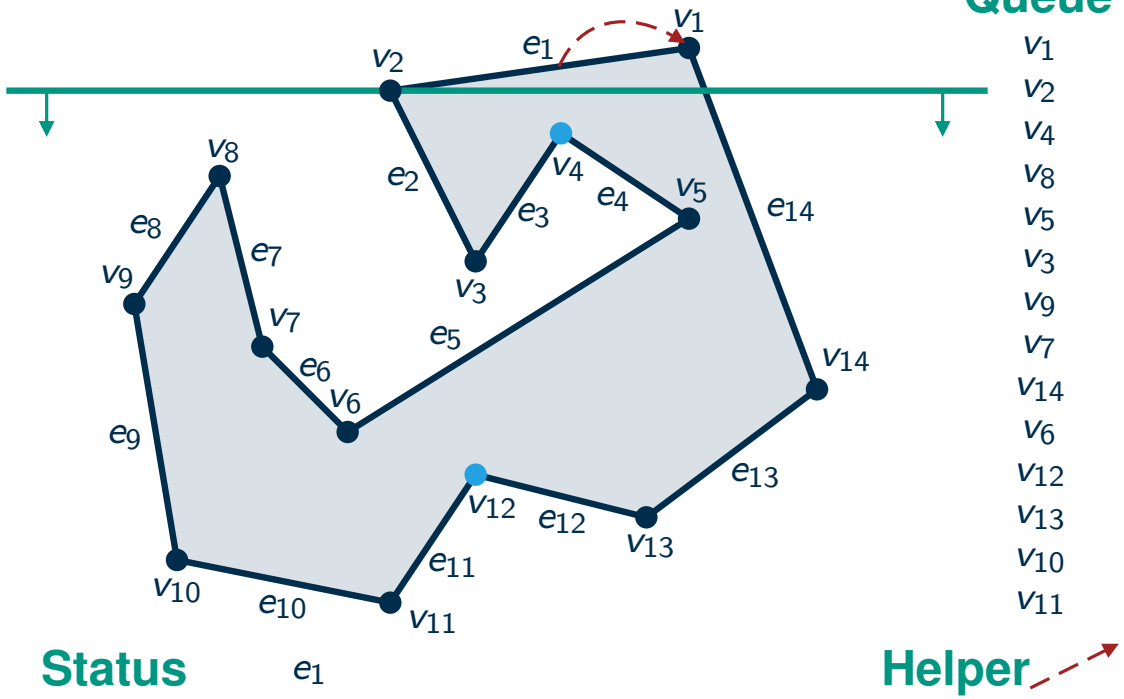

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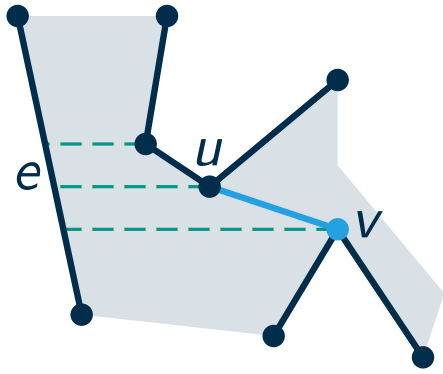

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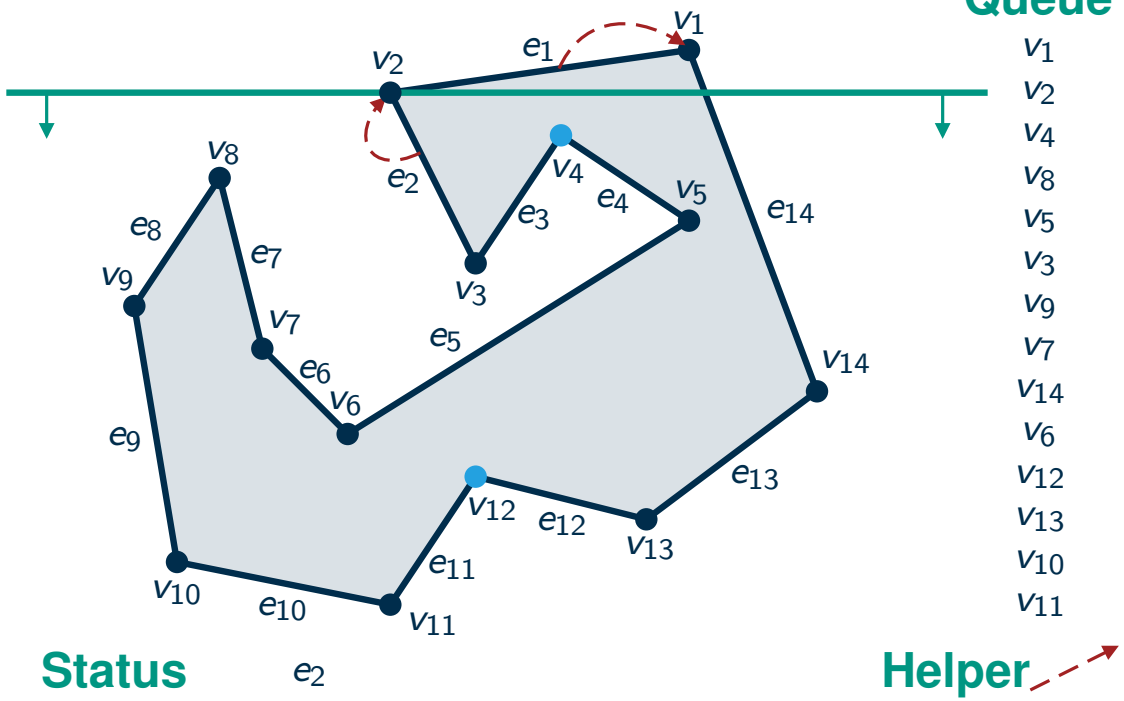

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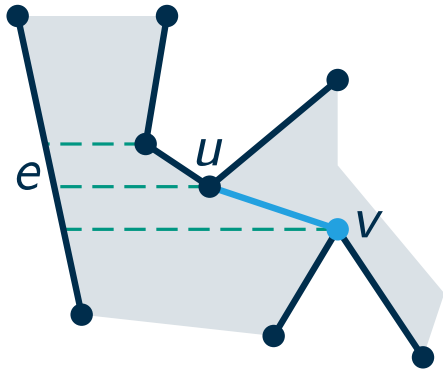

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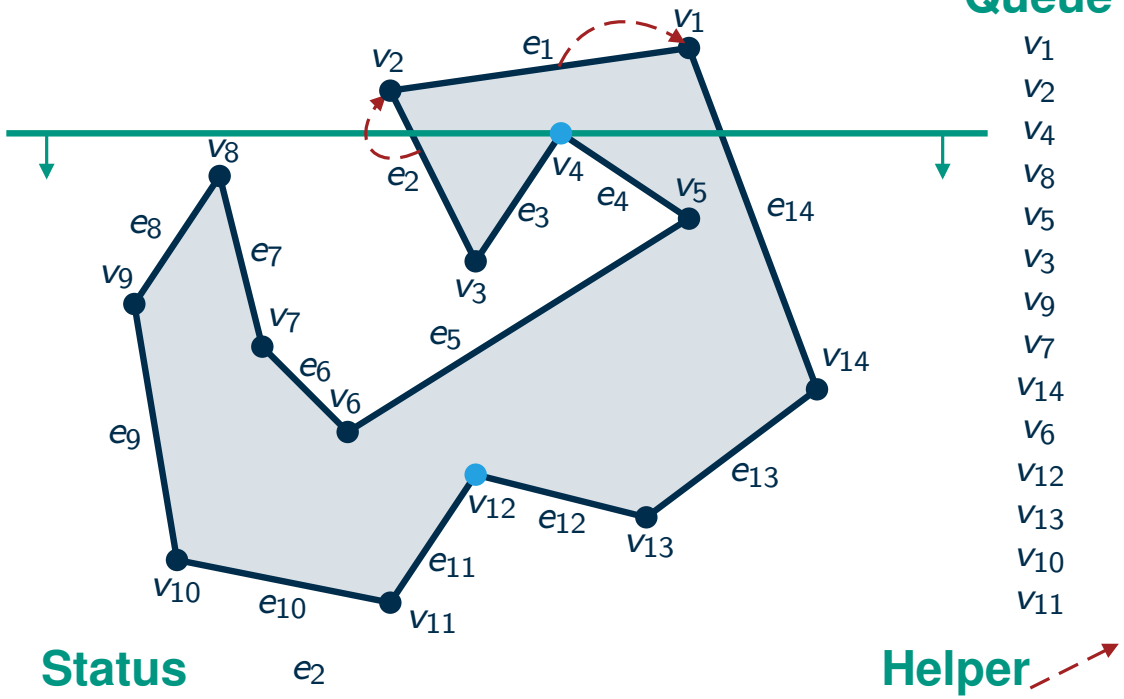

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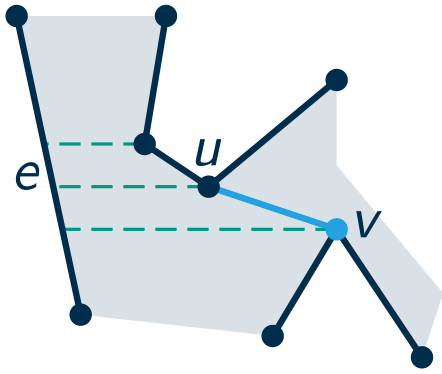

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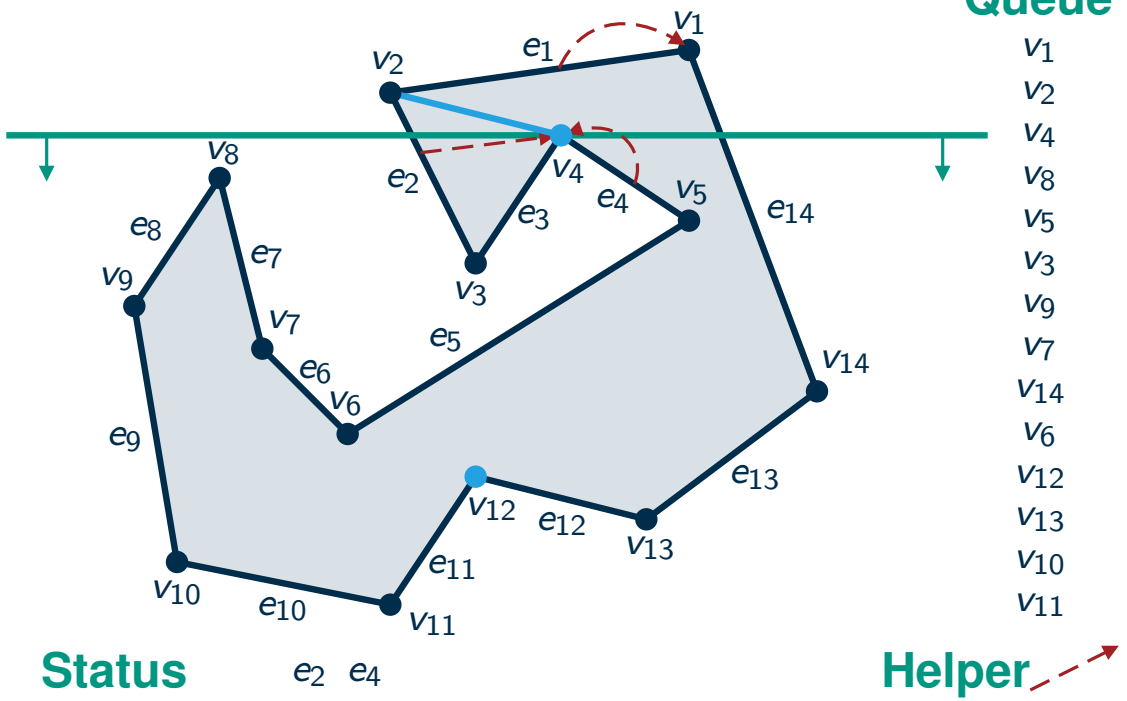

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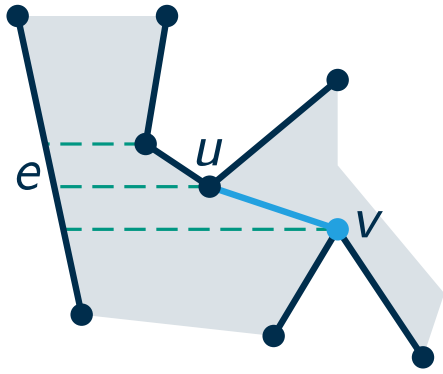

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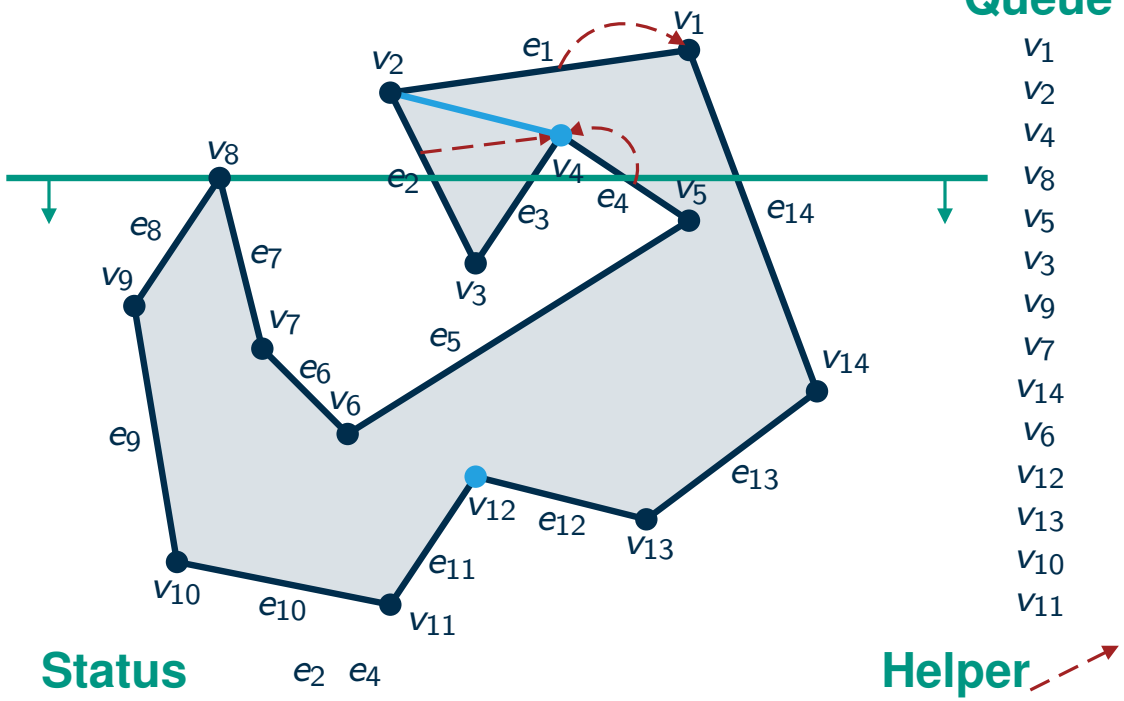
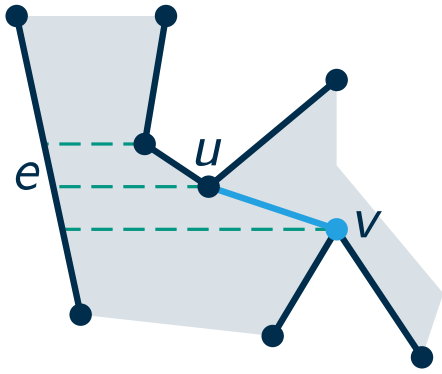


Eliminating Split Vertices



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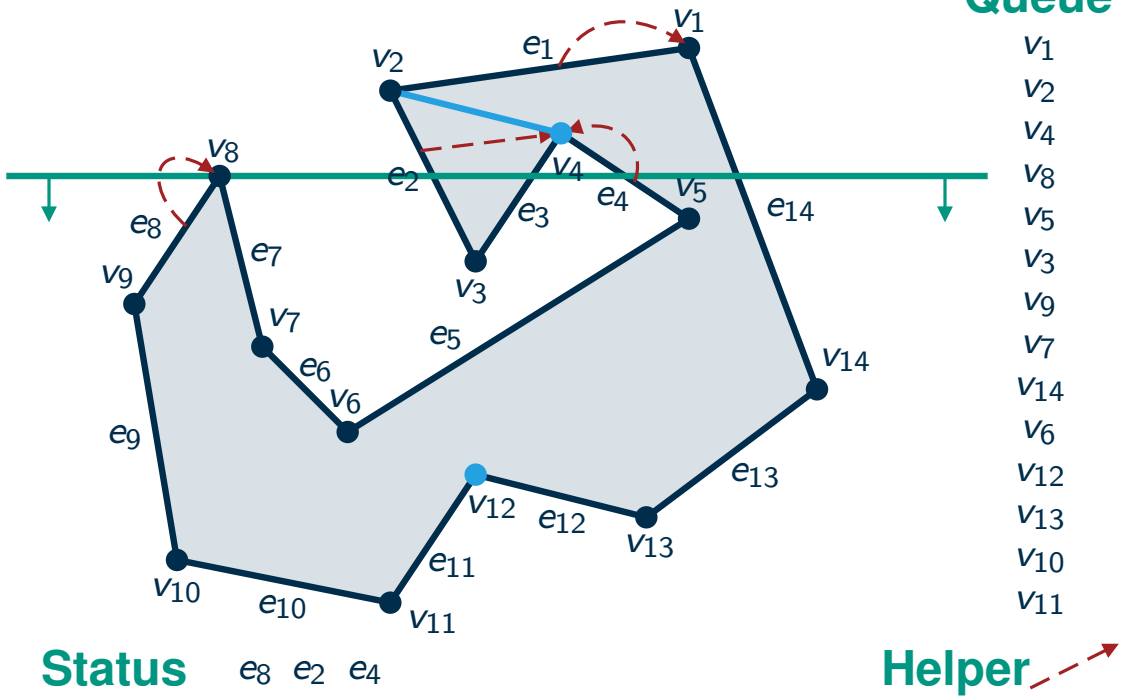

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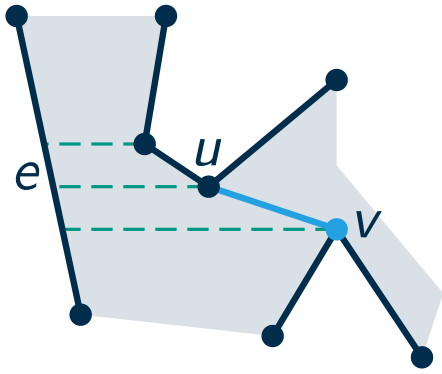

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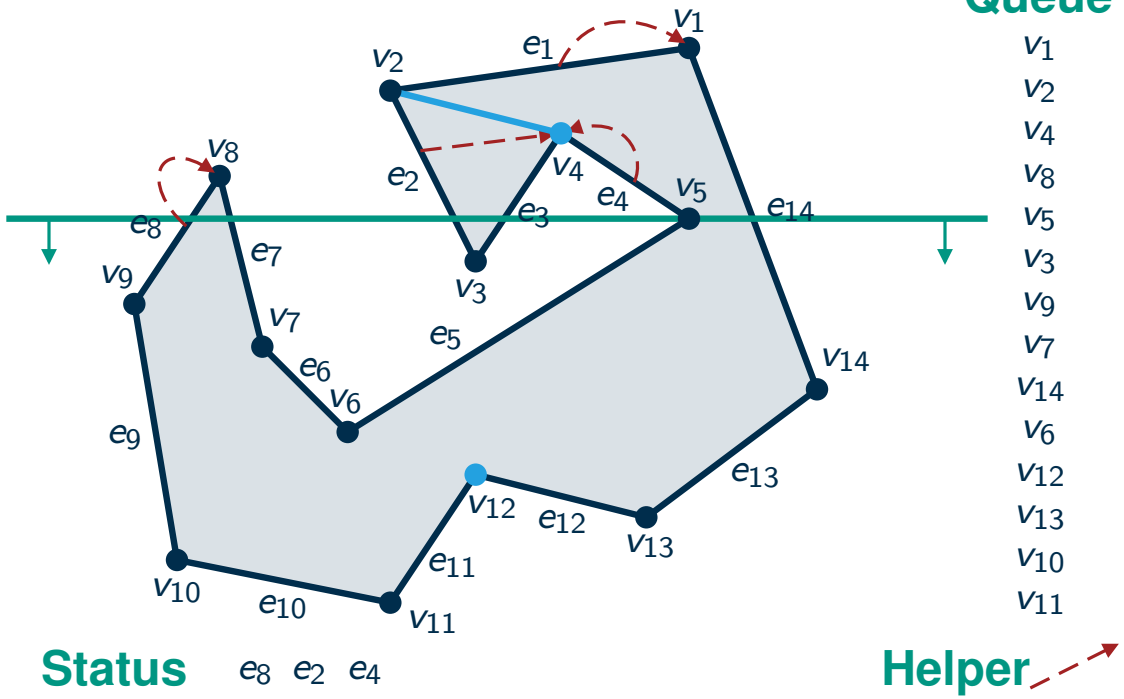

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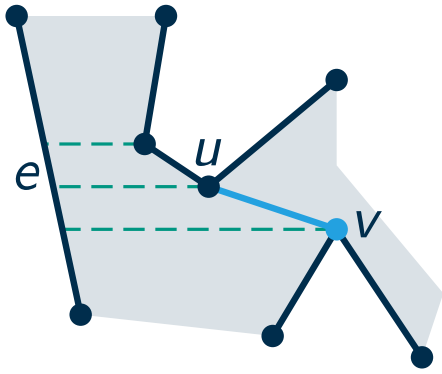

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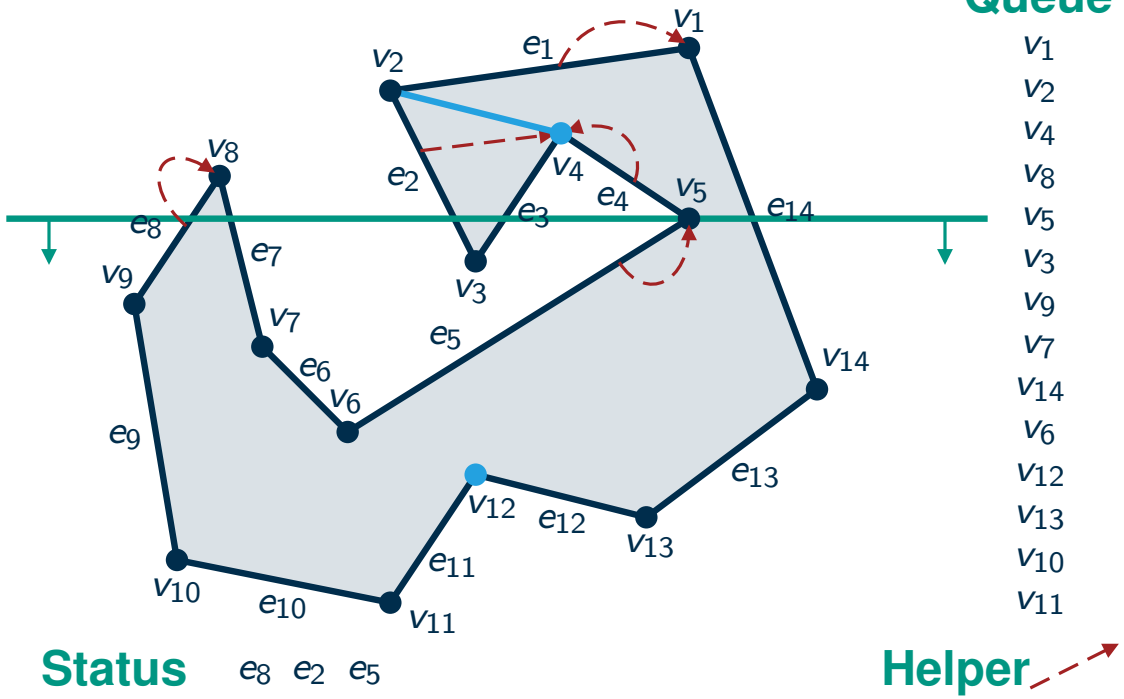

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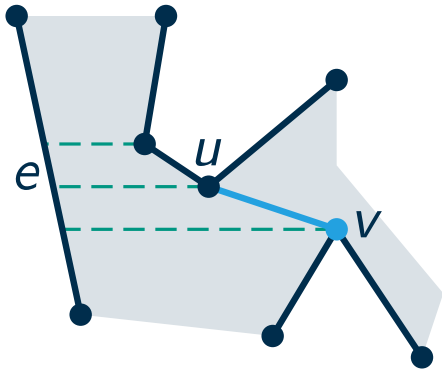

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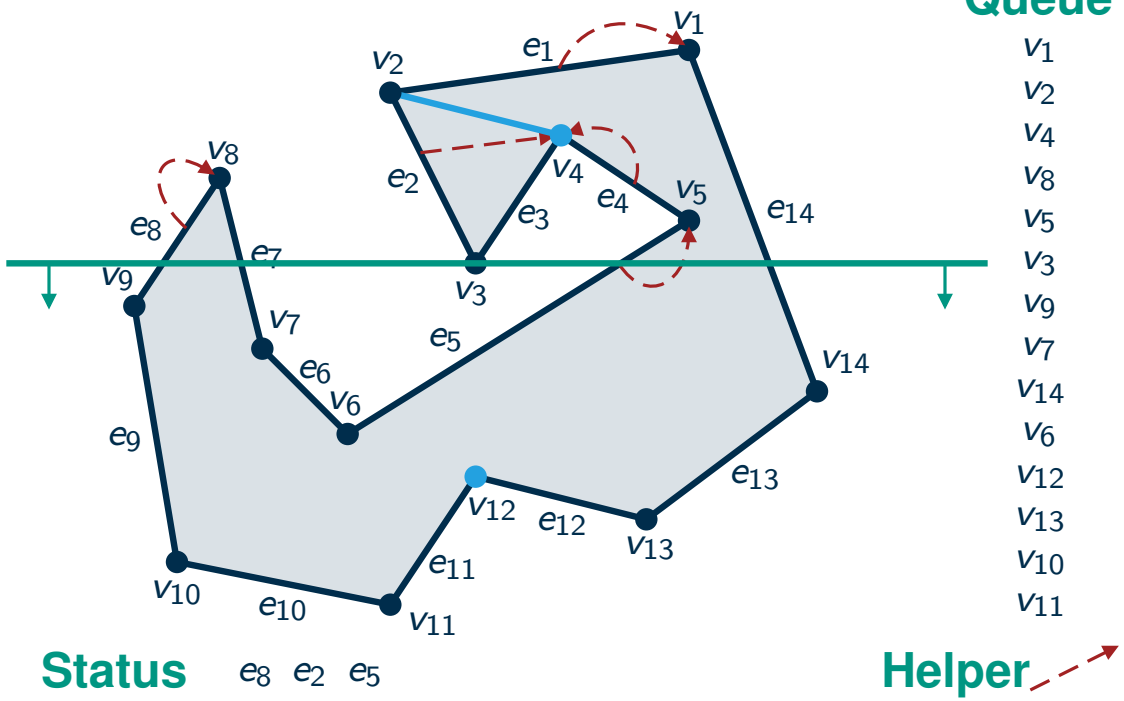

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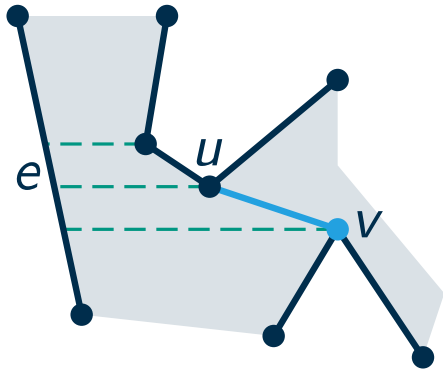

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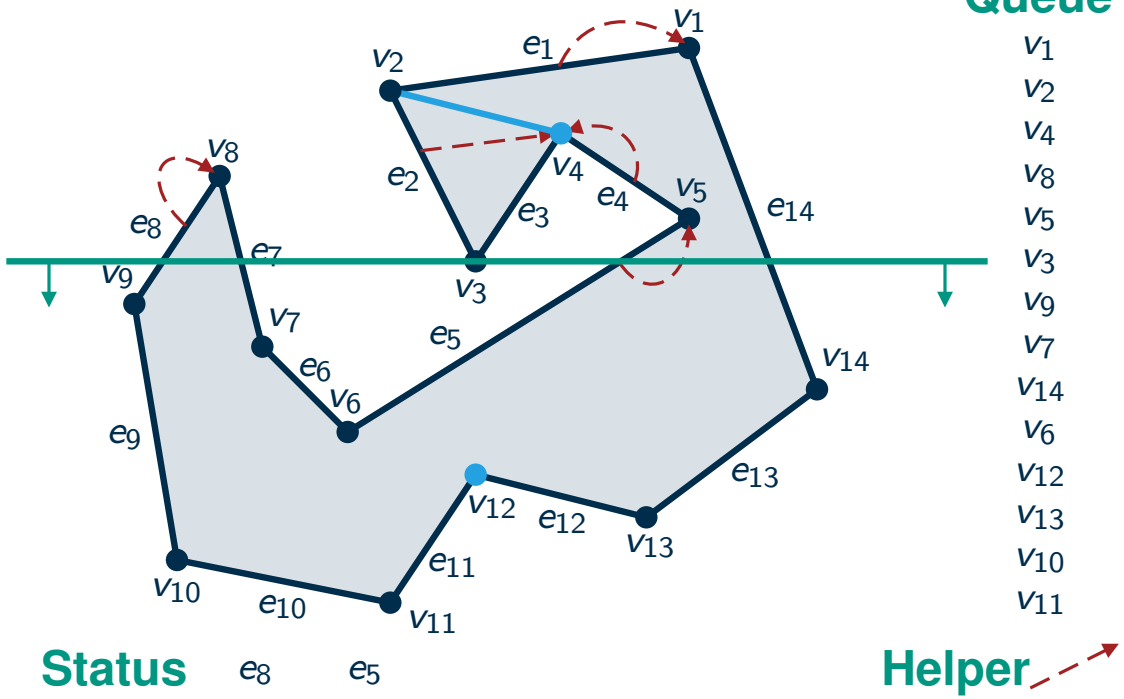

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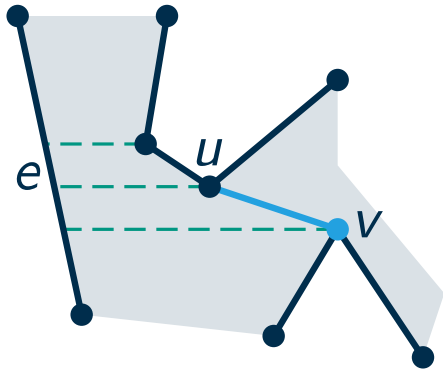

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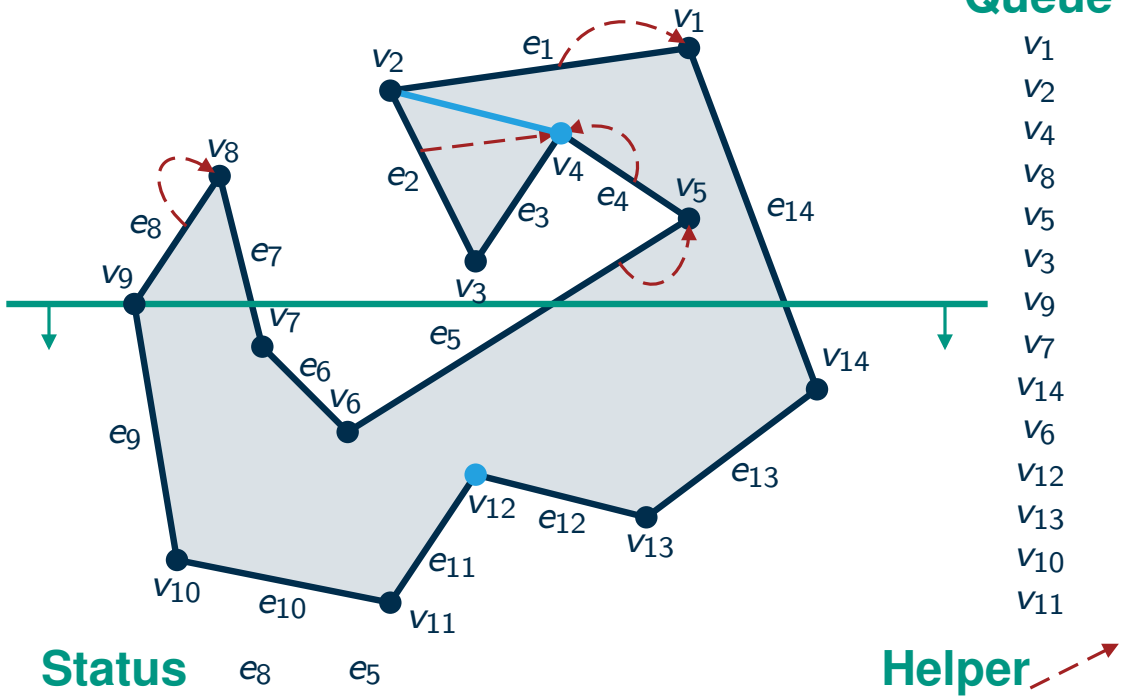

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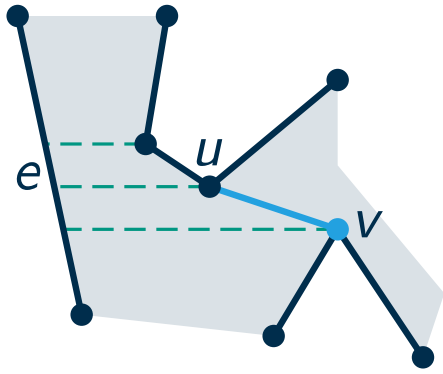

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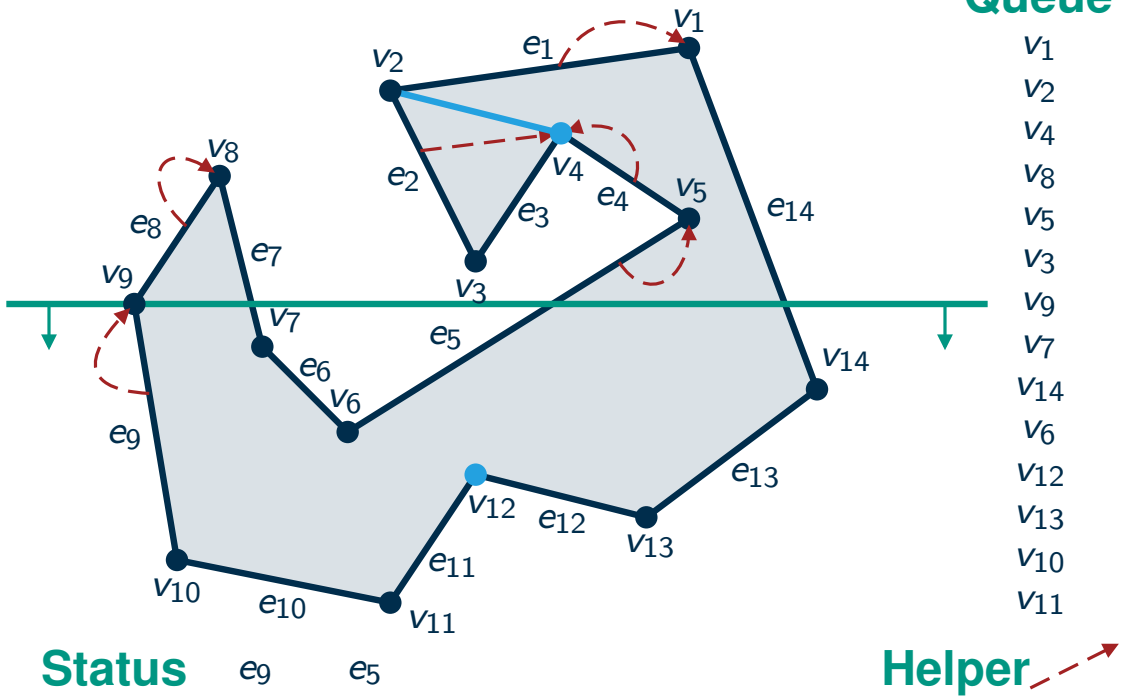

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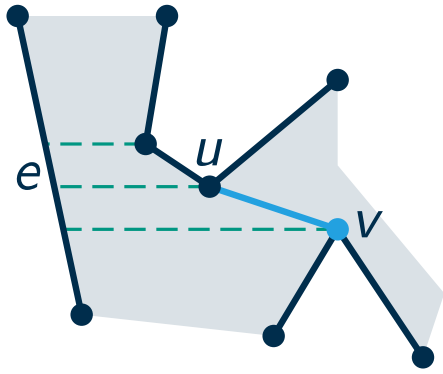

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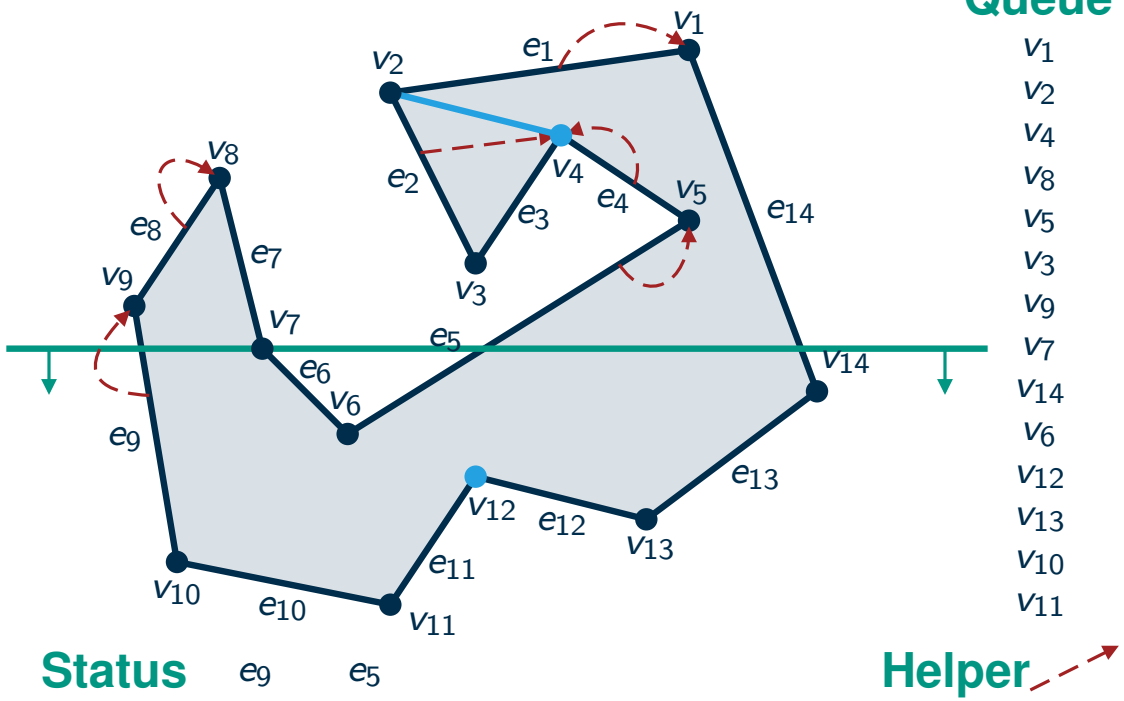

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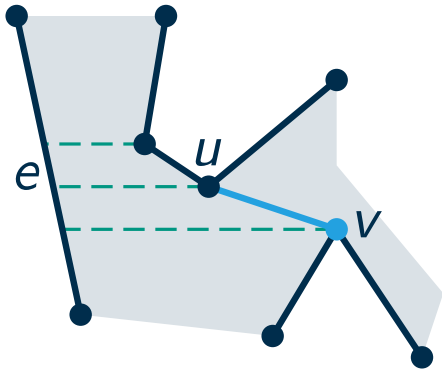

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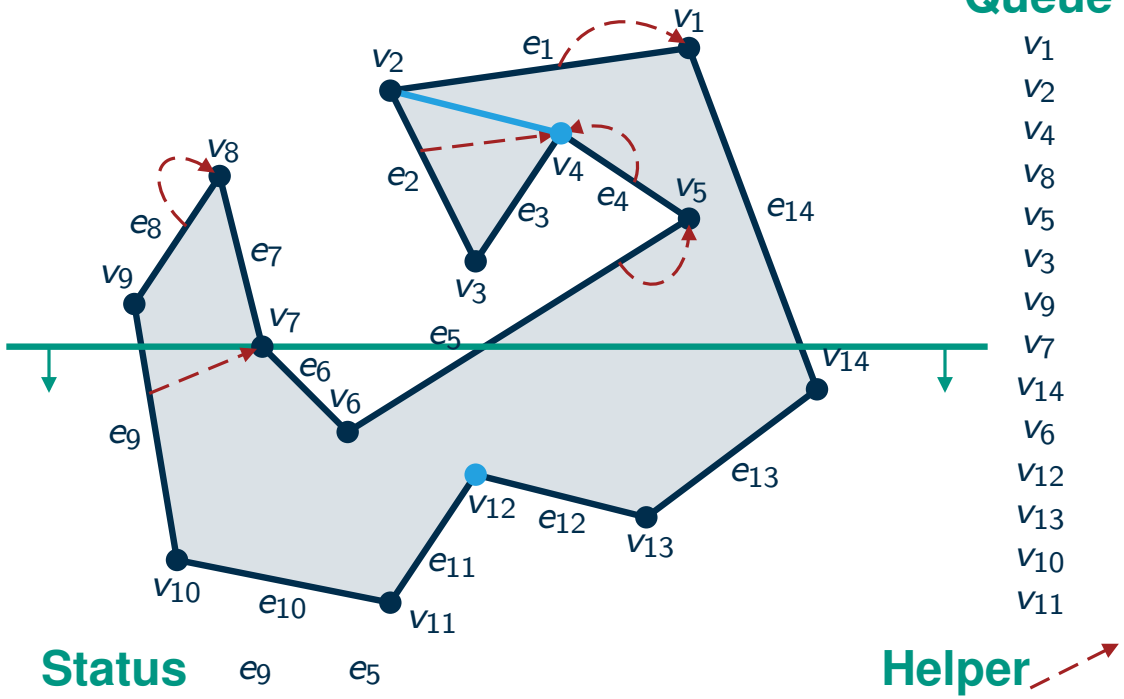

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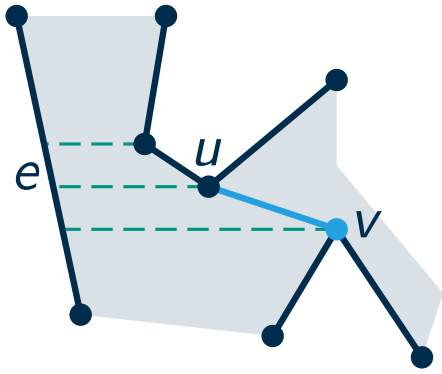

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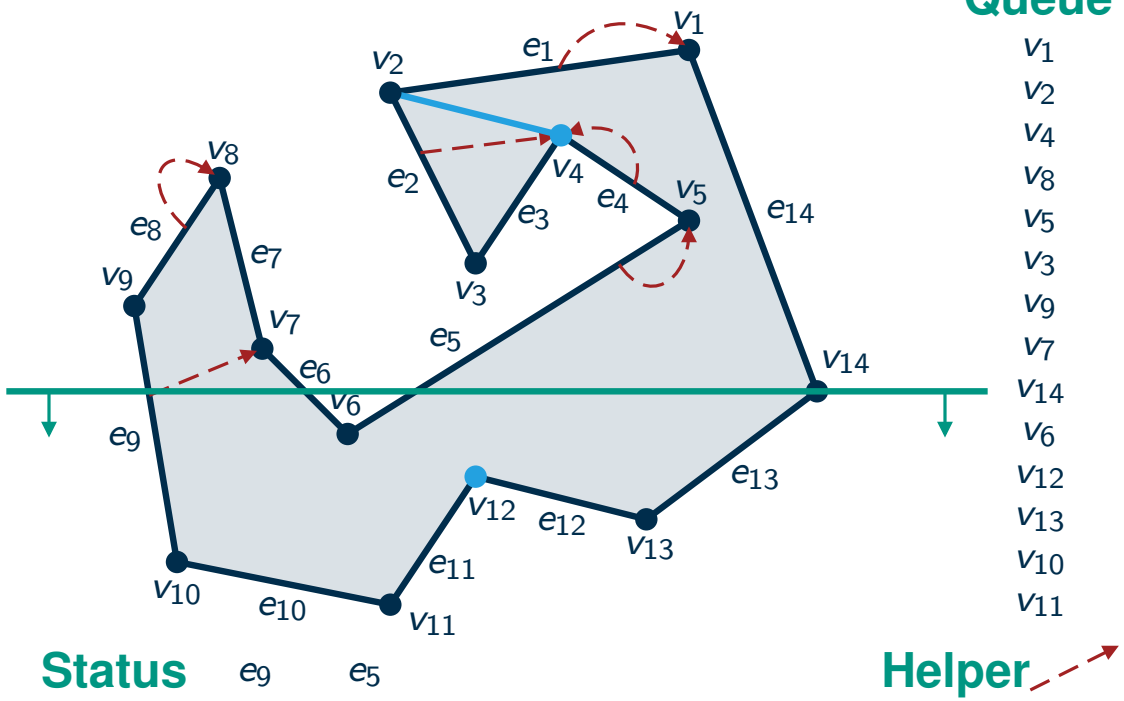

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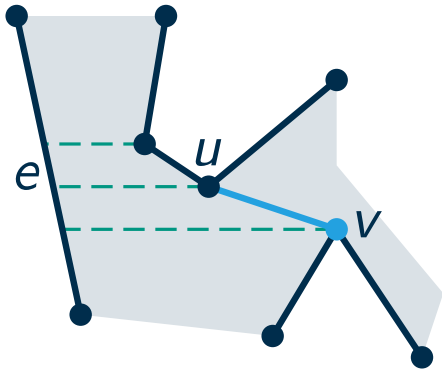

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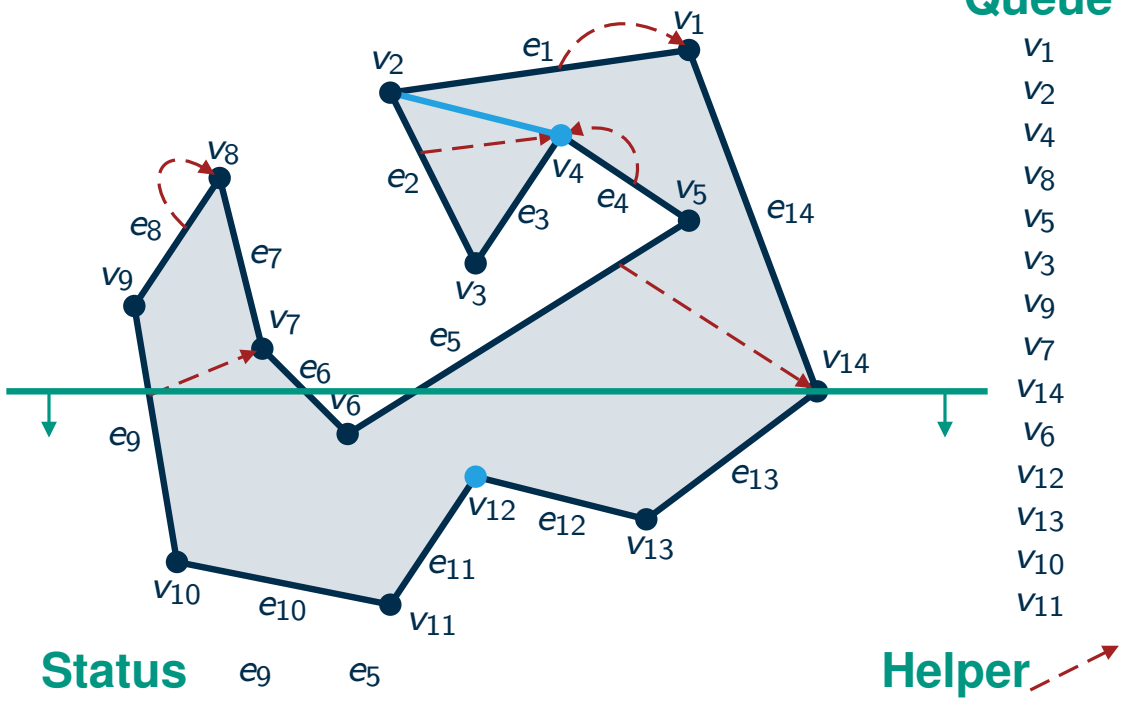

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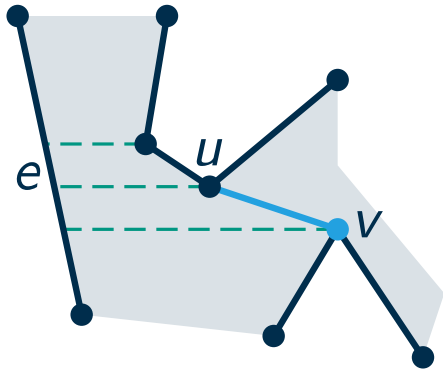

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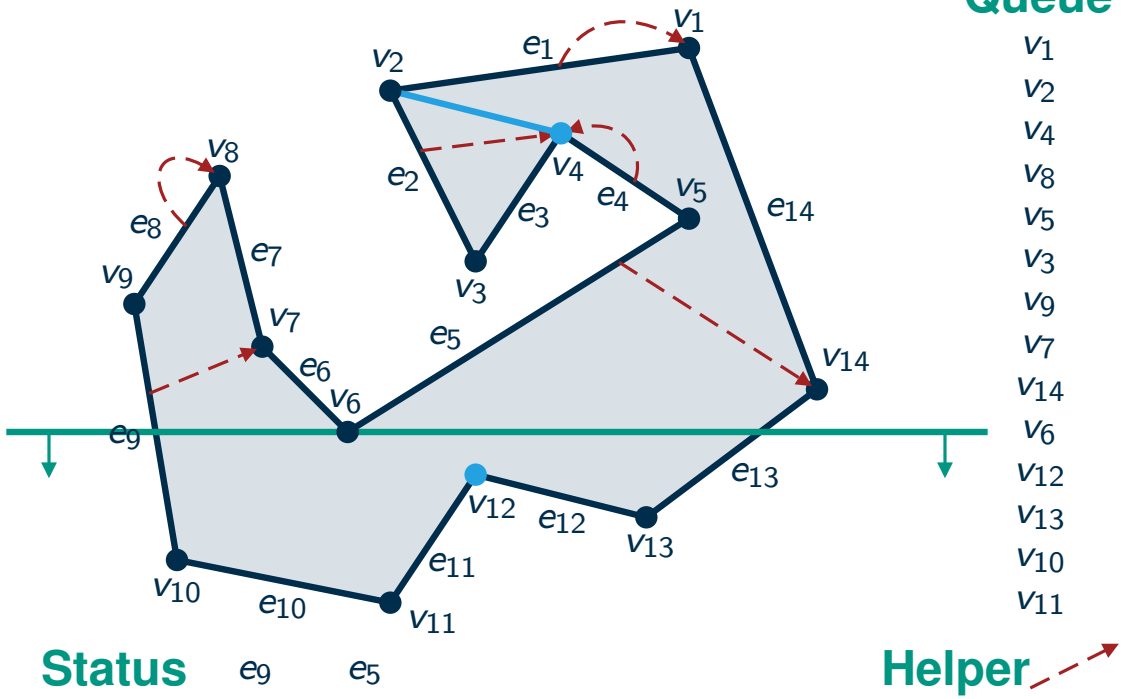

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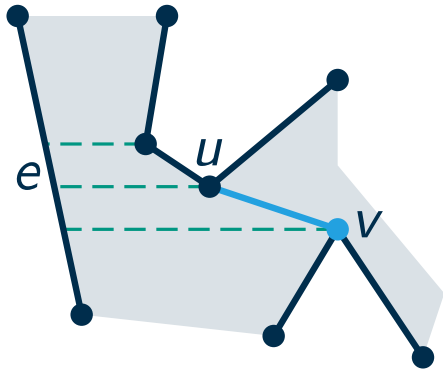

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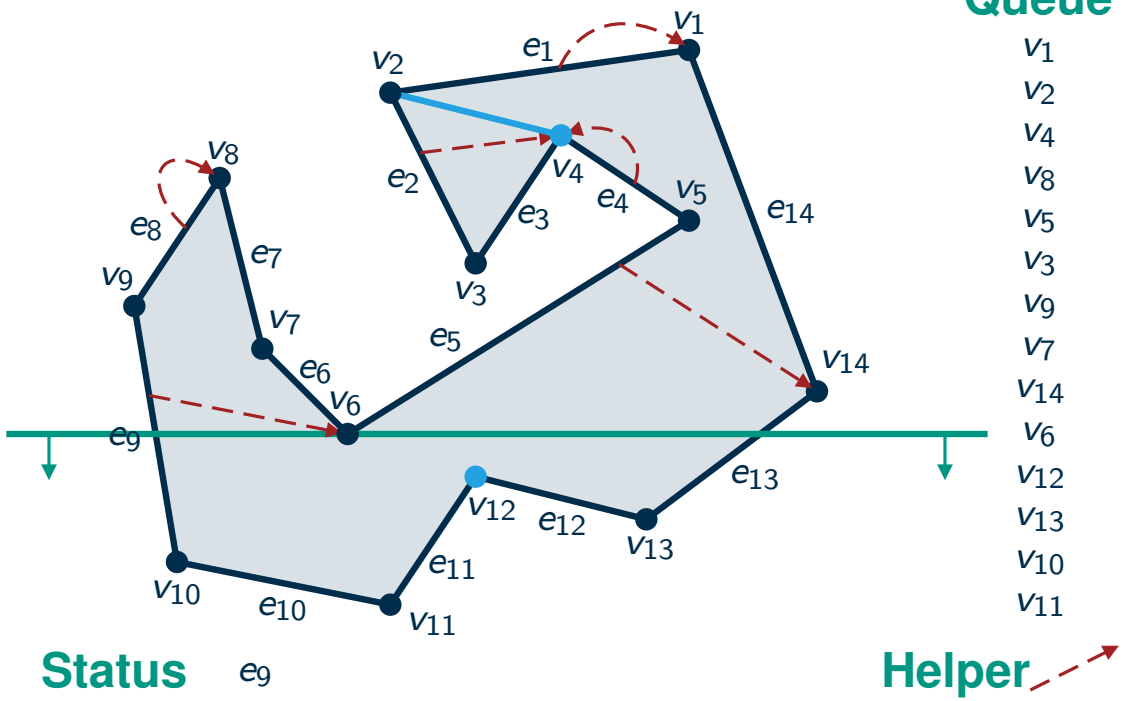

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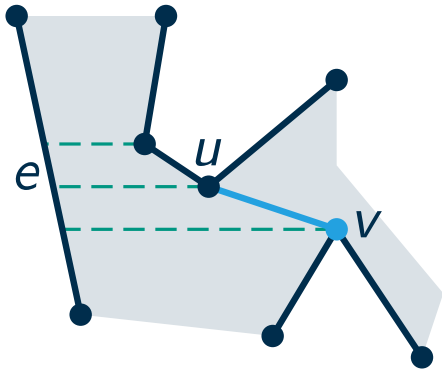

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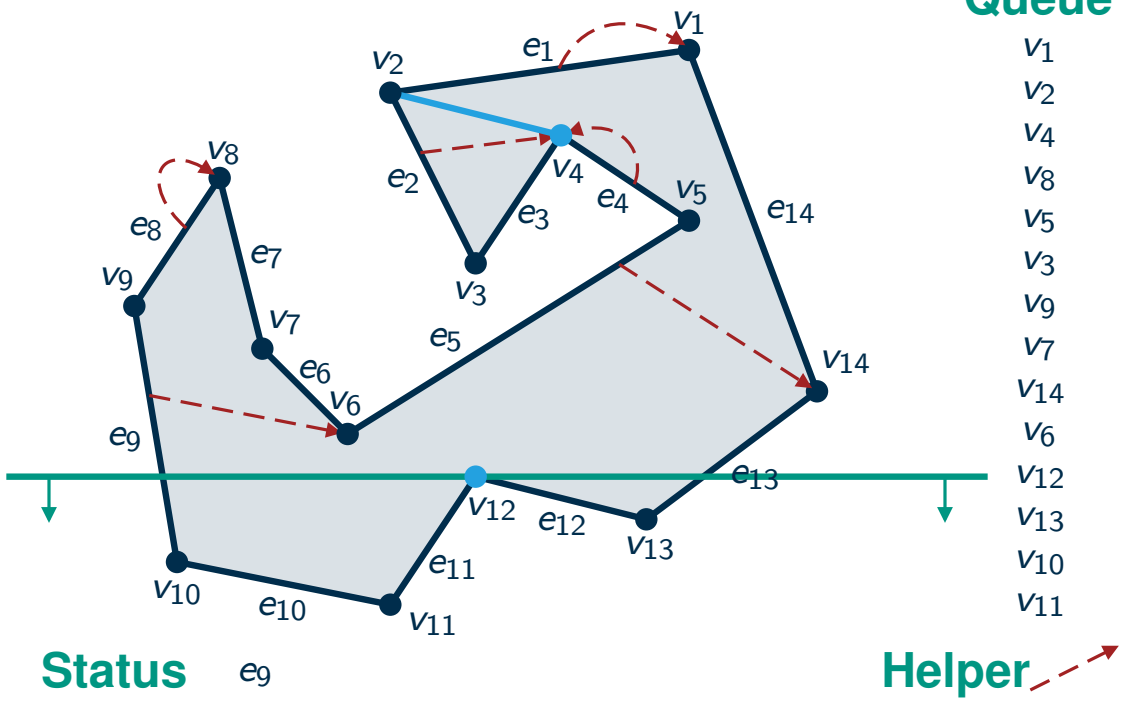

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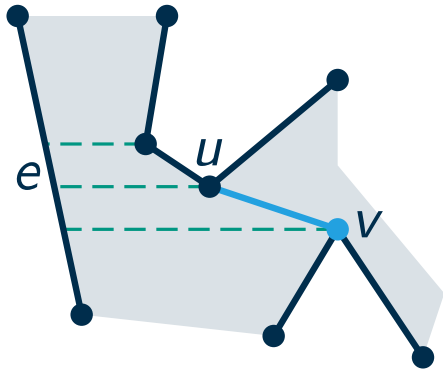

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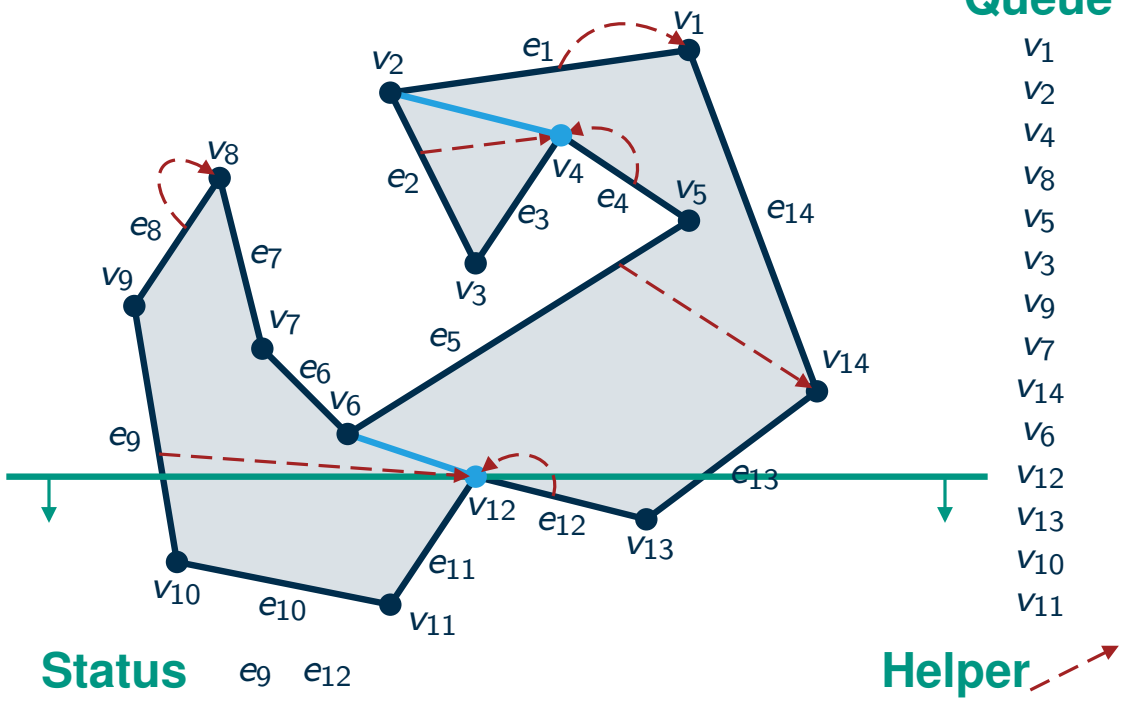

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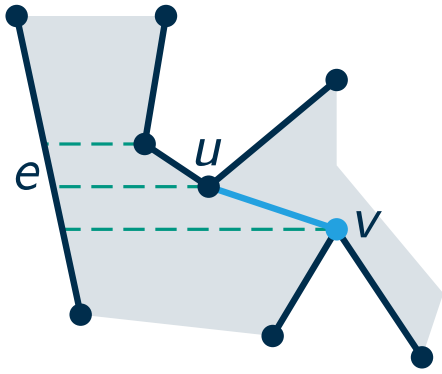

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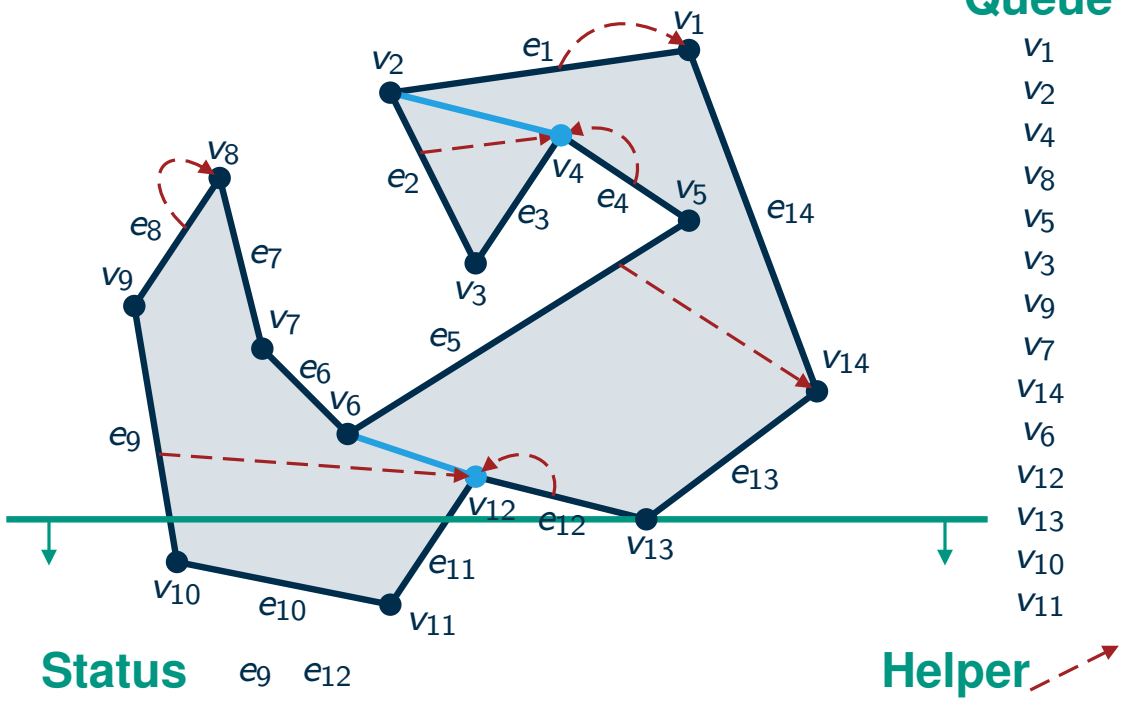

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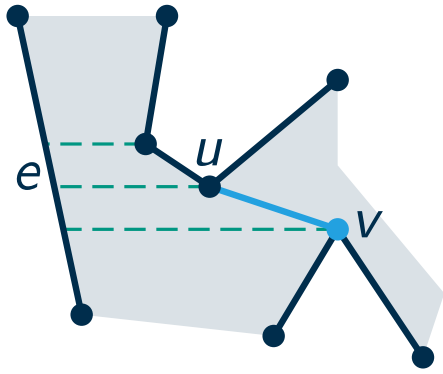

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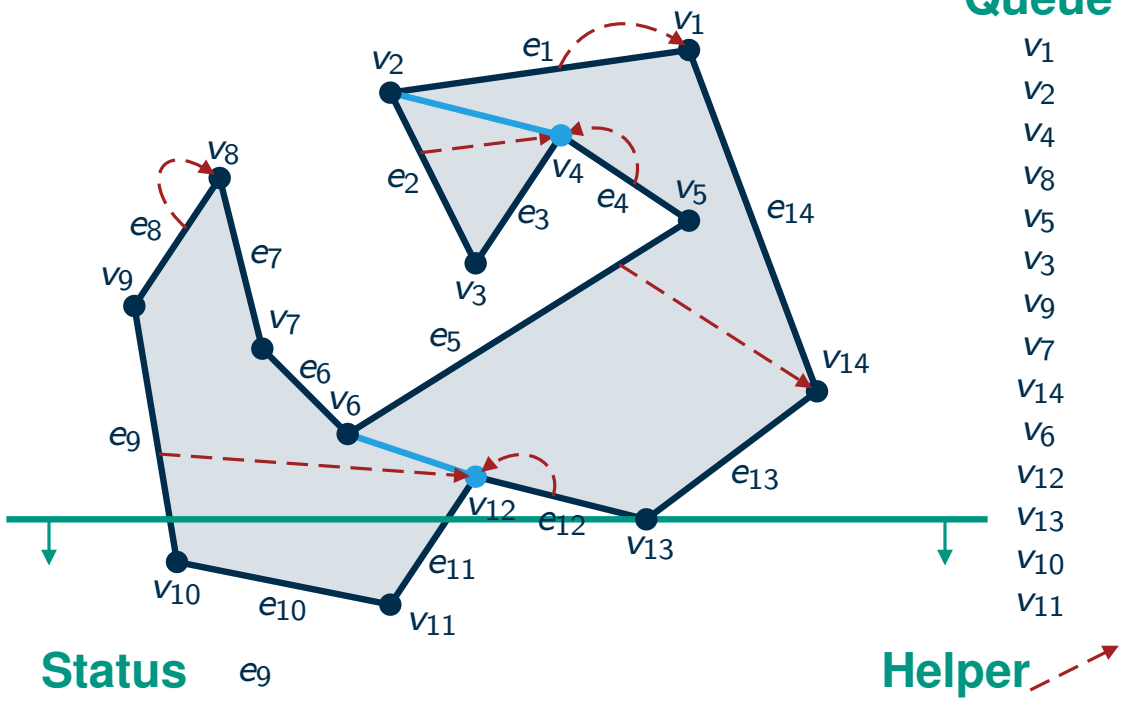

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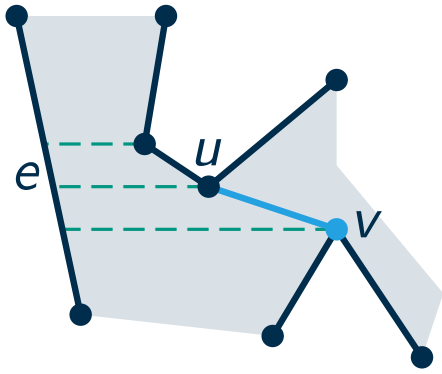

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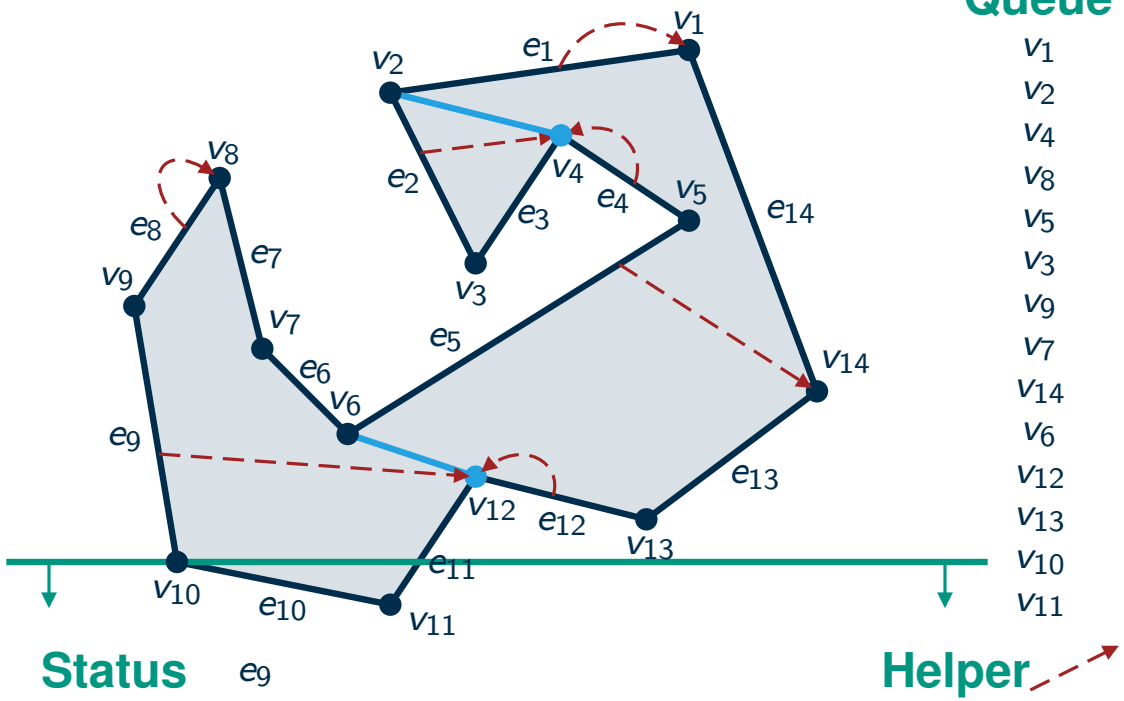

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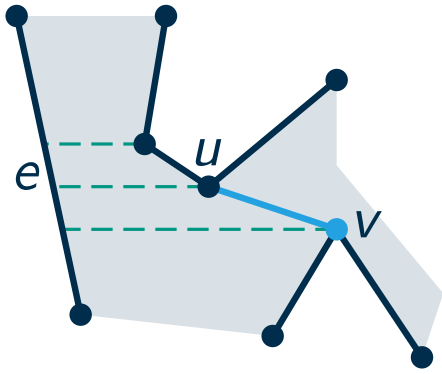

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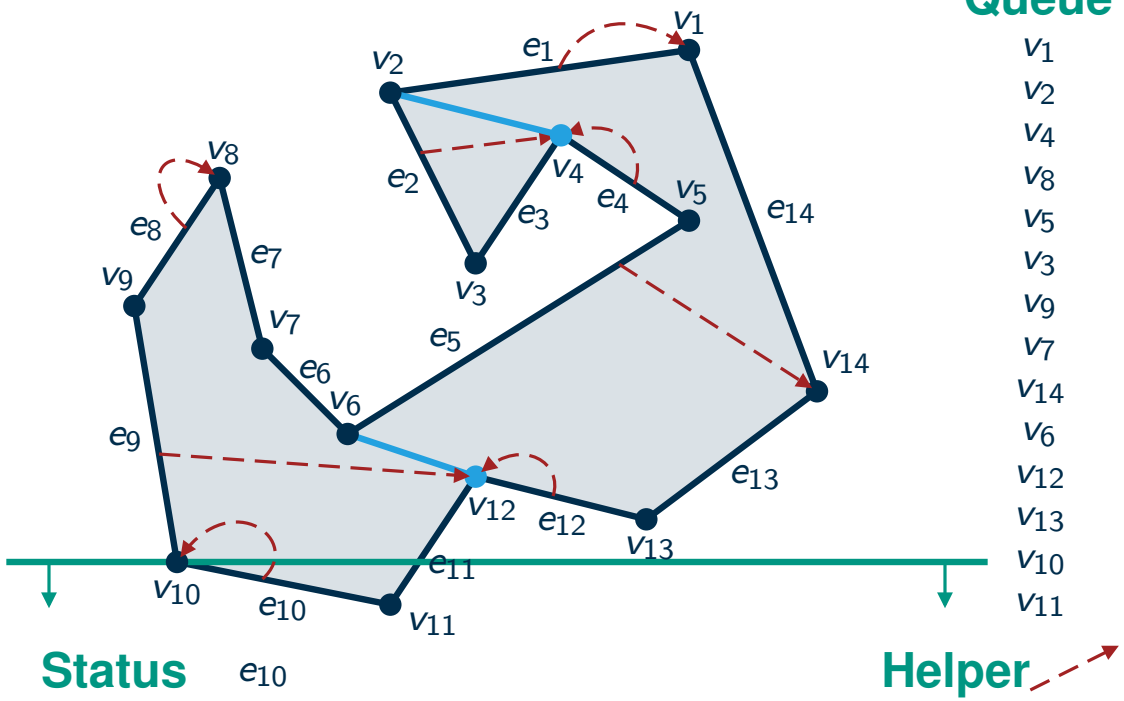

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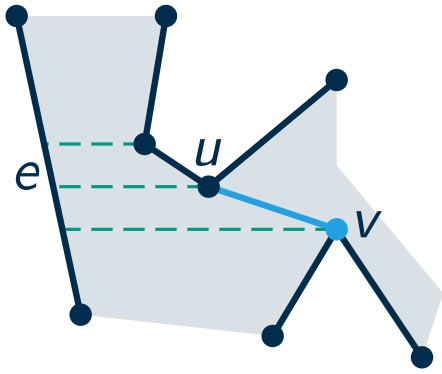

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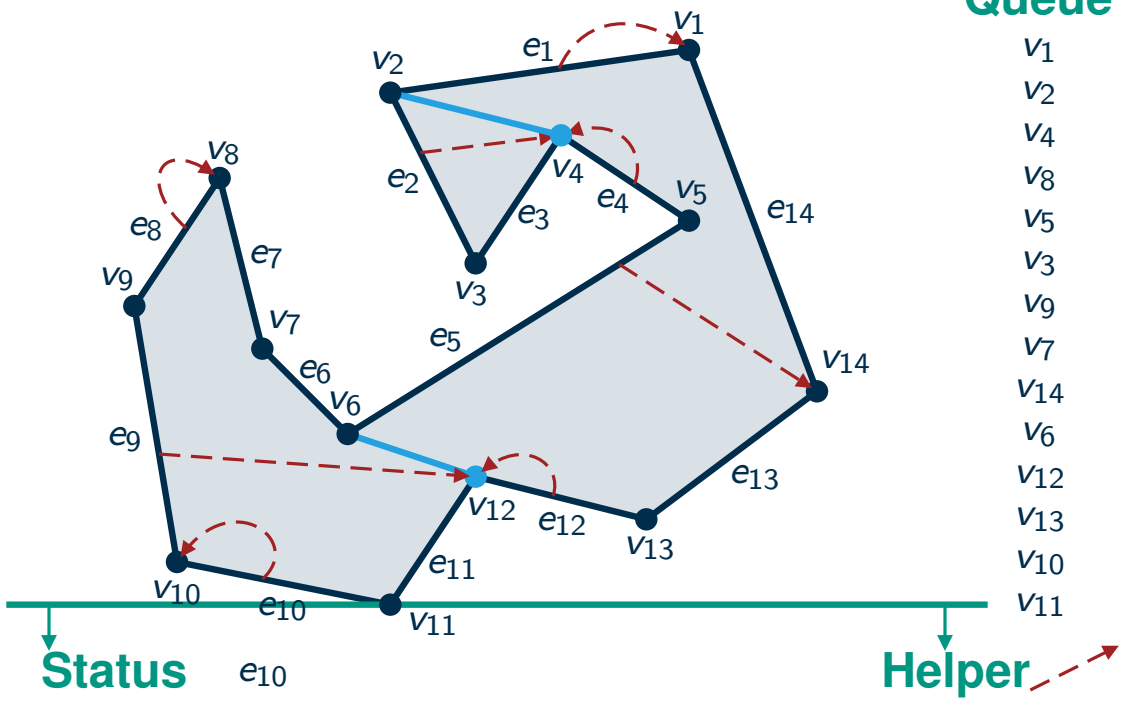

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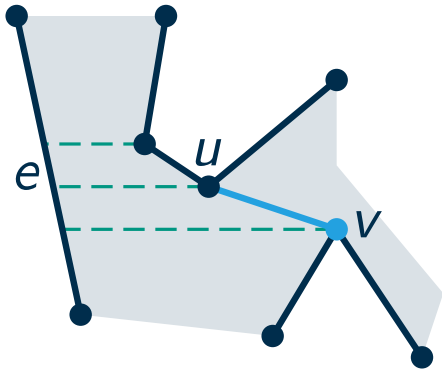

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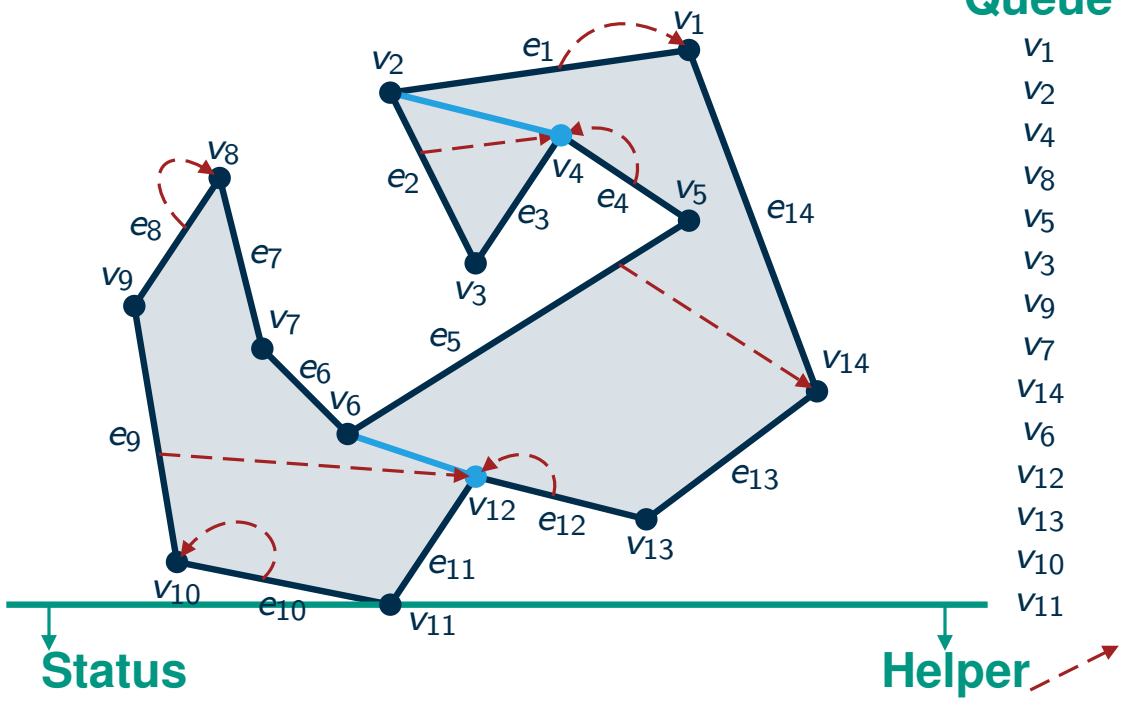

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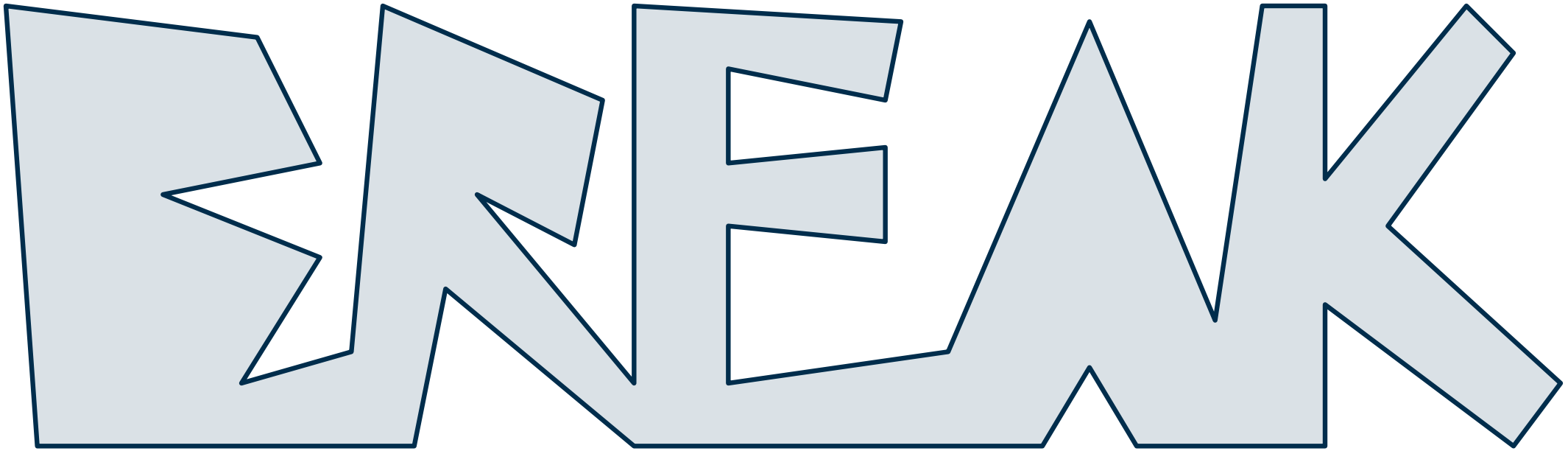

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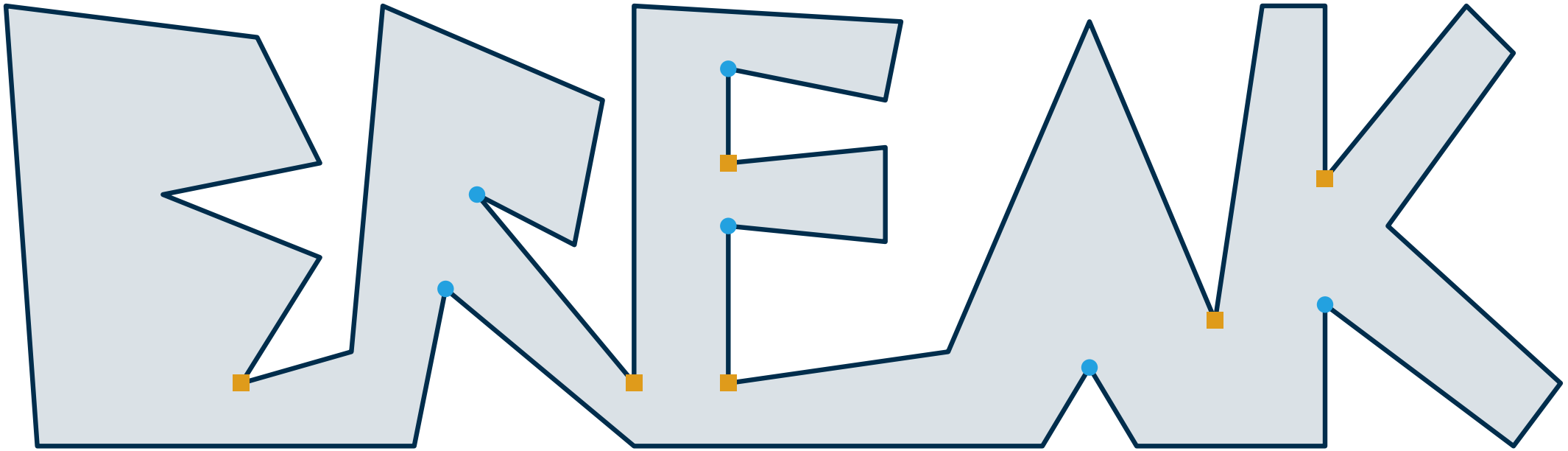
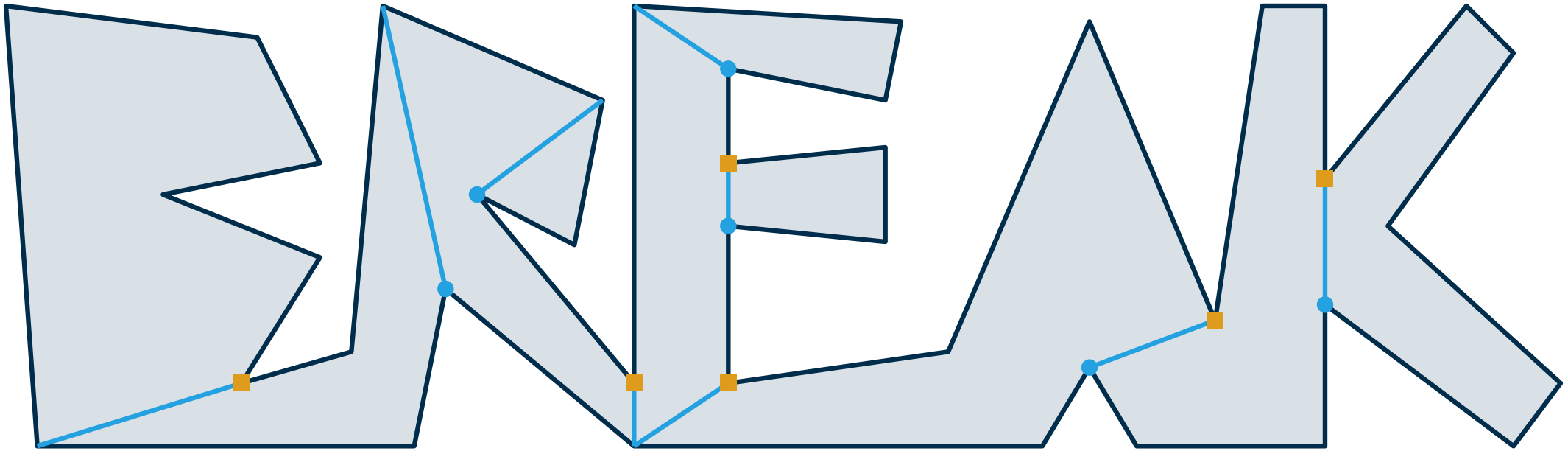


Short Break

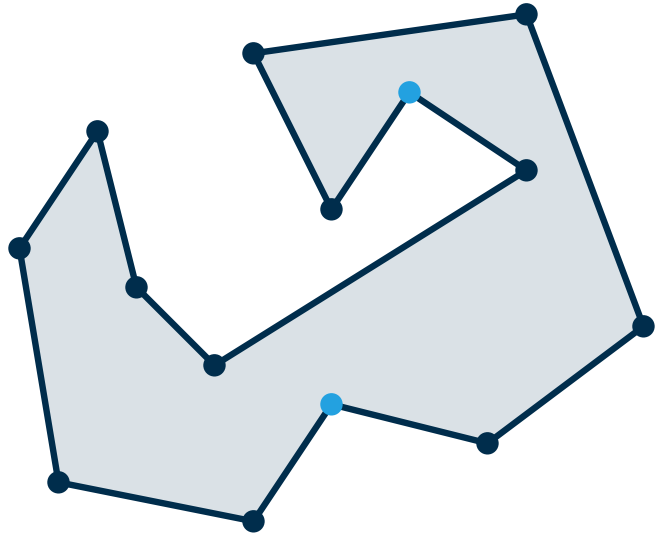
How many diagonals do we need at least to get *y*-monotone polygons?



Sweep-Line With Different Vertex Types

Different Vertex Types

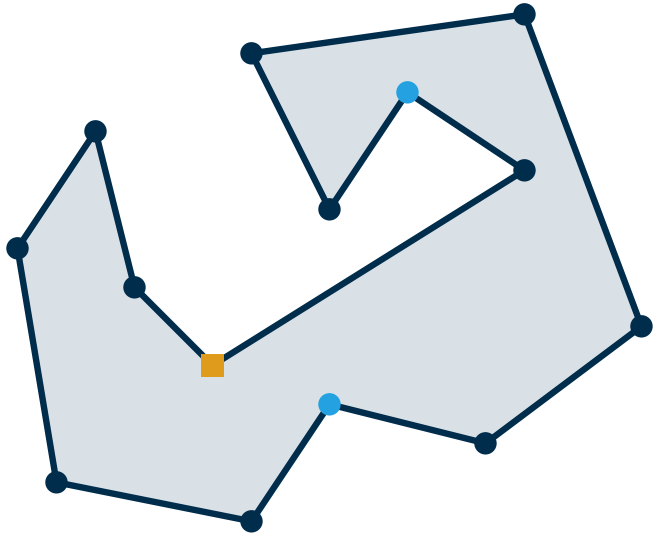
- split: edges below, polygon above



Sweep-Line With Different Vertex Types

Different Vertex Types

- split: edges below, polygon above
- merge: edges above, polygon below



Sweep-Line With Different Vertex Types

Different Vertex Types

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- merge: edges above, polygon below
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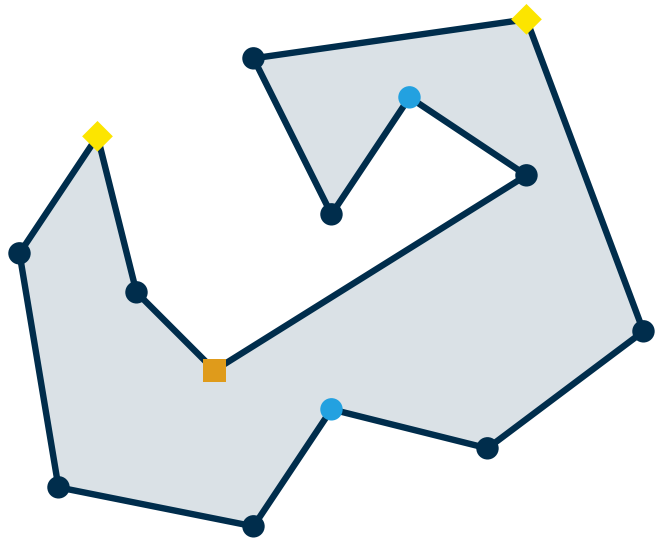

Sweep-Line With Different Vertex Types

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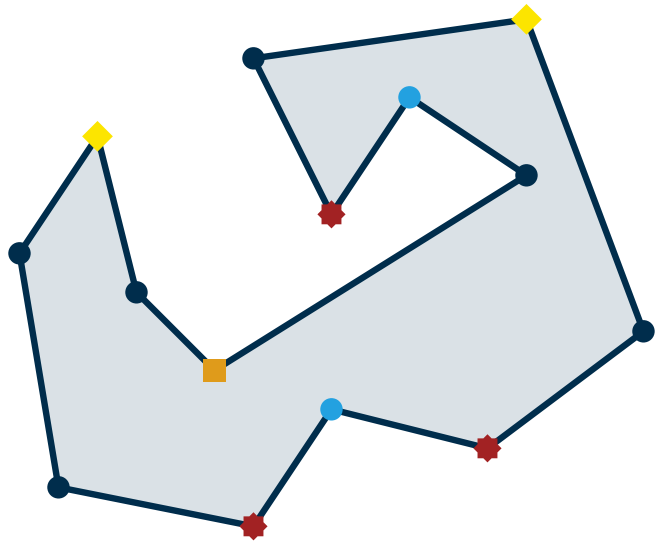
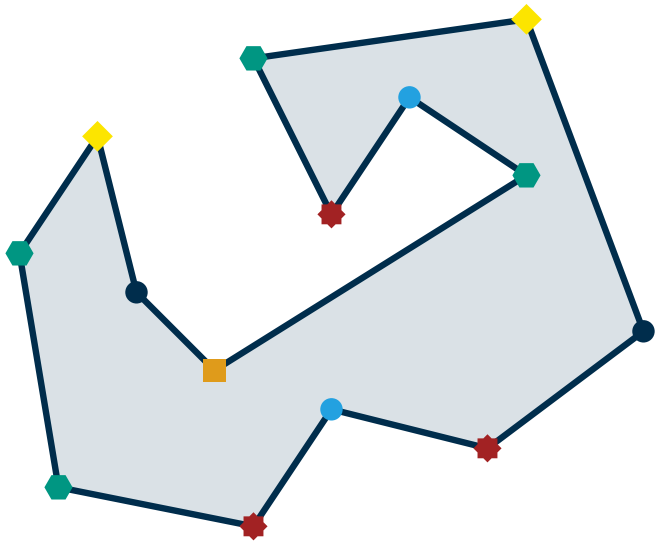


Sweep-Line With Different Vertex Types

Different Vertex Types

- split: edges below, polygon above
- merge: edges above, polygon below
- start: edges below, polygon below
- end: edges above, polygon above
- left: y-monoton, polygon right



Sweep-Line With Different Vertex Types

Different Vertex Types

- split: edges below, polygon above
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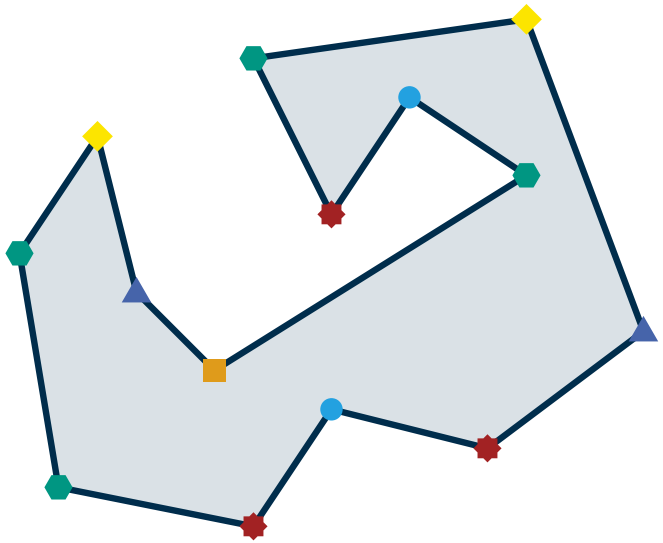
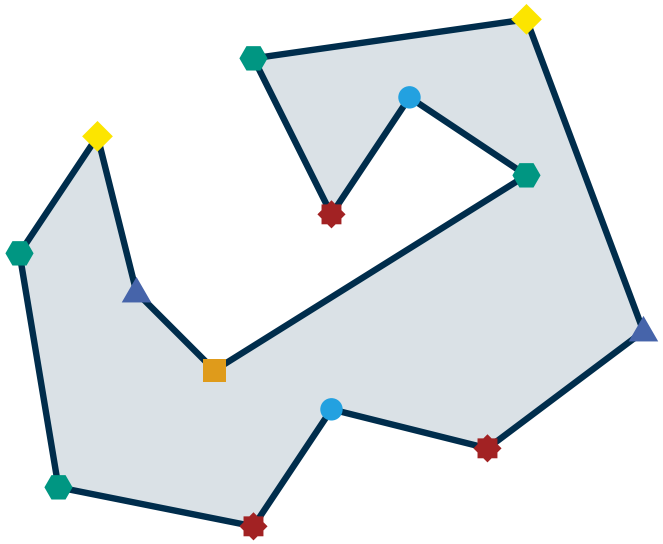
- end: edges above, polygon above
- left: y-monoton, polygon right
- right: y-monoton, polygon left



Sweep-Line With Different Vertex Types

function MAKEMONOTONE(P)
Input: polygon P (counterclockwise)
Output: diagonals, that make P y-monotone

$Q =$ vertices of P sorted by y-coordinate
$T =$ search tree
while $Q \neq \emptyset$
  $v = \min\{Q\}$ and $Q = Q - v$
  HANDLEVERTEX(v)



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  e_v^+ = edge after v in P
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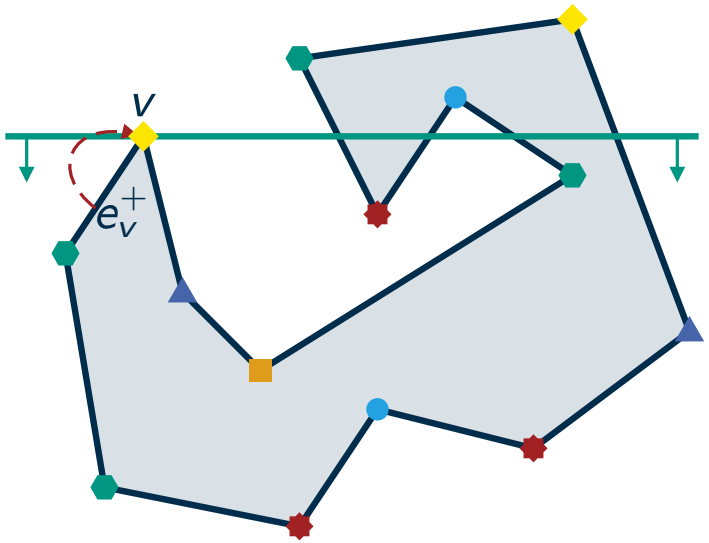

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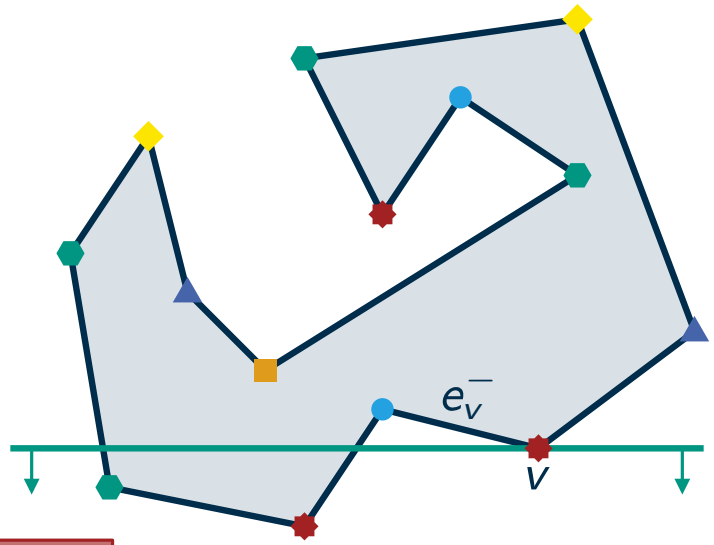

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e_v^+ = edge after v in P
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helper(e_v^+) = v ◇

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e_v^- = edge before v in P
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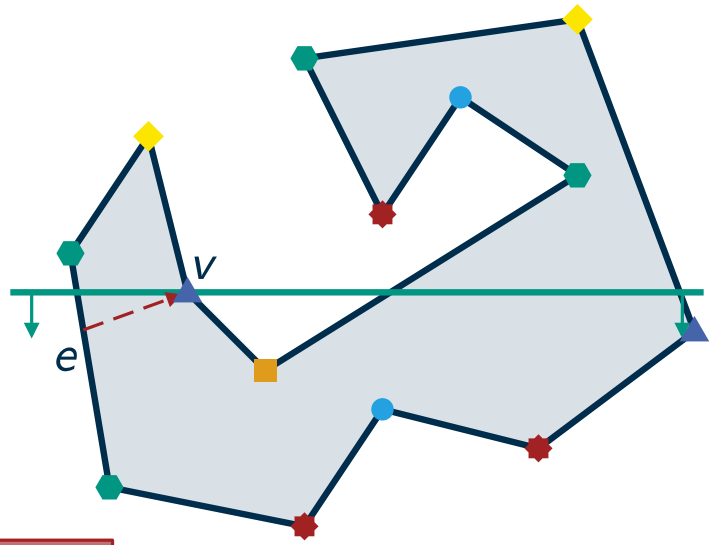

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HANDLERIGHTVERTEX(v)
- e = edge left of v in T
- helper(e) = v △

Sweep-Line With Different Vertex Types

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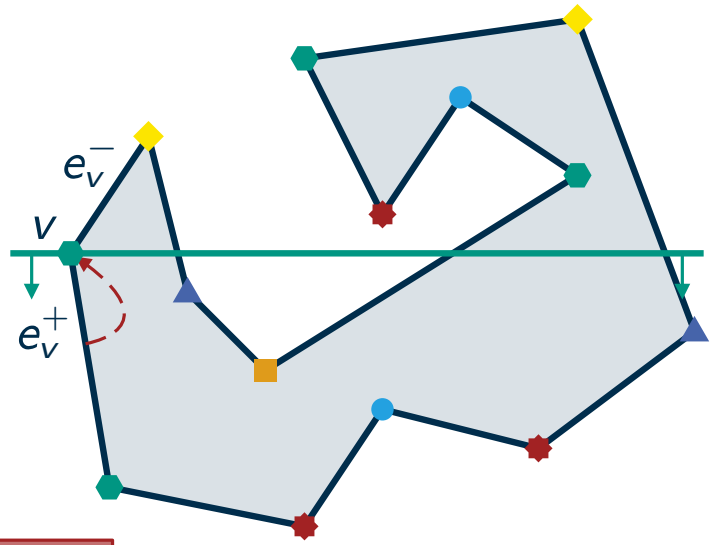

HANDLESTARTVERTEX(v)
- e_v^+ = edge after v in P
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- helper(e_v^+) = v

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- e_v^- = edge before v in P
- remove e_v^- from T

HANDLERIGHTVERTEX(v)
- e = edge left of v in T
- helper(e) = v

HANDLELEFTVERTEX(v)
- e_v^- = edge before v in P
- remove e_v^- from T
- e_v^+ = edge after v in P
- insert e_v^+ into T
- helper(e_v^+) = v

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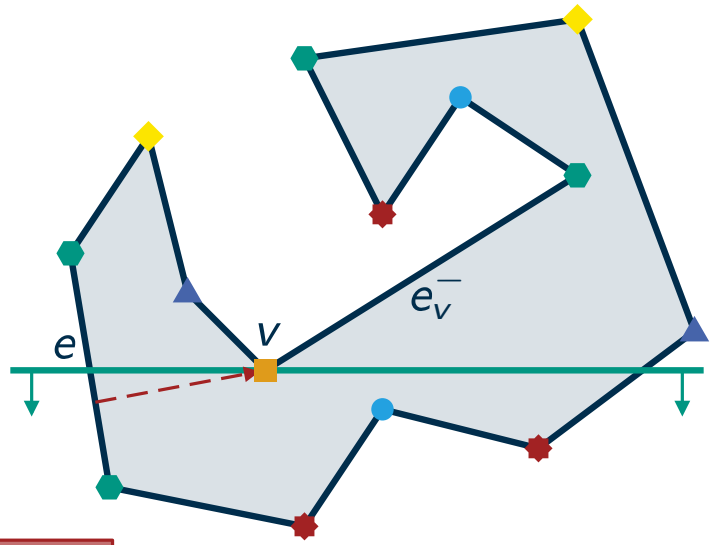

HANDLEMERGEVERTEX(v)
- e = edge left of v in T
- helper(e) = v
- e_v^- = edge before v in P
- remove e_v^- from T

HANDLESTARTVERTEX(v)
- e_v^+ = edge after v in P
- insert e_v^+ into T
- helper(e_v^+) = v

HANDLEENDVERTEX(v)
- e_v^- = edge before v in P
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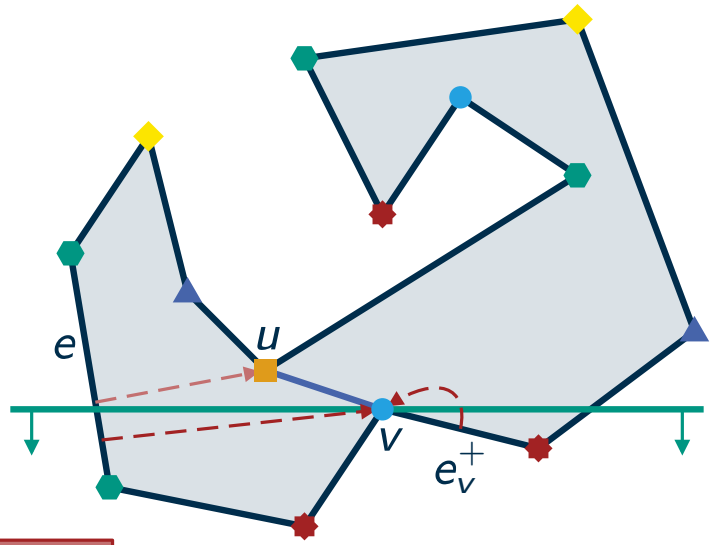

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- e_v^- = edge before v in P
- remove e_v^- from T
- e_v^+ = edge after v in P
- insert e_v^+ into T
- helper(e_v^+) = v

HANDLESPLITVERTEX(v)
- e = edge left of v in T
- u = helper(e)
- **output** diagonal uv
- helper(e) = v
- e_v^+ = edge after v in P
- insert e_v^+ into T
- helper(e_v^+) = v

Sweep-Line With Different Vertex Types

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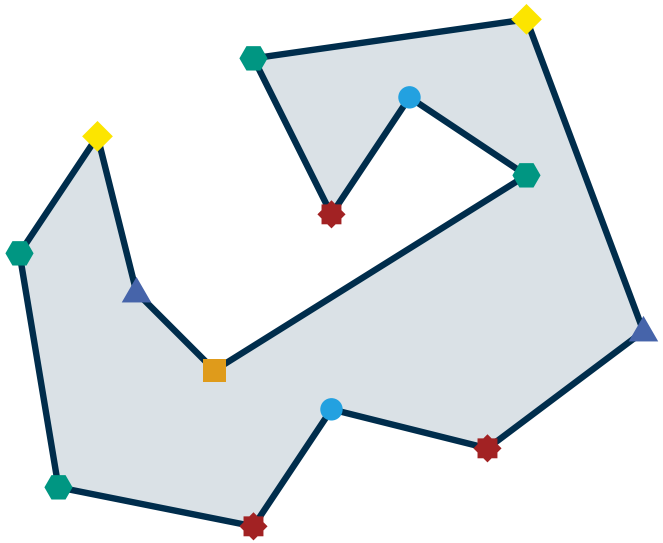

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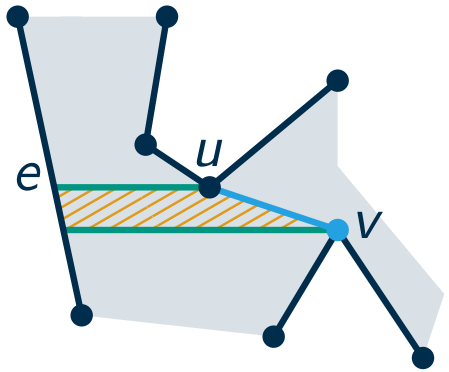
We Create No Intersections

Recall

- the inserted diagonals do not intersect the polygon
- core argument: the quadrilateral between uv and e contains no vertex



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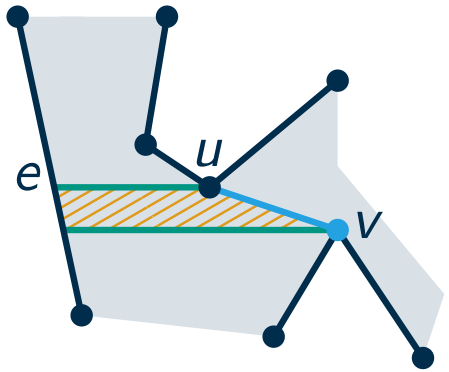

Can We Get An Intersection With A Previously Inserted Diagonal?

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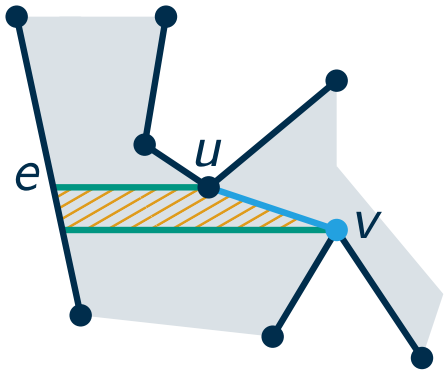
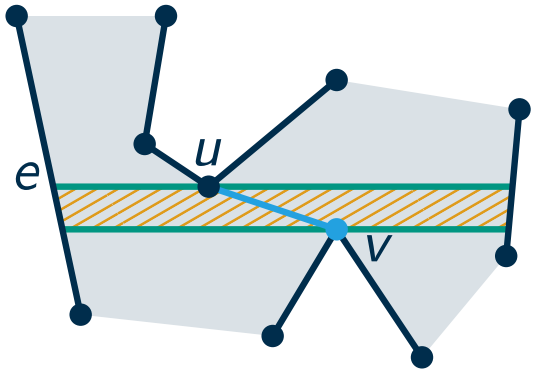


Can We Get An Intersection With A Previously Inserted Diagonal?

- extend the quadrilateral to the right
- same argument: the extended quadrilateral also contains no vertex



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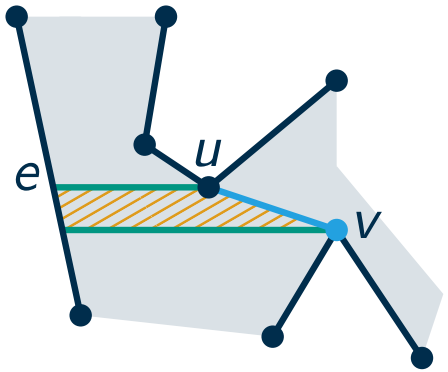

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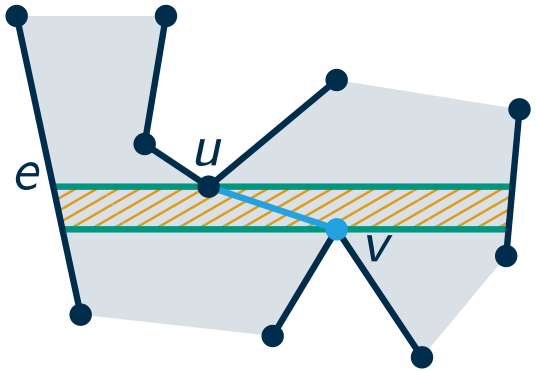

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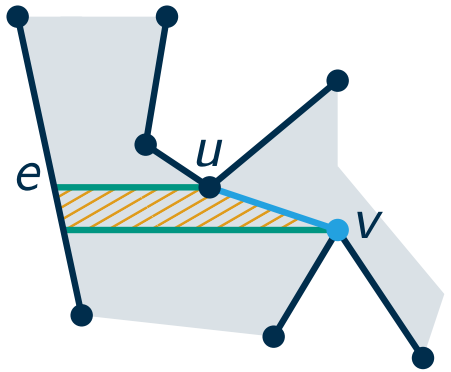

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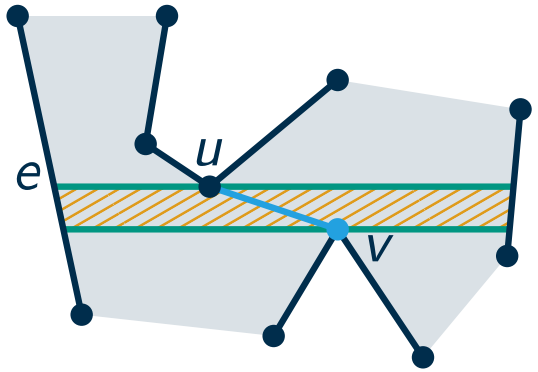
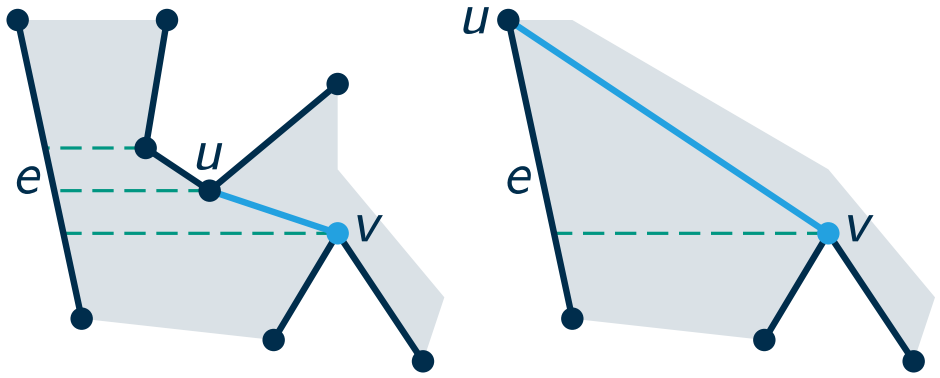


And What About Merge Vertices?



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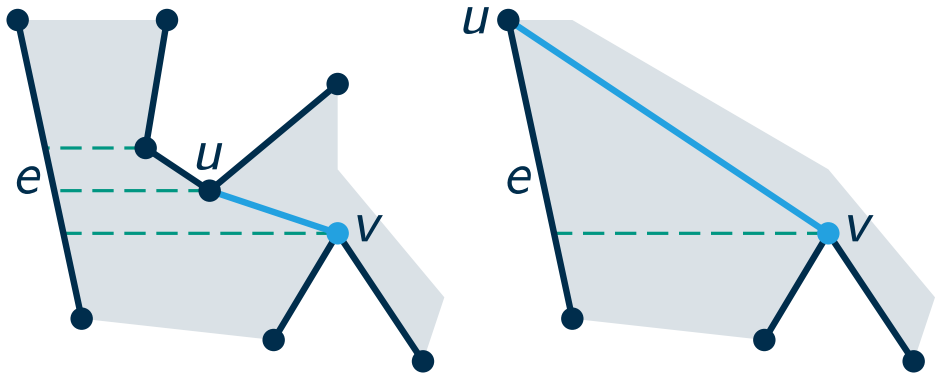

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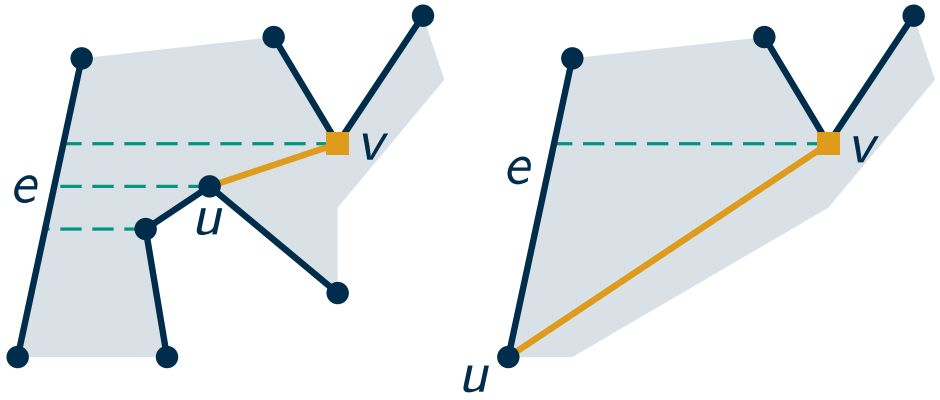

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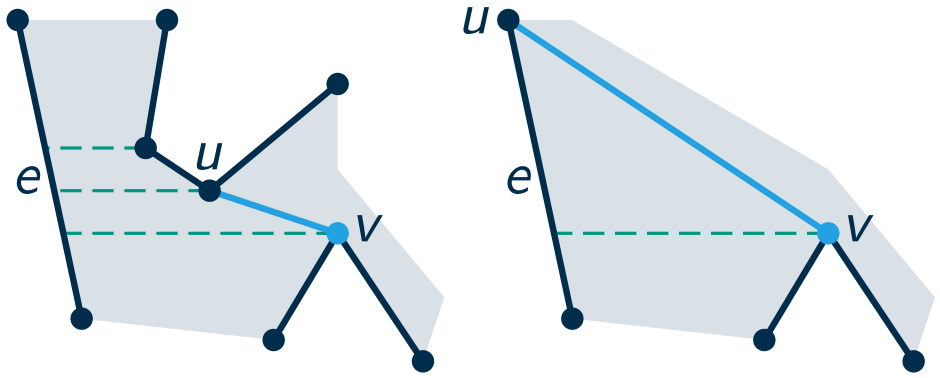

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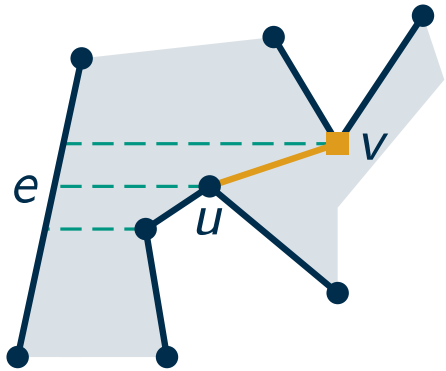

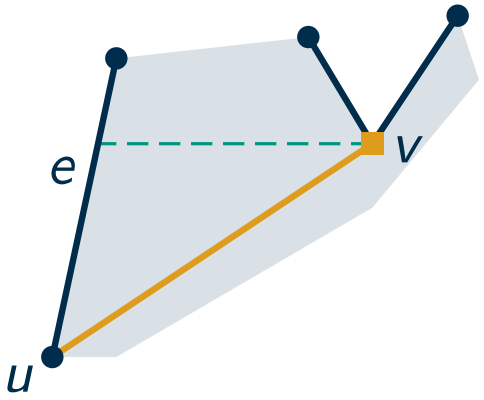

Handling A Merge Vertex *v*

- solution for the lazy theoretician:
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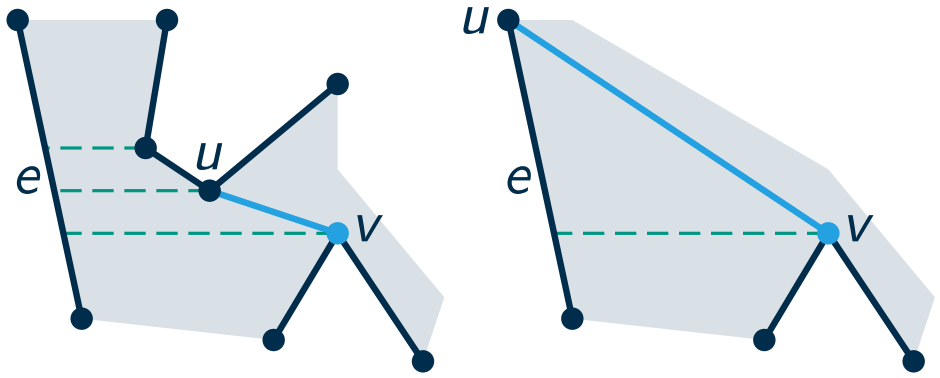

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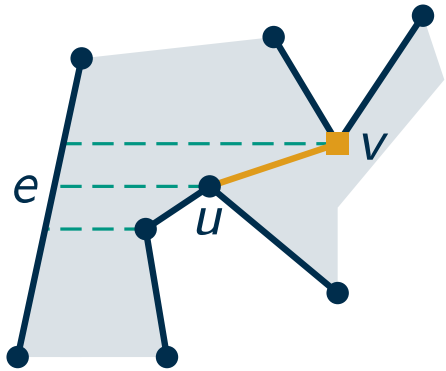

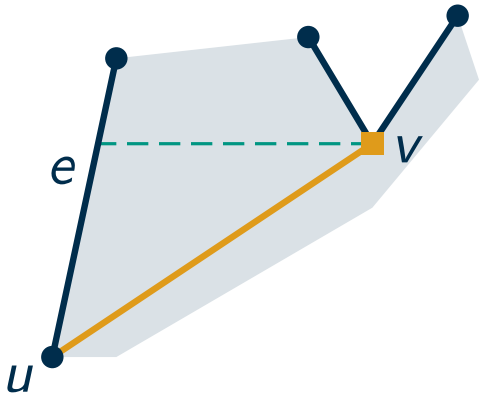

Handling A Merge Vertex *v*

- solution for the lazy theoretician:
  - just run it again in the opposite direction
- alternative:
  - observe: *v* is the helper of *e* when we process *u*
  - when *e* ends at *u* or gets a new helper *u*: current helper *v* is merge vertex → insert *uv*

Wrap-Up

Theorem **(subdivision into y-monotone pieces)**
A polygon with n vertices can be subdivided into y-monotone pieces in $O(n \log n)$ time.

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- concept of monotonicity
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- corresponding 3-dimensional problem is NP-hard

References

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