Computational Geometry

Introduction and Convex Hull

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What Is Computational Geometry?

Wikipedia

- Computational geometry is a branch of computer science devoted to the study of algorithms which can be stated in terms of geometry.
- Some purely geometrical problems arise out of the study of computational geometric algorithms, and such problems are also considered to be part of computational geometry.

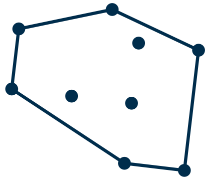
The Things We Deal With

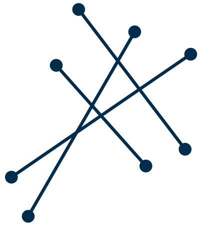
- points, lines, line segments, circles, polygons, ...
- but not: pixels

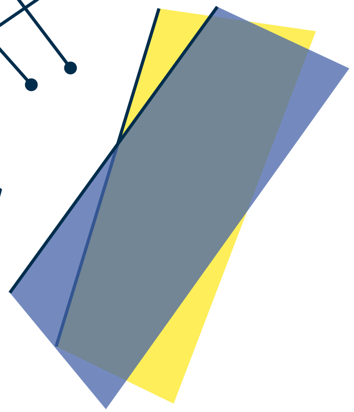
What Does That Mean Specifically?

Basic Toolbox

- convex hull
- line intersection
- triangulation
- plane intersection

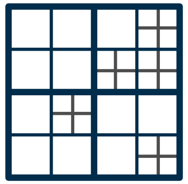


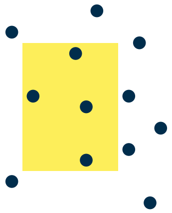


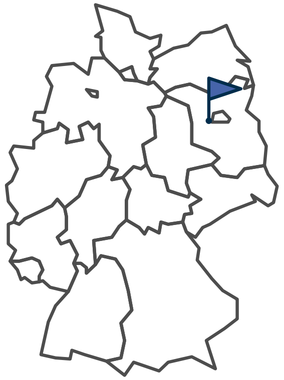


Geometric Data Structures

- orthogonal range searching
- space partitioning
- point location

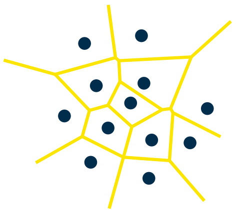


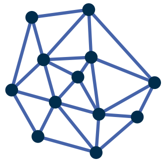


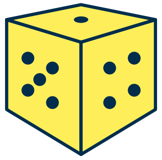


Advanced Toolbox

- Voronoi diagrams
- Delaunay triangulations
- randomized algorithms
- complexity

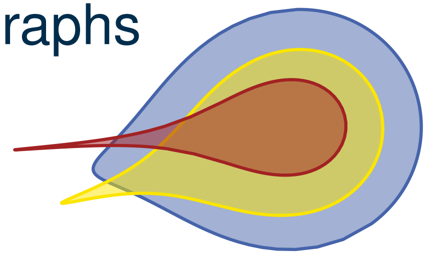


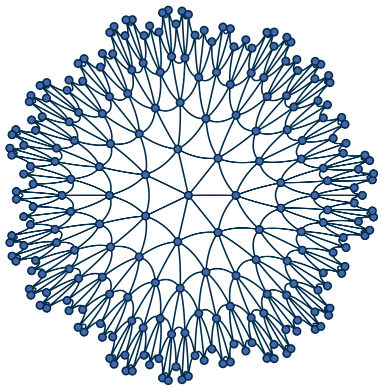




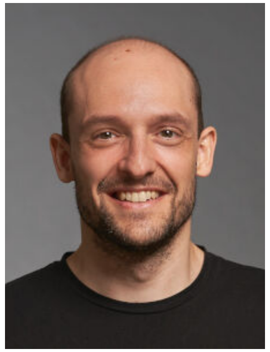
Related Topics

- What is geometry?
- hyperbolic geometry
- geometric graphs



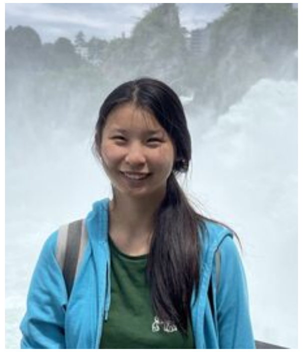


Before We Start


Thomas


Jean-Pierre

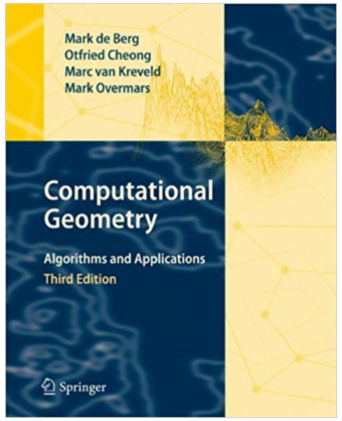

Marcus


Wendy

?
You

Materials & Infos

- slides, exercise sheets on our homepage: `https://scale.iti.kit.edu/teaching/2025ss/comput_geom/`
- Book: Computational Geometry
- Discord: `https://discord.gg/4jam9m7C` (or if you are already on our server: send `!help join` to the scale-bot)



Requirements

- good algorithmic understanding
- no (little) prior knowledge

Rough Schedule

week i							week $i+1$							week $i+2$							week $i+3$						
Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa	Su
(i even)			exercise sheet $\frac{i}{2}$														exercise sheet $\frac{i}{2}+1$										

Lecture
- lecture with slides
- new topics

Exercise Sheet
- hand in in groups of two or three
- graded by us

Active Session
- if it's not a Holiday
- training additional skills
- curiosities

Exercise Session (Week $i+1$)
- with Marcus, Wendy, Jean-Pierre
- recap
- support solving exercise sheets
- ???

Exam
- oral exam (20 min)
- admission only with exercise certificate

Exercise Certificate

Goal: $\frac{1}{2}$ of the points in total **and** $\frac{1}{4}$ on every exercise sheet

What If I Don't Find The Solution?

- you get points for explaining what you tried and why it did not work
- and: there are many ways to get support
  - talk to your peers
  - ask in the exercise session or on discord

What If I Can't Manage To Hand In An Exercise Sheet?

- sometimes, life can get in the way (for all sorts of reasons, e.g., sickness)
- talk to us, we'll find a solution

we don't want to make your life hard and we also don't bite
we just want you to learn something and have fun doing so

Our Goal

- you spend some time with the content of the lecture and write down your solution
- then, the exercise certificate should not be a big obstacle

Motivation

Different Mixtures Of Oil

- the exact ratio between different components depends on the oil spring
- goal: mix oil from different springs, such that the result is easy to process

Example

- oil contains components A and B
- two springs:
- third spring:

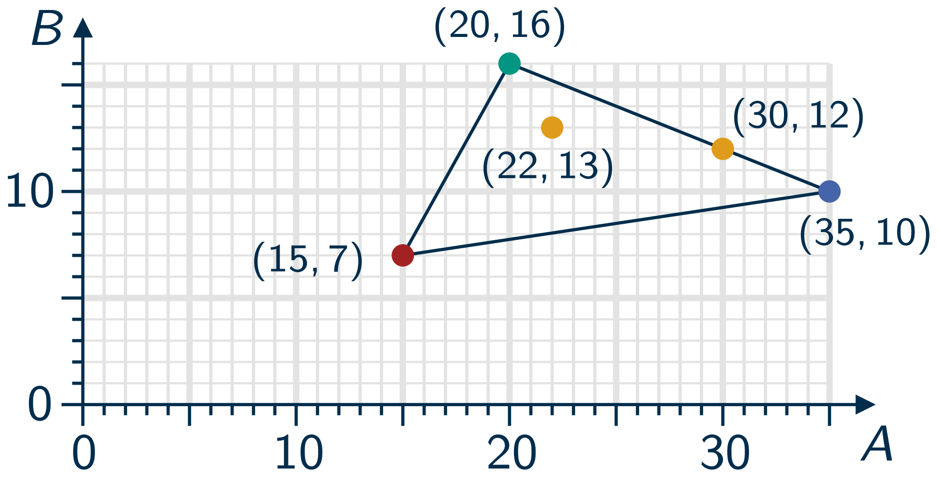
	A	B
spring 1	35%	10%
spring 2	20%	16%
spring 3	15%	7%

Can we achieve 30% A and 12% B? 2 : 1

What about 22% A and 13% B? 1 : 3 : 1

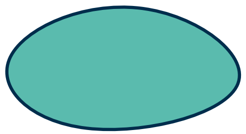
What Is The Relation To Geometry?

- ratios can be interpreted as points
- desired ratio is possible $\Leftrightarrow$ corresponding points lies "between" the available points

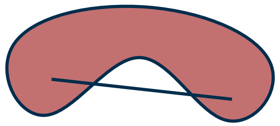


Convex Hull

Definition
A point set $P \subseteq \mathbb{R}^d$ is **convex** if for any two points $p, q \in P$, the line segment pq lies in P.

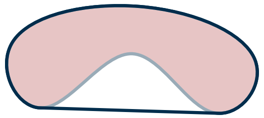


convex



not convex

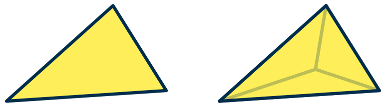
Definition
For $P \subseteq \mathbb{R}^d$, the **convex hull** $\boldsymbol{\mathcal{CH}(P)}$ is the minimal subset of $\mathbb{R}^d$ such that $\mathcal{CH}(P)$ is convex and $P \subseteq \mathcal{CH}(P)$.



Equivalent Definitions

- intersection of all convex sets in $\mathbb{R}^d$ that contain P
- union of all simplices with corners in P

simplices in different dimensions:

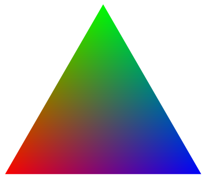


...

- set of all points that are convex combinations of points in P

convex combination: $\sum_{i=1}^{n} a_i \cdot p_i$ with $p_i \in P$, $a_i \in \mathbb{R}$, $a_i \geq 0$, and $\sum_{i=1}^{n} a_i = 1$

you might know this from the barycentric coordinate system

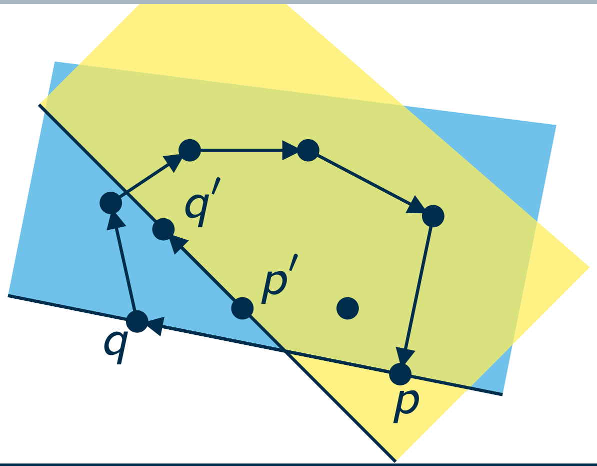


Convex Hull – Trivial Algorithm

Convex Hull Problem (2D): Given n points $P \subseteq \mathbb{R}^2$, compute the convex hull $\mathcal{CH}(P)$.

Notes And General Observations

- assumption: points are in general position
- boundary of $\mathcal{CH}(P)$ is a polygon $\rightarrow$ output is a sequence of points
- pq edge of $\mathcal{CH}(P) \Leftrightarrow$ all points of P lie in the half space right of pq



Trivial Algorithm

- iterate over all pairs of points $(p, q) \in P \times P$ (oriented)
  - check if all points of P lie to the right of pq
  - if yes: save the edge pq
- construct the polygon (as sequence of points) from the saved edges

When is pq an edge of $\mathcal{CH}(P)$?

Running Time: $\Theta(n^3)$

Convex Hull – Trivial Algorithm

Trivial Algorithm

- iterate over all pairs of points $(p, q) \in P \times P$ (oriented)
  - check if all points of P lie to the right of pq
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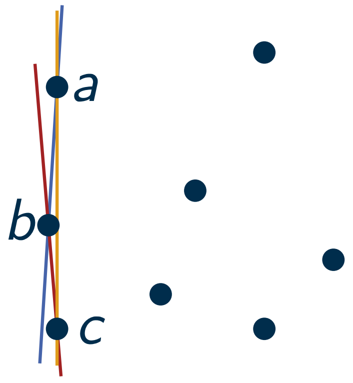
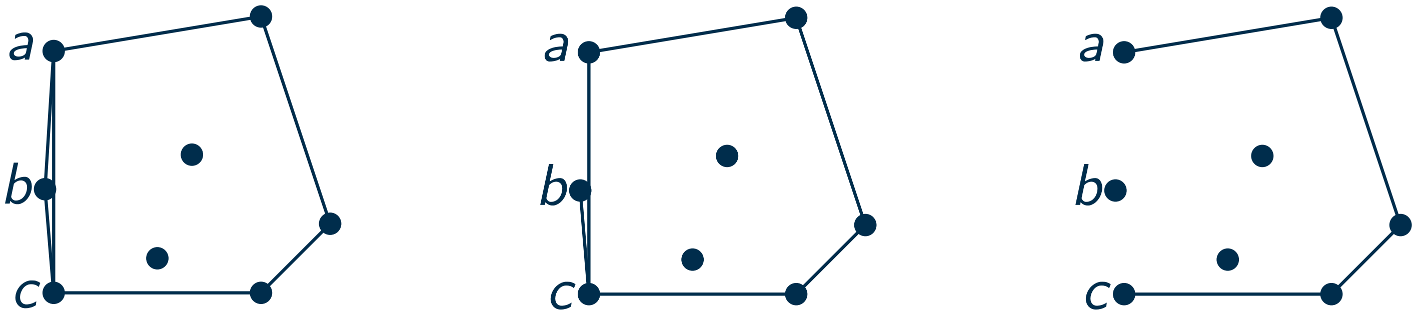
Problems

- the algorithm is slow
- the algorithm is not robust

Example For Lacking Robustness

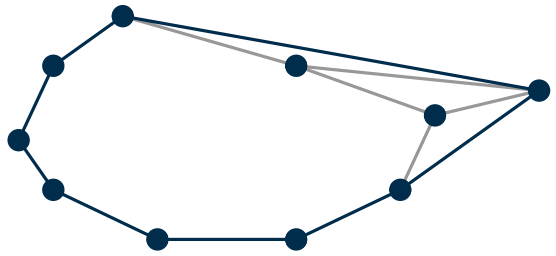


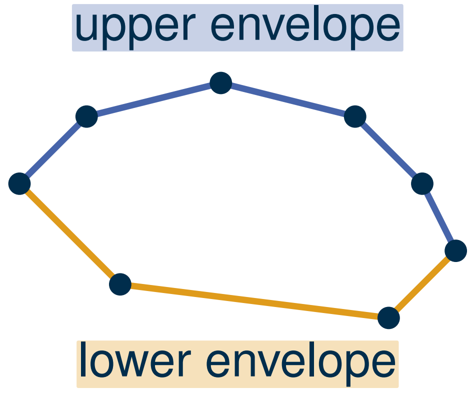
- three decisions “lies to the right of” are close
- wrong decision $\rightarrow$ output maybe not a polygon



Andrews Monotone Chain Algorithm

(variant of the Graham Scan)

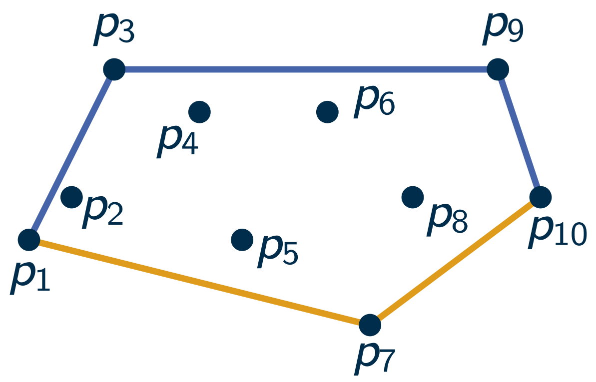




Idea: Iterative Approach

- add points one after another
- update convex hull in each step
- observe: convex hull makes only right bends
- order: from left to right
- for now: only the upper envelope

Example



L: p_1 p_3 p_9 p_{10}

analogously: lower envelope

Andrews Algorithm

- sort P (left to right): $p_1, \ldots, p_n$
- insert p_1 and p_2 into a L
- for each remaining point p_i:
  - append p_i to the back of L
  - while last three points form a left bend: remove the second-to-last point
- L is the upper envelop

Andrews Algorithm – Analysis

Andrews Algorithm

- sort P (left to right): $p_1, \ldots, p_n$
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Running Time: $O(n \log n)$

- $O(n \log n)$
- $O(1)$
- $O(n)$
  - $O(1)$
  - $O(1)$ (amortized) — happens at most once to each point
- $O(n)$

Andrews Algorithm – Analysis

Andrews Algorithm

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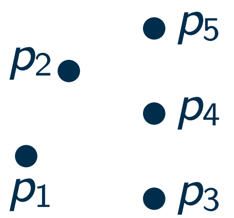
Running Time: $O(n \log n)$

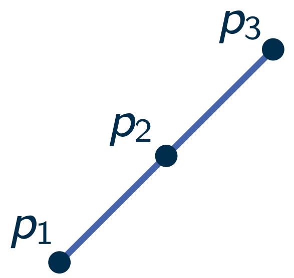
Special Case: Same x-Coordinate

- lexicographic order (first x, then y)
- make consistent with lower envelope



Special Case: Collinear Points

- $p2$ should not be part of the output
- check for right instead of left bend



Robustness

- resulting polygon maybe has a slight left bend
- a point may lie slightly outside the resulting polygon
- but: the result is always a polygon that is similar to $\mathcal{CH}(P)$

What if a check for left bend goes wrong?

Andrews Algorithm – Correctness

Andrews Algorithm

- sort P (left to right): $p_1, \ldots, p_n$
- insert p_1 and p_2 into a L
- for each remaining point p_i:
  - append p_i to the back of L
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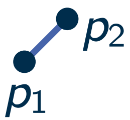
- L is the upper envelop

After Step i: L Goes From p_1 To p_i

- obvious, as the last point is never deleted

Lemma
In the end, L is the upper envelope of P.

- show: L connects p_1 with p_n, such that
  - L makes only right bends
  - every point in $P \setminus L$ lies below L
- induction over i for $P_i = \{p_1, \ldots, p_i\}$
- correct after the initialization ($i = 2$)



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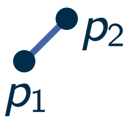
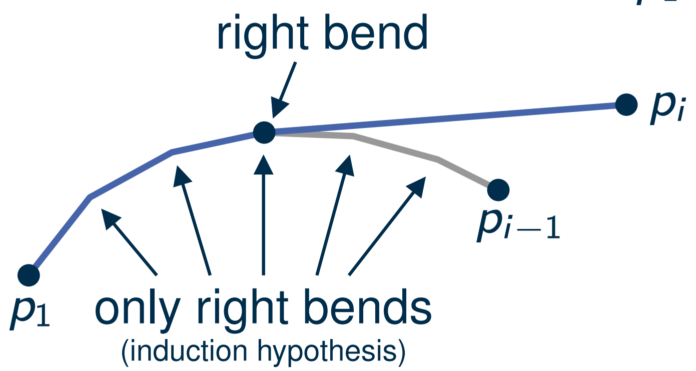


After Step i: L Has Only Right Bends

- after step i, L consists of two parts
  - prefix of the polygon L from the previous step $i-1$
  - edge to p_i

$\Rightarrow$ only right bends



Andrews Algorithm – Correctness

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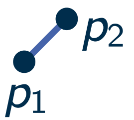
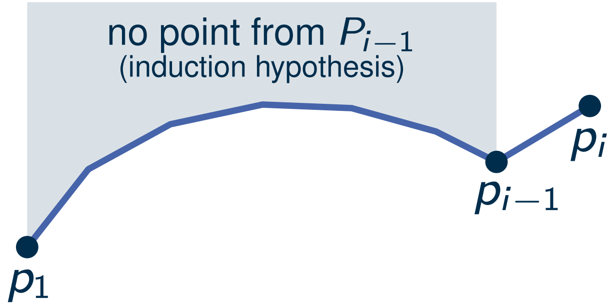
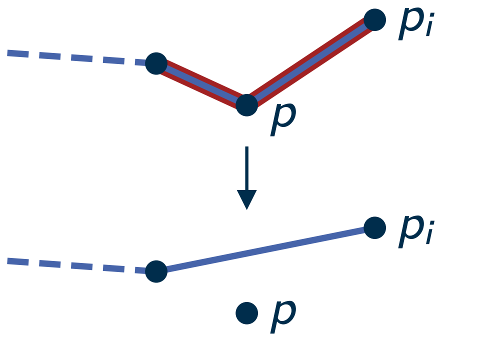


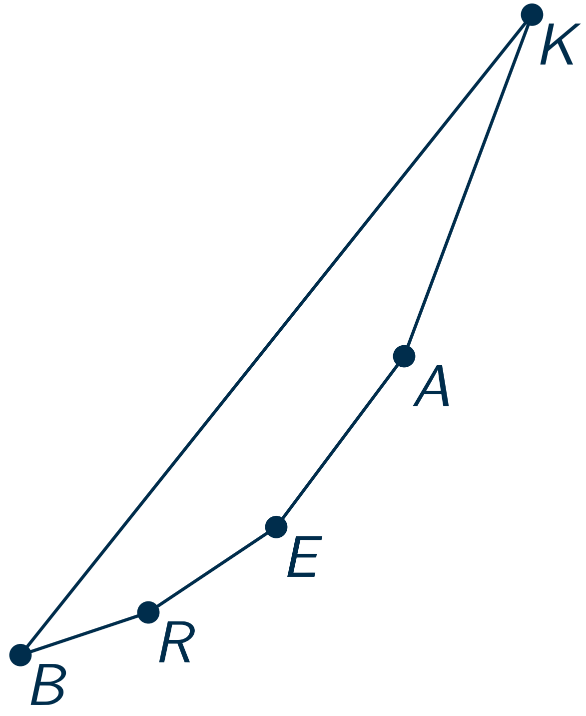
After Step i: Every Point In $P_i \setminus L$ Lies Below L

- still true after inserting p_i
- removing a point p from L moves L further up
- and afterwards, p itself lies below L





Compute The Convex Hull



Can We Be Faster?

Theorem
If the convex hull of n points can be computed in time $f(n)$, then we can sort n numbers in $O(f(n) + n)$ time.

Proof

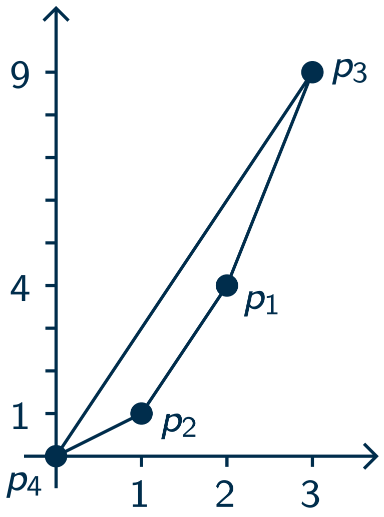
- given: n numbers $a_1, \ldots, a_n$
- construct n points $P = \{p_1, \ldots, p_n\}$ with $p_i = (a_i, a_i^2)$
- $\mathcal{CH}(P)$ contains the points sorted by a_i
- order can be obtained in $O(n)$ from $\mathcal{CH}(P)$

Lower Bound

- comparison based sorting: $\Omega(n \log n)$
- Andrews algorithm is optimal
  (unless you want to do crazy stuff with numbers)

Example

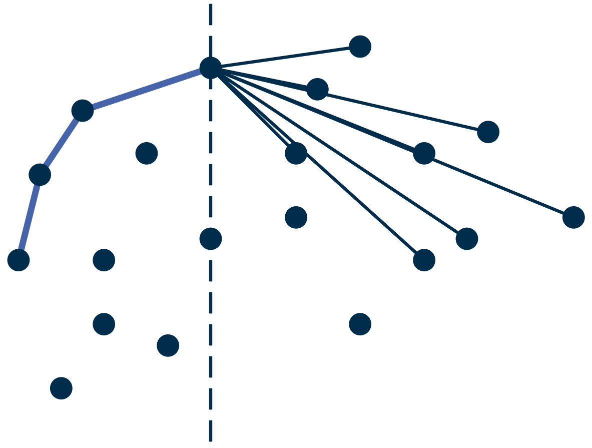
$a_1 = 2$, $a_2 = 1$, $a_3 = 3$, $a_4 = 0$



Gift Wrapping (Jarvis March)

Alternative Approach

- assumption: we already know parts of the upper envelope
- goal: find the next point on the upper envelope
- choose point with the smallest angle



Running Time

- each step: find minimum $\rightarrow O(n)$
- h steps, for $h = |\mathcal{CH}(P)|$

Theorem
The Gift Wrapping algorithm computes the convex hull of n points P in $O(hn)$ time, where h is the number of points of $\mathcal{CH}(P)$.

Comment

- such an algorithm is called **output sensitive**
- beats the lower bound on certain instances (small h)

Wrap-Up

What Have We Learned Today?

- algorithm for computing the convex hull in time $O(n \log n)$
- $\Omega(n \log n)$ lower bound
- output sensitive algorithm with running time $O(hn)$
- robustness is an important aspect in computational geometry
- initially assuming general position helps with algorithm design

What Else Is There?

- one can achieve running time $O(n \log h)$
- higher dimensions
- convex hull of a simple polygon can be computed in $O(n)$ time