Karlsruhe Institute of Technology

Hasso Plattner Institut
IT Systems Engineering | Universität Potsdam

Competitive Programming

Winter Term 23/24

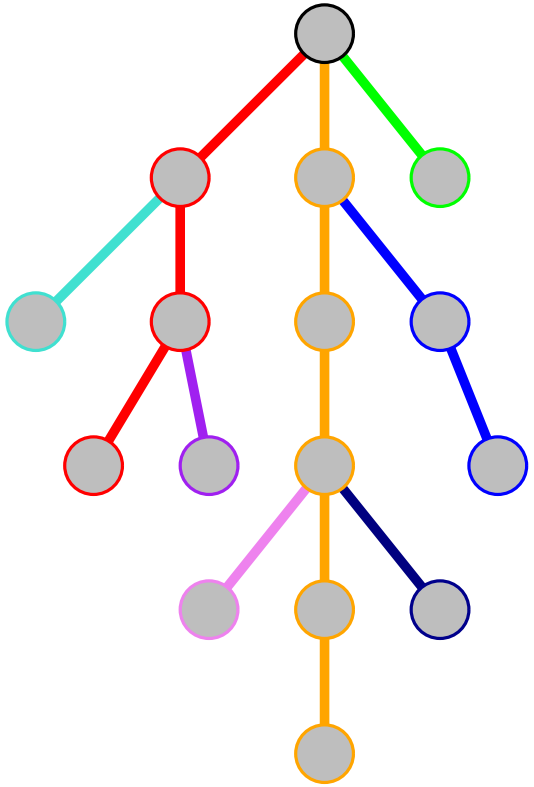
Trees

Kirill Simonov
Algorithm Engineering Group
HPI

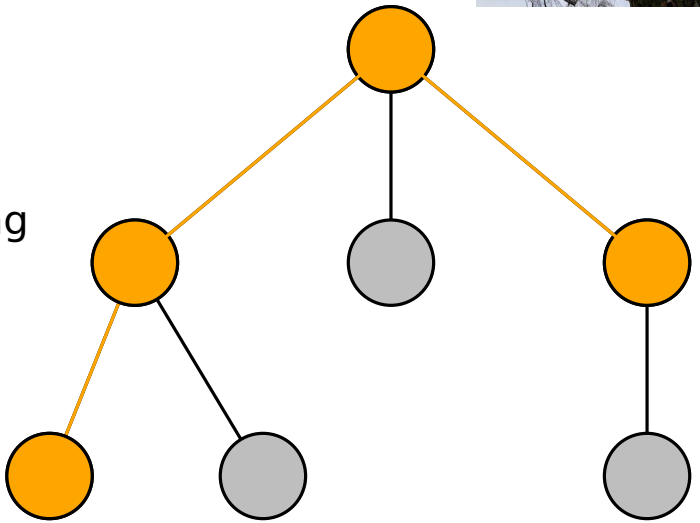
Michael Zündorf
Scalable Algorithms Group
KIT

Topics today

- today
  - lowest common ancestor
  - binary lifting
  - heavy-light decomposition
  - centroid decomposition



- not covered
  - fast LCA
  - tree rerooting



In this lecture

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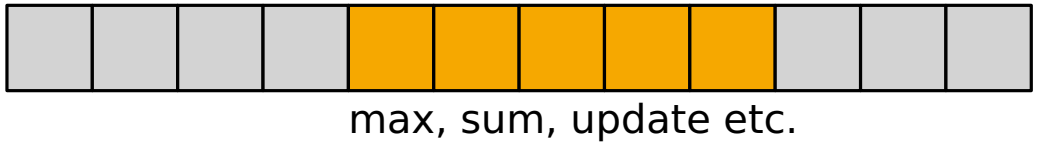
Queries on arrays:



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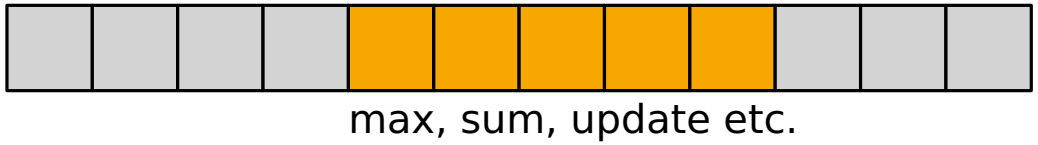

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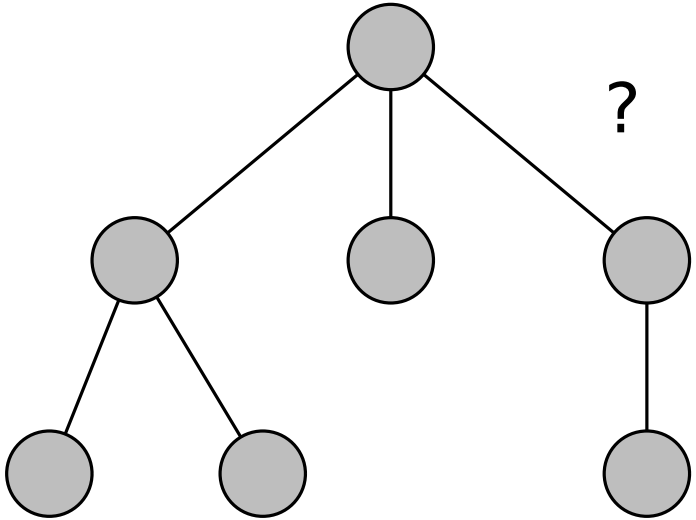

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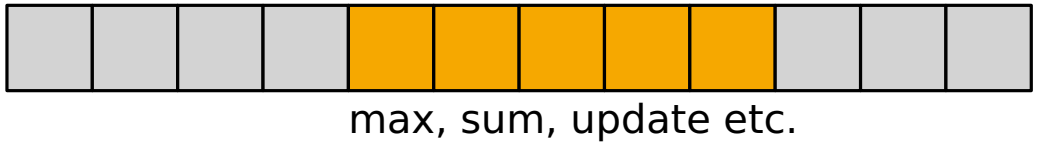

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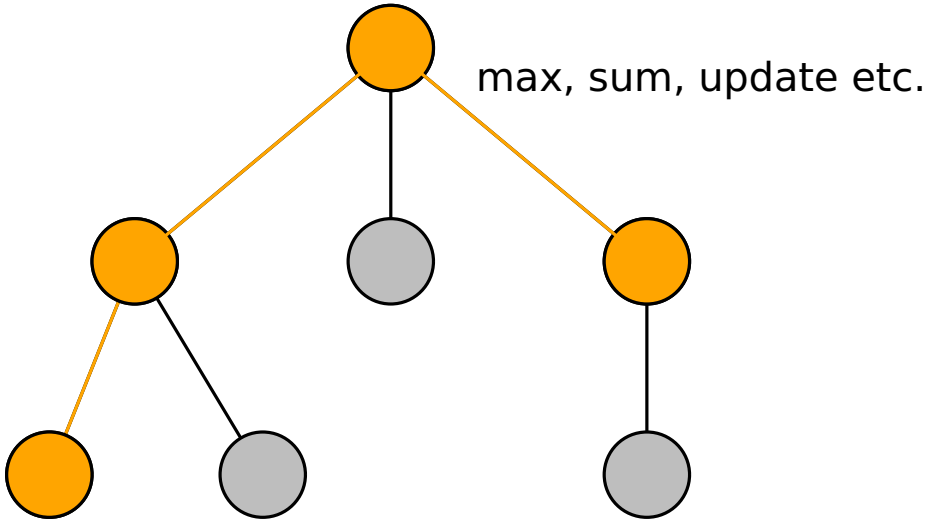

Lowest Common Ancestor

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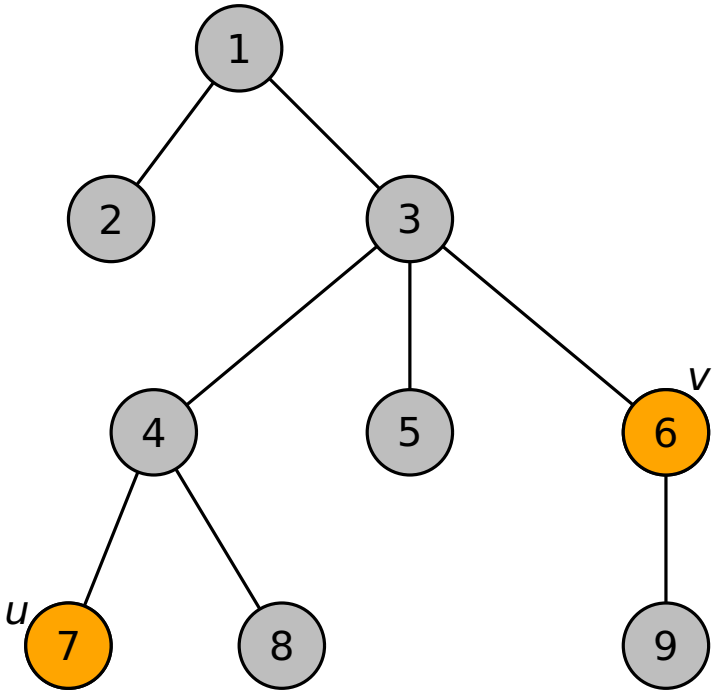

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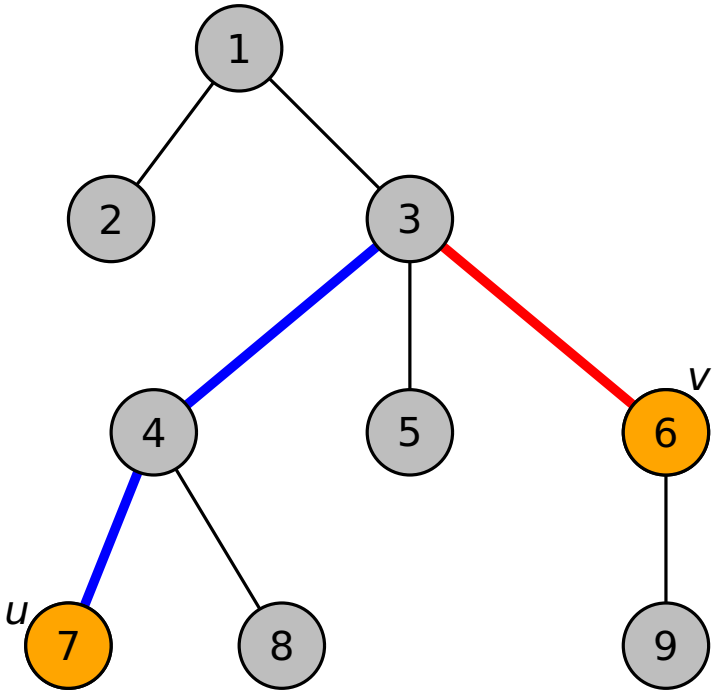
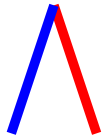


Idea: split path queries into two paths



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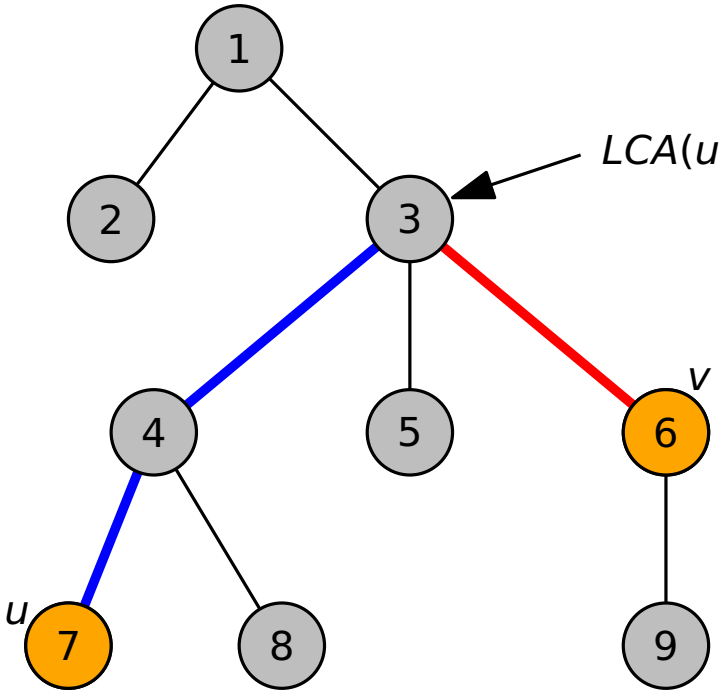

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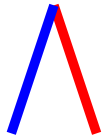

$LCA(u, v)$: lowest node on both paths to the root

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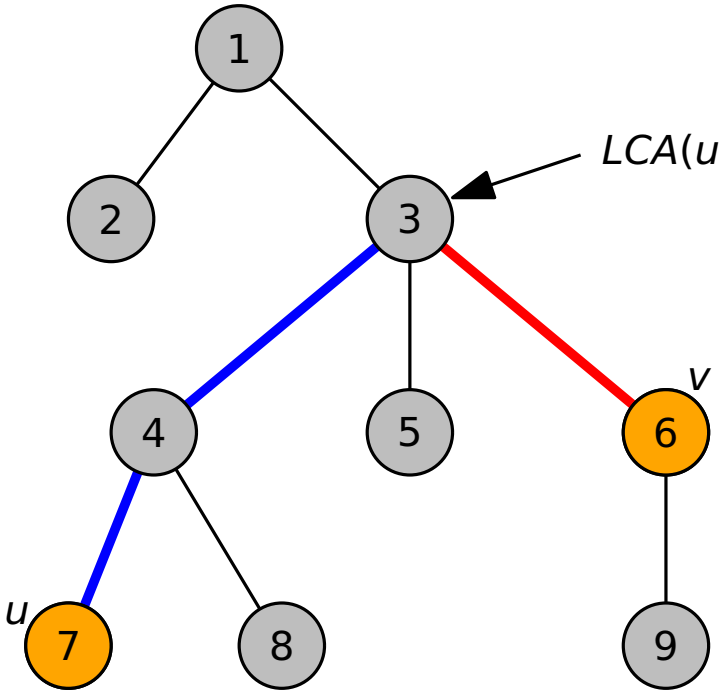

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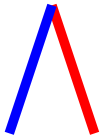

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How to find LCA? Build *Euler tour* of the tree:

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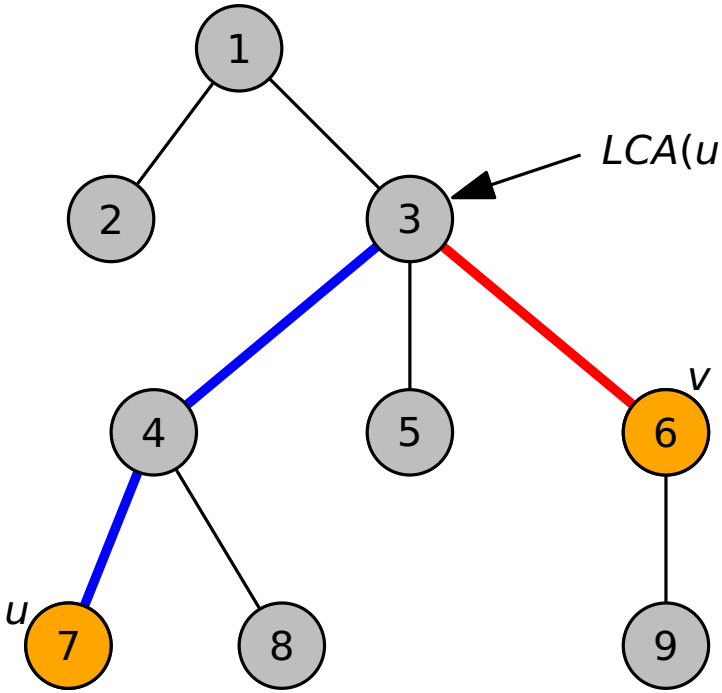

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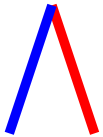

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$\boxed{1}$

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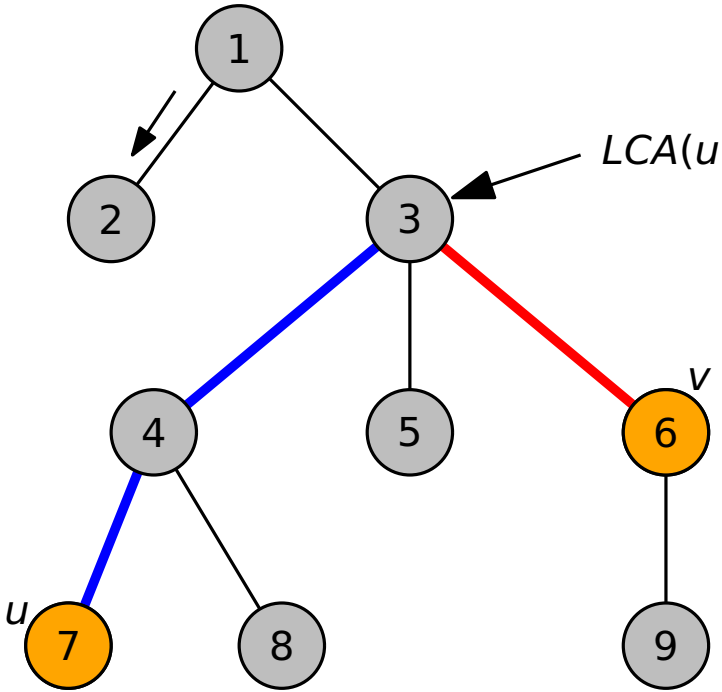

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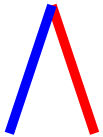

$LCA(u, v)$: lowest node on both paths to the root

How to find LCA? Build *Euler tour* of the tree:

1	2

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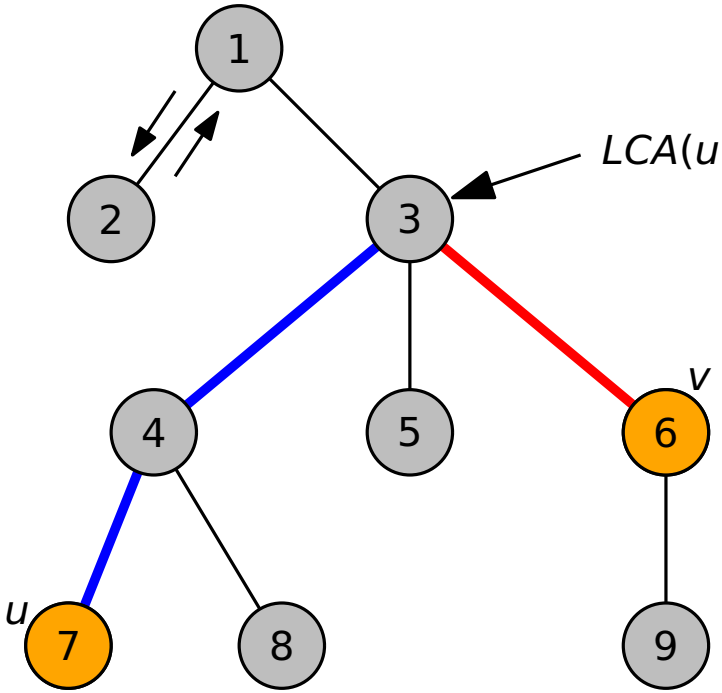

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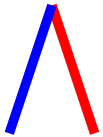

$LCA(u, v)$: lowest node on both paths to the root

How to find LCA? Build *Euler tour* of the tree:

1	2	1

Lowest Common Ancestor

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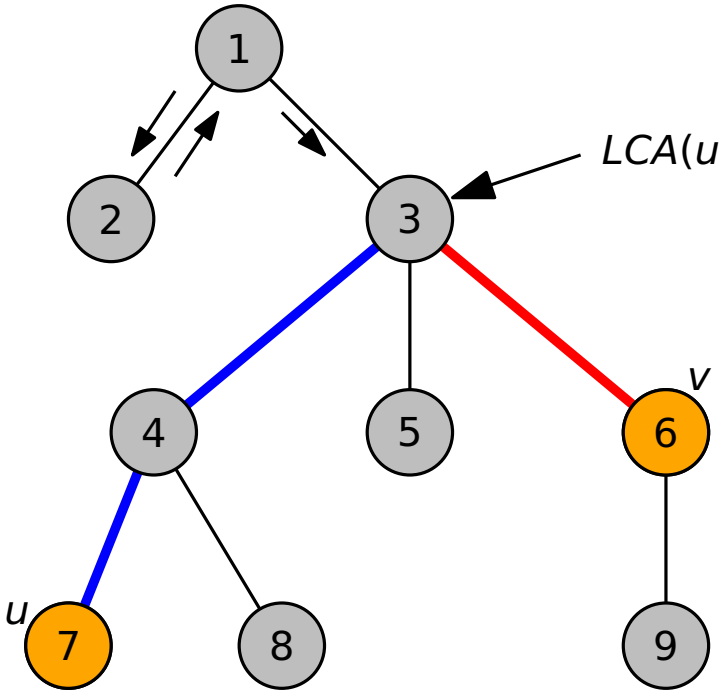

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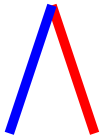

$LCA(u, v)$: lowest node on both paths to the root

How to find LCA? Build *Euler tour* of the tree:

1	2	1	3

Lowest Common Ancestor

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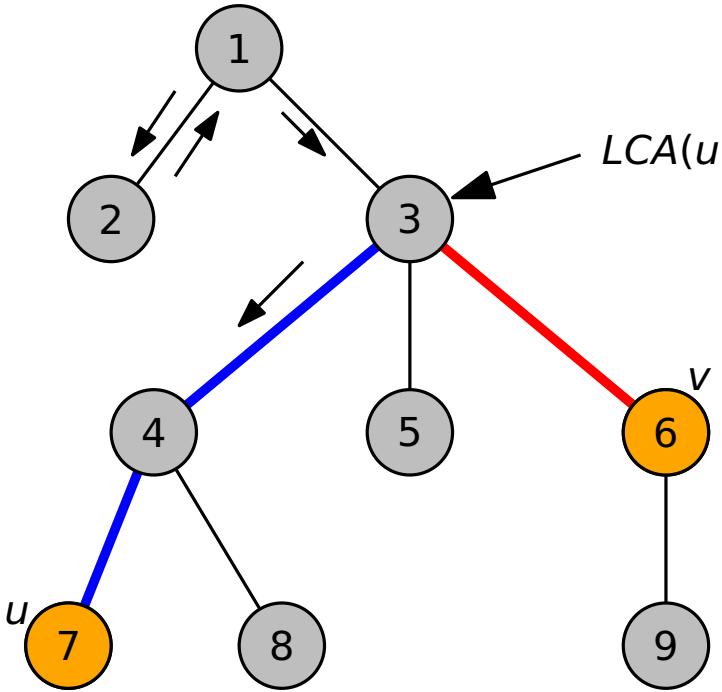

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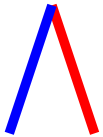

$LCA(u, v)$: lowest node on both paths to the root

How to find LCA? Build *Euler tour* of the tree:

1	2	1	3	4

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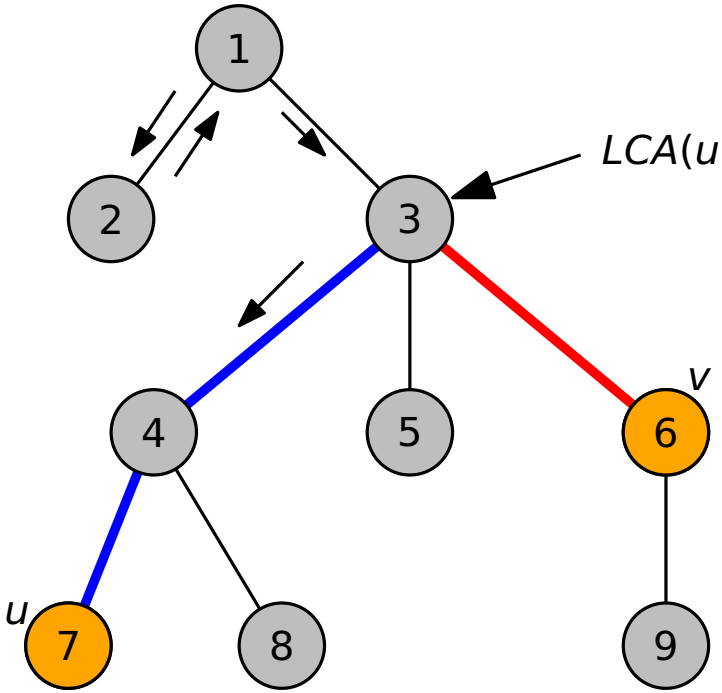

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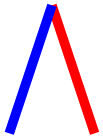

$LCA(u, v)$: lowest node on both paths to the root

How to find LCA? Build *Euler tour* of the tree:

1	2	1	3	4	7	4	8	4	3	5	3	6	9	6	3	1

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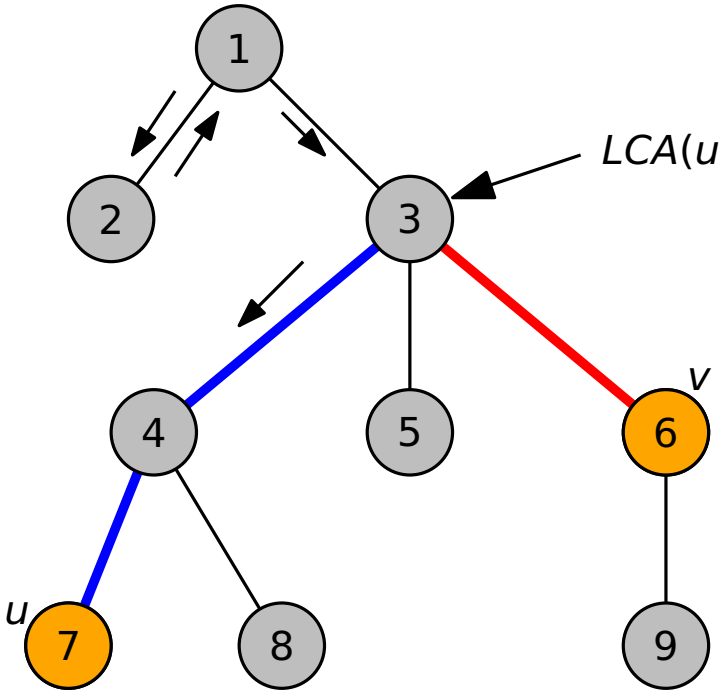

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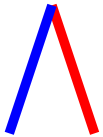

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Length?

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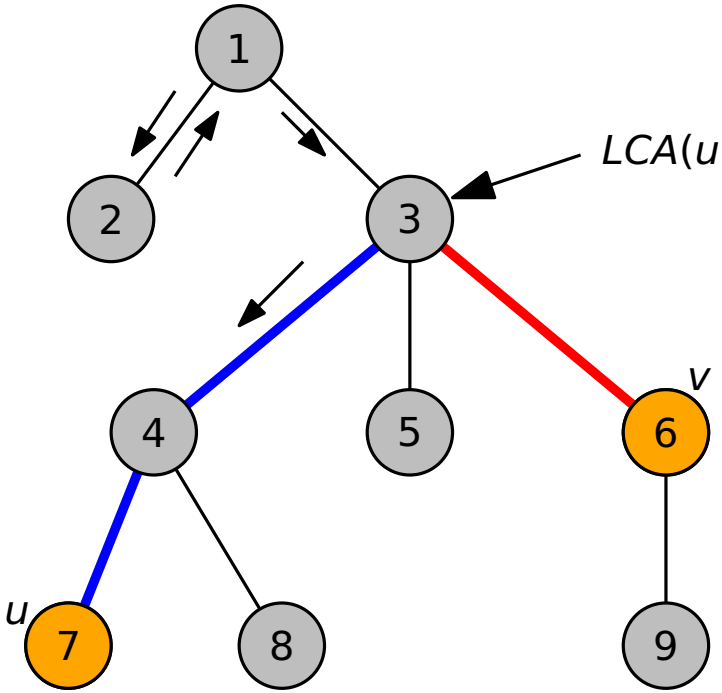

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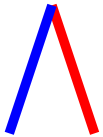

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How to find LCA? Build *Euler tour* of the tree:

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Length? $2n - 1$

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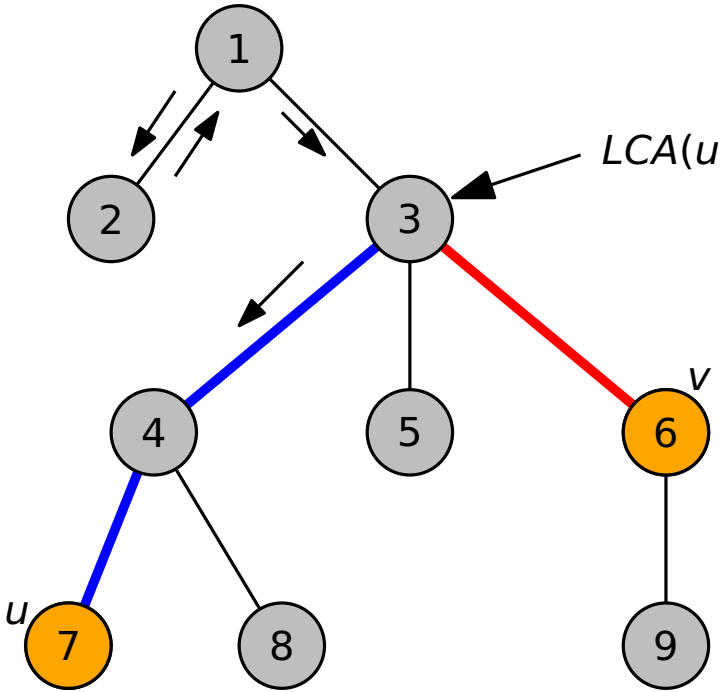

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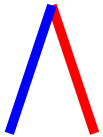

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Length? $2n - 1$

Subtrees?

Lowest Common Ancestor

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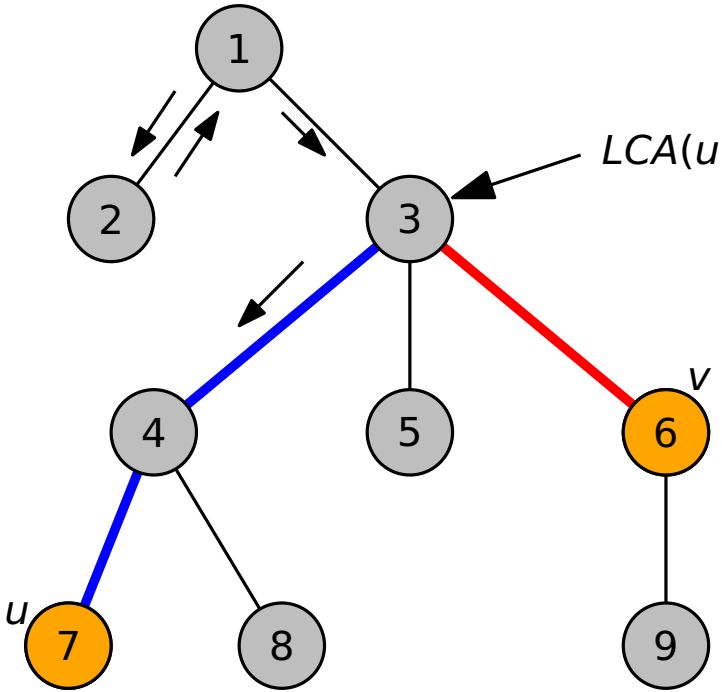

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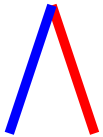

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Length? $2n - 1$

Subtrees? Subtrees are subarrays

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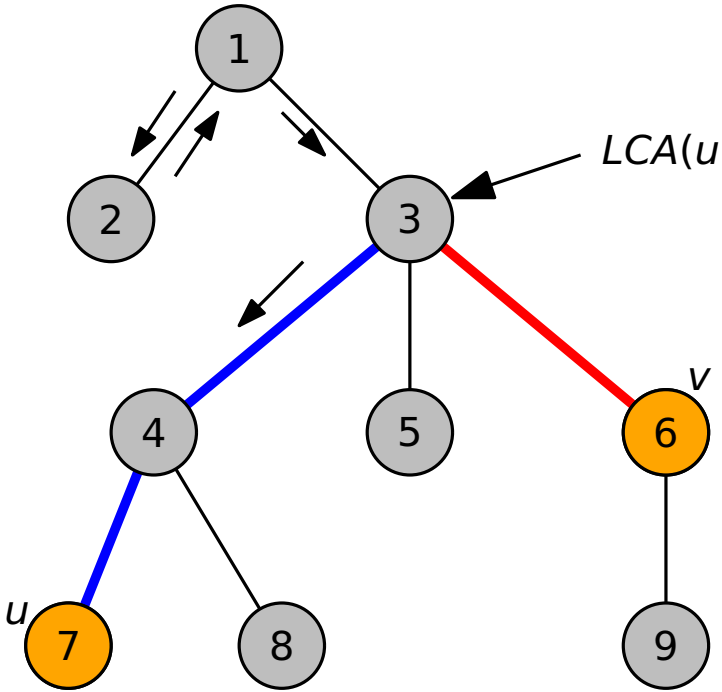

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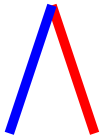

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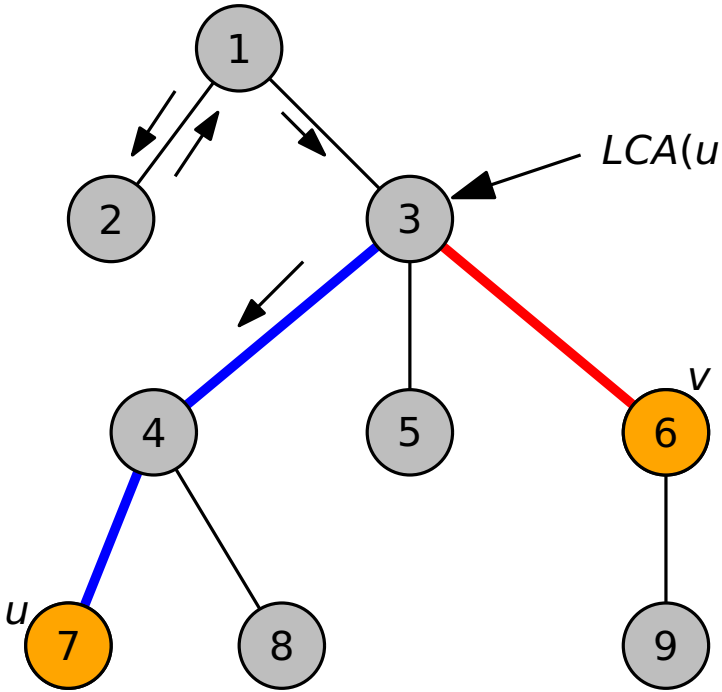

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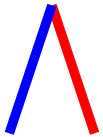

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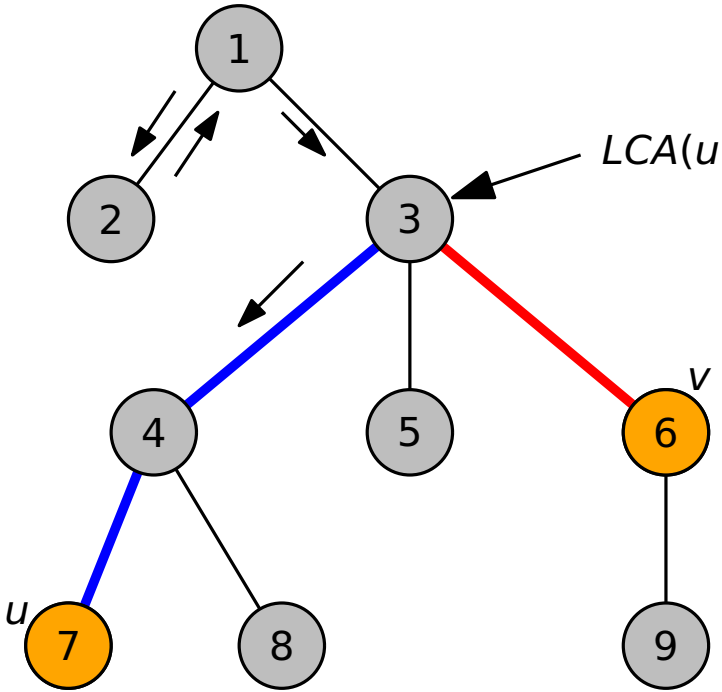

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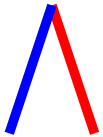

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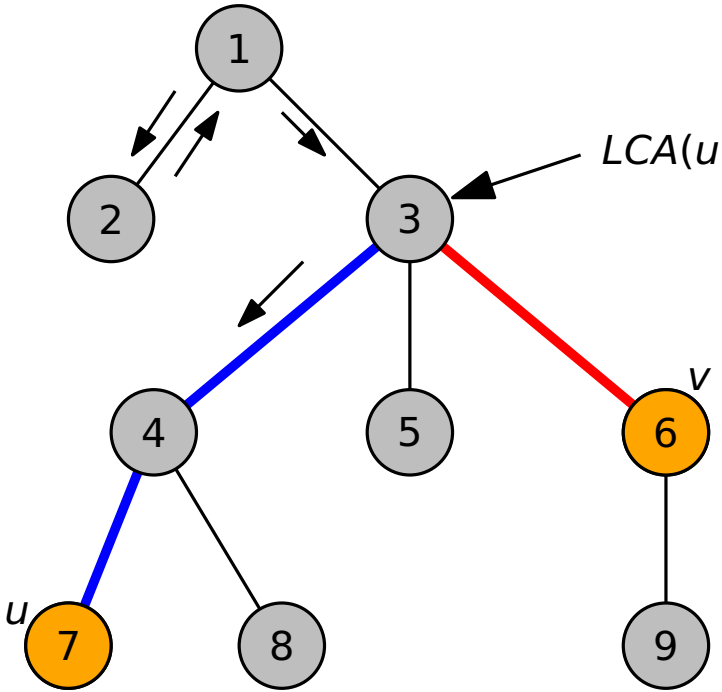

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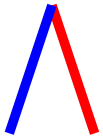

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Length? $2n - 1$

Subtrees? Subtrees are subarrays

Paths? Subarray contains path (plus subtrees)

LCA?

0	1	0	1	2	3	2	3	2	1	2	1	2	3	2	1	0

Lowest Common Ancestor

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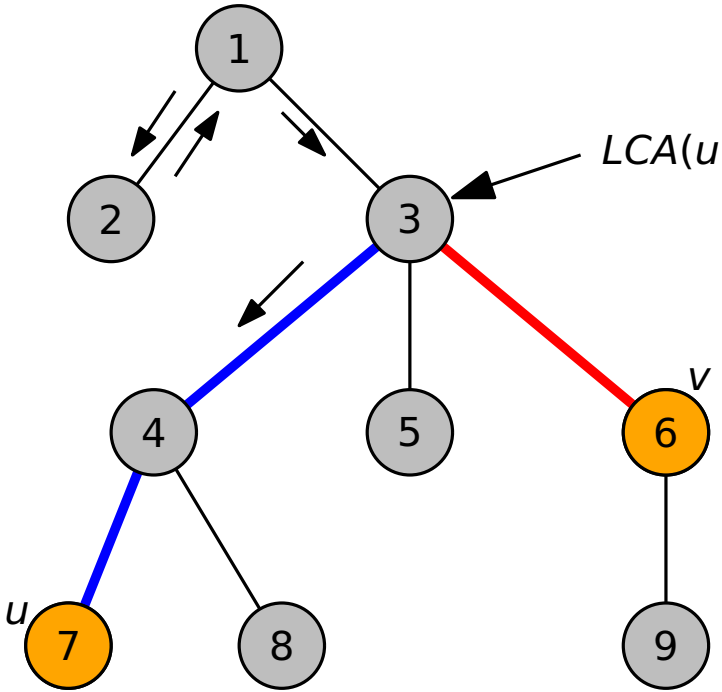

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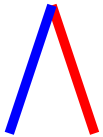

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Length? $2n - 1$

Subtrees? Subtrees are subarrays

Paths? Subarray contains path (plus subtrees)

LCA?

0	1	0	1	2	3	2	3	2	1	2	1	2	3	2	1	0

min

Lowest Common Ancestor

- Build Euler tour

Lowest Common Ancestor

- Build Euler tour

- Build data structure on tour

Lowest Common Ancestor

- Build Euler tour

- Build data structure on tour
- Answer LCA queries with data structure

Lowest Common Ancestor

- Build Euler tour

```
void dfs(Graph &adj, int v, int p = -1, int d = 0) {
    depth[v] = d;
    first[v] = euler.size();
    euler.push_back(v);
    for (auto nei : adj[node])
        if (nei != p) {
            dfs(adj, nei, v, d + 1);
            euler.push_back(v);
        }
}
```

- Build data structure on tour
- Answer LCA queries with data structure

Lowest Common Ancestor

- Build Euler tour

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Range minimum queries on static array

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Binary lifting

Binary lifting

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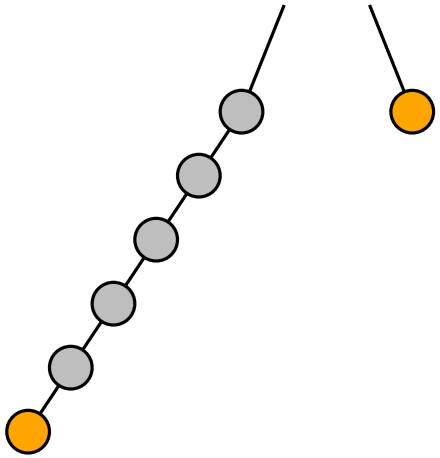

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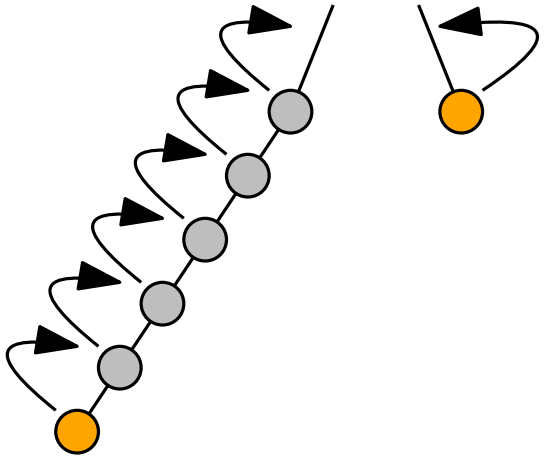

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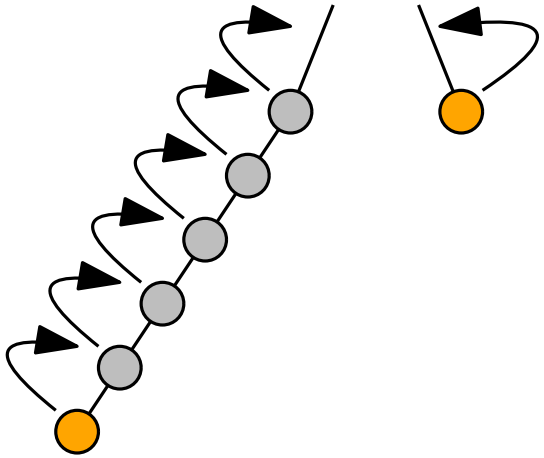

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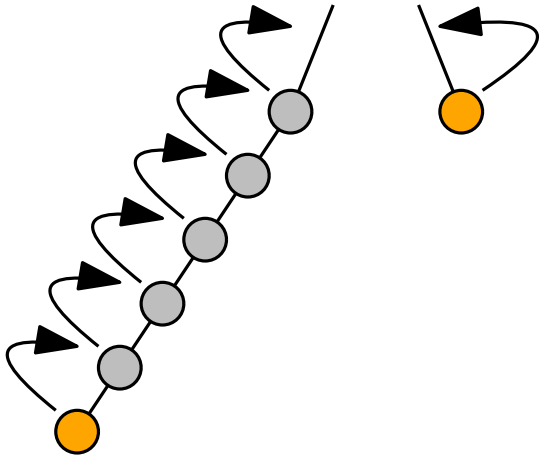

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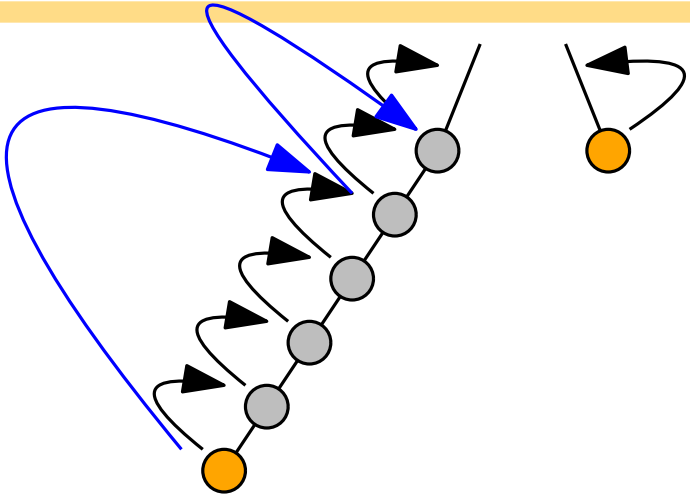

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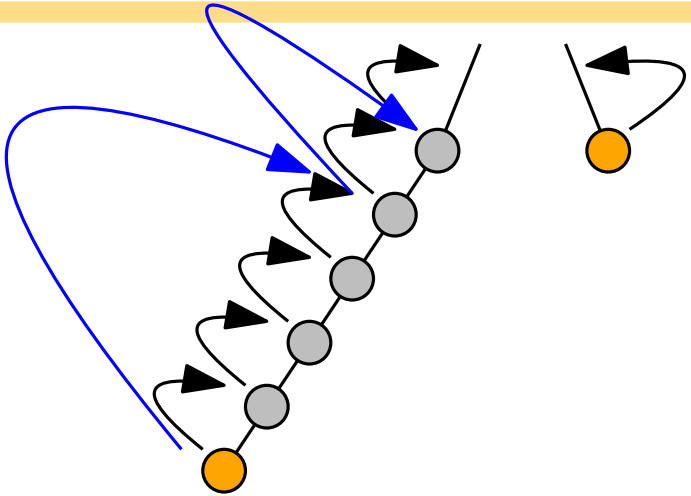

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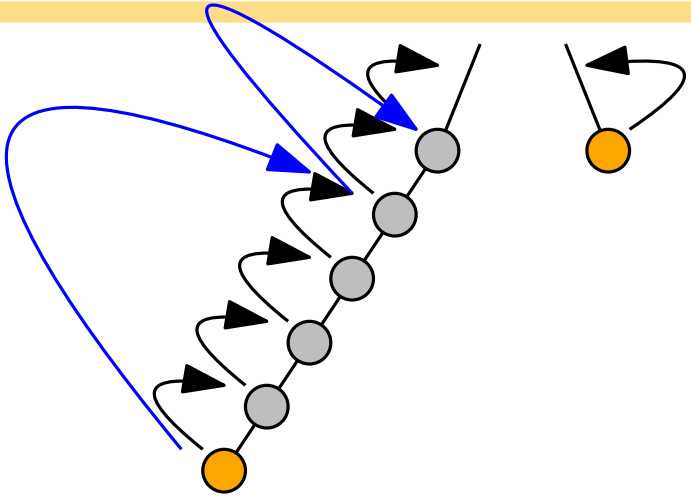

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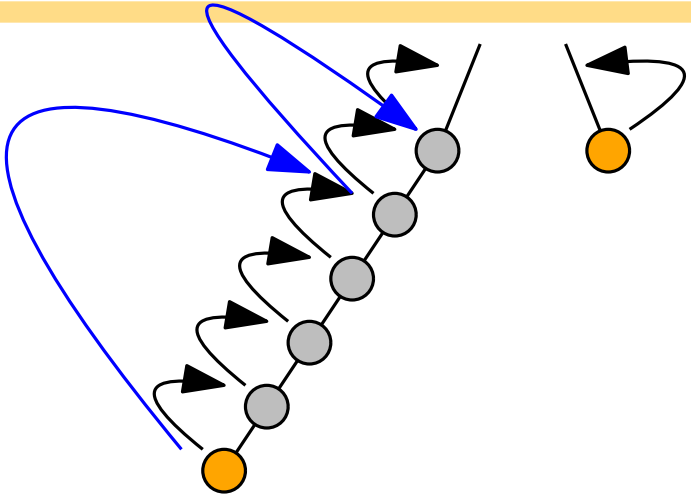

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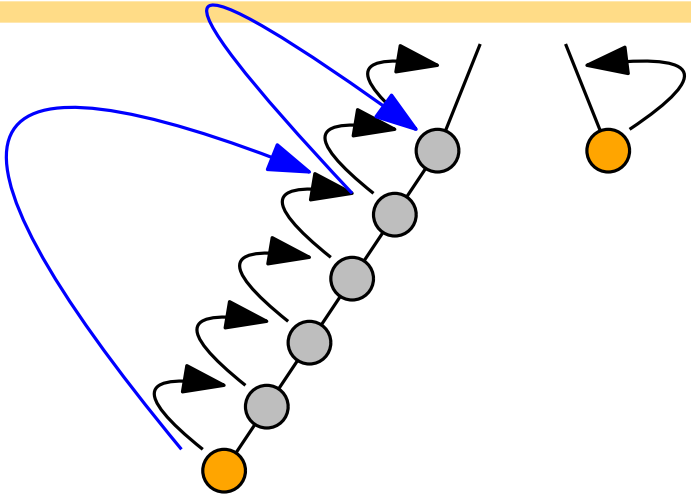

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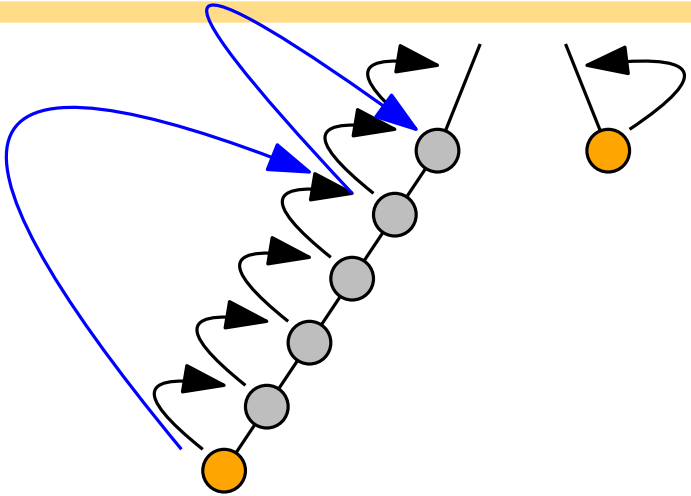

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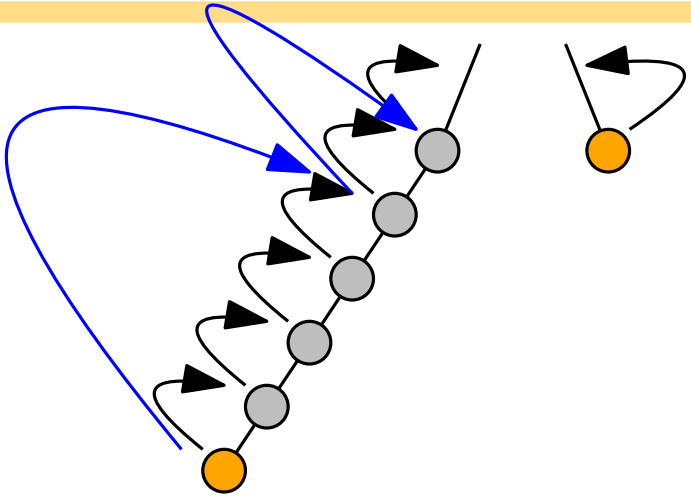

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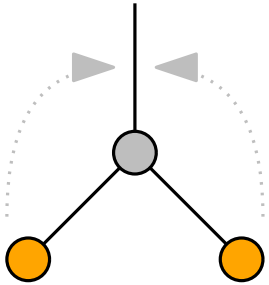

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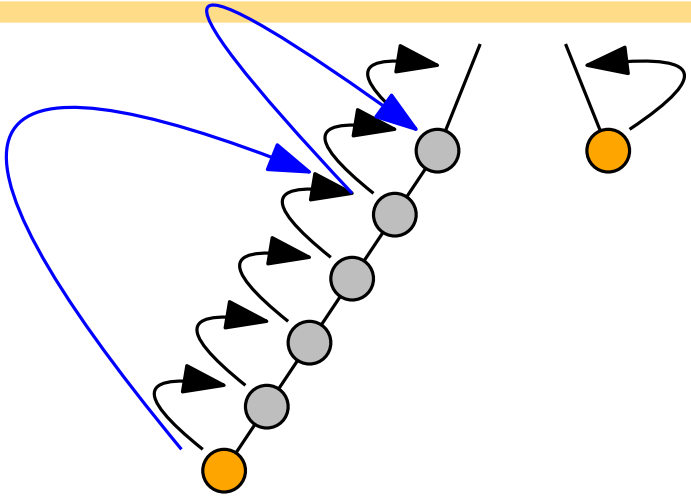

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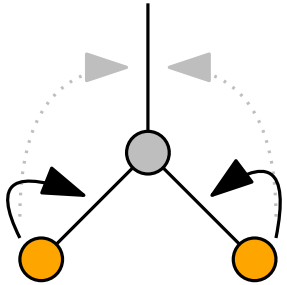

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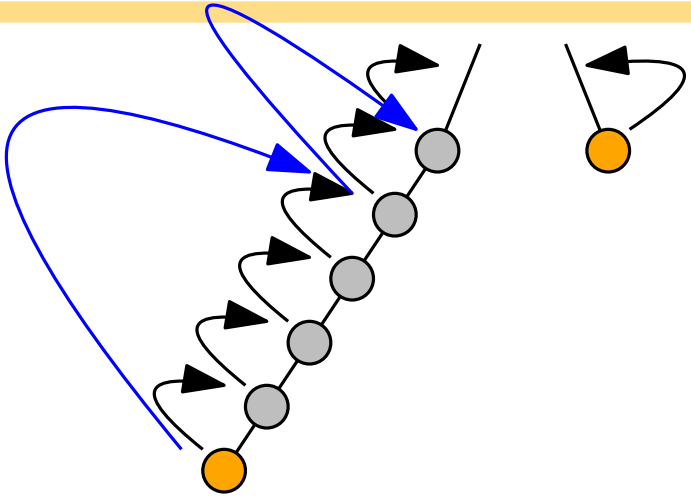

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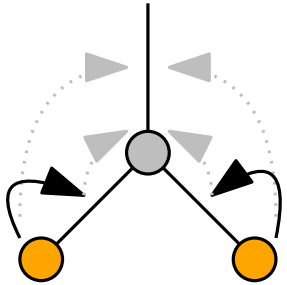

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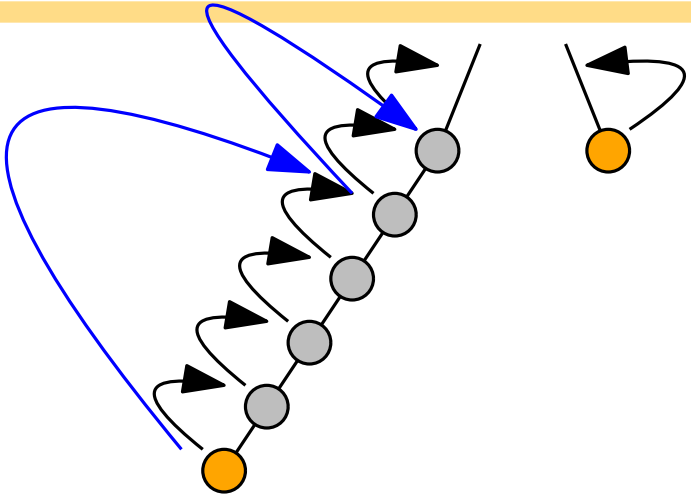

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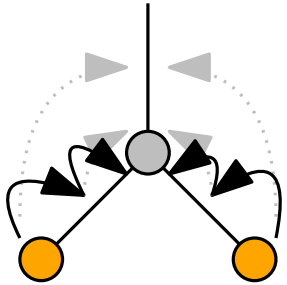

Ancestors for binary lifting

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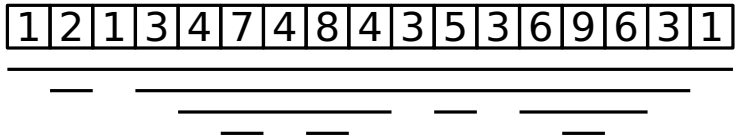

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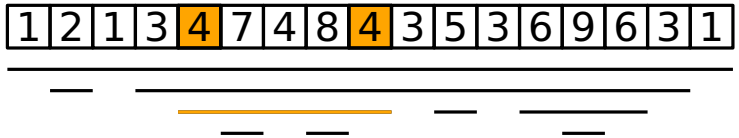

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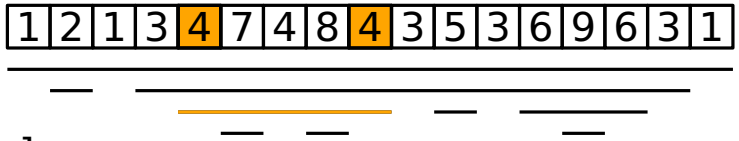

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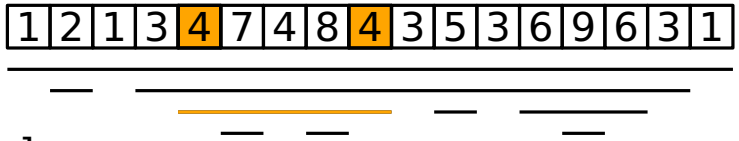

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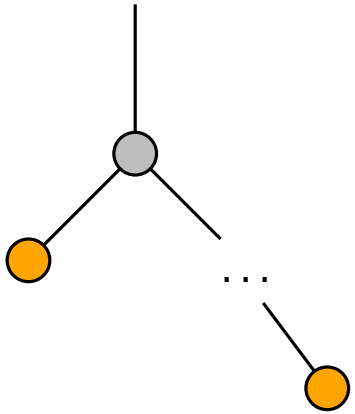

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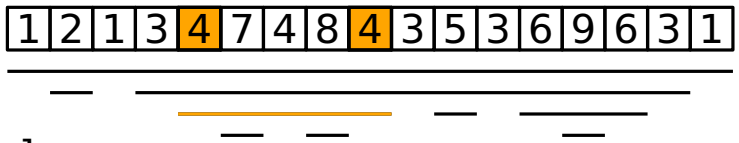

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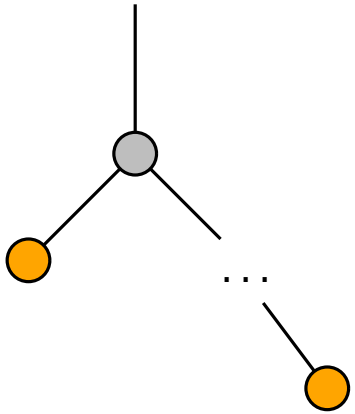

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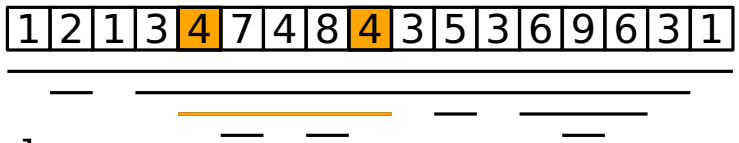

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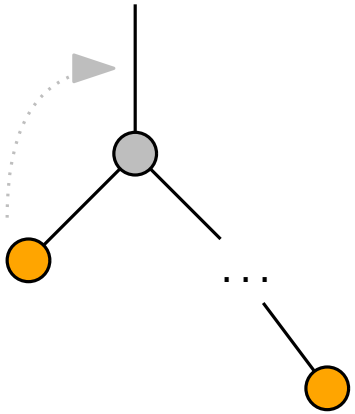

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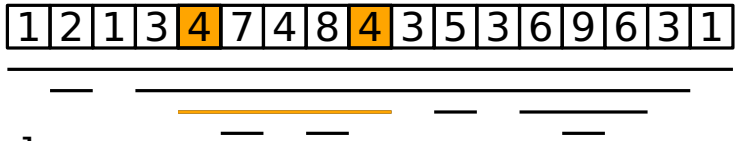

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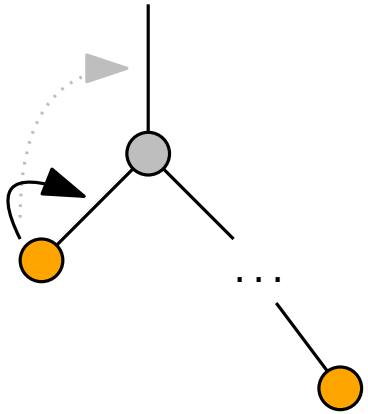

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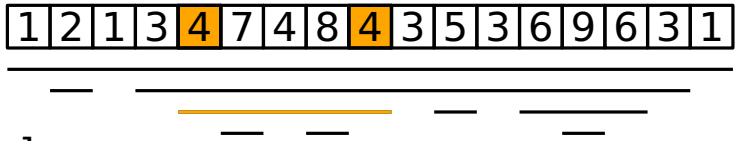

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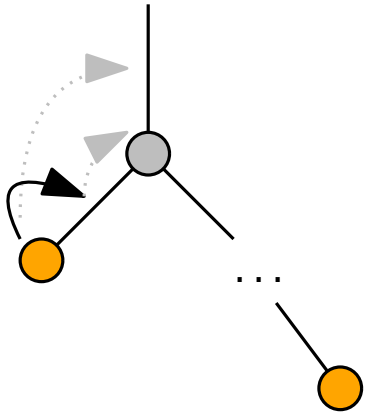

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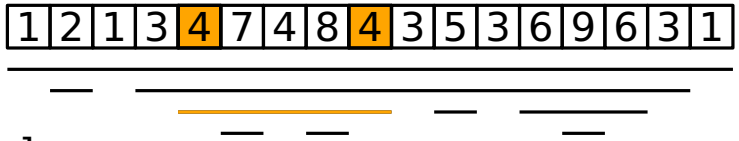

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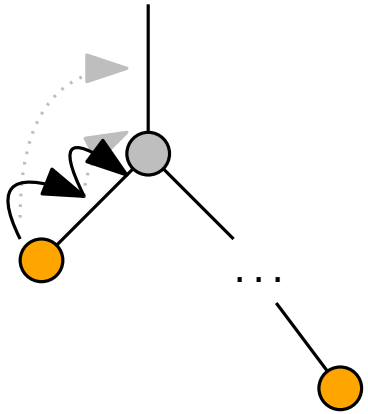

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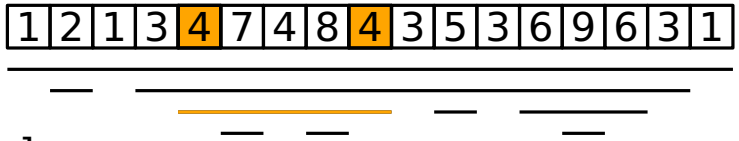
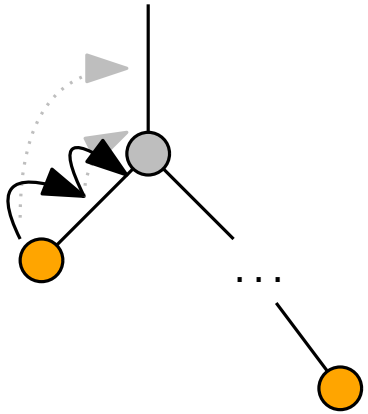


- Binary jumps (decreasing size) to find node below LCA
  - Jump starting at u if not ancestor of v
  - Parent of ending node is LCA



Ancestors for binary lifting

- Alternative LCA finding
- In Euler tour, let's save $first[v]$ and $last[v]$
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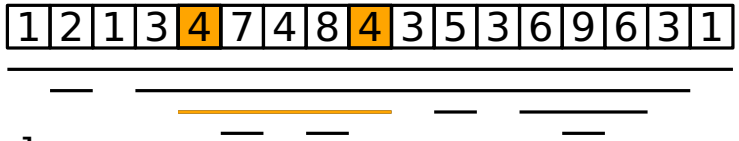

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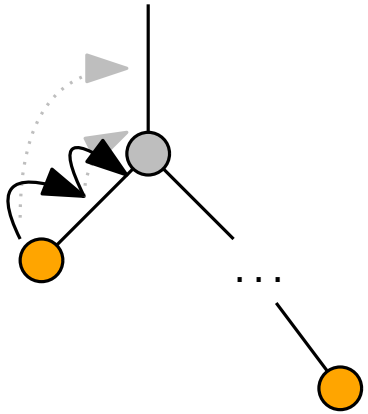

- Binary lifting in $O(n)$ memory is possible
  - https://codeforces.com/blog/entry/74847

LCA/binary lifting applications

LCA/binary lifting applications

- $LCA(u, v)$?

LCA/binary lifting applications

- $LCA(u, v)$? Euler tour/binary lifting

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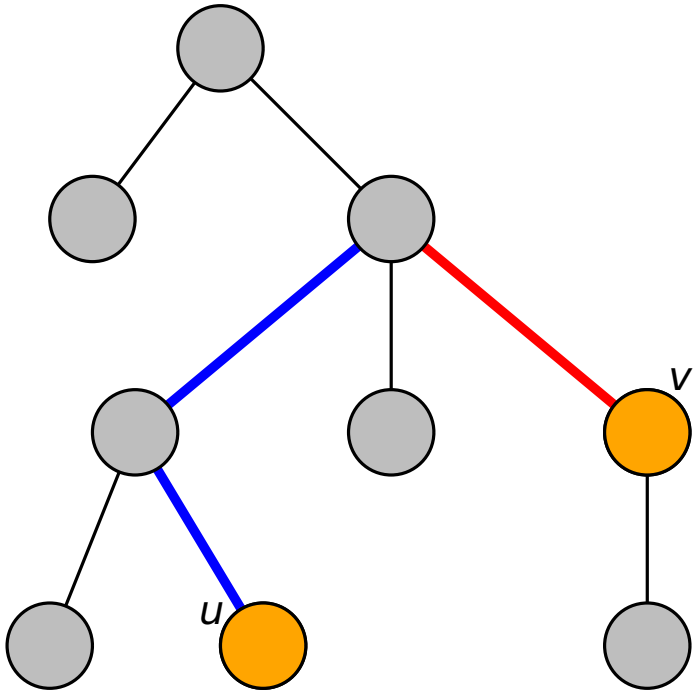

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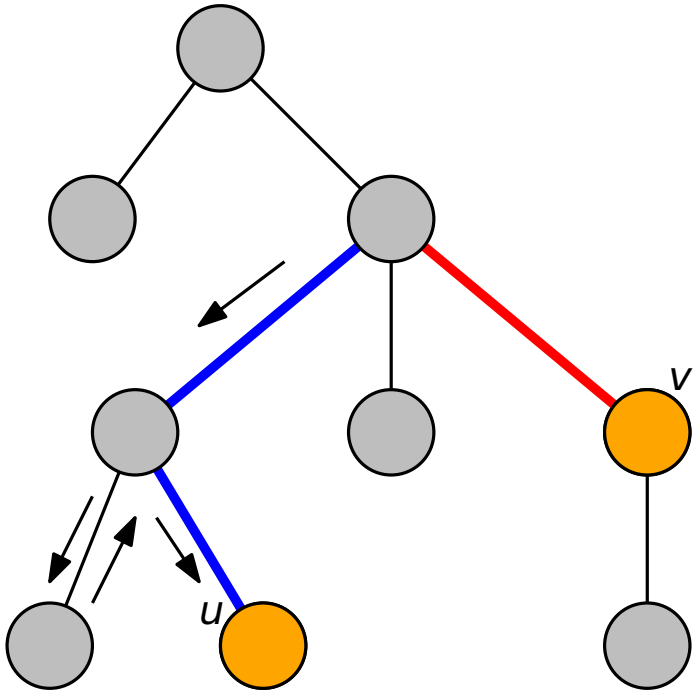

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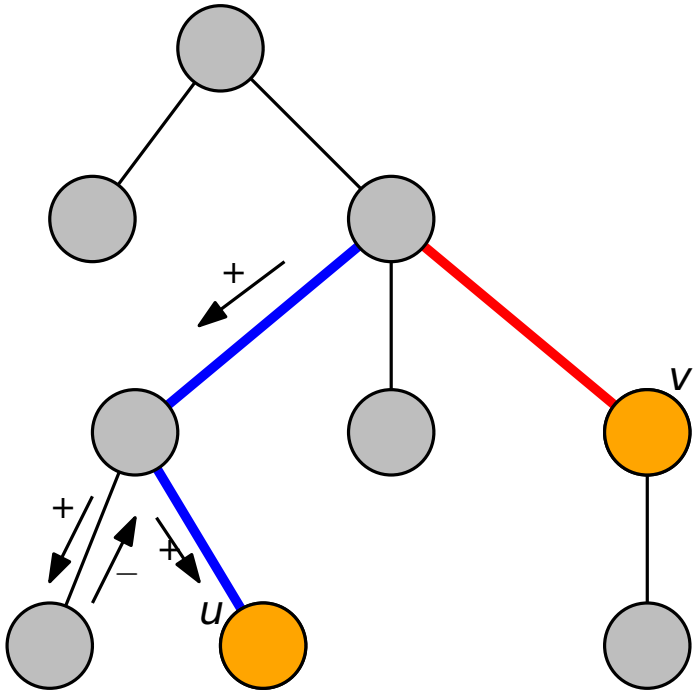
  Euler tour with two entry types (up + down)!



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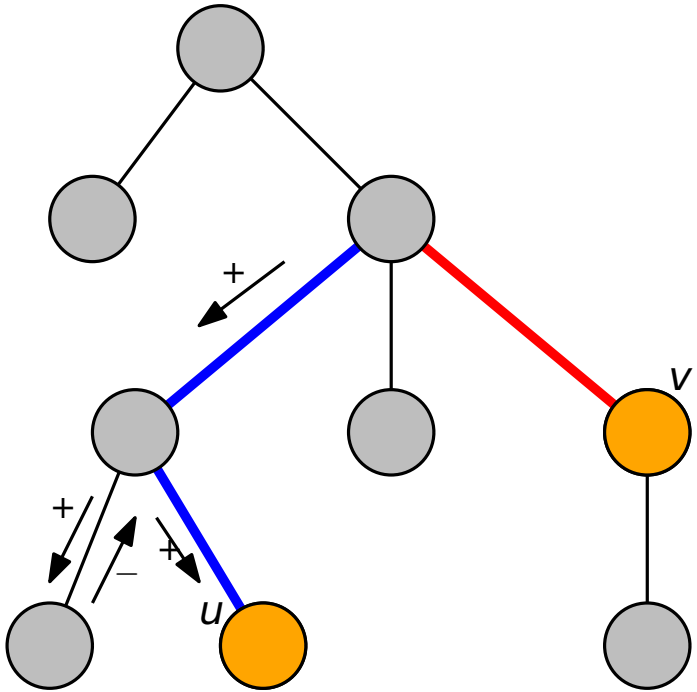

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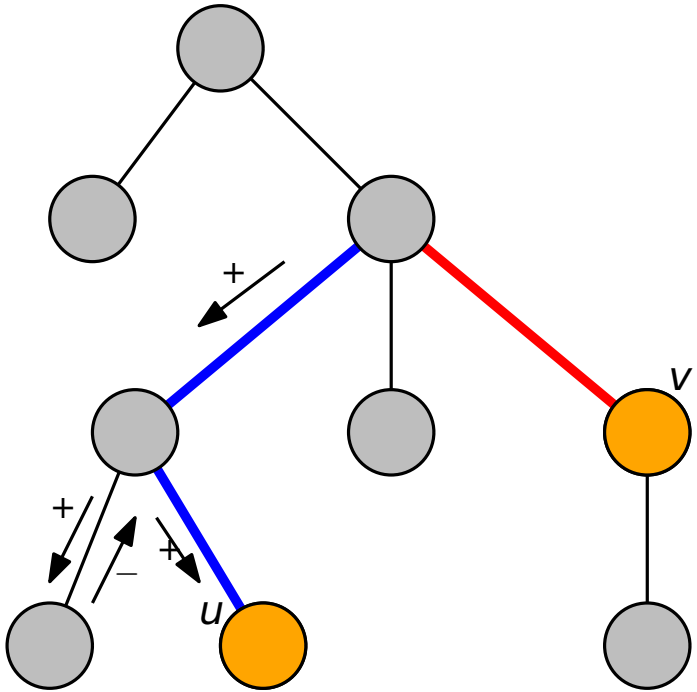
  Binary lifting with max information



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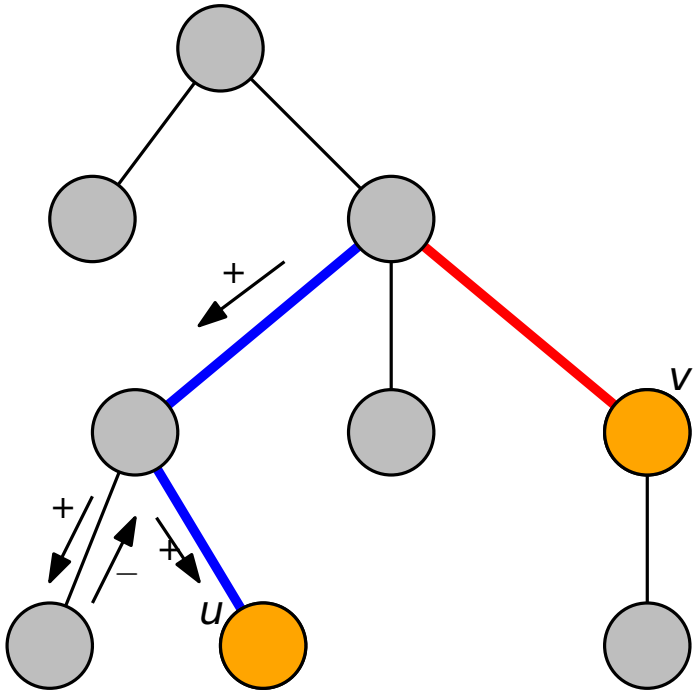
- max of edge weights with updates?



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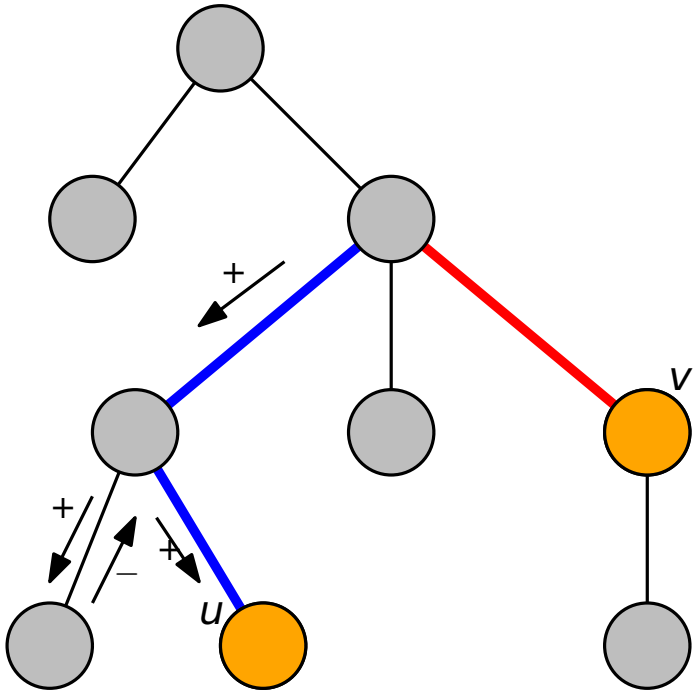

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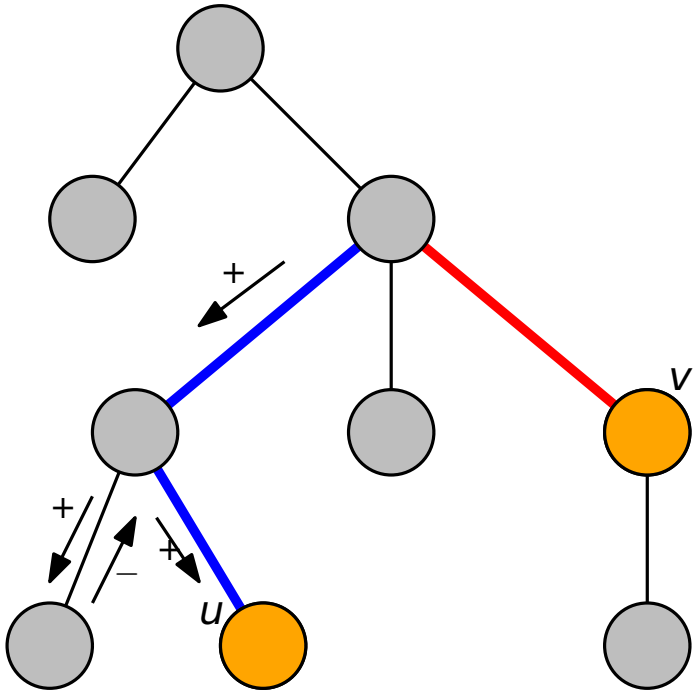
  Heavy-light decomposition!



Heavy light decomposition

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Idea: Partition tree into paths, such that we traverse $\mathcal{O}(\log n)$ paths per (u, v) query.

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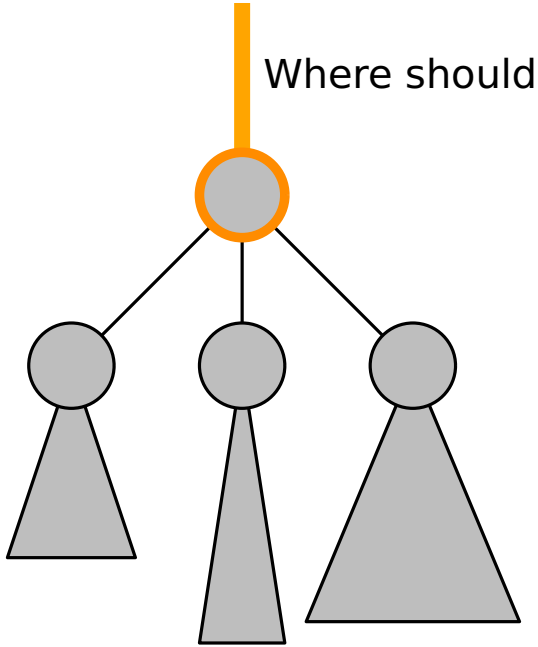

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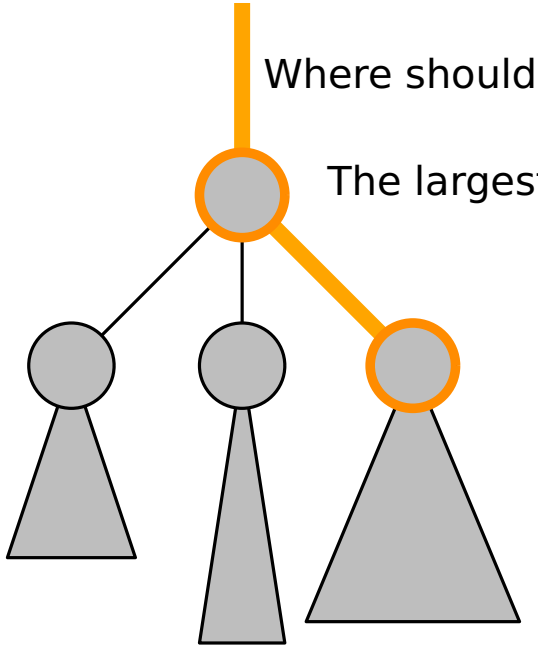

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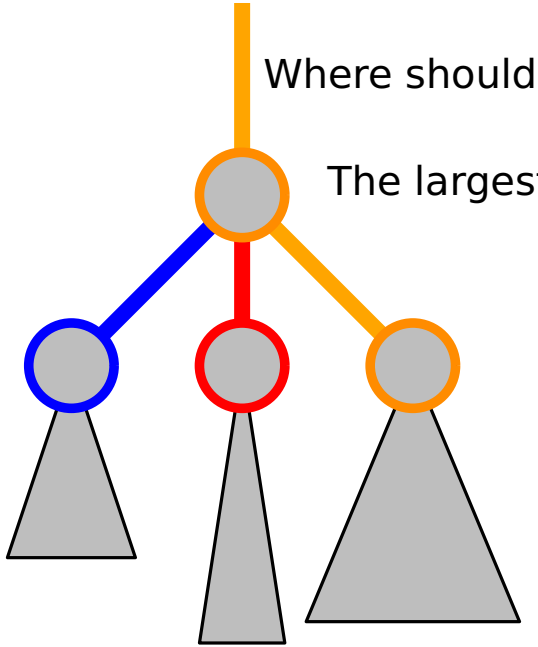

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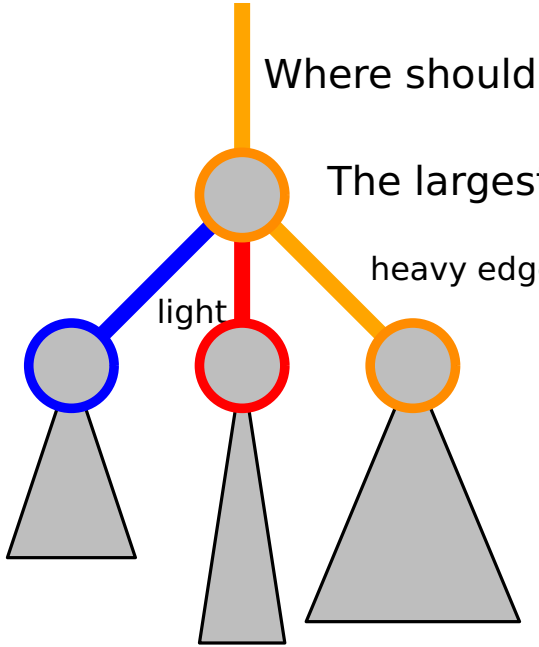

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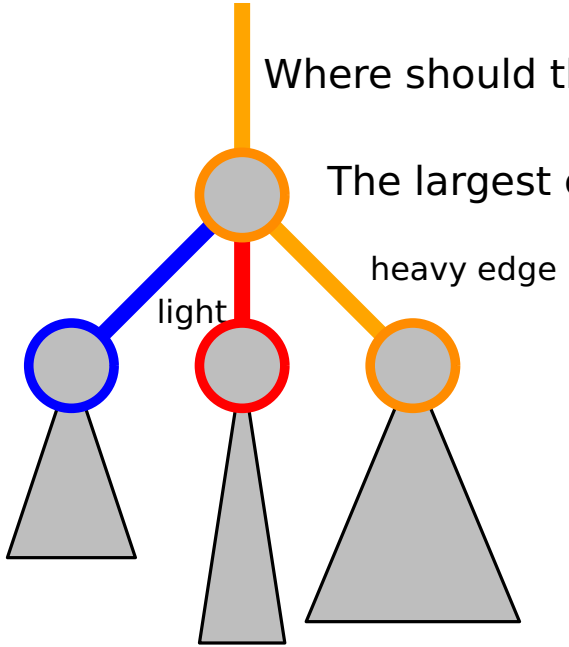

How many paths on the way from the root to any node?

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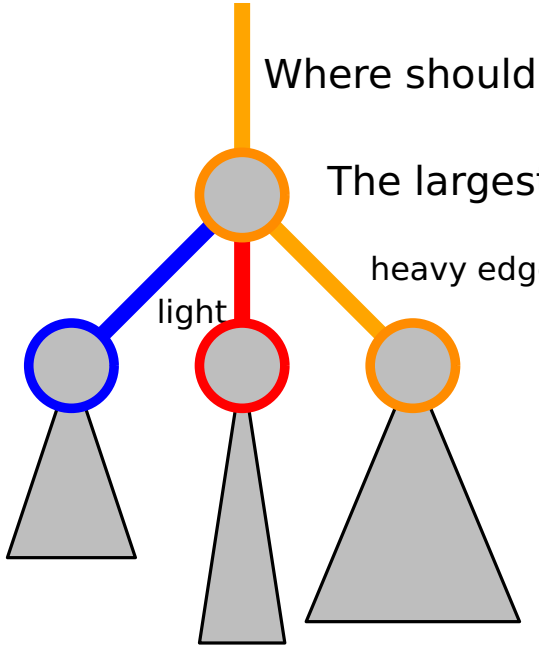

How many paths on the way from the root to any node?

- New path means light edge

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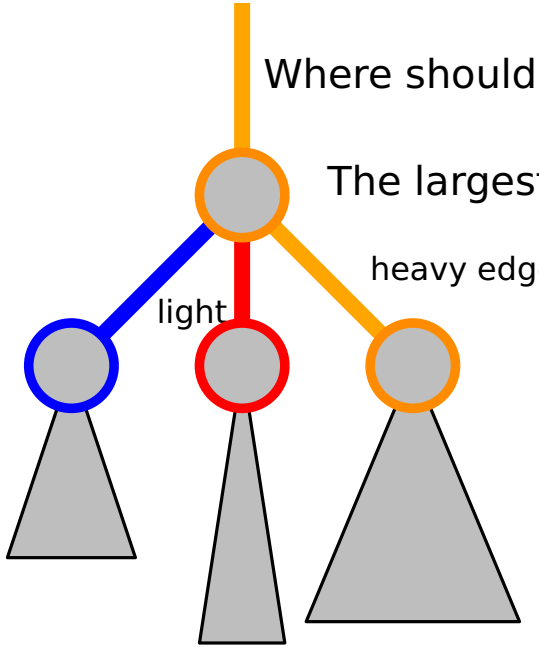

How many paths on the way from the root to any node?

- New path means light edge
- Light edge means not largest child

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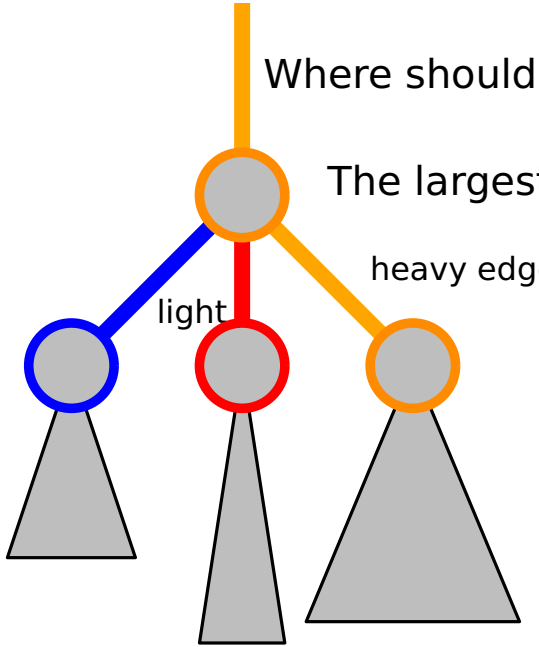

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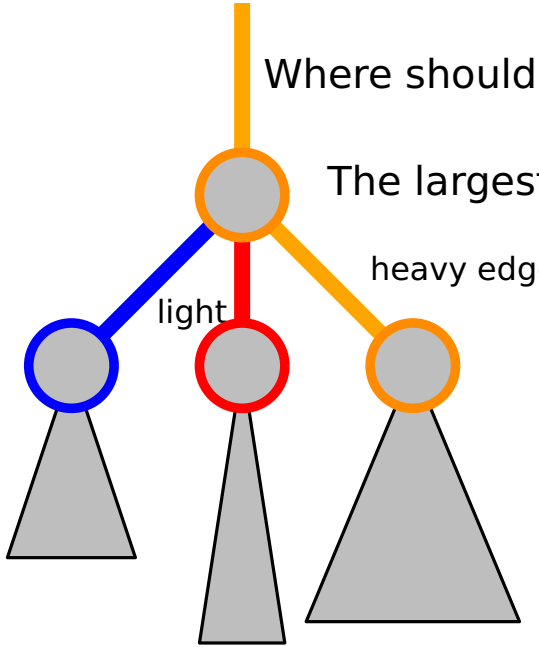

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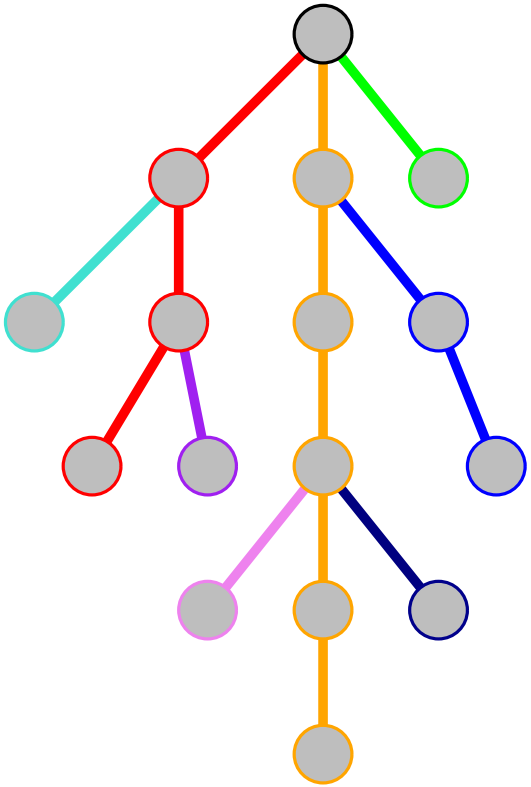
Every (u, v) path consists of $\mathcal{O}(\log n)$ (parts of) paths.



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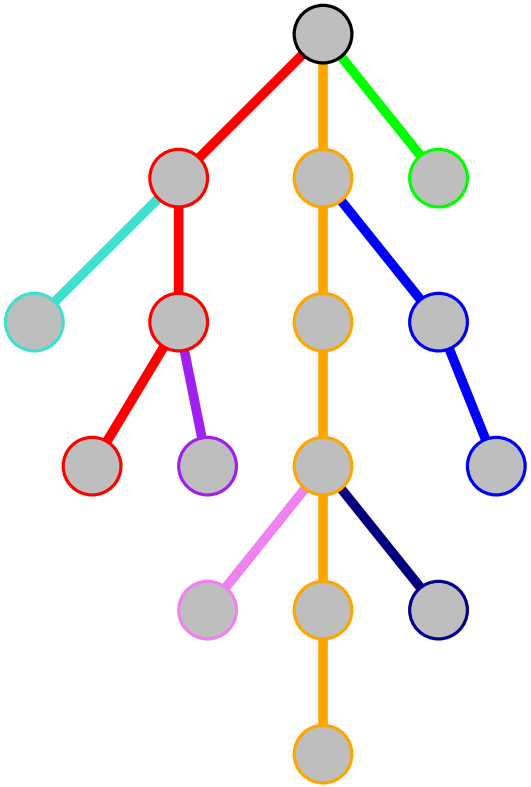

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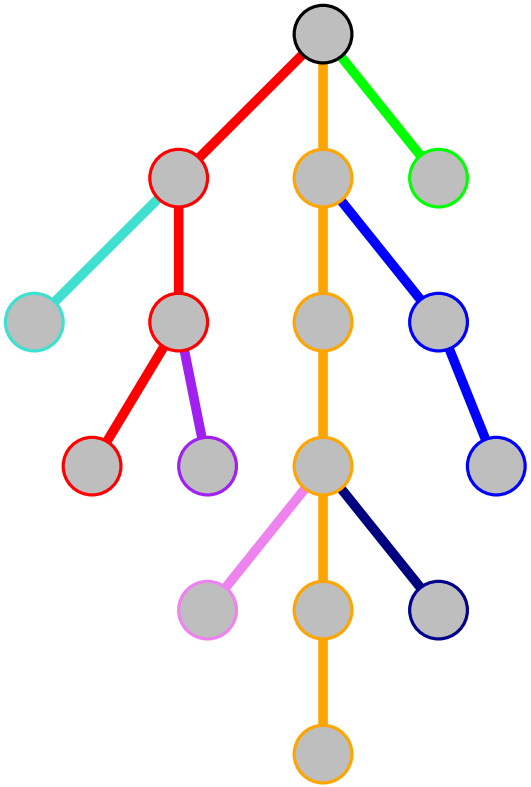
How does this help for queries? One data structure per path!



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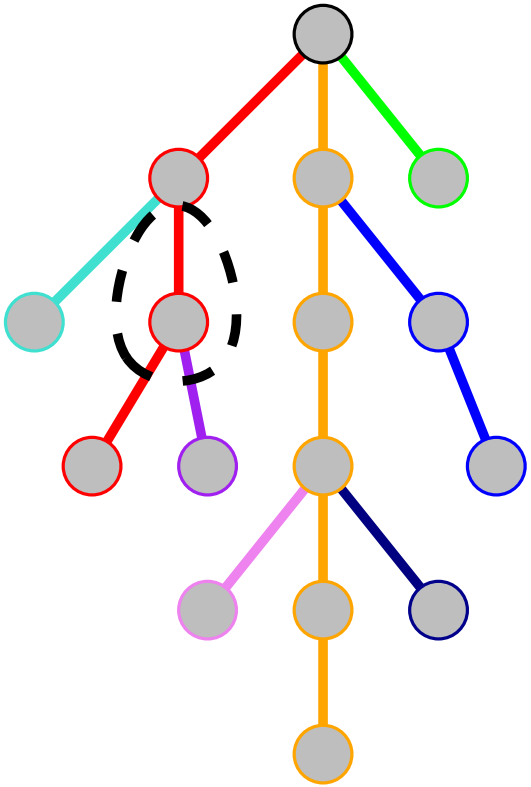

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Every (u, v) path consists of $\mathcal{O}(\log n)$ (parts of) paths.

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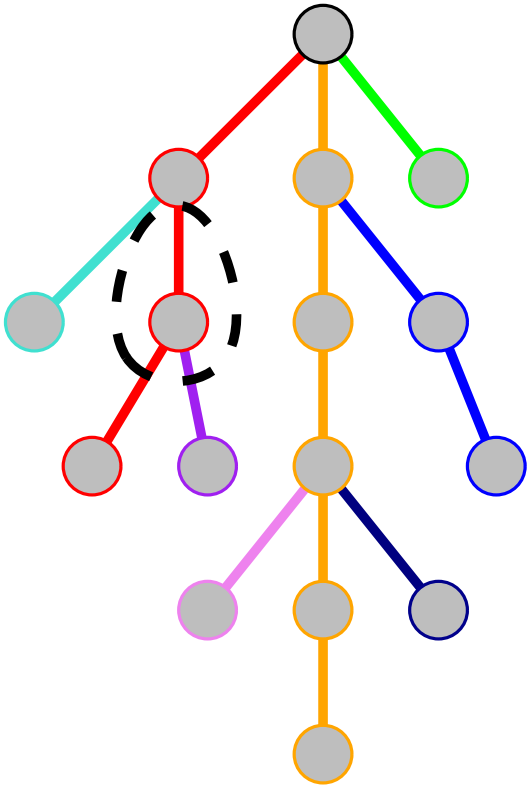
- Update single node/edge?
  Point update in 1 path



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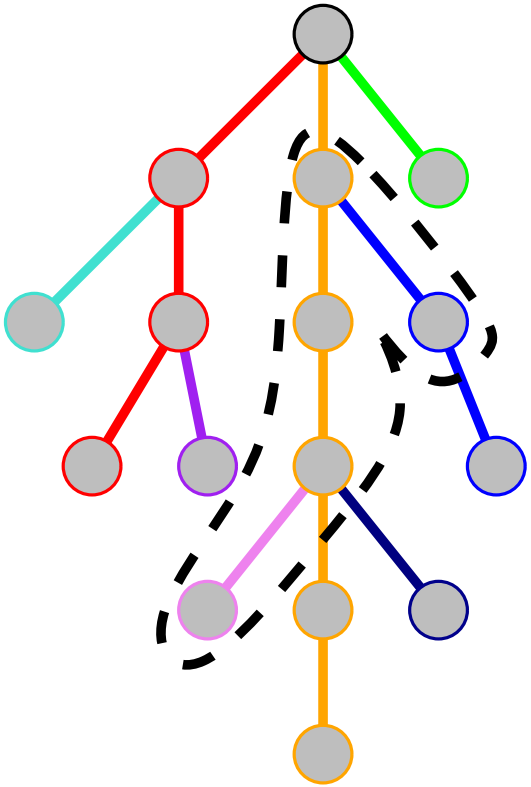

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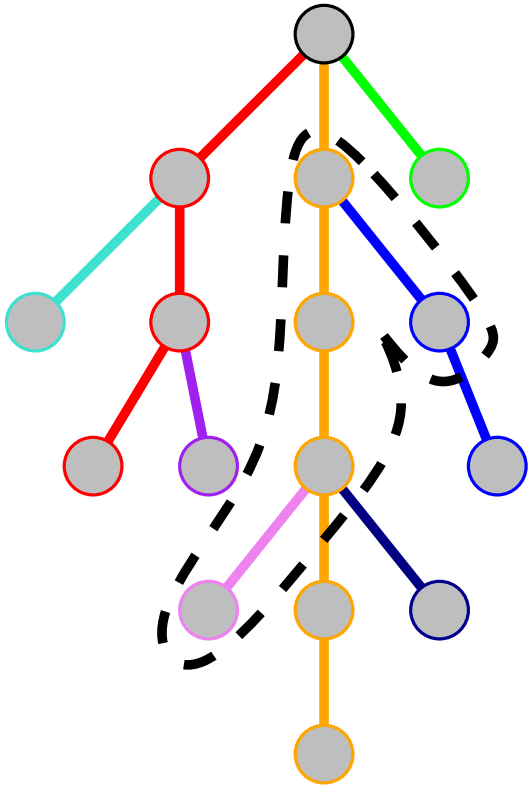
  Range update in $\mathcal{O}(\log n)$ paths



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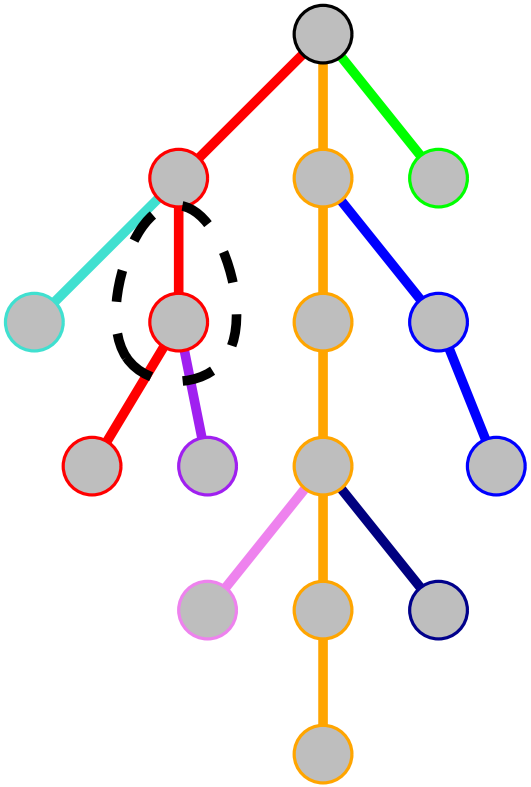

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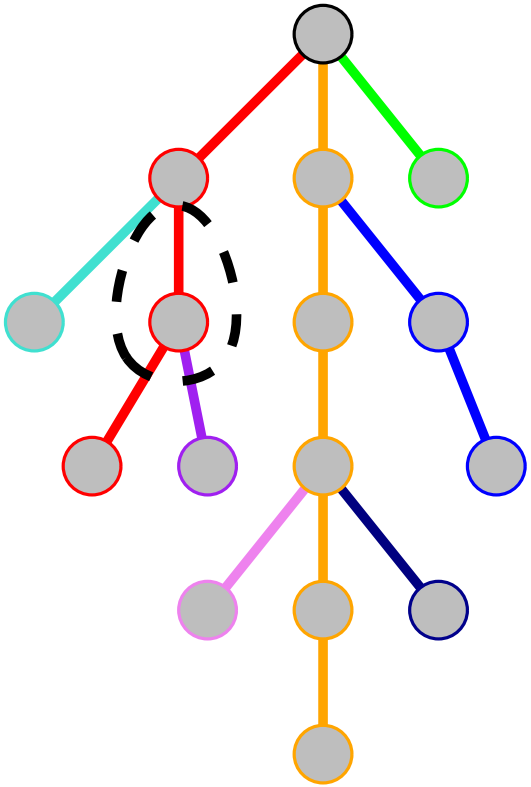

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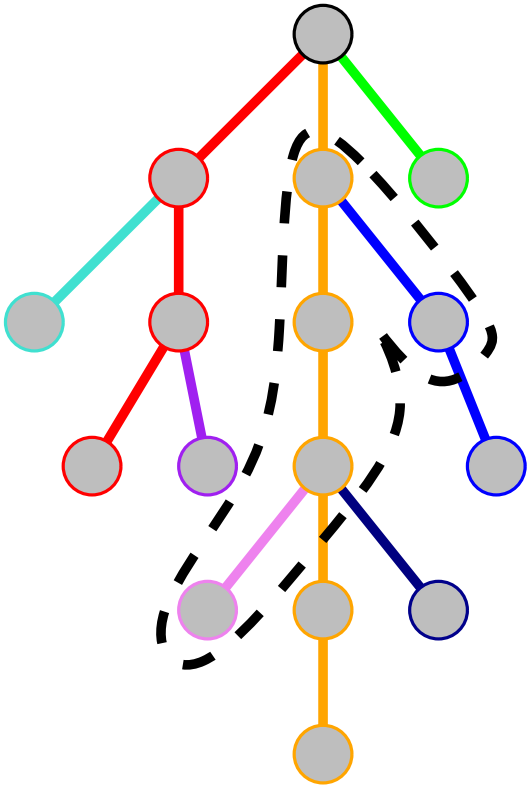

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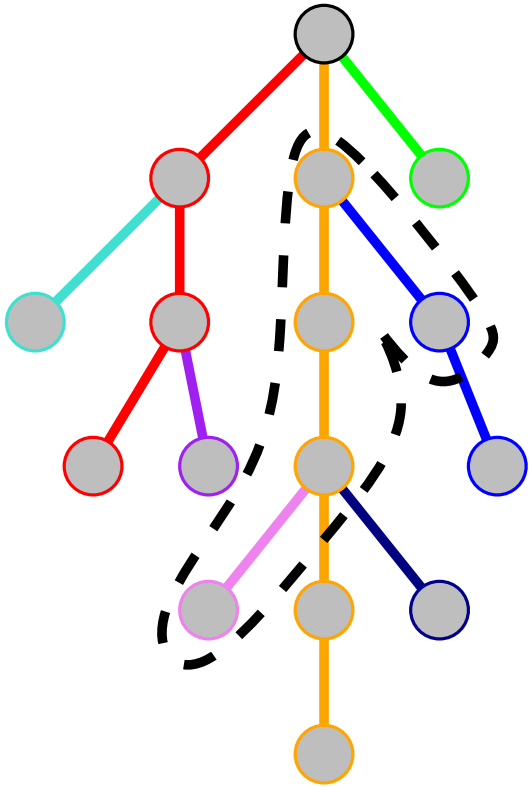

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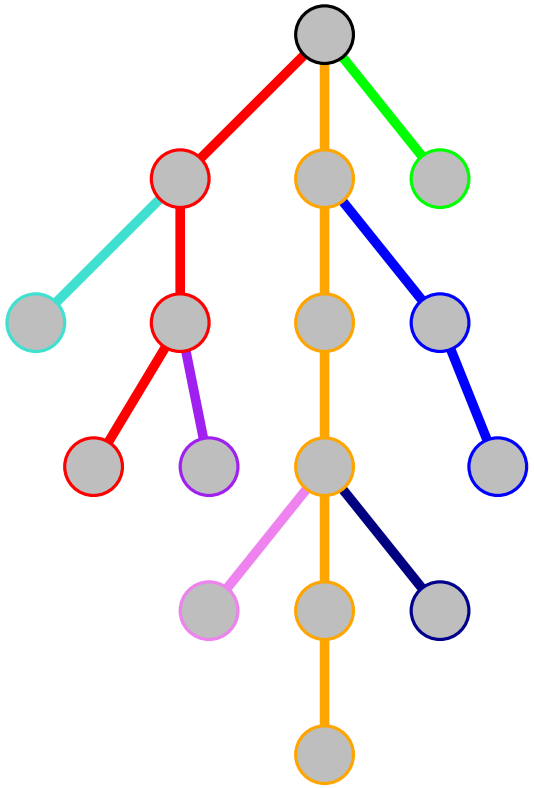
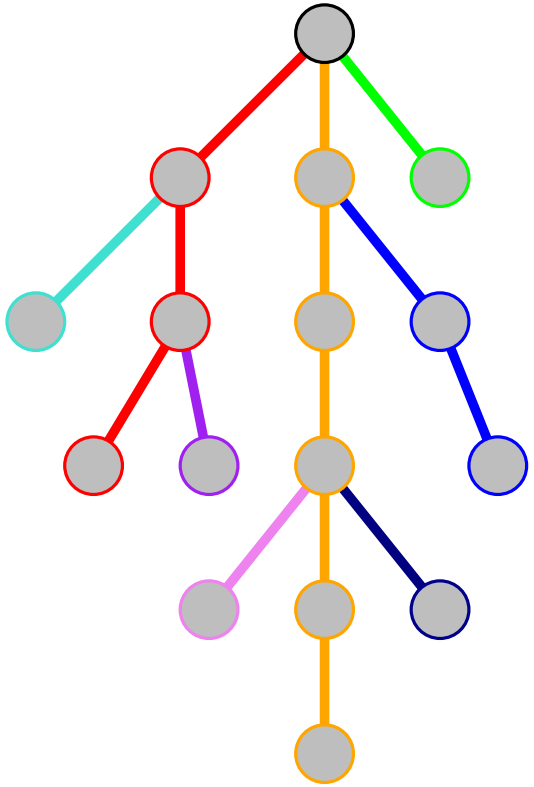


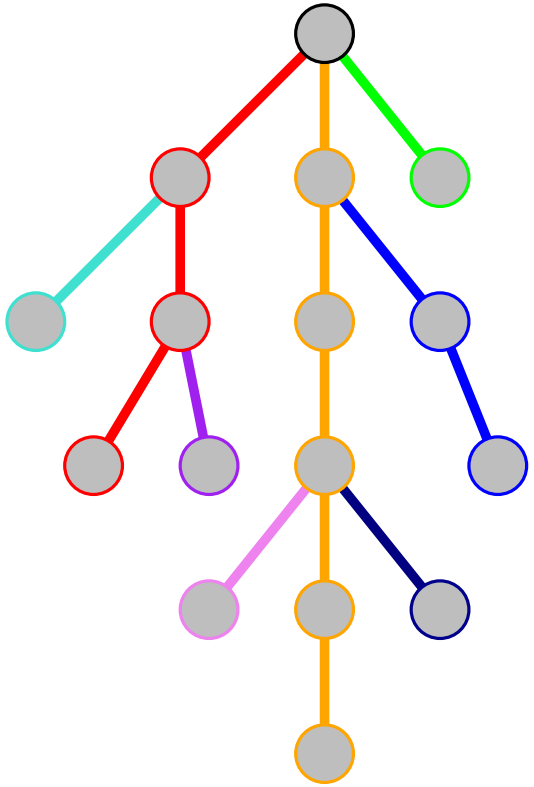
Heavy light decomposition

- One DS per path?



Heavy light decomposition

- ~~One DS per path?~~ One DS to rule them all!



Heavy light decomposition

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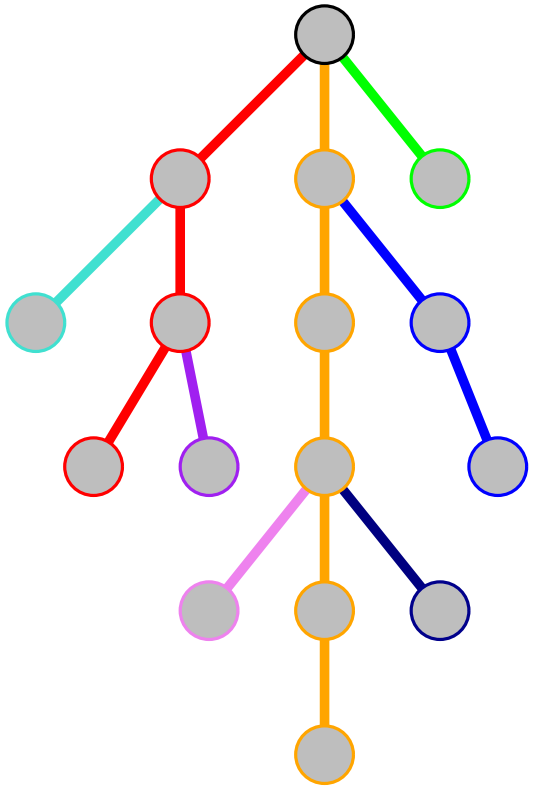

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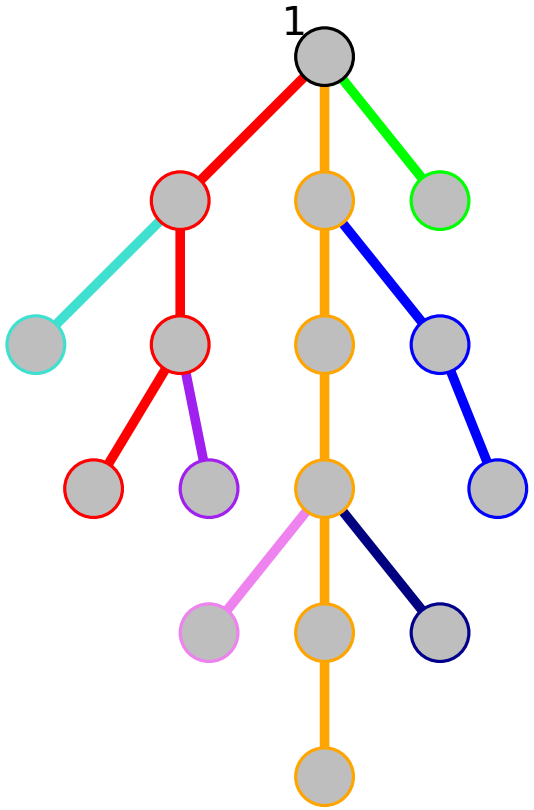

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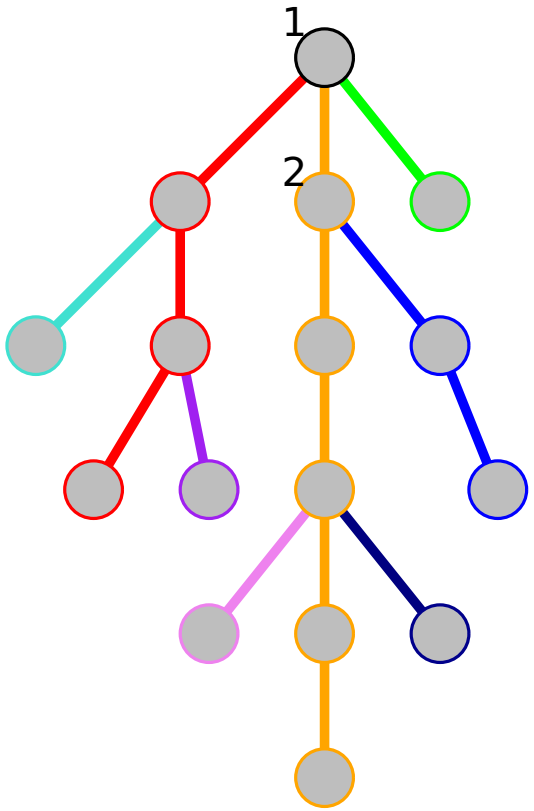

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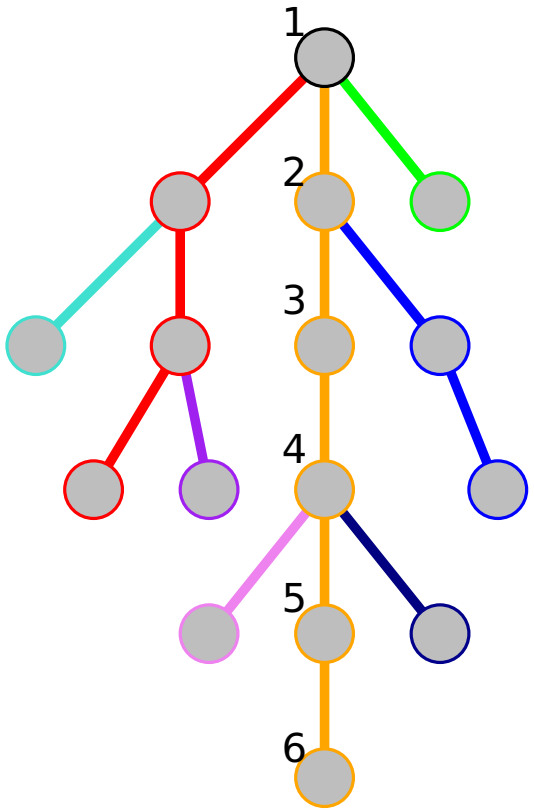

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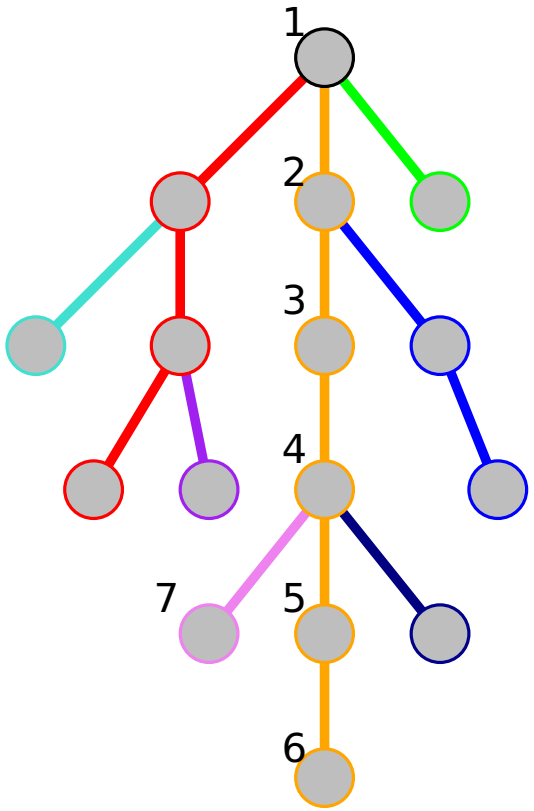

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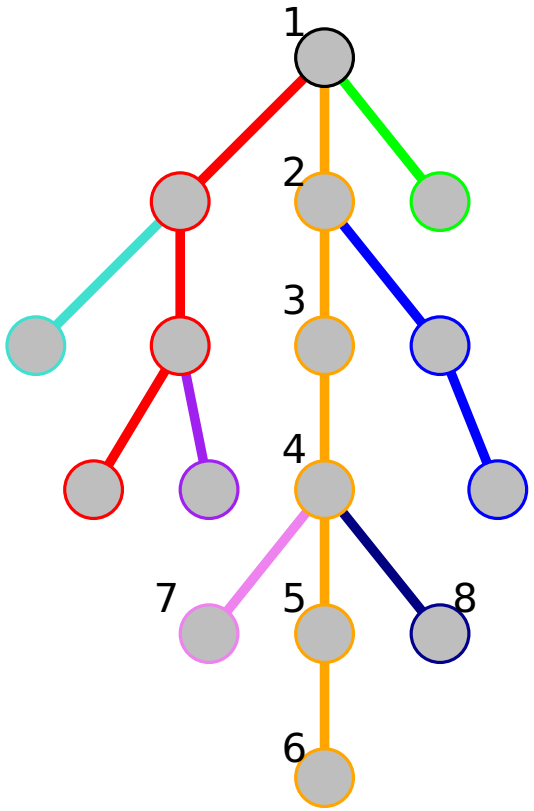

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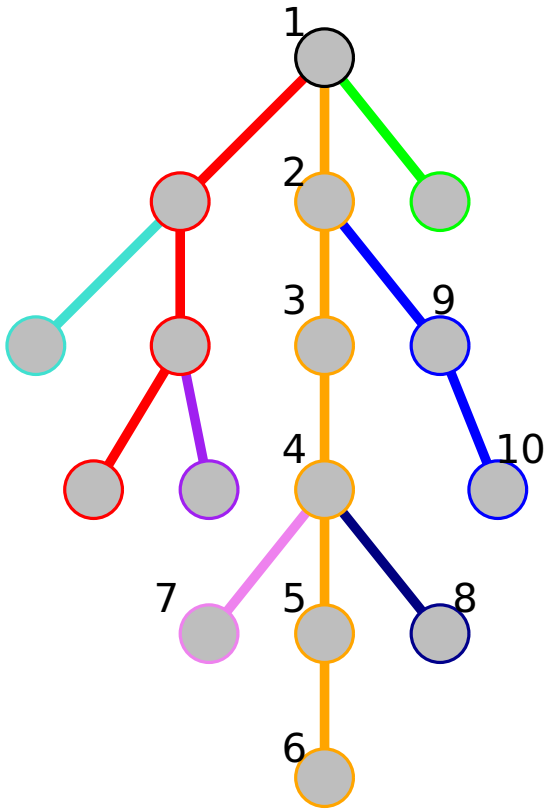

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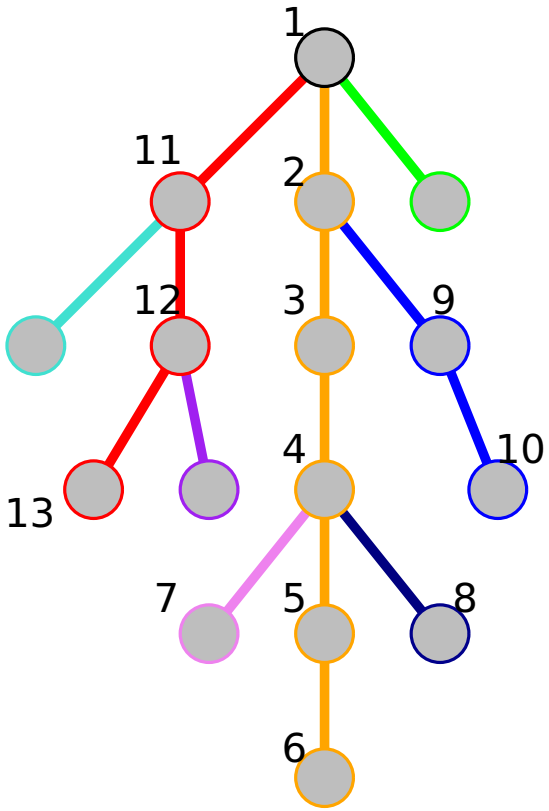

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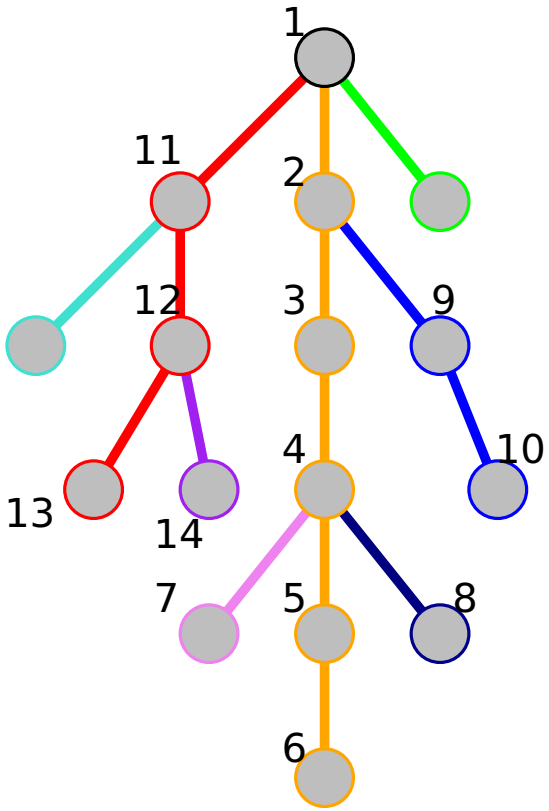

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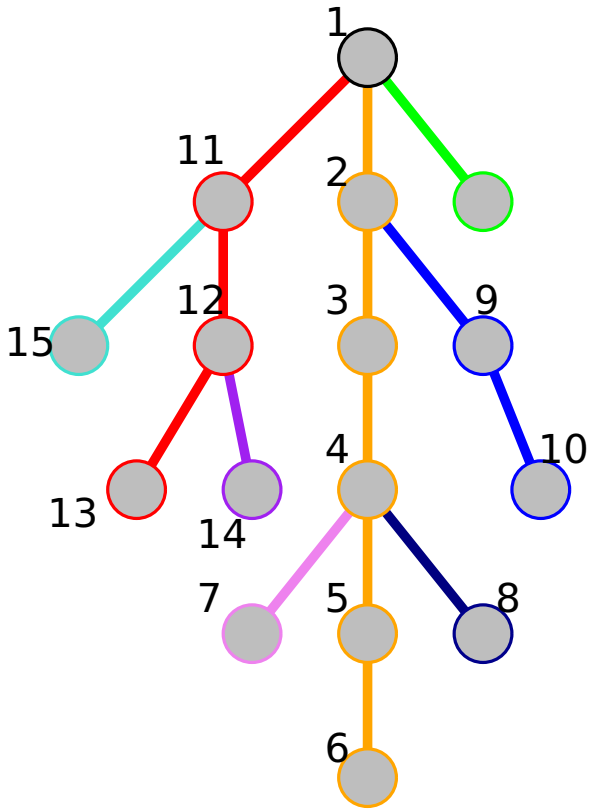

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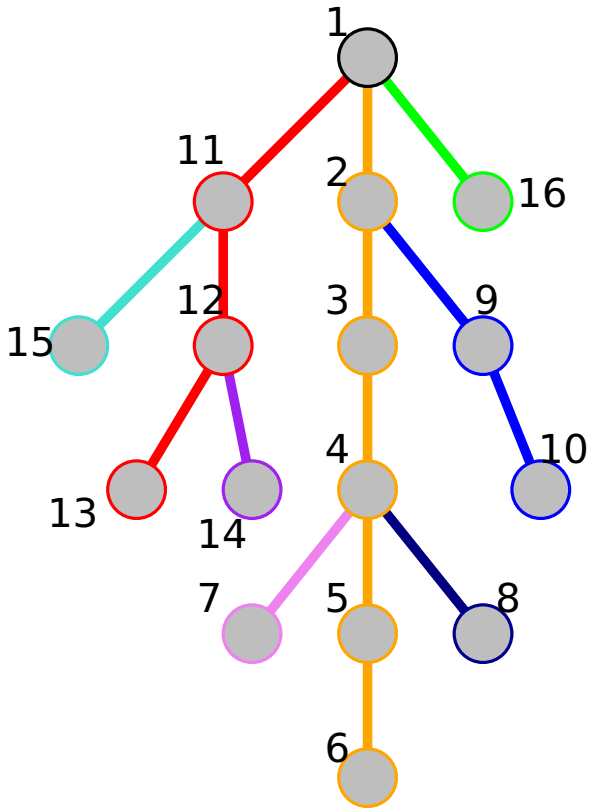

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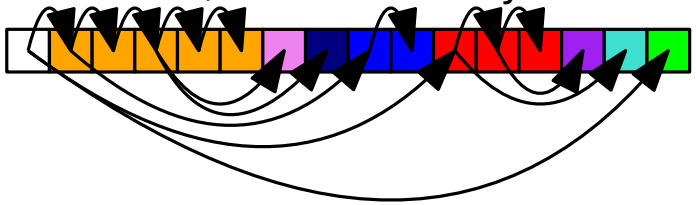

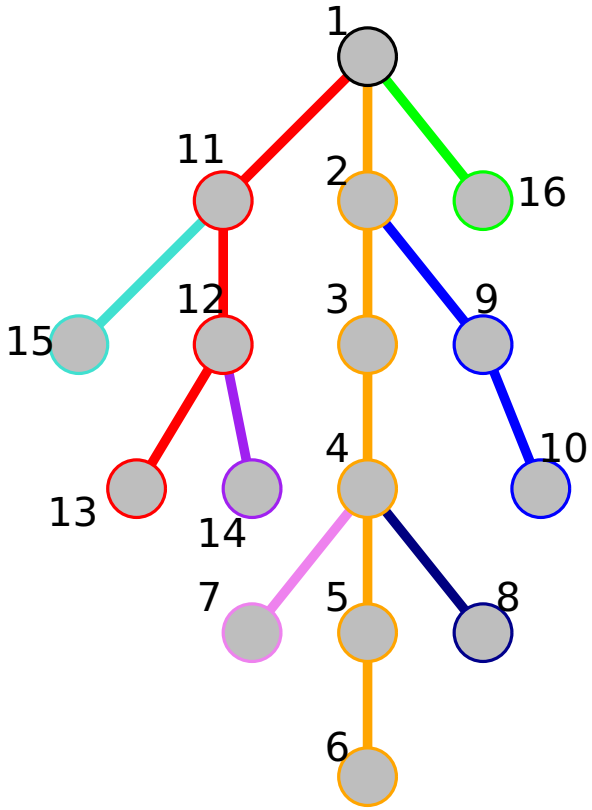

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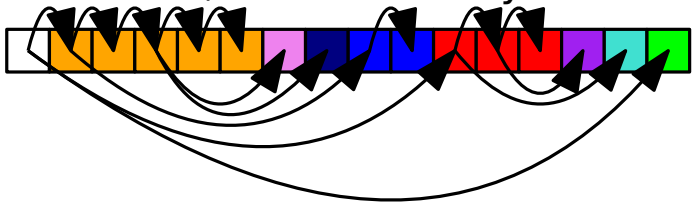

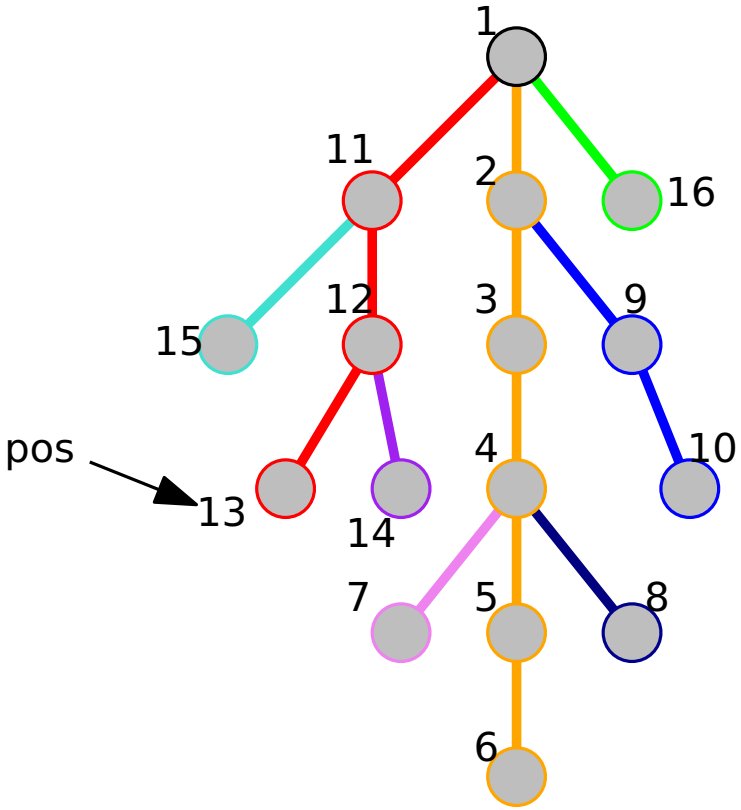

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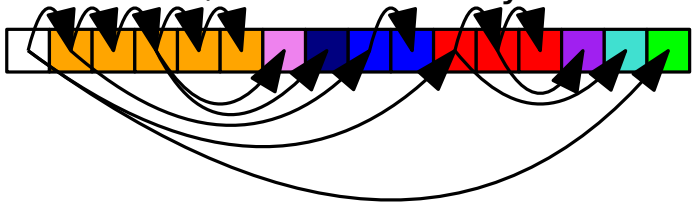

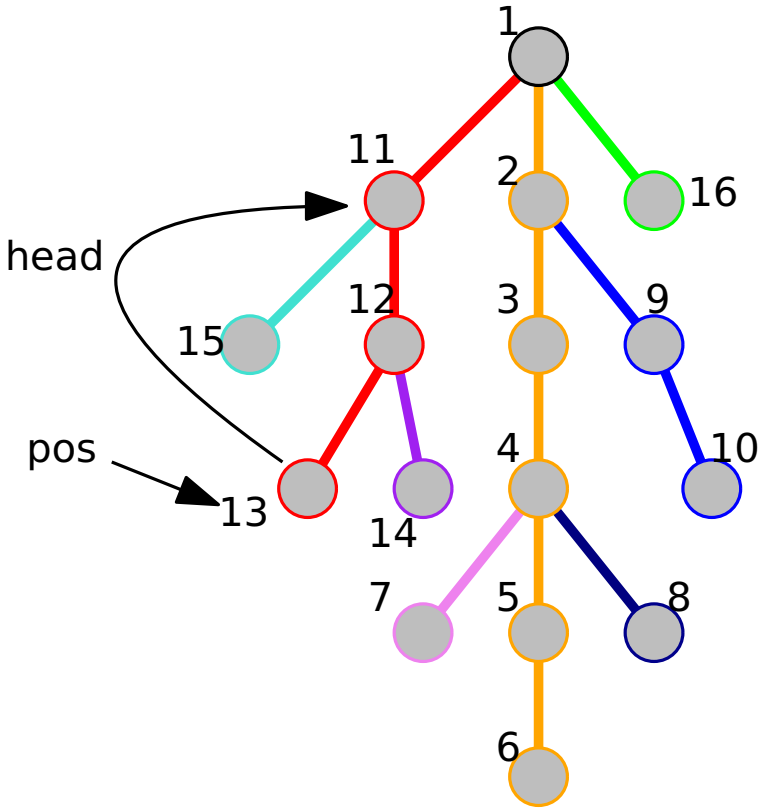

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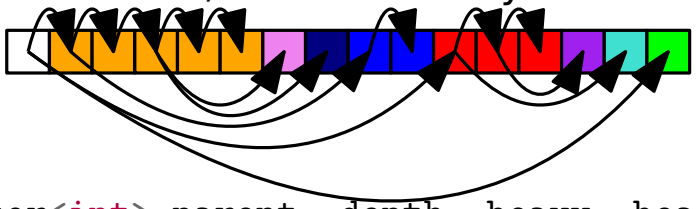
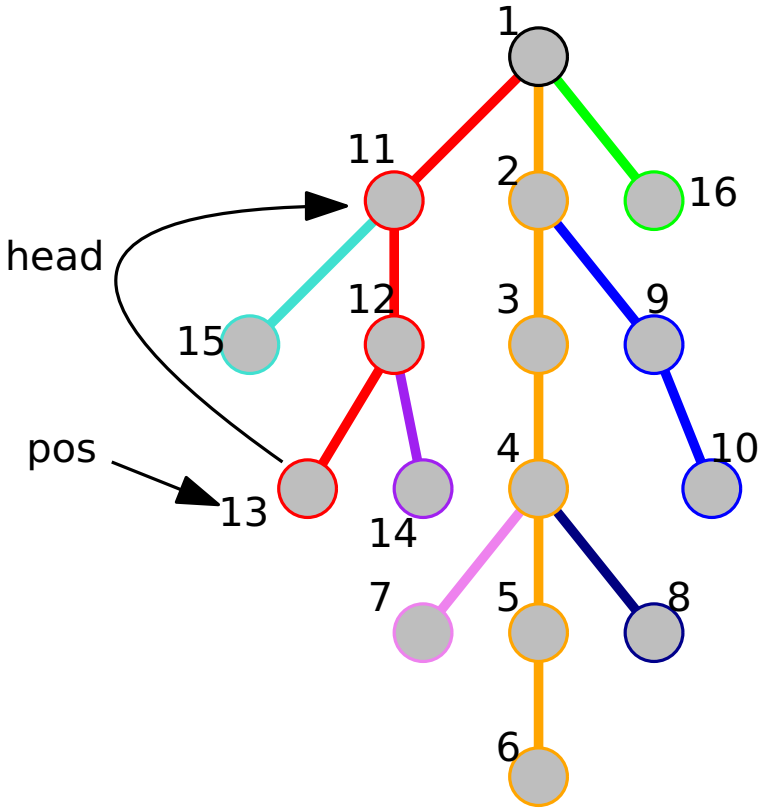


```
vector<int> parent, depth, heavy, head, pos;
int cur_pos = 0;

// TODO Run DFS first (parent, depth, heavy)
void decompose(int v, int h, Graph& adj) {
    head[v] = h, pos[v] = cur_pos++;
    if (heavy[v] != -1)
        decompose(heavy[v], h, adj);
    for (int c : adj[v])
        if (c != parent[v] && c != heavy[v])
            decompose(c, c, adj);
}
```



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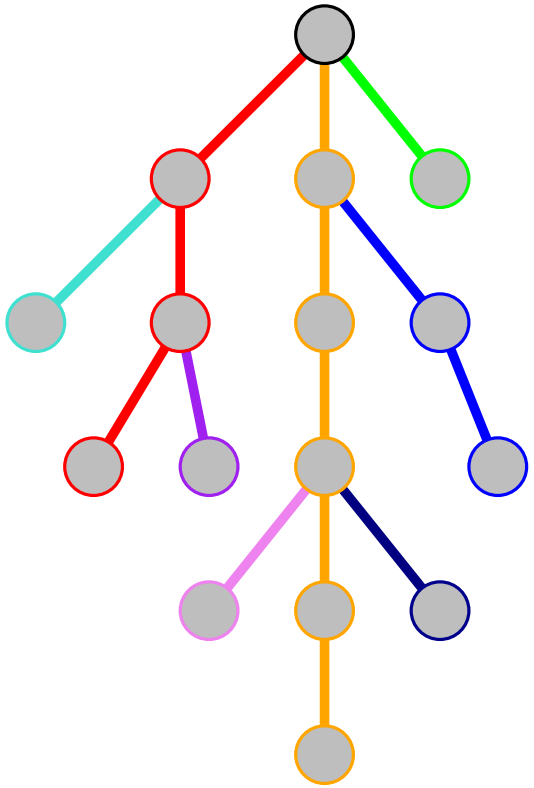

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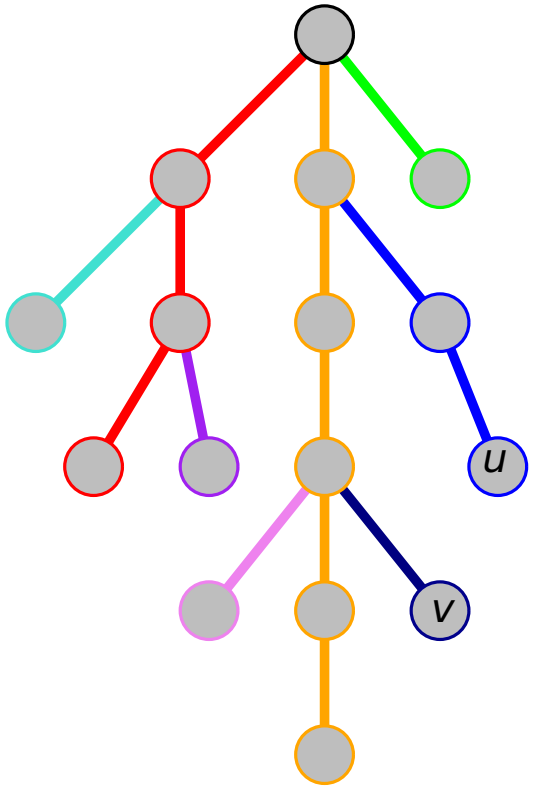

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void query(int u, int v) {
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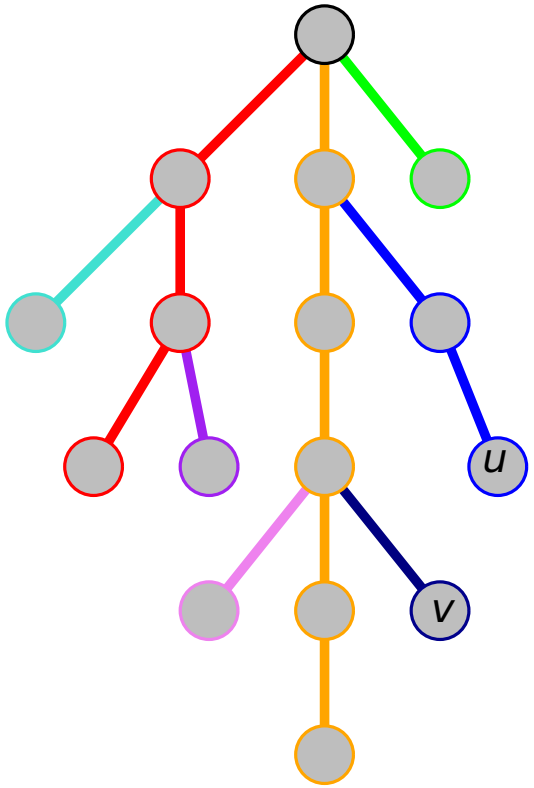

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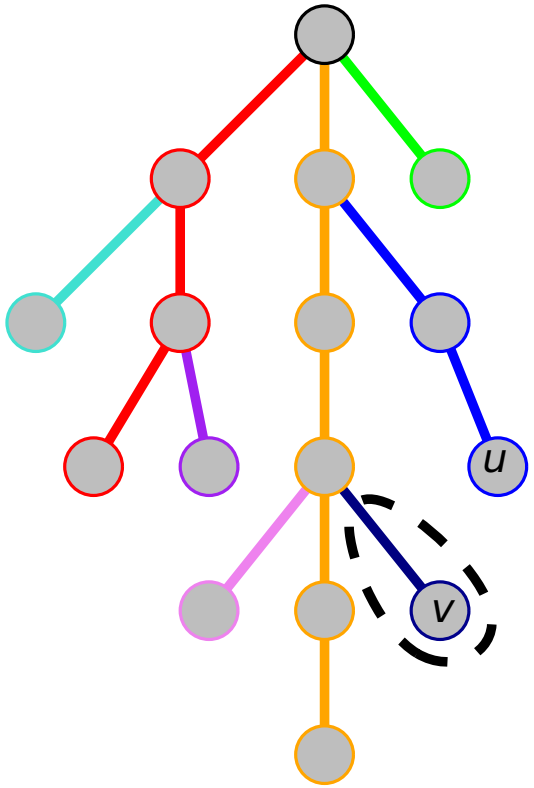

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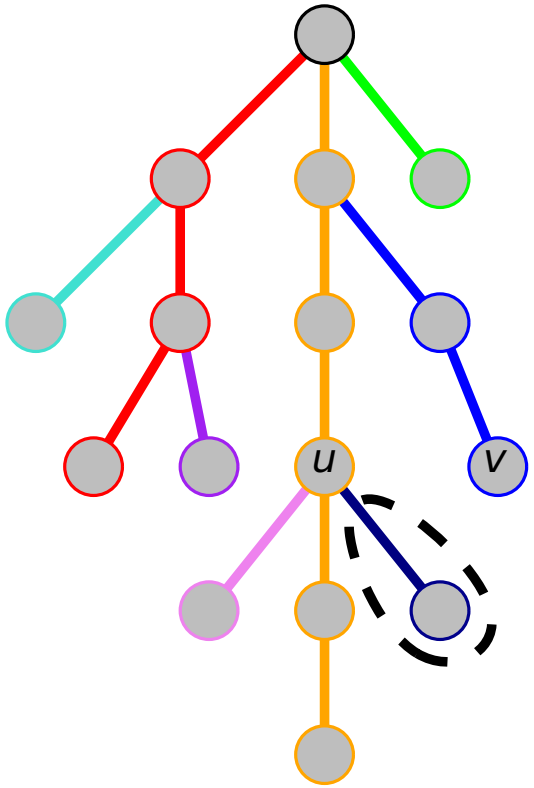

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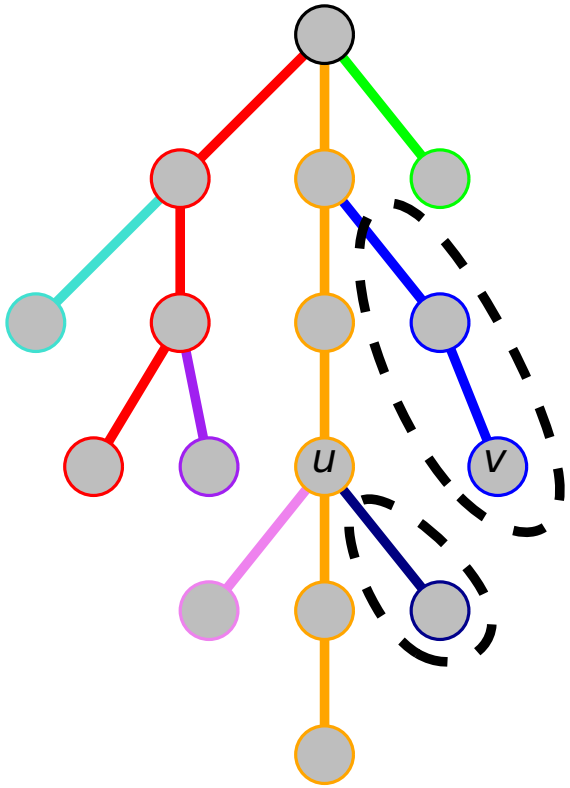

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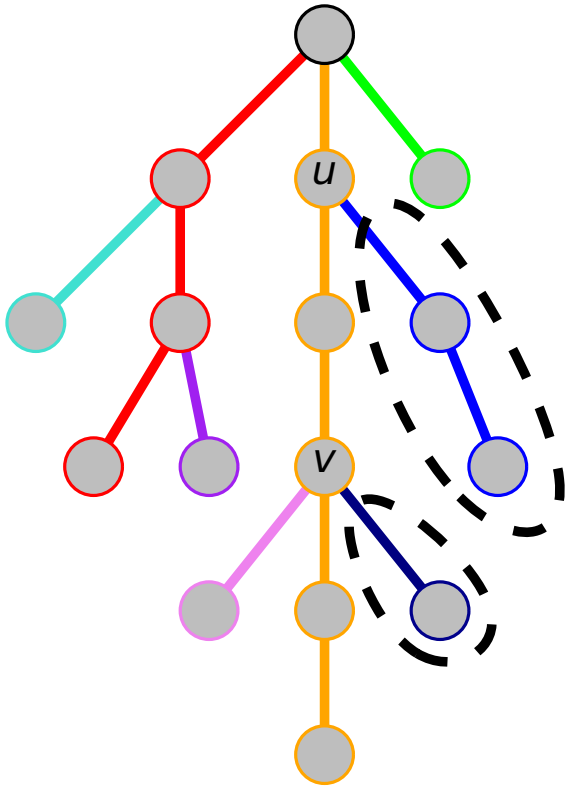

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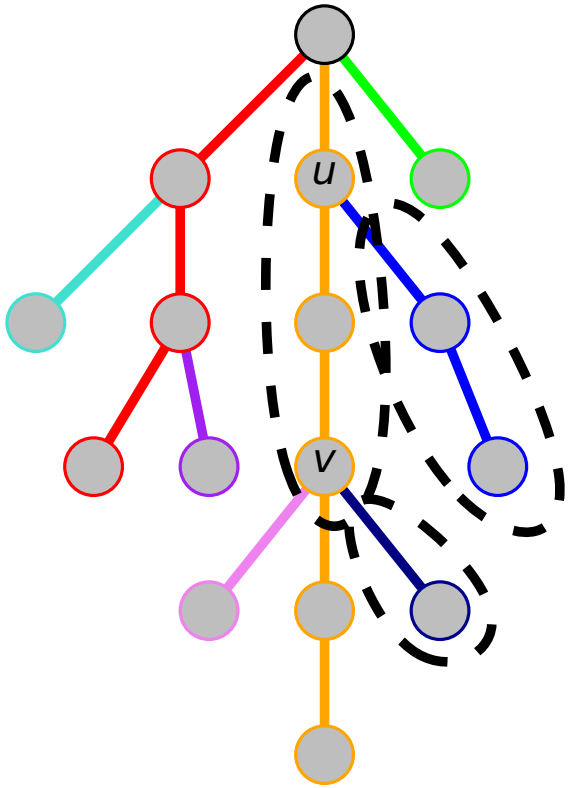

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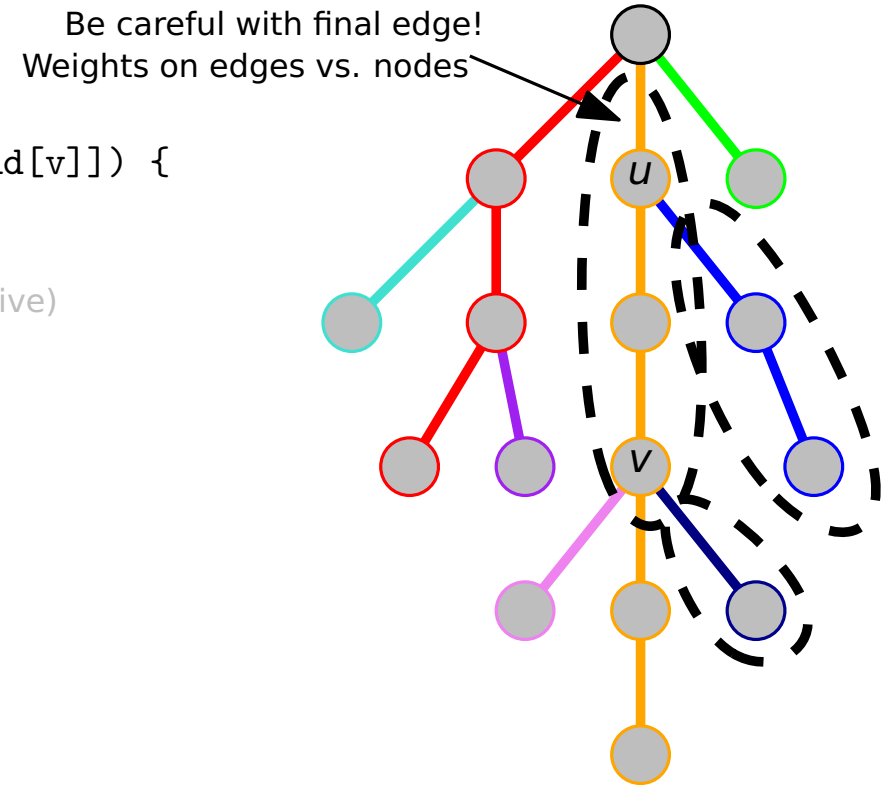

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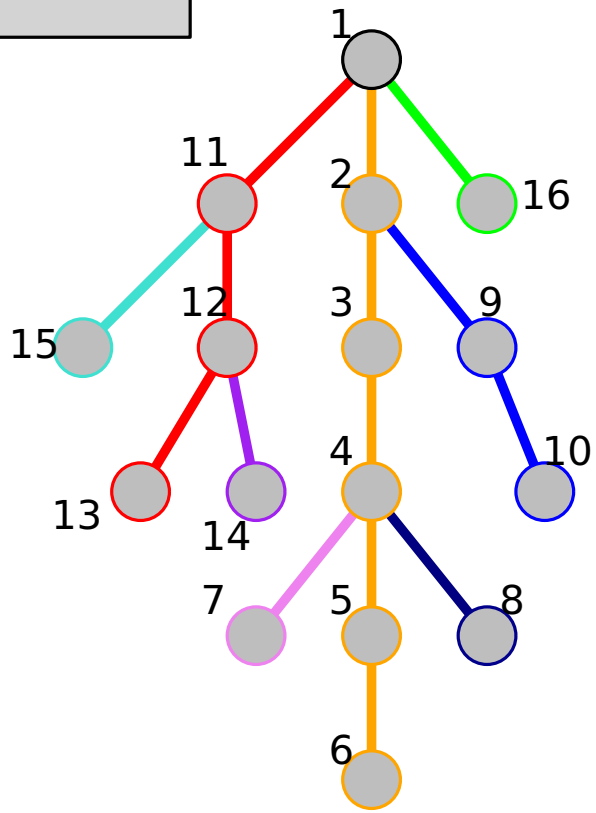

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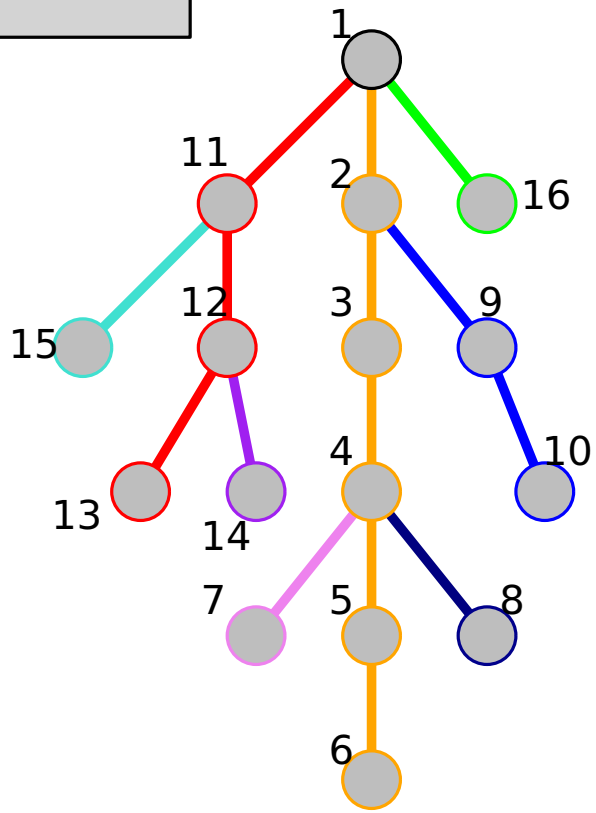
E.g., add 42 to all nodes in subtree of node 2



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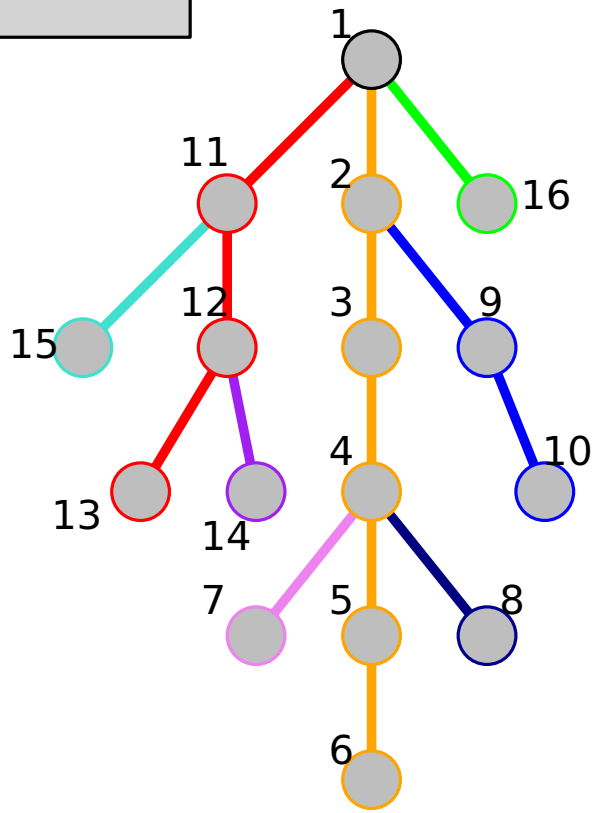

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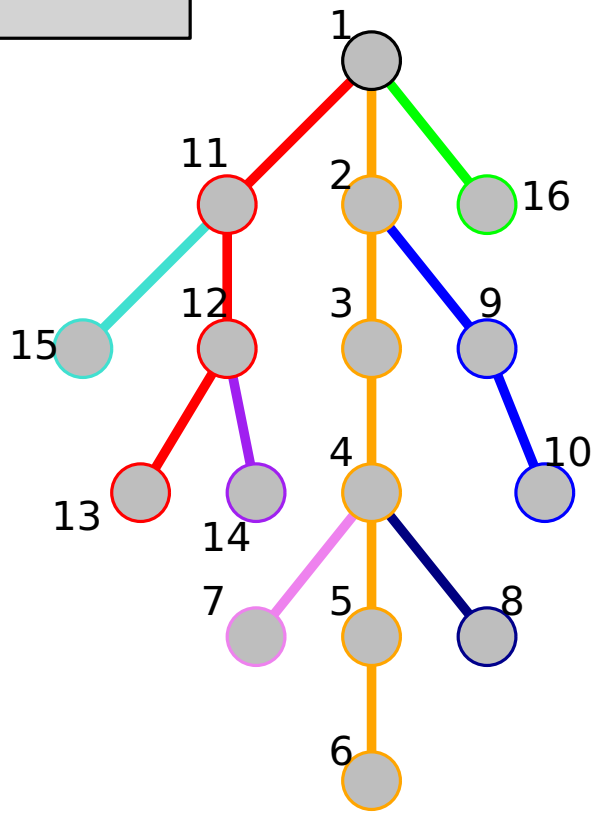
subtree size



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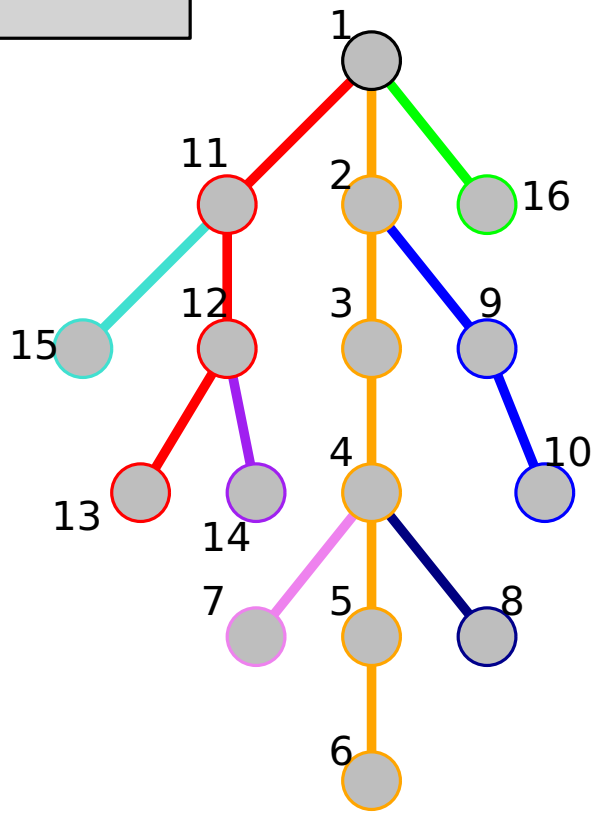

https://codeforces.com/blog/entry/53170

Centroid decomposition

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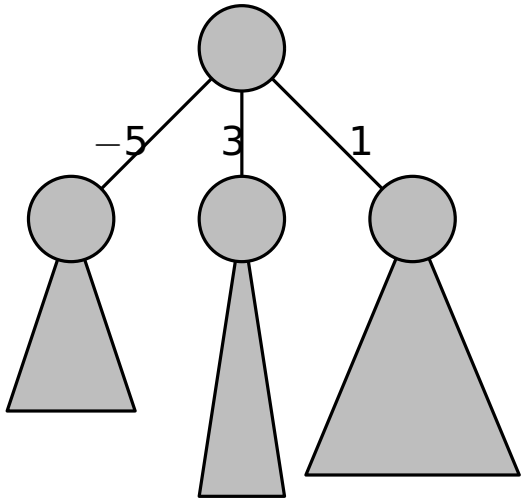

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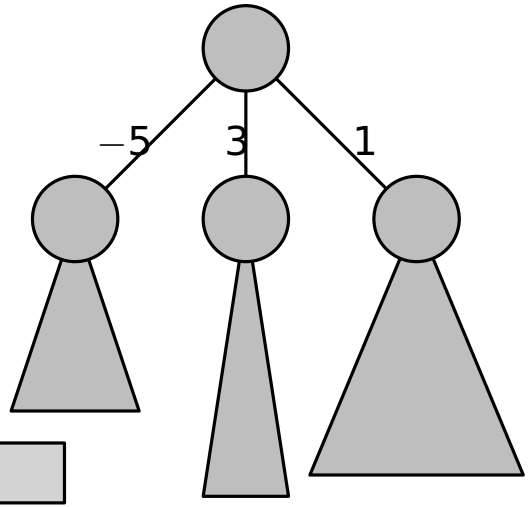

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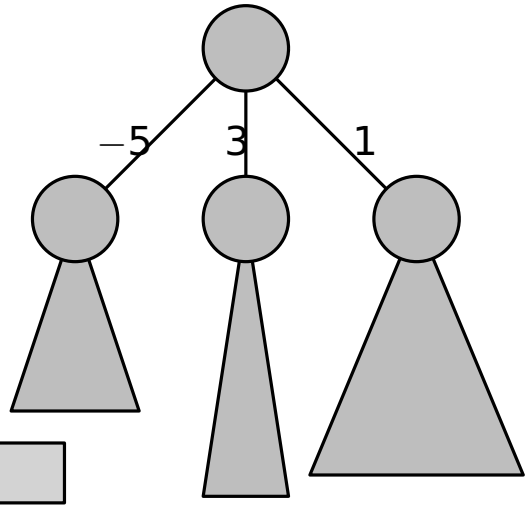

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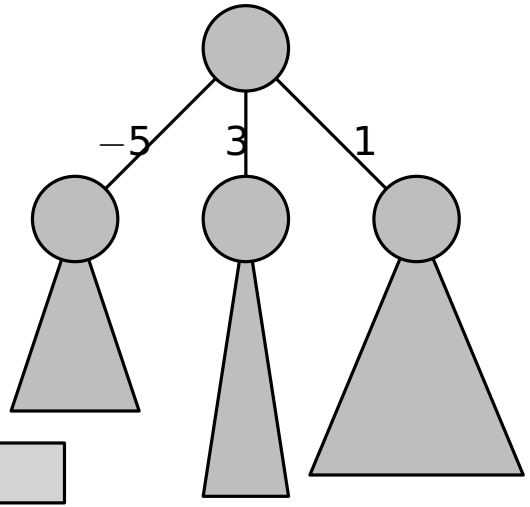

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. . . but only if we have large subtrees!

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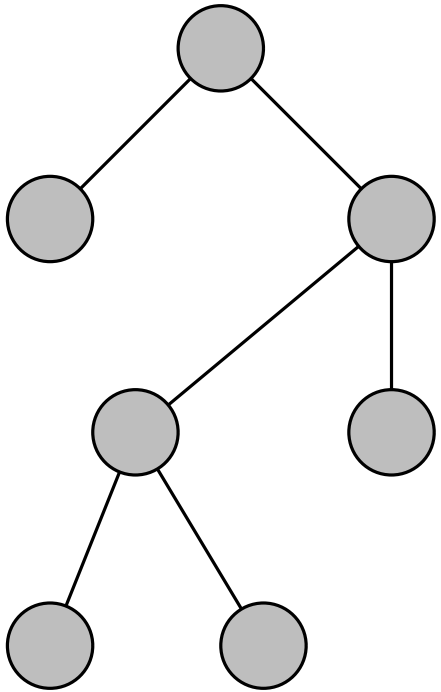

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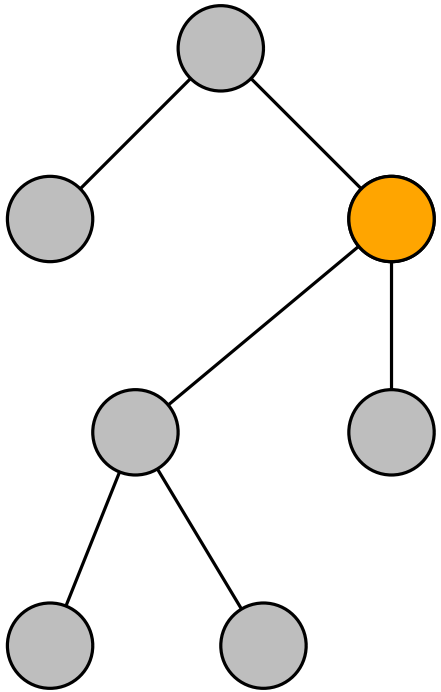

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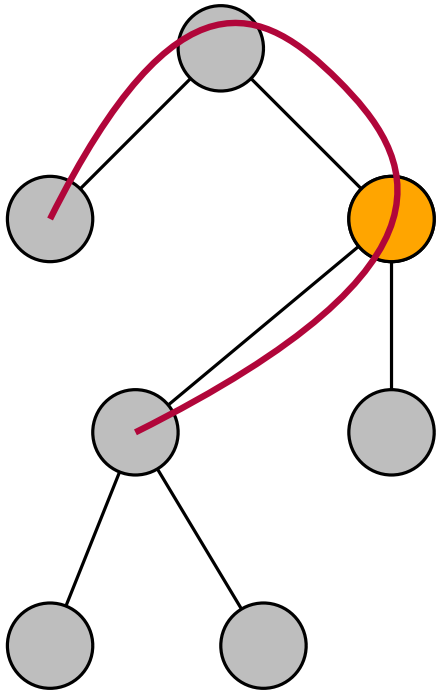

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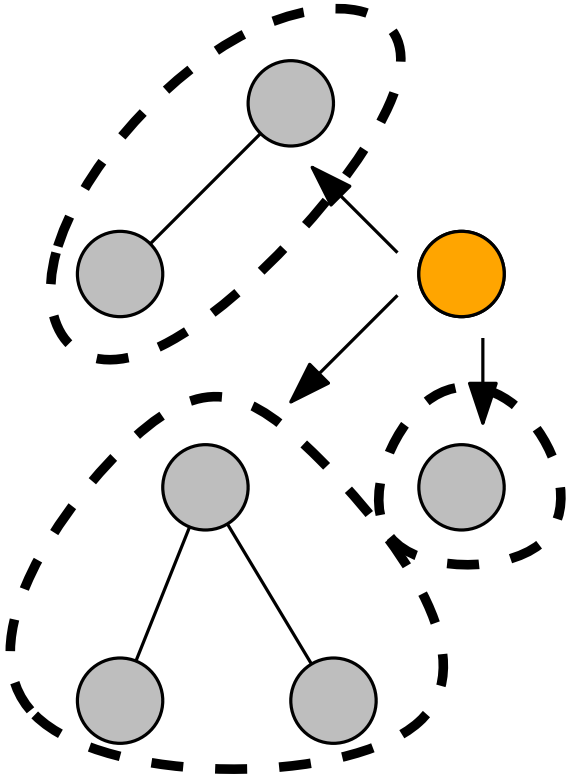

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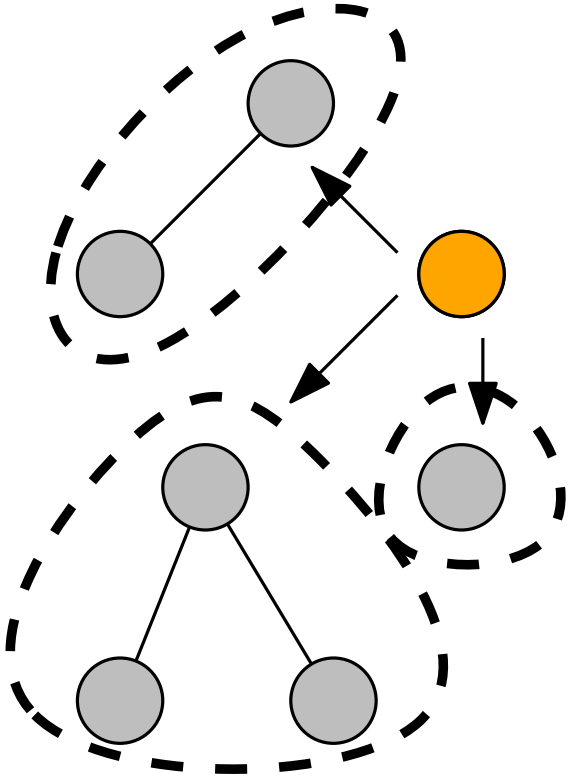

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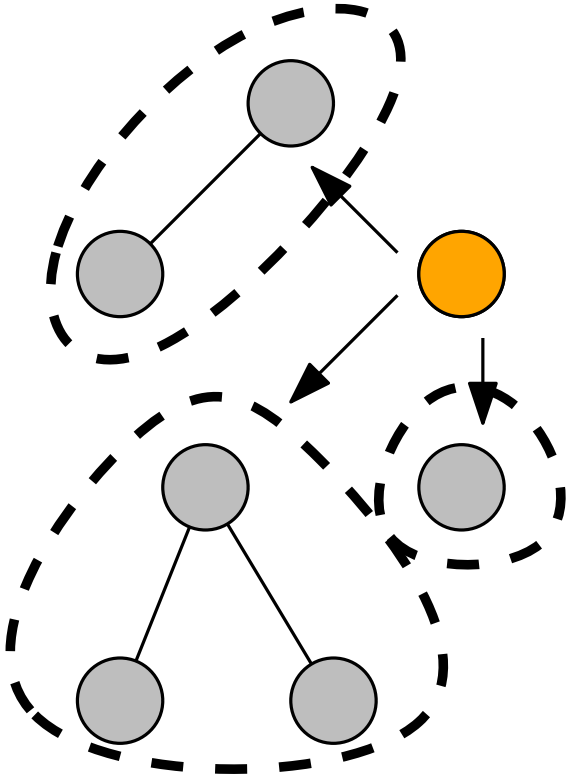

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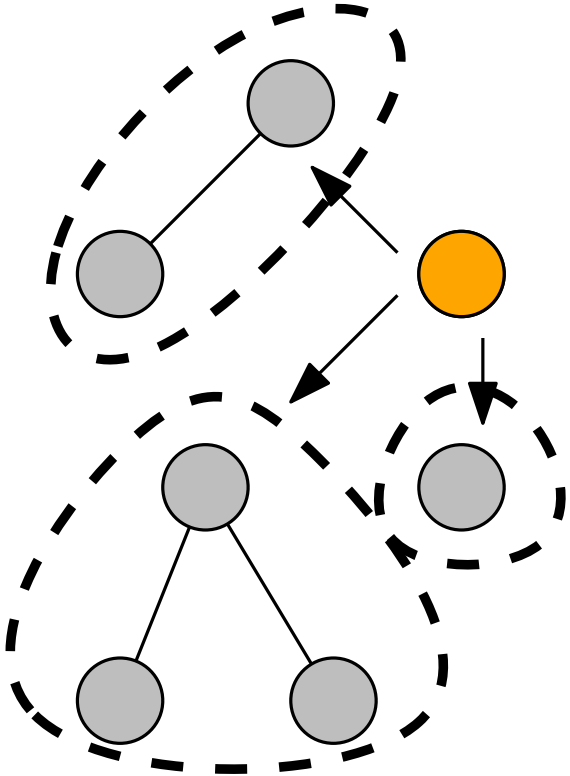

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How to choose the split node?

- Highest degree?
- Half-point of diameter path?
- Node with smallest subtrees?

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Per centroid in $\mathcal{O}$(size of current tree). Overall run time?

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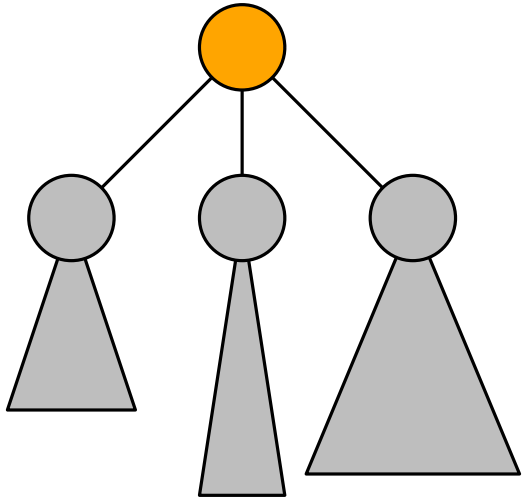

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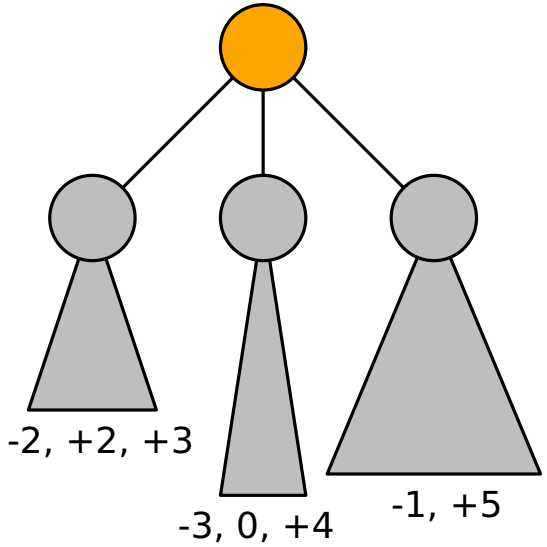

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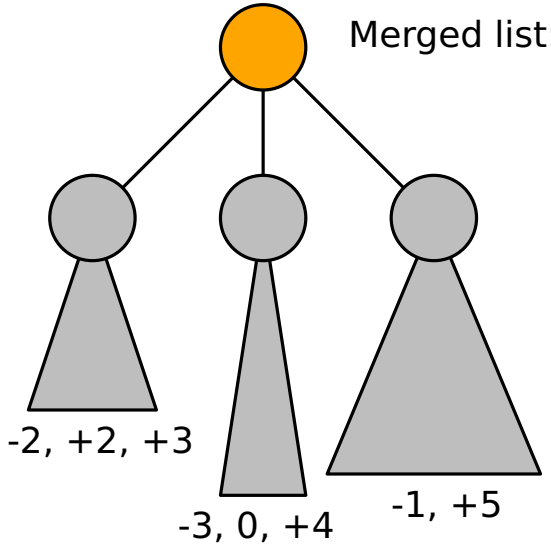

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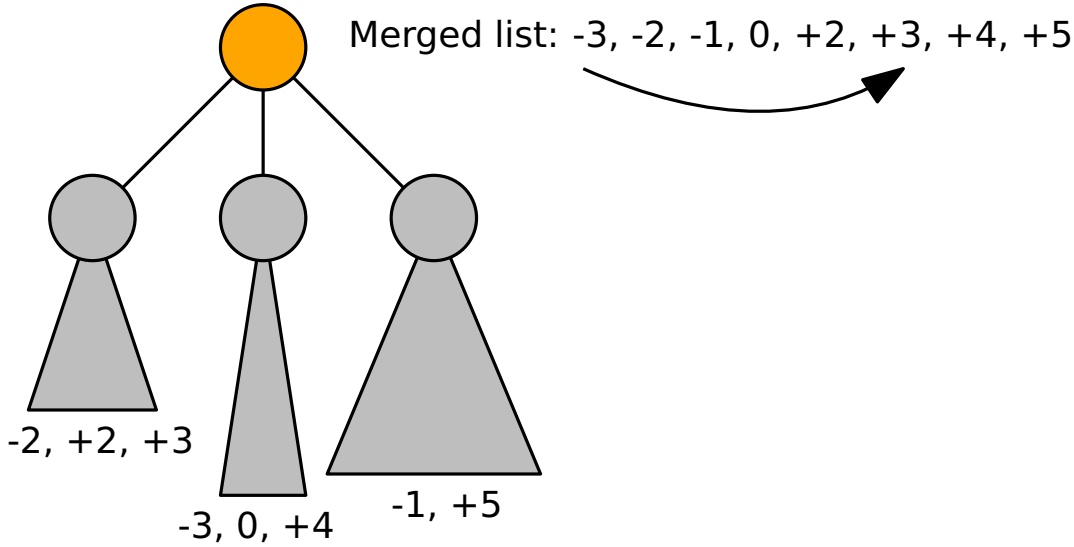

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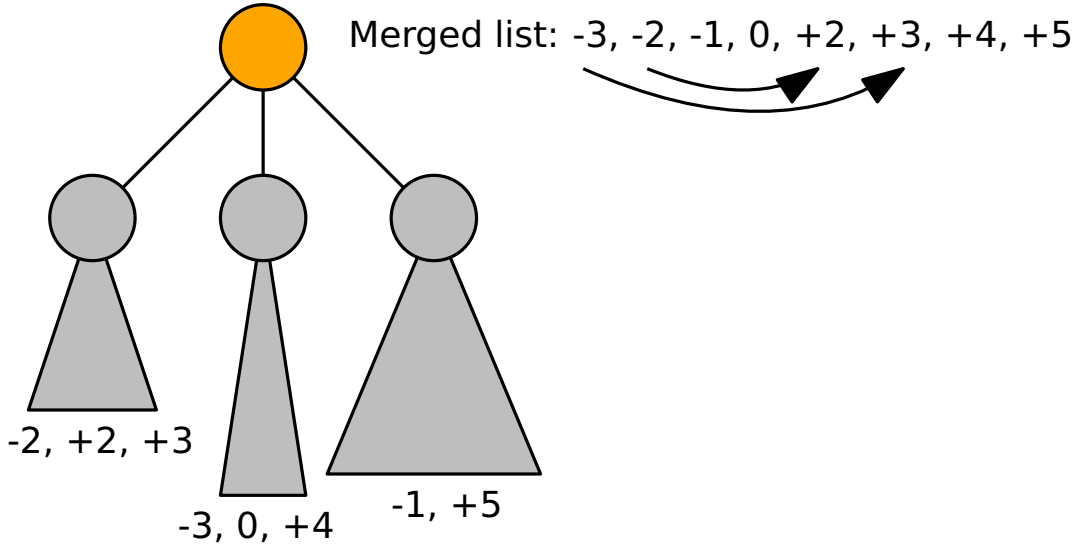

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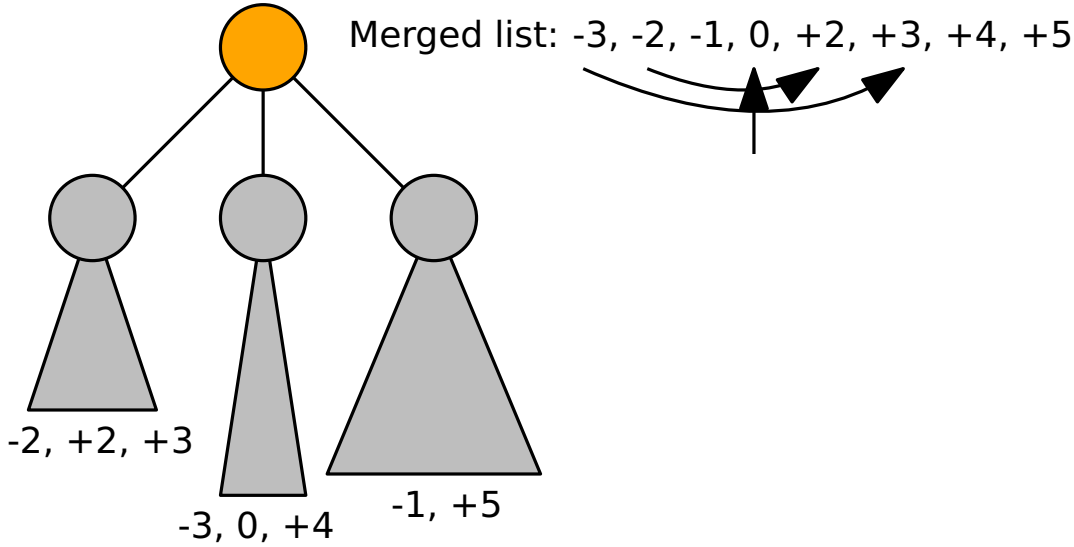

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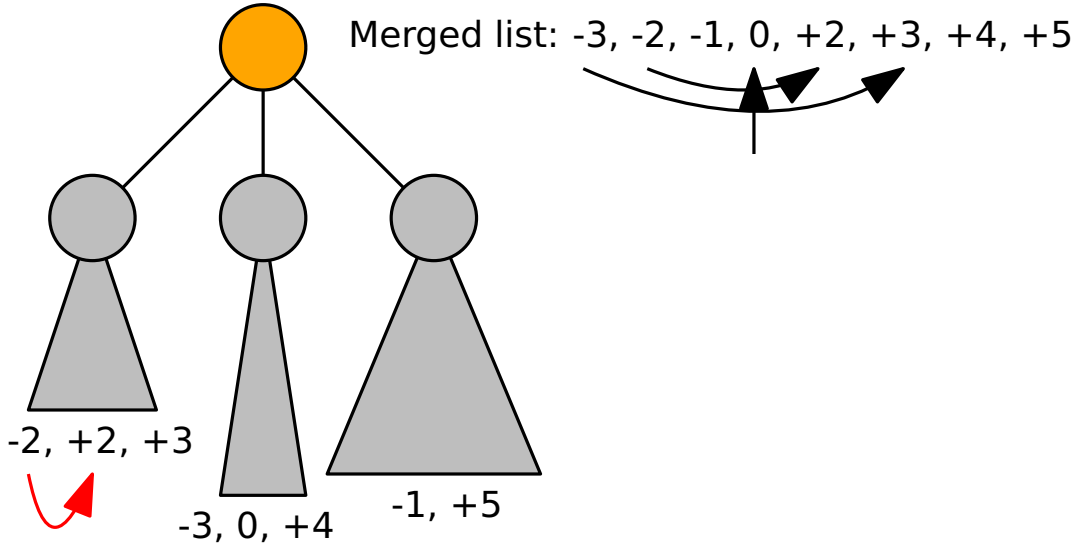

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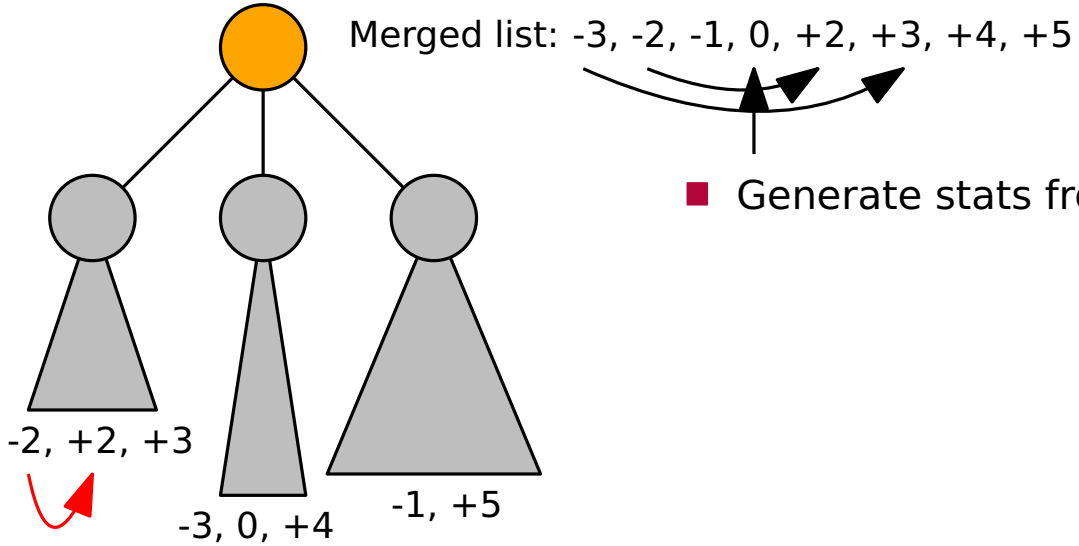

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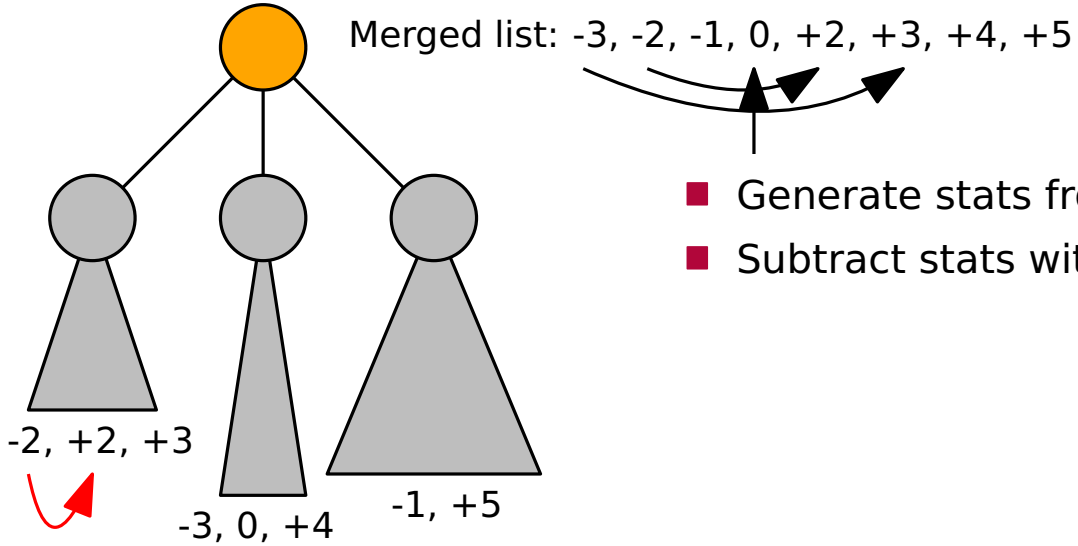

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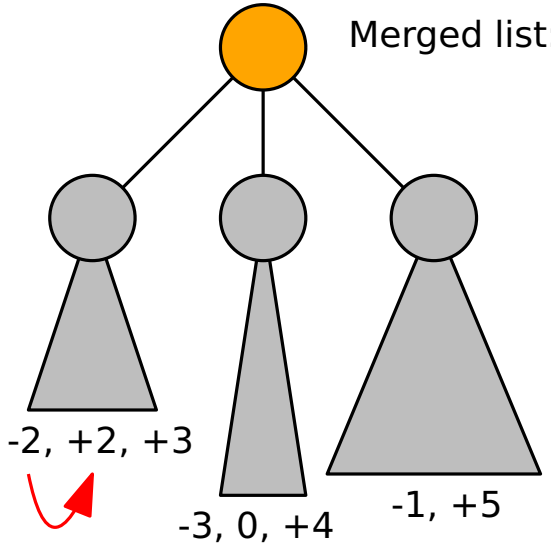

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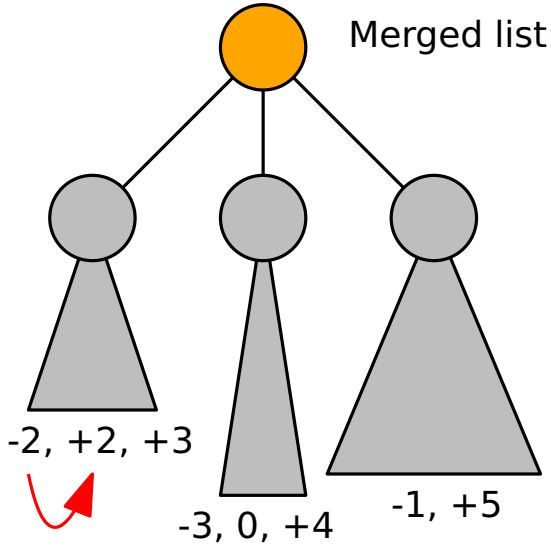

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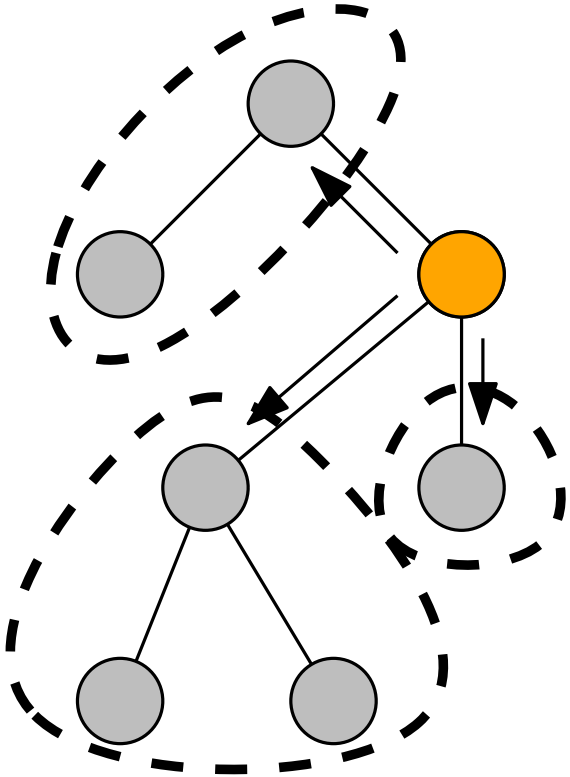
Edge weights 1, target x instead of 0

Centroid decomposition

- Calculate sizes via DFS
- Find centroid
- Calculate subtree stats through another DFS
- Combine stats
- Remove centroid from graph
- Recursively call for every neighbor



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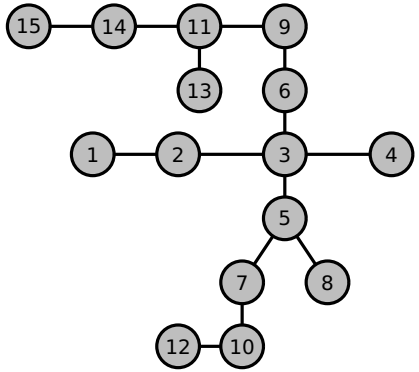

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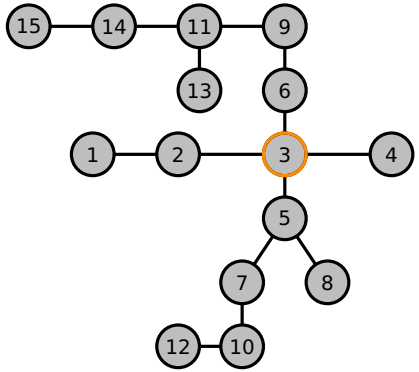

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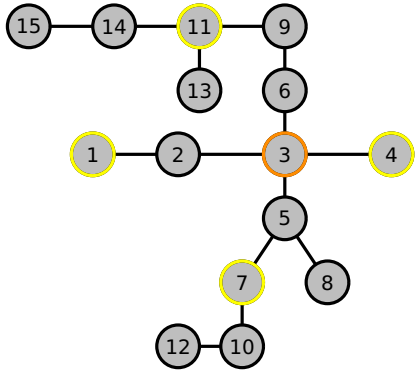

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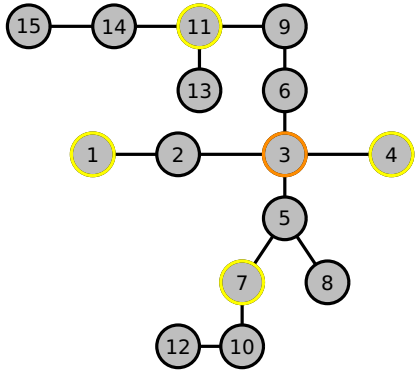

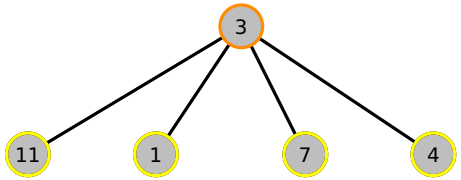

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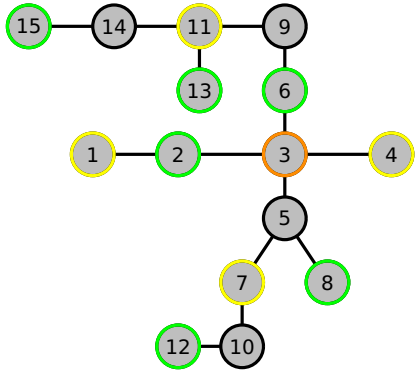

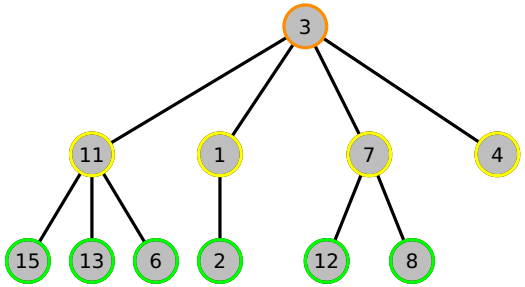

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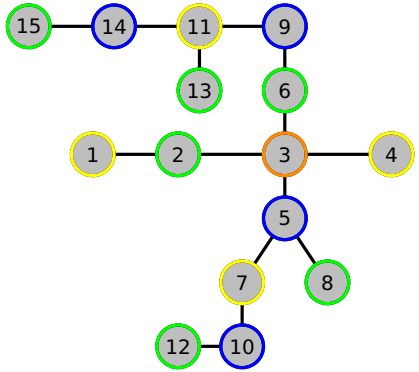

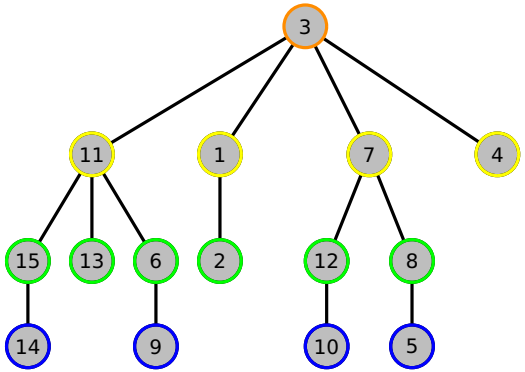

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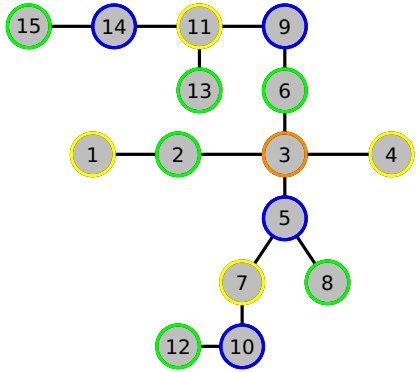

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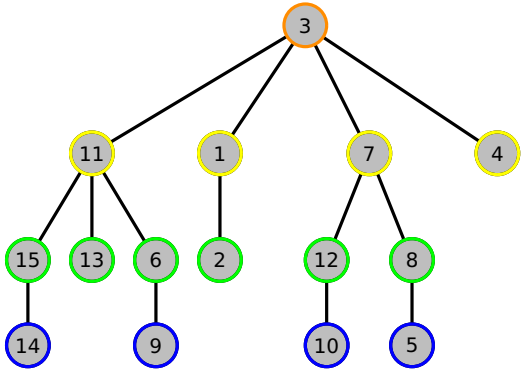

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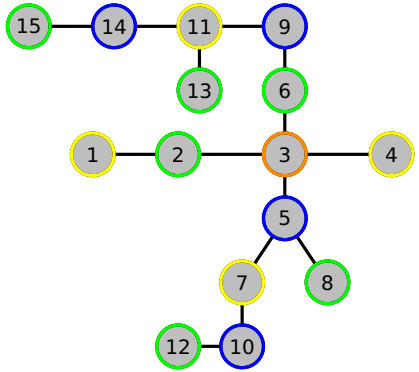

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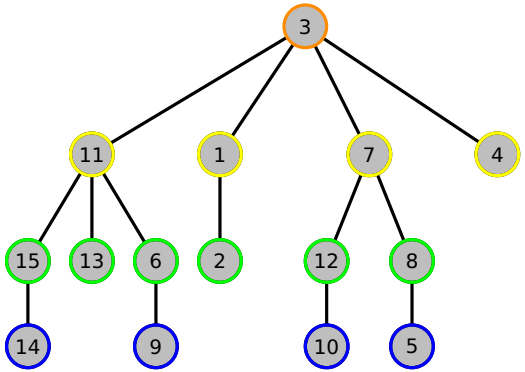

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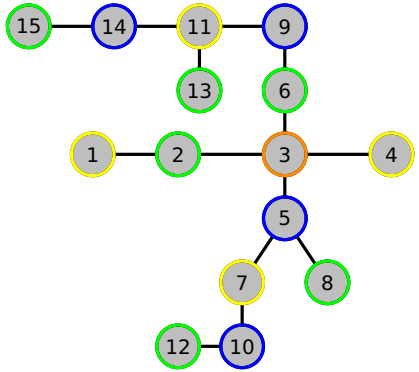

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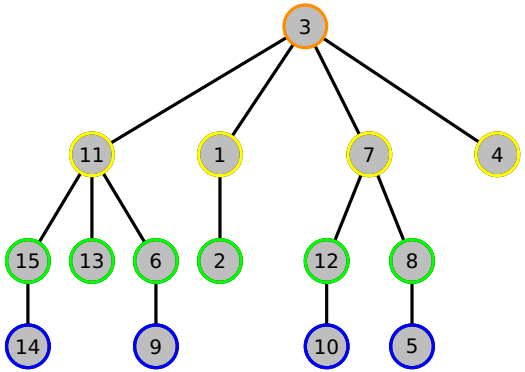

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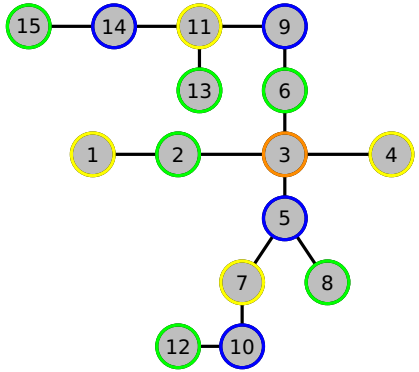

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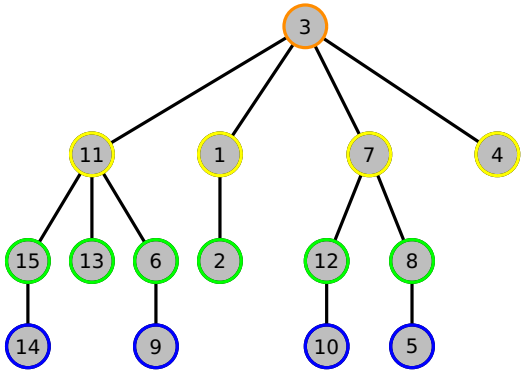

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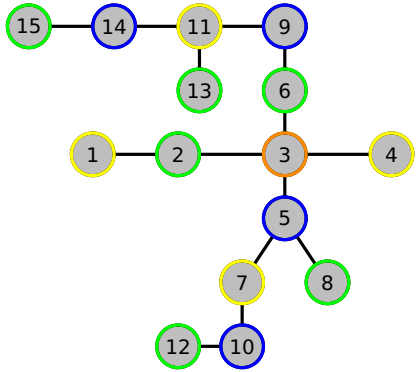

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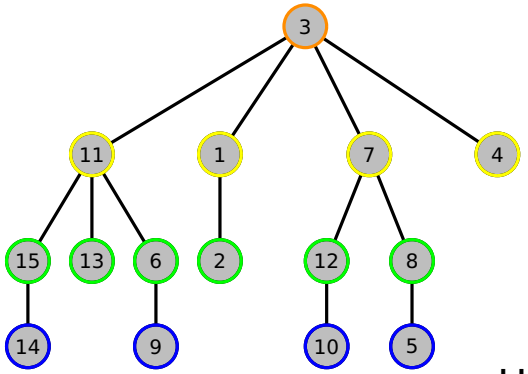

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How does this help for queries?

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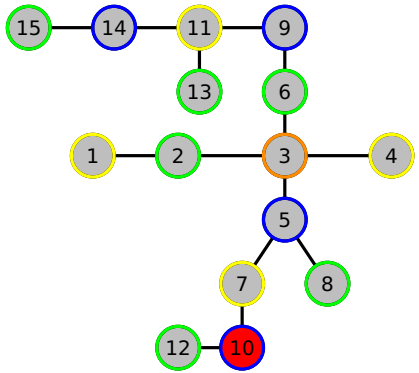

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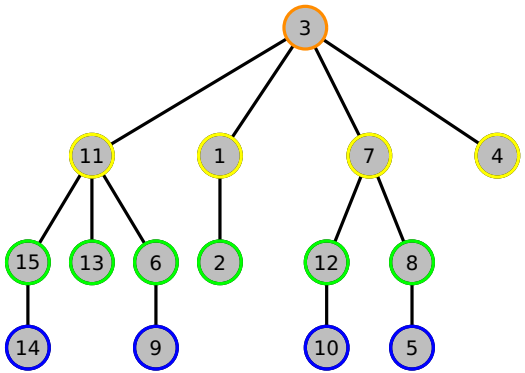

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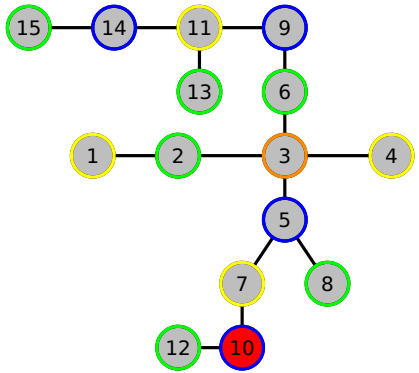

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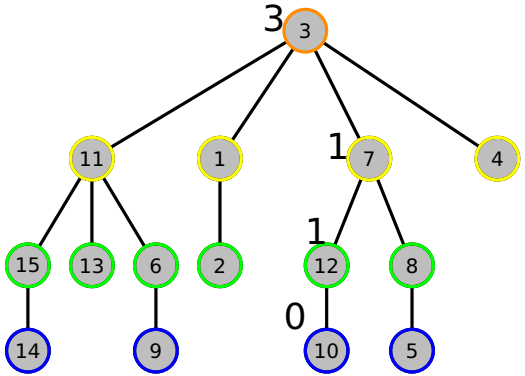

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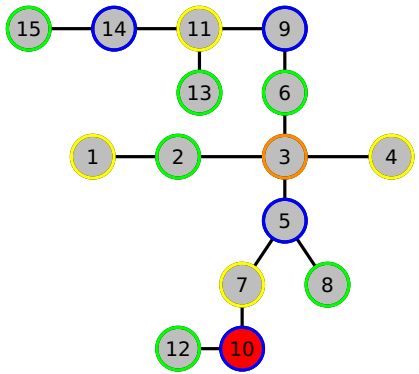

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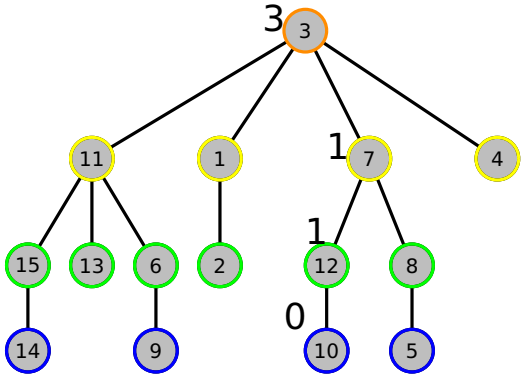

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- Query type 1: Update all CT ancestors

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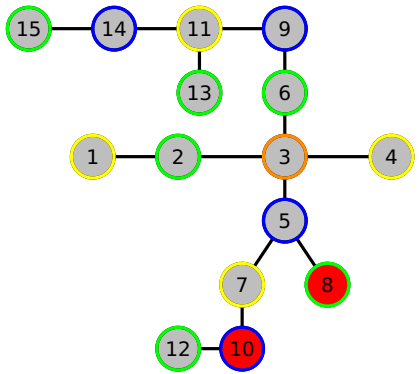
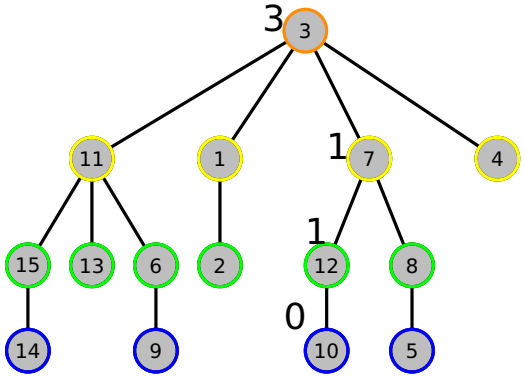


Centroid tree (CT)



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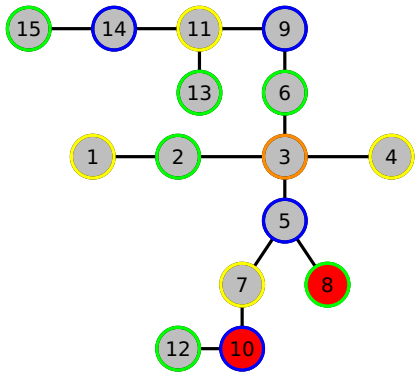

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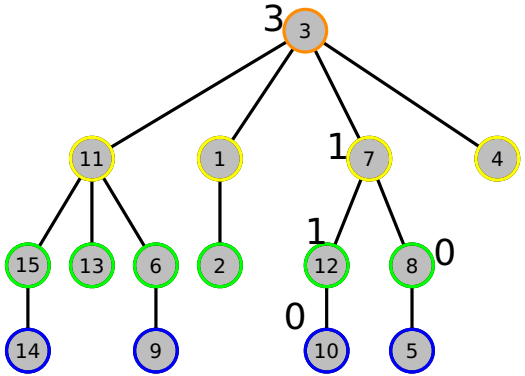

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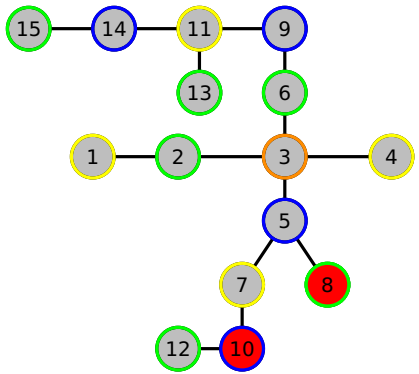

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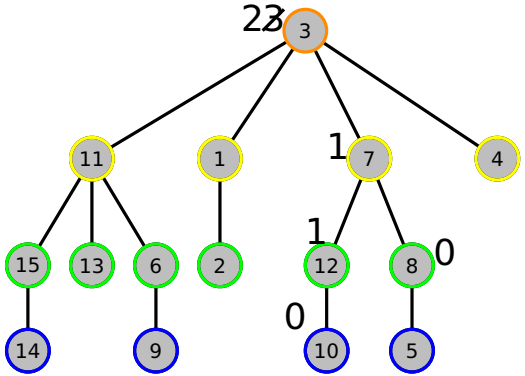

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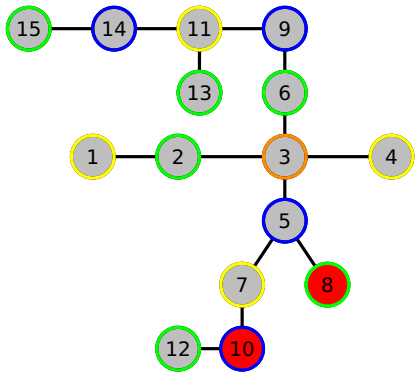

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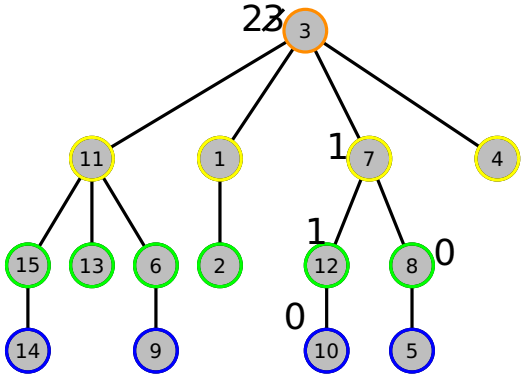

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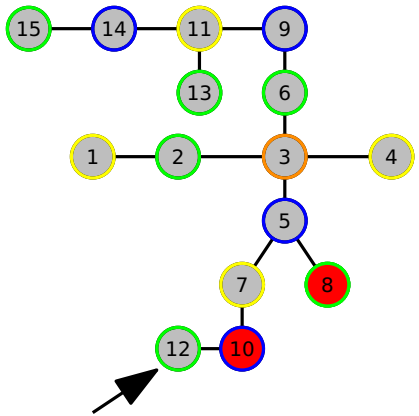

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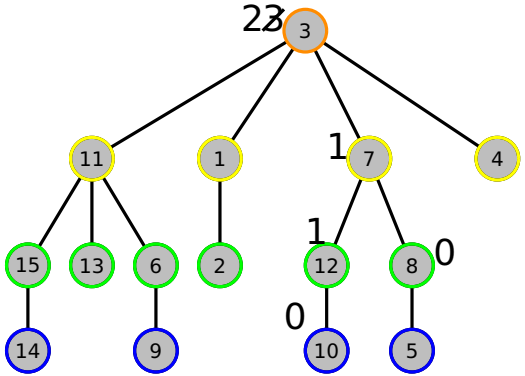

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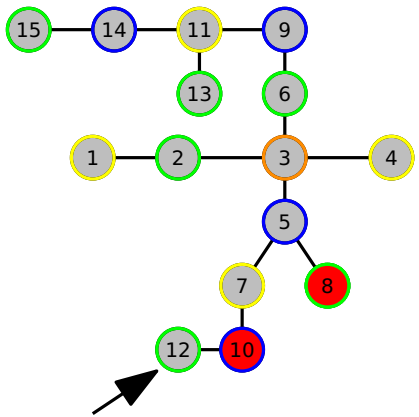

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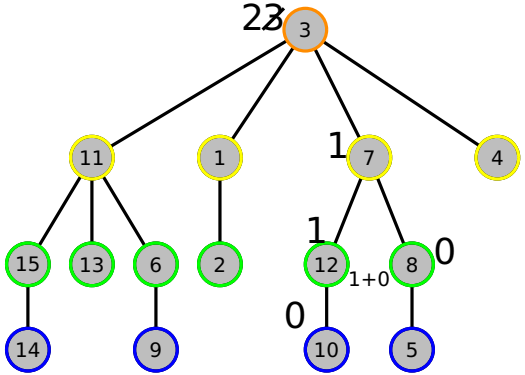

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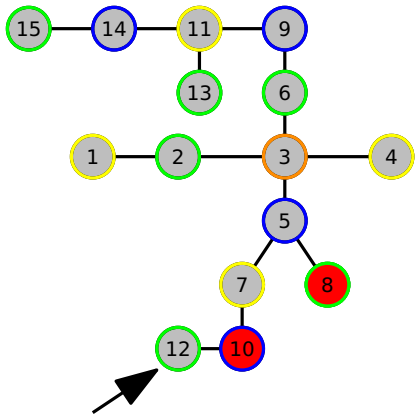

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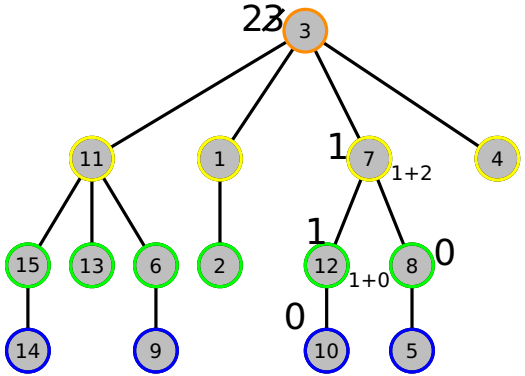

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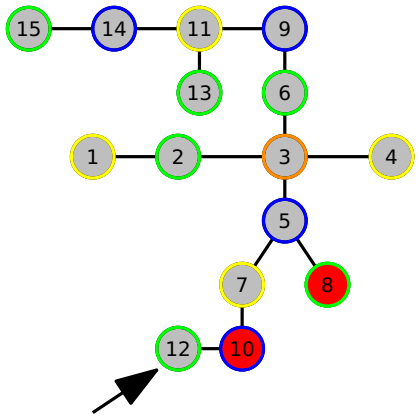

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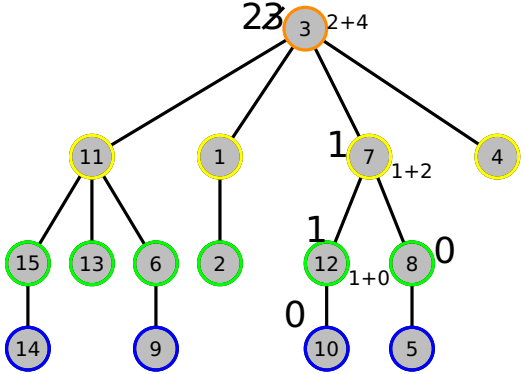

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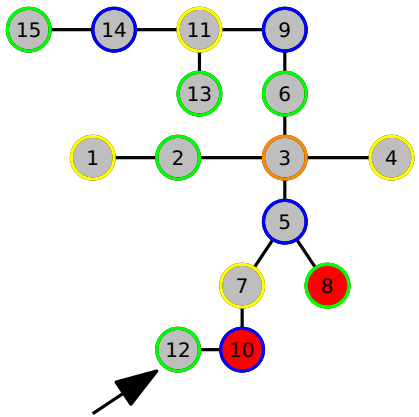

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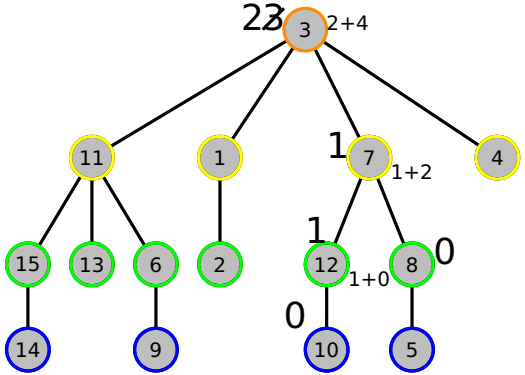

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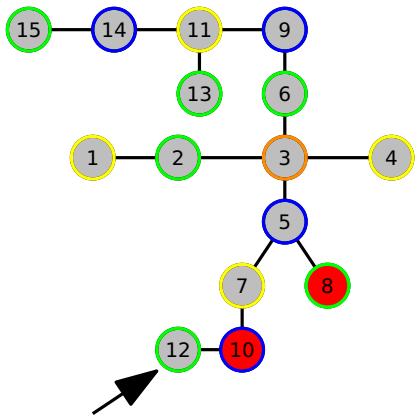

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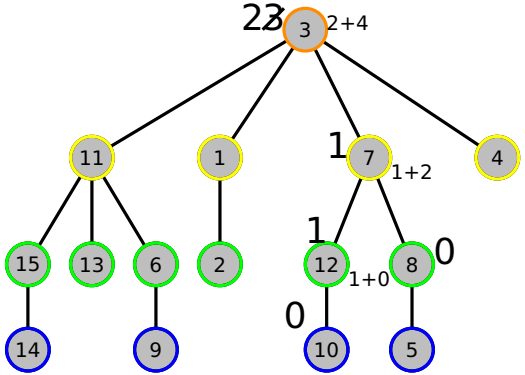

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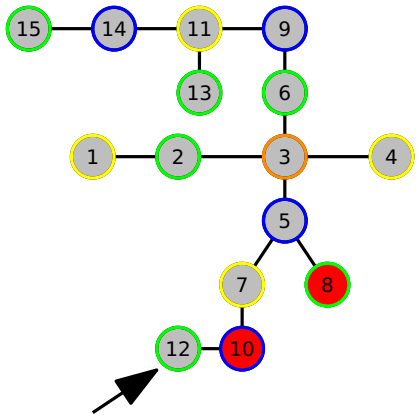

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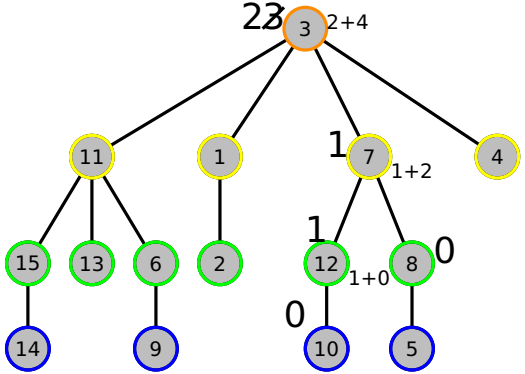

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Beware of “false LCAs” (non-simple paths)!

e.g., path from 12 via 3 to 8 with length 4+2 is not simple

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- Given node v, find sum of all node weights in radius k around v.
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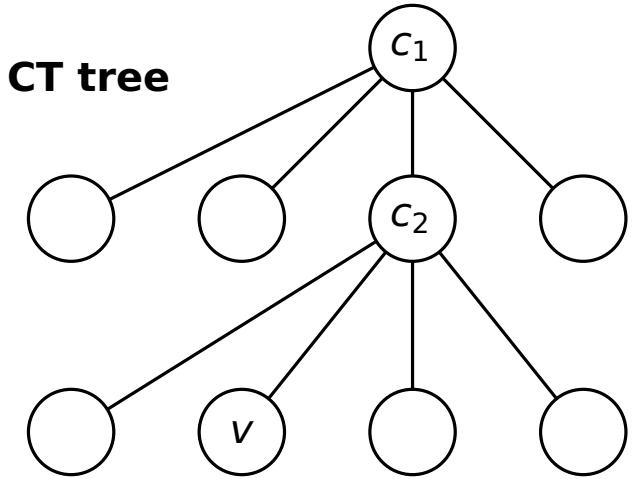

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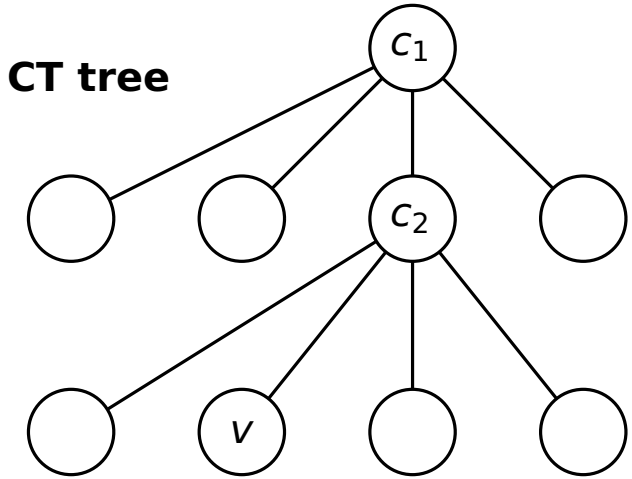

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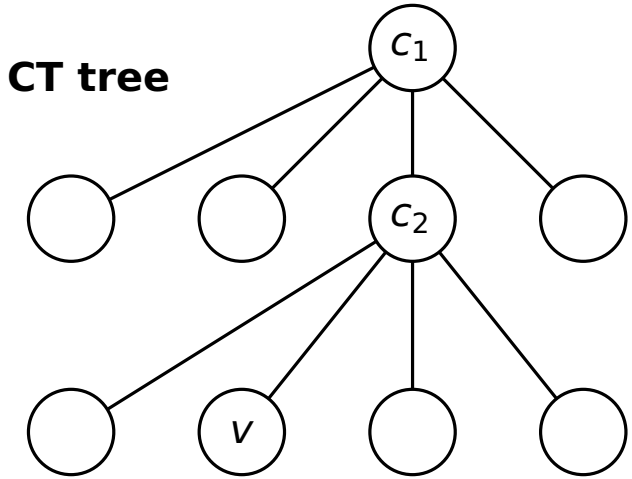

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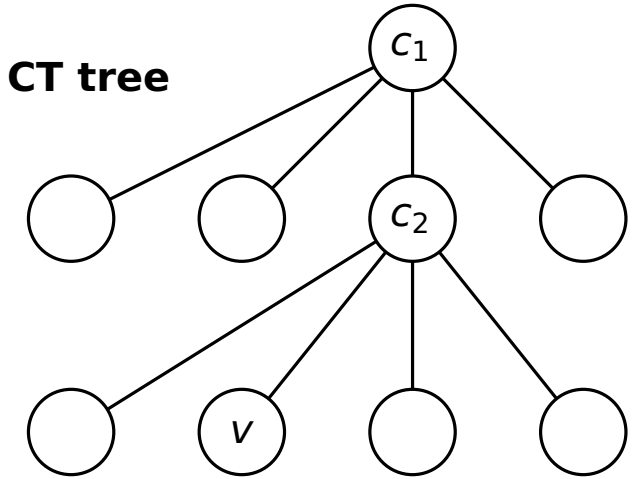

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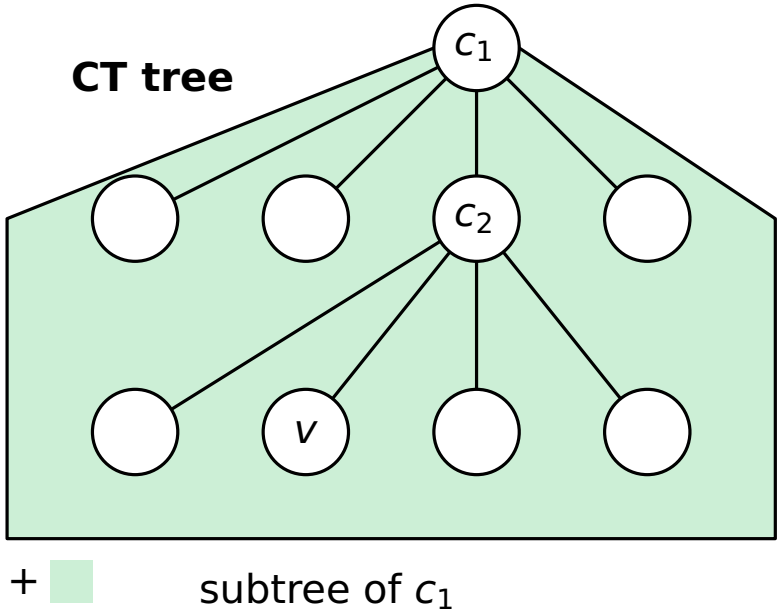

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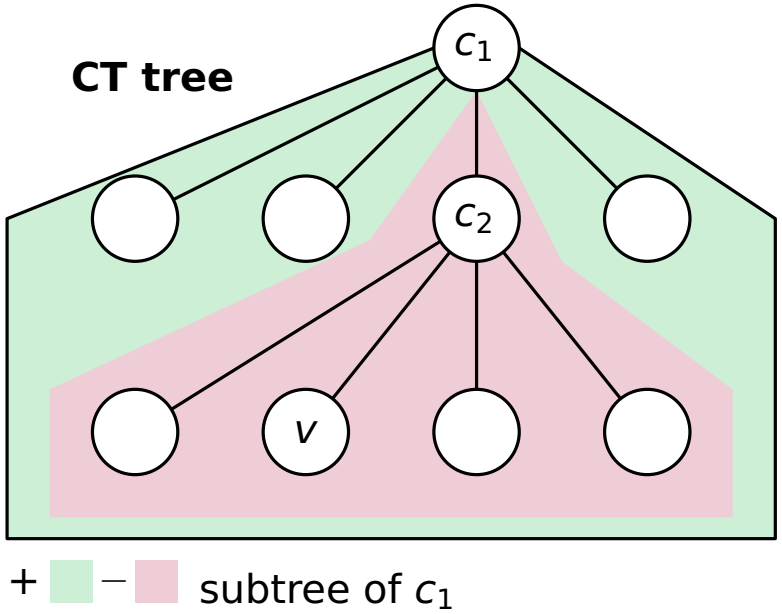

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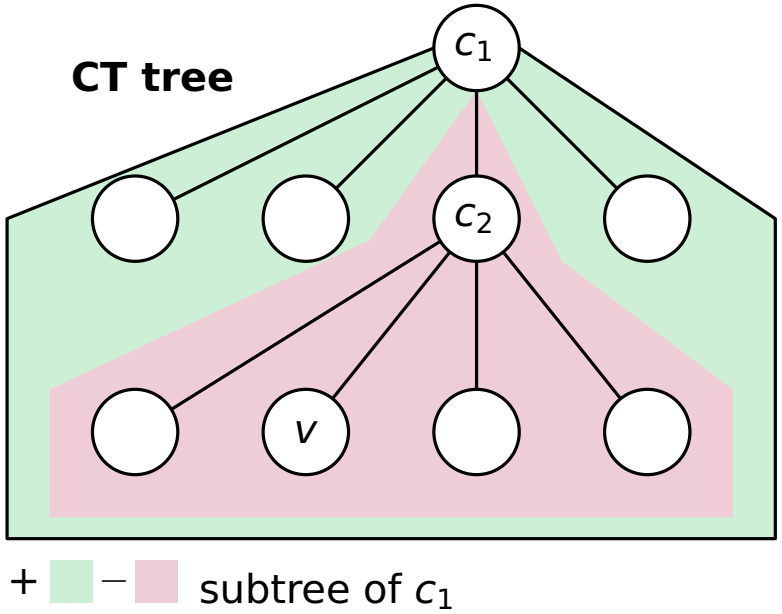

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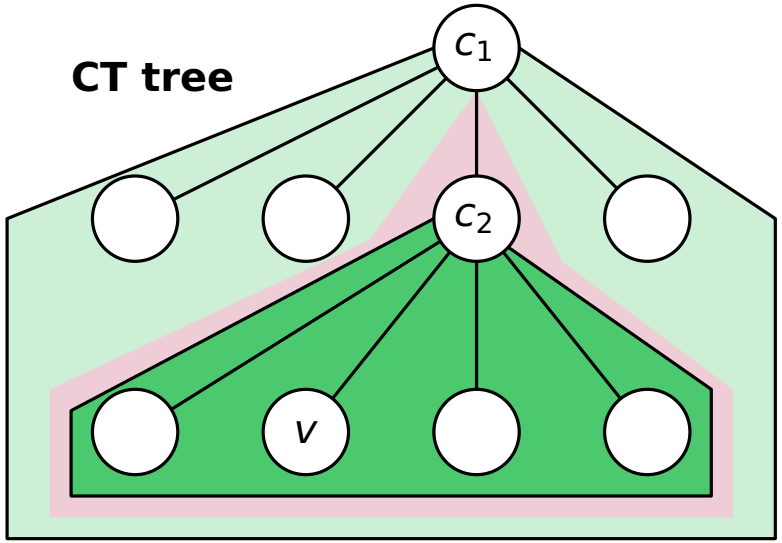

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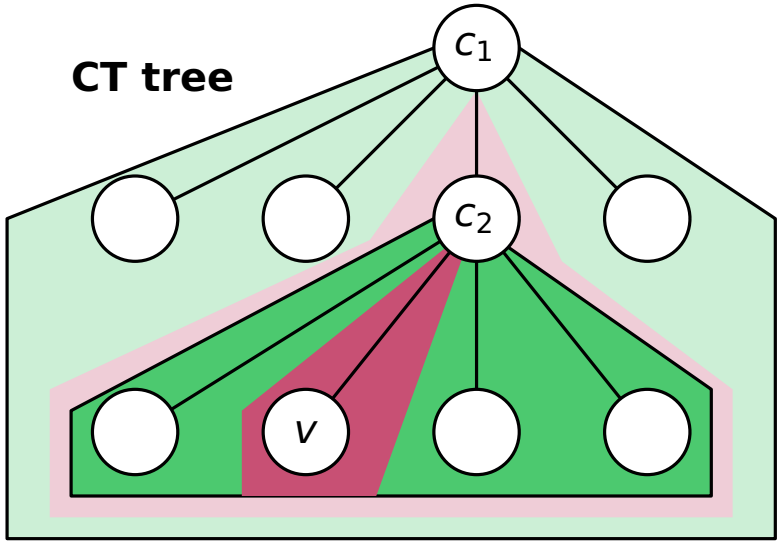

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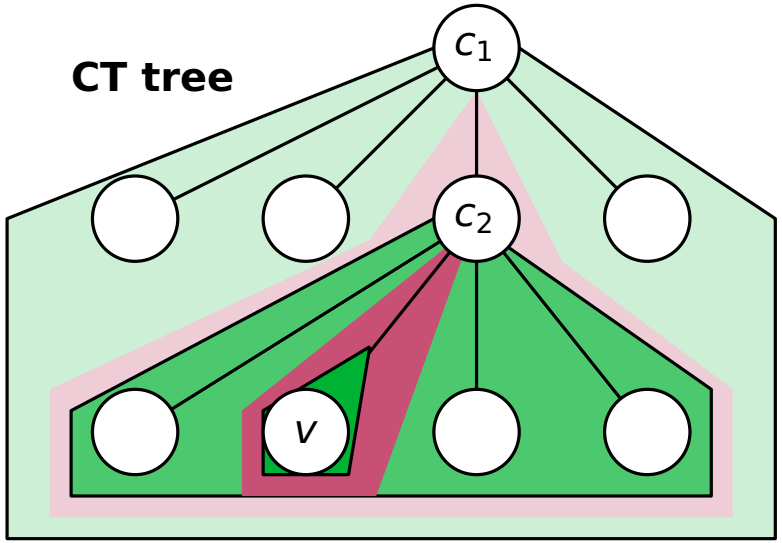

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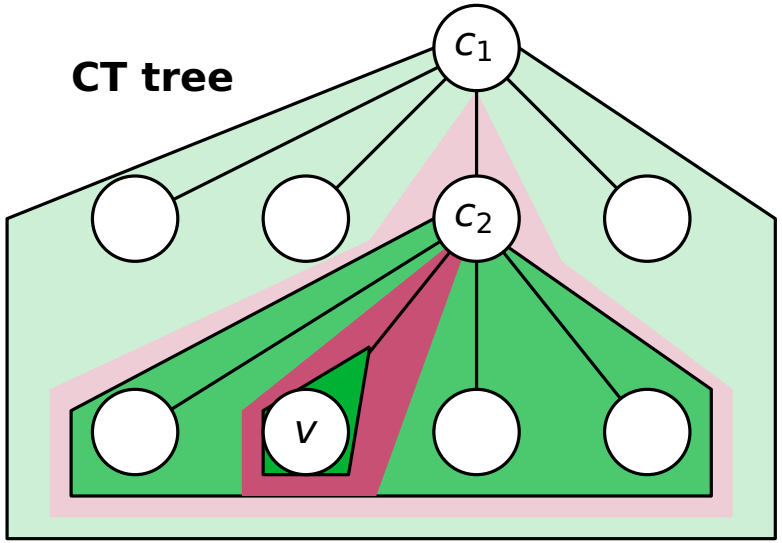

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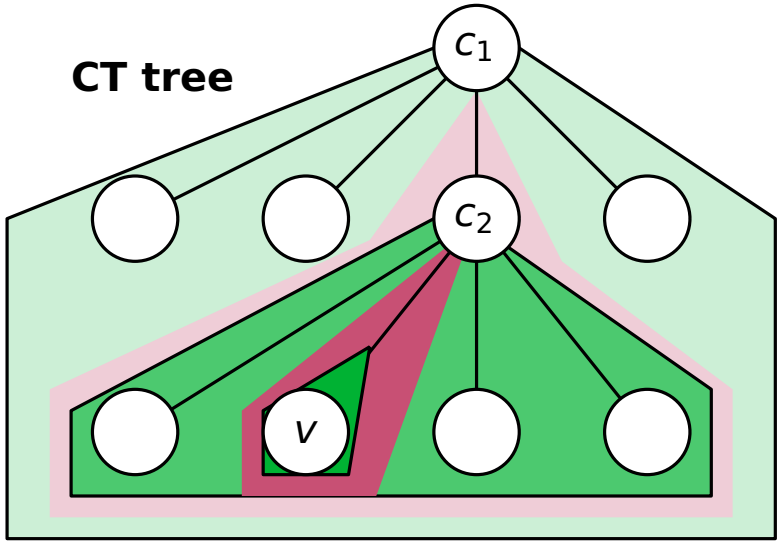

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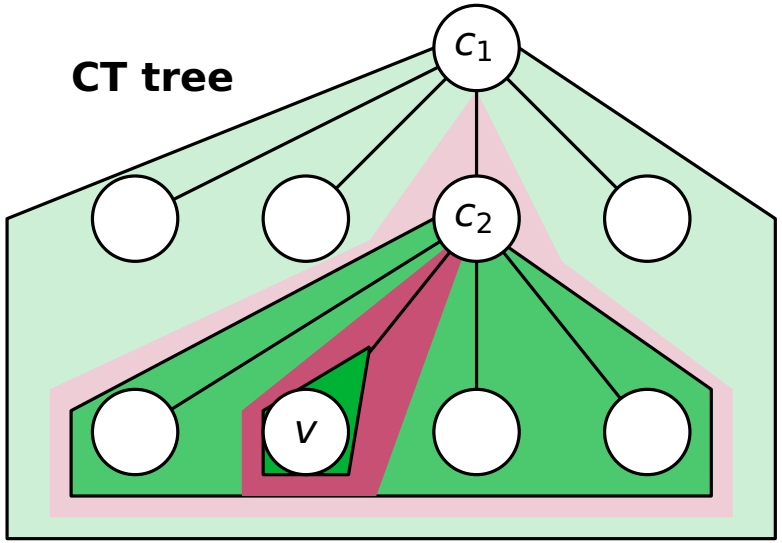

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