Computational Geometry

Introduction and Convex Hull

Thomas Bläsius

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The Things We Deal With

- points, lines, line segments, circles, polygons, . . .

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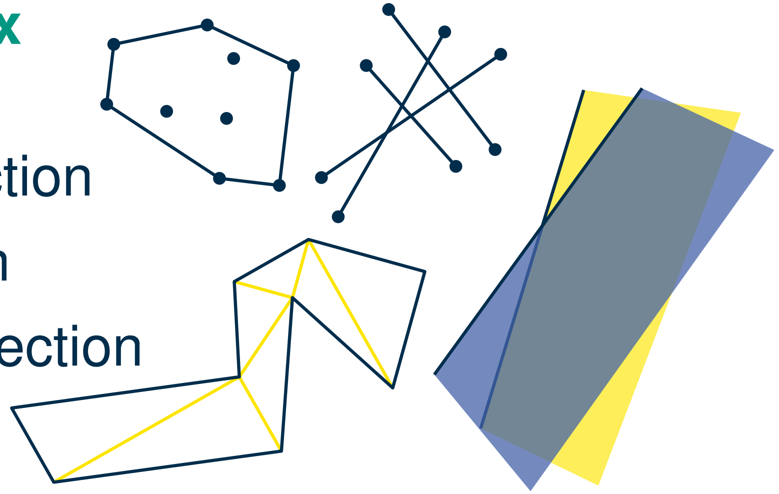
The Things We Deal With

- points, lines, line segments, circles, polygons, . . .
- but not: pixels

What Does That Mean Specifically?

Basic Toolbox

- convex hull
- line intersection
- triangulation
- plane intersection



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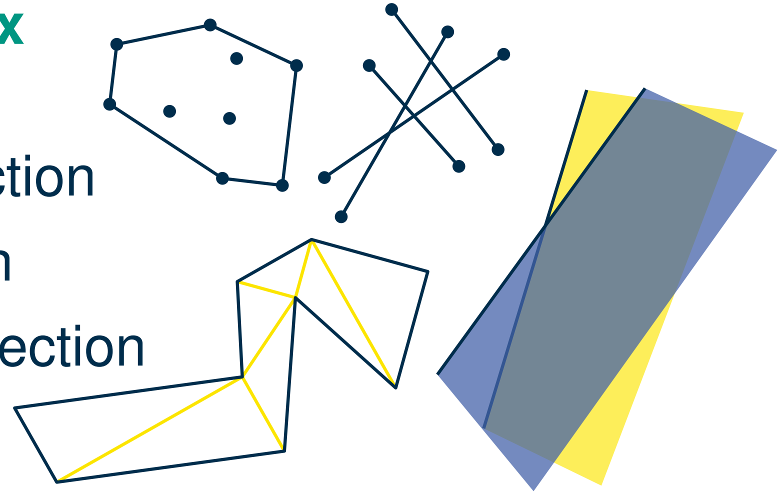
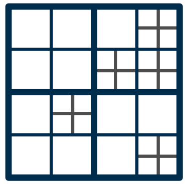
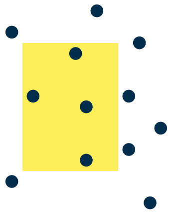
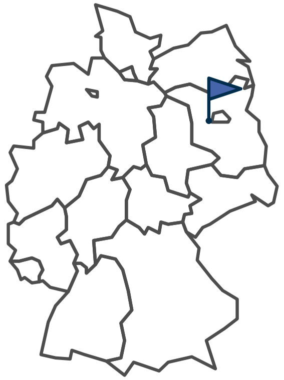


Geometric Data Structures

- orthogonal range searching
- space partitioning
- point location







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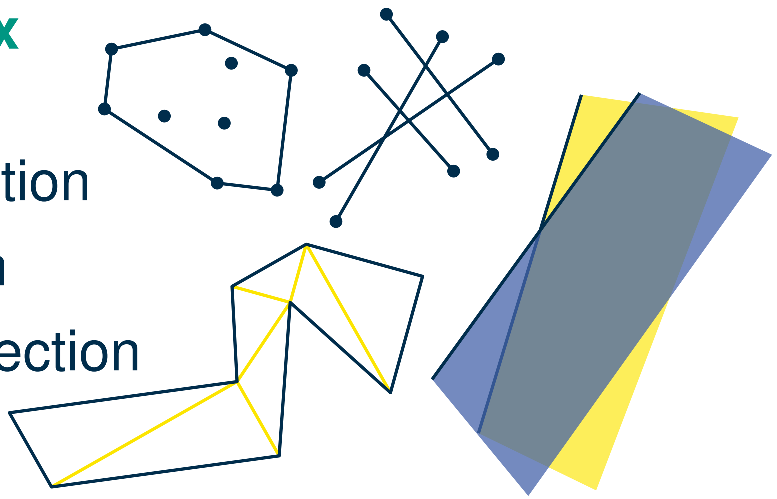

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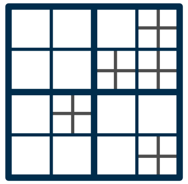
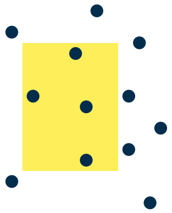
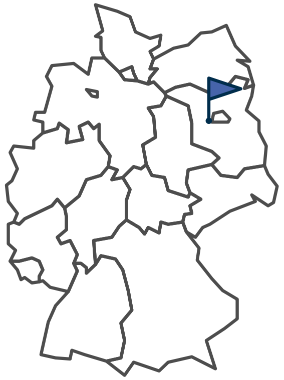
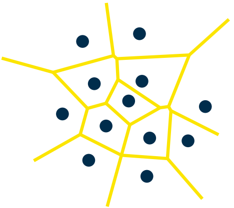
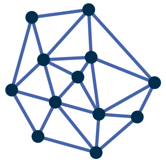


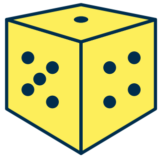




Advanced Toolbox
- Voronoi diagrams
- Delaunay triangulations
- randomized algorithms
- complexity







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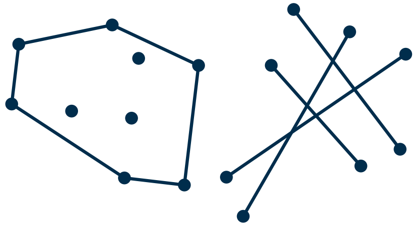

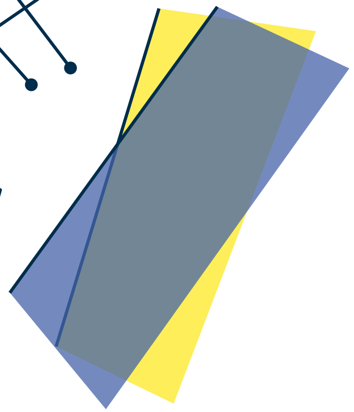

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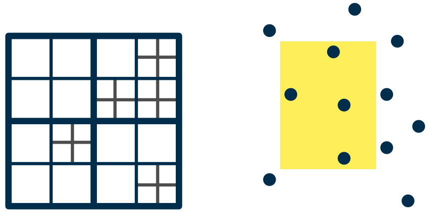

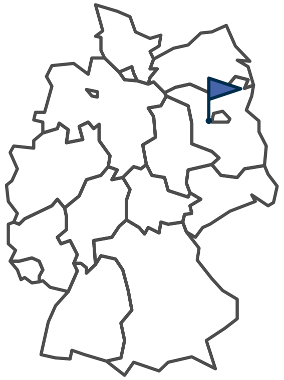

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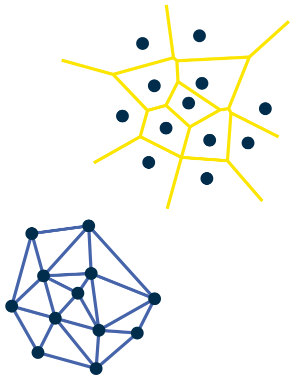
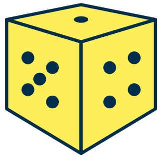
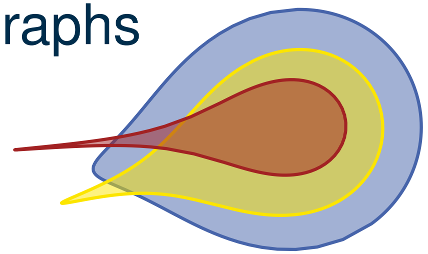
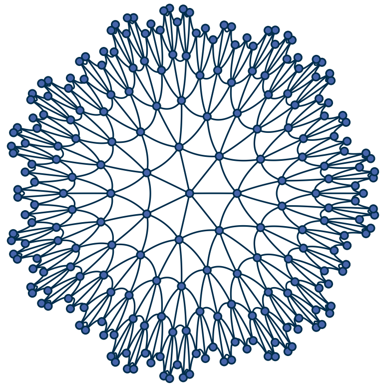




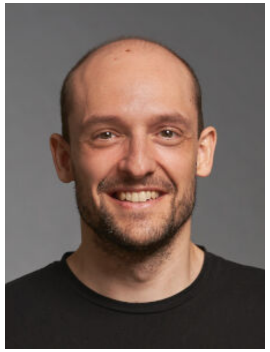
Related Topics

- What is geometry?
- hyperbolic geometry
- geometric graphs





Before We Start



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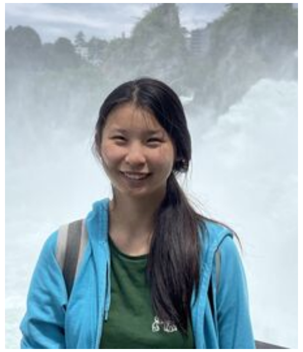

Wendy

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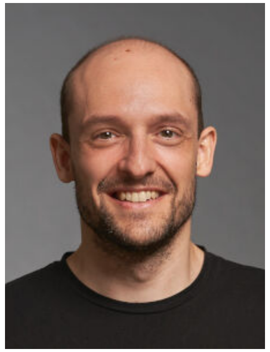
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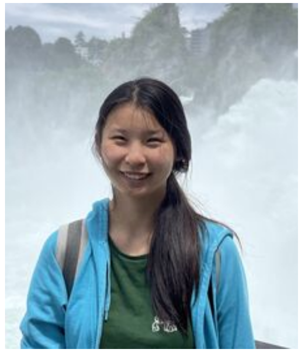
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You

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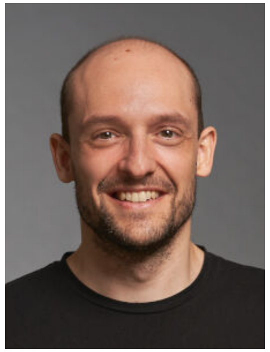
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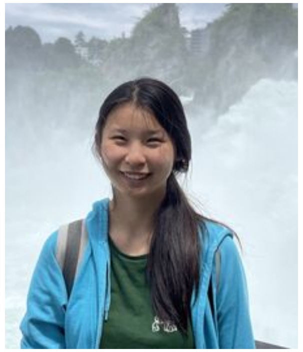
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Materials & Infos

- slides, exercise sheets on our homepage: `https://scale.iti.kit.edu/teaching/2025ss/comput_geom/`

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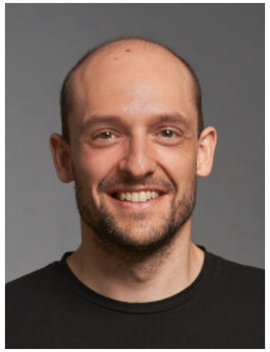
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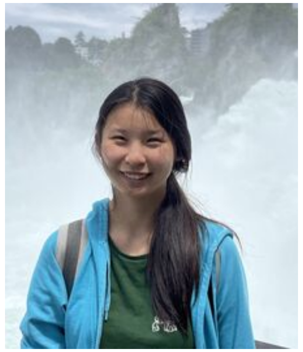
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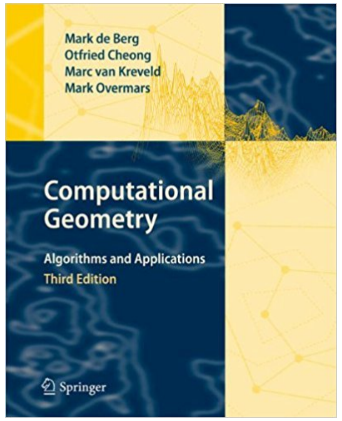

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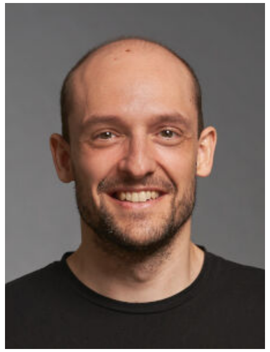

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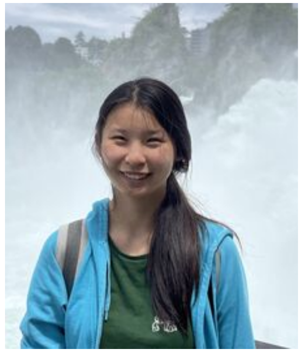

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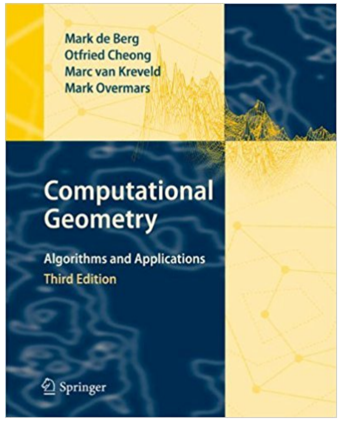

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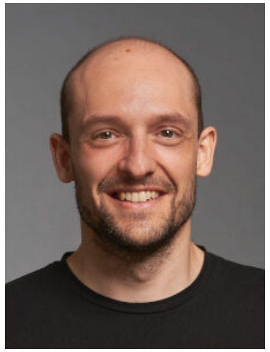

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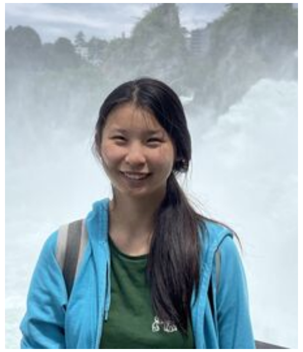

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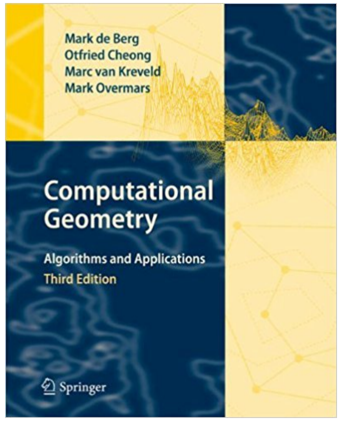


Requirements

- good algorithmic understanding
- no (little) prior knowledge

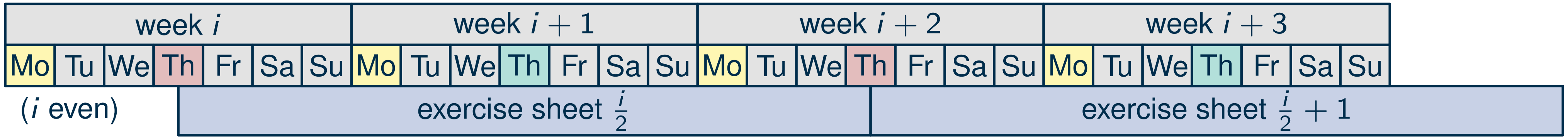
Rough Schedule

week i							week $i+1$							week $i+2$							week $i+3$						
Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa	Su
(i even)			exercise sheet $\frac{i}{2}$														exercise sheet $\frac{i}{2}+1$										

Lecture

- lecture with slides
- new topics

Rough Schedule



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- hand in in groups of two or three
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- recap
- support solving exercise sheets
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- training additional skills
- curiosities

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Exam
- oral exam (20 min)
- admission only with exercise certificate

Exercise Certificate

Goal: $\frac{1}{2}$ of the points in total **and** $\frac{1}{4}$ on every exercise sheet

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What If I Don't Find The Solution?

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we don’t want to make your life hard and we also don’t bite
we just want you to learn something and have fun doing so

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Our Goal

- you spend some time with the content of the lecture and write down your solution
- then, the exercise certificate should not be a big obstacle

Motivation

Different Mixtures Of Oil

- the exact ratio between different components depends on the oil spring
- goal: mix oil from different springs, such that the result is easy to process

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spring 1	35%	10%
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- oil contains components A and B
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- third spring:

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spring 3	15%	7%

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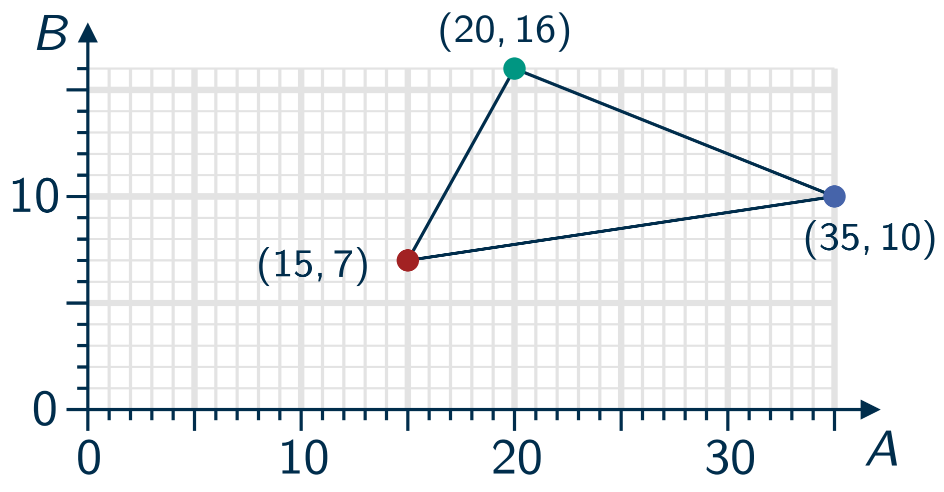

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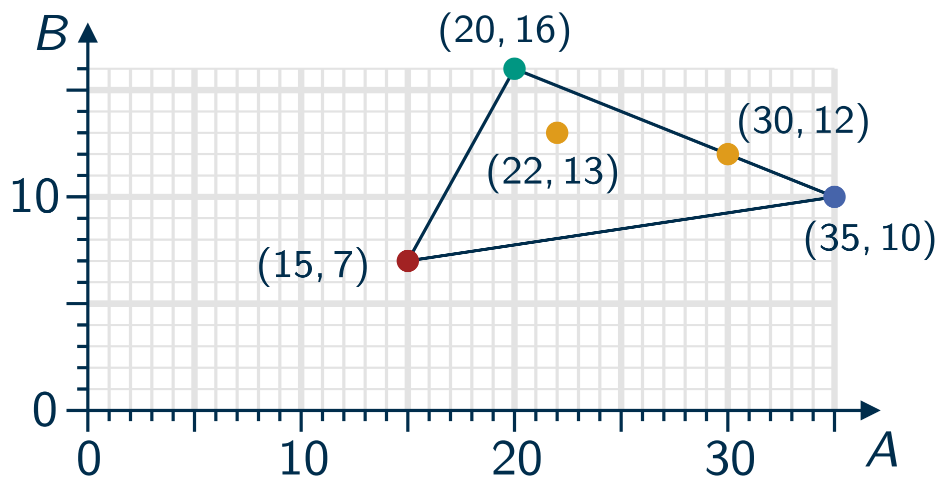
What Is The Relation To Geometry?

- ratios can be interpreted as points
- desired ratio is possible ⇔ corresponding points lies "between" the available points



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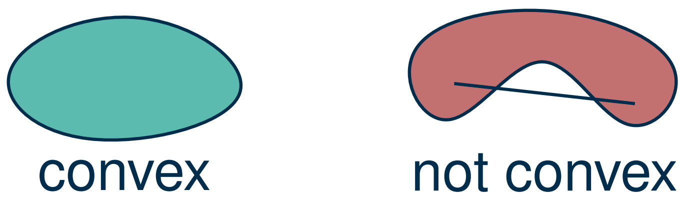
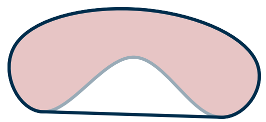


Definition
For $P \subseteq \mathbb{R}^d$, the **convex hull** $\boldsymbol{\mathcal{CH}(P)}$ is the minimal subset of $\mathbb{R}^d$ such that $\mathcal{CH}(P)$ is convex and $P \subseteq \mathcal{CH}(P)$.



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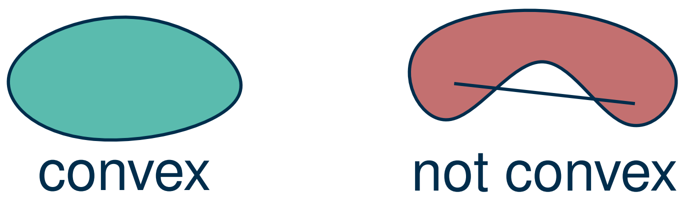

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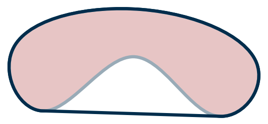

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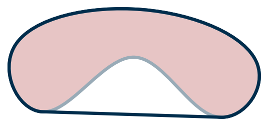

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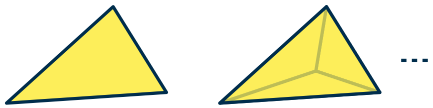
simplices in different dimensions:



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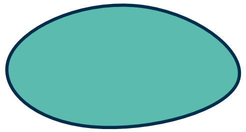
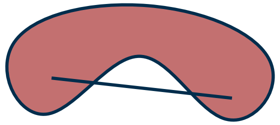

convex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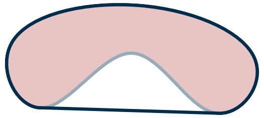

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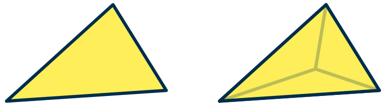
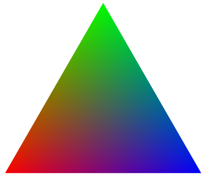
simplices in different dimensions:  ...

- set of all points that are convex combinations of points in P

convex combination: $\sum_{i=1}^{n} a_i \cdot p_i$ with $p_i \in P$, $a_i \in \mathbb{R}$, $a_i \geq 0$, and $\sum_{i=1}^{n} a_i = 1$

you might know this from the barycentric coordinate system



Convex Hull – Trivial Algorithm

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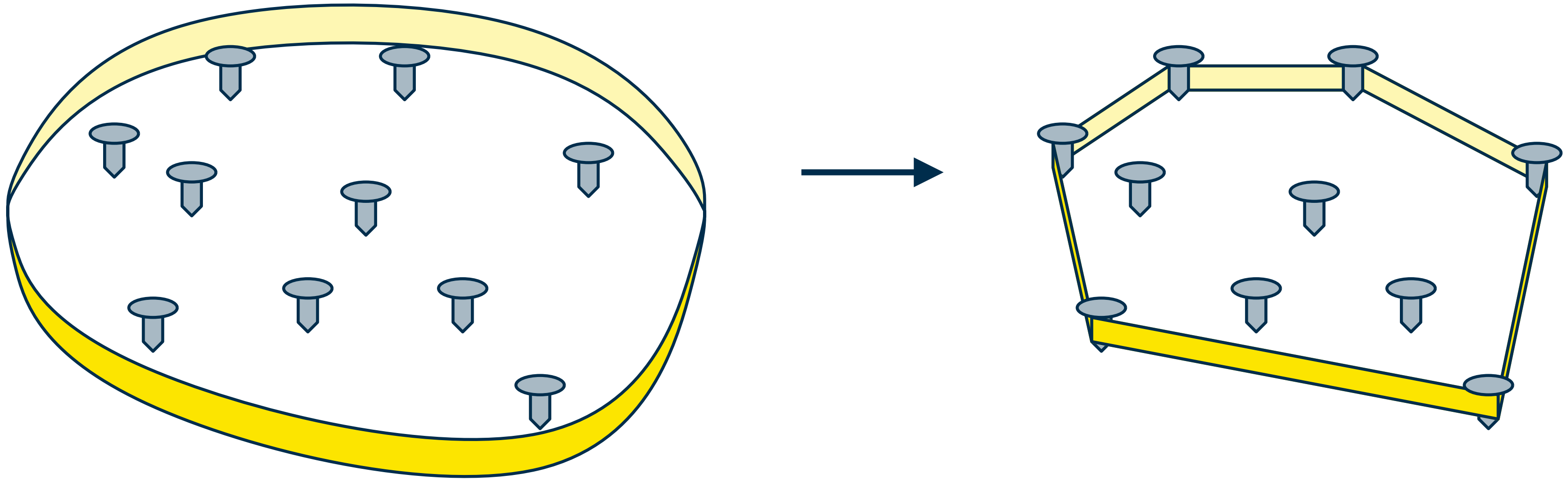
Intuition



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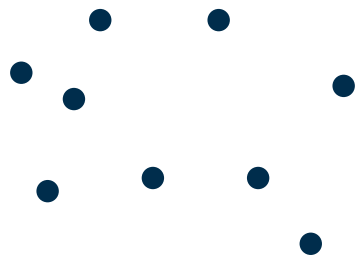

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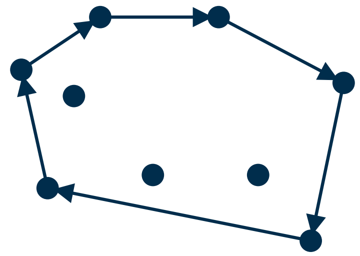

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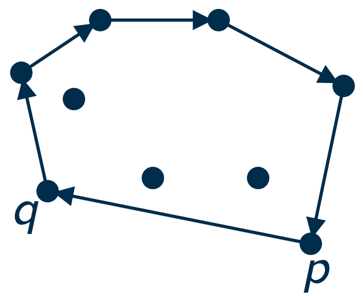

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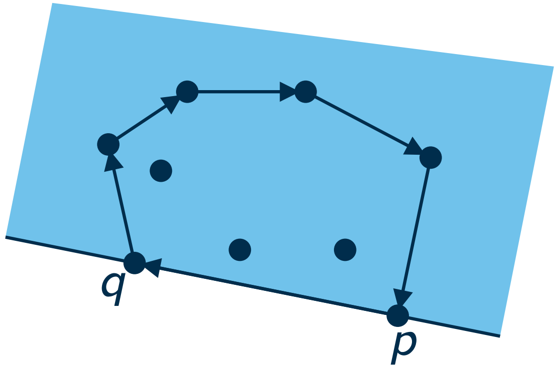

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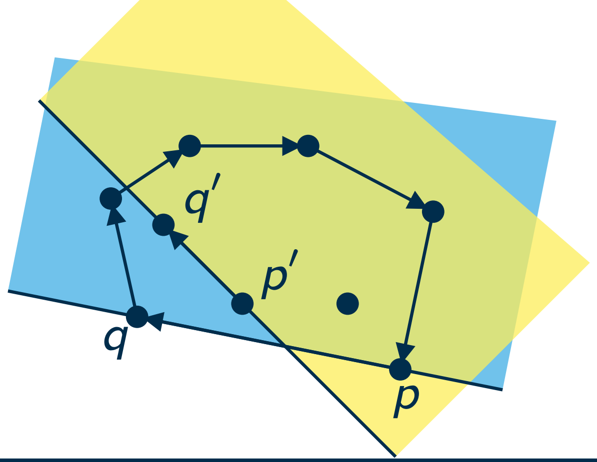

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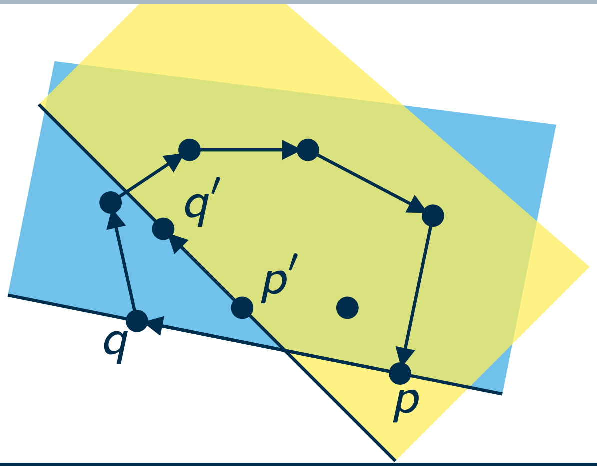

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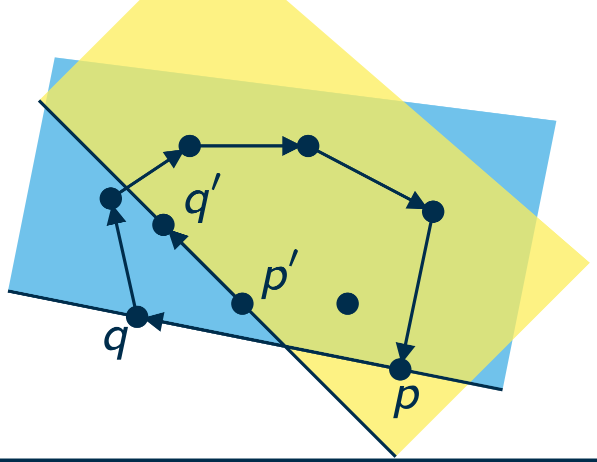

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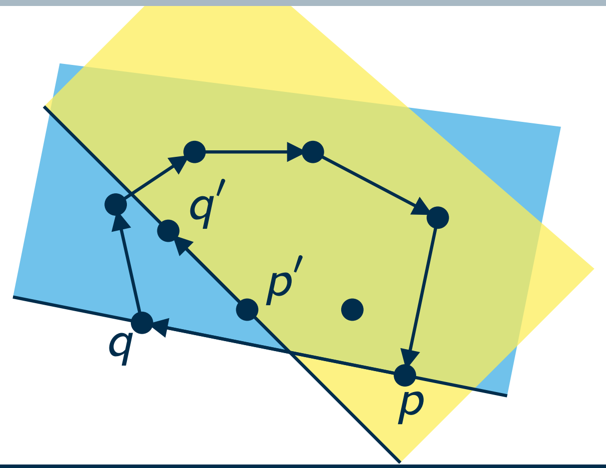

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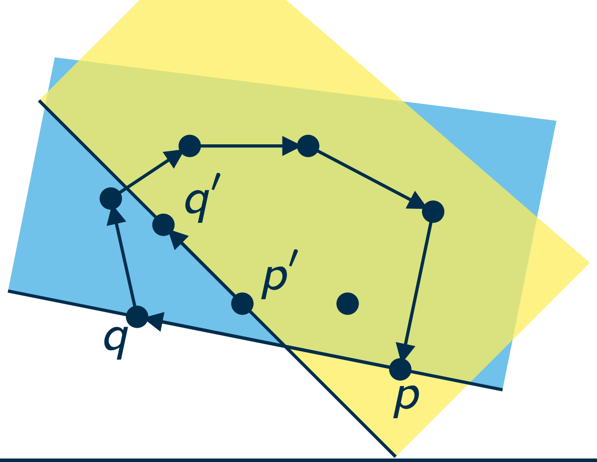

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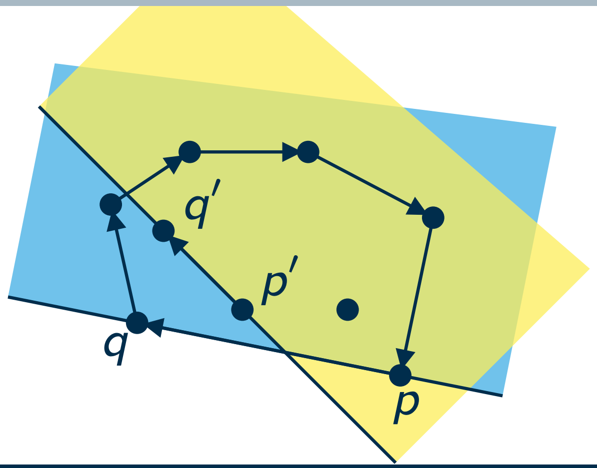

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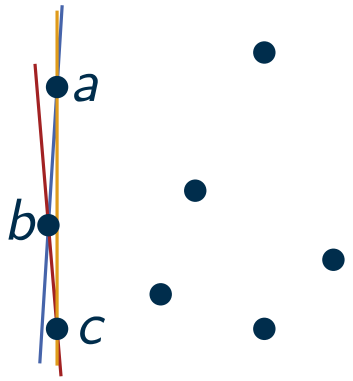

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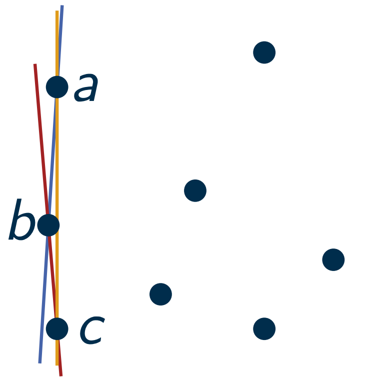
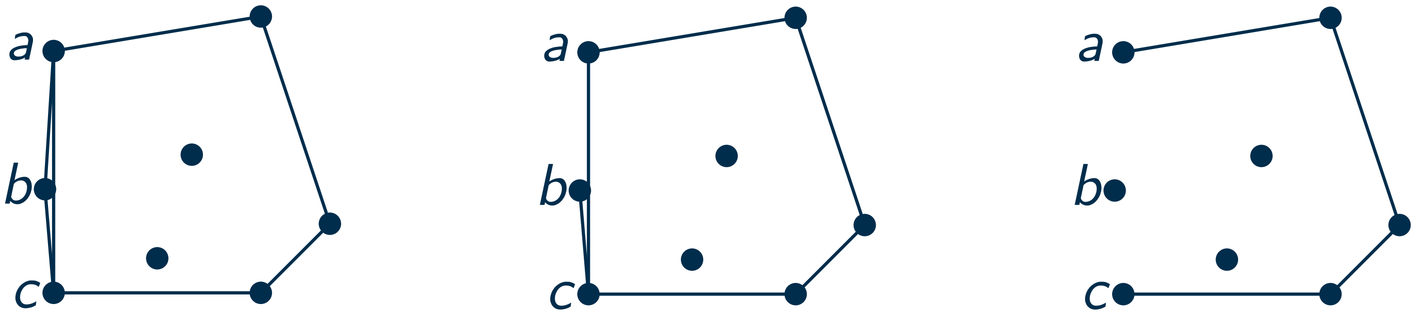


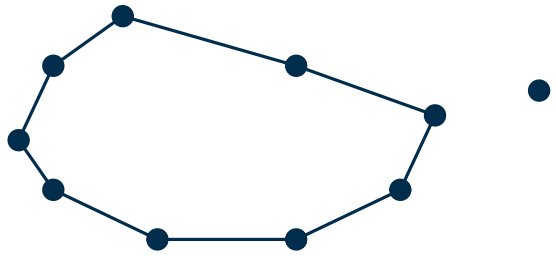
- three decisions “lies to the right of” are close
- wrong decision $\rightarrow$ output maybe not a polygon



Andrews Monotone Chain Algorithm
(variant of the Graham Scan)

Idea: Iterative Approach

- add points one after another
- update convex hull in each step



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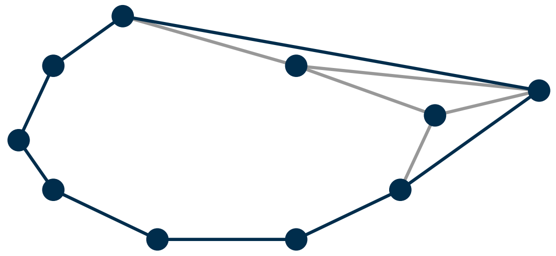

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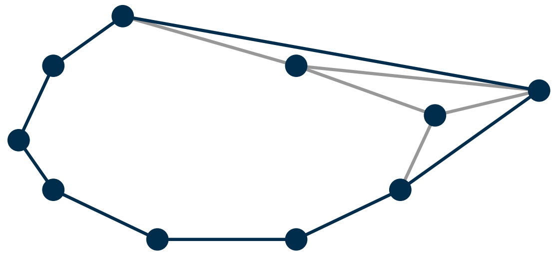

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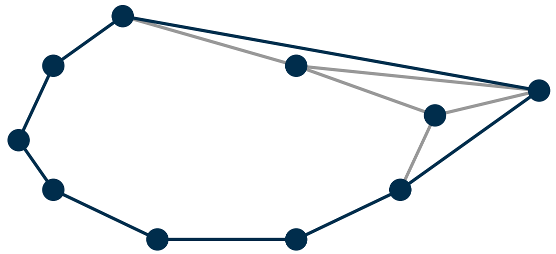

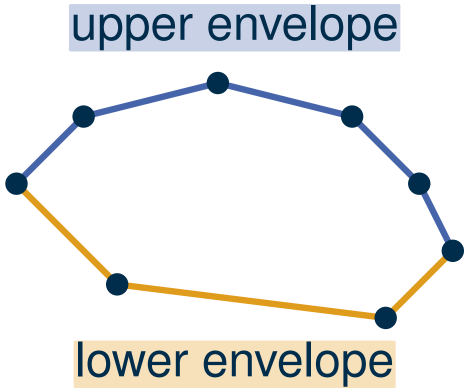

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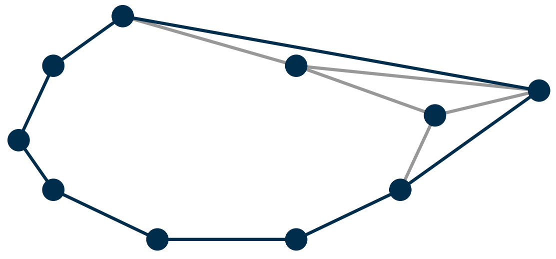

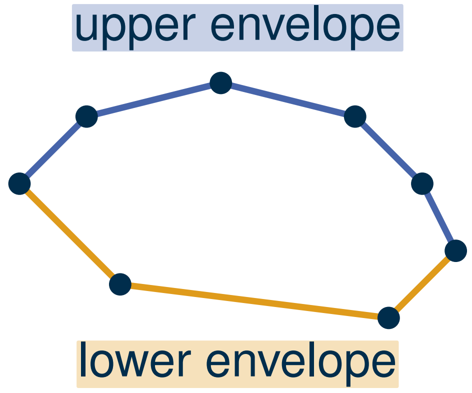

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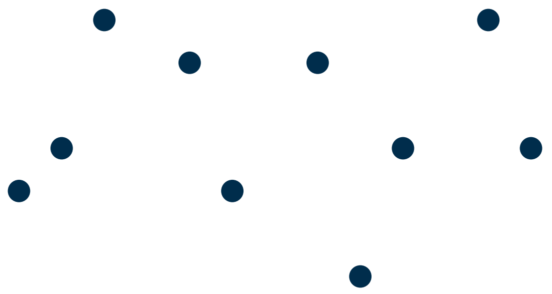

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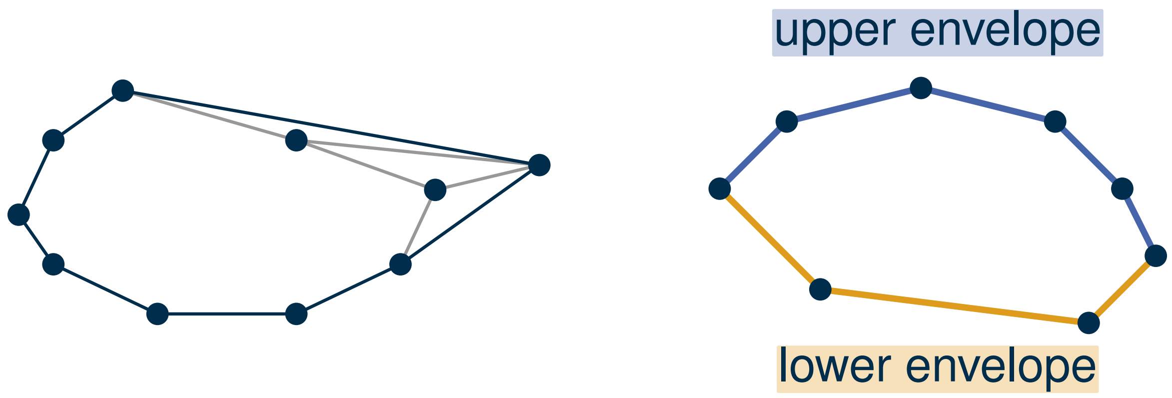

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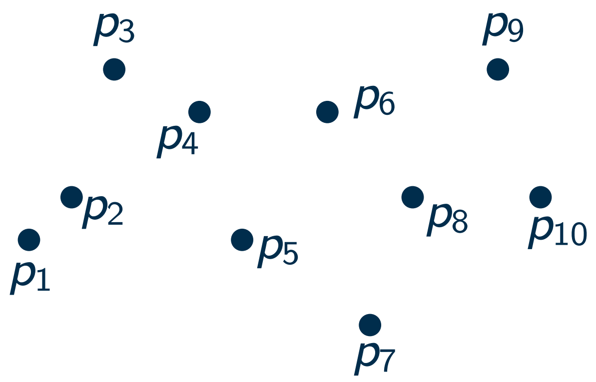

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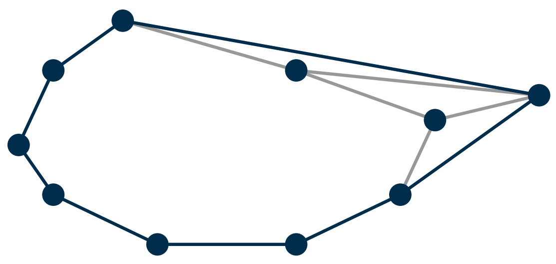

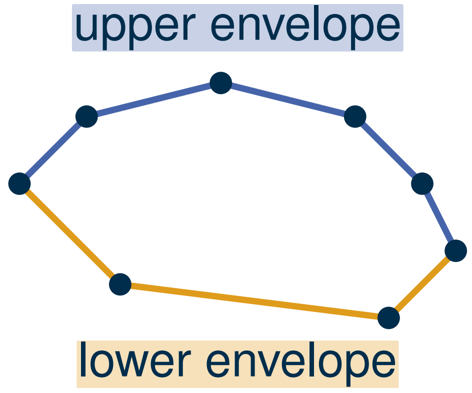

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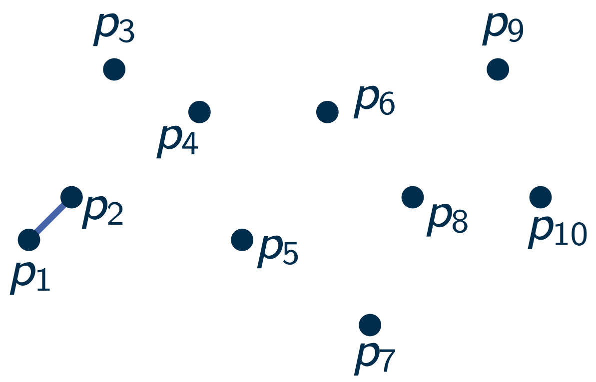

L: p_1 p_2

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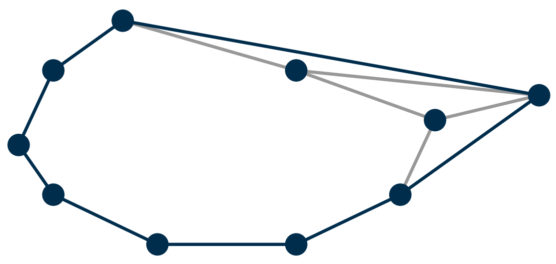

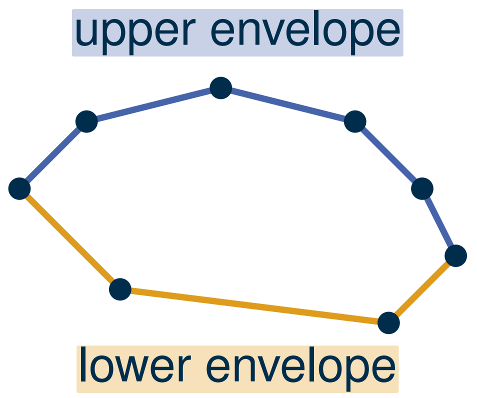

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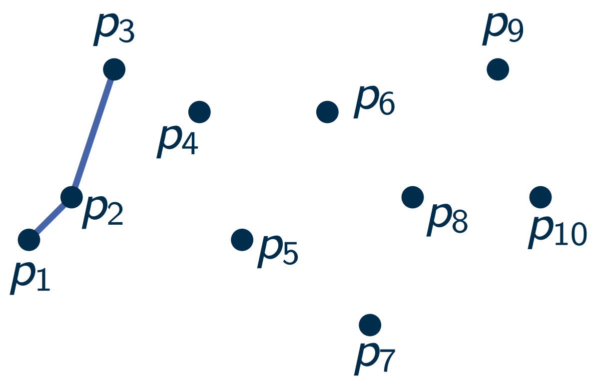

L: p_1 p_2 p_3

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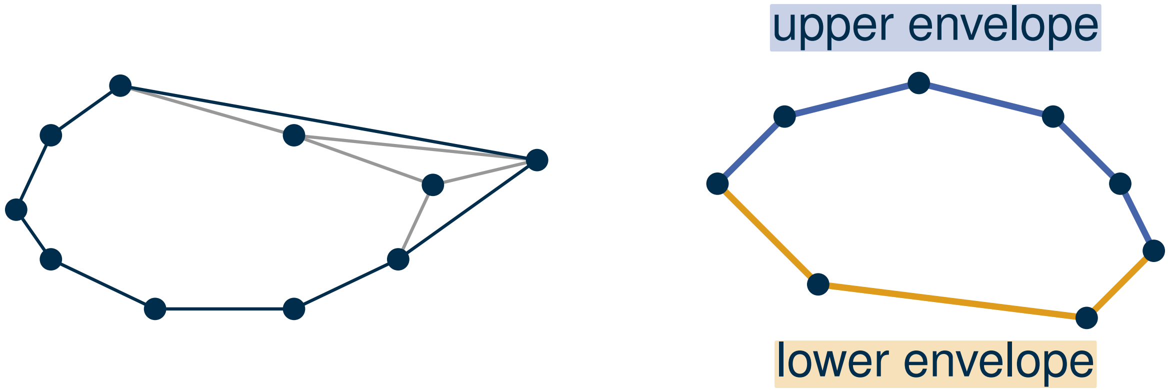

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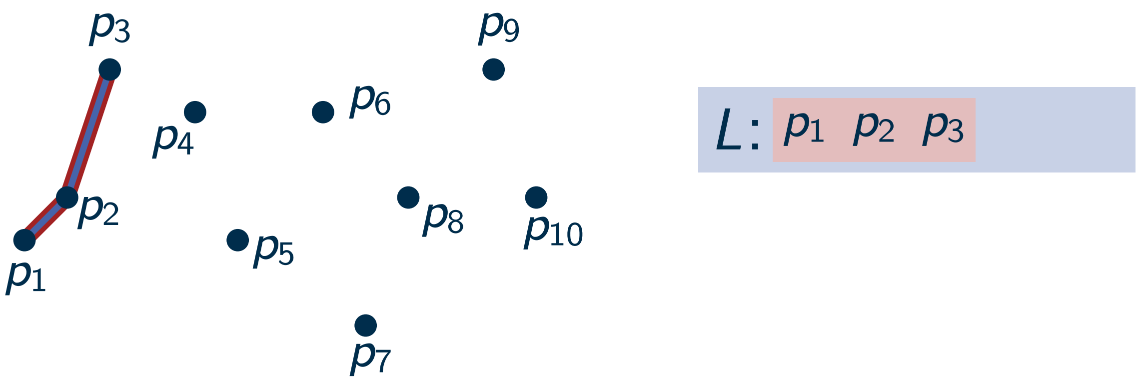

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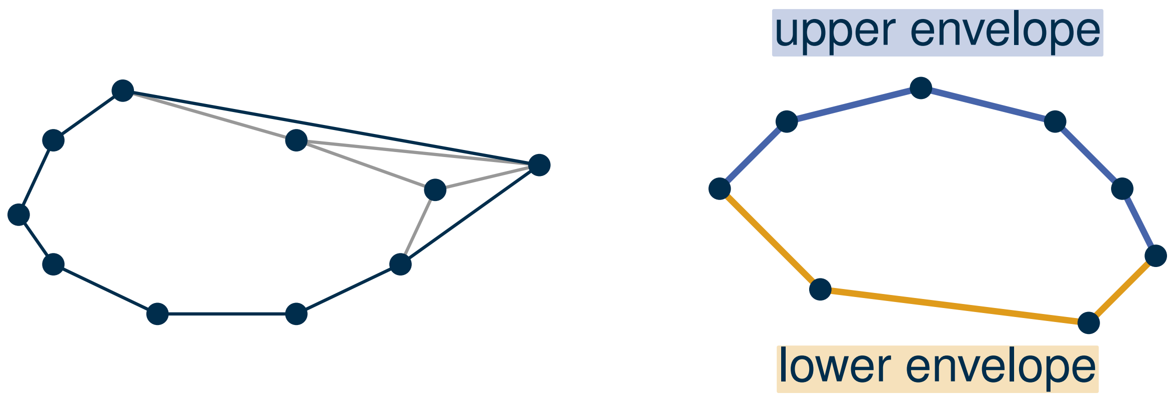

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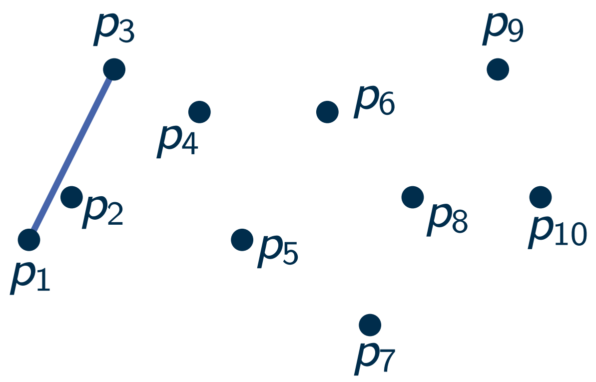

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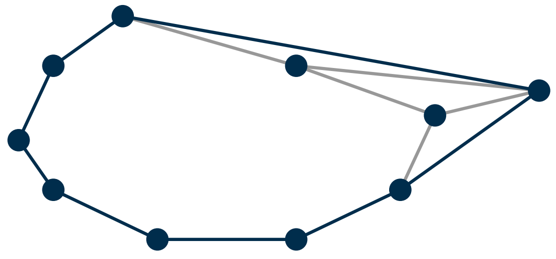
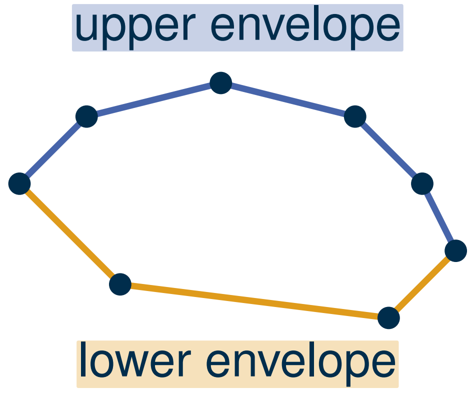
Andrews Monotone Chain Algorithm

(variant of the Graham Scan)





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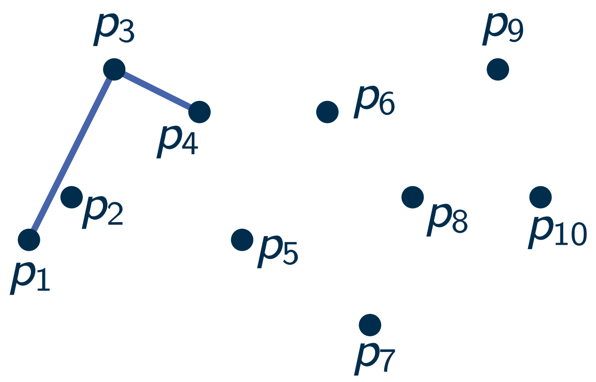

L: p_1 p_3 p_4

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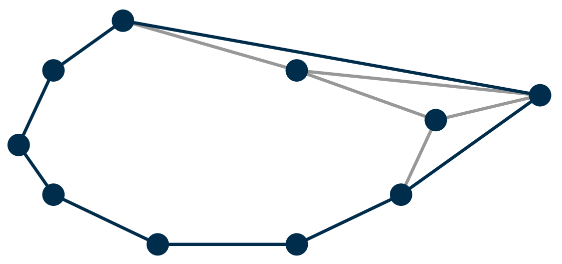

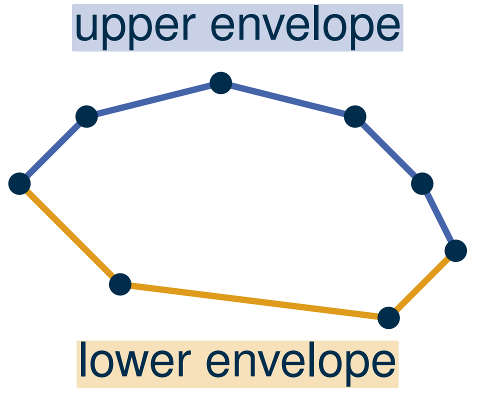

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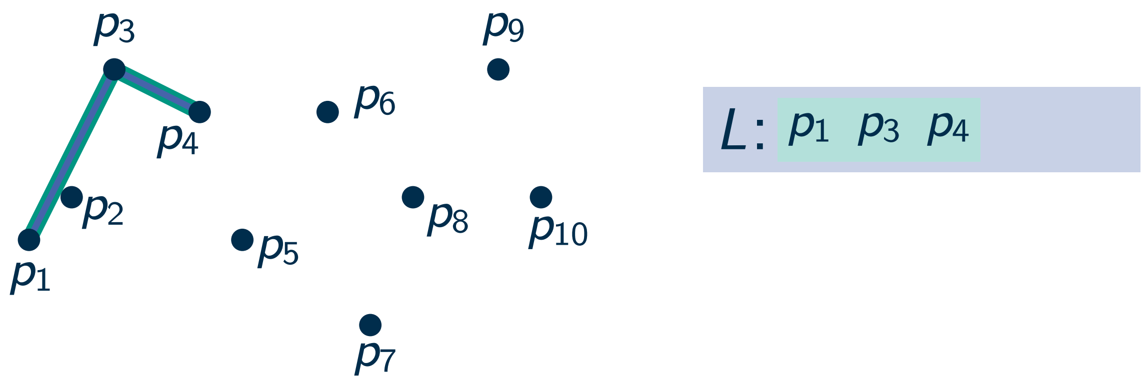

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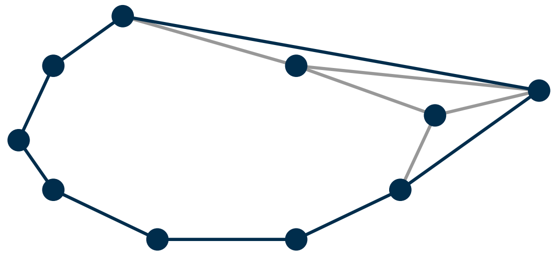

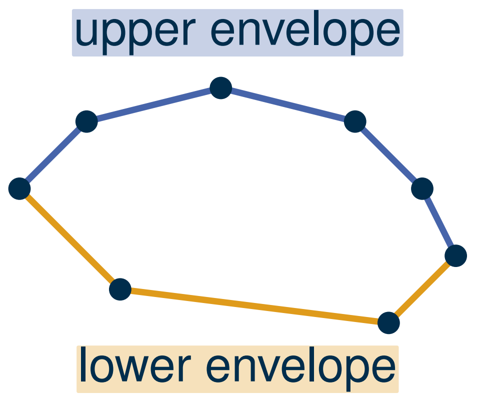

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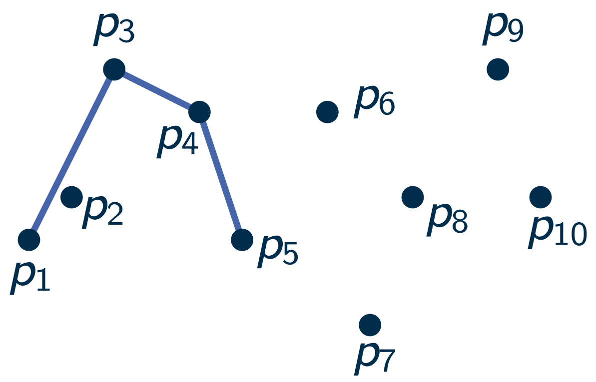

L: p_1 p_3 p_4 p_5

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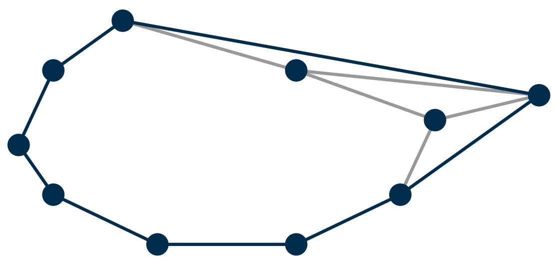

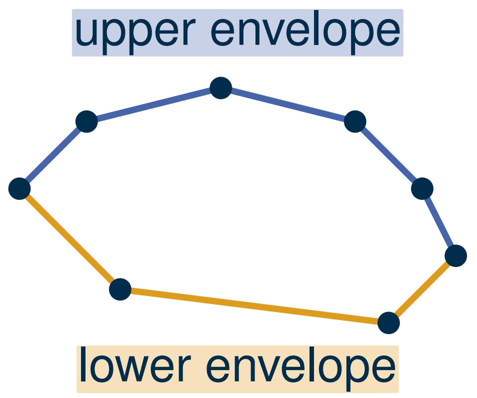

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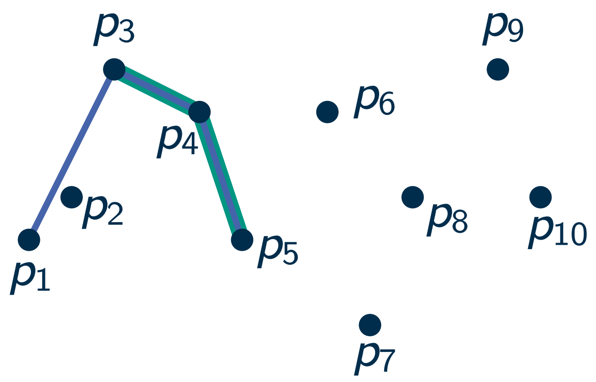

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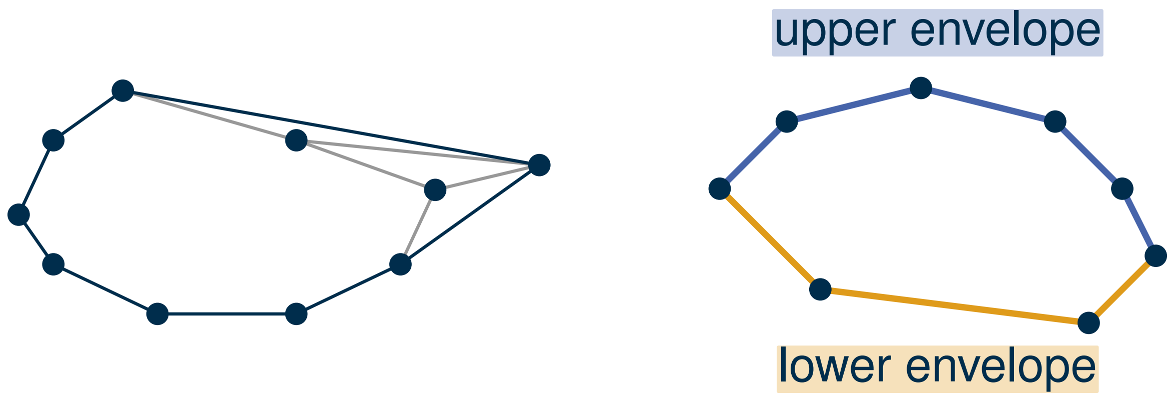

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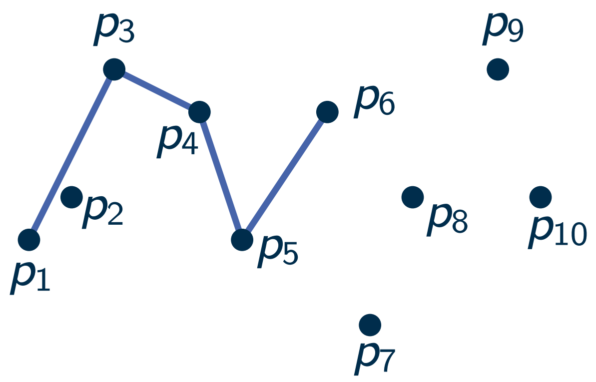

L: p_1 p_3 p_4 p_5 p_6

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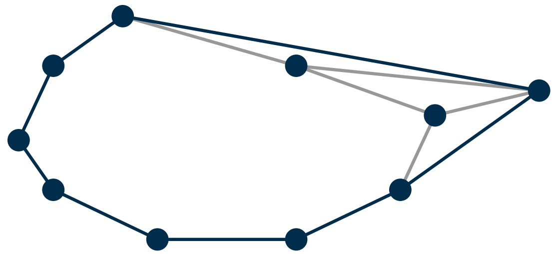

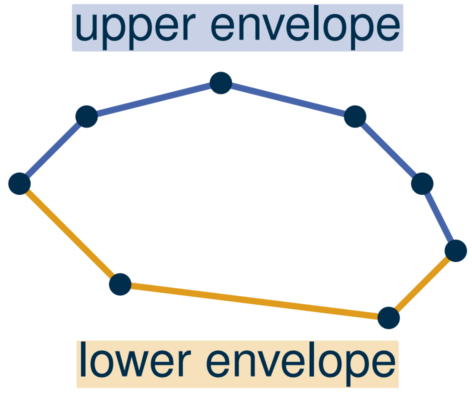

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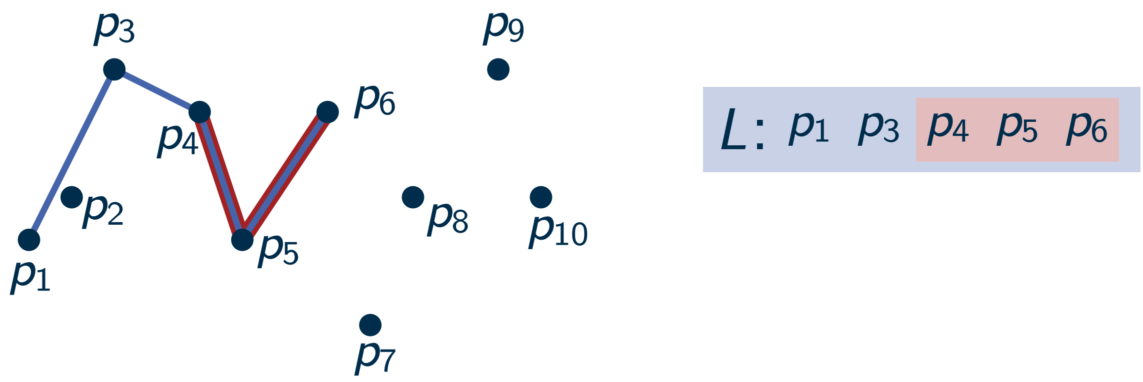

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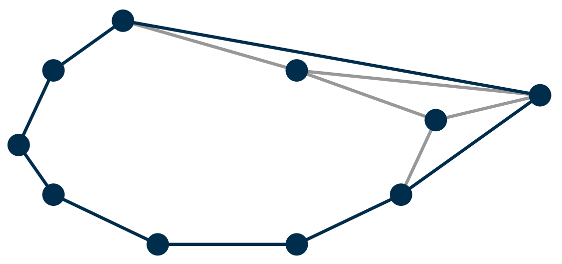

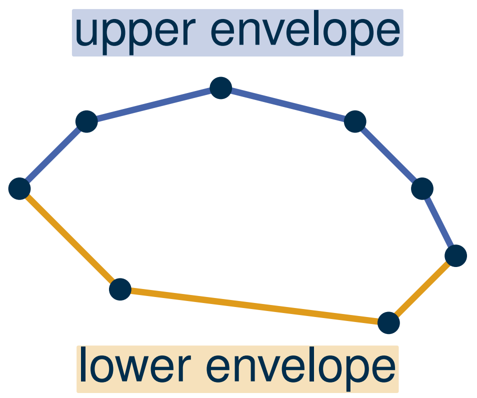

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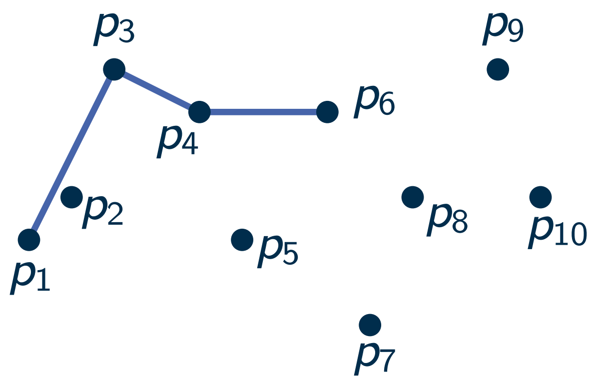

L: p_1 p_3 p_4 p_6

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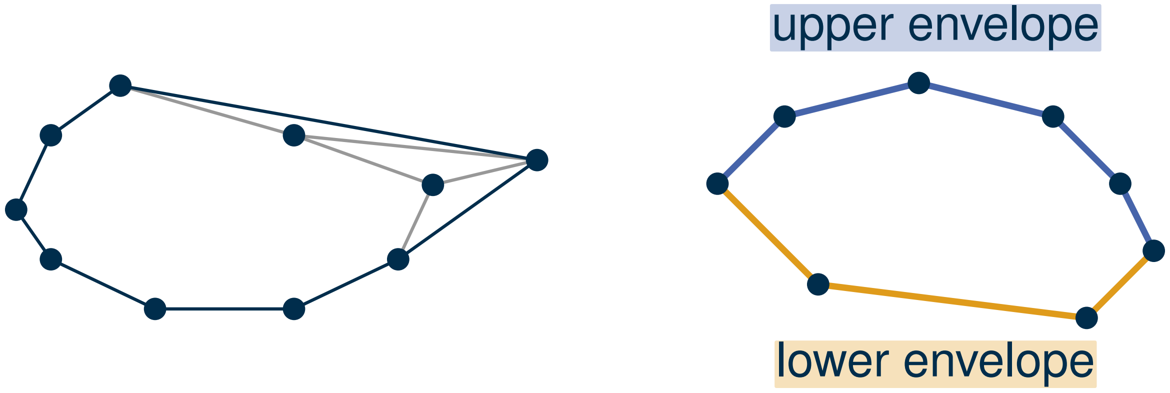

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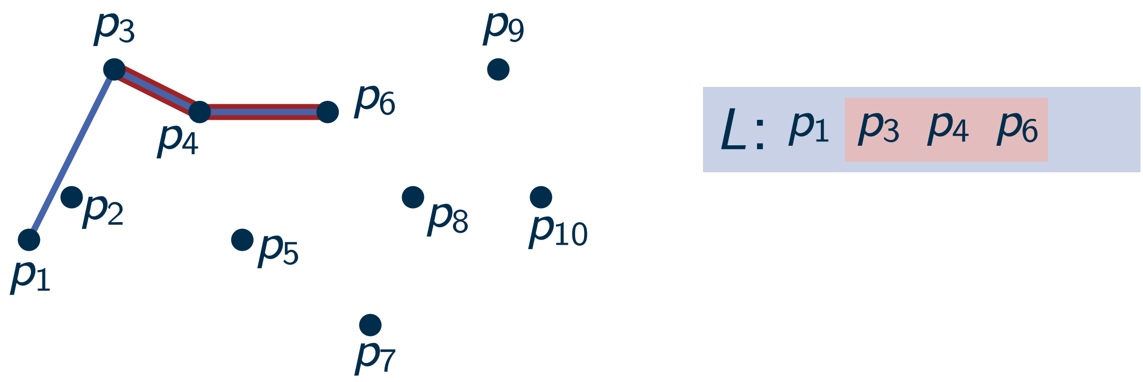

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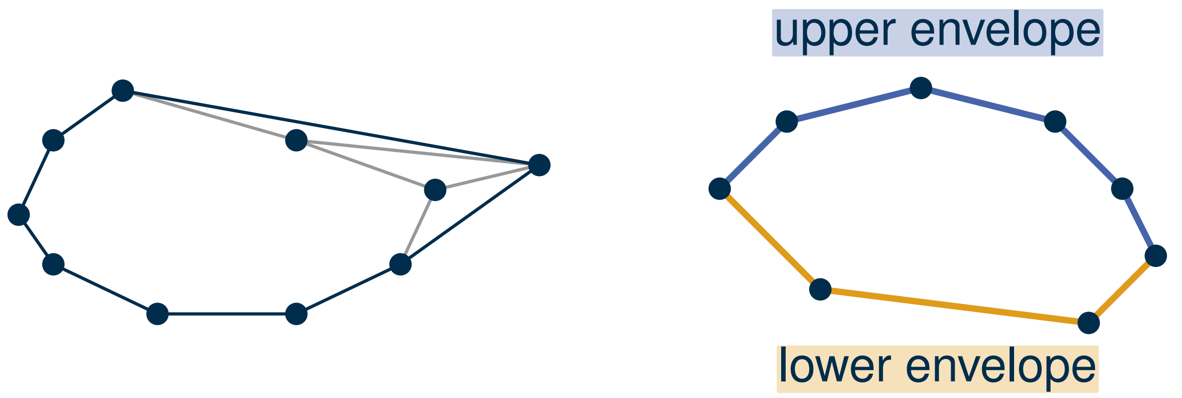

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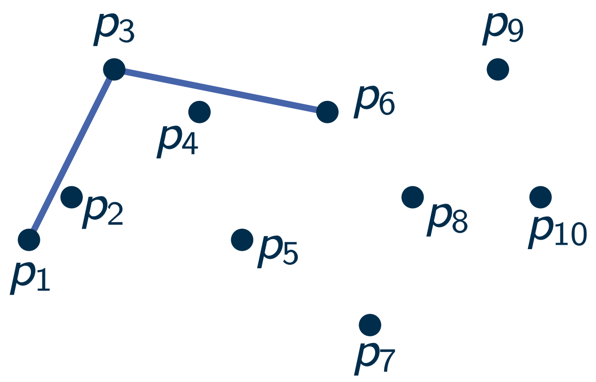

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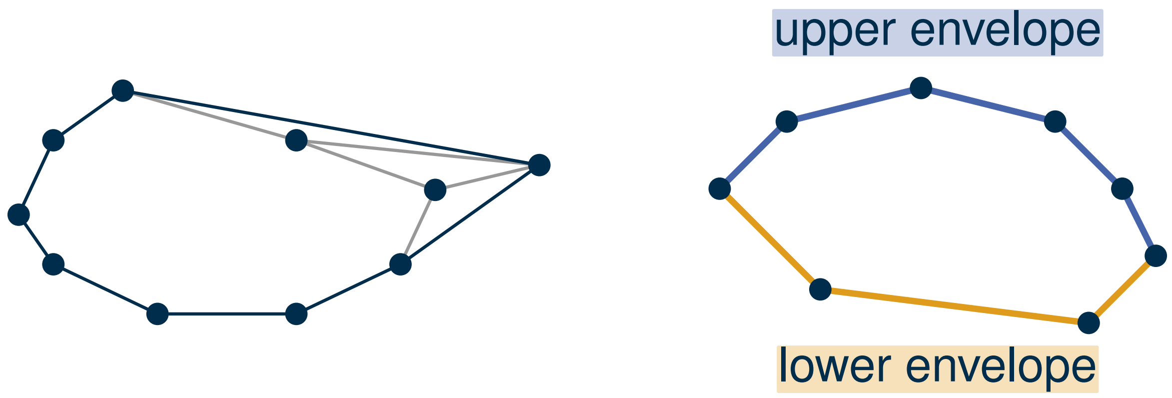

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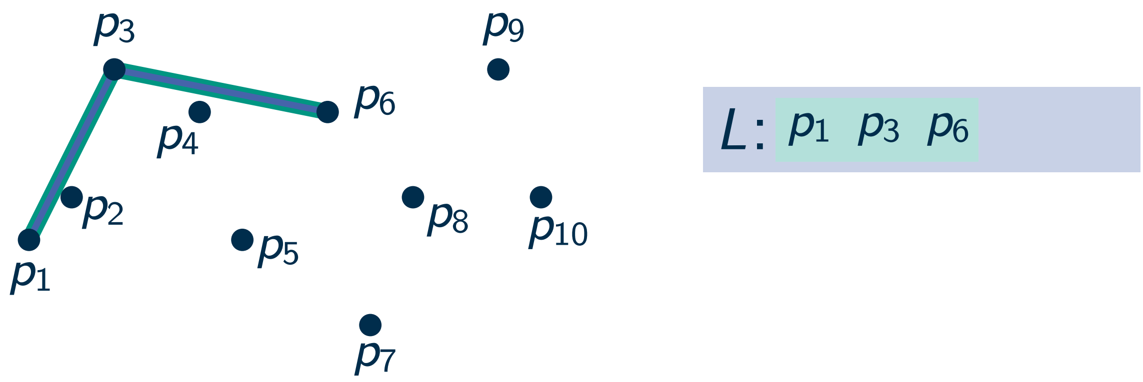

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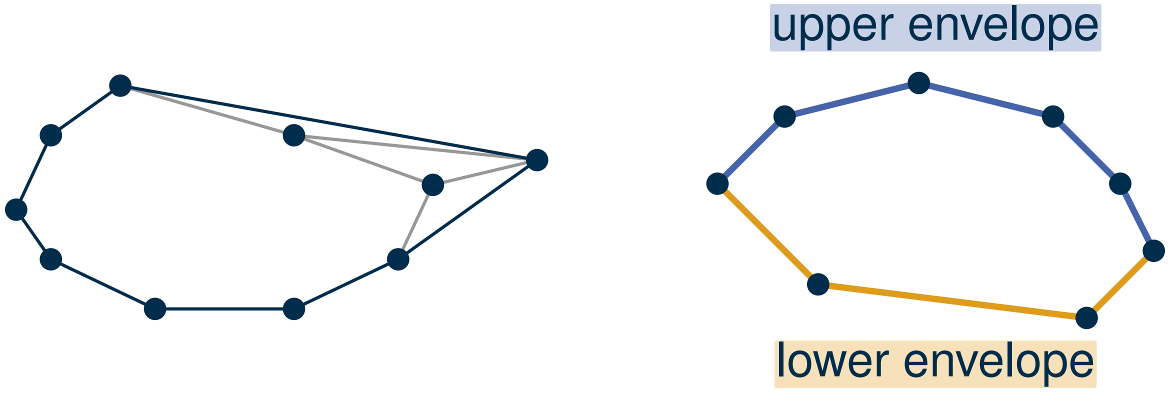

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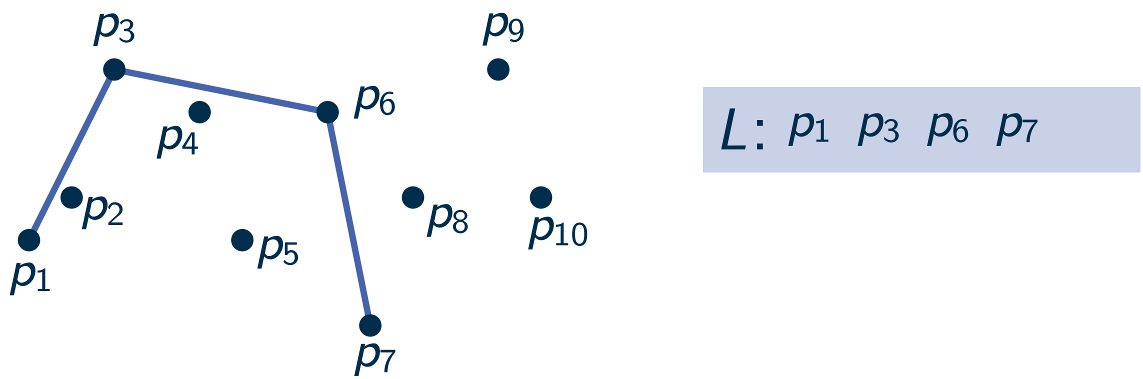

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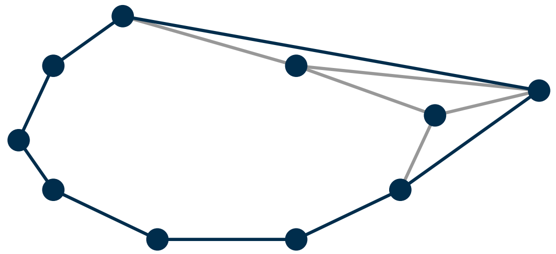

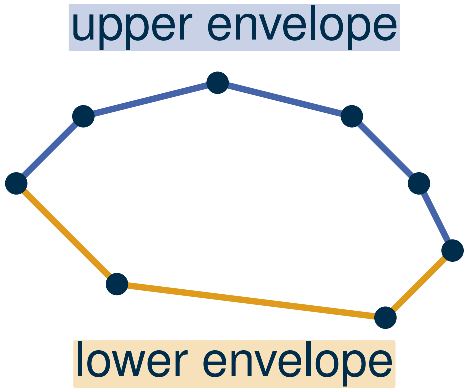

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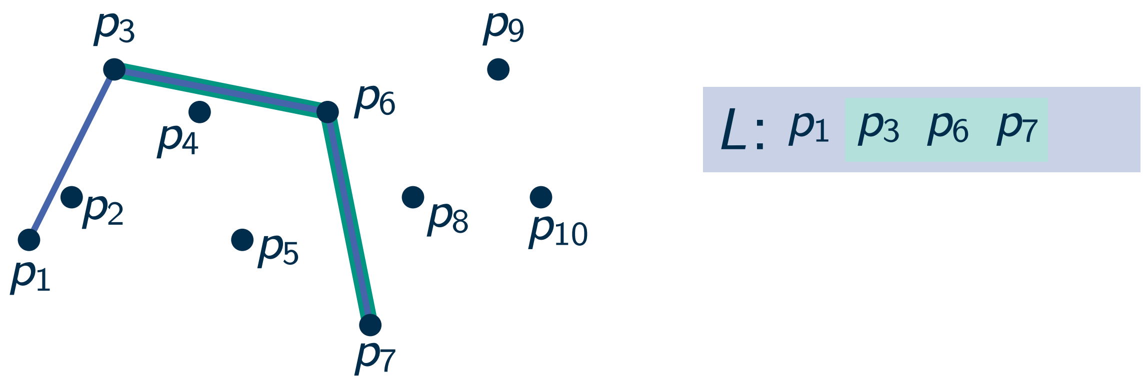

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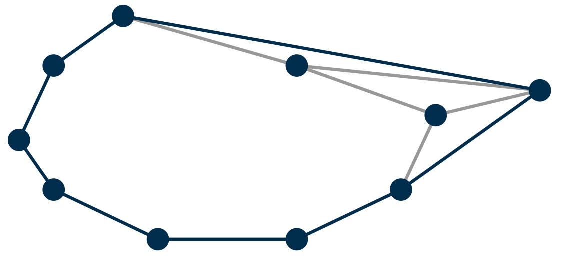

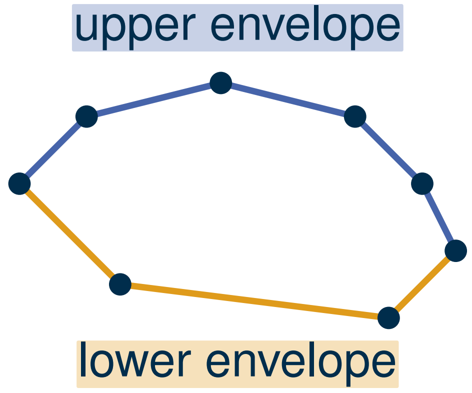

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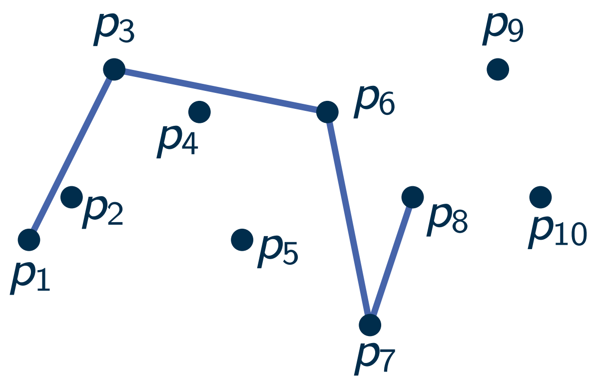

L: p_1 p_3 p_6 p_7 p_8

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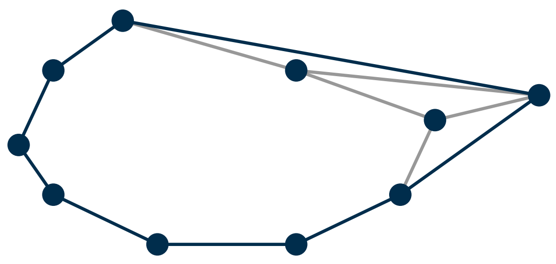

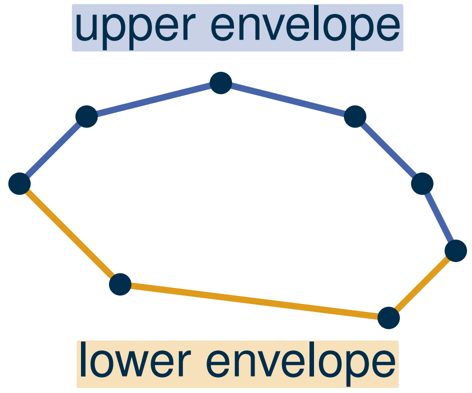

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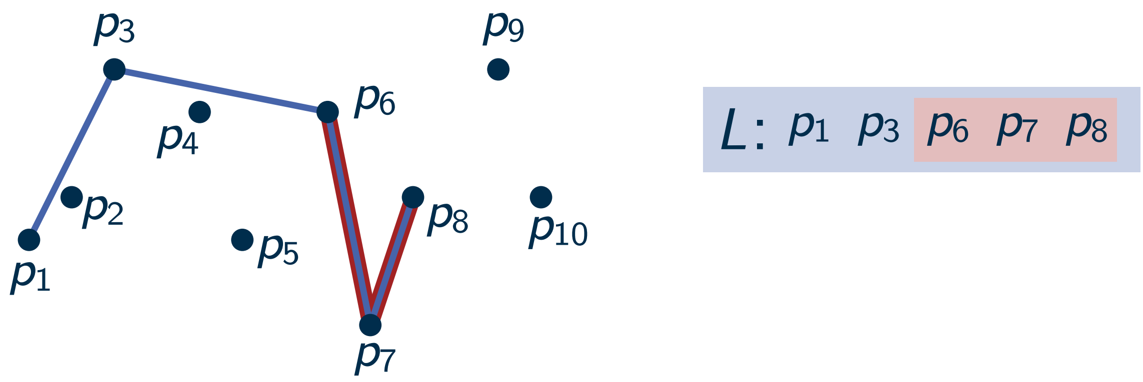

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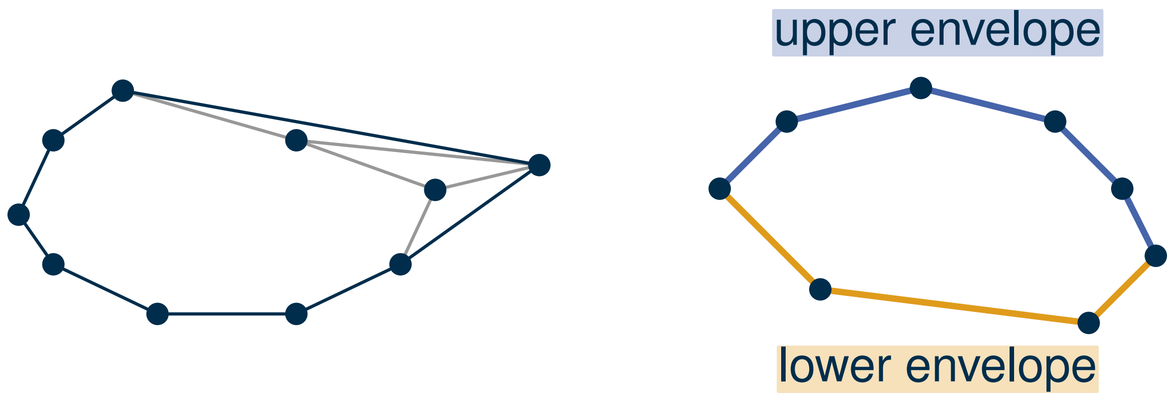

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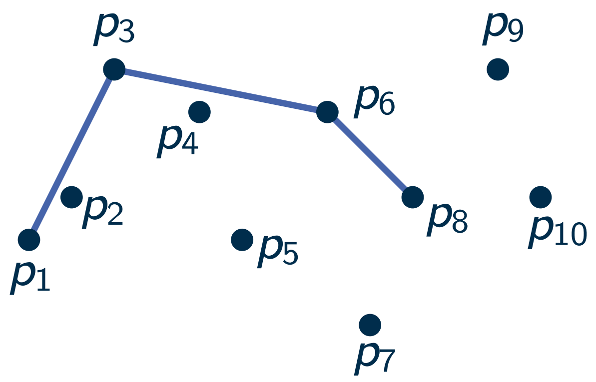

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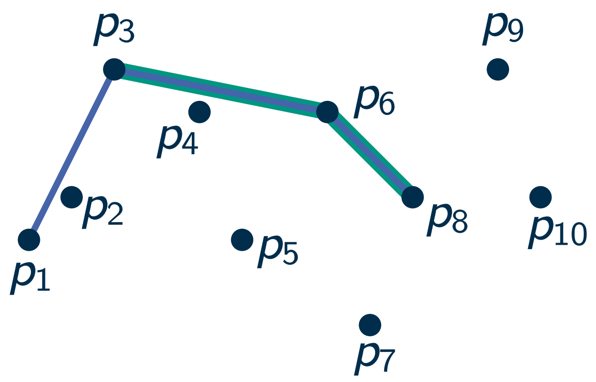

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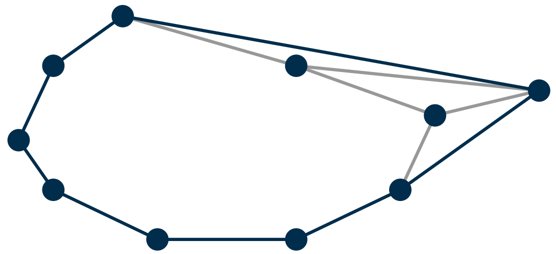

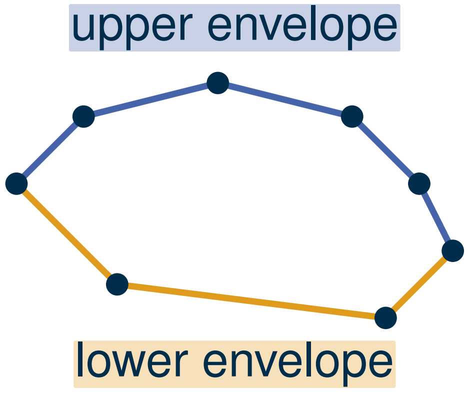

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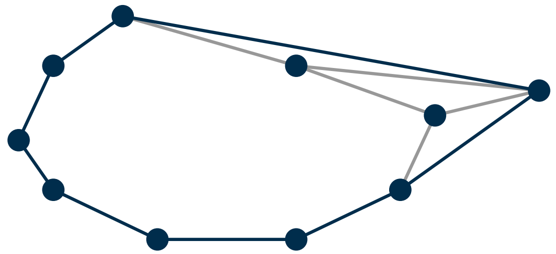

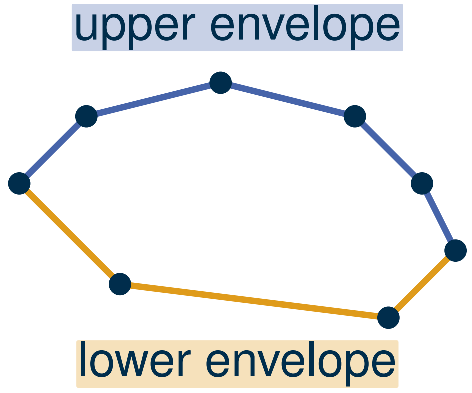

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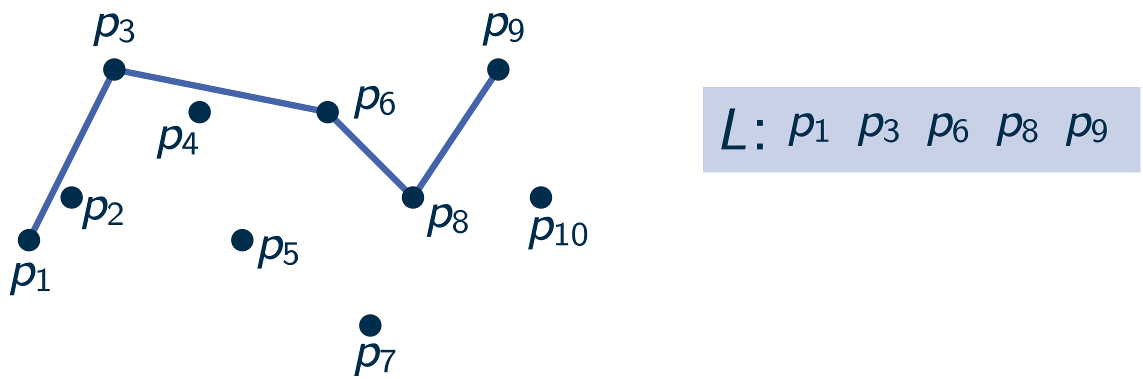

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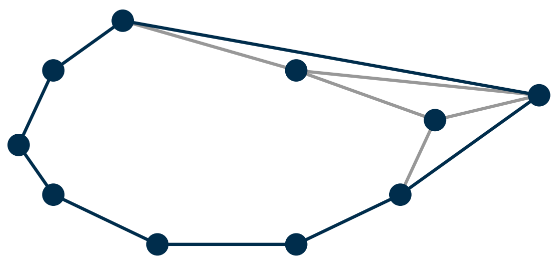

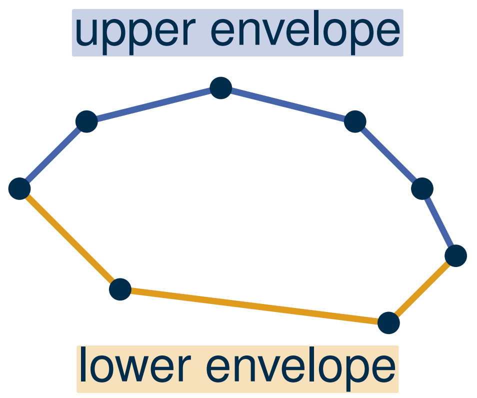

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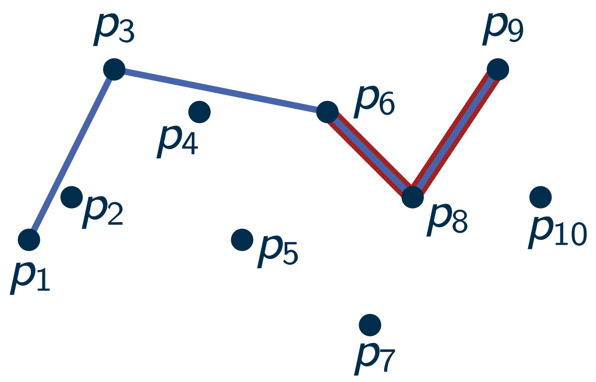

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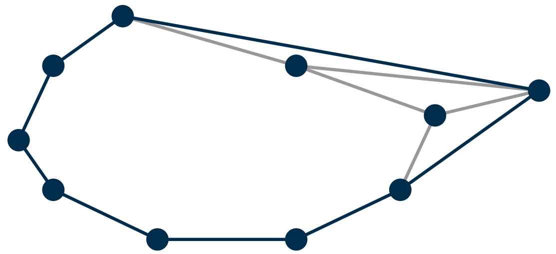
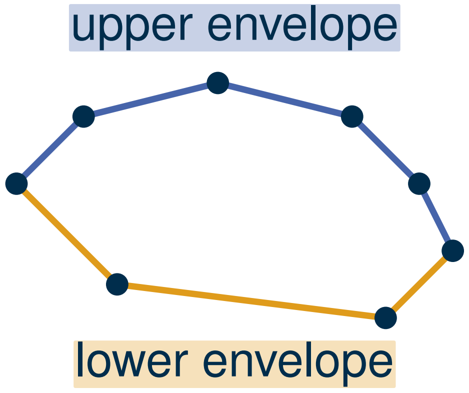
Andrews Monotone Chain Algorithm

(variant of the Graham Scan)





Idea: Iterative Approach

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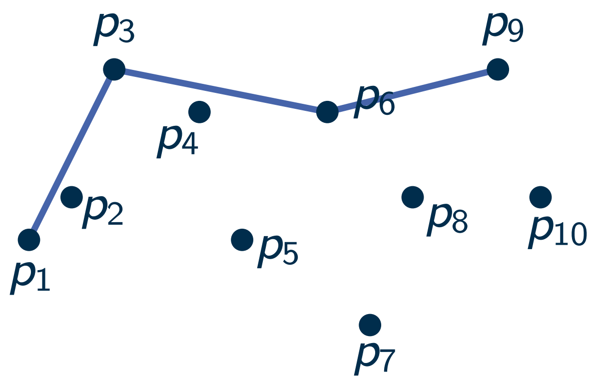

L: p_1 p_3 p_6 p_9

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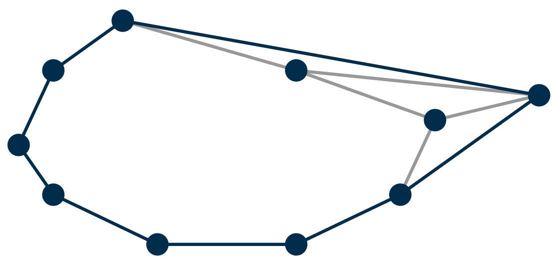

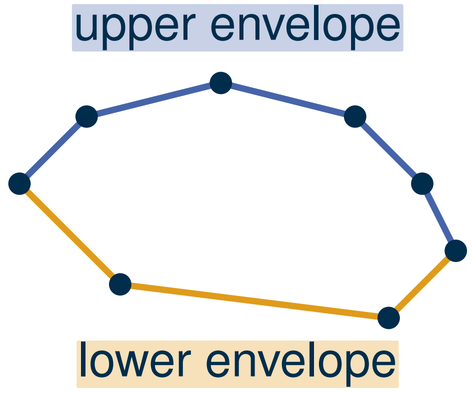

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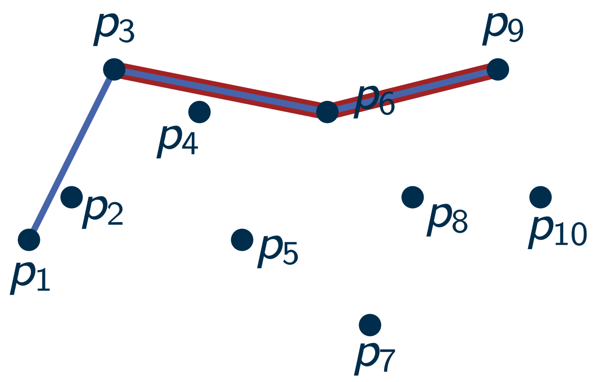

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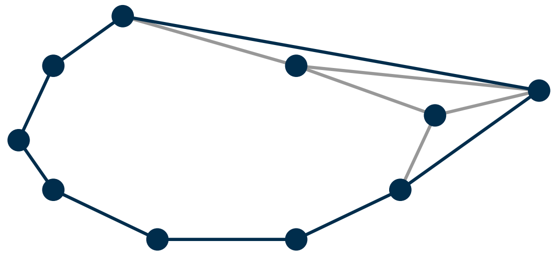

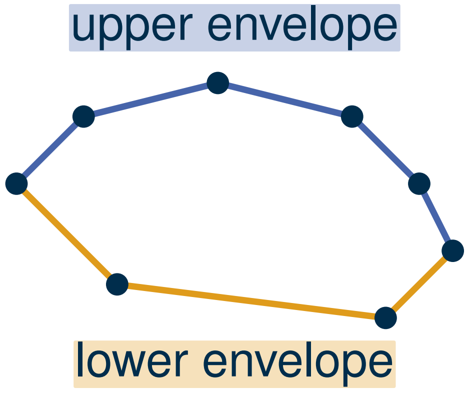

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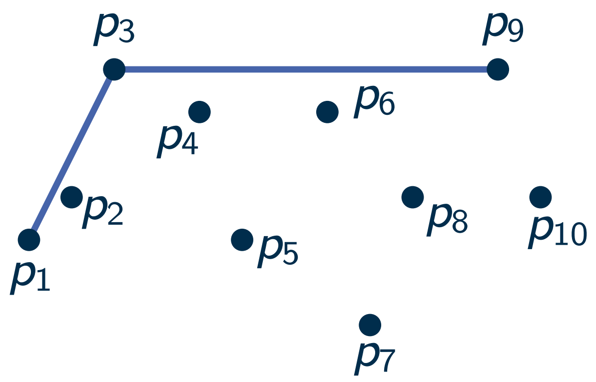

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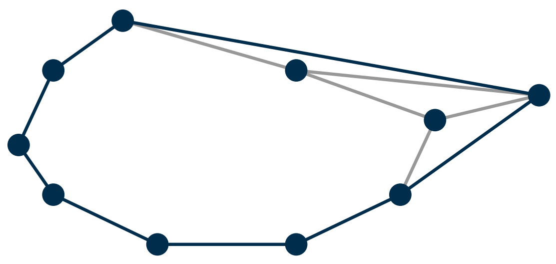

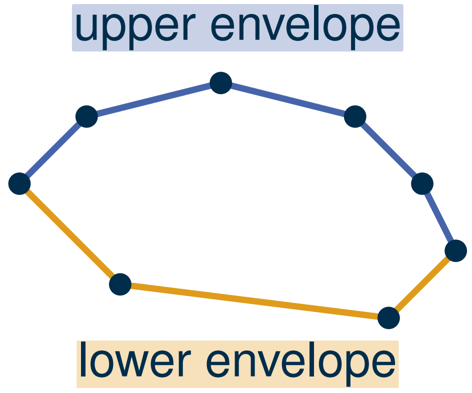

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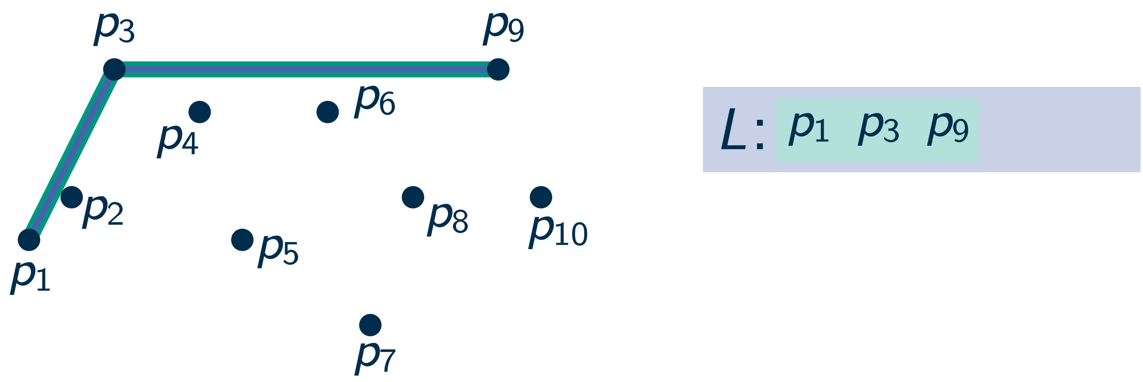

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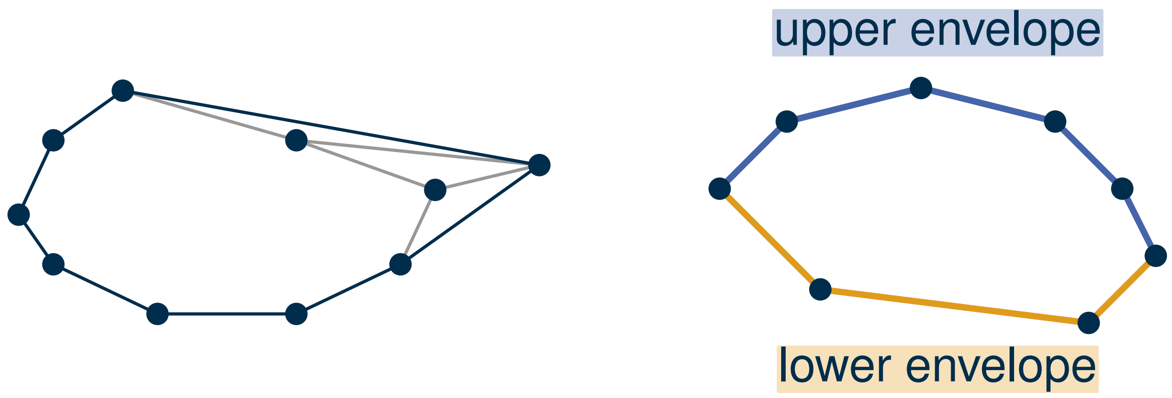

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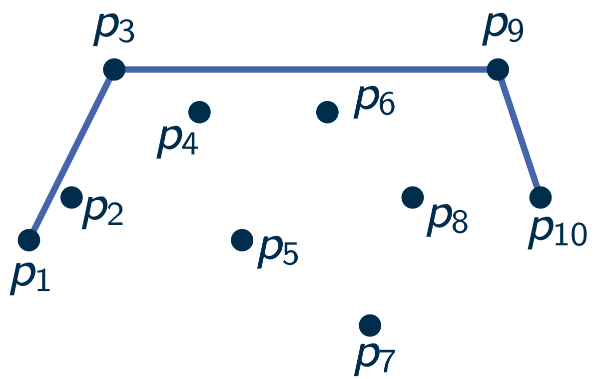

L: p_1 p_3 p_9 p_{10}

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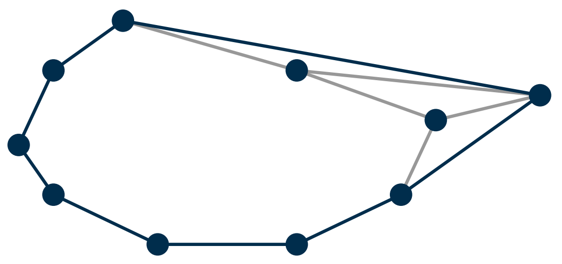

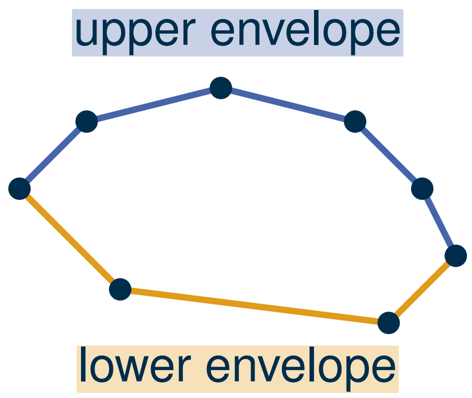

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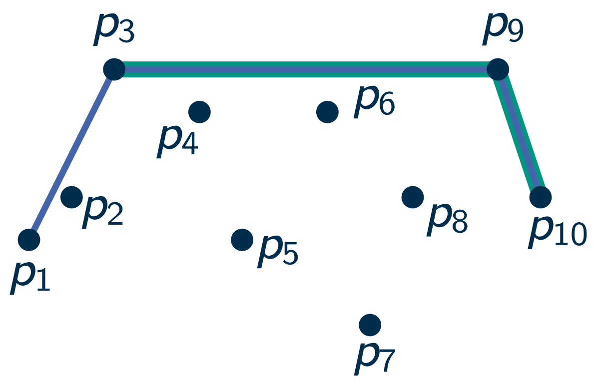

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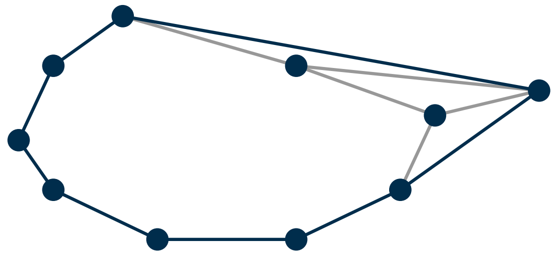

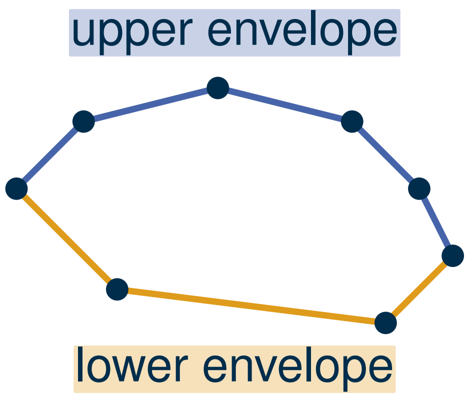

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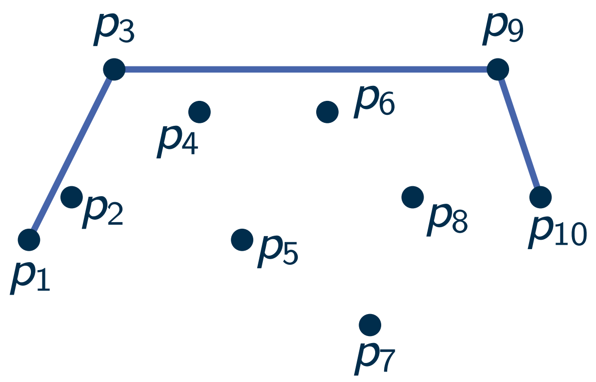

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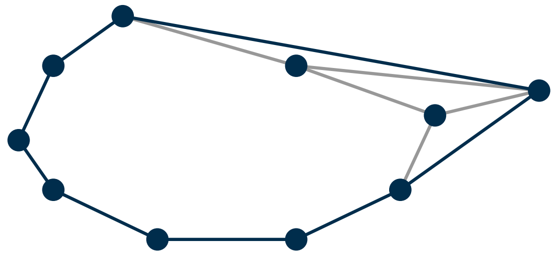

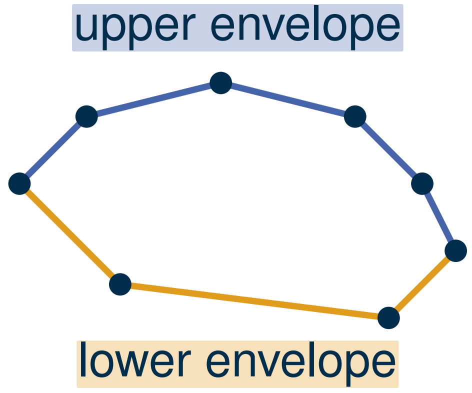

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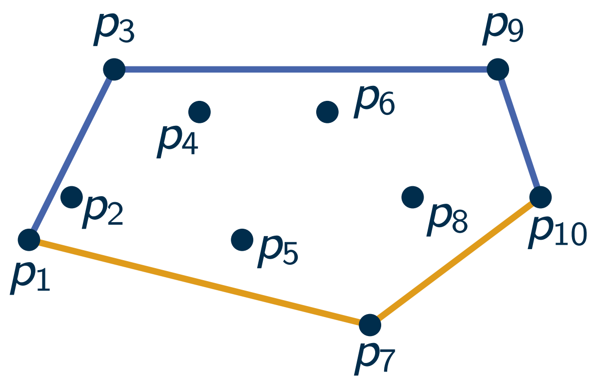

L: p_1 p_3 p_9 p_{10}

analogously: lower envelope

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Andrews Algorithm – Analysis

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Running Time:

Andrews Algorithm – Analysis

Andrews Algorithm	**Running Time:**
■ sort P (left to right): $p_1, \dots, p_n$	$O(n \log n)$
■ insert p_1 and p_2 into a L	$O(1)$
■ for each remaining point p_i:	$O(??)$
– append p_i to the back of L	$O(1)$
– while last three points form a left bend: remove the second-to-last point	$O(??)$
■ L is the upper envelop	$O(n)$

Andrews Algorithm – Analysis

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■ L is the upper envelop	$O(n)$

Andrews Algorithm – Analysis

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Running Time: $O(n \log n)$

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  - $O(1)$ (amortized) — happens at most once to each point
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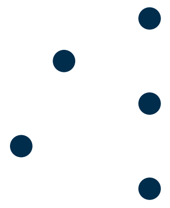
Running Time: $O(n \log n)$

Special Case: Same x-Coordinate



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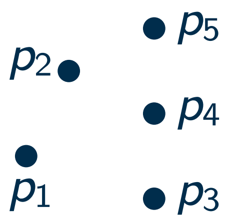
Special Case: Same x-Coordinate

- lexicographic order (first x, then y)
- make consistent with lower envelope



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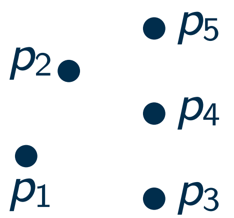
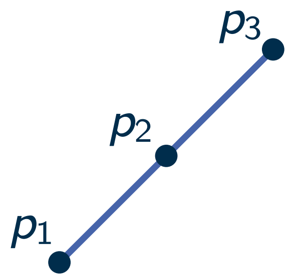


Special Case: Collinear Points



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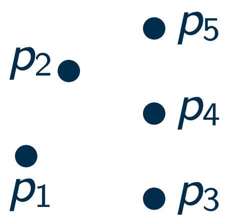
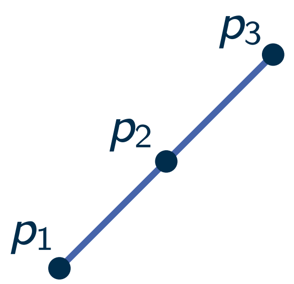


Special Case: Collinear Points

- $p2$ should not be part of the output
- check for right instead of left bend



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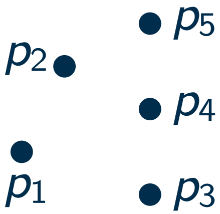

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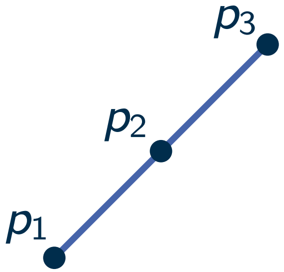

Robustness

What if a check for left bend goes wrong?

Andrews Algorithm – Analysis

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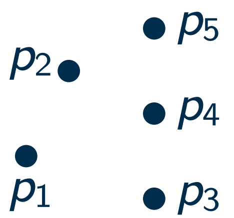

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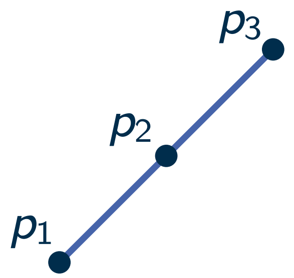

Robustness

- resulting polygon maybe has a slight left bend
- a point may lie slightly outside the resulting polygon

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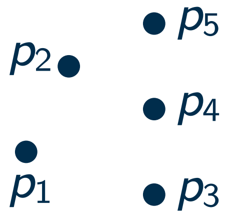

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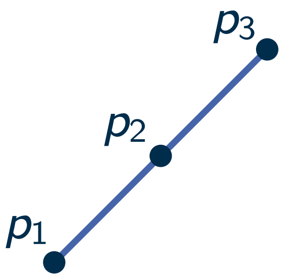

- $p2$ should not be part of the output
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Robustness

- resulting polygon maybe has a slight left bend
- a point may lie slightly outside the resulting polygon
- but: the result is always a polygon that is similar to $\mathcal{CH}(P)$

What if a check for left bend goes wrong?

Andrews Algorithm – Correctness

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Lemma
In the end, L is the upper envelope of P.

Andrews Algorithm – Correctness

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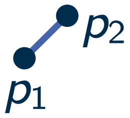
- correct after the initialization ($i = 2$)



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After Step i: L Goes From p_1 To p_i

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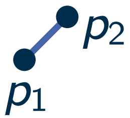

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- L is the upper envelop

After Step i: L Goes From p_1 To p_i

- obvious, as the last point is never deleted

Lemma
In the end, L is the upper envelope of P.

- show: L connects p_1 with p_n, such that
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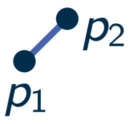

Andrews Algorithm – Correctness

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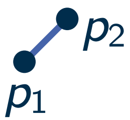

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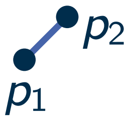
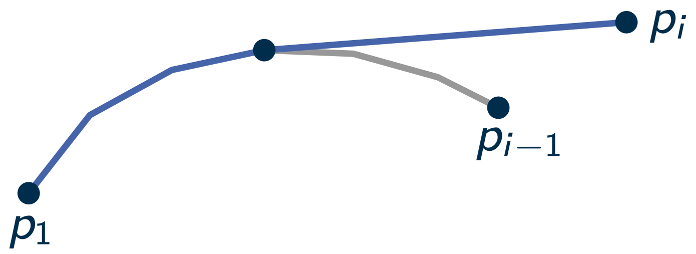


After Step i: L Has Only Right Bends

- after step i, L consists of two parts
  - prefix of the polygon L from the previous step $i-1$
  - edge to p_i



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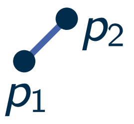

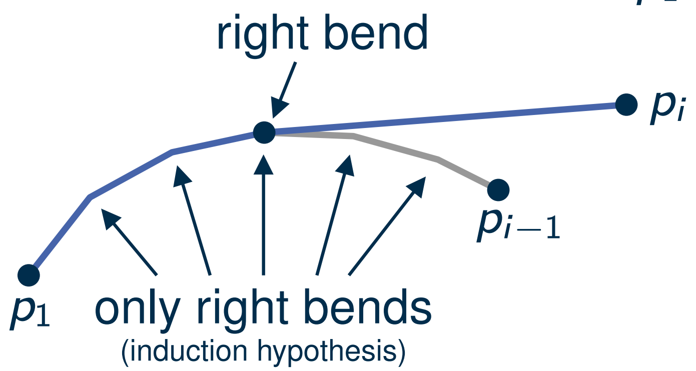

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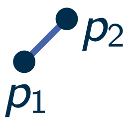

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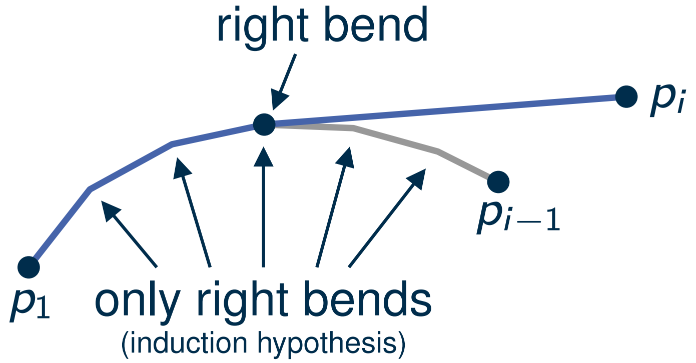
$\Rightarrow$ only right bends



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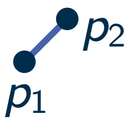

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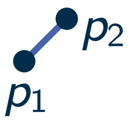

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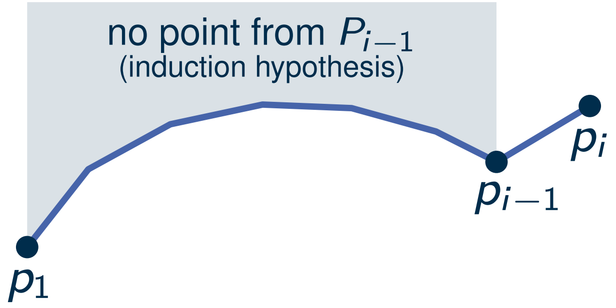

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- still true after inserting p_i
- removing a point p from L moves L further up
- and afterwards, p itself lies below L

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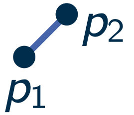

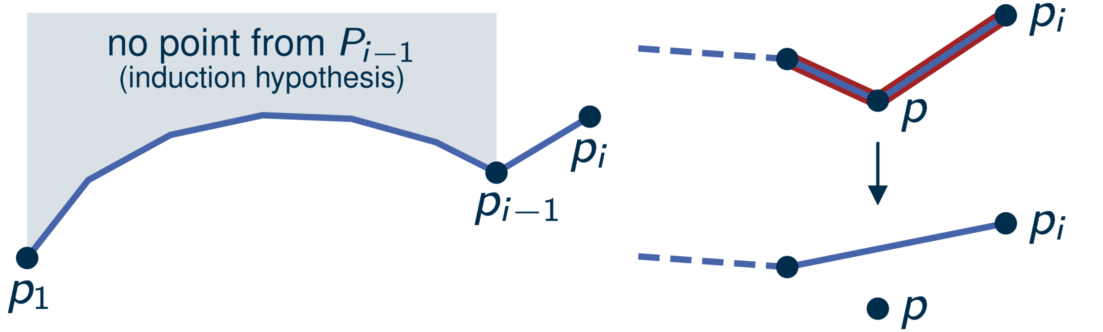

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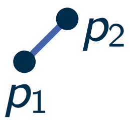
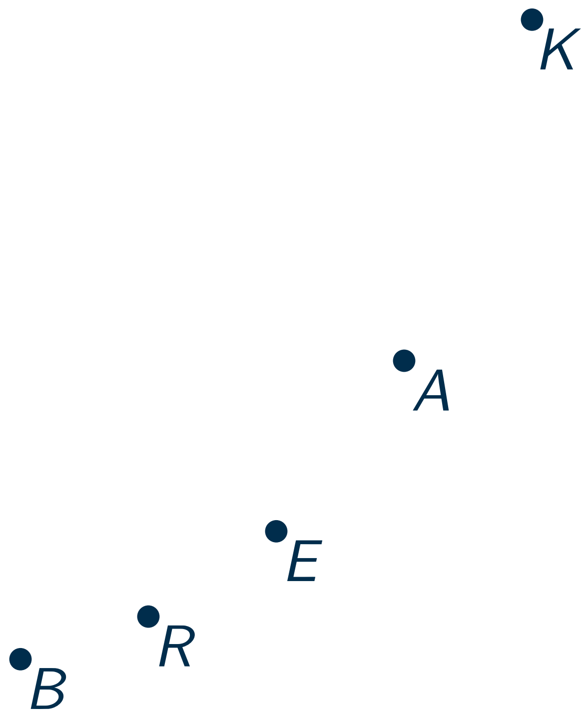


Theorem
Andrews algorithm computes the convex hull of n points in $O(n \log n)$ time.

Compute The Convex Hull



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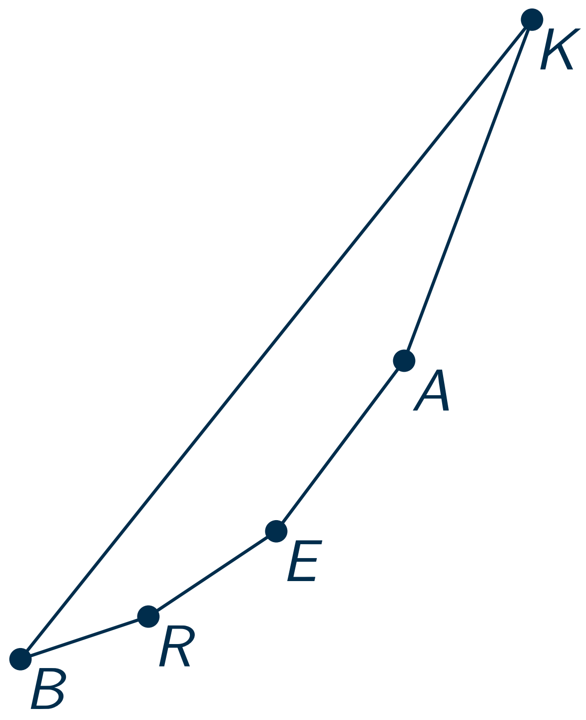

Can We Be Faster?

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Theorem
If the convex hull of n points can be computed in time $f(n)$, then we can sort n numbers in $O(f(n) + n)$ time.

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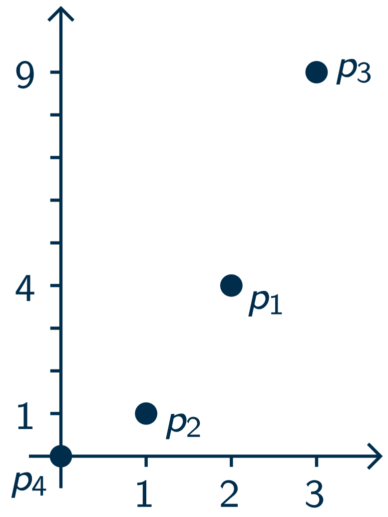
Example

$a_1 = 2$, $a_2 = 1$, $a_3 = 3$, $a_4 = 0$



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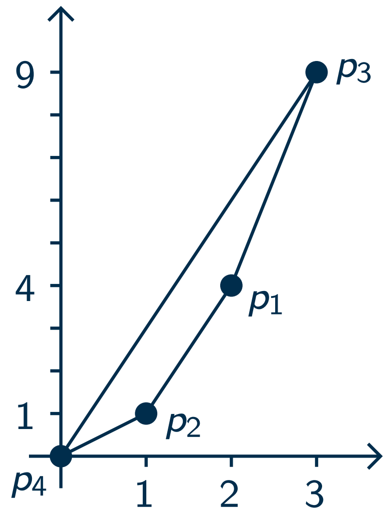

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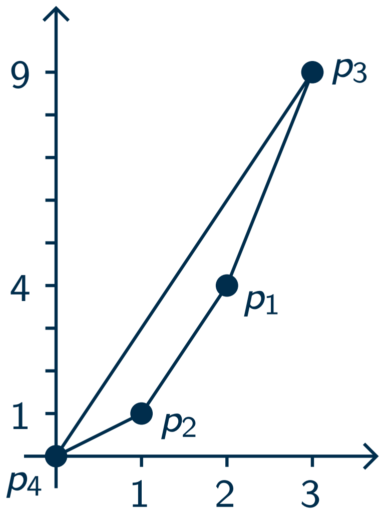

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Lower Bound

- comparison based sorting: $\Omega(n \log n)$
- Andrews algorithm is optimal
  (unless you want to do crazy stuff with numbers)

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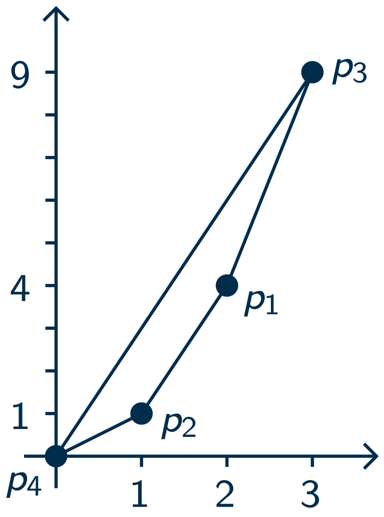
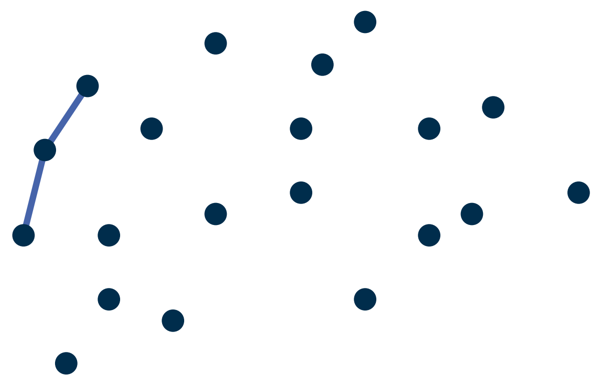


Gift Wrapping (Jarvis March)

Alternative Approach

- assumption: we already know parts of the upper envelope
- goal: find the next point on the upper envelope



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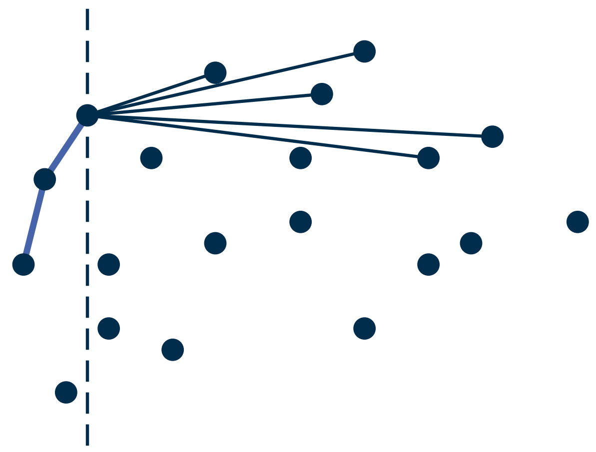

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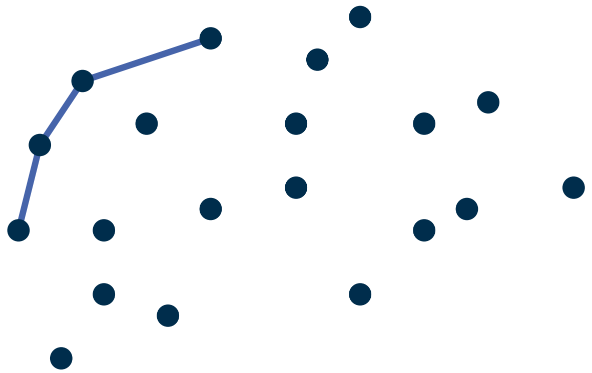

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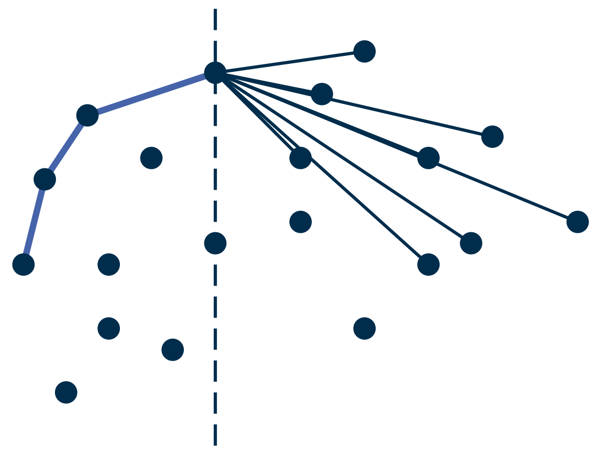

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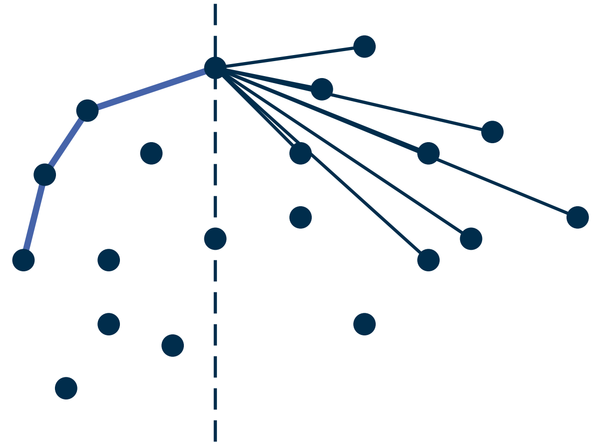

Running Time

- each step: find minimum $\rightarrow O(n)$

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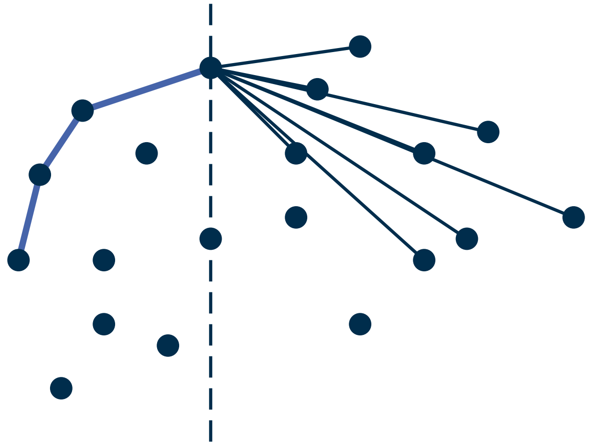

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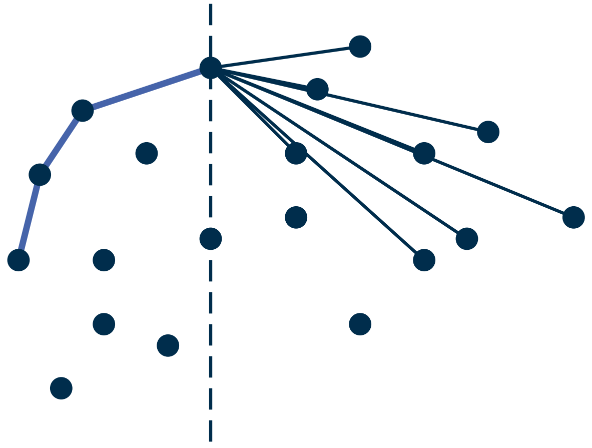

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Theorem
The Gift Wrapping algorithm computes the convex hull of n points P in $O(hn)$ time, where h is the number of points of $\mathcal{CH}(P)$.

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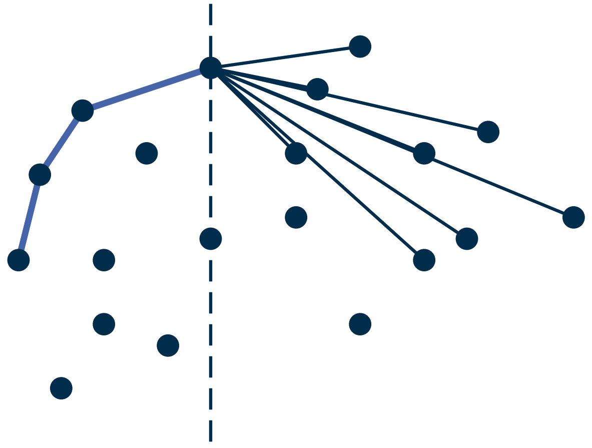

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- such an algorithm is called **output sensitive**

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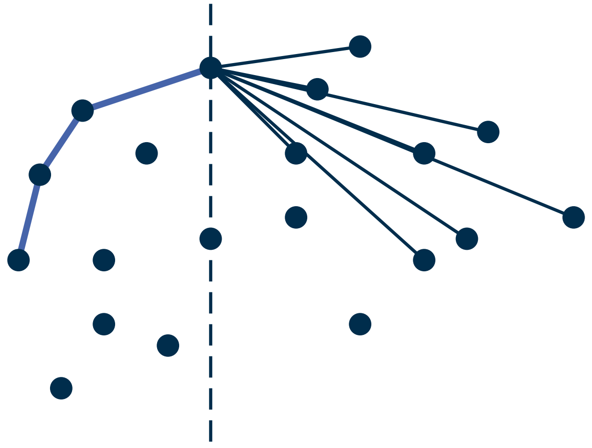

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Comment

- such an algorithm is called **output sensitive**
- beats the lower bound on certain instances (small h)

Wrap-Up

What Have We Learned Today?

- algorithm for computing the convex hull in time $O(n \log n)$

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What Else Is There?

- one can achieve running time $O(n \log h)$

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- output sensitive algorithm with running time $O(hn)$
- robustness is an important aspect in computational geometry
- initially assuming general position helps with algorithm design

What Else Is There?

- one can achieve running time $O(n \log h)$
- higher dimensions
- convex hull of a simple polygon can be computed in $O(n)$ time