Computational Geometry

Polygon Triangulation

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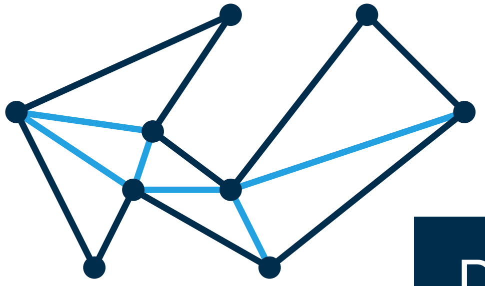
Polygon Triangulation



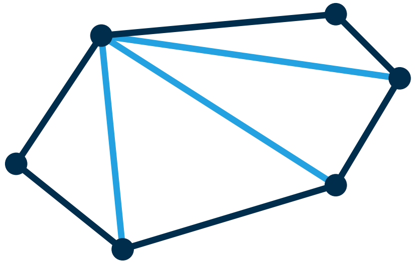
Definition
A **triangulation** of a polygon P is a planar subdivision of P such that each face is a triangle.

Does this always exist?

Problem
Given P, find diagonals that triangulate P.

Let's Simplify First

- convex polygons are easy to triangulate
- idea: subdivide P into convex pieces then triangulate those pieces
- problem: finding a convex subdivision is not much easier



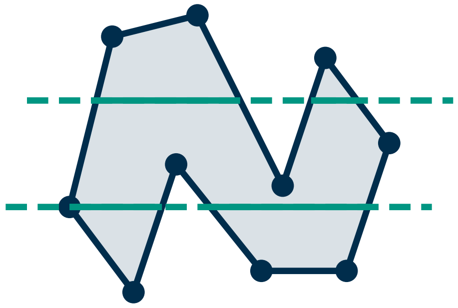
Our Plan

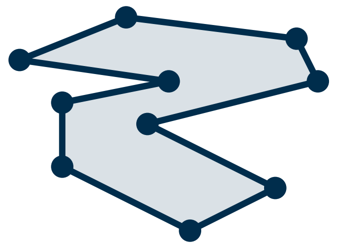
- find a weaker condition than convexity
- subdividing P into pieces with this property becomes easier
- triangulating the pieces becomes more difficult

y-Monotone Polygons

Definition
A polygon is ***y*-monotone** if the intersection with every horizontal line is connected.

not y-monotone **y-monotone**





disclaimer: I will not be super consistent whether "polygon" refers to its interior or its boundary; but it will be always clear from the context

Remark
- convex polygons are monotone in every direction

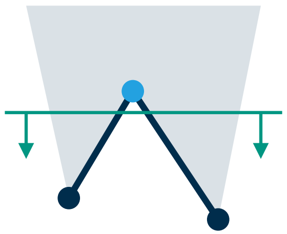
x- and y-monotone $\Rightarrow$ convex?

Our Plan
- subdivide arbitrary polygon in $O(n \log n)$ time in y-monotone pieces $\rightarrow$ today
- triangulate a y-monotone polygon in $O(n)$ time $\rightarrow$ exercise sheet

What Makes A Polygon Not y-Monotone?

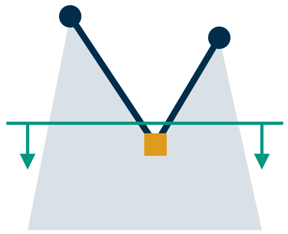
Split Vertex

- edges lie below
- polygon lies above
- polygon splits (coming from above)

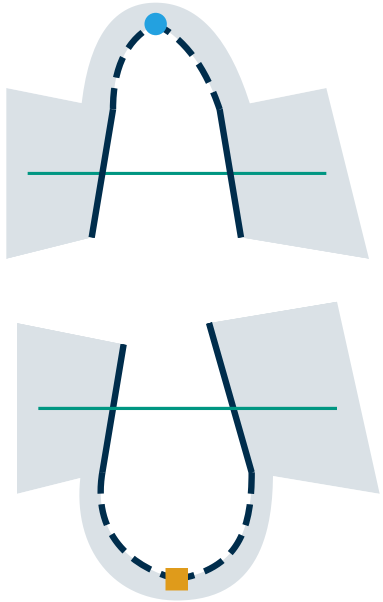


Merge Vertex

- edges lie above
- polygon lies below
- polygons parts merge



proof by picture



Observation

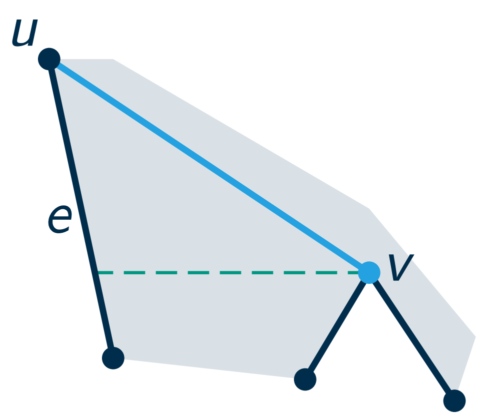
- a merge or split vertex exists $\Rightarrow$ the polygon is not y-monotone

Lemma **(y-monotonicity)**
A polygon is y-monotone if and only if it has no split or merge vertex.

Goal

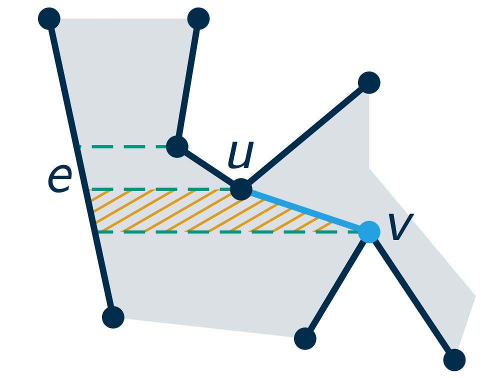
- eliminate all split and merge vertices by inserting diagonals
- upwards for split vertices and downwards for merge vertices

Eliminating Split Vertices



Idea For Split Vertex v

- idea: connect v to vertex u that is above v and close to v
- e: edge to the left of v ("to the left of": the next edge you hit when shooting a ray from v to the left)
- choose u to be the upper vertex of e



Issue (And How To Fix It)

- uv might intersect another edge of the polygon
- fix: choose for u the lowest vertex above v such that e is to the left of u
- we call u the **helper** of e (note: it depends on v)

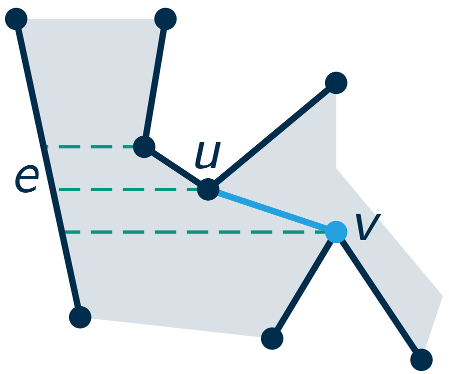
Lemma **(the helper is helpful)**
Let v be a split vertex, e the edge left of v, and u the helper of e (wrt v). Then uv does not intersect an edge of the polygon (except in u and v).

Proof

- the quadrilateral between uv and e contains no vertex
- no edge intersects uv

Why?

Eliminating Split Vertices



Observations

- goal for split vertex v: find edge e to the left of v and helper of e
- e lies (partially) above v
- the helper of e lies above v

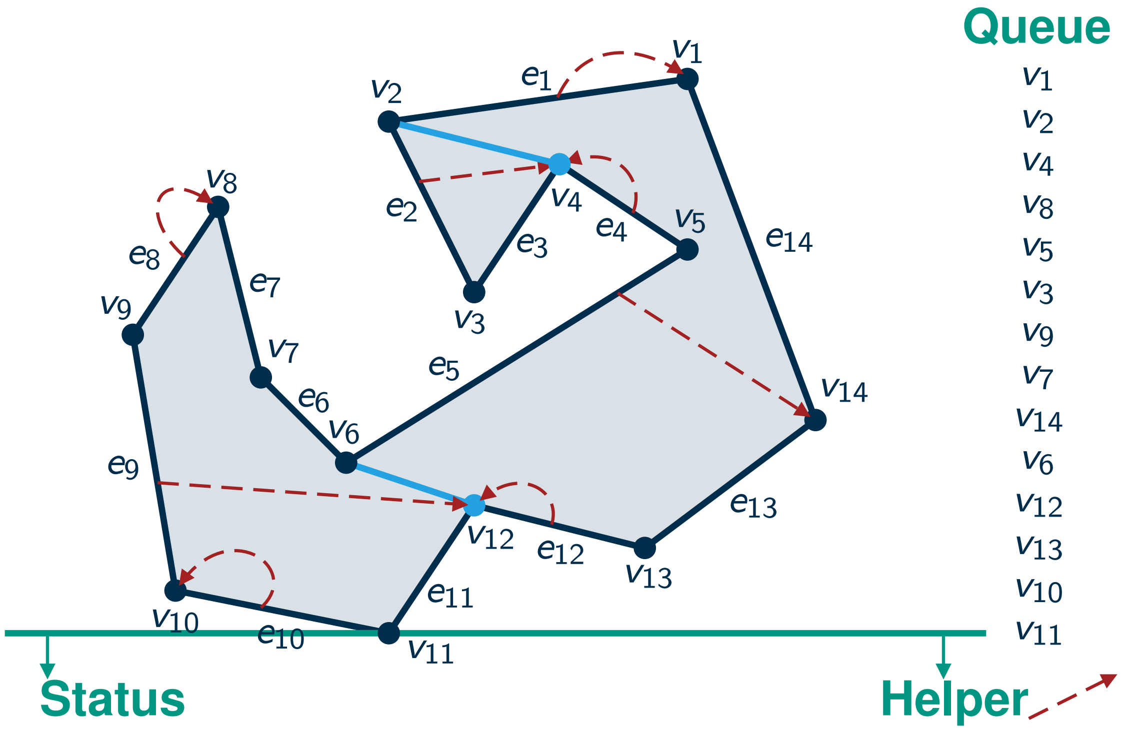
$\Rightarrow$ sweep line seems to be a good idea (horizontal sweep line ℓ from top to bottom)

Event Queue

- vertices of the polygon
- sorted by y-coordinate (or lexicographic by yx)

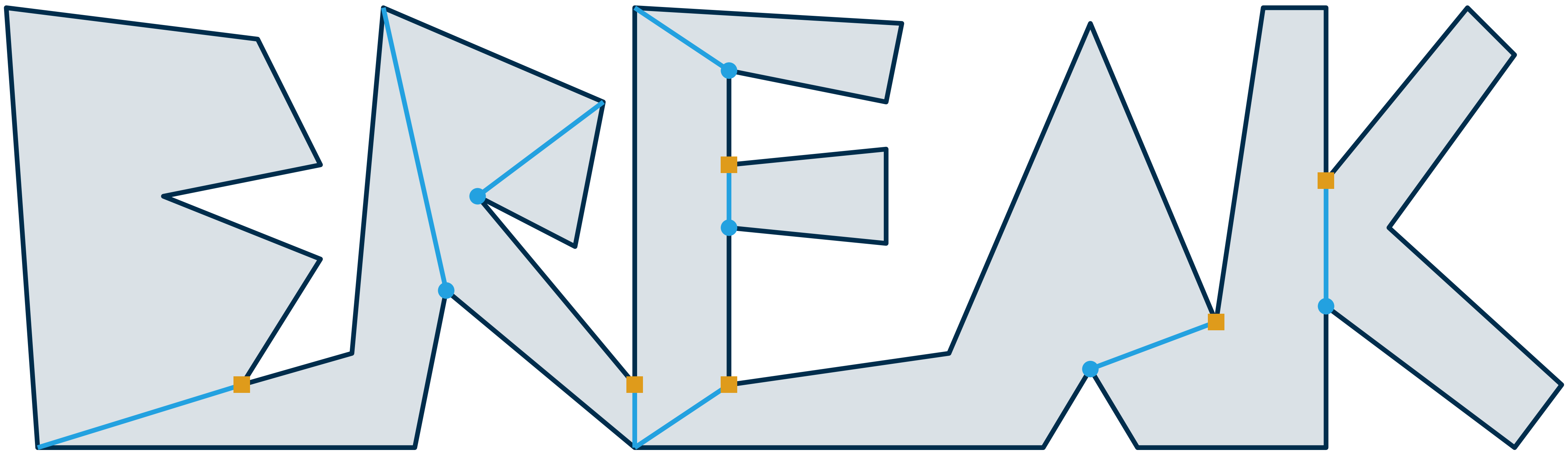
Sweep Line Status

- edges that intersect ℓ sorted by x-coordinate
- edges that have the polygon to their right suffice
- current helper for every edge



Short Break

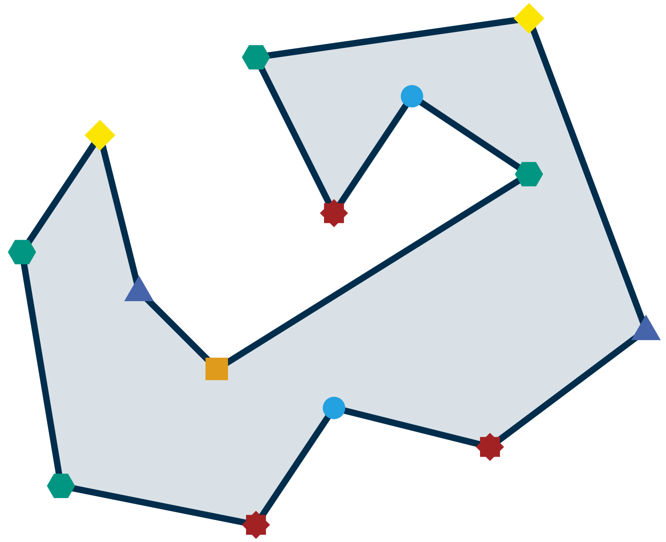
How many diagonals do we need at least to get y-monotone polygons?



Sweep-Line With Different Vertex Types

Different Vertex Types

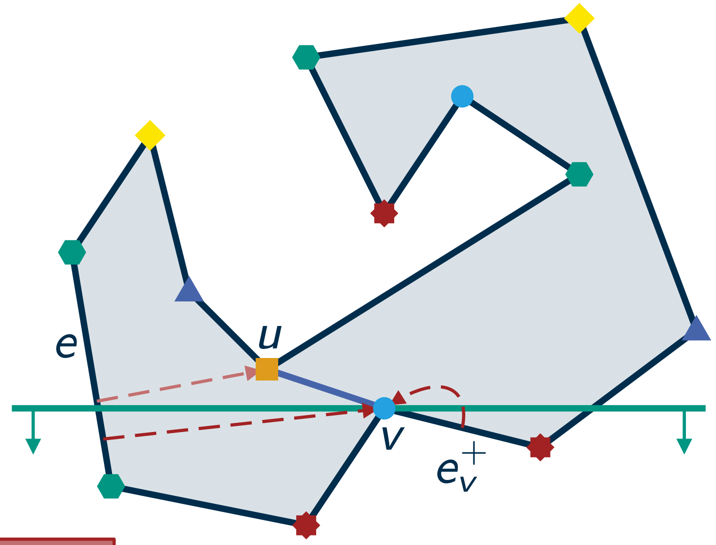
- split: edges below, polygon above
- merge: edges above, polygon below
- start: edges below, polygon below
- end: edges above, polygon above
- left: y-monoton, polygon right
- right: y-monoton, polygon left



Sweep-Line With Different Vertex Types

function MAKEMONOTONE(P)
Input: polygon P (counterclockwise)
Output: diagonals, that make P y-monotone

```
Q = vertices of P sorted by y-coordinate
T = search tree
while Q ≠ ∅
    v = min{Q} and Q = Q − v
    HANDLEVERTEX(v)
```



HANDLESTARTVERTEX(v)

```
    e_v^+ = edge after v in P
    insert e_v^+ into T
    helper(e_v^+) = v
```

HANDLEENDVERTEX(v)

```
    e_v^- = edge before v in P
    remove e_v^- from T
```

HANDLERIGHTVERTEX(v)

```
    e = edge left of v in T
    helper(e) = v
```

HANDLELEFTVERTEX(v)

```
    e_v^- = edge before v in P
    remove e_v^- from T
    e_v^+ = edge after v in P
    insert e_v^+ into T
    helper(e_v^+) = v
```

HANDLEMERGEVERTEX(v)

```
    e = edge left of v in T
    helper(e) = v
    e_v^- = edge before v in P
    remove e_v^- from T
```

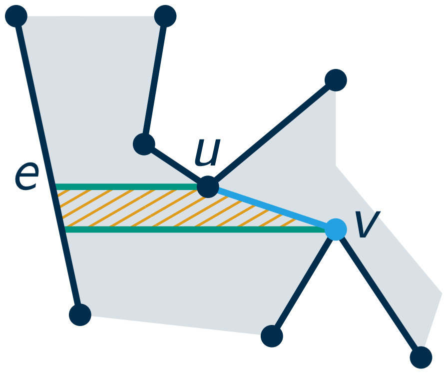
HANDLESPLITVERTEX(v)

```
    e = edge left of v in T
    u = helper(e)
    output diagonal uv
    helper(e) = v
    e_v^+ = edge after v in P
    insert e_v^+ into T
    helper(e_v^+) = v
```

We Create No Intersections

Recall

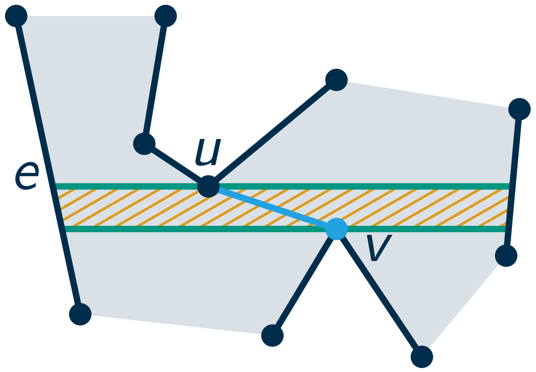
- the inserted diagonals do not intersect the polygon
- core argument: the quadrilateral between uv and e contains no vertex



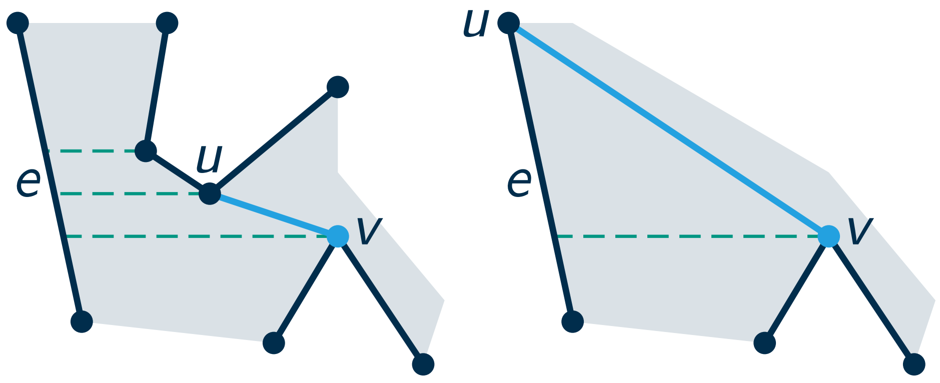
Can We Get An Intersection With A Previously Inserted Diagonal?

- extend the quadrilateral to the right
- same argument: the extended quadrilateral also contains no vertex
- both vertices of a previously inserted diagonal lie above v

$\Rightarrow$ uv does not intersect a previous diagonal



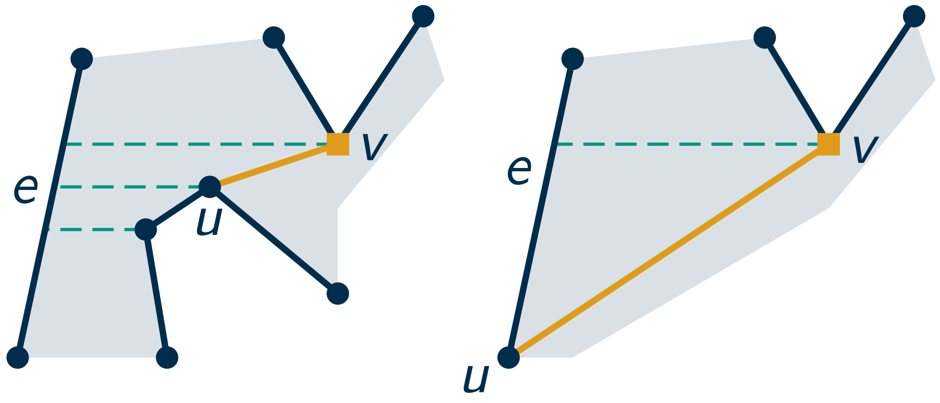
And What About Merge Vertices?



Recall: Split Vertex v

- e: edge to the left of v
- u: lowest vertex above v that has e to its left (helper of e)
- if it doesn't exist: choose u as the upper vertex of e
- connect v with u

mirroring translates v into a merge vertex



Handling A Merge Vertex v

- solution for the lazy theoretician:
  - just run it again in the opposite direction
- alternative:
  - observe: v is the helper of e when we process u
  - when e ends at u or gets a new helper u:
    current helper v is merge vertex $\rightarrow$ insert uv

Wrap-Up

Theorem **(subdivision into y-monotone pieces)**
A polygon with n vertices can be subdivided into y-monotone pieces in $O(n \log n)$ time.

What Else Have We Learned Today?

- additional application of the sweep line technique
- concept of monotonicity
- splitting a complicated problem into two simpler subproblems

What Else Is There?

- lower bound of $\Omega(n \log n)$ if the polygon can have holes
- $O(n \log \log n)$, $O(n \log^* n)$, and even $O(n)$ is possible, if the polygon has no holes
- corresponding 3-dimensional problem is NP-hard

References

$O(n \log \log n)$

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$O(n \log^* n)$

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$O(n)$

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