Computational Geometry

Duality

Thomas Bläsius

Duality

point $\boldsymbol{p}$: (p_x, p_y) | **line $\boldsymbol{\ell}$:** $y = \ell_x \cdot x - \ell_y$

Duality

point p: (p_x, p_y)

How to choose p_y depending on p_x, such that $p \in \ell$?

line ℓ: $y = \ell_x \cdot x - \ell_y$

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duality: lines and points are interchangeable

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point $\boldsymbol{\ell^\star}$: (ℓ_x, ℓ_y)

$$p \in \ell \Leftrightarrow \ell^\star \in p^\star$$

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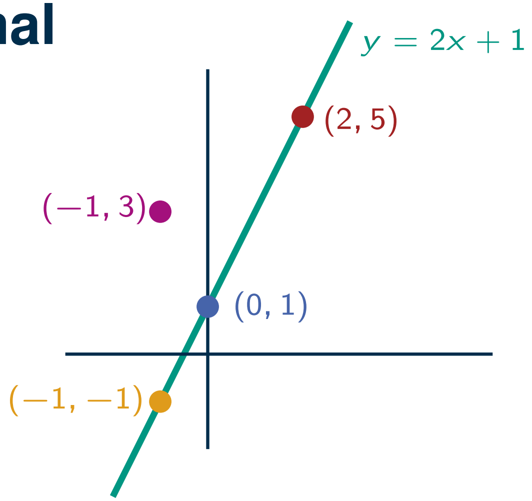
primal



dual

What does the dual look like?

If the point p lies above the line ℓ, where does $\ell^\star$ lie with respect to $p^\star$?

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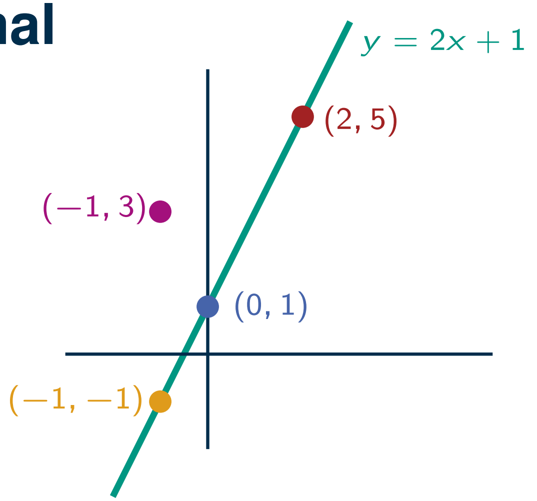

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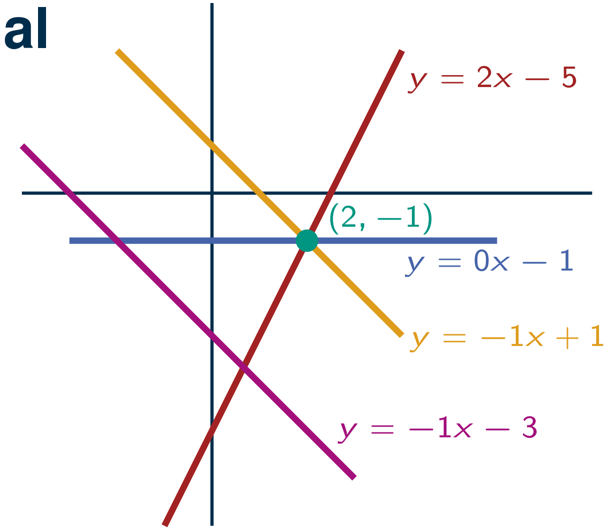

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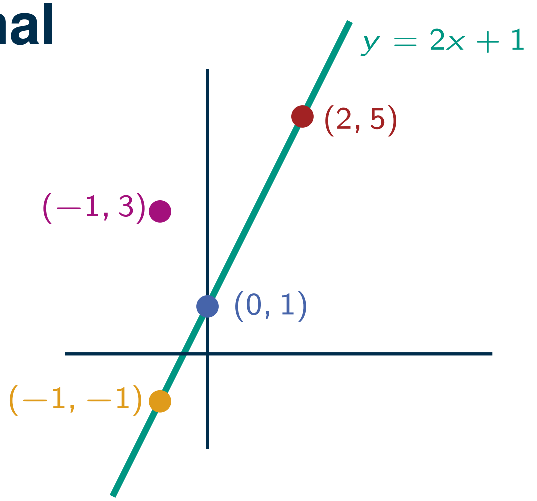

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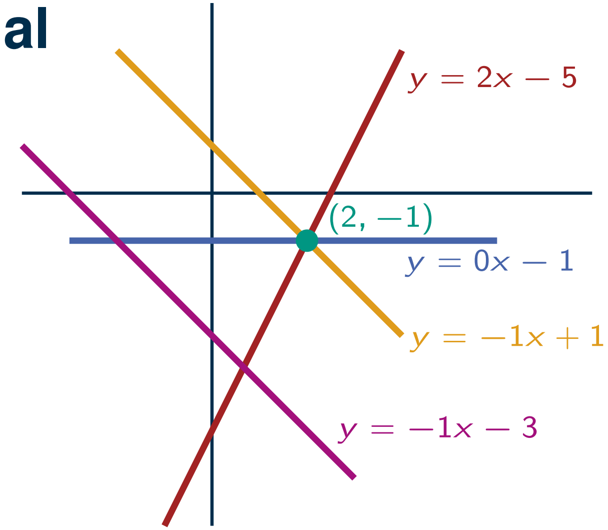


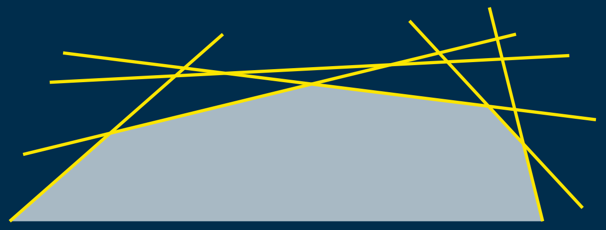
If the point p lies above the line ℓ, where does $\ell^\star$ lie with respect to $p^\star$?

p above $\ell \Leftrightarrow \ell^\star$ above $p^\star$

Using Duality

Region Below Lines

- given: set of lines L
- find: region below all lines



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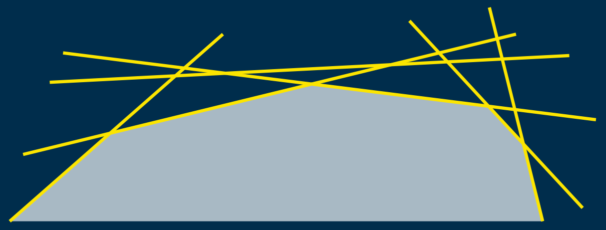

Minimum Triangle

- given: set of points P
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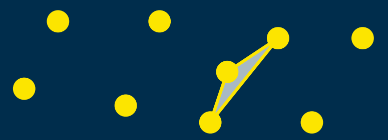

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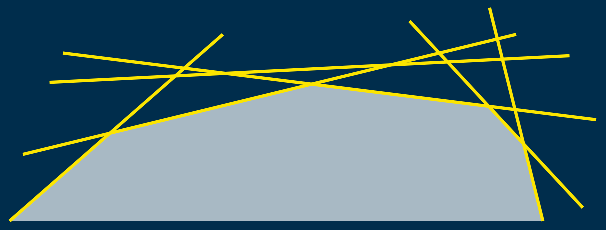

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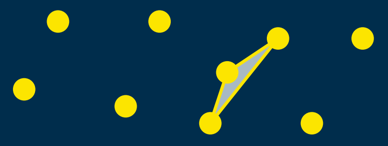
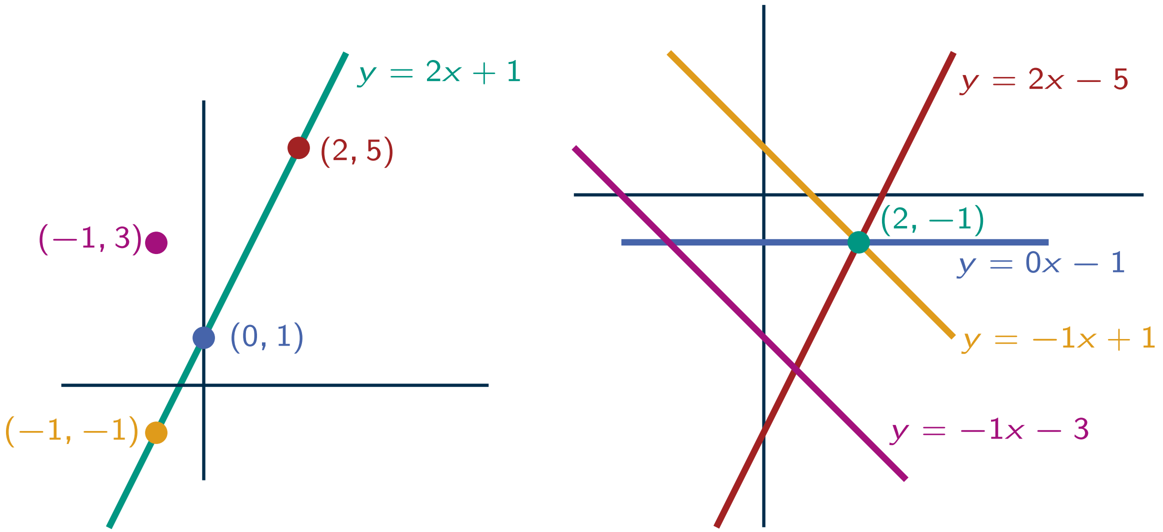


$$\boldsymbol{p}\text{:}\ (p_x, p_y) \quad \leftrightarrow \quad \boldsymbol{p}^\star\text{:}\ y = p_x \cdot x - p_y$$
$$\boldsymbol{\ell}\text{:}\ y = \ell_x \cdot x - \ell_y \quad \leftrightarrow \quad \boldsymbol{\ell}^\star\text{:}\ (\ell_x, \ell_y)$$
$$p \in \ell \Leftrightarrow \ell^\star \in p^\star$$
$$p \text{ above } \ell \Leftrightarrow \ell^\star \text{ above } p^\star$$



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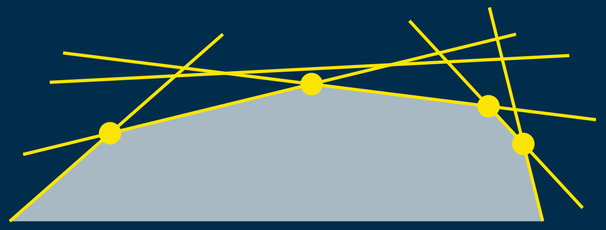

Primal (Reformulation)

- given: set of lines L
- find: set of points P such that $p \in P$ iff:
  - p is intersection between lines in L
  - p lies below all other lines in L

half-plane intersection

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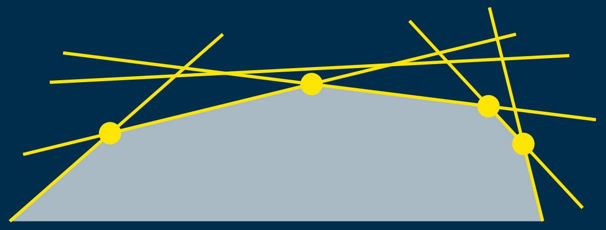

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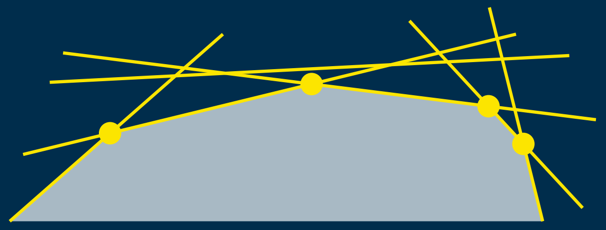

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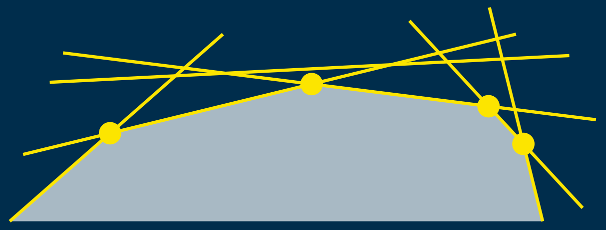

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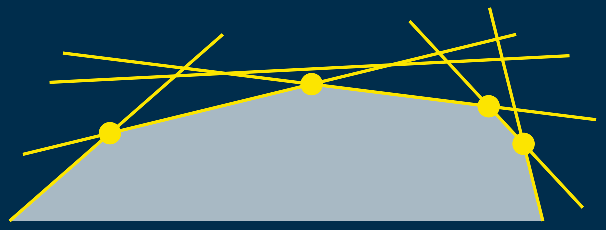

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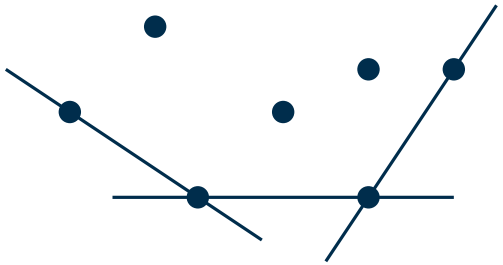

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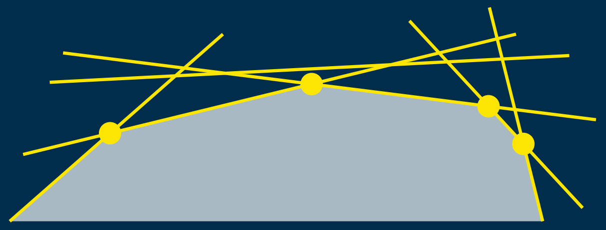

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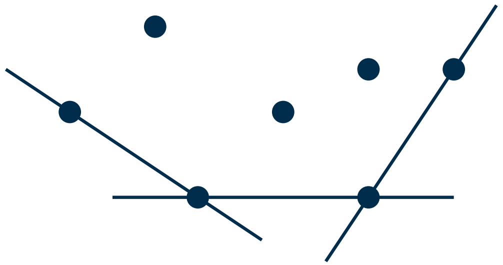

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lower envelope
$\rightarrow$ convex hull

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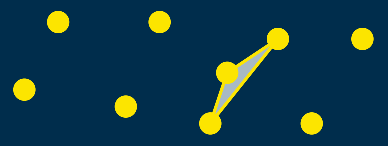
Minimum Triangle

- given: set of points P
- find: three points forming min-area triangle



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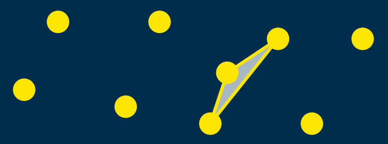

Area Of A Triangle

- area $= \frac{1}{2} \cdot$ altitude $\cdot$ base

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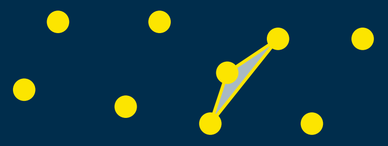

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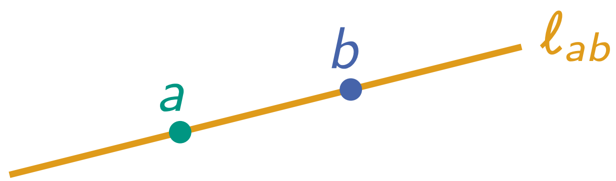

Primal Plan

- for every $a \in P$ and $b \in P$, let ℓ_{ab} be the line through a and b

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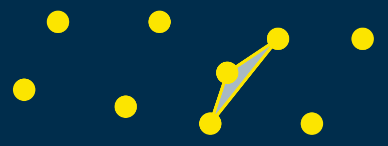

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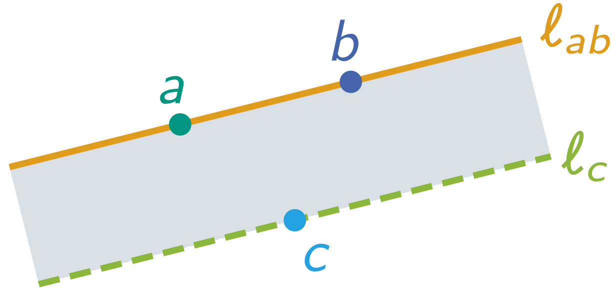

Primal Plan

- for every $a \in P$ and $b \in P$, let ℓ_{ab} be the line through a and b
- find $c \in P$ such that no other point in P lies between ℓ_{ab} and ℓ_c

(ℓ_c: line through c, parallel to ℓ_{ab})

Using Duality

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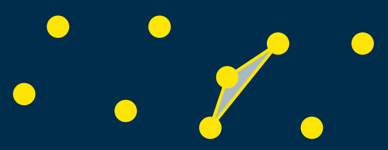

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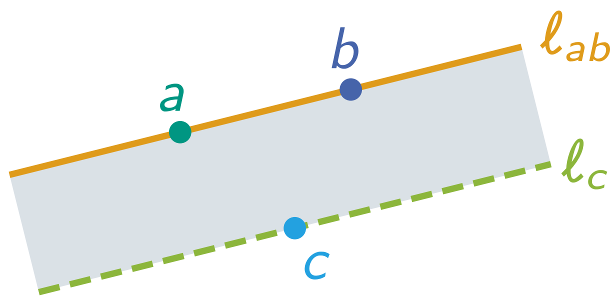

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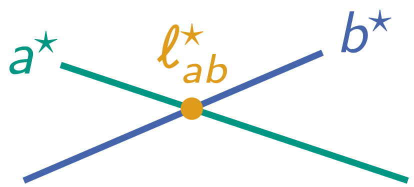

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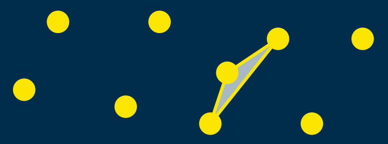

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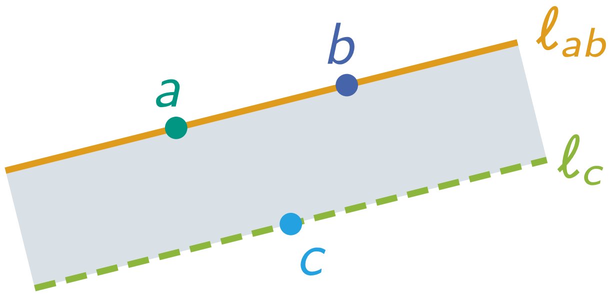

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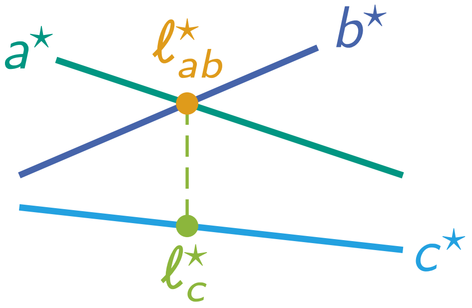
($\ell_c^\star$: point on $c^\star$, same x-coordinate as $\ell_{ab}^\star$)



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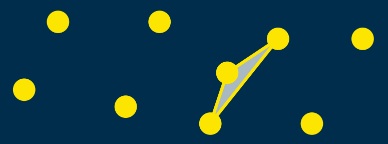

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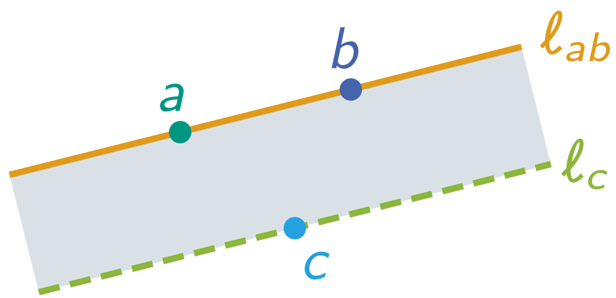

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Dualization

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  ($\ell^\star_c$: point on $c^\star$, same x-coordinate as $\ell^\star_{ab}$)
- can be solved with the sweep-line algo for line intersection