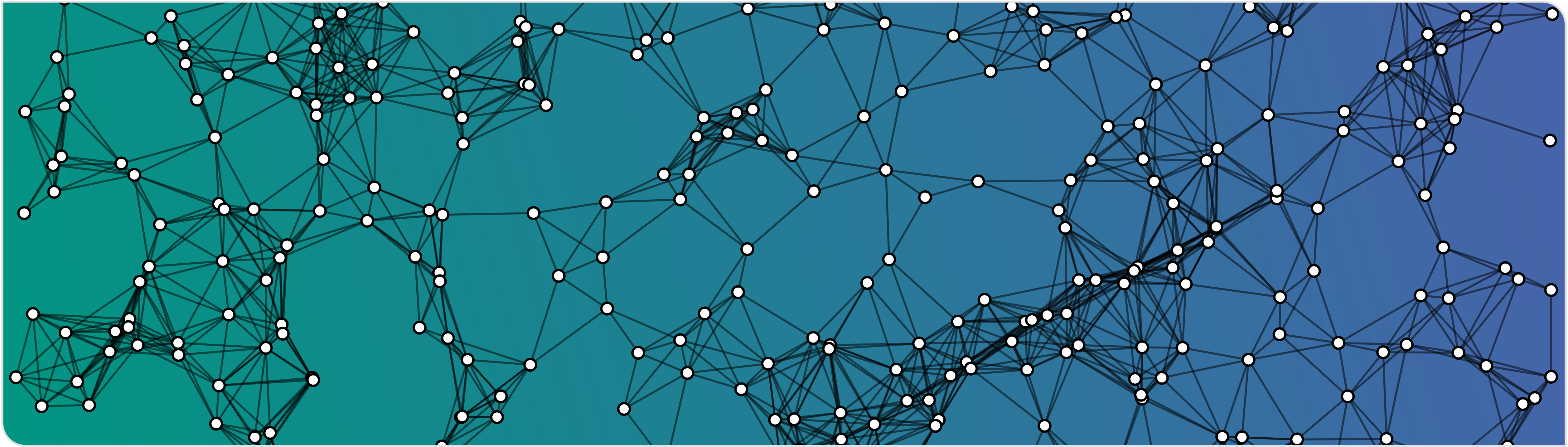
Karlsruher Institut für Technologie

Seminar Algorithmentechnik – Combinatorial Problems on *H*-Graphs

Kilian Krause



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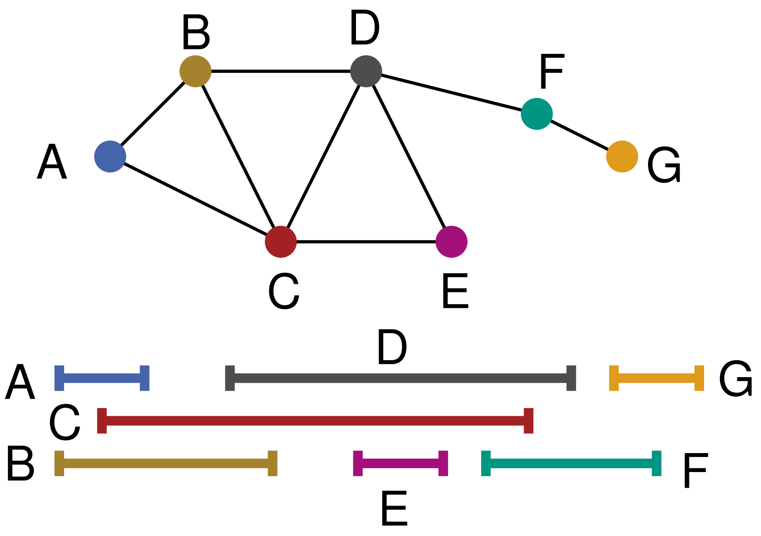

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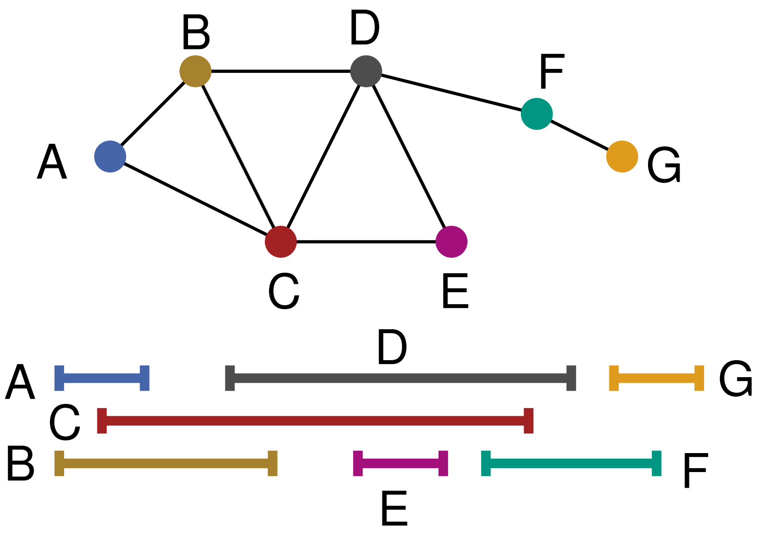
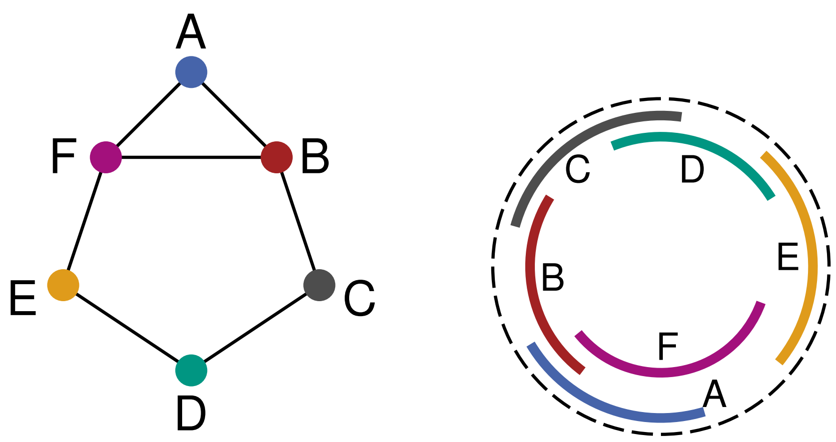


Circular-arc graph



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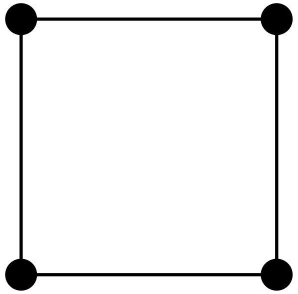

C_4

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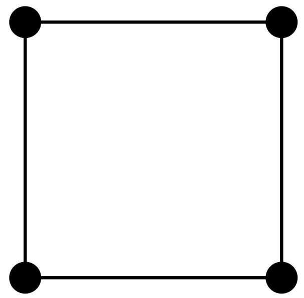

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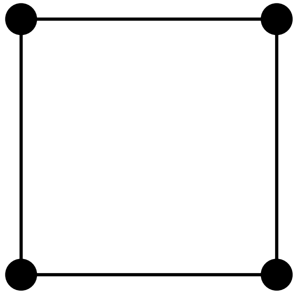

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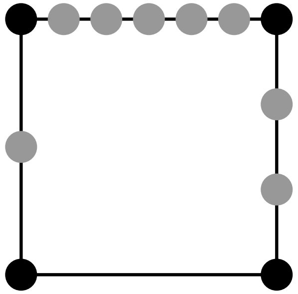

subdivision of C_4

H-graphs

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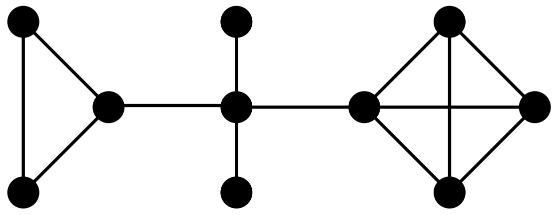

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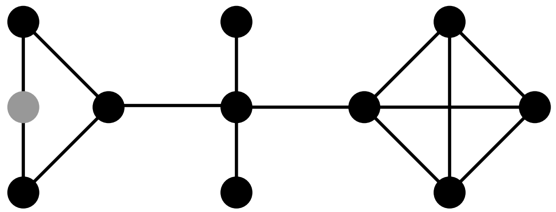

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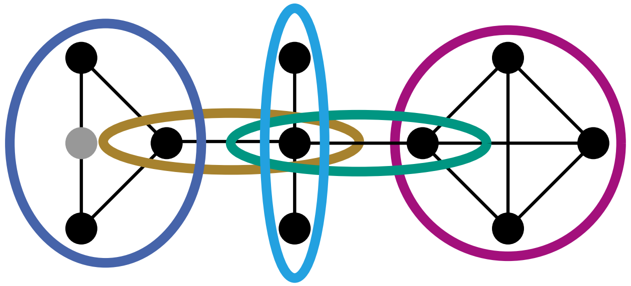

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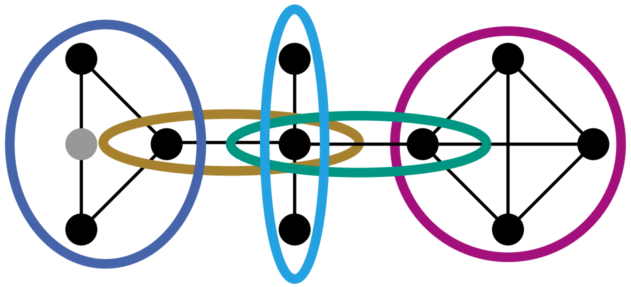

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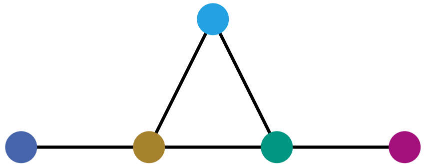

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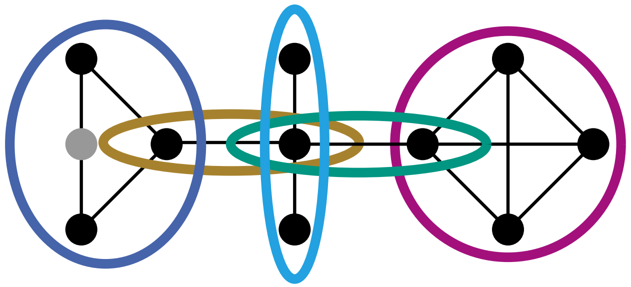

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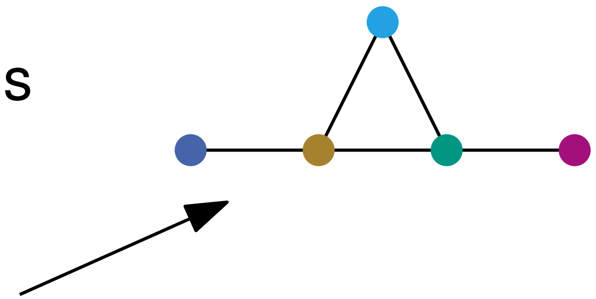

H-graph

Interval graphs as *H*-graphs

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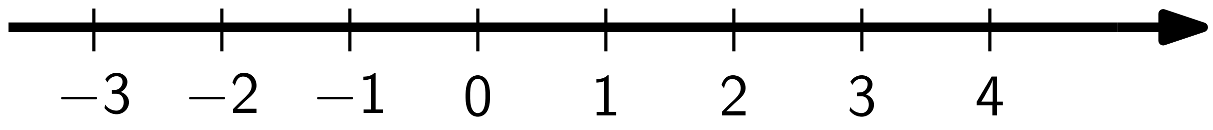

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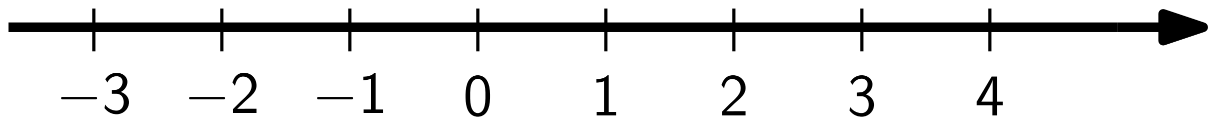

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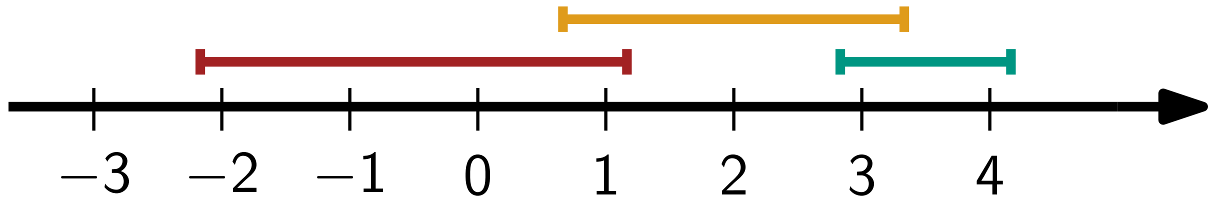

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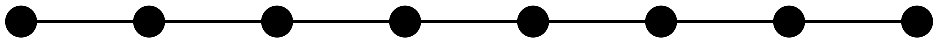

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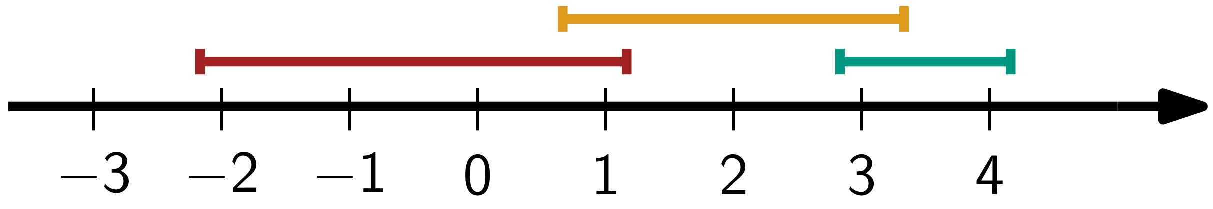

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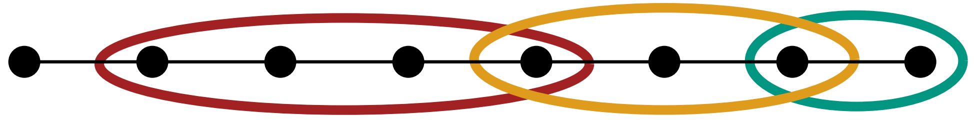
we can consider subpaths of the path



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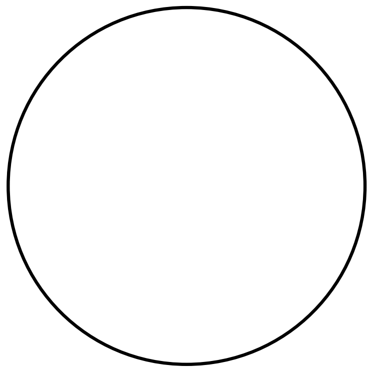

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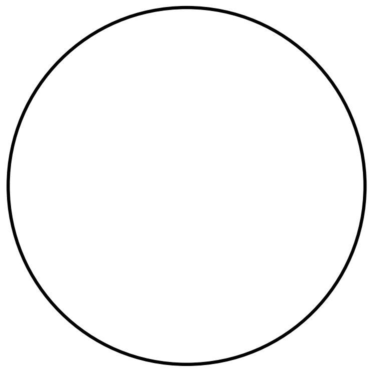

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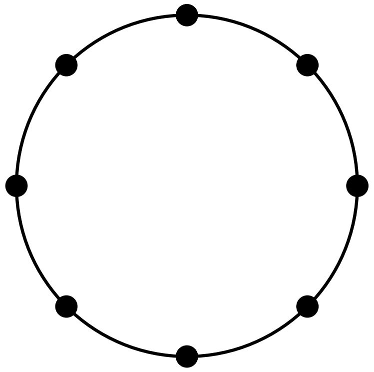

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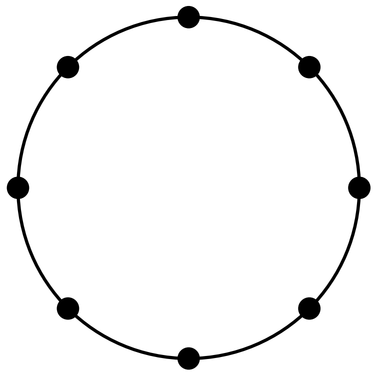

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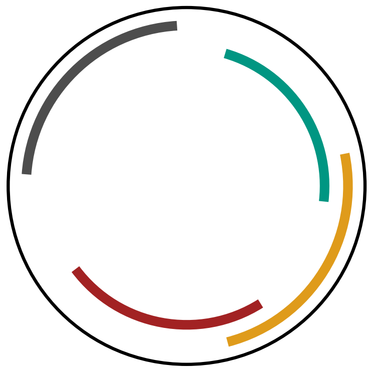
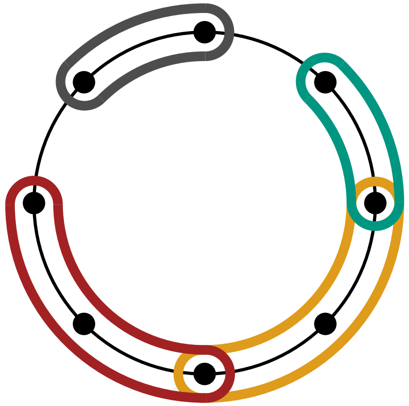
Instead of considering a circle — we can consider a graph-theoretic cycle

Instead of circular-arcs — we can consider subpaths of the cycle $^{(*)}$





$(*)$ subpaths or the cycle itself

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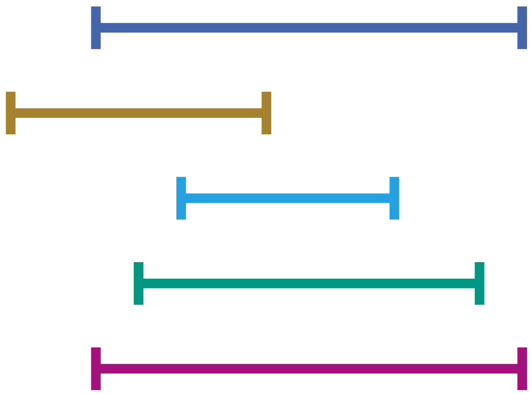

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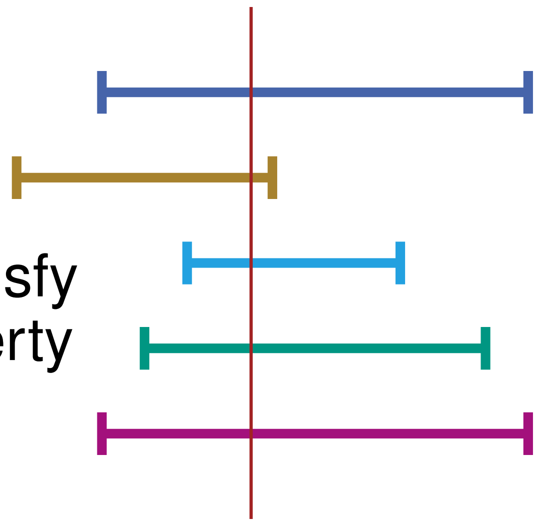

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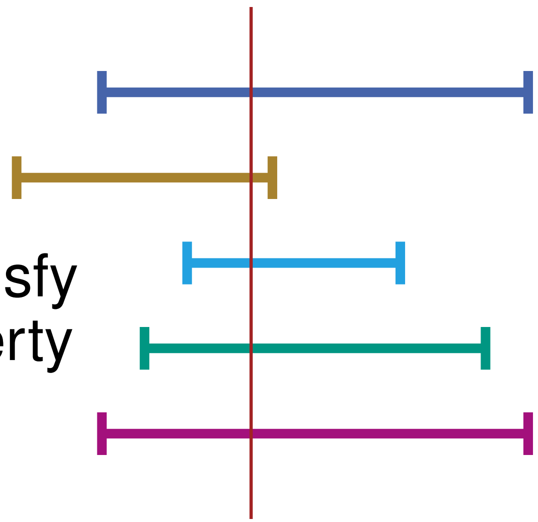

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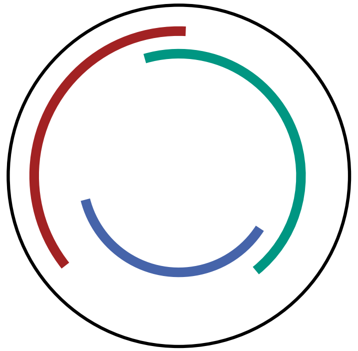

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A *Helly* H-graph G is a graph that admits an H-representation which satisfies the Helly property.

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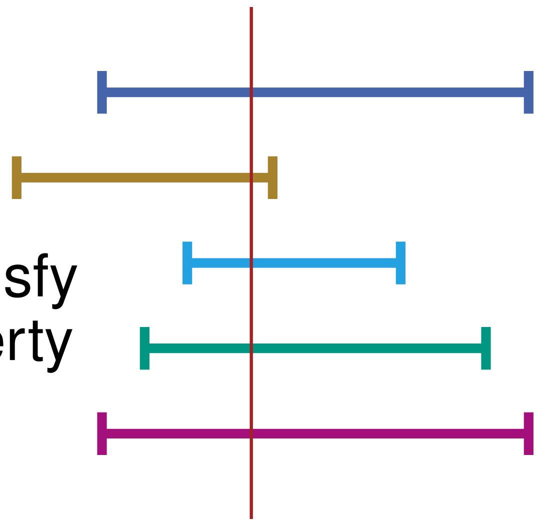

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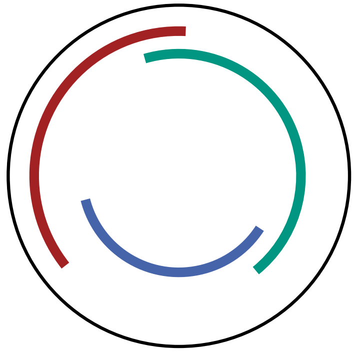

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  - $\Rightarrow$ the corresponding subgraphs in H' pairwise intersect
  - $\Rightarrow \bigcap_{v \in C} H'_v \neq \emptyset$
  - $\Rightarrow$ C corresponds to a node x_C of H'

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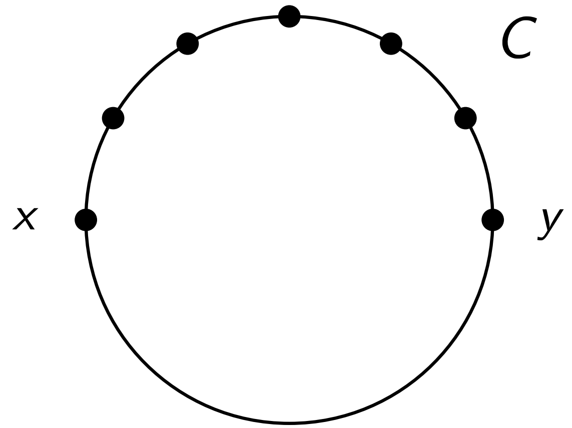

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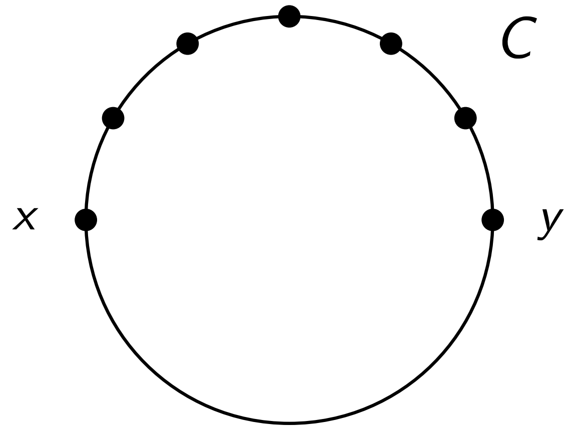

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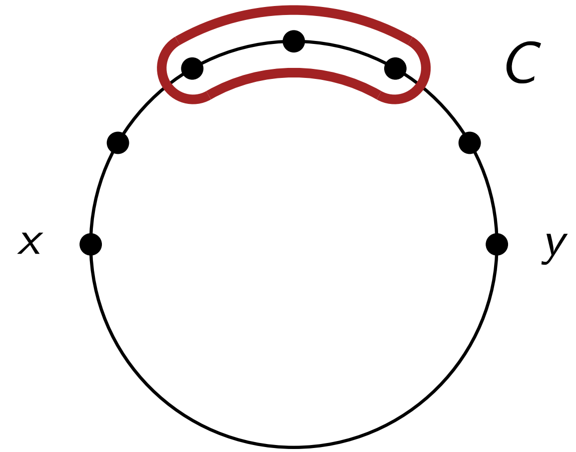

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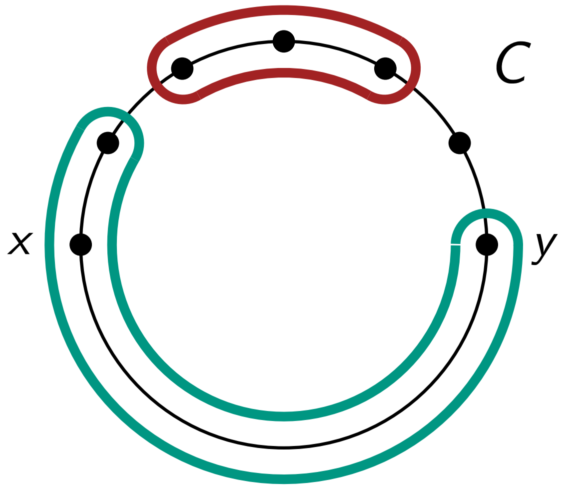
- Otherwise: $C_v = H'_v|_P + xy$



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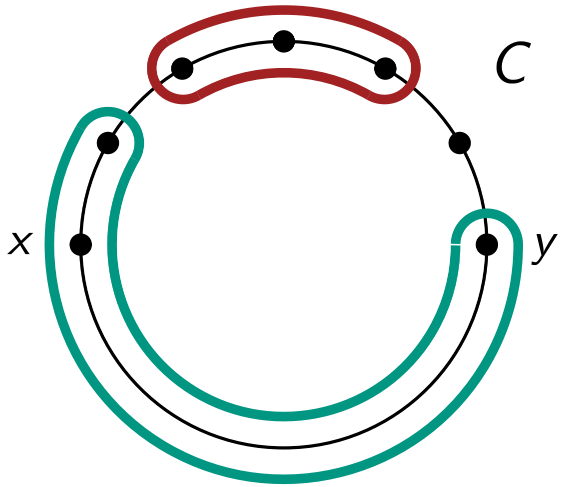

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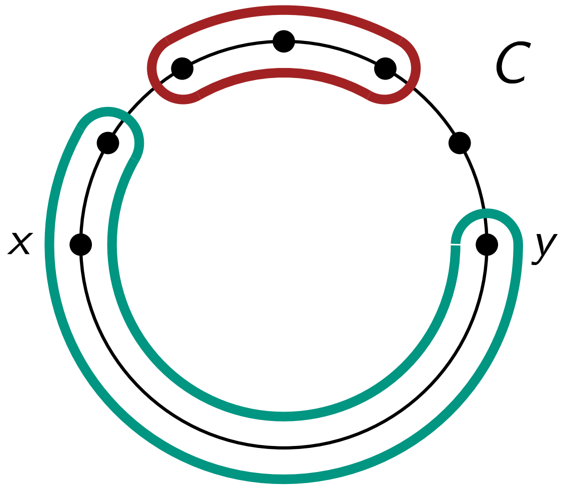

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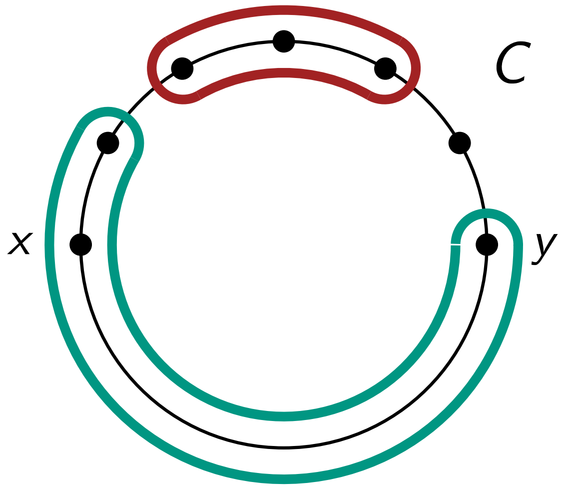

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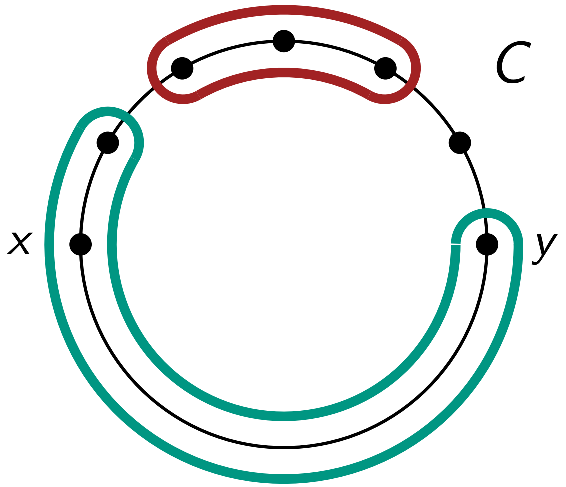

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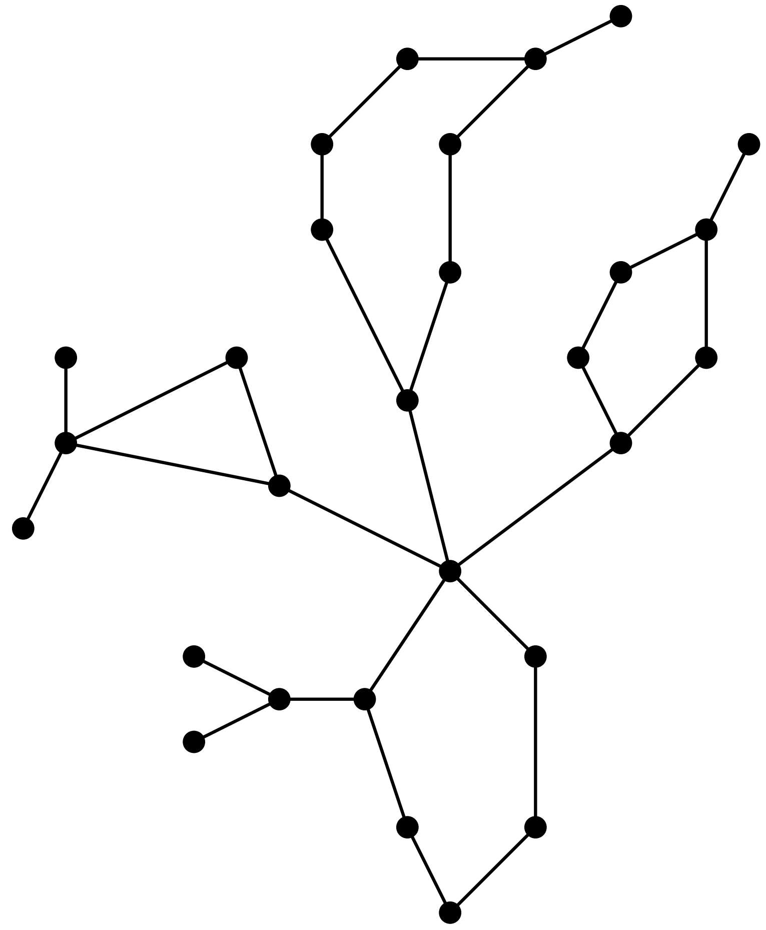

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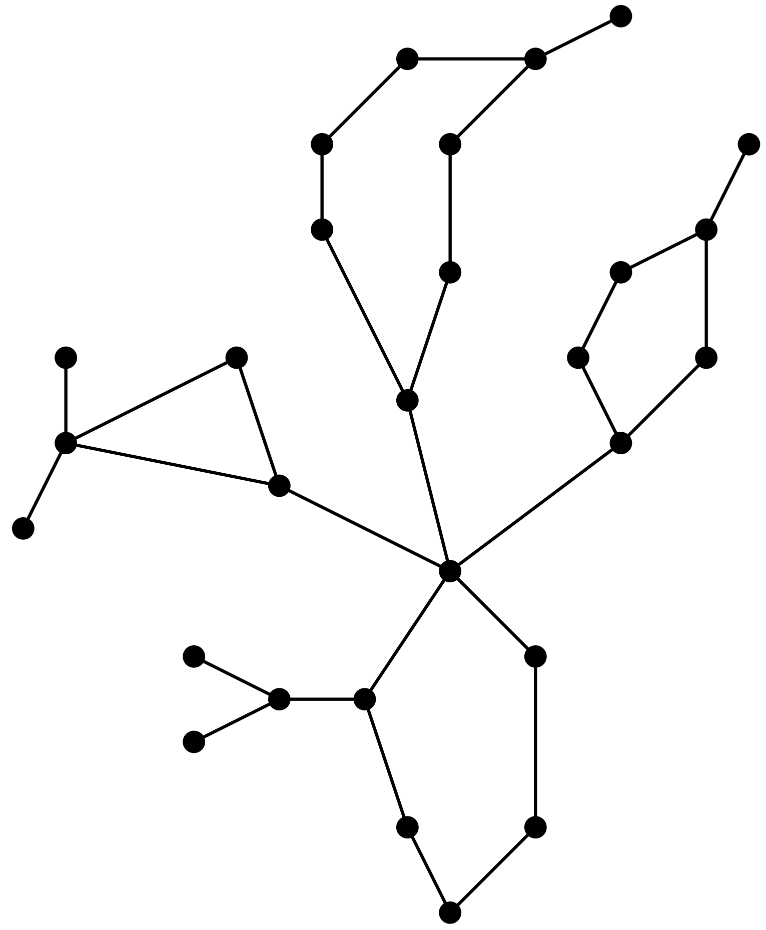

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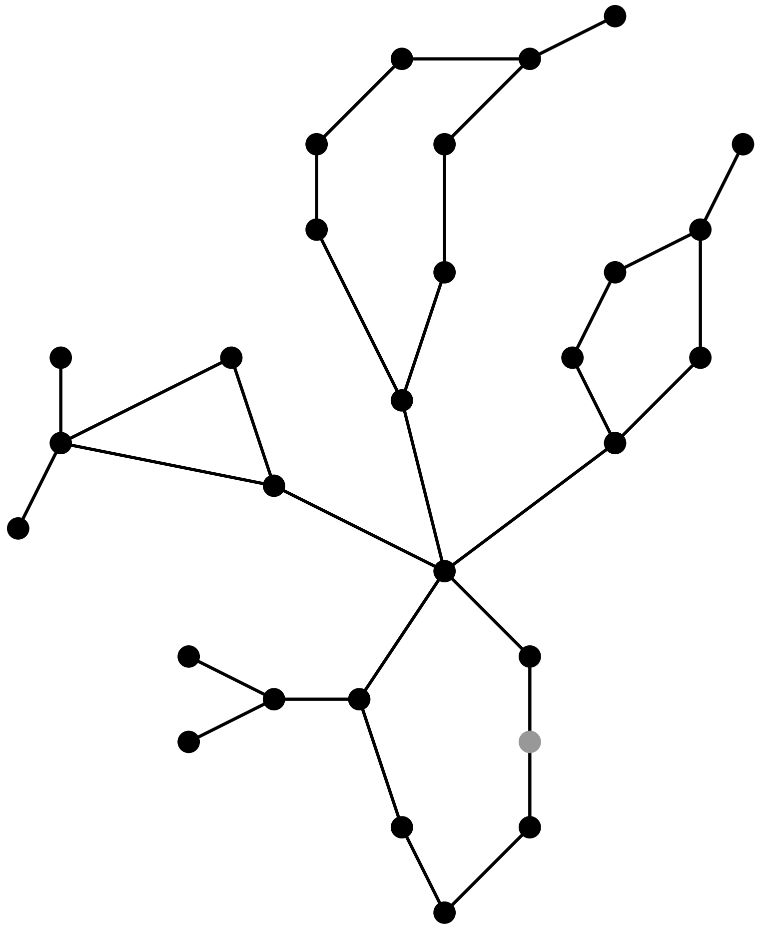

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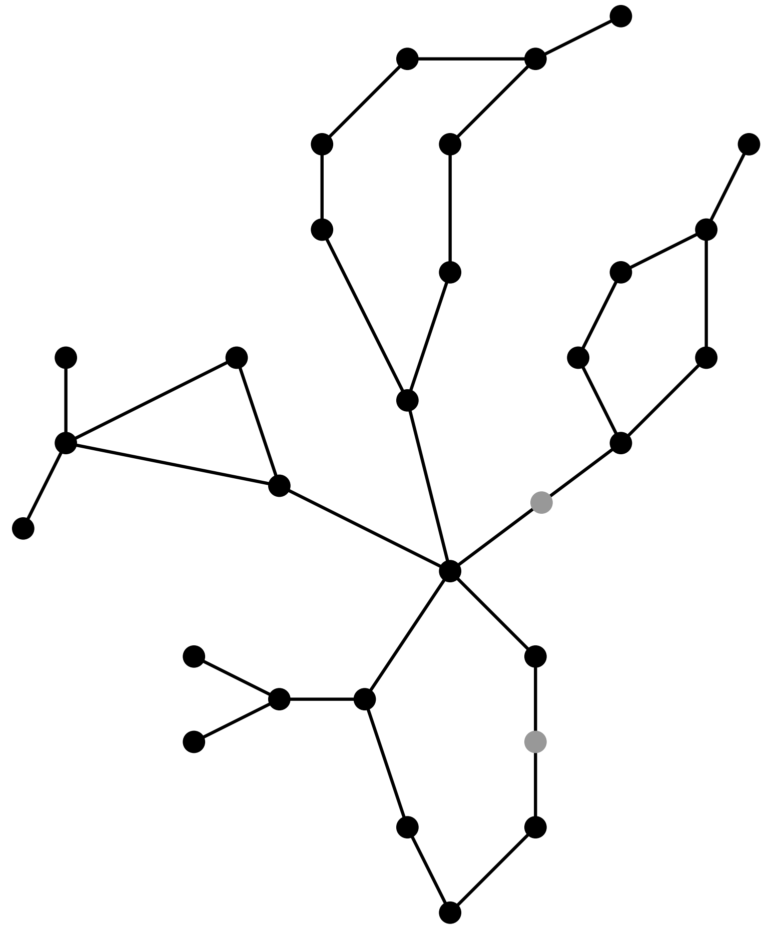

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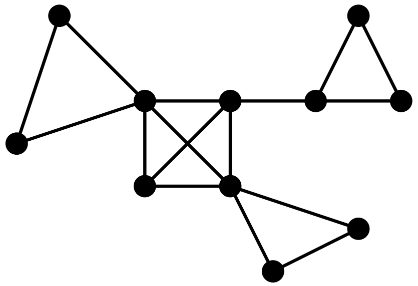

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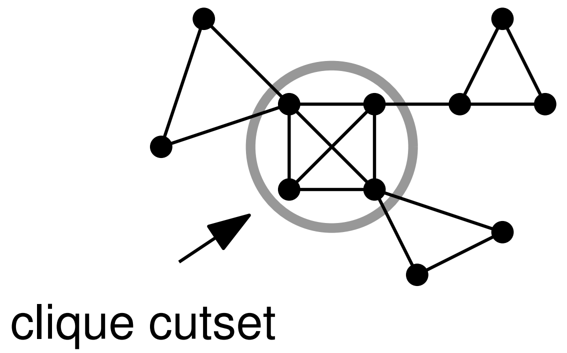

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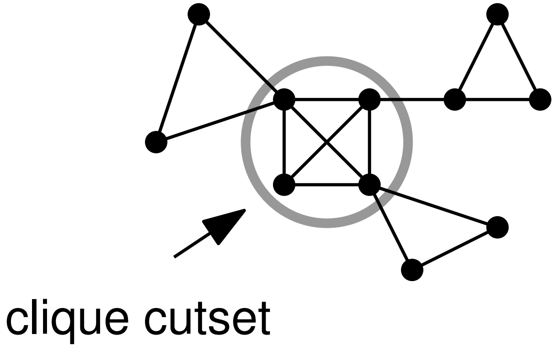

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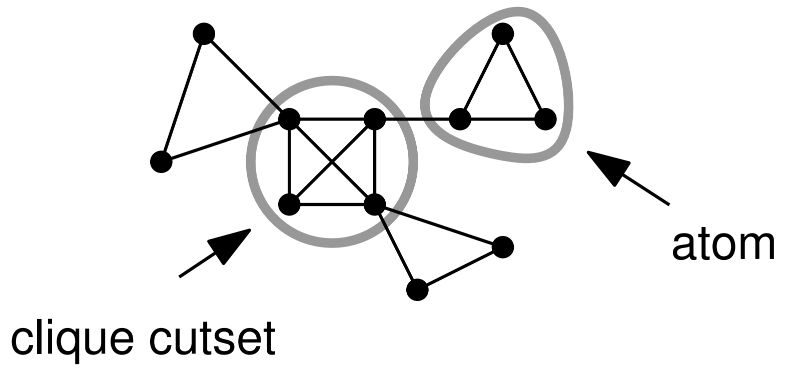

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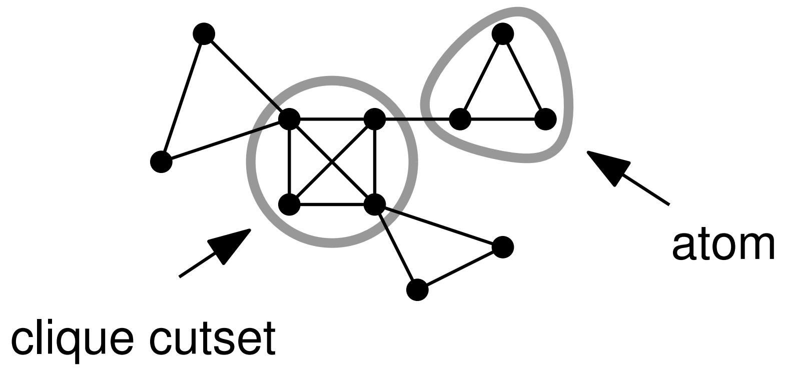

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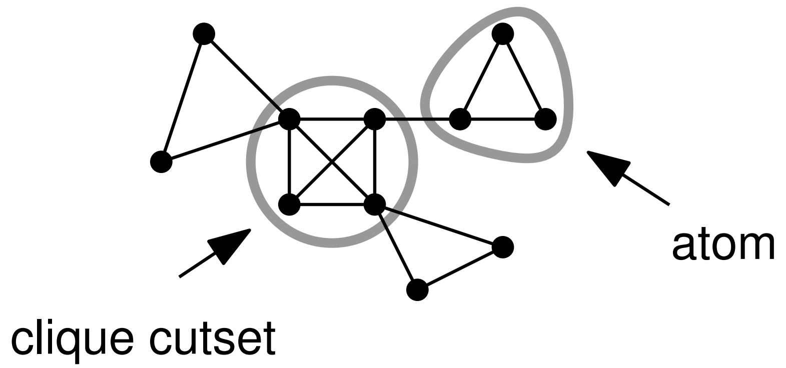

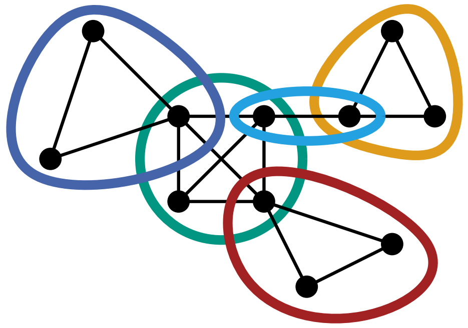

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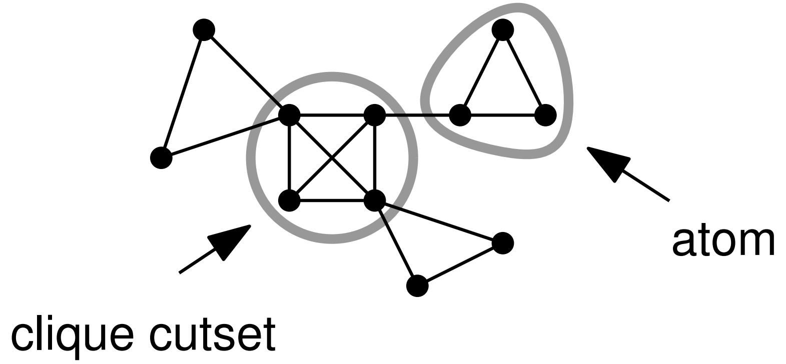

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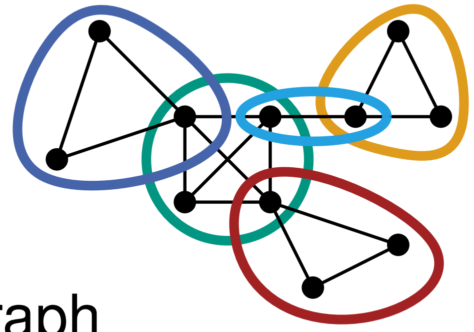

Fact:
A clique-cutset decomposition $\{A_1, \dots, A_k\}$ (with $k \leq n$) of a graph G can be computed in polynomial time, s.t. a maximum clique in G is contained in some A_i.

Atoms of cactus-graphs

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- If H' is a path or cycle, we are done
- Otherwise iteratively repeat this process until we obtain a path or cycle

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Fact:
The clique problem can be solved in polynomial time on circular-arc graphs.

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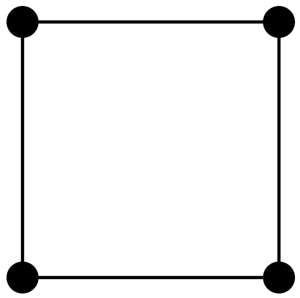

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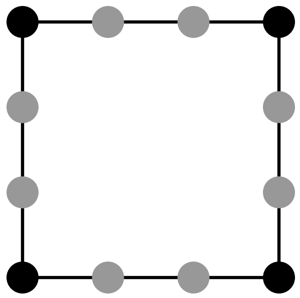

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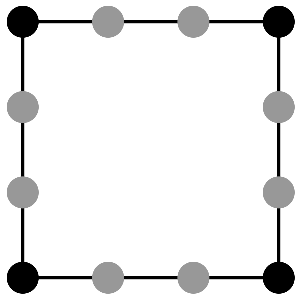

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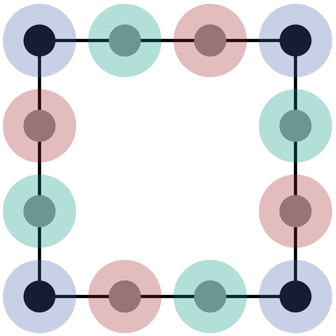

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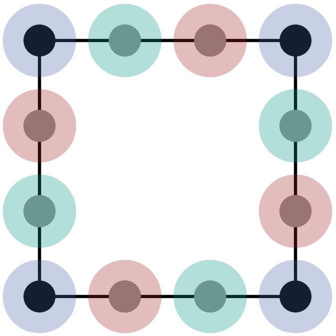

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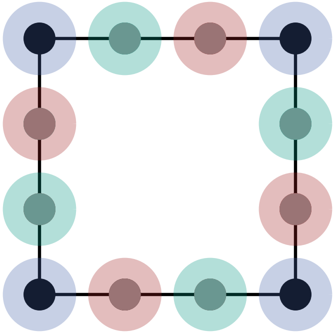

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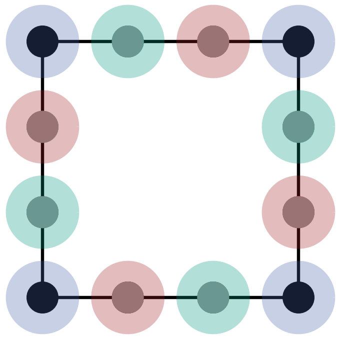

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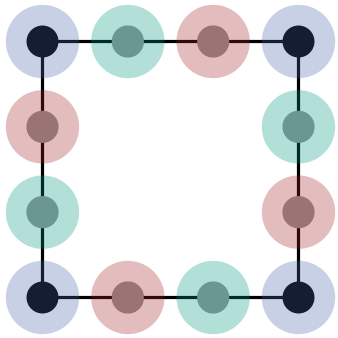

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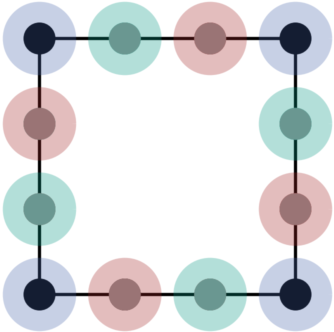

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- The clique problem is APX-hard on $\overline{\text{SUBD}_2}$

Graphs containing the double triangle as a minor

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Karlsruher Institut für Technologie

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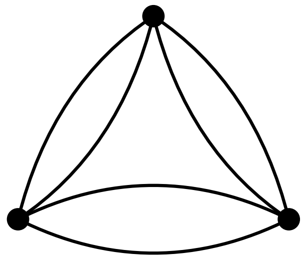

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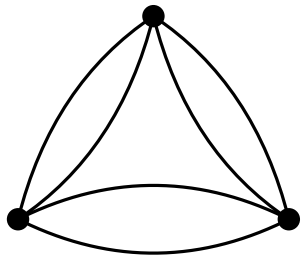

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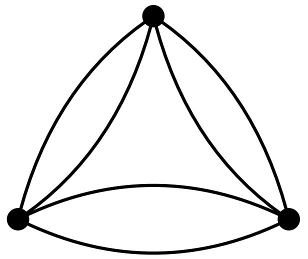

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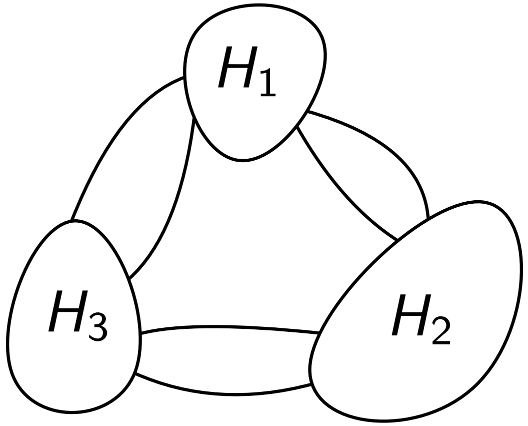

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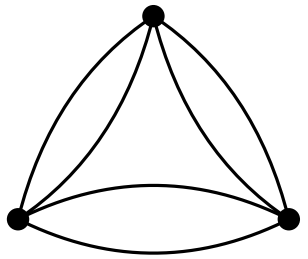

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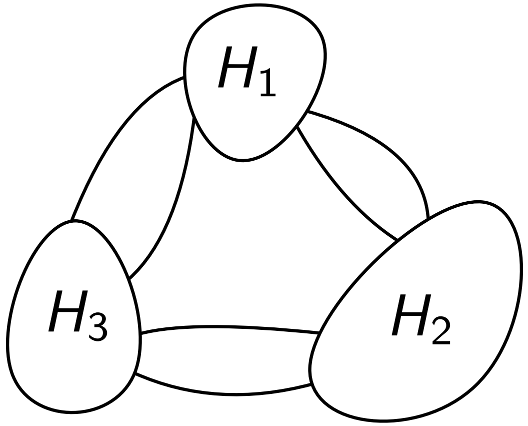

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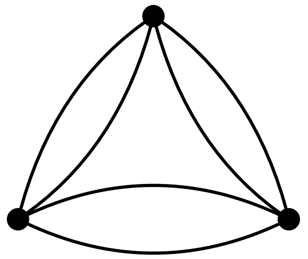

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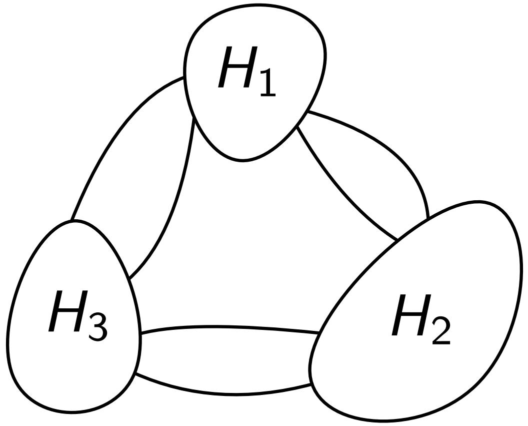

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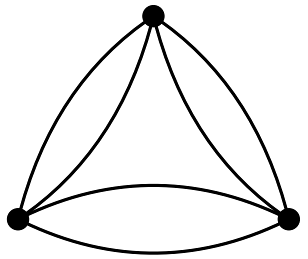

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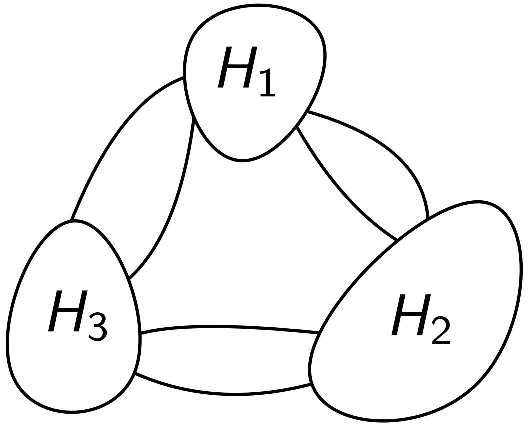

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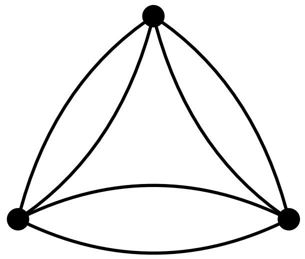

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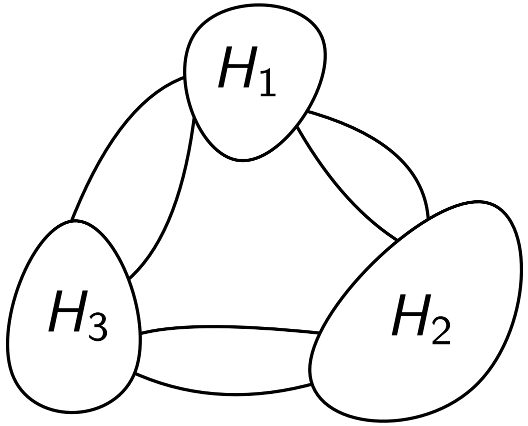

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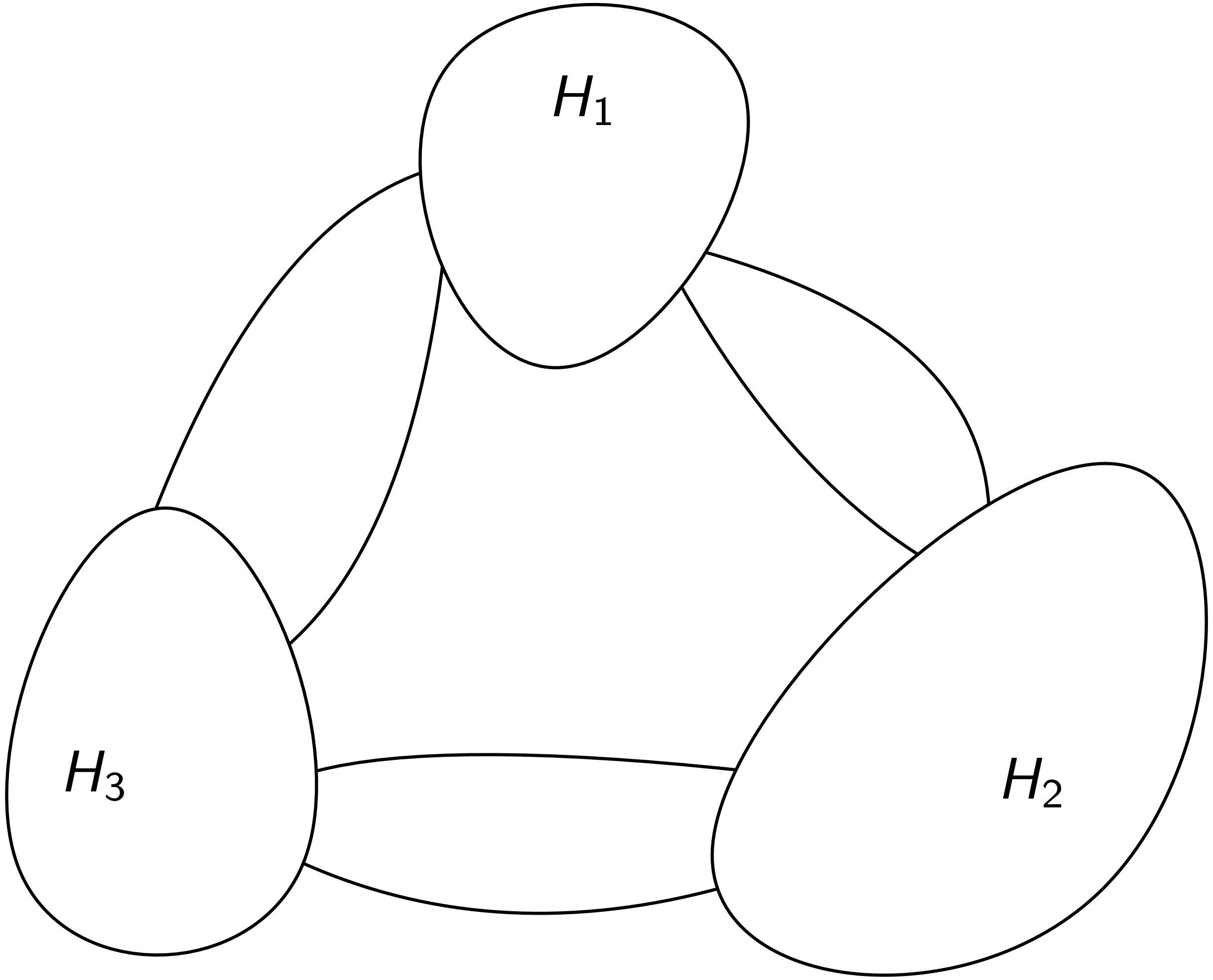

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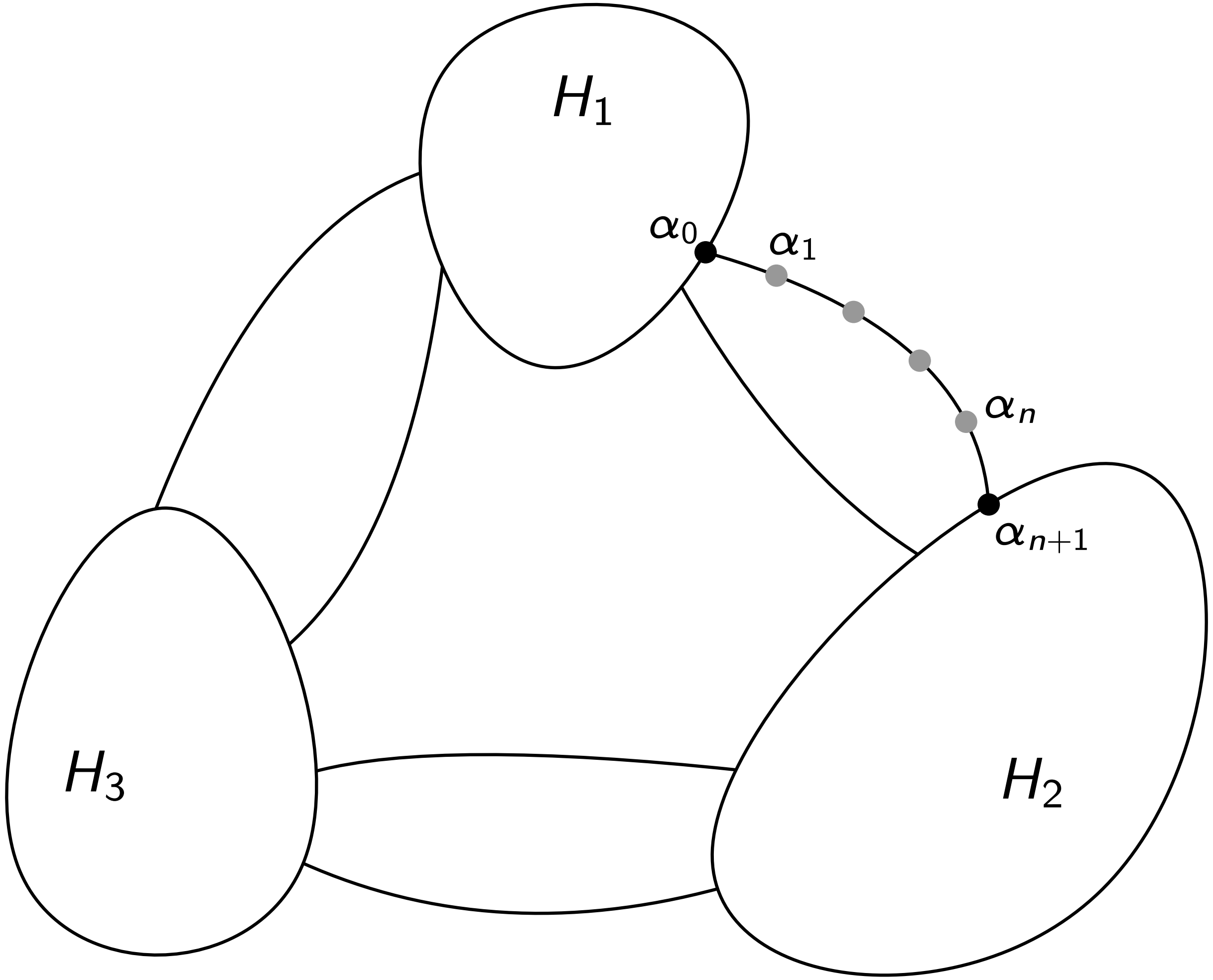

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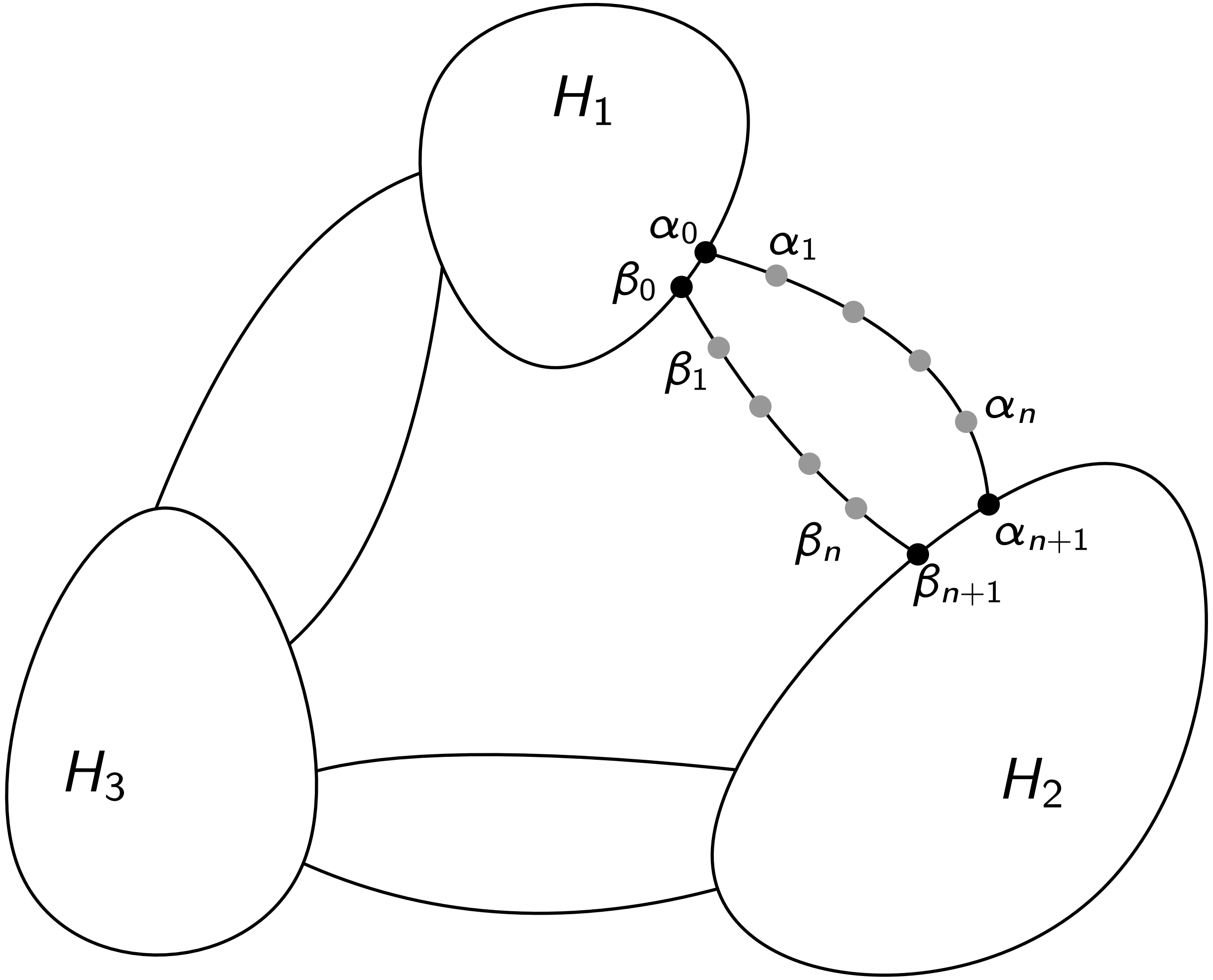

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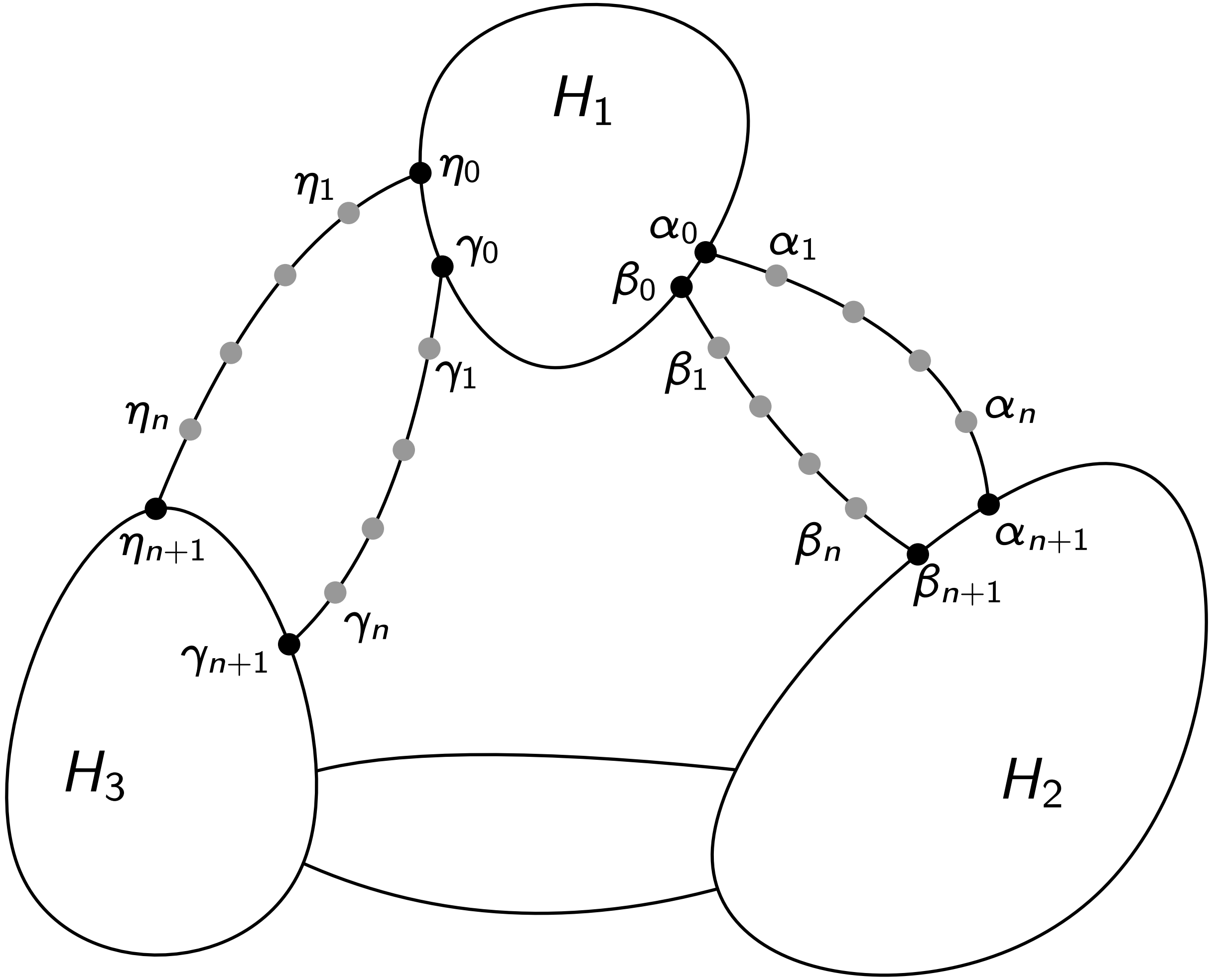

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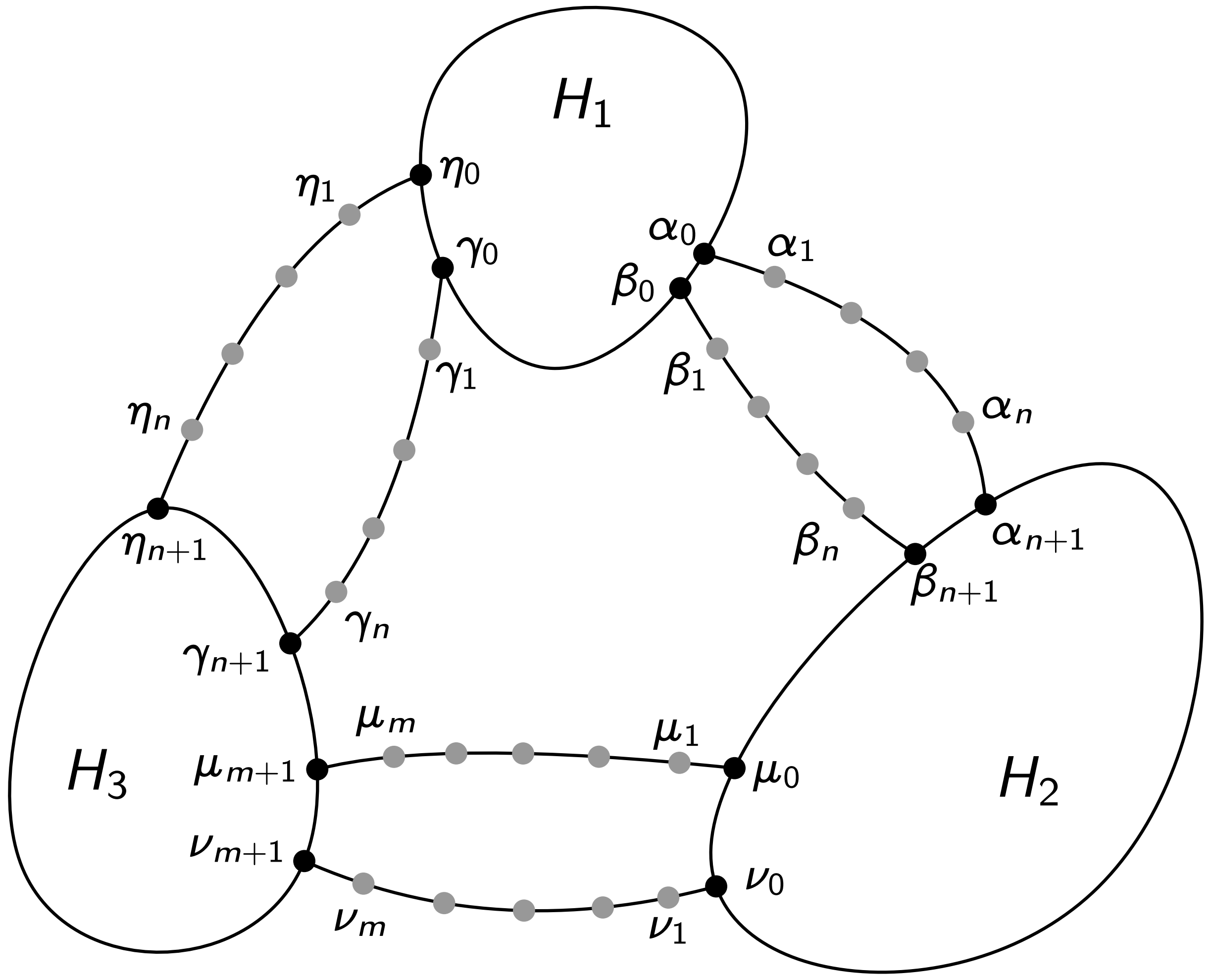

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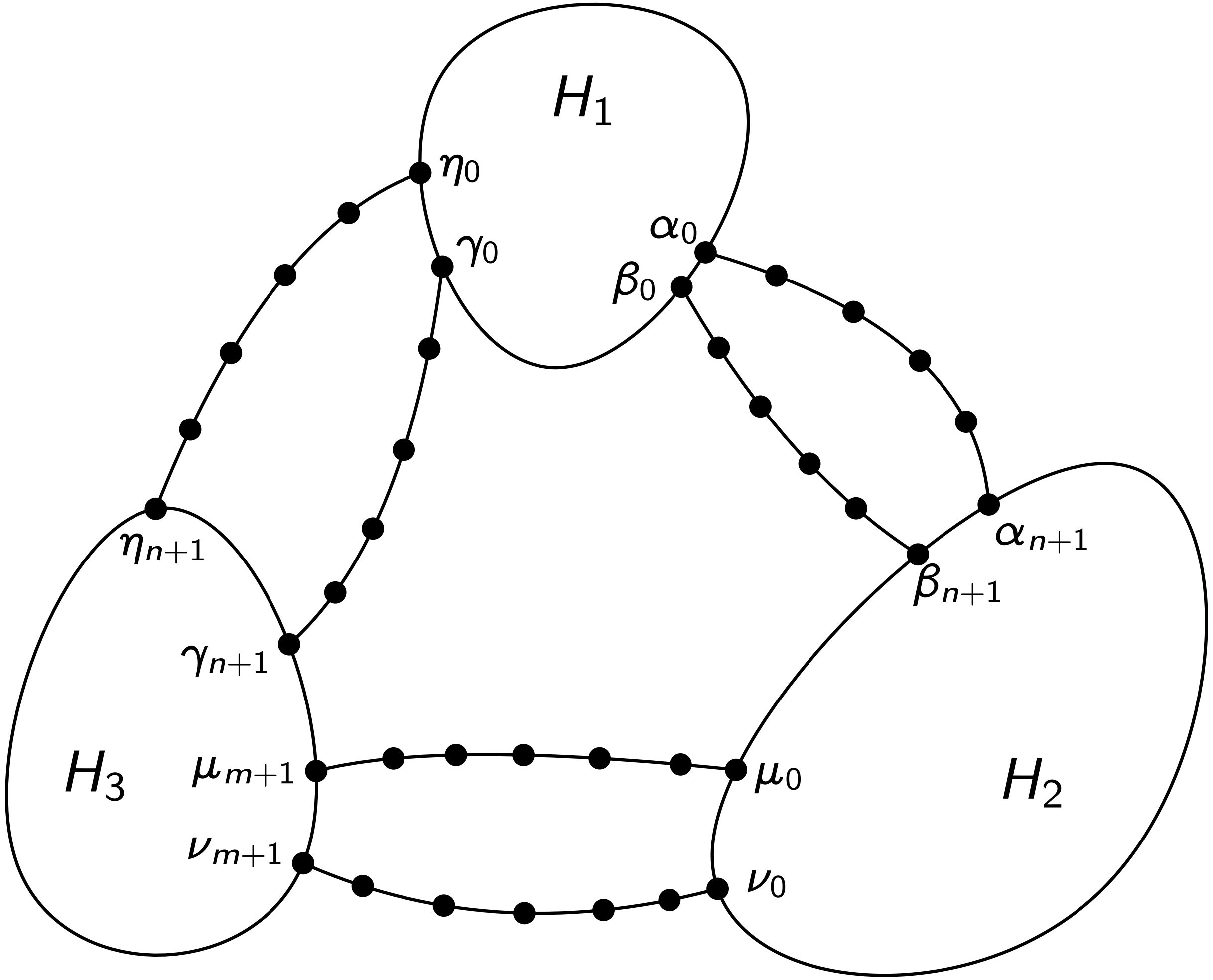

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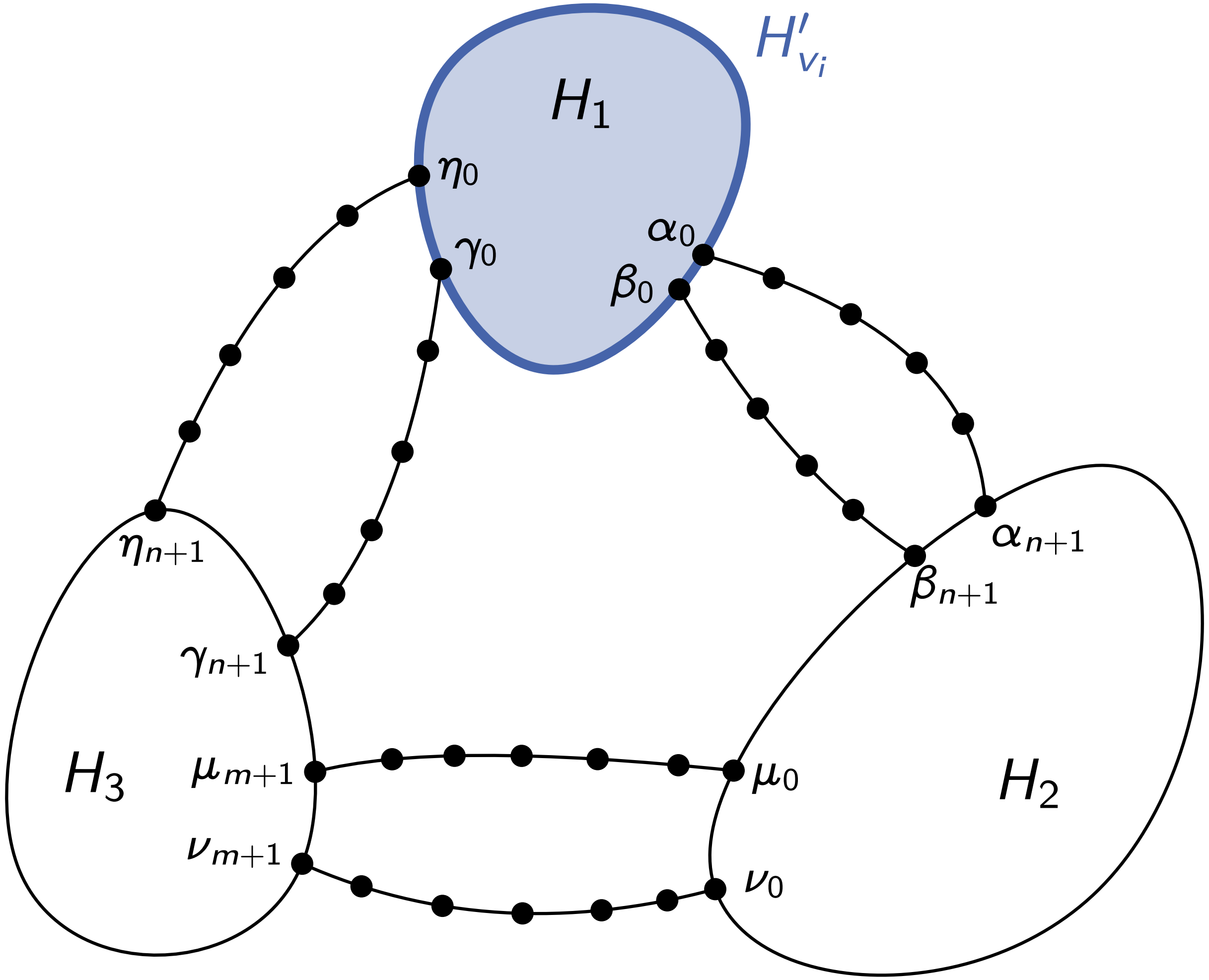

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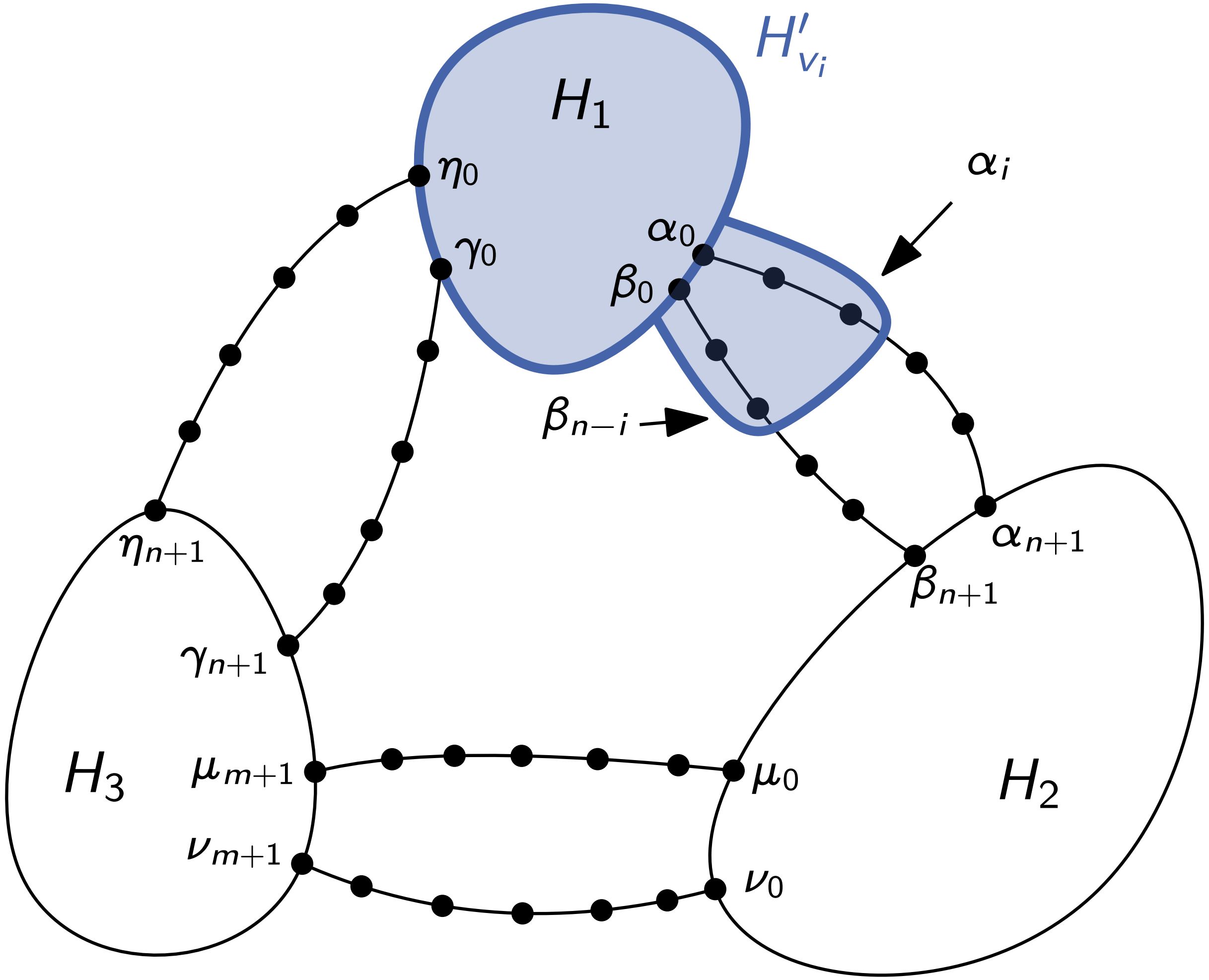

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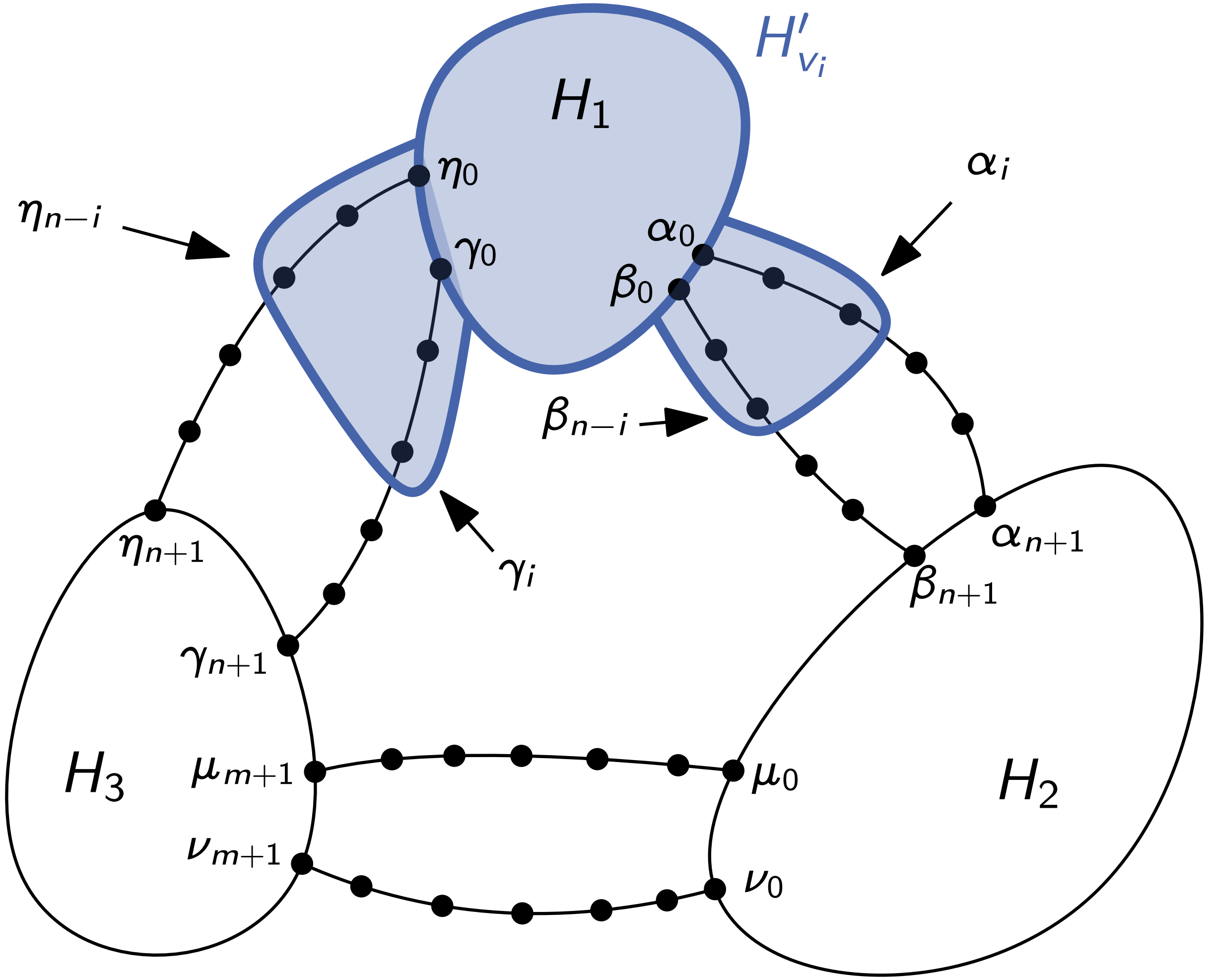

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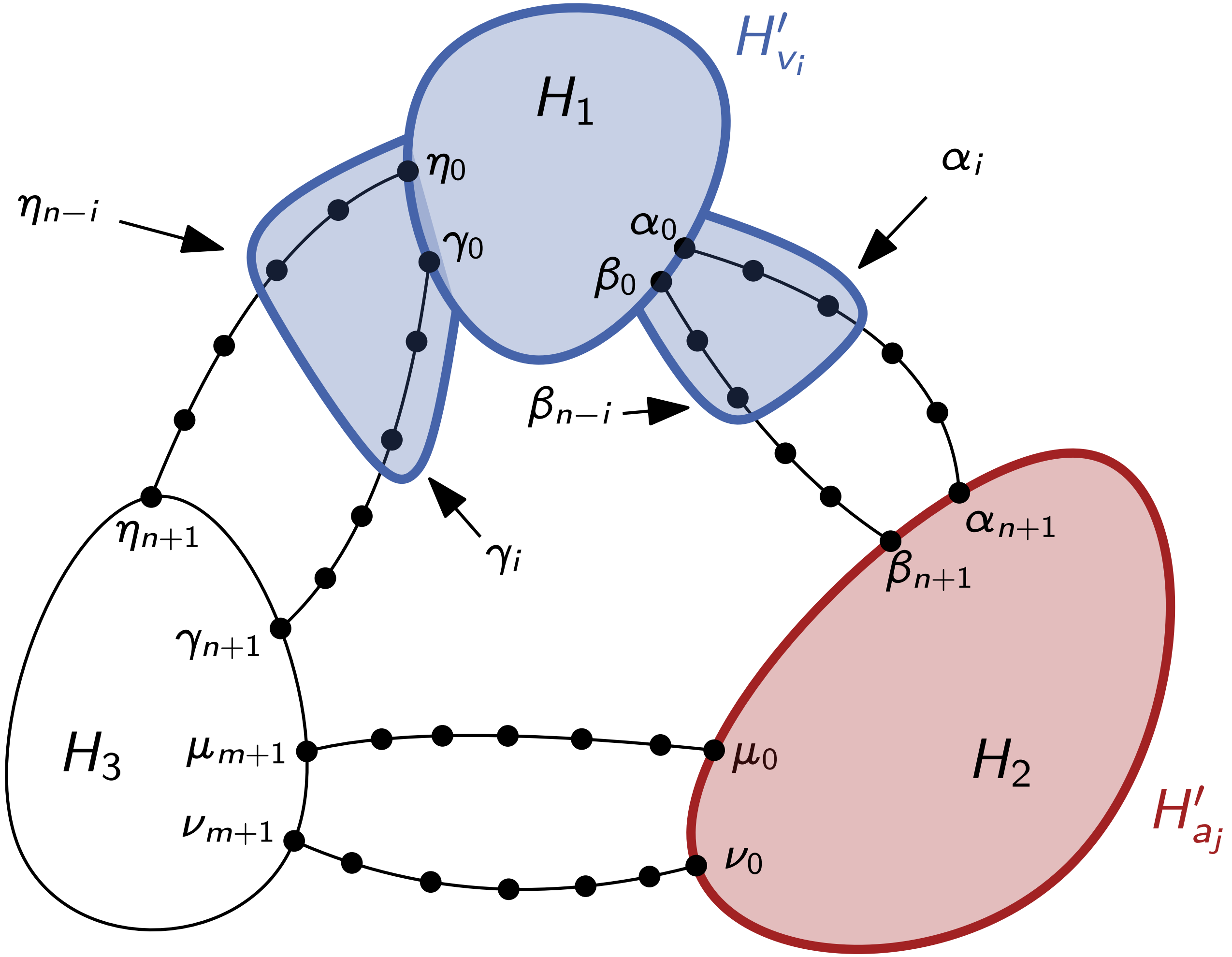

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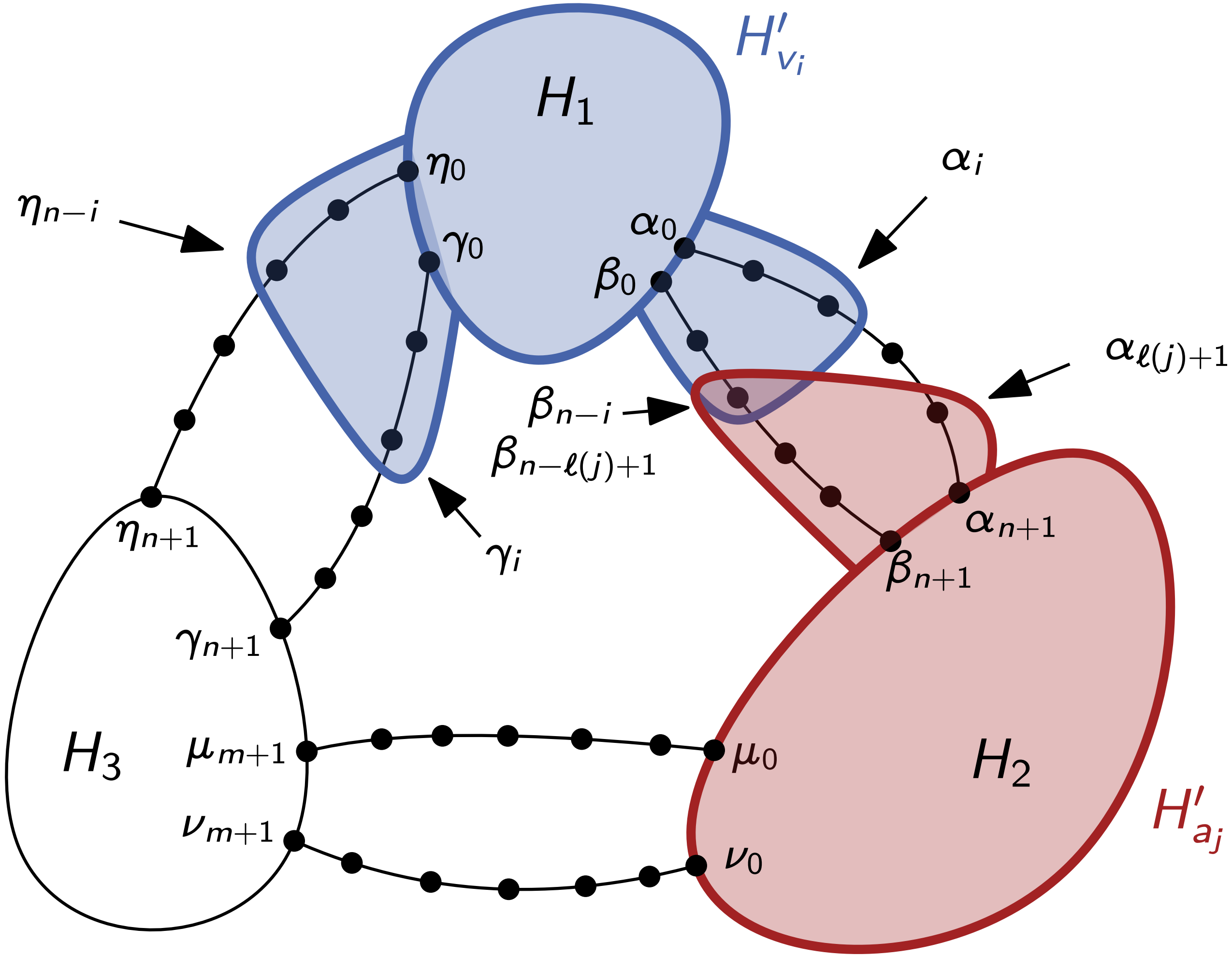

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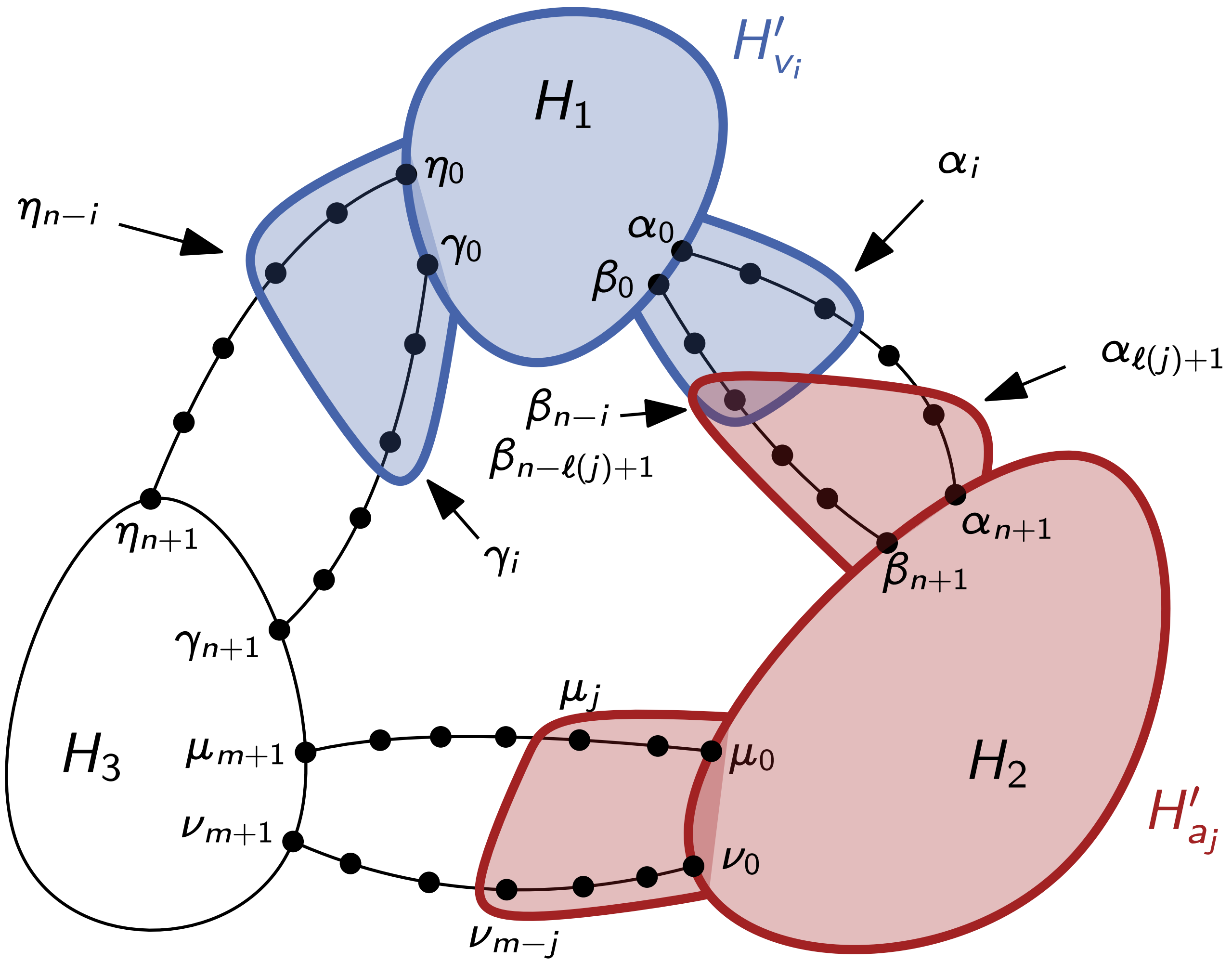

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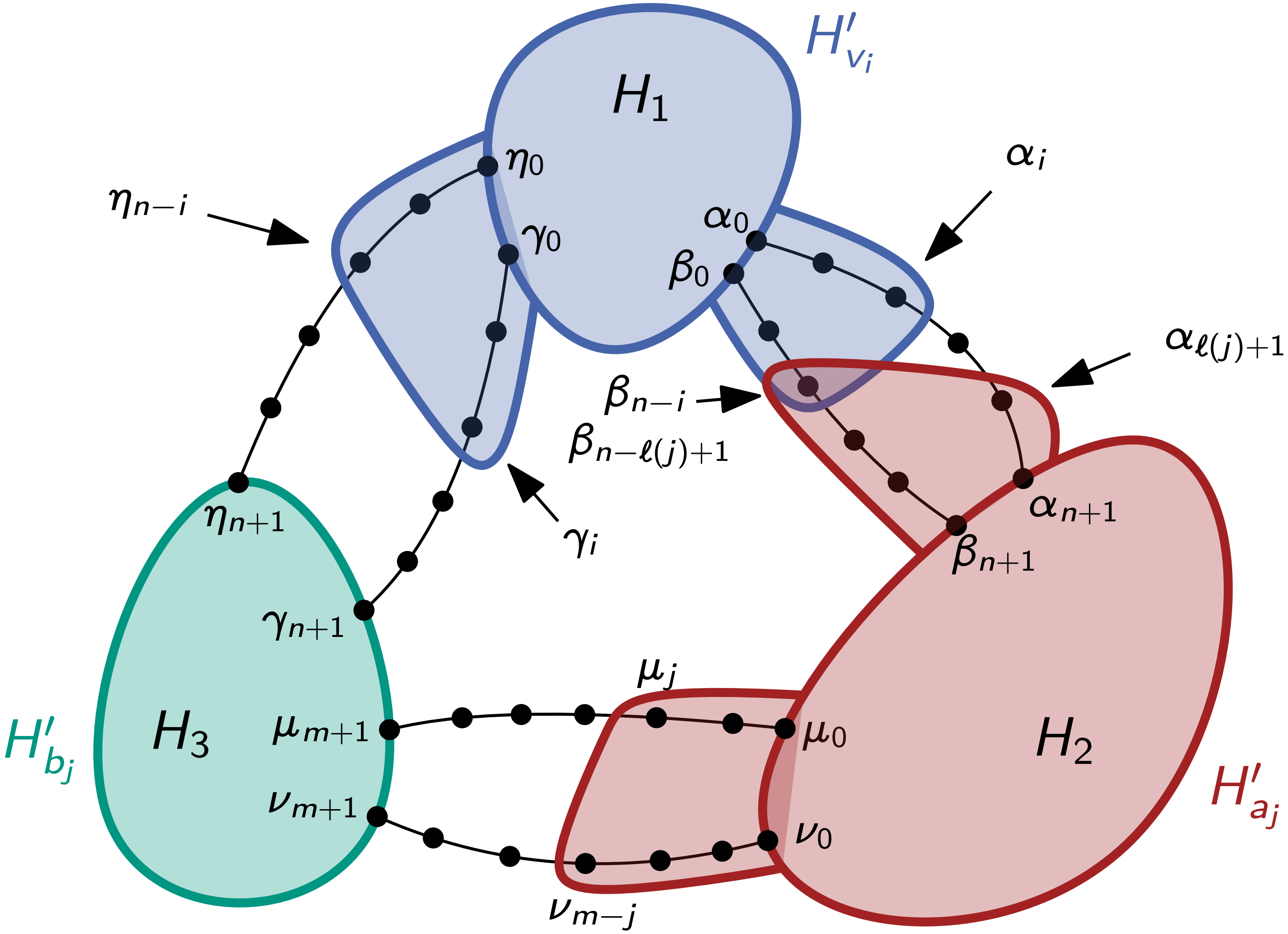

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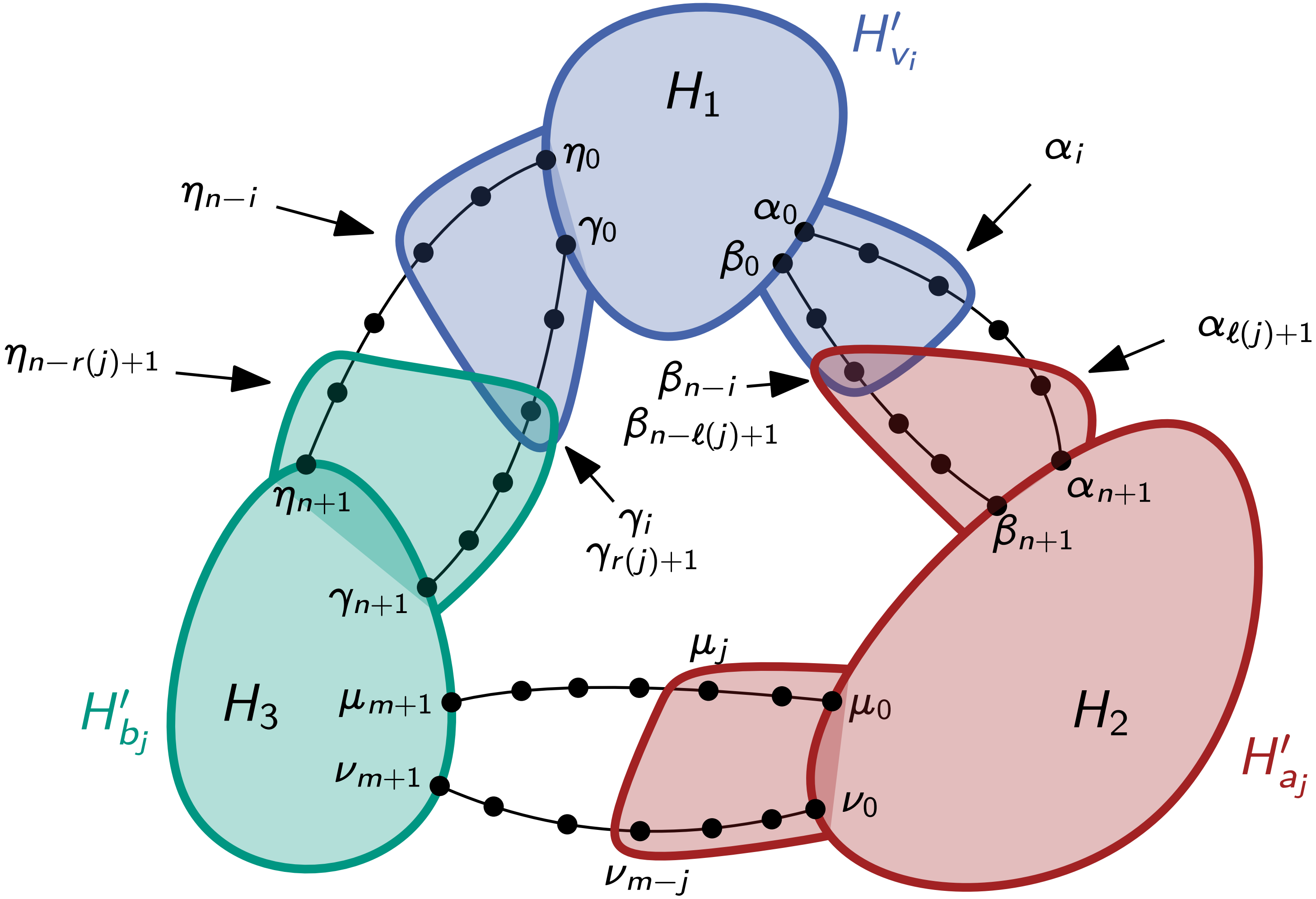

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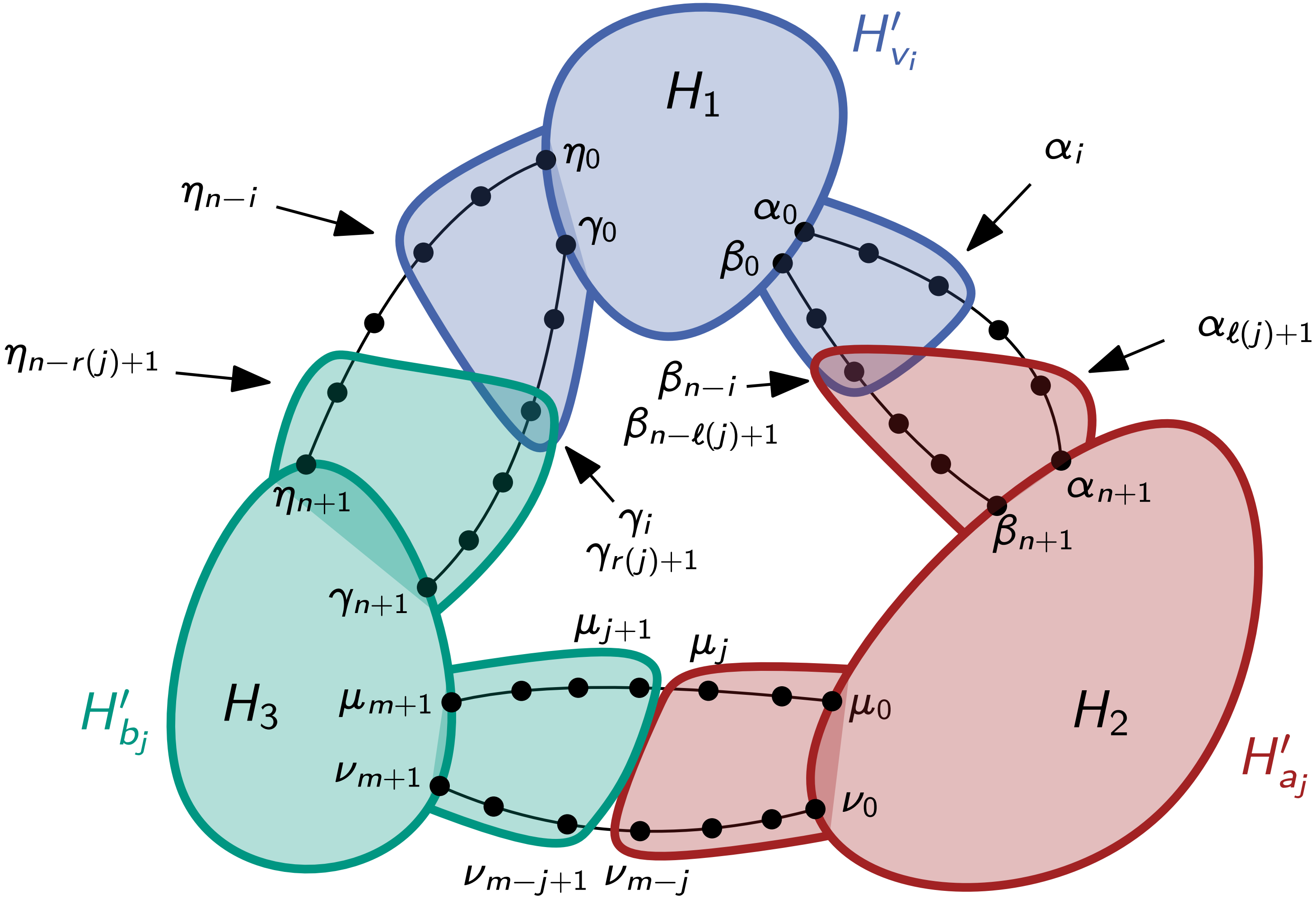

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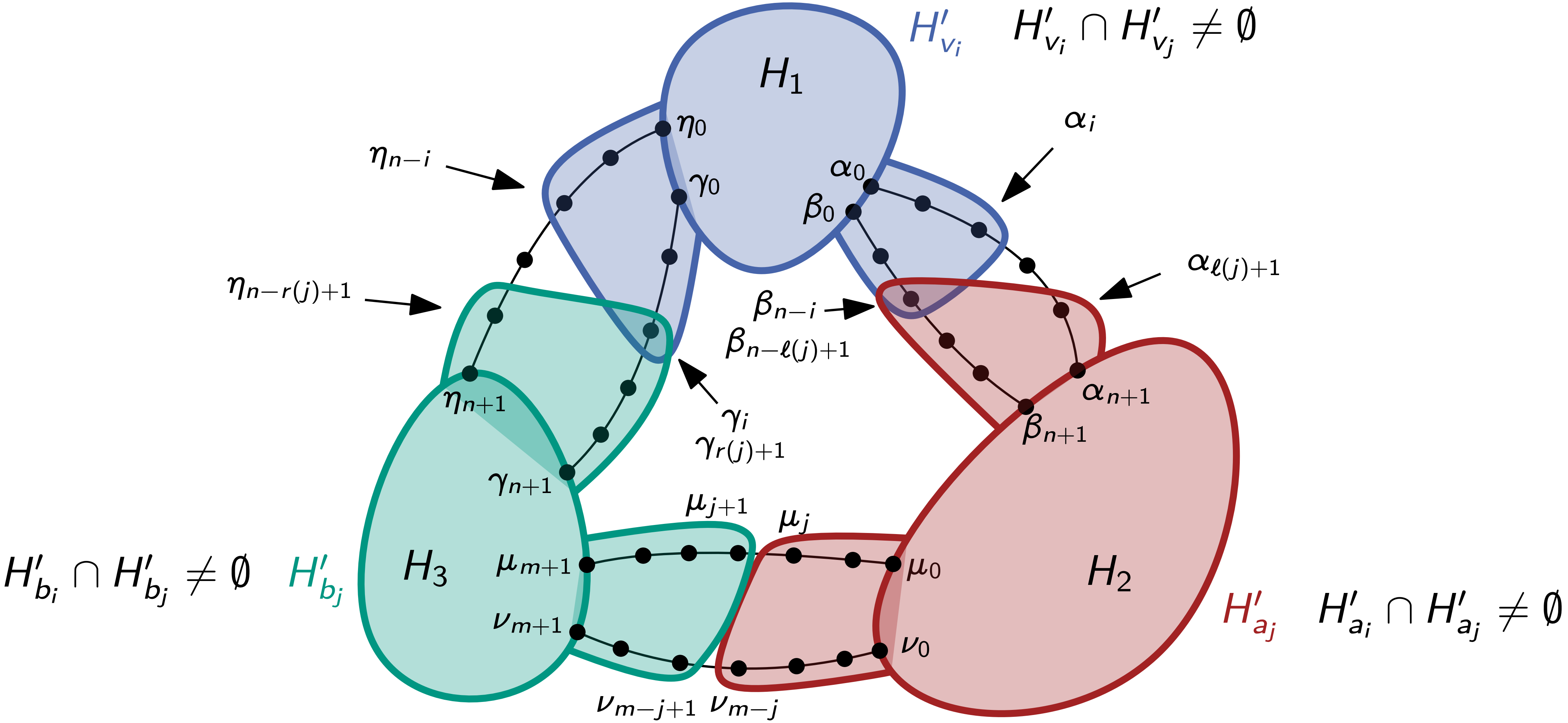

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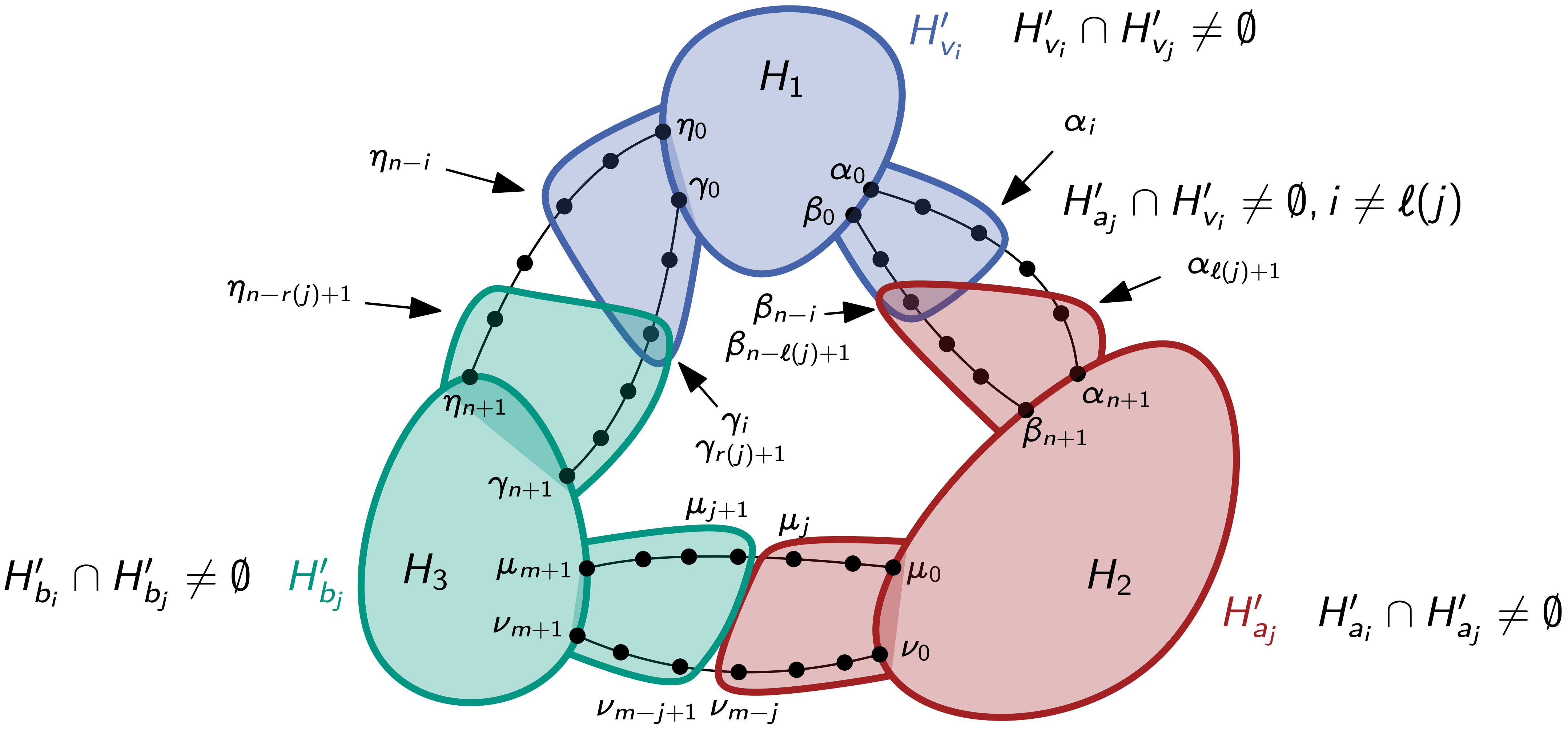

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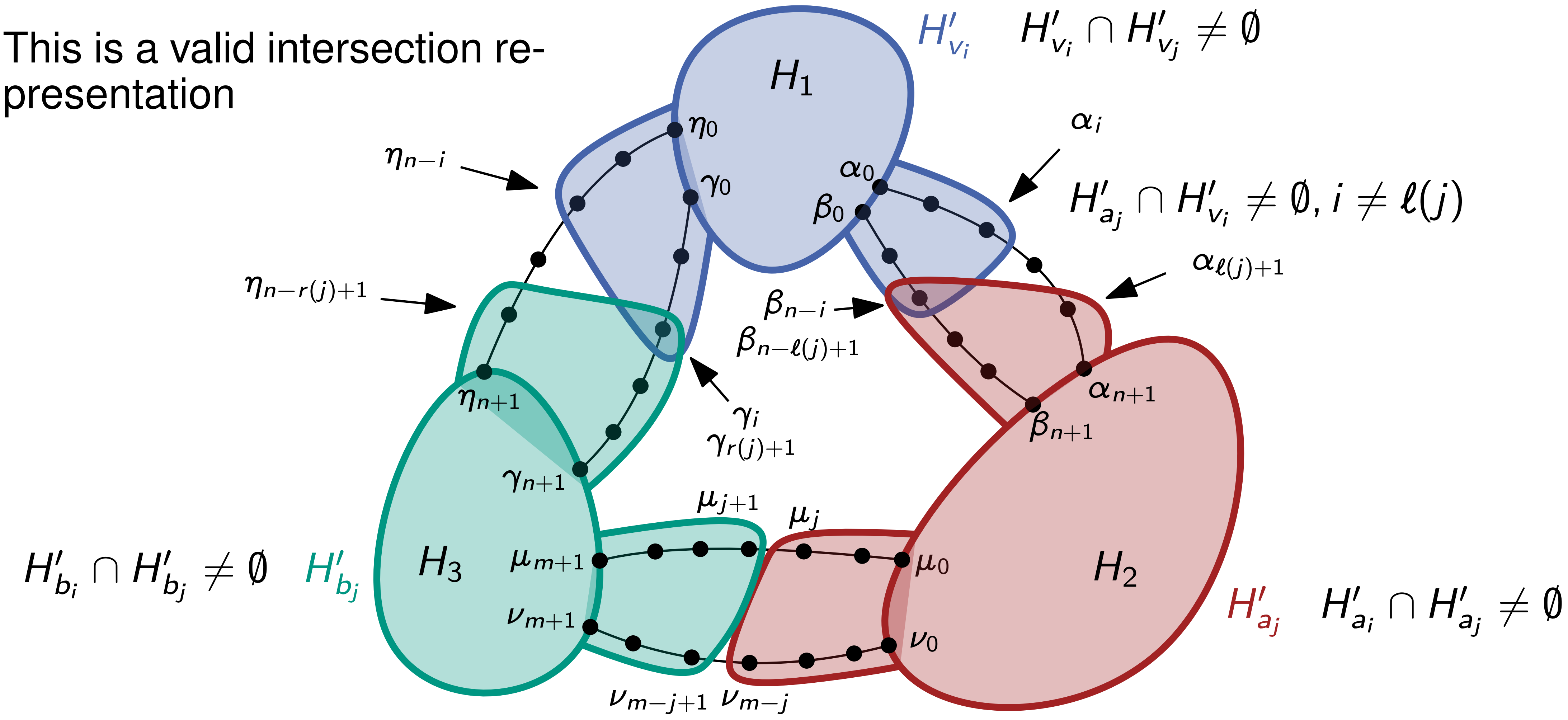
This is a valid intersection representation



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Corollary
Let H be a graph containing the double triangle as a minor. Then the clique problem is APX-hard on the class of all H-graphs and this class is also isomorphism-complete.

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What is the time complexity of the isomorphism problem on H-graphs, if $H = K_3$, that is the class of circular-arc graphs?

Thank you!

Questions?