Coupling Degrees in $G(n, p)$

- Consider a $G(n, p)$ containing vertices u and v
- We are interested in the probability $q = \Pr[\deg(v) = d \wedge \deg(u) = d] - \Pr[\deg(v) = d]^2$
- Consider $\underbrace{Y_1, Y_2 \sim \mathrm{Bin}(n-2, p), \text{ and } X_1, X_2 \sim \mathrm{Ber}(p)}_{\text{In a shared probability space and all independent}}$ and the coupling

$$\begin{tikzcd} \deg(v) \arrow[r, leftrightarrow, "\text{dependent}"'] \arrow[d, phantom, "\overset{d}{=}"] & \deg(v) \arrow[d, phantom, "\overset{d}{=}"] \\ X_1+Y_1 \arrow[r, leftrightarrow, "\text{independent}"] & X_2+Y_2 \end{tikzcd}$$

- Then $\Pr[\deg(v) = d] = \Pr[X_1 + Y_1 = d] = \Pr[X_2 + Y_2 = d]$
- Thus $q = \Pr[\deg(v) = d \wedge \deg(u) = d] - \Pr[X_1 + Y_1 = d] \Pr[X_2 + Y_2 = d]$
- Now define random variables as tuples of random variables: $Z = (\deg(v), \deg(u))$ and $Z' = (X_1 + Y_1, X_2 + Y_2)$, and consider the coupling

$$\begin{tikzcd} Z = (\deg(v), \deg(u)) \arrow[r, leftrightarrow, "\text{independent}"'] \arrow[d, phantom, "\overset{d}{=}"] & (X_1 + Y_1, X_2 + Y_2) = Z' \arrow[d, phantom, "\overset{d}{=}"] \\ (X_1+Y_1, X_1 + Y_2) \arrow[r, leftrightarrow, "\text{dependent}"] & (X_1 + Y_1, X_2 + Y_2) \end{tikzcd}$$

- Then $q = \Pr[X_1 + Y_1 = d \wedge X_1 + Y_2 = d] - \Pr[X_1 + Y_1 = d] \Pr[X_2 + Y_2 = d]$