Probability & Computing

Probability Amplification

The Segmentation Problem

Input

- Set P of points in a feature space (e.g., $\mathbb{R}^d$)
- Similarity measure $\sigma\colon P \times P \mapsto \mathbb{R}_+$

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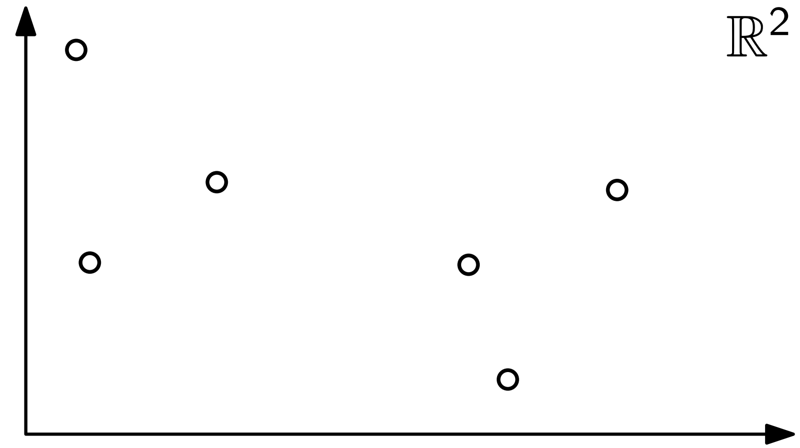

Example

- six points in $\mathbb{R}^2$
- σ is the inversed Euclidean distance

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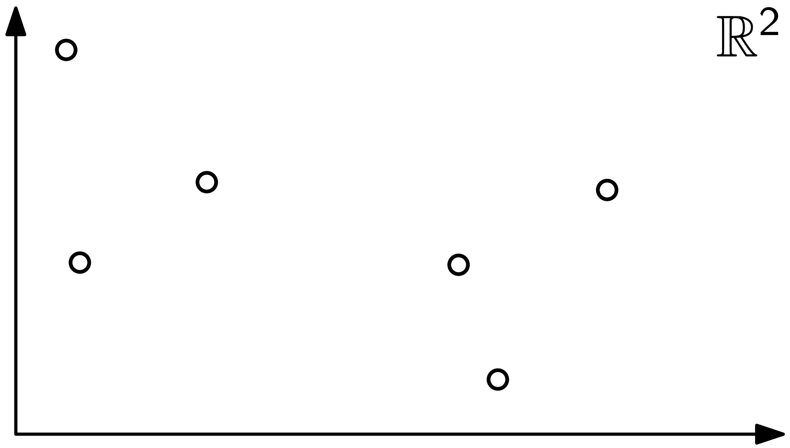
Output: $P_1, \ldots, P_k$ such that

- Points within a P_i have high similarity
- Points in distinct P_i, P_j have low similarity



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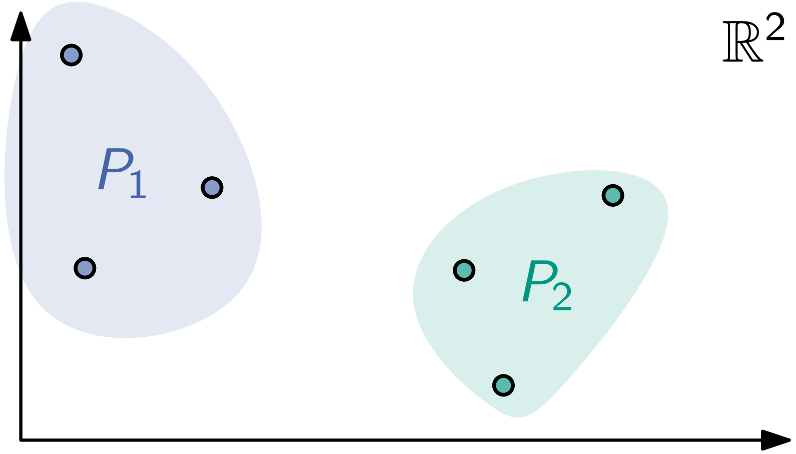

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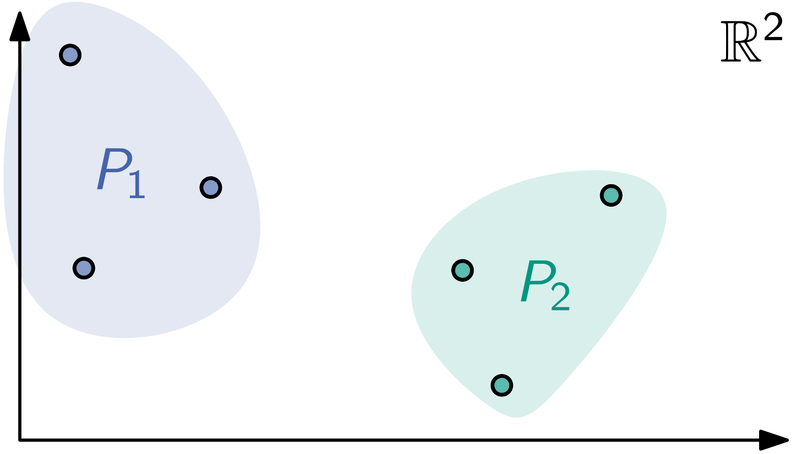

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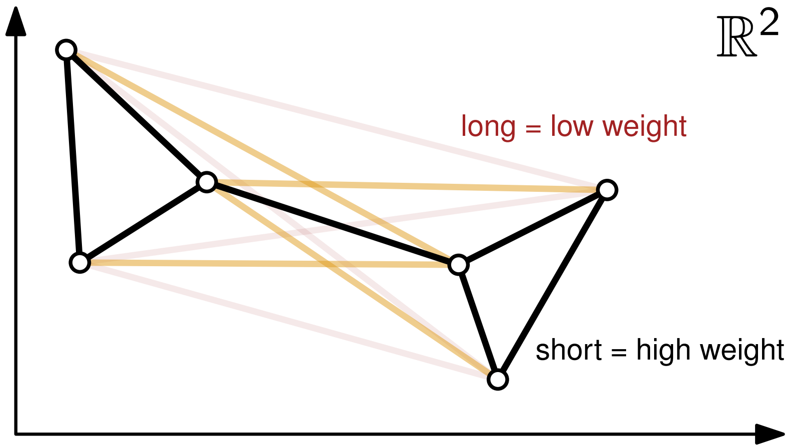
Approach: Model as graph

- Each point is a node
- Edges between all node pairs, with the weight given by the similarity of the two nodes



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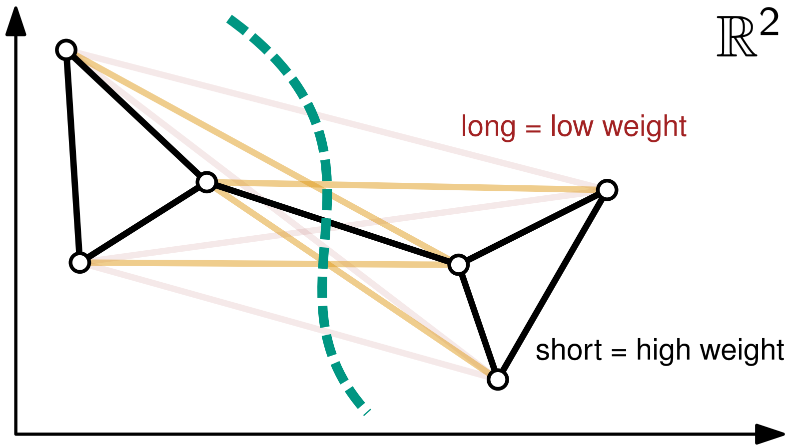

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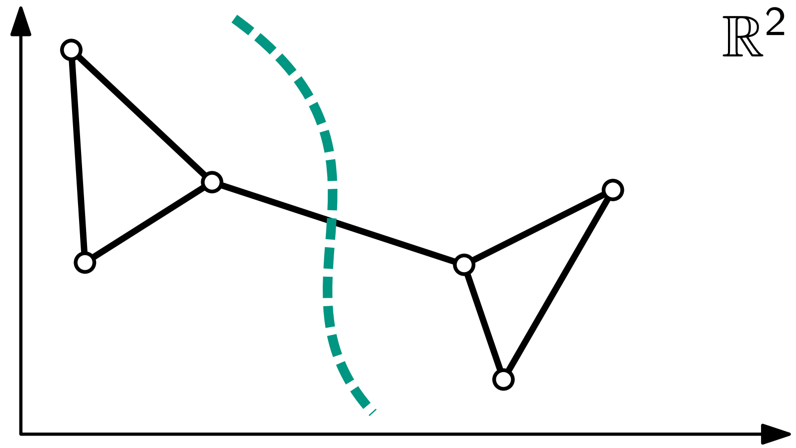

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Today

$k = 2$ and $\sigma\colon P \times P \mapsto \{0, 1\}$

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Cuts

- $G = (V, E)$ an unweighted, undirected, connected graph
- *Cut*: partition of V into parts V_1, V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$.
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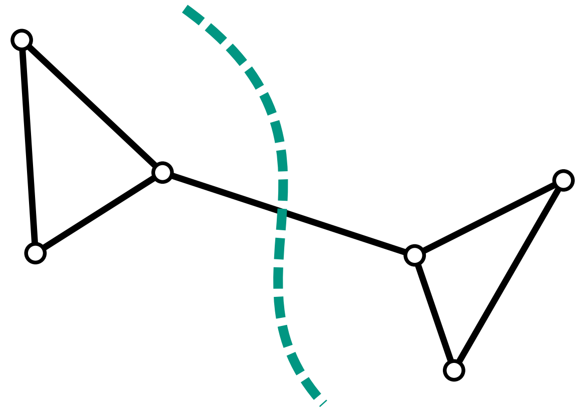

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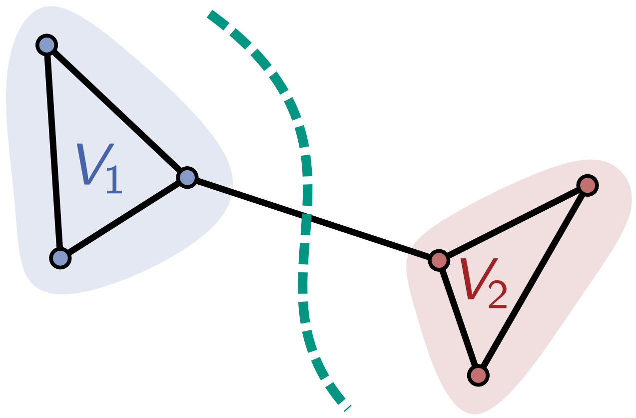

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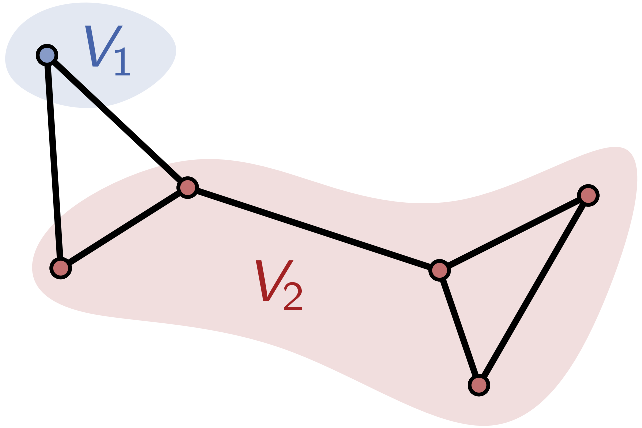

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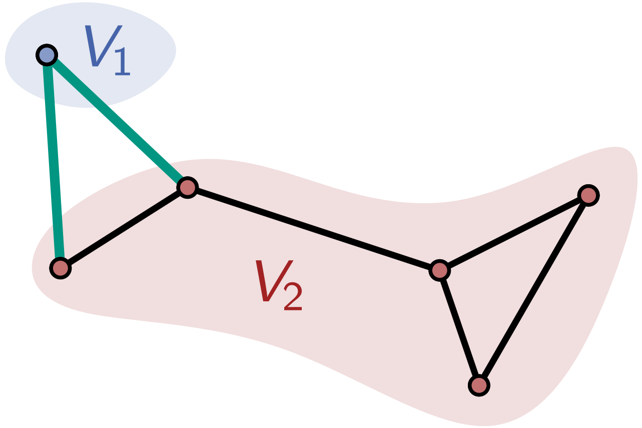

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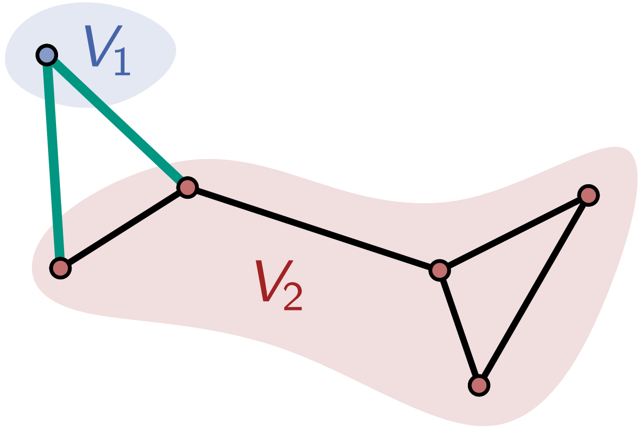

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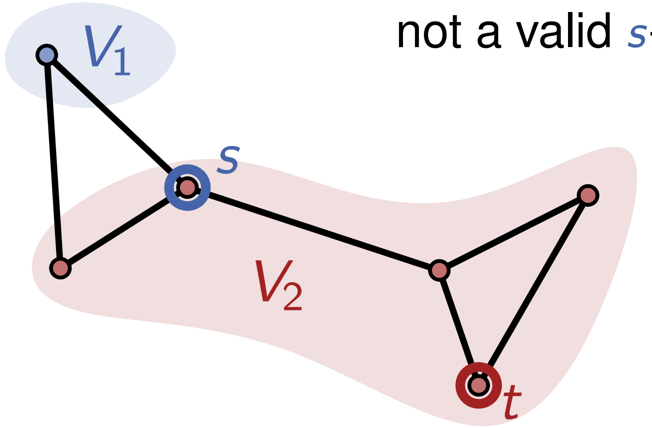

not a valid s-t-cut

Excursion: Cuts with Terminals

- each part contains exectly one of a specified vertex set

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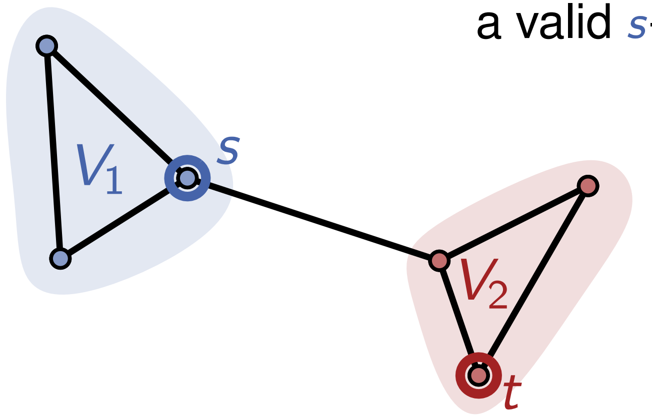

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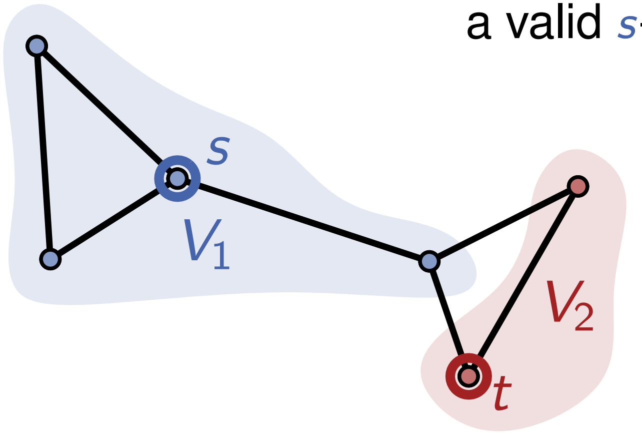

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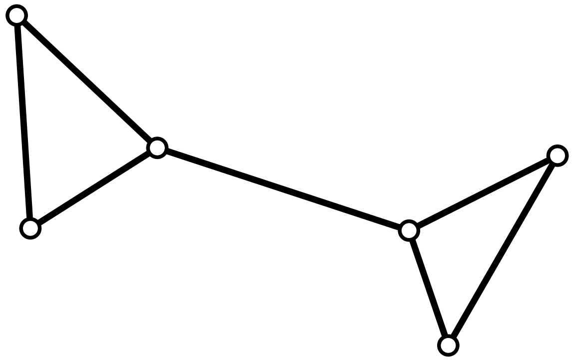

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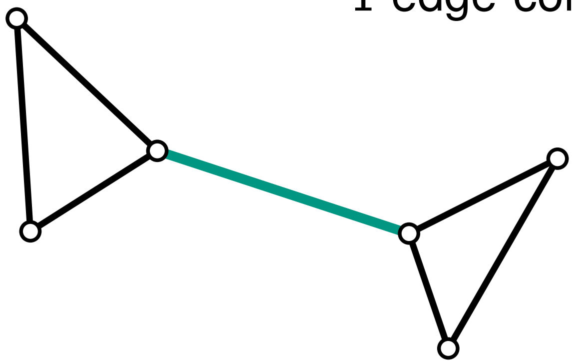

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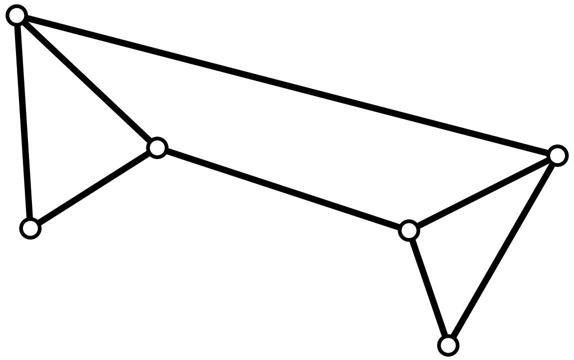

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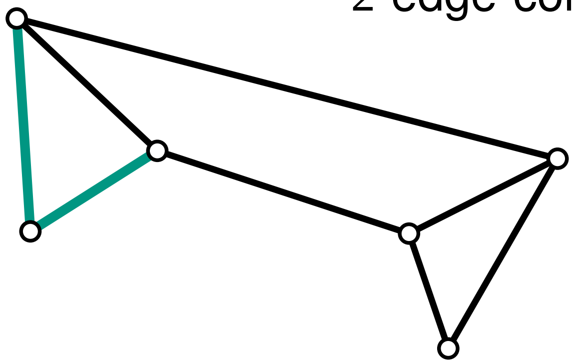

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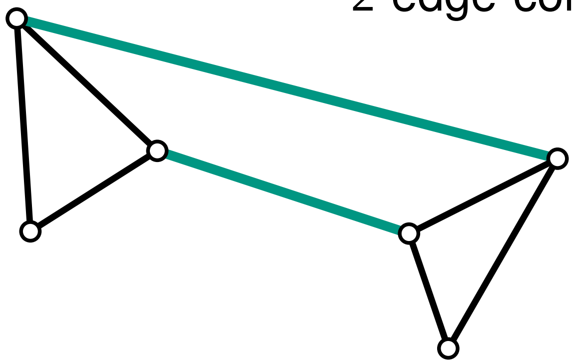
2-edge-connected



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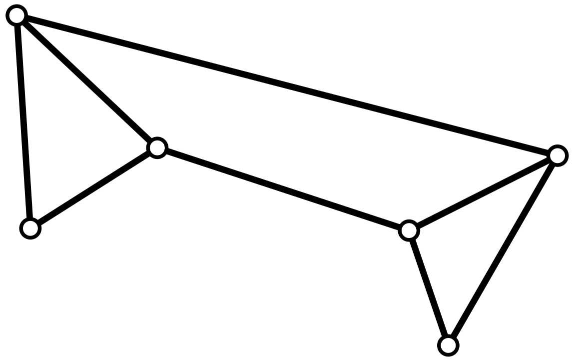

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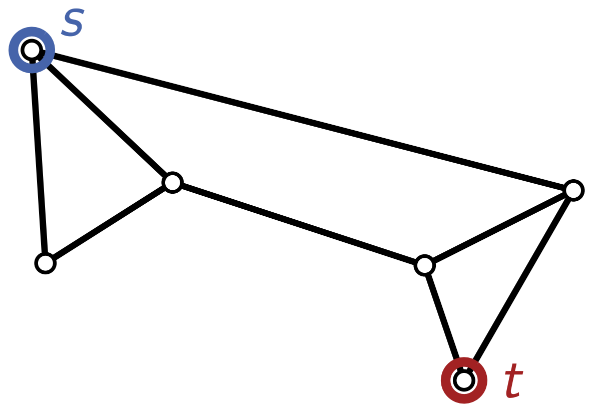

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- assign *flow* to edges s.t.
  - in-flow $=$ out-flow for all vertices (not s and t)
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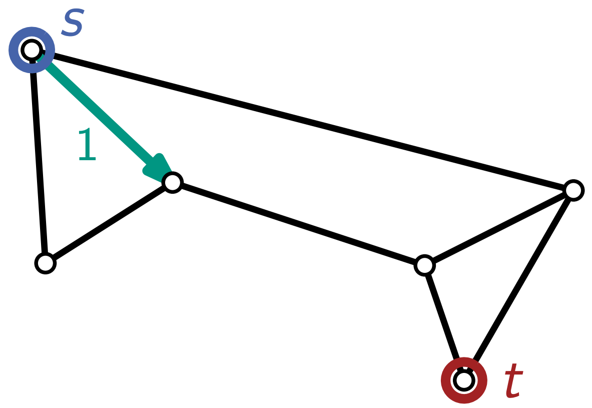

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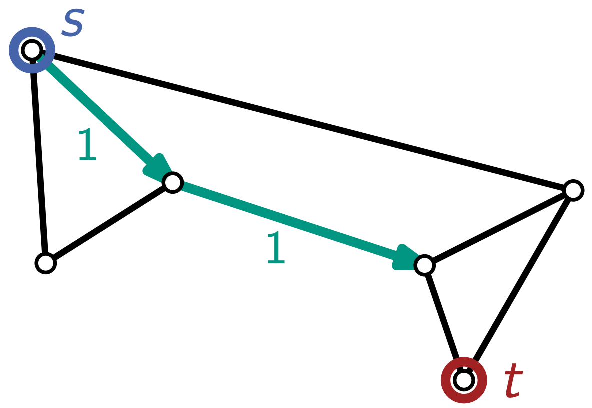

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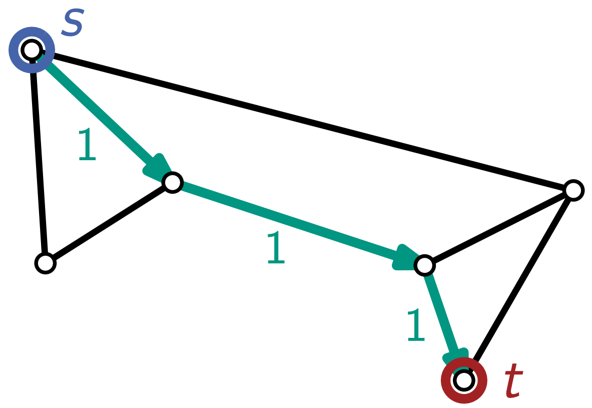

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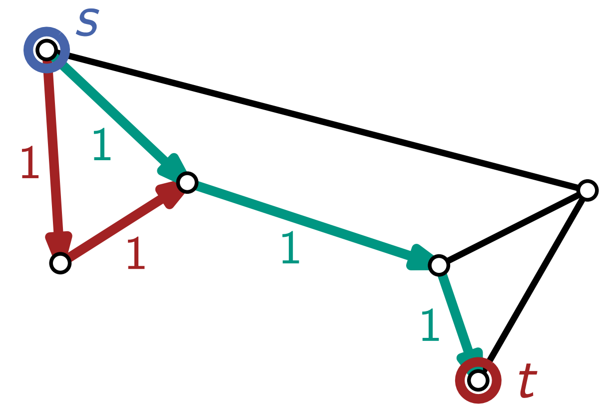

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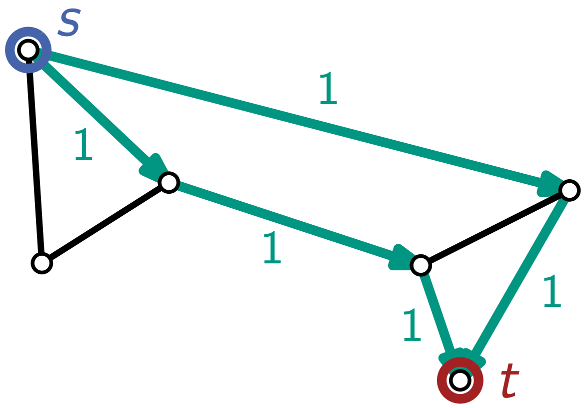

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The Edge-Connectivity Problem

Cuts

- $G = (V, E)$ an unweighted, undirected, connected graph
- *Cut*: partition of V into parts V_1, V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$. (in general one can consider more than two parts)
- *Cut-set*: set of edges with one endpoint in V_1 and the other in V_2
- *Weight*: size of the cut-set (or sum of weights in a weighted graph)

Excursion: Cuts with Terminals

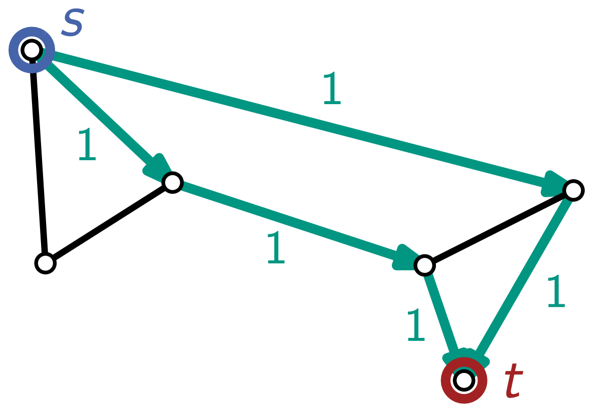
- each part contains exectly one of a specified vertex set

k-Edge-Connectivity

- *k-edge-connected*: a minimum cut has weight at least k (we cannot disconnect the graph by removing less than k edges)

Edge-Connectivity

- max. k such that G is k-edge-connected (exactly the weight of a min-cut)



Excursion: Flows

- given source s and target t
- assign *flow* to edges s.t.
  - in-flow $=$ out-flow for all vertices (not s and t)
  - flow of an edge bounded by edge-capacity (here: ≤ 1)
  - flow in t is maximized

Thm. Max-Flow = Min-Cut.

Deterministic Algorithms for Edge-Connectivity

Flow-based

- Compute max-flow between all vertex pairs $\rightarrow O(n^2 \cdot T_{\text{max-flow}})$

Deterministic Algorithms for Edge-Connectivity

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Enter: The Power of Randomness!

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111000

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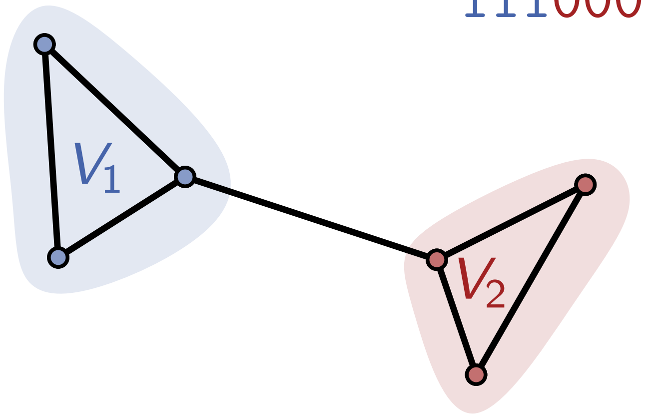

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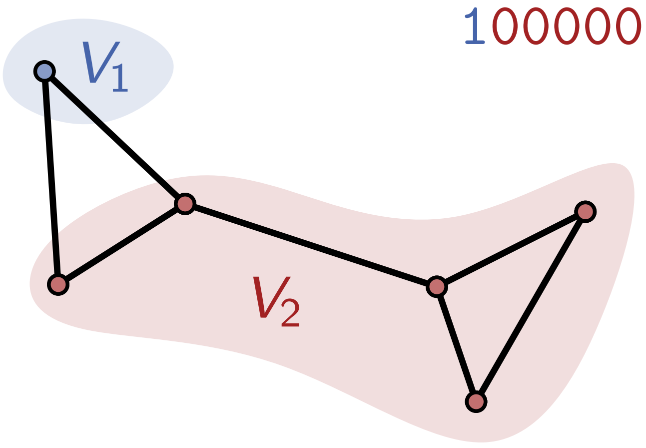

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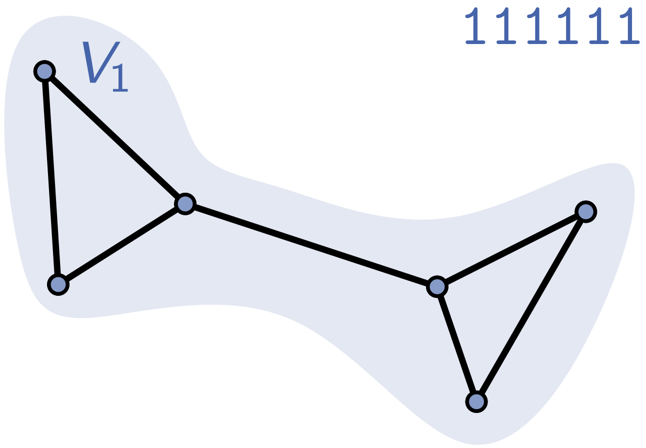

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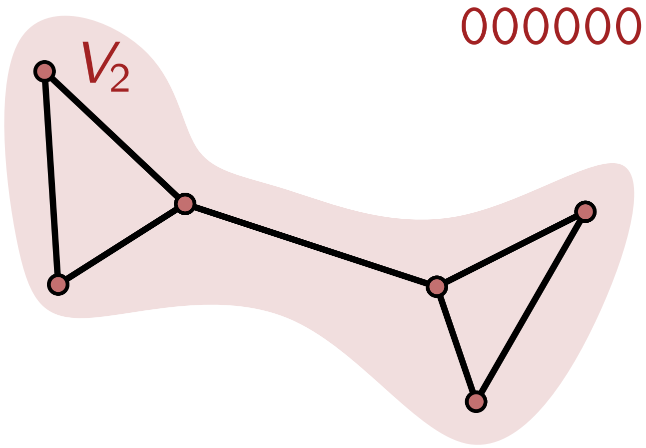

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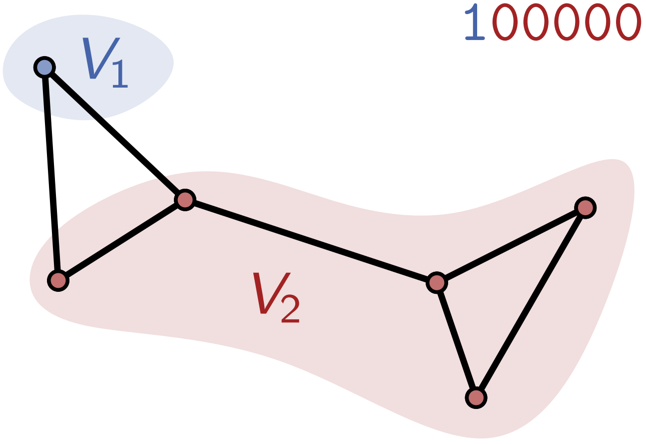
100000



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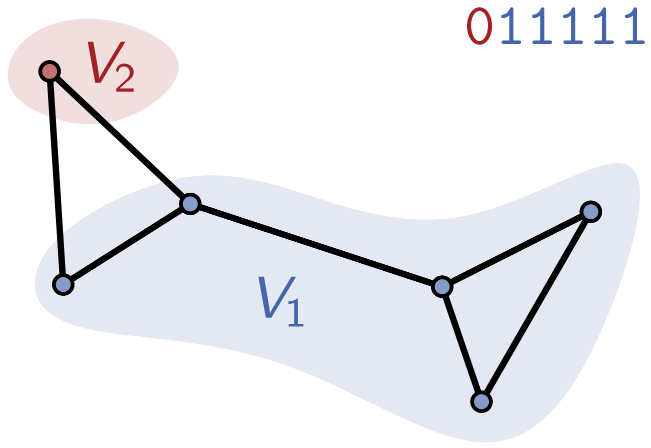

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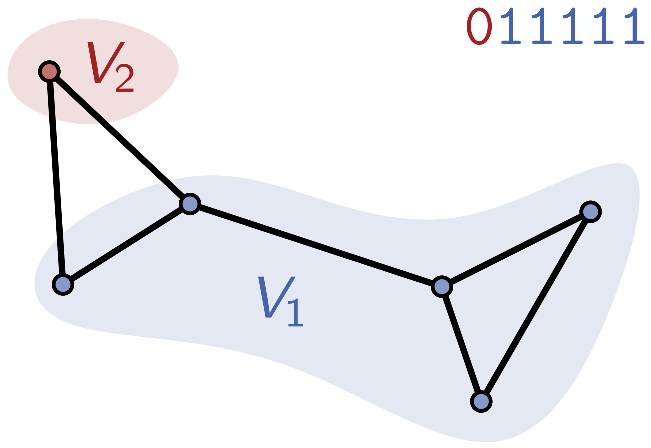

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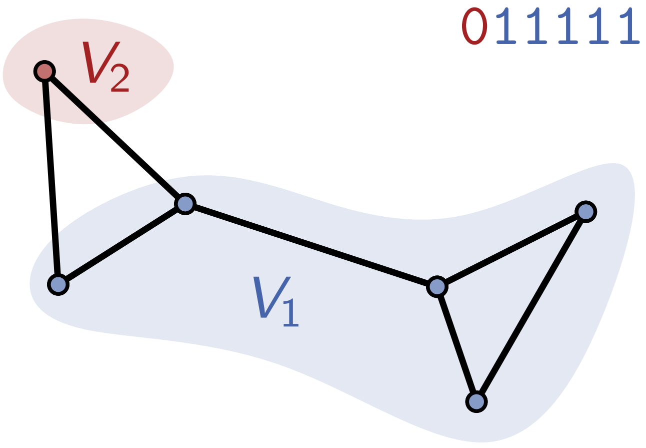

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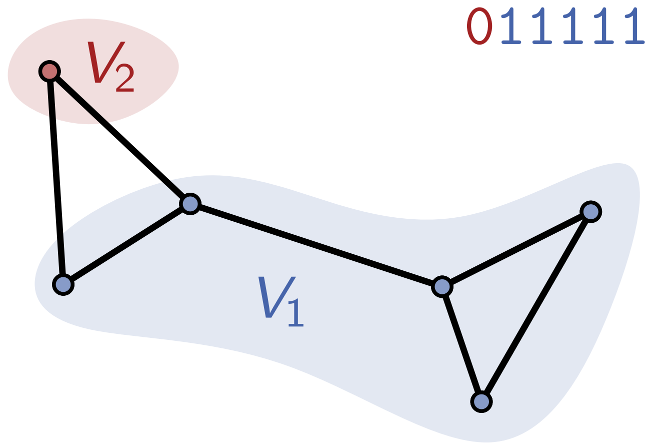

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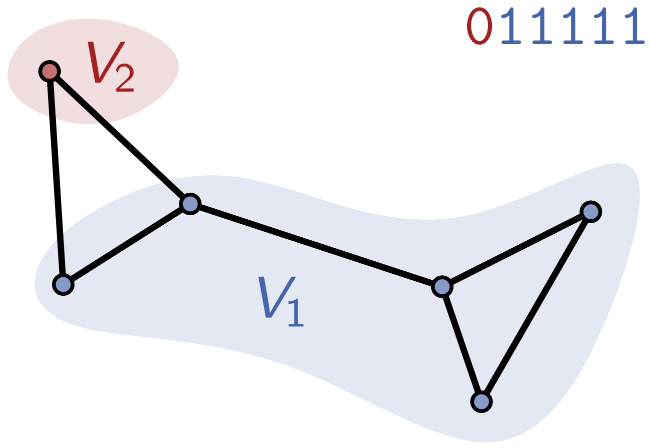

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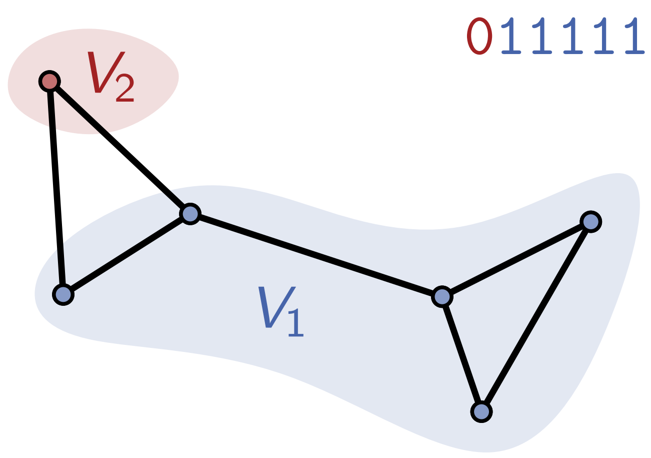

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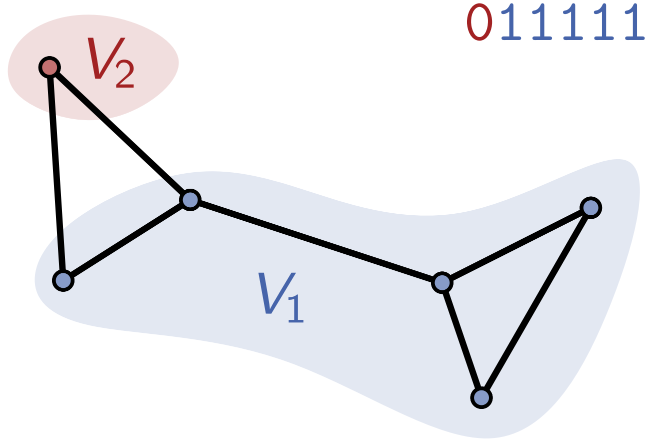

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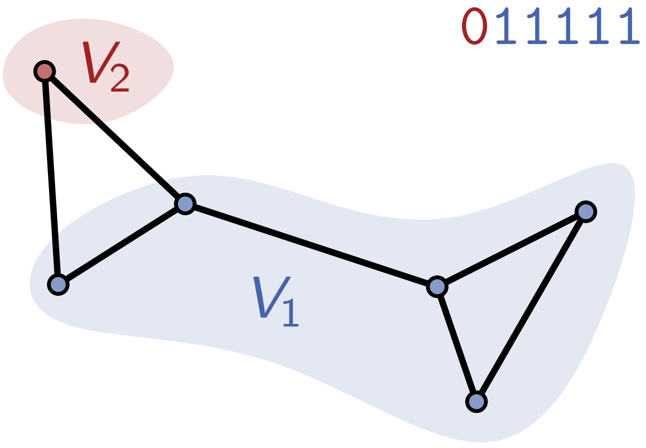

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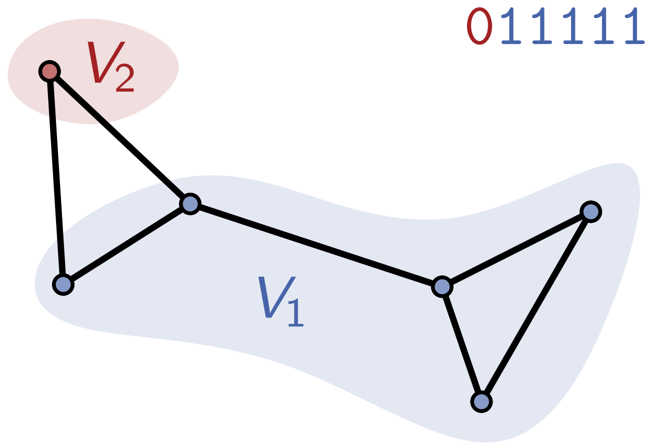

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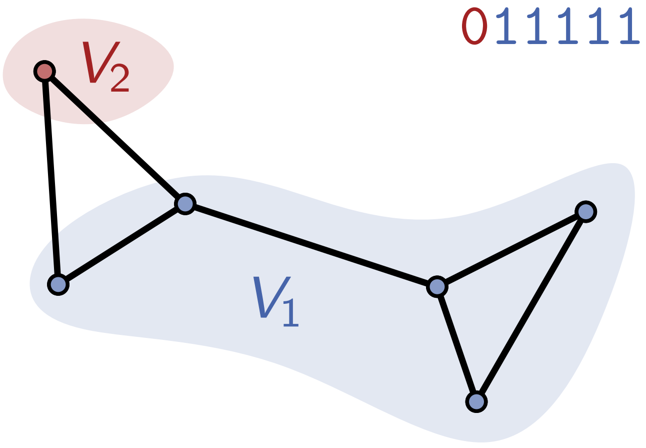

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random number from $\{1, \ldots, 2^n - 2\}$? $\rightarrow$ exponential in input size

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Excursion: Uniform Non-Identical Bit Strings

`[For educational purposes only!]`

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unibs(n)

$b := 00\ldots0$ // n zeros
$k :=$ **rand**($\{1, \ldots, n-1\}$) // number of 1s
$P :=$ **randSet**($\{1, \ldots, n\}, k$) // positions of 1s
$b[P] = 1$ // set 1s in b
return b

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Assumptions: We can sample . . .

- uniformly from $\{0, ..., O(n+m)\}$ in $O(1)$ time
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Not possible in theory. Reasonable in practice.

- 2-step process: choose k & choose k 1s in n bits

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  b := 00...0 // n zeros
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```

How to sample k?

- uniform?

$$\Pr[1000] = 1/3 \cdot 1/4 = 1/12$$

$$\Pr[1100] = 1/3 \cdot 1/6 = 1/18$$

$n = 4$, $2^n - 2$:

1000, 0100, 0010, 0001	$k = 1$
1100, 1010, 1001, 0110, 0101, 0011	$k = 2$
1110, 1101, 1011, 0111	$k = 3$

Excursion: Uniform Non-Identical Bit Strings

[For educational purposes only!]

- **Goal**: Choose uniformly at random from the length n bit-strings that are not 0^n or 1^n
- Number of valid bit-strings:

$$2^n - 2 = \left(\sum_{k=0}^{n} \binom{n}{k}\right) - 2 = \sum_{k=1}^{n-1} \binom{n}{k}$$

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How to sample k?
- uniform?

$$\left.\begin{aligned} \Pr[1000] &= 1/3 \cdot 1/4 = 1/12 \\ \Pr[1100] &= 1/3 \cdot 1/6 = 1/18 \end{aligned}\right\} \neq 1/14$$

$n = 4$

$$\underbrace{\left.\begin{array}{c}1000\\0100\\0010\\0001\end{array}\right\} k=1 \quad \left.\begin{array}{c}1100\\1010\\1001\\0110\\0101\\0011\end{array}\right\} k=2 \quad \left.\begin{array}{c}1110\\1101\\1011\\0111\end{array}\right\} k=3}_{2^n - 2}$$

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- choose k with prob $\binom{n}{k}/(2^n - 2)$

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- choose k with prob $\binom{n}{k}/(2^n - 2)$
- Reduce to uniform using *Inverse Transform Sampling*

$n = 4$

$$\left.\begin{matrix}1000\\0100\\0010\\0001\end{matrix}\right\} k = 1 \qquad \left.\begin{matrix}1100\\1010\\1001\\0110\\0101\\0011\end{matrix}\right\} k = 2 \qquad \left.\begin{matrix}1110\\1101\\1011\\0111\end{matrix}\right\} k = 3$$

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How to sample P?

$n = 4$

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Excursion-Excursion: Reservoir Sampling

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```
randSet({1, ..., n}, k)
  r := [1, ..., k] // reservoir
  for i from k + 1 to n do
    j := unif({1, ..., i})
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randSet($\{1, \dots, n\}, k$)
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1	2	3	4	5	6	7	8

1	2	3

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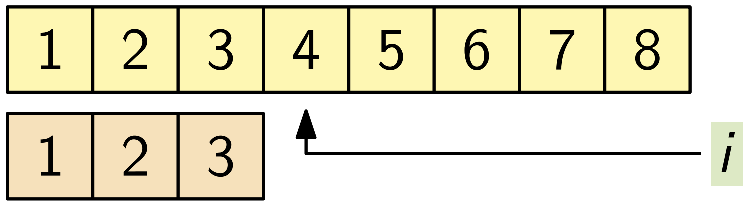

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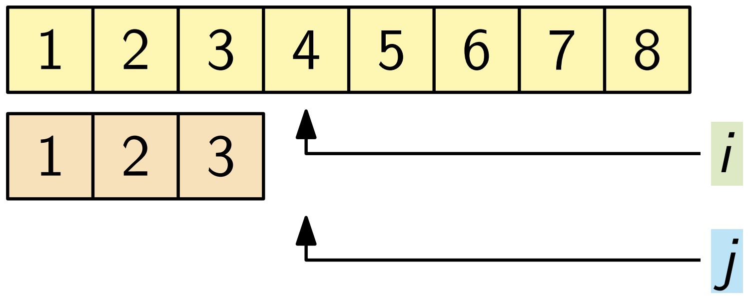

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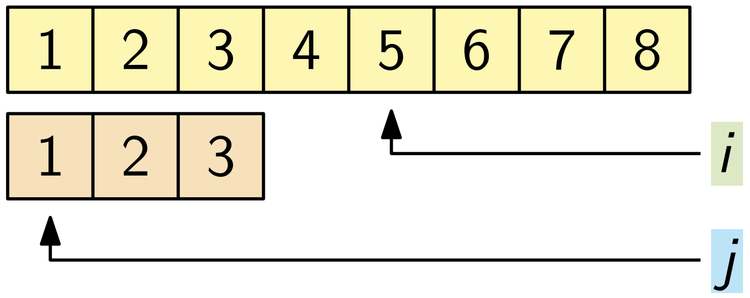

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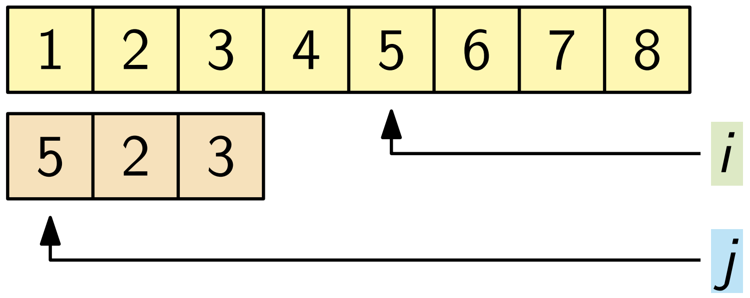

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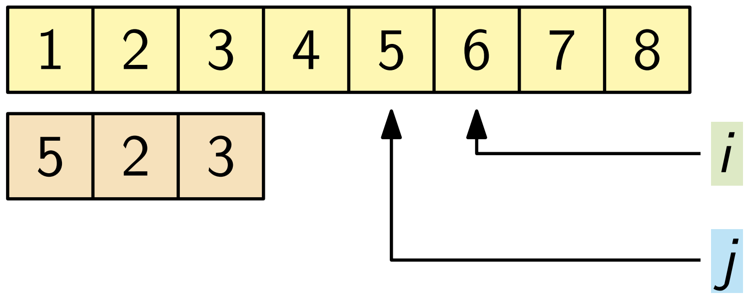

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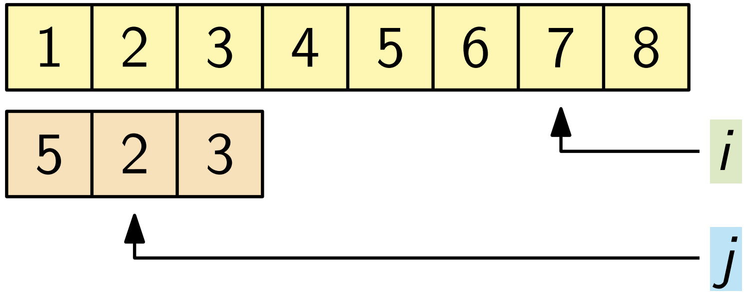

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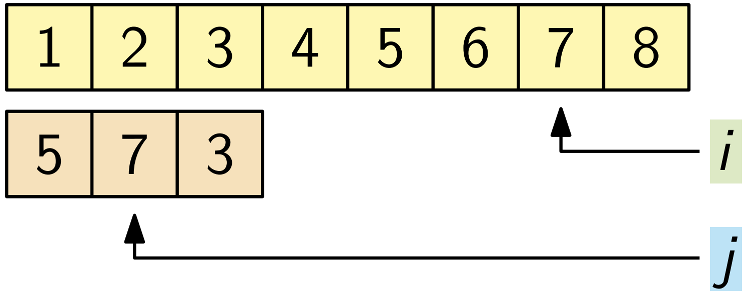

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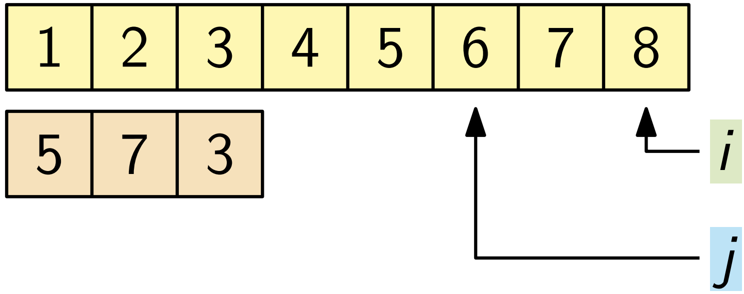

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$r := [1, \ldots, k]$ // reservoir // $O(k)$
for i from $k+1$ to n **do** // $O(n-k)$
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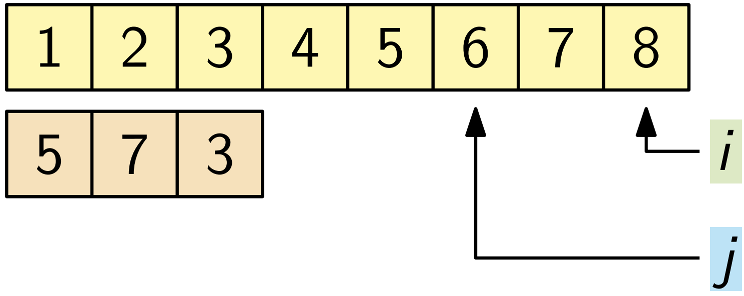

- **Running time**: $O(n)$

Excursion: Uniform Non-Homogeneous Bit Strings

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unibs(n)

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$b := 00\ldots0$ // n zeros // $O(n)$

$k :=$ **rand**$(\{1, \ldots, n-1\})$ // number of 1s // $O(\log(n))$ via Inverse Transform Sampling

$P :=$ **randSet**$(\{1, \ldots, n\}, k)$ // positions of 1s // $O(n)$ via Reservoir Sampling

$b[P] = 1$ // set 1s in b // $O(k) \subseteq O(n)$

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k := rand({1, ..., n − 1}) // number of 1s              // O(log(n)) via Inverse Transform Sampling
P := randSet({1, ..., n}, k) // positions of 1s         // O(n) via Reservoir Sampling
b[P] = 1 // set 1s in b                                 // O(k) ⊆ O(n)
return b
```

Under our assumptions, we can sample a length n bit string that is not 0^n or 1^n uniformly at random in time $O(n)$.

Simple Randomized Cut

Karlsruhe Institute of Technology

- Simple idea: choose a cut uniformly at random among all possible cuts and return it.
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- How many min-cuts? $\rightarrow$ pessimistic assumption: 1

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Amplification

- Repeat the algorithm to obtain t independent random cuts, return the smallest

  $\Pr[\text{“minimum found”}] \geq 1 - \left(1 - 1/(2^{n-1} - 1)\right)^t$

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$$\Pr[\text{“minimum found”}] \geq 1 - \Big(\underbrace{1 -}_{\text{not}} \underbrace{1/(2^{n-1} - 1)}_{\text{minimum}}\Big)^t$$

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- $2^{n-1} - 1$ cuts in a graph with n nodes
- How many min-cuts? $\rightarrow$ pessimistic assumption: 1

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Amplification

- Repeat the algorithm to obtain t independent random cuts, return the smallest

$$\Pr[\text{“minimum found”}] \geq 1 - \big(\underbrace{1 -}_{\text{not}} \underbrace{1/(2^{n-1} - 1)}_{\text{minimum}}\big)^{\overset{t}{\downarrow\, t \text{ times}}}$$

Simple Randomized Cut

- Simple idea: choose a cut uniformly at random among all possible cuts and return it.
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$1 + x \leq e^x$ for $x \in \mathbb{R}$

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Probability Amplification

Definition: A **Monte Carlo Algorithm** is a randomized algorithm that terminates deterministically and whose output is correct only with a certain probability $p \in (0, 1)$.

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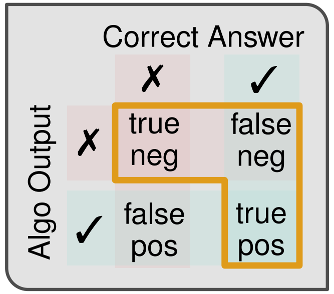

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Karlsruhe Institute of Technology

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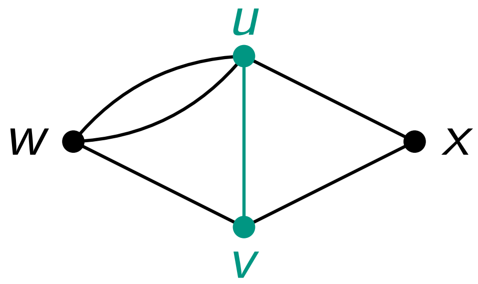
For Simple Randomized Cut we had to pay with exponentially large running time . . .

Karger’s Algorithm

Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops



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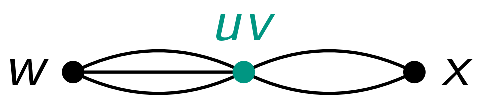

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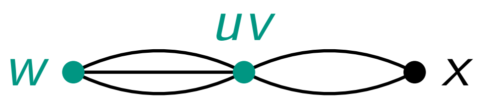

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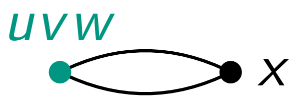

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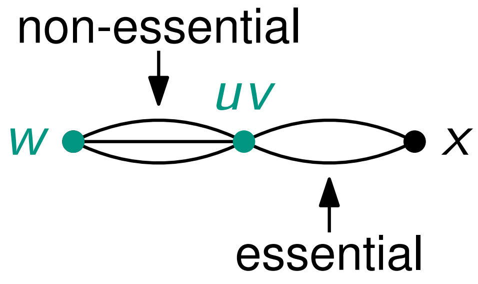

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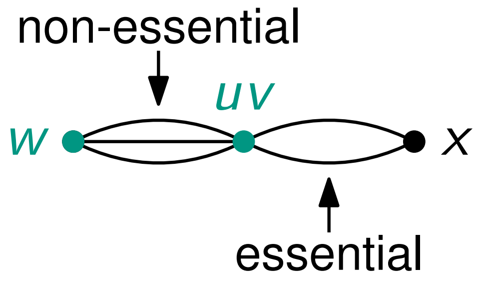

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> **for** $i = 1$ to $n - 2$ **do**
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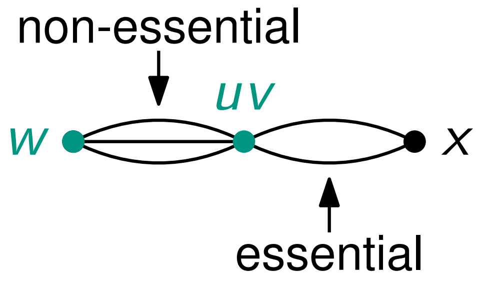

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- Running time in $O(n^2)$
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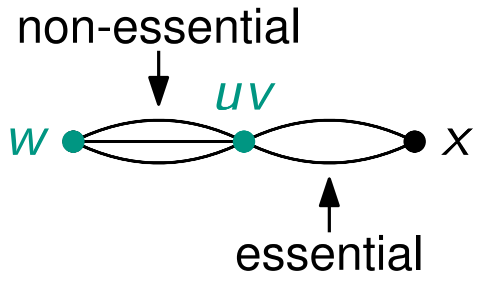

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Success Probability

Observation: A cut in G_i is a cut in G_0.

Because endpoints of removed edges (self-loops) are within the same part in a cut in G_i.

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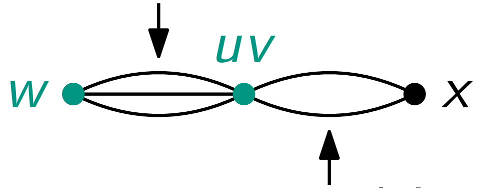

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```
for i = 1 to n − 2 do              // O(n)
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Success Probability

Observation: A cut in G_i is a cut in G_0.

- Consider min-cut with cut set C and $|C| = k$

Karger's Algorithm

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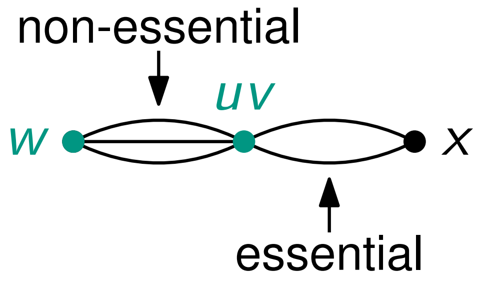

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Success Probability

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- Consider min-cut with cut set C and $|C| = k$
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$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m}$

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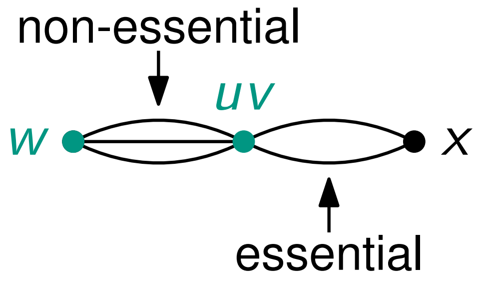

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o.w. ↯

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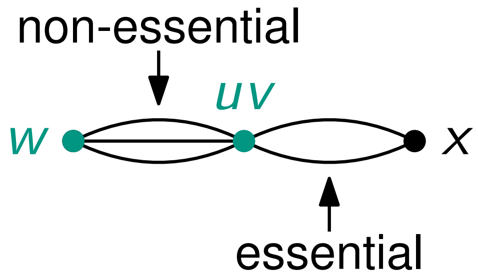
Observation: min-degree $\geq k$

$\Pr[\mathcal{E}_1] = 1 - \frac{k}{m}$

Karger’s Algorithm

Edge Contraction



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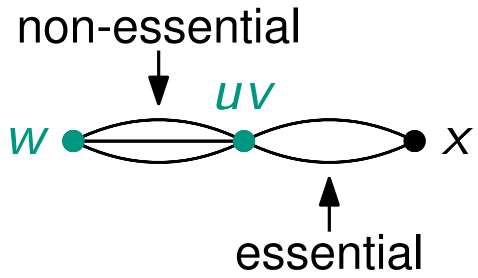

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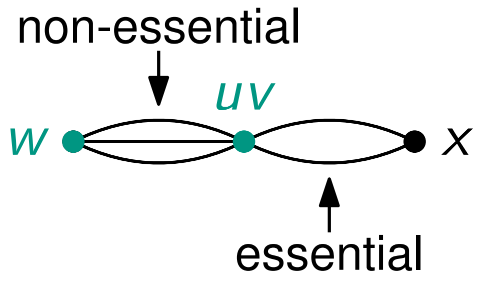

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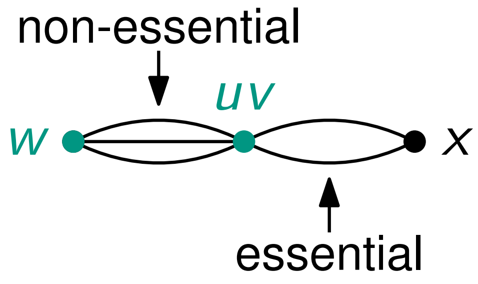

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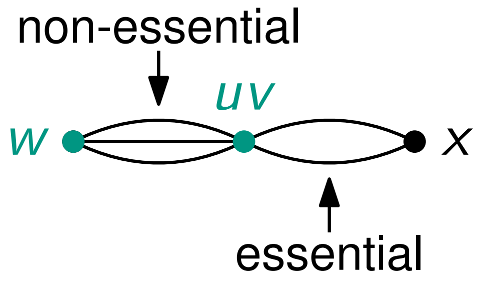

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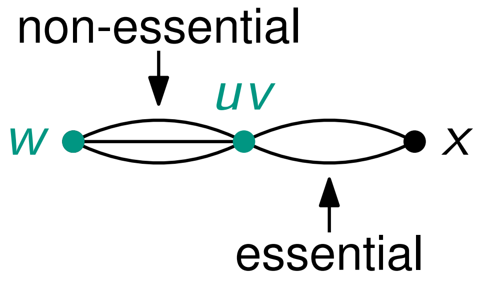
none of the k edges of C contracted

do not contract k edges in an $n - 1$-node graph

Karger's Algorithm

Edge Contraction

- Merge two adjacent nodes in a multigraph without self-loops
- A (multi) graph with two nodes has a unique cut



Contraction Algorithm

- Motivation: distinguish $\overbrace{\textit{non-essential}}^{\text{not part of a min-cut}}$ as well as *essential* edges } part of a min-cut
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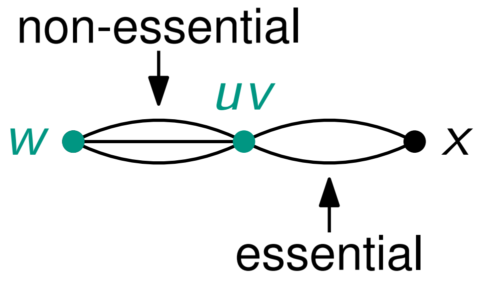

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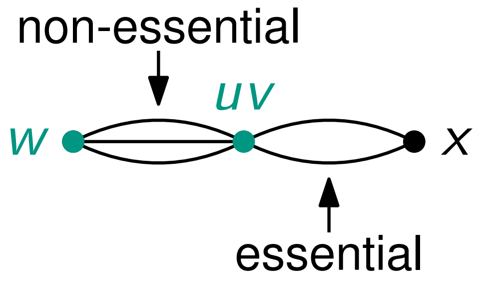

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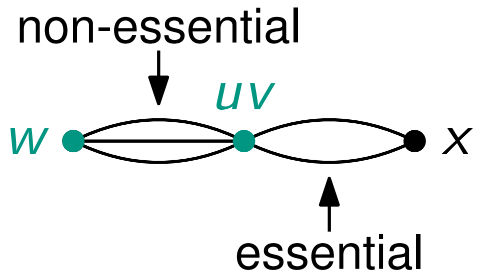

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$1 = \frac{n-i}{n-i}$

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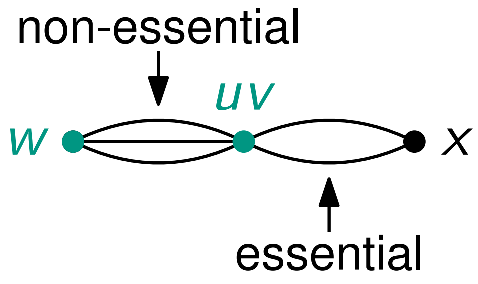

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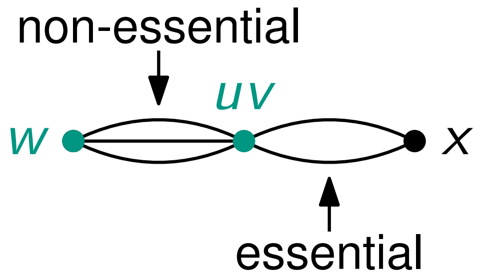

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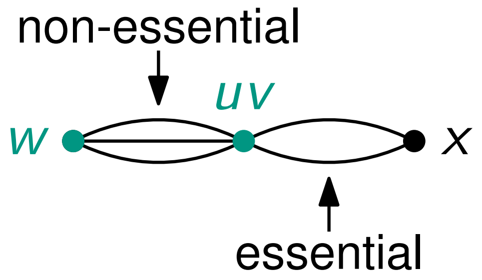

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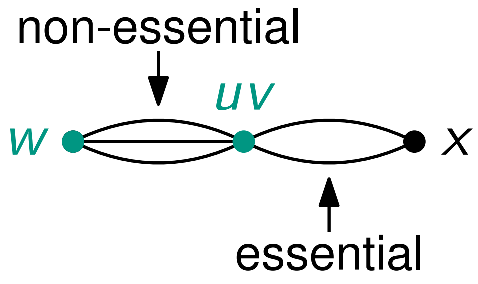

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```
for i = 1 to n − 2 do            // O(n)
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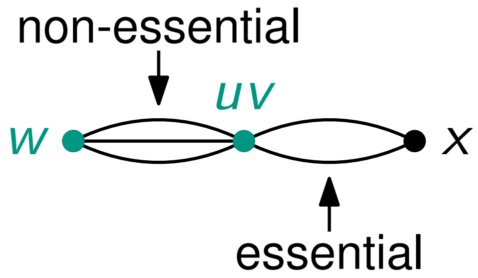

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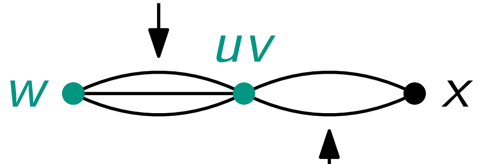

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$$\geq \frac{2}{n\,(n-1)}$$

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Theorem: On a graph with n nodes, Karger's algorithm runs in $O(n^2)$ time and returns a minimum cut with probability at least $2/(n(n-1))$.

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for $t = \frac{n(n-1)}{2} \log(n)$

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Amplified prob. $\geq 1 - e^{-pt}$

Corollary: On a graph with n nodes, $O(n^2 \log(n))$ Karger repetitions run in $O(n^4 \log(n))$ total time and return a min-cut with high probability. Much better than exp. time Simple Randomized Cut!

Sidenote: Number of minimum cuts

- Let $C_1, \ldots, C_\ell$ be all the min-cuts in G and $\underbrace{\mathcal{E}^i_{n-2}}_{\text{disjoint, since the algorithm returns only one cut}}$ for $i \in [\ell]$ be the event that C_i is returned by Karger's algorithm
- Just seen: $\Pr[\mathcal{E}^i_{n-2}] \geq \frac{2}{n(n-1)}$

Karger's Algorithm Amplified

Karlsruhe Institute of Technology

Theorem: On a graph with n nodes, Karger's algorithm runs in $O(n^2)$ time and returns a minimum cut with probability at least $2/(n(n-1))$.

$$\Pr[\text{“min-cut found”}] \geq 1 - \exp\left(-\frac{2t}{n(n-1)}\right) = 1 - \frac{1}{n}$$

for $t = \frac{n(n-1)}{2}\log(n)$

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Observation: A graph on n nodes contains at most $\frac{n(n-1)}{2}$ minimum cuts.

More Amplification: Karger-Stein

Karlsruhe Institute of Technology

Motivation

- Probability that a min-cut survives i contractions

$$\Pr[\mathcal{E}_i] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}]$$

$$\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right) \cdots \left(1 - \frac{2}{n-i+2}\right)\left(1 - \frac{2}{n-i+1}\right)$$

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$$= \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-4}{n-2}\right) \cdots \left(\frac{n-i+2-2}{n-i+2}\right)\left(\frac{n-i+1-2}{n-i+1}\right)$$

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$$= \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right) \cdots \left(\frac{n-i}{\cancel{n-i+2}}\right)\left(\frac{n-i-1}{\cancel{n-i+1}}\right)$$

$$= \frac{(n-i)(n-i-1)}{n(n-1)}$$

More Amplification: Karger-Stein

Karlsruhe Institute of Technology

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- With increasing number of steps the probability for a min-cut to survive decreases

More Amplification: Karger-Stein

Motivation

- Probability that a min-cut survives *i* contractions

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- With increasing number of steps the probability for a min-cut to survive decreases
- Idea: stop when a min-cut is still likely to exist and recurse

More Amplification: Karger-Stein

Karlsruhe Institute of Technology

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- With increasing number of steps the probability for a min-cut to survive decreases
- Idea: stop when a min-cut is still likely to exist and recurse
- After $t = n - n/\sqrt{2} - 1$ steps we have

$$\Pr[\mathcal{E}_t] = \frac{(n - (n - n/\sqrt{2} - 1))(n - (n - n/\sqrt{2} - 1) - 1)}{n(n-1)}$$

More Amplification: Karger-Stein

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- Probability that a min-cut survives i contractions

$$\begin{aligned}
\Pr[\mathcal{E}_i] &= \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}] \\
&\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right)\cdots\left(1 - \frac{2}{n-i+2}\right)\left(1 - \frac{2}{n-i+1}\right) \\
&= \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right)\cdots\left(\frac{n-i}{\cancel{n-i+2}}\right)\left(\frac{n-i-1}{\cancel{n-i+1}}\right) \\
&= \frac{(n - i)(n - i - 1)}{n(n-1)}
\end{aligned}$$

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$$= \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right)\cdots\left(\frac{n-i}{\cancel{n-i+2}}\right)\left(\frac{n-i-1}{\cancel{n-i+1}}\right)$$

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More Amplification: Karger-Stein

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$$\Pr[\mathcal{E}_i] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 \mid \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_i \mid \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}]$$

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- Probability that no mistake made after t steps still large

More Amplification: Karger-Stein

Motivation

- Probability that a min-cut survives i contractions

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- With increasing number of steps the probability for a min-cut to survive decreases

KargerStein$(G_0 = (V_0, E_0))$
- **if** $|V_0| = 2$ **then** return unique cut
- **for** $i = 1$ to $t = |V_0| - \frac{|V_0|}{\sqrt{2}} - 1$ **do**
  - $e :=$ **unif**(E_{i-1})
  - $G_i = G_{i-1}$.**contract**(e)
- $C_1 :=$ **KargerStein**(G_t) // inde-
- $C_2 :=$ **KargerStein**(G_t) // pendent // runs
- **return** smaller of C_1, C_2

- Idea: stop when a min-cut is still likely to exist and recurse
- After $t = n - n/\sqrt{2} - 1$ steps we have

$$\Pr[\mathcal{E}_t] = \frac{(\cancel{n-n} + n/\sqrt{2} + 1)(\cancel{n-n} + n/\sqrt{2} + \cancel{1-1})}{n(n-1)} = \frac{n^2/2 + n/\sqrt{2}}{n(n-1)} = \frac{\cancel{n}(n/2 + 1/\sqrt{2})}{\cancel{n}(n-1)} = \frac{1}{2} \cdot \underbrace{\frac{n+\sqrt{2}}{n-1}}_{\geq 1} \geq \frac{1}{2}$$

- Probability that no mistake made after t steps still large

Karger-Stein: Running Time

KargerStein$(G_0 = (V_0, E_0))$

	KargerStein$(G_0 = (V_0, E_0))$
// $O(1)$	**if** $\|V_0\| = 2$ **then** return unique cut
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// $O(n)$	$G_i = G_{i-1}.$**contract**(e)
	$C_1 :=$ **KargerStein**(G_t) // inde-
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	return smaller of C_1, C_2

Karger-Stein: Running Time

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Recursion

- After $t = n - n/\sqrt{2} - 1$ steps the number of nodes is $n/\sqrt{2} + 1$

$$T(n) = 2T\left(\frac{n}{\sqrt{2}} + 1\right) + O(n^2)$$

```
          KargerStein(G_0 = (V_0, E_0))
// O(1)     if |V_0| = 2 then return unique cut
// O(n)     for i = 1 to t = |V_0| - |V_0|/√2 - 1 do
// O(1)       e := unif(E_{i-1})
// O(n)       G_i = G_{i-1}.contract(e)
            C_1 := KargerStein(G_t)   // inde-
            C_2 := KargerStein(G_t)   // pendent
                                      // runs
            return smaller of C_1, C_2
```

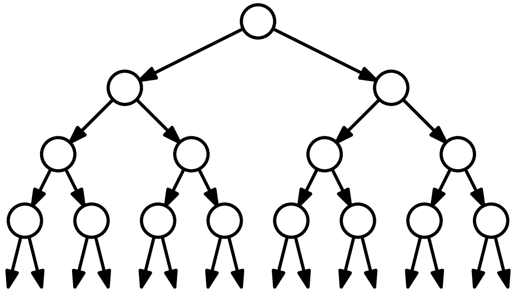
Karger-Stein: Running Time

Recursion

- After $t = n - n/\sqrt{2} - 1$ steps the number of nodes is $n/\sqrt{2} + 1$

$$T(n) = 2T\left(\frac{n}{\sqrt{2}} + 1\right) + O(n^2)$$

Recursion tree



KargerStein$(G_0 = (V_0, E_0))$

- **if** $|V_0| = 2$ **then** return unique cut — // $O(1)$
- **for** $i = 1$ to $t = |V_0| - \frac{|V_0|}{\sqrt{2}} - 1$ **do** — // $O(n)$
  - $e :=$ **unif**(E_{i-1}) — // $O(1)$
  - $G_i = G_{i-1}$.**contract**(e) — // $O(n)$
- $C_1 :=$ **KargerStein**(G_t) — // inde-
- $C_2 :=$ **KargerStein**(G_t) — // pendent // runs
- **return** smaller of C_1, C_2

Karger-Stein: Running Time

Karlsruhe Institute of Technology

Recursion

- After $t = n - n/\sqrt{2} - 1$ steps the number of nodes is $n/\sqrt{2} + 1$

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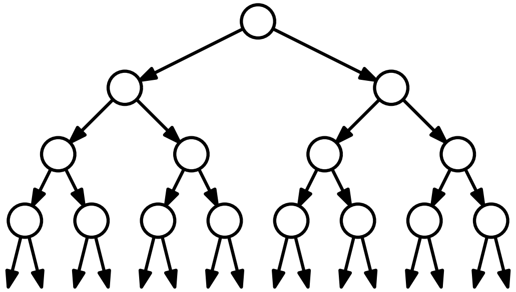
Recursion tree

- Layers: $\log_{\sqrt{2}}(n)$



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Karger-Stein: Running Time

Recursion

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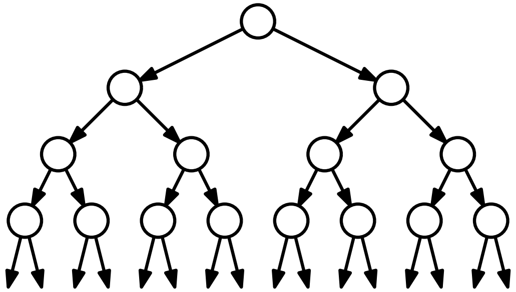
Recursion tree

- Layers: $\log_{\sqrt{2}}(n)$
- Nodes on layer j: 2^j



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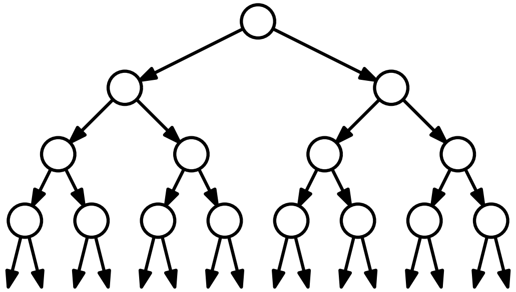

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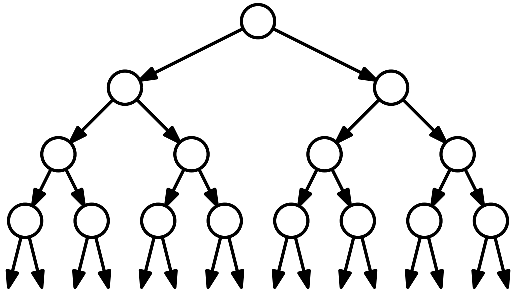

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return smaller of C_1, C_2

$$T(n) = \sum_{j=1}^{\log_{\sqrt{2}}(n)} 2^j \cdot O\left(\left(\frac{n}{\sqrt{2}^j}\right)^2\right) = O\left(n^2 \cdot \sum_{j=1}^{\log_{\sqrt{2}}(n)} \frac{2^j}{2^j}\right) = O(n^2 \log_{\sqrt{2}}(n)) = O(n^2 \log(n))$$

Karger-Stein: Success Probability

- After $t = n - n/\sqrt{2} - 1$ steps we have $\Pr[\mathcal{E}_t] \geq 1/2$ (t was chosen to achieve exactly that)

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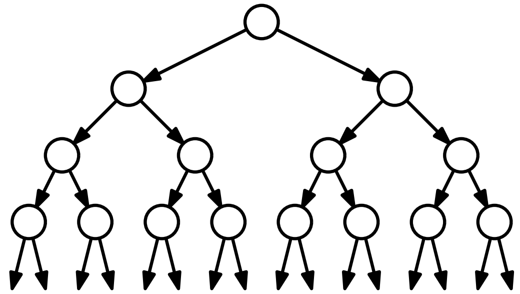

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Recursion tree

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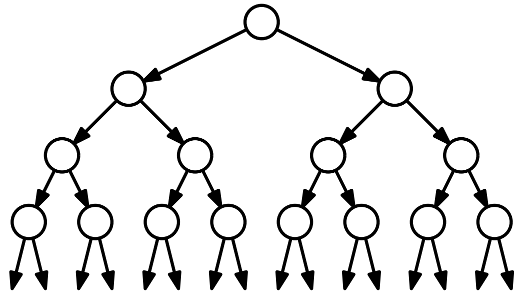

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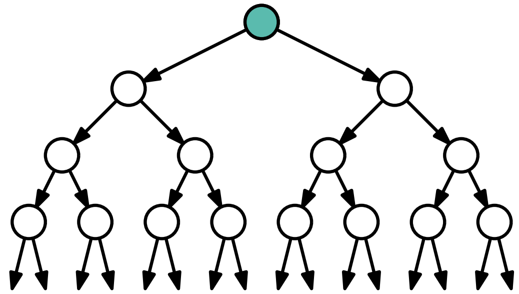

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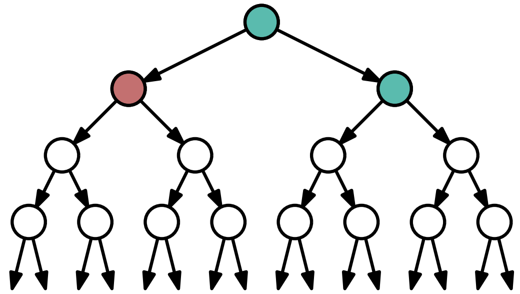

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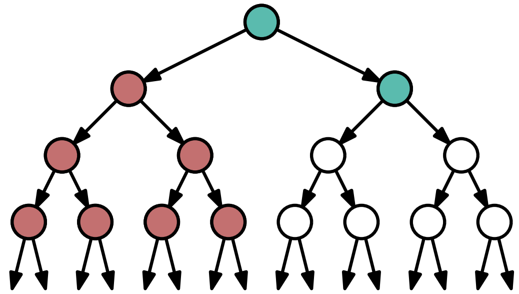

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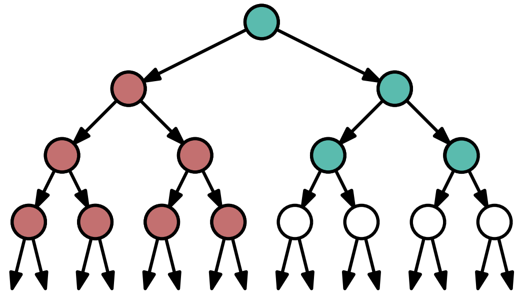

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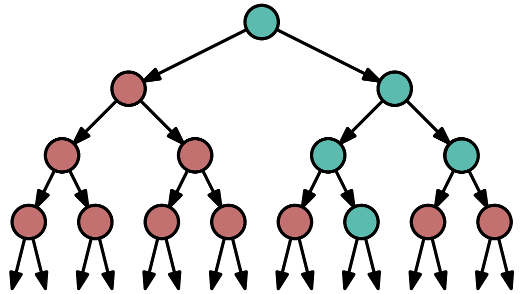

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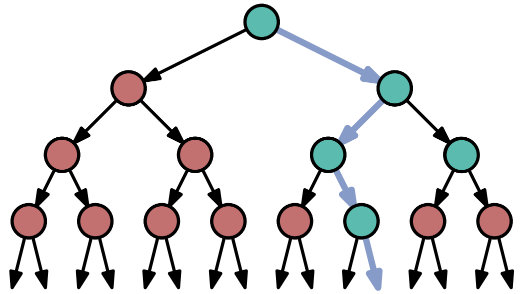

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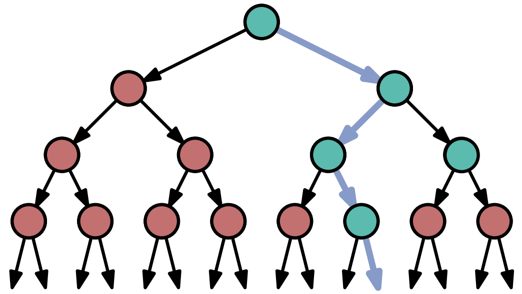

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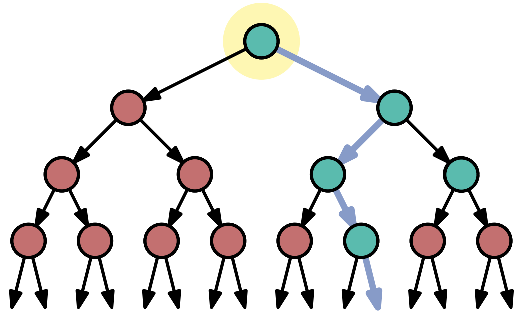

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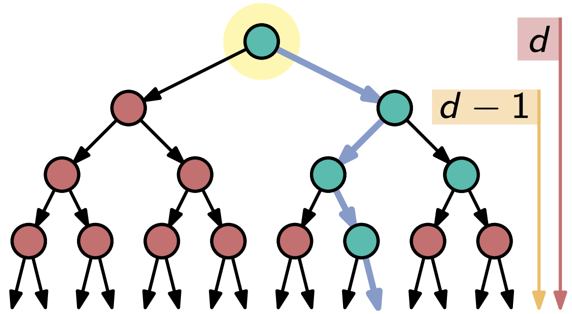

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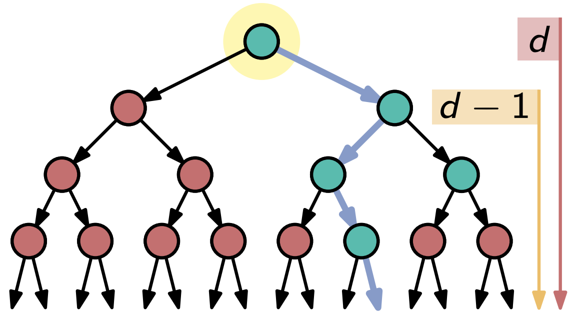

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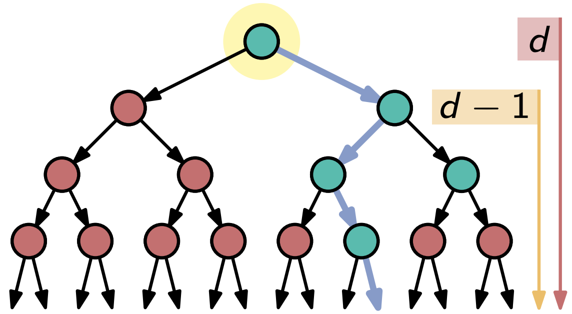

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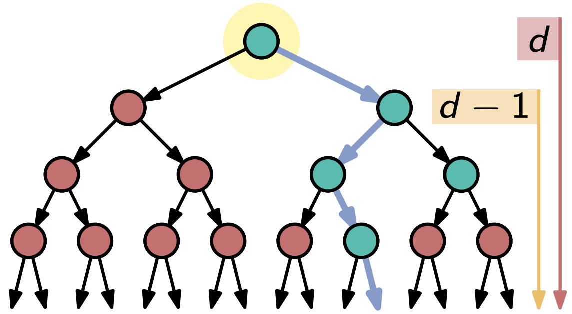

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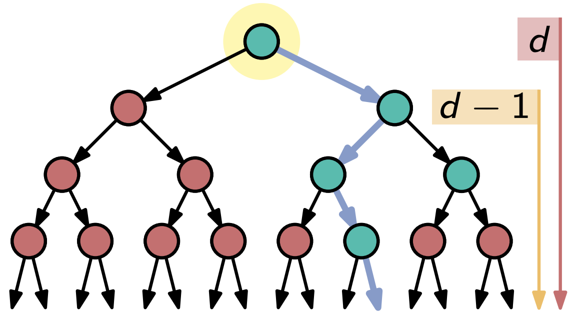

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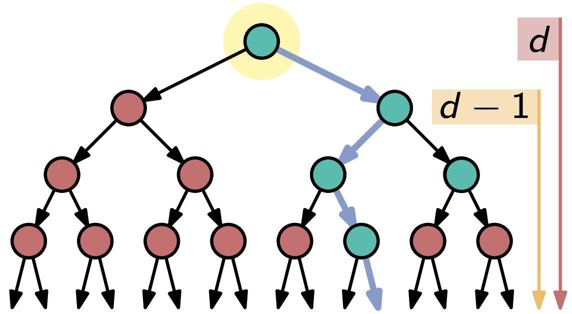

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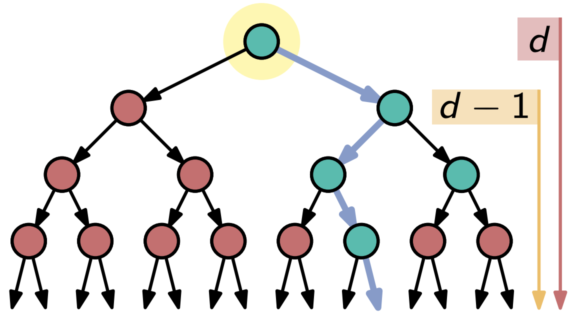

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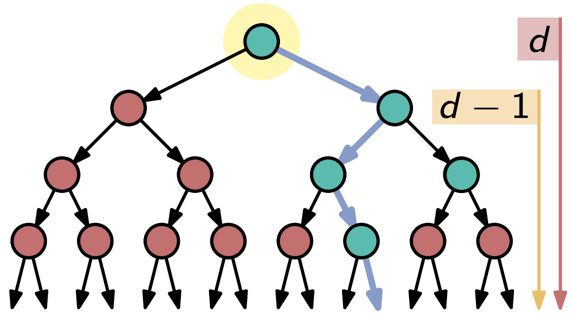

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&= \tfrac{1}{2} \cdot \left(2\Pr[\mathcal{P}_{d-1}] - \Pr[\mathcal{P}_{d-1}]^2\right) \\
&= \Pr[\mathcal{P}_{d-1}] - \tfrac{1}{2}\Pr[\mathcal{P}_{d-1}]^2
\end{aligned}$$

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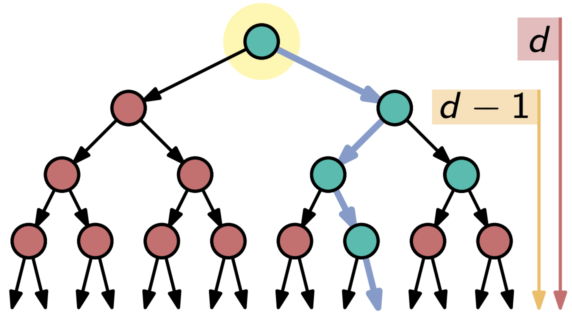

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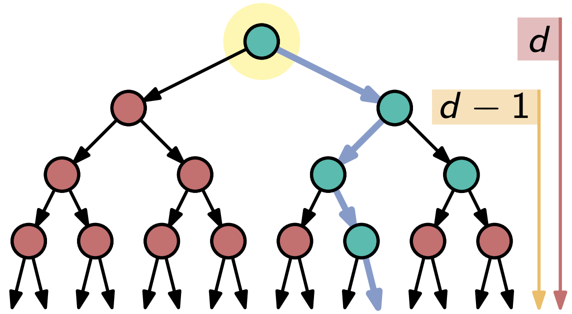

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Claim $\Pr[\mathcal{P}_d] \geq \frac{1}{d+2}$

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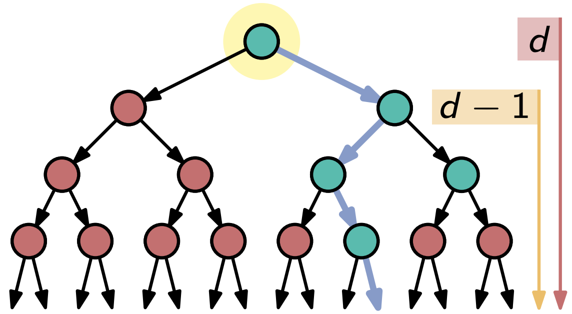

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Karger-Stein: Success Probability

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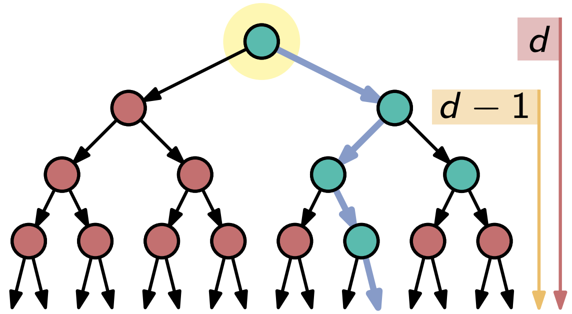
Recursion tree



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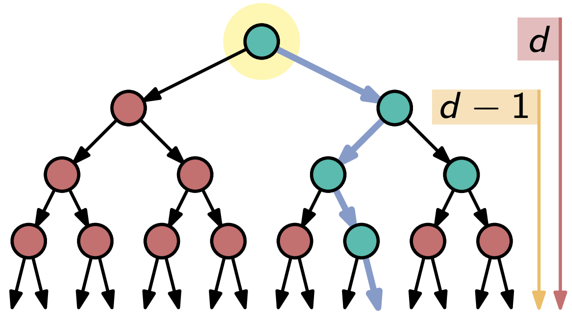

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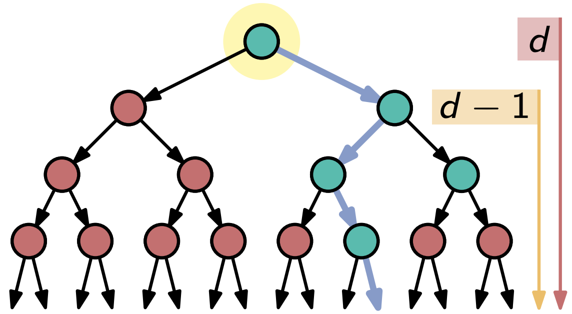

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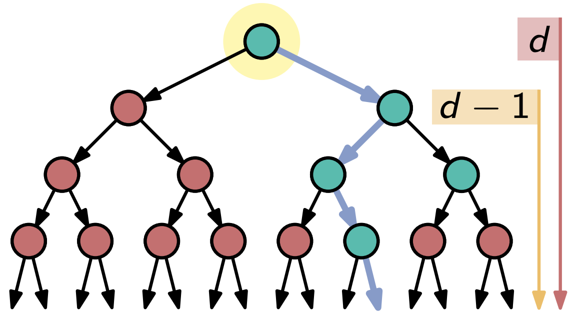

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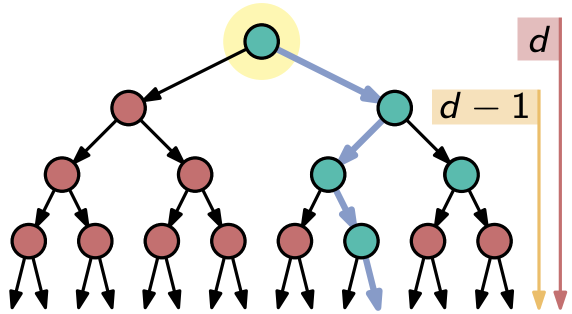

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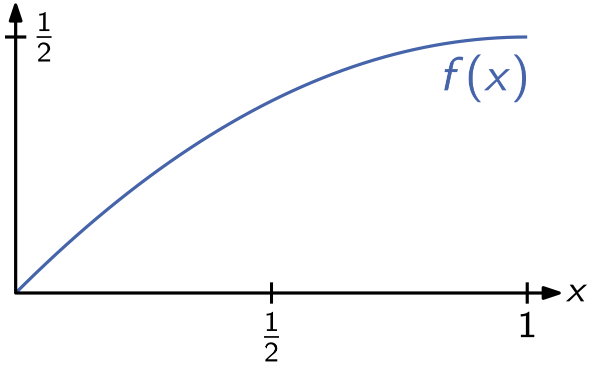

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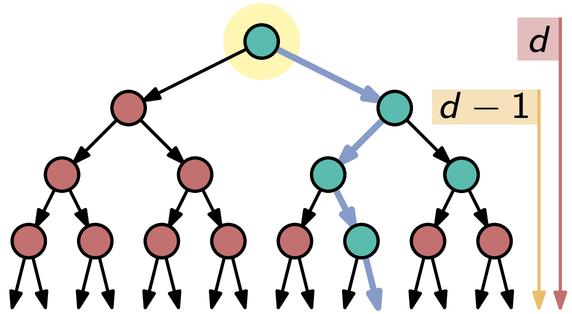

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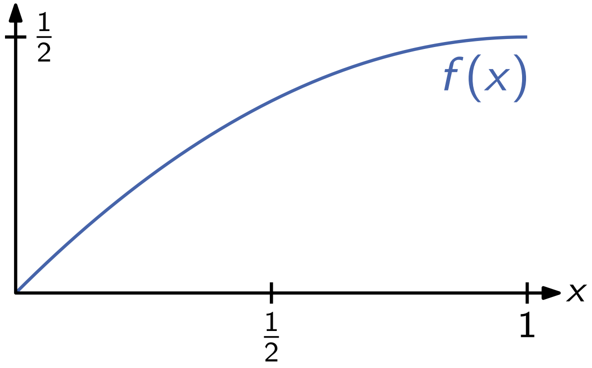

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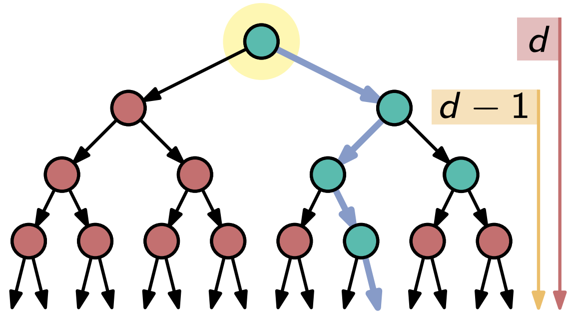

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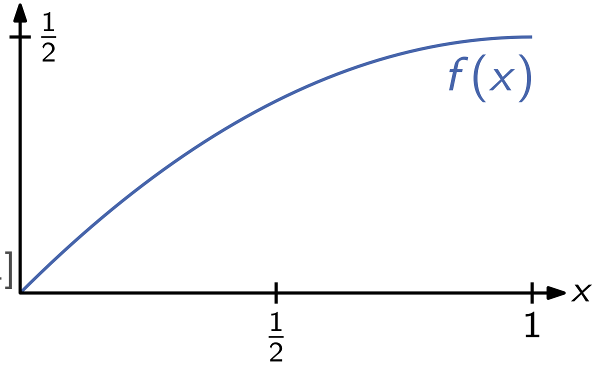

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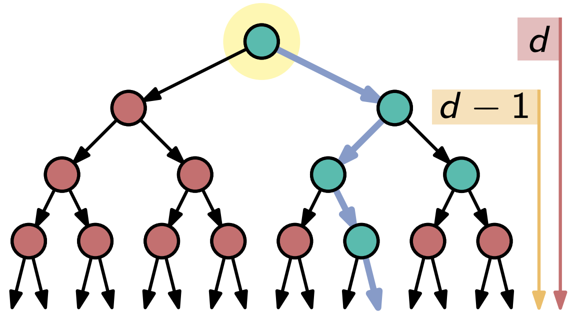

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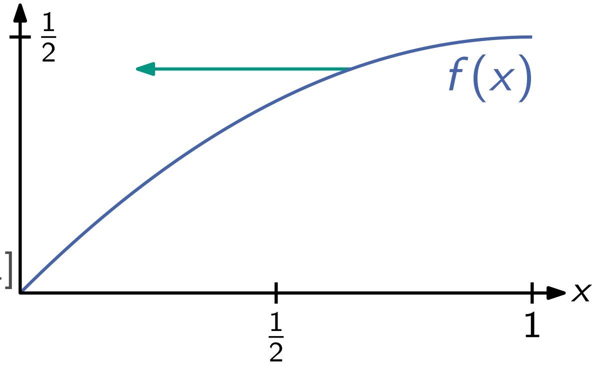

smaller x

Karger-Stein: Success Probability

Karlsruhe Institute of Technology

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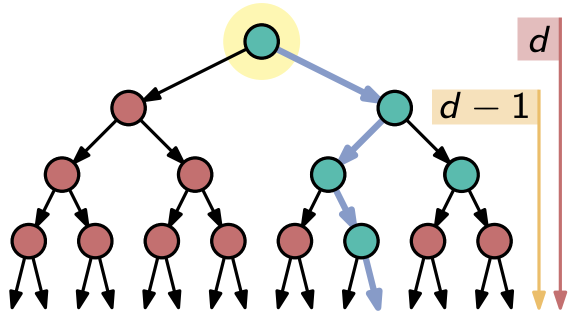

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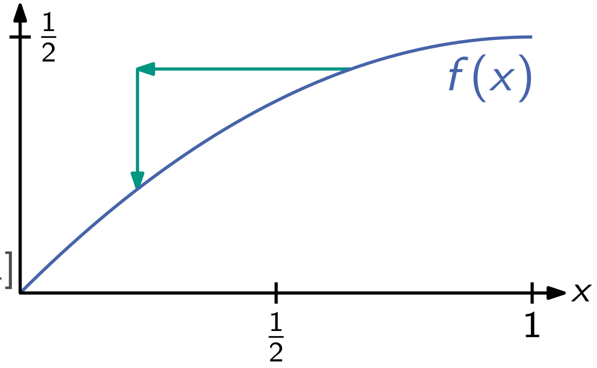

smaller x yields smaller $f(x)$

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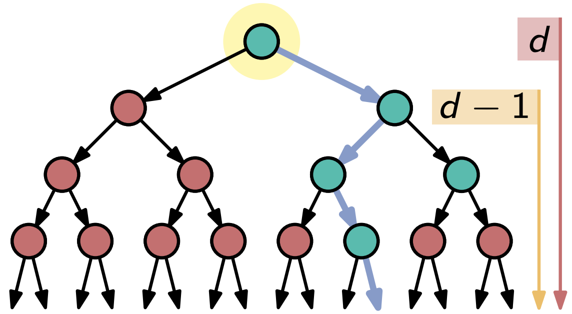

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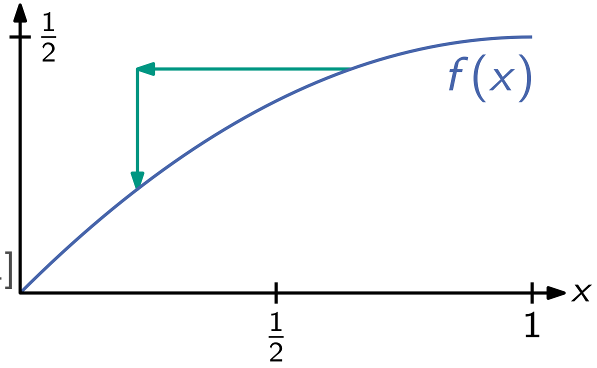

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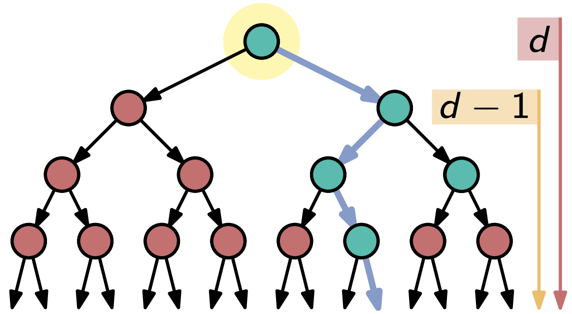

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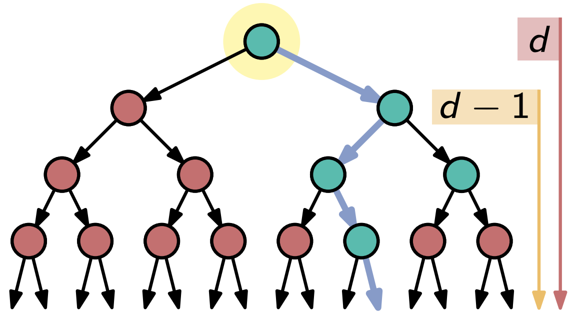

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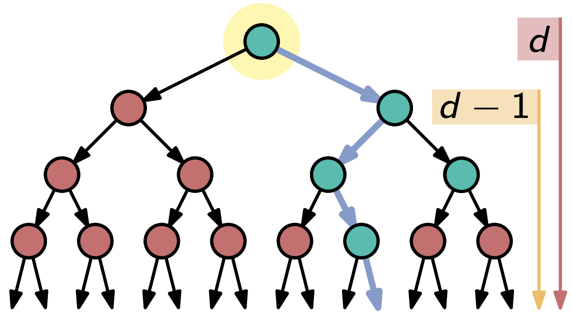

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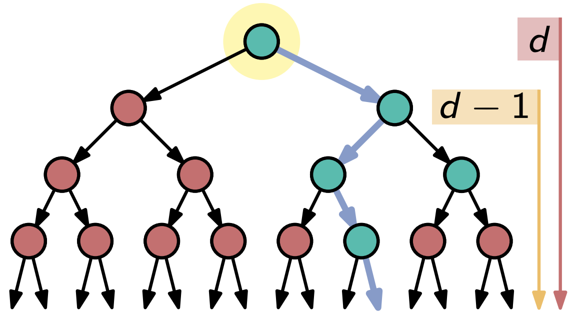

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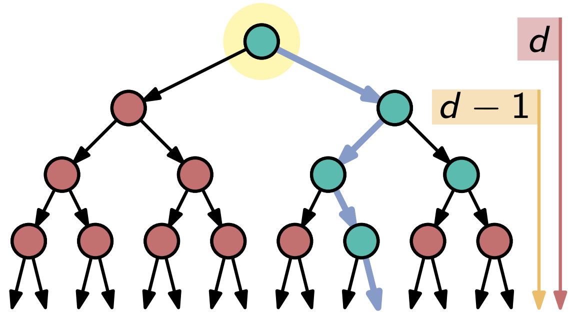
Recursion tree



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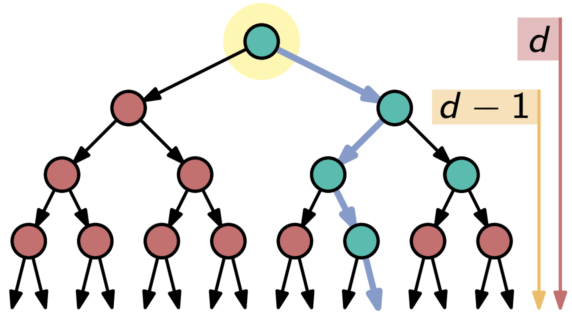

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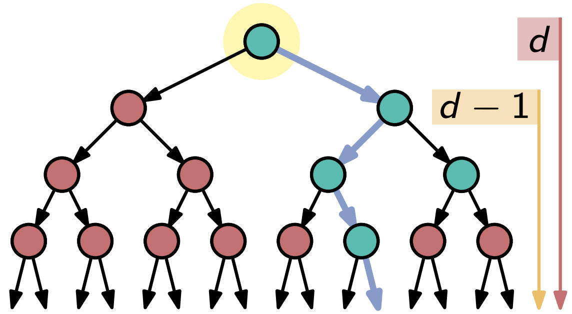

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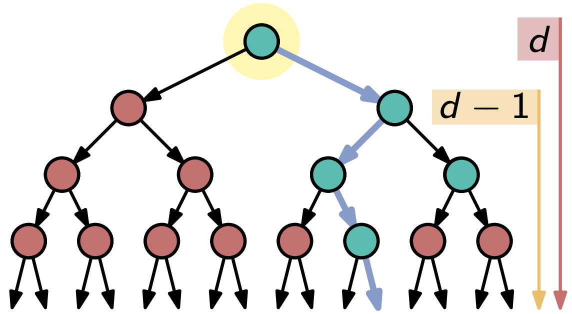

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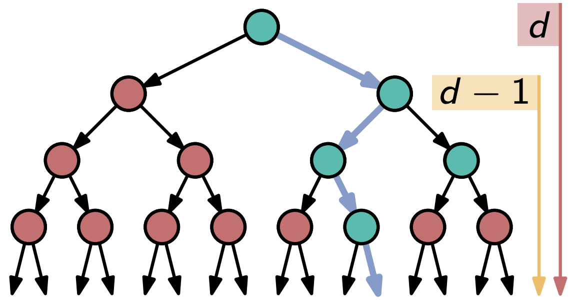

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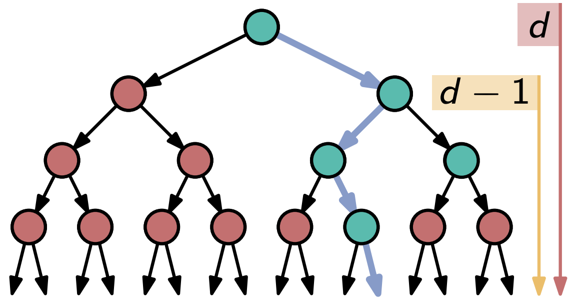

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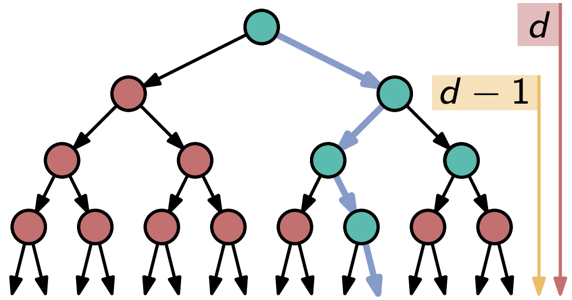

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- Compared to $O(n^4 \log(n))$ for Karger
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Conclusion

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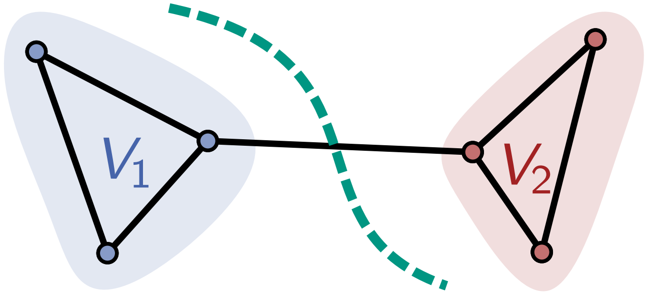
- Many deterministic flow-based algorithms
- . . . with worst-case running times in $\Omega(n^3)$



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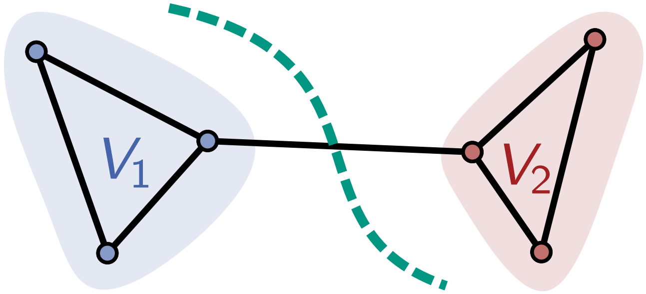
- Karger's edge-contraction algorithm



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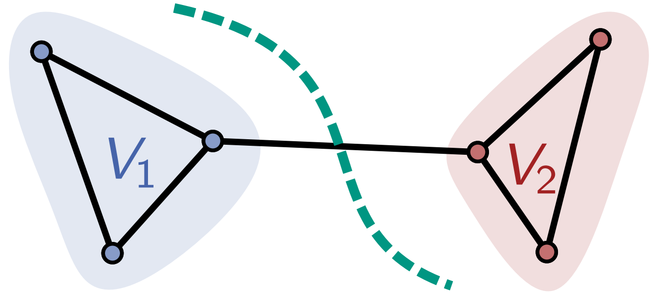

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- Monte Carlo algorithms with and without biases
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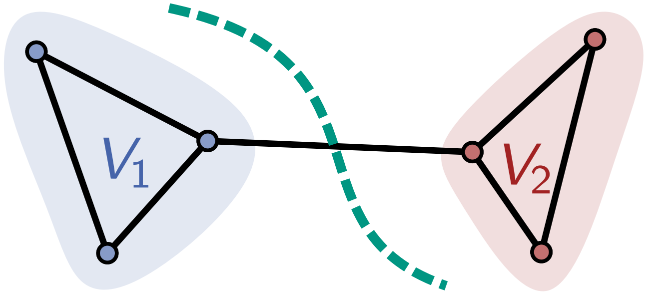

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Outlook

"Minimum cuts in near-linear time", Karger, J.Acm. '00

Success w.h.p. in time $O(m \log^3(n))$

"Faster algorithms for edge connectivity via random 2-out contractions", Ghaffari & Nowicki & Thorup, SODA'20

Success w.h.p. in time $O(m \log(n))$ and $O(m + n \log^3(n))$