

# **Probability & Computing**

#### **Probabilistic Method**



www.kit.edu

**The Problem** 

• Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)





# Karlsruhe Institute of Technology

# **Complete Coloring**

#### **The Problem**

- Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)
- A *k*-clique is a complete subgraph with *k* vertices



# Karlsruhe Institute of Technology

## Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

**The Problem** 

**Complete Coloring** 

• A *k*-clique is a complete subgraph with *k* vertices



3-cliques

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

# **Complete Coloring**

#### **The Problem**

- Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)
- A *k*-clique is a complete subgraph with *k* vertices







# **Complete Coloring**

#### The Problem

• Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue





Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

# **Complete Coloring**

#### The Problem

• Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue





Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

# **Complete Coloring**

#### **The Problem**

Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?



colorings



# **Complete Coloring**

#### The Problem

• Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?





# **Complete Coloring**

#### The Problem

Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?





# **Complete Coloring**

## The Problem

Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?





# **Complete Coloring**

## The Problem

Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?





# **Complete Coloring**

## The Problem

• Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?





# **Complete Coloring**

## The Problem

• Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?

## The Solution?

Brute-force algorithm?





# **Complete Coloring**

## The Problem

Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?

# The Solution?

- Brute-force algorithm?
  - $n = 6 \Rightarrow 2^{n(n-1)} = 2^{30} = 1,073,741,824$  possible colorings





# **Complete Coloring**

## The Problem

Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?

# The Solution?

- Brute-force algorithm?
  - $n = 6 \Rightarrow 2^{n(n-1)} = 2^{30} = 1,073,741,824$  possible colorings
  - $k = 3 \Rightarrow {6 \choose 3} = 20$  triangles to check  $\Rightarrow 60$  edges per coloring





•  $k = 3 \Rightarrow \binom{6}{3} = 20$  triangles to check  $\Rightarrow 60$  edges per coloring

- A k-clique is a complete subgraph with k vertices A coloring of the graph assigns each edge one of two colors: red or blue
  - In a graph with n vertices, does there exist a coloring with *no* monochromatic *k*-clique?

## The Solution?

The Problem

Brute-force algorithm?

**Complete Coloring** 

- $n = 6 \Rightarrow 2^{n(n-1)} = 2^{30} = 1,073,741,824$  possible colorings







#### **The Problem**

• Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?

## The Solution?

- Brute-force algorithm?
  - $n = 6 \Rightarrow 2^{n(n-1)} = 2^{30} = 1,073,741,824$  possible colorings
  - $k = 3 \Rightarrow {6 \choose 3} = 20$  triangles to check  $\Rightarrow 60$  edges per coloring
  - What about n = 1000 and k = 20?





naive implementation: 20min

no coloring exists

## The Problem

Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?

# The Solution?

- Brute-force algorithm?
  - $n = 6 \Rightarrow 2^{n(n-1)} = 2^{30} = 1,073,741,824$  possible colorings
  - $k = 3 \Rightarrow \binom{6}{3} = 20$  triangles to check  $\Rightarrow 60$  edges per coloring
  - What about n = 1000 and k = 20?

Randomized algorithm?



naive implementation: 20min no coloring exists

## The Problem

Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)

- A *k*-clique is a complete subgraph with *k* vertices
- A coloring of the graph assigns each edge one of two colors: red or blue
- In a graph with *n* vertices, does there *exist* a coloring with *no* monochromatic *k*-clique?

# The Solution?

- Brute-force algorithm?
  - $n = 6 \Rightarrow 2^{n(n-1)} = 2^{30} = 1,073,741,824$  possible colorings
  - $k = 3 \Rightarrow {6 \choose 3} = 20$  triangles to check  $\Rightarrow 60$  edges per coloring
  - What about n = 1000 and k = 20?
- Randomized algorithm?
  - How often shall we try before assuming that no coloring exists?





naive implementation: 20min no coloring exists

Algorithm

• For each edge independently, choose one of the colors with probability 1/2





Algorithm

 $\blacksquare$  For each edge independently, choose one of the colors with probability 1/2



- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique





- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique





- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique





- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques





- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques





## Algorithm

- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques

Karlsruhe Institute of Technolog



- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques





- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques





- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques





- $\blacksquare$  For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic



# Algorithm For each edge independently, choose one of the colors with probability 1/2

- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

# Randomized Coloring

# H<sub>i</sub>





- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?
  - Consider the first edge that gets colored





#### Algorithm

3

- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?
  - Consider the first edge that gets colored (we do not care which color it is, but...)





- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?
  - Consider the first edge that gets colored (we do not care which color it is, but...)
  - The  $\binom{k}{2} 1$  remaining edges need to get the same color




#### **Randomized Coloring**

#### Algorithm

- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?
  - Consider the first edge that gets colored (we do not care which color it is, but...)
  - The  $\binom{k}{2} 1$  remaining edges need to get the same color

$$\Pr[X_i=1] = \left(\frac{1}{2}\right)^{\binom{k}{2}-\frac{1}{2}}$$





#### **Randomized Coloring**

#### Algorithm

- For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?
  - Consider the first edge that gets colored (we do not care which color it is, but...)
  - The  $\binom{k}{2} 1$  remaining edges need to get the same color

$$\Pr[X_{i} = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2} - 1} = 2^{-\binom{k}{2} + 1}$$





## Algorithm

**Randomized Coloring** 

- $\blacksquare$  For each edge independently, choose one of the colors with probability 1/2
- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic

• What is  $\Pr[X = 1]$ ?

- What is  $\Pr[X_i = 1]$ ?
  - Consider the first edge that gets colored (we do not care which color it is, but...)  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right]$
  - The <sup>k</sup><sub>2</sub> -1 remaining edges need to get the same color

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2} - 1} = 2^{-\binom{k}{2} + 1}$$





**Randomized Coloring** 

- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X_i = 1]$ ? Union bound  $\binom{n}{k}$  Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [X_i]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2} - 1} = 2^{-\binom{k}{2} + 1}$$





- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $=\binom{n}{k}2^{-\binom{k}{2}+1}$
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{-\binom{k}{2}+1}$$







- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $=\binom{n}{k}2^{-\binom{k}{2}+1} \leq \frac{n^{k}}{k!}2^{-\frac{k(k-1)}{2}+1}$
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2} - 1} = 2^{-\binom{k}{2} + 1}$$







- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum^{n} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1}$
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{-\binom{k}{2}+1}$$

#### **Randomized Coloring**



**Randomized Coloring** 

- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1}$
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{-\binom{k}{2}+1}$$





- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

3

- Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{\nu!} 2 \cdot 2^{-\frac{k^{2}-k}{2}}$
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

 $-2^{-\binom{k}{2}+1}$ 

 $\Pr[X_i = 1] = (\frac{1}{2})^{\binom{k}{2} - 1}$ 

#### **Randomized Coloring**





- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}}$
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2} - 1} = 2^{-\binom{k}{2} + 1}$$

## **Randomized Coloring**





**Randomized Coloring** 

- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2^{-\frac{k^{2}-k}{2}}$
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2} - 1} = 2^{-\binom{k}{2} + 1}$$





- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$
- The  $\binom{k}{2} 1$  remaining edges need to get the same color

 $=2^{-\binom{n}{2}+1}$ 

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2} - \frac{1}{2}}$$





- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot \frac{2^{-\frac{k^{2}-k}{2}}}{2} = \frac{n^{k}}{k!} 2 \cdot \left(2^{-\frac{k}{2}}\right)^{k} \cdot 2^{\frac{k}{2}}$
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2} - 1} = 2^{-\binom{k}{2} + 1}$$







- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $=\binom{n}{k}2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!}2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!}2\cdot2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!}2\cdot(2^{-\frac{k}{2}})^{k}\cdot2^{\frac{k}{2}}$
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_{i} = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2} - 1} = 2^{-\binom{k}{2} + 1}$$





- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?
  - What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$
  - The  $\binom{k}{2} 1$  remaining edges need to get the same color

 $=2^{-\binom{n}{2}+1}$ 

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

Algorithm

**Randomized Coloring** 



simplify by assuming  $k > 2 \log(n)$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum \Pr[X_i = 1]$  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$ simplify by assuming  $k \ge 2$  lo  $\Rightarrow 2^{-\frac{k}{2}}$ **COOPED** (we do not care which color it is, but...)
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$=2^{-\binom{k}{2}+1}$$

 $\Pr[X_i = 1] = (\frac{1}{2})^{\binom{k}{2} - 1}$ 

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms





simplify by assuming  $k > 2 \log(n)$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

• Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)

• What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ 

• The  $\binom{k}{2} - 1$  remaining edges need to get the same color  $\Pr[X_i = 1] = (\frac{1}{2})^{\binom{k}{2} - 1}$ 

 $=2^{-\binom{k}{2}+1}$ 

## **Randomized Coloring**



 $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$ simplify by assuming  $k \ge 2$  lo  $2^{-\frac{k}{2}}$ simplify by assuming  $k > 2 \log(n)$  $\Rightarrow 2^{-\frac{k}{2}} < \frac{1}{2}$ 





- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?
  - Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum \Pr[X_i = 1]$  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$ **COOPED** (we do not care which color it is, but...)

 $\leq \frac{1}{k!} 2\sqrt{2}^k$ 

• What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ 

• The  $\binom{k}{2}$  -1 remaining edges need to get the same color

 $-2^{-\binom{k}{2}+1}$ 

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2} - 1}$$

# **Randomized Coloring**

Algorithm







simplify by assuming  $k > 2 \log(n)$ 

- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum \Pr[X_i = 1]$  $\left| = \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}} \right|$   $\leq \frac{1}{k!} 2\sqrt{2}^{k} \le \frac{2\sqrt{2}^{k}}{e(\frac{k}{2})^{k}}$   $\Rightarrow 2^{-\frac{k}{2}} < \frac{1}{k!} 2\sqrt{2}^{k} \le \frac{2\sqrt{2}^{k}}{e(\frac{k}{2})^{k}}$ **COOPED** (we do not care which color it is, but...)
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

 $=2^{-\binom{k}{2}+1}$ 

 $\Pr[X_i = 1] = (\frac{1}{2})^{\binom{k}{2} - 1}$ 

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

simplify by assuming  $k \ge 2\log(n)$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

• Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{\frac{k!}{2}} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$ 

 $\leq rac{1}{k!} 2\sqrt{2}^k \leq rac{2\sqrt{2}^k}{e(rac{k}{2})^k}$ 

• What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ 

• The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_i=1] = \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

 $-2^{-\binom{\kappa}{2}+1}$ 

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms



simplify by assuming  $k > 2 \log(n)$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

• Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$  $\le \frac{1}{k!} 2\sqrt{2}^{k} \le \frac{2\sqrt{2}^{k}}{e(\frac{k}{2})^{k}} = \frac{2}{e} \left(\frac{\sqrt{2}e}{k}\right)^{k}$  simplify by assuming  $k \ge 2 \log \frac{1}{2} + 2^{-\frac{k}{2}} < \frac{1}{2} + 2^{-\frac{k}{2}} <$ **COOPED** (we do not care which color it is, but...)

• What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ 

• The  $\binom{k}{2}$  -1 remaining edges need to get the same color

 $=2^{-1}1^{-1}$ 

121

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2}-1}$$

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms





simplify by assuming  $k \ge 2 \log(n)$  $\Rightarrow 2^{-\frac{k}{2}} < \frac{1}{2}$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

• Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum \Pr[X_i = 1]$  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$  $\le \frac{1}{k!} 2\sqrt{2}^{k} \le \frac{2\sqrt{2}^{k}}{e(\frac{k}{e})^{k}} = \frac{2}{e} \left(\frac{\sqrt{2}e}{k}\right)^{k}$  simplify by assuming  $k \ge 2$  lo  $\Rightarrow 2^{-\frac{k}{2}} \le 2^{-\frac{k}{2}} \le$ **COOPED** (we do not care which color it is, but...)

• What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ 

• The  $\binom{k}{2}$  -1 remaining edges need to get the same color

$$\Pr[X_i = 1] = \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{-\binom{k}{2}+1}$$

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms



simplify by assuming  $k \ge 2 \log(n)$  $\Rightarrow 2^{-\frac{k}{2}} < \frac{1}{2}$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?
  - What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$  $\le \frac{1}{k!} 2\sqrt{2}^{k} \le \frac{2\sqrt{2}^{k}}{e(\frac{k}{e})^{k}} = \frac{2}{e} \left(\frac{\sqrt{2}e}{k}\right)^{k}$  simplify by assuming  $k \ge 2$  lo  $\Rightarrow 2^{-\frac{k}{2}} \le \frac{2\sqrt{2}}{e(\frac{k}{e})^{k}} = \frac{2}{e} \left(\frac{\sqrt{2}e}{k}\right)^{k}$ **COOPED** (we do not care which color it is, but...)
  - The  $\binom{k}{2}$  -1 remaining edges need to get the same color

 $=2^{-\binom{k}{2}+1}$ 

 $\Pr[X_i = 1] = (\frac{1}{2})^{\binom{k}{2} - 1}$ 

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

#### **Randomized Coloring**

#### Algorithm

3



simplify by assuming  $k \geq 2\log(n)$  $\Rightarrow 2^{-rac{k}{2}} < rac{1}{2}$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

- What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$   $\le \frac{1}{k!} 2\sqrt{2}^{k} \le \frac{2\sqrt{2}^{k}}{e(\frac{k}{e})^{k}} = \frac{2}{e} \left(\frac{\sqrt{2}e}{k}\right)^{k} < 1$ simplify by assuming  $k \ge 2$  low  $\Rightarrow 2^{-\frac{k}{2}} \le 2^{-\frac{k$ **COOPED** (we do not care which color it is, but...)
- The  $\binom{k}{2}$  -1 remaining edges need to get the same color

 $-2^{-\binom{k}{2}+1}$ 

$$\Pr[X_i=1] = \left(\frac{1}{2}\right)^{\binom{k}{2}-1}$$

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms



 $\Rightarrow 2^{-\frac{k}{2}} < \frac{1}{2}$ 

simplify by assuming  $k \ge 2 \log(n)$  $\Rightarrow 2^{-rac{k}{2}} < rac{1}{2}$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic

• What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ 

 $\left| \leq \frac{1}{k!} 2\sqrt{2}^{k} \leq \frac{2\sqrt{2}^{k}}{e(\frac{k}{e})^{k}} = \frac{2}{e} \left( \frac{\sqrt{2}e}{k} \right)^{k} < 1 \right|$ 

• What is  $\Pr[X_i = 1]$ ?

Algorithm

- Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{\frac{k!}{2}} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{\frac{k!}{2}} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{\frac{k!}{2}} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$
- The  $\binom{k}{2} 1$  remaining edges need to get the same color

 $=2^{-\binom{k}{2}+1}$ 

 $\Pr[X_i = 1] = (\frac{1}{2})^{\binom{k}{2} - 1}$ 

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms





simplify by assuming  $k \ge 2\log(n)$  $\Rightarrow 2^{-\frac{k}{2}} < \frac{1}{2}$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic

• What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ 

 $\left| \leq \frac{1}{k!} 2\sqrt{2}^k \leq \frac{2\sqrt{2}^k}{e(\frac{k}{e})^k} = \frac{2}{e} \left( \frac{\sqrt{2}e}{k} \right)^k < 1 \right|$ 

- What is  $\Pr[X_i = 1]$ ?
  - Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum \Pr[X_i = 1]$  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{\frac{k!}{2}} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$ **COOPED** (we do not care which color it is, but...)
  - The  $\binom{k}{2}$  -1 remaining edges need to get the same color

 $=2^{-\binom{k}{2}+1}$ 

 $\Pr[X_i = 1] = (\frac{1}{2})^{\binom{k}{2} - 1}$ 

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

## **Randomized Coloring**

#### Algorithm



simplify by assuming  $k \ge 2 \log(n)$  $\Rightarrow 2^{-\frac{k}{2}} < \frac{1}{2}$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

• What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $= \binom{n}{k} 2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!} 2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!} 2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!} 2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$ 

 $\leq \frac{1}{k!} 2\sqrt{2}^{k} \leq \frac{2\sqrt{2}^{k}}{e(\frac{k}{e})^{k}} = \frac{2}{e} \left( \underbrace{\frac{\sqrt{2}e}{k}}_{<1 < 1}^{k} < 1 \right)$ 

 $\Rightarrow \Pr[X = 0] = 1 - \Pr[X = 1] > 0$ 

• The  $\binom{k}{2} - 1$  remaining edges need to get the same color

$$\Pr[X_i=1] = \left(\frac{1}{2}\right)^{\binom{k}{2}-1}$$

 $-2^{-\binom{k}{2}+1}$ 

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms



simplify by assuming  $k \ge 2 \log(n)$  $\Rightarrow 2^{-\frac{k}{2}} < \frac{1}{2}$ 



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?
  - What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $=\binom{n}{k}2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!}2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!}2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!}2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$

 $\leq \frac{1}{k!} 2\sqrt{2}^{k} \leq \frac{2\sqrt{2}^{k}}{e(\frac{k}{e})^{k}} = \frac{2}{e} \left( \underbrace{\frac{\sqrt{2}e}{k}}_{k} \right)^{k} < 1$ 

 $\Rightarrow \Pr[X = 0] = 1 - \Pr[X = 1] > 0$ 

• The  $\binom{k}{2} - 1$  remaining edges need to get the same color

$$\Pr[X_i=1] = \left(\frac{1}{2}\right)^{\binom{k}{2}-1}$$

 $-2^{-\binom{k}{2}+1}$ 

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

## **Randomized Coloring**

#### Algorithm



simplify by assuming  $k \ge 2\log(n)$ 

with the desired property!

It may happen that the algorithm returns a coloring



- Let X be the indicator variable with X = 1 if and only if the resulting coloring contains a monochromatic k-clique
- Let  $H_1, ..., H_{\binom{n}{k}}$  be all the different k-cliques
- Let  $X_i$  be the indicator variable with  $X_i = 1$  if and only if  $H_i$  is monochromatic
- What is  $\Pr[X_i = 1]$ ?

Algorithm

• What is  $\Pr[X = 1]$ ? union bound  $\binom{n}{k}$ • Consider the first edge that gets  $\Pr[X = 1] = \Pr\left[\exists_{i \in [\binom{n}{k}]} : X_i = 1\right] \leq \sum_{i \in [\binom{n}{k}]} \Pr[X_i = 1]$ **COOPED** (we do not care which color it is, but...)  $=\binom{n}{k}2^{-\binom{k}{2}+1} \le \frac{n^{k}}{k!}2^{-\frac{k(k-1)}{2}+1} = \frac{n^{k}}{k!}2 \cdot 2^{-\frac{k^{2}-k}{2}} = \frac{n^{k}}{k!}2 \cdot (2^{-\frac{k}{2}})^{k} \cdot 2^{\frac{k}{2}}$ 

 $\leq \frac{1}{k!} 2\sqrt{2}^{k} \leq \frac{2\sqrt{2}^{k}}{e(\frac{k}{e})^{k}} = \frac{2}{e} \left( \underbrace{\frac{\sqrt{2}e}{k}}_{1 \leq 1} \right)^{k} < 1$ 

 $\Rightarrow \Pr[X = 0] = 1 - \Pr[X = 1] > 0$ 

• The  $\binom{k}{2} - 1$  remaining edges need to get the same color

$$\Pr[X_i=1] = \left(\frac{1}{2}\right)^{\binom{k}{2}-1}$$

 $-2^{-\binom{\kappa}{2}+1}$ 

tical Informatics, Algorithm Engineering & Scalable Algorithms



 $H_i$ 

simplify by assuming  $k \ge 2\log(n)$ 

It may happen that the algorithm returns a coloring

with the desired property! not very confident...





**The Probability Space** 



#### **The Probability Space**

• What is the sample space of the algorithm?

• Each edge is red or blue with prob. 1/2



#### **The Probability Space**

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings



#### **The Probability Space**

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings





#### **The Probability Space**

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings





#### **The Probability Space**

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings





#### **The Probability Space**

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings




#### **The Probability Space**

• What is the sample space of the algorithm?

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings

• Each occurs with equal probability  $1/2^{\binom{n}{2}}$ 





#### **The Probability Space**

• What is the sample space of the algorithm?

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings
- Each occurs with equal probability  $1/2^{\binom{n}{2}}$

- X = 0 ⇒ coloring returned by algorithm contains *no* monochromatic *k*-clique
- Pr[*X* = 0] > 0





#### **The Probability Space**

• What is the sample space of the algorithm?

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings
- Each occurs with equal probability  $1/2^{\binom{n}{2}}$

- X = 0 ⇒ coloring returned by algorithm contains *no* monochromatic *k*-clique
- Pr[*X* = 0] > 0
- Consequence: At least one such coloring in the sample space!





#### **The Probability Space**

• What is the sample space of the algorithm?

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings
- Each occurs with equal probability  $1/2^{\binom{n}{2}}$

- X = 0 ⇒ coloring returned by algorithm contains *no* monochromatic *k*-clique
- Pr[*X* = 0] > 0
- Consequence: At least one such coloring in the sample space!





#### **The Probability Space**

• What is the sample space of the algorithm?

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings
- Each occurs with equal probability  $1/2^{\binom{n}{2}}$

- X = 0 ⇒ coloring returned by algorithm contains *no* monochromatic *k*-clique
- Pr[*X* = 0] > 0
- Consequence: At least one such coloring in the sample space! Unclear where.





#### **The Probability Space**

• What is the sample space of the algorithm?

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings
- Each occurs with equal probability  $1/2^{\binom{n}{2}}$

- X = 0 ⇒ coloring returned by algorithm contains *no* monochromatic *k*-clique
- Pr[*X* = 0] > 0
- Consequence: At least one such coloring in the sample space! Unclear where. But we know deterministically that it exists!





#### **The Probability Space**

• What is the sample space of the algorithm?

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings
- Each occurs with equal probability  $1/2^{\binom{n}{2}}$

- X = 0 ⇒ coloring returned by algorithm contains *no* monochromatic *k*-clique
- Pr[*X* = 0] > 0
- Consequence: At least one such coloring in the sample space! Unclear where. But we know deterministically that it exists! algo algo





#### **The Probability Space**

• What is the sample space of the algorithm?

- Each edge is red or blue with prob. 1/2
- $\binom{n}{2}$  edges  $\Rightarrow 2^{\binom{n}{2}}$  possible colorings
- Each occurs with equal probability  $1/2^{\binom{n}{2}}$

### **Just Shown**

- X = 0 ⇒ coloring returned by algorithm contains *no* monochromatic *k*-clique
- Pr[*X* = 0] > 0
- Consequence: At least one such coloring in the sample space! Unclear where. But we know deterministically that it exists! No need to actually run the algorithm to find it!





#### Recap

• G = (V, E) an unweighted, undirected, connected graph



#### Recap

• G = (V, E) an unweighted, undirected, connected graph

• Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$ 





#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$





#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set



# Karlsruhe Institute of Technology

# **Application: Cuts**

### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- Cut-set: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?



- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- Cut-set: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?





- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- Cut-set: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

### **Random Process**



# Karlsruhe Institute of Technology

# **Application: Cuts**

### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- Cut-set: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

#### Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

# **Application: Cuts**

#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- *Cut*: partition of *V* into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$





#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?
- Random Process
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$





#### Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

# **Application: Cuts**

#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- *Cut*: partition of *V* into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$



#### Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

# **Application: Cuts**

#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- *Cut*: partition of *V* into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$





#### Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

# **Application: Cuts**

#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- *Cut*: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$





# **Application: Cuts**

#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2?

### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 





#### Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

# **Application: Cuts**

#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- *Cut*: partition of *V* into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$







5

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$





5

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$



# Recap

**Application: Cuts** 

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$

**Positive Probability** 





## Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

## **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

# **Positive Probability**

Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set



#### **Recap** • G = (V, E) an unweighted, undirected, connected graph

**Application: Cuts** 

- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2?

### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

### **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut





# Recap

**Application: Cuts** 

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

# **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

# **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut
- To show:  $\Pr[X \ge \frac{m}{2}] > 0$



#### **Recap** • G = (V, E) an unweighted, undirected, connected graph

**Application: Cuts** 

- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut
- To show:  $\Pr[X \ge \frac{m}{2}] > 0$

$$\Pr[X \ge \frac{m}{2}] = \Pr\left[\sum_{i=1}^{m} X_i \ge \frac{m}{2}\right]$$

exists by proving that it has a positive

probability of occuring from a random process.



# G = (V, E) an unweighted, undirected

**Application: Cuts** 

- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut
- To show:  $\Pr[X \ge \frac{m}{2}] > 0$

d, connected graph  

$$V_2 = \emptyset$$
 and  $V_1 \cup V_2 = V$   
int in V, and the other in V

$$\Pr[X \ge \frac{m}{2}] = \Pr\left[\sum_{i=1}^{m} X_i \ge \frac{m}{2}\right] = ???$$



#### **Recap** • G = (V, E) an unweighted, undirected, connected graph

- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut
- To show:  $\Pr[X \ge \frac{m}{2}] > 0$

$$\Pr[X \ge \frac{m}{2}] = \Pr\left[\sum_{i=1}^{m} X_i \ge \frac{m}{2}\right] = ???$$

Depends on the graph?





**Probabilistic Method**: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### **Recap** • G = (V, E) an unweighted, undirected, connected graph

**Application: Cuts** 

- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?
- Random Process
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut
- To show:  $\Pr[X \ge \frac{m}{2}] > 0$

$$\Pr[X \ge \frac{m}{2}] = \Pr\left[\sum_{i=1}^{m} X_i \ge \frac{m}{2}\right] = ???$$

- Depends on the graph?
- The X<sub>i</sub> are not even independent...



#### **Recap** $C = (V \in E)$ an unweighter

**Application: Cuts** 

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

### Random Process

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

# **Positive Probability**

5

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut
- To show:  $\Pr[X \ge \frac{m}{2}] > 0$

 $\Pr[X \ge \frac{m}{2}] = \Pr\left[\sum_{i=1}^{m} X_i \ge \frac{m}{2}\right] = ???$ 

• The  $X_i$  are not even independent...

Depends on the graph?





# **Recap**

**Application: Cuts** 

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2? **Random Process**
- Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$

# **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut
- To show:  $\Pr[X \ge \frac{m}{2}] > 0$

$$\Pr[X \ge \frac{m}{2}] = \Pr\left[\sum_{i=1}^{m} X_i \ge \frac{m}{2}\right] = ???$$

The 
$$X_i$$
 are not even independent...  
 $e_1 \xrightarrow{e_2}_{e_3} X_2 = X_3 = 1 \Rightarrow X_1 = 1$ 

exists by proving that it has a positive



# **Recap**

**Application: Cuts** 

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

# **Positive Probability**

- Consider edges  $e_1, ..., e_m$  and let  $X_i$  be the indicator that is 1 iff  $e_i$  is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut
- To show:  $\Pr[X \ge \frac{m}{2}] > 0$

$$r[X > \frac{m}{2}] = \Pr\left[\sum_{i=1}^{m} X_i > \frac{m}{2}\right] = ???$$

The 
$$X_i$$
 are not even independent...  
 $e_1 \xrightarrow{e_2} X_2 = X_3 = 1 \Rightarrow X_1 = 1$ 



**Probabilistic Method**: Show that something exists by proving that it has a positive probability of occuring from a random process.
# Recap

**Application: Cuts** 

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- Cut-set: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there exist a cut of weight at least m/2? **Probabilistic Method**: Show that something

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges  $e_1, ..., e_m$  and let  $X_i$  be the indicator that is 1 iff  $e_i$  is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut
- To show:  $\Pr[X \ge \frac{m}{2}] > 0$

$$\operatorname{Pr}[X > \frac{m}{2}] = \operatorname{Pr}\left[\sum_{i=1}^{m} X_i > \frac{m}{2}\right] = ???$$

The 
$$X_i$$
 are not even independent...  
 $e_1 \xrightarrow{e_2} X_2 = X_3 = 1 \Rightarrow X_1 = 1$ 

exists by proving that it has a positive

probability of occuring from a random process.





**Theorem**: Let X be a random variable taking values in a set S. Then,  $Pr[X \ge \mathbb{E}[X]] > 0$  and  $Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ ) **Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )

**Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )

**Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x]$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ ) **Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x < \mathbb{E}[X]} x \cdot \Pr[X = x] + \sum_{x \in S, x \ge \mathbb{E}[X]} x \cdot \Pr[X = x]$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ ) **Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x < \mathbb{E}[X]} x \cdot \Pr[X = x] + \sum_{x \in S, x \ge \mathbb{E}[X]} x \cdot \Pr[X = x]$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )

**Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x < \mathbb{E}[X]} x \cdot \Pr[X = x] + \sum_{x \in S, x \ge \mathbb{E}[X]} x \cdot \Pr[X = x] = 0$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )

**Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x < \mathbb{E}[X]} x \cdot \Pr[X = x] + \sum_{x \in S, x \ge \mathbb{E}[X]} x \cdot \Pr[X = x] = 0$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )

**Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x < \mathbb{E}[X]} x \cdot \Pr[X = x]$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )

**Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x \in \mathbb{E}[X]} x \cdot \Pr[X = x]$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )

**Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x < \mathbb{E}[X]} x \cdot \Pr[X = x]$$
$$< \sum_{x \in S, x < \mathbb{E}[X]} \mathbb{E}[X] \cdot \Pr[X = x]$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )

**Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x < \mathbb{E}[X]} x \cdot \Pr[X = x]$$
$$< \sum_{x \in S, x < \mathbb{E}[X]} \mathbb{E}[X] \cdot \Pr[X = x]$$
$$= \mathbb{E}[X] \cdot \sum_{x \in S, x < \mathbb{E}[X]} \Pr[X = x]$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

- There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )
- **Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)
- Towards a contradiction assume  $Pr[X \ge \mathbb{E}[X]] = 0$

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x < \mathbb{E}[X]} x \cdot \Pr[X = x]$$
$$< \sum_{x \in S, x < \mathbb{E}[X]} \mathbb{E}[X] \cdot \Pr[X = x]$$
$$= \mathbb{E}[X] \cdot \sum_{x \in S, x < \mathbb{E}[X]} \Pr[X = x]$$
$$< 1$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )

**Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x < \mathbb{E}[X]} x \cdot \Pr[X = x]$$

$$< \sum_{x \in S, x < \mathbb{E}[X]} \mathbb{E}[X] \cdot \Pr[X = x]$$

$$= \mathbb{E}[X] \cdot \sum_{x \in S, x < \mathbb{E}[X]} \Pr[X = x]$$

$$\leq \mathbb{E}[X] \stackrel{\checkmark}{\leq} 1$$



**Theorem**: Let X be a random variable taking values in a set S. Then,  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

• There always exists at least one sample that yields  $X \ge \mathbb{E}[X]$  ( $X \le \mathbb{E}[X]$ )

**Proof** ( $\Pr[X \ge \mathbb{E}[X]] > 0$ , the other works analogous)

$$\mathbb{E}[X] = \sum_{x \in S} x \cdot \Pr[X = x] = \sum_{x \in S, x < \mathbb{E}[X]} x \cdot \Pr[X = x]$$

$$\neq \leq \sum_{x \in S, x < \mathbb{E}[X]} \mathbb{E}[X] \cdot \Pr[X = x]$$

$$= \mathbb{E}[X] \cdot \sum_{x \in S, x < \mathbb{E}[X]} \Pr[X = x]$$

$$\leq \mathbb{E}[X]$$



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut
- To show:  $\Pr[X \ge \frac{m}{2}] > 0$

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\left[\Pr[X \ge \mathbb{E}[X]] > 0\right]$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.



#### Recap

7

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there exist a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\Pr[X \ge \mathbb{E}[X]] > 0$ 

 $\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{m} X_i] = \sum_{i=1}^{m} \mathbb{E}[X_i]$ 



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there exist a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\left[\Pr[X \ge \mathbb{E}[X]] > 0\right]$ 

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{m} X_i] = \sum_{i=1}^{m} \mathbb{E}[X_i]$$
$$= m \cdot \Pr[X_i = 1]$$

**Probabilistic Method**: Show that something

probability of occuring from a random process.



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\left[\Pr[X \ge \mathbb{E}[X]] > 0\right]$ 

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{m} X_i] = \sum_{i=1}^{m} \mathbb{E}[X_i]$$
$$= m \cdot \Pr[X_i = 1]^?$$

**Probabilistic Method**: Show that something

probability of occuring from a random process.



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with m edges, does there *exist* a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\left[\Pr[X \ge \mathbb{E}[X]] > 0\right]$ 

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{m} X_i] = \sum_{i=1}^{m} \mathbb{E}[X_i]$$
$$= m \cdot \Pr[X_i = 1]$$

e; o—o

**Probabilistic Method**: Show that something

probability of occuring from a random process.



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\Pr[X \ge \mathbb{E}[X]] > 0$ 

**Probabilistic Method**: Show that something

probability of occuring from a random process.



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

#### **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\left[\Pr[X \ge \mathbb{E}[X]] > 0\right]$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

 $\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{m} X_i] = \sum_{i=1}^{m} \mathbb{E}[X_i]$  $= m \cdot \Pr[X_i = 1]$  $\frac{1}{4} \qquad \frac{1}{4} \qquad \frac{1}{4} \qquad \frac{1}{4} \qquad \frac{1}{4}$ Pr



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- Cut-set: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges  $e_1, ..., e_m$  and let  $X_i$  be the indicator that is 1 iff  $e_i$  is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\Pr[X \ge \mathbb{E}[X]] > 0$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{m} X_i] = \sum_{i=1}^{m} \mathbb{E}[X_i]$$
$$= m \cdot \Pr[X_i = 1]$$
$$e_i \circ \bullet \circ \bullet \circ \bullet \circ \bullet$$
$$\Pr[X_i = 1]$$
$$\Pr[X_i = \frac{1}{4}, \frac{$$

 $\mathbf{v}$  1



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

Pr

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\Pr[X \ge \mathbb{E}[X]] > 0$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

 $= m \cdot \Pr[X_i = 1]$ 

 $\frac{1}{4} \qquad \frac{1}{4} \qquad \frac{1}{4} \qquad \frac{1}{4} \qquad \frac{1}{4} = \frac{1}{2}$ 

 $\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{m} X_i] = \sum_{i=1}^{m} \mathbb{E}[X_i]$ 



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set

• Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\left[\Pr[X \ge \mathbb{E}[X]] > 0\right]$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{m} X_i] = \sum_{i=1}^{m} \mathbb{E}[X_i]$$
$$= m \cdot \Pr[X_i = 1] = \frac{m}{2}$$
$$e_i \circ \bullet \circ \bullet \circ \circ \circ \circ \circ$$
$$\Pr[X_i = 1] = \frac{m}{2}$$
$$\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} = \frac{1}{2}$$



#### Recap

- G = (V, E) an unweighted, undirected, connected graph
- Cut: partition of V into  $V_1$ ,  $V_2$  s.t.  $V_1 \cap V_2 = \emptyset$  and  $V_1 \cup V_2 = V$
- *Cut-set*: set of edges with one endpoint in  $V_1$  and the other in  $V_2$
- Weight: size of the cut-set
- Question now: In a graph with *m* edges, does there *exist* a cut of weight at least m/2?

#### **Random Process**

• Add each vertex to one of the two sets with equal prob.  $\frac{1}{2}$ 

## **Positive Probability**

- Consider edges e<sub>1</sub>, ..., e<sub>m</sub> and let X<sub>i</sub> be the indicator that is 1 iff e<sub>i</sub> is in the cut-set
- $X = \sum_{i=1}^{m} X_i$  is the weight of the cut

• To show:  $\Pr[X \ge \frac{m}{2}] > 0$   $\Pr[X \ge \mathbb{E}[X]] > 0$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

**The Problem** 

Two vertices in a graph are *independent*, if they are not adjacent





**The Problem** 

Two vertices in a graph are *independent*, if they are not adjacent





#### The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent





#### **The Problem**

8

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent





#### **The Problem**

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G





#### **The Problem**

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)





**Theorem**: Let *G* be a graph with *n* vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

• Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

The Problem

**Application: Independent Sets** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent



8





 $\alpha(G) = 4$ 

The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.




The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof

Random Process





The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let *G* be a graph with *n* vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

# $d = \frac{24}{9}$



The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let *G* be a graph with *n* vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.



 $d = \frac{24}{9} \Rightarrow$  Survival rate:  $\frac{3}{8}$ 



8 Maximilian Katzmann, Stefan Walzer – Probability & Computing

### **Application: Independent Sets**

The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 





The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 



The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let *G* be a graph with *n* vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 





## Two vertices in a graph are *independent*, if they are not adjacent

**Application: Independent Sets** 

- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

The Problem

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 





The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let *G* be a graph with *n* vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 





The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 





The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let *G* be a graph with *n* vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

 $\mathbf{O}$ 





#### 8 Maximilian Katzmann, Stefan Walzer – Probability & Compu

## The Problem

**Application: Independent Sets** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 





**The Problem** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 





**The Problem** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let *G* be a graph with *n* vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

• Let d = 2m/n be the average degree of G

• Independently, delete each vertex with probability  $1 - \frac{1}{d}$ 





The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

- Let d = 2m/n be the average degree of G
- Independently, delete each vertex with probability  $1 \frac{1}{d}$
- Afterwards, for each remaining edge, delete one endpoint chosen uniformly at random



 $d = \frac{24}{9} \Rightarrow$  Survival rate:  $\frac{3}{8}$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

- Let d = 2m/n be the average degree of G
- Independently, delete each vertex with probability  $1 \frac{1}{d}$
- Afterwards, for each remaining edge, delete one endpoint chosen uniformly at random





**Probabilistic Method**: Show that something

probability of occuring from a random process.

exists by proving that it has a positive

The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

- Let d = 2m/n be the average degree of G
- Independently, delete each vertex with probability  $1 \frac{1}{d}$
- Afterwards, for each remaining edge, delete one endpoint chosen uniformly at random



**Probabilistic Method**: Show that something

probability of occuring from a random process.

exists by proving that it has a positive



The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

- Let d = 2m/n be the average degree of G
- Independently, delete each vertex with probability  $1 \frac{1}{d}$
- Afterwards, for each remaining edge, delete one endpoint chosen uniformly at random







The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

- Let d = 2m/n be the average degree of G
- Independently, delete each vertex with probability  $1 \frac{1}{d}$
- Afterwards, for each remaining edge, delete one endpoint chosen uniformly at random







The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Random Process

- Let d = 2m/n be the average degree of G
- Independently, delete each vertex with probability  $1 \frac{1}{d}$
- Afterwards, for each remaining edge, delete one endpoint chosen uniformly at random
- Note that the remaining vertices form an independent set





**The Problem** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof

Positive Probability

Random Process: d = 2m/nStep 1: Delete v with prob.  $1 - \frac{1}{d}$ Step 2: Delete one endpoint of each e





The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Positive Probability

Random Process: d = 2m/nStep 1: Delete *v* with prob.  $1 - \frac{1}{d}$ Step 2: Delete one endpoint of each *e* 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

•  $X_V$ : number of vertices that survive the first step





**The Problem** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Positive Probability

Random Process: d = 2m/nStep 1: Delete *v* with prob.  $1 - \frac{1}{d}$ Step 2: Delete one endpoint of each *e* 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

- $X_V$ : number of *vertices* that survive the first step
- X<sub>E</sub>: number of edges that survive the first step





**The Problem** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

#### Proof

Positive Probability

Random Process: d = 2m/nStep 1: Delete v with prob.  $1 - \frac{1}{d}$ Step 2: Delete one endpoint of each e

- $X_V$ : number of *vertices* that survive the first step
- X<sub>E</sub>: number of *edges* that survive the first step
- Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex





The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Random Process: d = 2m/n

#### Proof

Positive Probability

Step 1: Delete v with prob.  $1 - \frac{1}{d}$ Step 2: Delete one endpoint of each e

- $X_V$ : number of vertices that survive the first step
- $X_E$ : number of *edges* that survive the first step
- Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex
- Size of resulting independent set S is  $\geq X_V X_E$
- $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V X_E \ge n^2/(4m)]$





**The Problem** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof

**E-Argument**:  $\Pr[X > \mathbb{E}[X]] > 0$ Positive Probability

Random Process: d = 2m/nStep 1: Delete v with prob.  $1 - \frac{1}{d}$ Step 2: Delete one endpoint of each e

**Probabilistic Method**: Show that something exists by proving that it has a positive probability of occuring from a random process.

 $d = \frac{24}{9} \Rightarrow$  Survival rate:  $\frac{3}{8}$ 

 $\mathbf{O}$ 

- $X_V$ : number of *vertices* that survive the first step
- X<sub>E</sub>: number of edges that survive the first step
- Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex
- Size of resulting independent set S is  $\geq X_V X_E$
- $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V X_E \ge n^2/(4m)]$



The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ . Proof  $\mathbb{E}$ -Argument:  $\Pr[X \ge \mathbb{E}[X]] > 0$ Random Process: d = 2m/n**Probabilistic Method**: Show that something Step 1: Delete v with prob.  $1 - \frac{1}{d}$ exists by proving that it has a positive Positive Probability Step 2: Delete one endpoint of each e probability of occuring from a random process.  $X_V$ : number of *vertices* that survive the first step  $\mathbf{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ ) X<sub>E</sub>: number of edges that survive the first step • Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex • Size of resulting independent set S is  $\geq X_V - X_E$ •  $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V - X_E \ge n^2/(4m)]$ 





#### 8 Maximilian Katzmann, Stefan Walzer – Probability & Computing

## **Application: Independent Sets**

The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof  $\mathbb{E}$ -Argument:  $\Pr[X \ge \mathbb{E}[X]] > 0$ Random Process: d = 2m/n**Probabilistic Method**: Show that something Step 1: Delete v with prob.  $1 - \frac{1}{d}$ exists by proving that it has a positive Positive Probability Step 2: Delete one endpoint of each e probability of occuring from a random process. •  $X_V$ : number of *vertices* that survive the first step •  $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ )  $X_E :$  number of *edges* that survive the first step • Edge  $\{u, v\}$  survives if both u, v do • Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex • Size of resulting independent set S is  $\geq X_V - X_E$ •  $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V - X_E \ge n^2/(4m)]$ 





#### 8 Maximilian Katzmann, Stefan Walzer – Probability & Computing

### **Application: Independent Sets**

The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof  $\mathbb{E}$ -Argument:  $\Pr[X \ge \mathbb{E}[X]] > 0$ Random Process: d = 2m/n**Probabilistic Method**: Show that something Step 1: Delete v with prob.  $1 - \frac{1}{d}$ exists by proving that it has a positive Positive Probability Step 2: Delete one endpoint of each e probability of occuring from a random process. •  $X_V$ : number of *vertices* that survive the first step •  $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ )  $X_E :$  number of *edges* that survive the first step • Edge  $\{u, v\}$  survives if both u, v do • Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex  $\bullet \mathbb{E}[X_E] = m \cdot \frac{1}{d^2}$ • Size of resulting independent set S is  $\geq X_V - X_E$ •  $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V - X_E \ge n^2/(4m)]$ 





The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof  $\mathbb{E}$ -Argument:  $\Pr[X \ge \mathbb{E}[X]] > 0$ Random Process: d = 2m/n**Probabilistic Method**: Show that something Step 1: Delete v with prob.  $1 - \frac{1}{d}$ exists by proving that it has a positive Positive Probability Step 2: Delete one endpoint of each e probability of occuring from a random process. •  $X_V$ : number of *vertices* that survive the first step •  $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ )  $X_E :$  number of *edges* that survive the first step • Edge  $\{u, v\}$  survives if both u, v do • Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex  $\blacksquare \mathbb{E}[X_E] = \mathbf{m} \cdot \frac{1}{d^2} = \frac{nd}{2} \cdot \frac{1}{d^2}$ • Size of resulting independent set S is  $\geq X_V - X_E$ •  $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V - X_E \ge n^2/(4m)]$ 





**The Problem** 

Two vertices in a graph are *independent*, if they are not adjacent

- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof  $\mathbb{E}$ -Argument:  $\Pr[X \ge \mathbb{E}[X]] > 0$ Random Process: d = 2m/n**Probabilistic Method**: Show that something Step 1: Delete v with prob.  $1 - \frac{1}{d}$ exists by proving that it has a positive Positive Probability Step 2: Delete one endpoint of each e probability of occuring from a random process. •  $X_V$ : number of *vertices* that survive the first step •  $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ )  $X_E :$  number of *edges* that survive the first step • Edge  $\{u, v\}$  survives if both u, v do • Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex  $\blacksquare \mathbb{E}[X_E] = m \cdot \frac{1}{d^2} = \frac{nd}{2} \cdot \frac{1}{d^2} = \frac{n}{2d}$ • Size of resulting independent set S is  $\geq X_V - X_E$ •  $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V - X_E \ge n^2/(4m)]$ 





#### 8 Maximilian Katzmann, Stefan Walzer – Probability & Computing

#### Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

 $\blacksquare \mathbb{E}[X_V - X_E] = \mathbb{E}[X_V] - \mathbb{E}[X_E]$ 

# Application: Independent Sets

The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof<br/>Positive Probability $\mathbb{P}r[X \ge \mathbb{E}[X]] > 0$ <br/>Step 1: Delete v with prob.  $1 - \frac{1}{d}$ <br/>Step 2: Delete one endpoint of each eProbabilistic Method: Show that something<br/>exists by proving that it has a *positive*<br/>probability of occuring from a random process. $X_V$ : number of vertices that survive the first step<br/> $X_E$ : number of edges that survive the first step $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ ) $\mathbb{E}[X_E] = m \cdot \frac{1}{d^2} = \frac{nd}{2} \cdot \frac{1}{d^2} = \frac{n}{2d}$ 

- Size of resulting independent set S is  $\geq X_V X_E$
- $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V X_E \ge n^2/(4m)]$





#### 8 Maximilian Katzmann, Stefan Walzer – Probability & Computing

#### Probability & Computing

#### Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

0

### **Application: Independent Sets**

The Problem

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof  $\mathbb{E}$ -Argument:  $\Pr[X > \mathbb{E}[X]] > 0$ Random Process: d = 2m/n**Probabilistic Method**: Show that something Step 1: Delete v with prob.  $1 - \frac{1}{d}$ exists by proving that it has a positive Positive Probability Step 2: Delete one endpoint of each e probability of occuring from a random process. •  $X_V$ : number of *vertices* that survive the first step •  $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ ) X<sub>E</sub>: number of edges that survive the first step • Edge  $\{u, v\}$  survives if both u, v do • Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex  $\blacksquare \mathbb{E}[X_E] = m \cdot \frac{1}{d^2} = \frac{nd}{2} \cdot \frac{1}{d^2} = \frac{n}{2d}$ • Size of resulting independent set S is  $\geq X_V - X_E$  $\blacksquare \mathbb{E}[X_V - X_E] = \frac{\mathbb{E}[X_V] - \mathbb{E}[X_E]}{\mathbb{E}[X_V] - \mathbb{E}[X_E]}$ •  $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V - X_E \ge n^2/(4m)]$ 



 $d = \frac{24}{9} \Rightarrow$  Survival rate:  $\frac{3}{8}$ 



## 8

#### Maximilian Katzmann, Stefan Walzer - Probability & Computing

### **Application: Independent Sets**

**The Problem** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof  $\mathbb{E}$ -Argument:  $\Pr[X > \mathbb{E}[X]] > 0$ Random Process: d = 2m/n**Probabilistic Method**: Show that something Step 1: Delete v with prob.  $1 - \frac{1}{d}$ exists by proving that it has a positive Positive Probability Step 2: Delete one endpoint of each e probability of occuring from a random process. •  $X_V$ : number of *vertices* that survive the first step •  $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ ) X<sub>E</sub>: number of edges that survive the first step • Edge  $\{u, v\}$  survives if both u, v do

- Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex  $\blacksquare \mathbb{E}[X_E] = m \cdot \frac{1}{d^2} = \frac{nd}{2} \cdot \frac{1}{d^2} = \frac{n}{2d}$
- Size of resulting independent set S is  $\geq X_V X_E$
- $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V X_E \ge n^2/(4m)]$

•  $\mathbb{E}[X_V - X_E] = \frac{\mathbb{E}[X_V]}{\mathbb{E}[X_E]} = \frac{n}{d} - \frac{n}{2d}$ 





#### Maximilian Katzmann, Stefan Walzer - Probability & Computing 8

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

#### **Application: Independent Sets**

**The Problem** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof  $\mathbb{E}$ -Argument:  $\Pr[X \ge \mathbb{E}[X]] > 0$ Random Process: d = 2m/n**Probabilistic Method**: Show that something Step 1: Delete v with prob.  $1 - \frac{1}{d}$ exists by proving that it has a positive Positive Probability Step 2: Delete one endpoint of each e probability of occuring from a random process. •  $X_V$ : number of *vertices* that survive the first step •  $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ ) X<sub>E</sub>: number of edges that survive the first step • Edge  $\{u, v\}$  survives if both u, v do • Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex  $\blacksquare \mathbb{E}[X_E] = m \cdot \frac{1}{d^2} = \frac{nd}{2} \cdot \frac{1}{d^2} = \frac{n}{2d}$ • Size of resulting independent set S is  $\geq X_V - X_E$  $\blacksquare \mathbb{E}[X_V - X_E] = \mathbb{E}[X_V] - \mathbb{E}[X_E] = \frac{n}{d} - \frac{n}{2d}$ •  $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V - X_E \ge n^2/(4m)]$  $=\frac{11}{2d}$ 





Maximilian Katzmann, Stefan Walzer - Probability & Computing 8

## **Application: Independent Sets**

**The Problem** 

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and  $m \ge n/2$  edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Proof  $\mathbb{E}$ -Argument:  $\Pr[X \ge \mathbb{E}[X]] > 0$ Random Process: d = 2m/n**Probabilistic Method**: Show that something Step 1: Delete v with prob.  $1 - \frac{1}{d}$ exists by proving that it has a positive Positive Probability Step 2: Delete one endpoint of each e probability of occuring from a random process. •  $X_V$ : number of *vertices* that survive the first step •  $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ ) X<sub>E</sub>: number of edges that survive the first step • Edge  $\{u, v\}$  survives if both u, v do • Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex  $\blacksquare \mathbb{E}[X_E] = m \cdot \frac{1}{d^2} = \frac{nd}{2} \cdot \frac{1}{d^2} = \frac{n}{2d}$ • Size of resulting independent set S is  $\geq X_V - X_E$  $\blacksquare \mathbb{E}[X_V - X_E] = \mathbb{E}[X_V] - \mathbb{E}[X_E] = \frac{n}{d} - \frac{n}{2d}$ •  $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V - X_E \ge n^2/(4m)]$  $=\frac{n}{2d}=\frac{n}{2(2m/n)}$ 



# • $X_V$ : number of *vertices* that survive the first step

**E-Argument**:  $\Pr[X > \mathbb{E}[X]] > 0$ 

- X<sub>E</sub>: number of edges that survive the first step
- Step 2: each of the  $X_E$  edges removes  $\leq 1$  vertex  $\mathbf{I} = \mathbb{E}[X_E] = m \cdot \frac{1}{d^2} = \frac{nd}{2} \cdot \frac{1}{d^2} = \frac{n}{2d}$  $\blacksquare \mathbb{E}[X_V - X_E] = \mathbb{E}[X_V] - \mathbb{E}[X_E] = \frac{n}{d} - \frac{n}{2d}$
- Size of resulting independent set S is  $\geq X_V X_E$
- $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V X_E \ge n^2/(4m)]$

### **Application: Independent Sets**

#### The Problem

Positive Probability

Proof

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let *G* be a graph with *n* vertices and 
$$m \ge n/2$$
 edges. Then  $\alpha(G) \ge n^2/(4m)$ .

Random Process: 
$$d = 2m/n$$
  
Step 1: Delete v with prob.  $1 - \frac{1}{d}$   
Step 2: Delete one endpoint of each eProbabilistic Method: Show that something  
exists by proving that it has a *positive*  
probability of occuring from a random process.vive the first step  
ve the first step $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ )  
 $\mathbb{E}[dge \{u, v\}$  survives if both  $u, v$  do

**Probabilistic Mathad:** Show that comothing

 $=\frac{n}{2d}=\frac{n}{2(2m/n)}=\frac{n^2}{(4m)}$ 




## Positive Pr $X_V$ : num

 $\mathbb{E}$ -Argument:  $\Pr[X \ge \mathbb{E}[X]] > 0$ 

- *X<sub>E</sub>*: num
- Step 2: e
- Size of re
- $\Pr[|S| \ge n^2/(4m)] \ge \Pr[X_V X_E \ge n^2/(4m)] > 0 \checkmark$

Probabilistic Method: Show that something

 $=\frac{n}{2d}=\frac{n}{2(2m/n)}=\frac{n^2}{(4m)}$ 

# **Application: Independent Sets**

The Problem

Proof

- Two vertices in a graph are *independent*, if they are not adjacent
- An independent set of a graph is a subgraph whose vertices are pairwise independent
- Let  $\alpha(G)$  denote the size of a largest independent set in G (in general, determining  $\alpha(G)$  is NP-complete)

**Theorem**: Let G be a graph with n vertices and 
$$m \ge n/2$$
 edges. Then  $\alpha(G) \ge n^2/(4m)$ .

|| Random Process: d = 2m/n

Step 1: Delete v with prob. 
$$1 - \frac{1}{d}$$
  
Step 2: Delete one endpoint of each eexists by proving that it has a *positive*  
probability of occuring from a random process.ber of vertices that survive the first step  
ber of edges that survive the first step  
each of the  $X_E$  edges removes  $\leq 1$  vertex  
esulting independent set S is  $\geq X_V - X_E$  $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  (since each vertex survives with prob.  $\frac{1}{d}$ ) $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  $\mathbb{E}[X_V] = n \cdot \frac{1}{d}$  $\mathbb{E}[X_V] = \frac{nd}{2} \cdot \frac{1}{d^2} = \frac{n}{2d}$  $\mathbb{E}[X_V - X_E] = \mathbb{E}[X_V] - \mathbb{E}[X_E] = \frac{n}{d} - \frac{n}{2d}$ 







**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>



 $\Delta=2 \ \Rightarrow 8\Delta=16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

- **Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.
- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>



 $\Delta=2 \ \Rightarrow 8\Delta=16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>



 $\Delta=2 \ \Rightarrow 8\Delta=16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>



 $\Delta=2 \ \Rightarrow 8\Delta=16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

Positive Probability

• To show:  $\Pr["S independent"] > 0$ 



 $\Delta=2 \ \Rightarrow 8\Delta=16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

- To show:  $\Pr["S independent"] > 0$  (both endpoints in S)
- *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$

 $\Delta = 2 \implies 8\Delta = 16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

- To show:  $\Pr["S independent"] > 0$  (both endpoints in S)
- S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$



 $\Delta = 2 \implies 8\Delta = 16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

- To show:  $\Pr["S independent"] > 0$  (both endpoints in S)
- *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e]$

$$\Delta = 2 \Rightarrow 8\Delta = 16$$



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

- To show:  $\Pr["S independent"] > 0$  (both endpoints in S)
- *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e] = \prod_{e \in E} \Pr[\neg A_e]$

$$\Delta = 2 \Rightarrow 8\Delta = 16$$



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

- To show:  $\Pr["S independent"] > 0$  (both endpoints in S)
- *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$ Pr["*S* independent"] = Pr[ $\bigcap_{e \in E} \neg A_e$ ] =  $\prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e])$



 $\Delta = 2 \implies 8\Delta = 16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

Positive Probability

• To show:  $\Pr["S independent"] > 0$ 

(both endpoints in *S*)

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

• *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$ Pr["*S* independent"] = Pr[ $\bigcap_{e \in E} \neg A_e$ ] =  $\prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \frac{\Pr[A_e]}{\Pr[A_e]})$ 

 $\Delta = 2 \implies 8\Delta = 16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

Positive Probability

To show: Pr["*S* independent"] > 0

(both endpoints in *S*)

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

• *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e] = \prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \frac{\Pr[A_e]}{\Pr[A_e]})$ 



 $\Delta = 2 \implies 8\Delta = 16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

Positive Probability

• To show:  $\Pr["S independent"] > 0$ 

(both endpoints in *S*)

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

• *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$ Pr["*S* independent"] = Pr[ $\bigcap_{e \in E} \neg A_e$ ] =  $\prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e])$ 



 $\Delta = 2 \implies 8\Delta = 16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

Positive Probability

- To show:  $\Pr["S independent"] > 0$
- (both endpoints in *S*)

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

• *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$ Pr["*S* independent"] = Pr[ $\bigcap_{e \in E} \neg A_e$ ] =  $\prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e])$ 



$$\Delta = 2 \Rightarrow 8\Delta = 16$$

if both endpoints in the same  $V_i$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

## Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

- To show:  $\Pr["S independent"] > 0$  (both endpoints in S)
- *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$ Pr["*S* independent"] = Pr[ $\bigcap_{e \in E} \neg A_e$ ] =  $\prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e]) \ge \prod_{e \in E} (1 - \frac{1}{k^2})$



 $\Delta = 2 \implies 8\Delta = 16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

- To show:  $\Pr["S independent"] > 0$  (both endpoints in S)
- *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$   $\Pr["S \text{ independent"}] = \Pr[\bigcap_{e \in E} \neg A_e] = \prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e]) \ge \prod_{e \in E} (1 - \frac{1}{k^2}) > 0$  $\leq \frac{1}{k} \cdot \frac{1}{k}$



 $\Delta = 2 \implies 8\Delta = 16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

## Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each V<sub>i</sub>

- To show:  $\Pr["S independent"] > 0$  (both endpoints in S)
- *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e] = \prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e]) \ge \prod_{e \in E} (1 - \frac{1}{k^2}) > 0 \checkmark$



 $\Delta = 2 \implies 8\Delta = 16$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

## Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each  $V_i$

Positive Probability

• To show: 
$$Pr["S independent"] > 0$$
 (both endpoints in S)

• *S* is independent iff no edge 
$$e = \{u, v\}$$
 has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$   
 $\Pr["S \text{ independent"}] = \Pr[\bigcap_{e \in E} \neg A_e] \neq \prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e]) \ge \prod_{e \in E} (1 - \frac{1}{k^2}) > 0 \checkmark$   
The events are not independent!

cvents are not mu

$$\Delta = 2 \Rightarrow 8\Delta = 16$$

k > 1



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each  $V_i$

- To show:  $\Pr["S independent"] > 0$ (both endpoints in S)
- S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e] \neq \prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e]) \ge \prod_{e \in E} (1 - \frac{1}{k^2}) > 0 \checkmark$ The events are not independent!





```
\Delta = 2 \implies 8\Delta = 16
```



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

## Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let *S* be the set obtained by independently choosing one vertex uniformly at random from each  $V_i$

*Positive Probability* 

- To show:  $\Pr["S independent"] > 0$ (both endpoints in S)
- S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e] \neq \prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e]) \ge \prod_{e \in E} (1 - \frac{1}{k^2}) > 0 \checkmark$ The events are not independent!

The events are not independent!



 $\cdot \quad \Pr[A_{e_1}] = \frac{1}{k^2}$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

#### Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let *S* be the set obtained by independently choosing one vertex uniformly at random from each  $V_i$

*Positive Probability* 

- To show: Pr["*S* independent"] > 0 (both endpoints in S)
- S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e] \neq \prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e]) \ge \prod_{e \in E} (1 - \frac{1}{k^2}) > 0 \checkmark$ The events are not independent!

The events are not independent!



 $\Delta = 2 \implies 8\Delta = 16$ 

 $\Pr[A_{e_1}|A_{e_2} \cap A_{e_3}]$ 

 $\cdot \quad \Pr[A_{e_1}] = \frac{1}{k^2}$ 

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let S be the set obtained by independently choosing one vertex uniformly at random from each  $V_i$

*Positive Probability* 

- To show: Pr["*S* independent"] > 0 (both endpoints in S)
- S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e] \neq \prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e]) \ge \prod_{e \in E} (1 - \frac{1}{k^2}) > 0 \checkmark$ The events are not independent!

The events are not independent!



 $\Delta = 2 \implies 8\Delta = 16$ 

 $\Pr[A_{e_1} | A_{e_2} \cap A_{e_3}]$ 

 $\Pr[A_{e_1}] = \frac{1}{k^2}$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

## Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let *S* be the set obtained by independently choosing one vertex uniformly at random from each  $V_i$

*Positive Probability* 

- To show: Pr["*S* independent"] > 0 (both endpoints in S)
- S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e] \neq \prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e]) \ge \prod_{e \in E} (1 - \frac{1}{k^2}) > 0 \checkmark$ The events are not independent!

The events are not independent!



 $\Delta = 2 \implies 8\Delta = 16$ 

 $\Pr[A_{e_1}] = \frac{1}{k^2}$ 

 $\Pr[A_{e_1} | A_{e_2} \cap A_{e_2}]$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

## Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Let *S* be the set obtained by independently choosing one vertex uniformly at random from each  $V_i$

*Positive Probability* 

- To show:  $\Pr["S independent"] > 0$ (both endpoints in S)
- S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e] \neq \prod_{e \in E} \Pr[\neg A_e] = \prod_{e \in E} (1 - \Pr[A_e]) \ge \prod_{e \in E} (1 - \frac{1}{k^2}) > 0 \checkmark$ The events are not independent!

 $\Pr[A_{e_1} | A_{e_2} \cap A_{e_3}] = 1$  The probability of an event is affected by the outcomes of other events. Dependence...

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 



 $\Delta = 2 \implies 8\Delta = 16$ 

 $\Pr[A_{e_1}] = \frac{1}{k^2}$ 

## To be or not to be... independent



#### Independence

**Definition**: Event *A* is **independent of an event** *B* if Pr[A | B] = Pr[A]. ( $Pr[A \cap B] = Pr[A]Pr[B]$ )

## To be or not to be... independent



#### Independence

**Definition**: Event *A* is **independent of an event** *B* if Pr[A | B] = Pr[A]. ( $Pr[A \cap B] = Pr[A] Pr[B]$ )

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

## To be or not to be... independent



#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

## Example

• Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$ 

Graph 30-02


#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

### Example

• Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$ 

Graph



#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

### Example

• Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$ 

Graph



#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

### Example

• Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$ 

Graph



**Definition**: Event *A* is **independent of an event** *B* if Pr[A | B] = Pr[A]. ( $Pr[A \cap B] = Pr[A] Pr[B]$ )

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

### Example

• Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$ 





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color •  $A = A_{12}$ ,  $B = A_{23}$ :





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color
- $A = A_{12}, B = A_{23}$ :  $Pr[A_{12}]$





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color
- $A = A_{12}, B = A_{23}$ :  $Pr[A_{12}] = \frac{1}{2}$





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. ( $Pr[A \cap B] = Pr[A]Pr[B]$ )

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

### Example

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color

•  $A = A_{12}, B = A_{23}$ :  $Pr[A_{12}] = \frac{1}{2}$  $Pr[A_{12} \mid A_{23}]$ 





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

### Example

10

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color

• 
$$A = A_{12}, B = A_{23}$$
:  
 $Pr[A_{12}] = \frac{1}{2}$   
 $Pr[A_{12} \mid A_{23}] = \frac{Pr[A_{12} \cap A_{23}]}{Pr[A_{23}]}$ 



| Pr            | Graph        | 1 | 2 | 3 | <i>A</i> <sub>12</sub> | <i>A</i> <sub>13</sub>  | A <sub>23</sub>   |
|---------------|--------------|---|---|---|------------------------|---|---|
| $\frac{1}{8}$ | 3 <b>0</b> 1 | 0 | 0 | 0 | <                      | ✓   | ✓   |
| $\frac{1}{8}$ |              | 0 | 0 | 0 | $\checkmark$           | X   | X   |
| $\frac{1}{8}$ |              | 0 | 0 | 0 | X                      | $\checkmark$  | X   |
| $\frac{1}{8}$ |              | 0 | 0 | 0 | X                      | X   | $\checkmark$  |
| $\frac{1}{8}$ | <b>Å</b>     | 0 | 0 | 0 | X                      | X   | $\checkmark$  |
| $\frac{1}{8}$ | <b>Å</b>     | 0 | 0 | 0 | X                      | ✓   | X   |
| $\frac{1}{8}$ |              | 0 | 0 | 0 | ✓                      | ×   | ×   |
| $\frac{1}{8}$ | <u>Å</u>     | 0 | 0 | 0 | <b>√</b>               | <ul> <li>Image: A start of the start of</li></ul> | <ul> <li>Image: A start of the start of</li></ul> |
|               |              |   |   |   |                        |   |   |

#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color

• 
$$A = A_{12}, B = A_{23}$$
:  
 $Pr[A_{12}] = \frac{1}{2}$   
 $Pr[A_{12} \mid A_{23}] = \frac{Pr[A_{12} \cap A_{23}]}{Pr[A_{23}]}$ 



| Pr            | Graph                     | 1 | 2 | 3        | $A_{12}$  | $A_{12}$     | A22          |
|---------------|---------------------------|---|---|----------|---|--------------|--------------|
|               |                           | - | 2 | <u> </u> | <i>/ 12</i>   | , 12         | <i>1</i> 23  |
| 8             | 30-02                     | 0 | 0 | 0        | $\checkmark$  |              | $\checkmark$ |
| $\frac{1}{8}$ |                           | 0 | 0 | 0        | $\checkmark$  | X            | X            |
| $\frac{1}{8}$ |                           | 0 | 0 | 0        | ×   | $\checkmark$ | X            |
| $\frac{1}{8}$ | <b>~</b>                  | 0 | 0 | 0        | ×   | ×            | ✓            |
| $\frac{1}{8}$ | $\sim$                    | 0 | 0 | 0        | ×   | X            | ✓            |
| $\frac{1}{8}$ | $\overset{\bullet}{\sim}$ | 0 | 0 | 0        | X   | ✓            | ×            |
| $\frac{1}{8}$ |                           | 0 | 0 | 0        | ✓   | X            | X            |
| $\frac{1}{8}$ | 00                        | 0 | 0 | 0        | <ul> <li>Image: A start of the start of</li></ul> | ✓            | <            |

#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color







#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color

• 
$$A = A_{12}, B = A_{23}$$
:  
 $\Pr[A_{12}] = \frac{1}{2}$   
 $\Pr[A_{12} \mid A_{23}] = \frac{\Pr[A_{12} \cap A_{23}]}{\Pr[A_{23}]} = \frac{1/4}{1/2}$ 





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color

• 
$$A = A_{12}, B = A_{23}$$
:  
 $Pr[A_{12}] = \frac{1}{2}$   
 $Pr[A_{12} | A_{23}] = \frac{Pr[A_{12} \cap A_{23}]}{Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$ 



| Pr            | Graph    | 1 | 2 | 3 | <i>A</i> <sub>12</sub> | <i>A</i> <sub>13</sub> | A <sub>23</sub> |
|---------------|----------|---|---|---|------------------------|------------------------|-----------------|
| $\frac{1}{8}$ | 30-02    | 0 | 0 | 0 | 1                      | <                      | 1               |
| $\frac{1}{8}$ | <b>~</b> | 0 | 0 | 0 | ✓                      | X                      | ×               |
| $\frac{1}{8}$ |          | 0 | 0 | 0 | ×                      | $\checkmark$           | ×               |
| $\frac{1}{8}$ | <b>Å</b> | 0 | 0 | 0 | X                      | X                      | $\checkmark$    |
| $\frac{1}{8}$ | 00       | 0 | 0 | 0 | X                      | X                      | $\checkmark$    |
| $\frac{1}{8}$ | $\sim$   | 0 | 0 | 0 | X                      | ✓                      | X               |
| $\frac{1}{8}$ |          | 0 | 0 | 0 | ✓                      | X                      | X               |
| $\frac{1}{8}$ | 00       | 0 | 0 | 0 | <b>√</b>               | $\checkmark$           | <               |

#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

### Example

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color

•  $A = A_{12}, B = A_{23}$ :  $\Pr[A_{12}] = \frac{1}{2}$   $\Pr[A_{12} \mid A_{23}] = \frac{\Pr[A_{12} \cap A_{23}]}{\Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$ (same holds for all choices of *A* and *B*)





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

### Example

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color

•  $A = A_{12}, B = A_{23}$ :  $Pr[A_{12}] = \frac{1}{2}$   $Pr[A_{12} | A_{23}] = \frac{Pr[A_{12} \cap A_{23}]}{Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$ (same holds for all choices of *A* and *B*) • All  $A_{ii}$  are *pairwise* independent





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color
- $A = A_{12}, B = A_{23}$ :  $\Pr[A_{12}] = \frac{1}{2}$   $\Pr[A_{12} \mid A_{23}] = \frac{\Pr[A_{12} \cap A_{23}]}{\Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$ (same holds for all choices of *A* and *B*)
   All  $A_{ii}$  are *pairwise* independent





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color
- $A = A_{12}, B = A_{23}$ : •  $Pr[A_{12}] = \frac{1}{2}$ •  $Pr[A_{12} | A_{23}] = \frac{Pr[A_{12} \cap A_{23}]}{Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$ (same holds for all choices of *A* and *B*) • All  $A_{ij}$  are *pairwise* independent





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color
- $A = A_{12}, B = A_{23}$ :  $\Pr[A_{12}] = \frac{1}{2}$   $\Pr[A_{12} \mid A_{23}] = \frac{\Pr[A_{12} \cap A_{23}]}{\Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$ (same holds for all choices of *A* and *B*)
  All  $A_{ii}$  are *pairwise* independent





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. (Pr[A \cap B] = Pr[A] Pr[B])

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color
- $A = A_{12}, B = A_{23}$ :  $Pr[A_{12}] = \frac{1}{2}$   $Pr[A_{12} | A_{23}] = \frac{Pr[A_{12} \cap A_{23}]}{Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$ (same holds for all choices of A and B)
   All  $A_{ii}$  are pairwise independent
    $A = A_{12}, \mathcal{E} = \{A_{13}, A_{23}\}$ :  $Pr[A_{12} | A_{13} \cap A_{23}]$  $= \frac{Pr[A_{12} \cap A_{13} \cap A_{23}]}{Pr[A_{13} \cap A_{23}]}$





#### Independence

**Definition**: Event *A* is **independent of an event** *B* if Pr[A | B] = Pr[A]. ( $Pr[A \cap B] = Pr[A] Pr[B]$ )

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

### Example

10

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that *i* and *j* have the same color
- $A = A_{12}, B = A_{23}$ :  $Pr[A_{12}] = \frac{1}{2}$   $Pr[A_{12} | A_{23}] = \frac{Pr[A_{12} \cap A_{23}]}{Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$ (same holds for all choices of A and B)
   All  $A_{ii}$  are pairwise independent
    $A = A_{12}, \mathcal{E} = \{A_{13}, A_{23}\}$ :  $Pr[A_{12} | A_{13} \cap A_{23}]$  $= \frac{Pr[A_{12} \cap A_{13} \cap A_{23}]}{Pr[A_{13} \cap A_{23}]}$





#### Independence

**Definition**: Event *A* is **independent of an event** *B* if Pr[A | B] = Pr[A]. ( $Pr[A \cap B] = Pr[A] Pr[B]$ )

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that *i* and *j* have the same color
- $A = A_{12}, B = A_{23}$ :  $Pr[A_{12}] = \frac{1}{2}$   $Pr[A_{12} | A_{23}] = \frac{Pr[A_{12} \cap A_{23}]}{Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$ (same holds for all choices of A and B)
   All  $A_{ii}$  are pairwise independent
    $A = A_{12}, \mathcal{E} = \{A_{13}, A_{23}\}$ :  $Pr[A_{12} | A_{13} \cap A_{23}]$  $= \frac{Pr[A_{12} \cap A_{13} \cap A_{23}]}{Pr[A_{13} \cap A_{23}]}$





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. ( $Pr[A \cap B] = Pr[A]Pr[B]$ )

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color
- A = A<sub>12</sub>, B = A<sub>23</sub>: Pr[A<sub>12</sub>] = <sup>1</sup>/<sub>2</sub> Pr[A<sub>12</sub> | A<sub>23</sub>] = <sup>Pr[A<sub>12</sub> ∩ A<sub>23</sub>]/<sub>Pr[A<sub>23</sub>]</sub> = <sup>1/4</sup>/<sub>1/2</sub> = <sup>1</sup>/<sub>2</sub> (same holds for all choices of A and B)
   All A<sub>ii</sub> are pairwise independent
   A = A<sub>12</sub>, E = {A<sub>13</sub>, A<sub>23</sub>}: Pr[A<sub>12</sub>, A<sub>13</sub> ∩ A<sub>23</sub>] = <sup>Pr[A<sub>13</sub> ∩ A<sub>23</sub>]</sup>/<sub>Pr[A<sub>13</sub> ∩ A<sub>23</sub>]</sub> = <sup>1/4</sup>/<sub>1/4</sub> = 1
  </sup>





#### Independence

**Definition**: Event A is **independent of an event** B if Pr[A | B] = Pr[A]. ( $Pr[A \cap B] = Pr[A]Pr[B]$ )

**Definition**: Event *A* is **independent of a set of events**  $\mathcal{E}$  if for all subsets  $\mathcal{E}' = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{E}$  we have  $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$ .

- Triangle, independently color each vertex red/blue with prob.  $\frac{1}{2}$
- Let  $A_{ij}$  for i < j be the event that i and j have the same color
- A = A<sub>12</sub>, B = A<sub>23</sub>: Pr[A<sub>12</sub>] = <sup>1</sup>/<sub>2</sub>
   Pr[A<sub>12</sub> | A<sub>23</sub>] =  $\frac{\Pr[A_{12} \cap A_{23}]}{\Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$  (same holds for all choices of A and B)
   All A<sub>ij</sub> are *pairwise* independent
   A = A<sub>12</sub>, E = {A<sub>13</sub>, A<sub>23</sub>}: Pr[A<sub>12</sub> | A<sub>13</sub> \cap A<sub>23</sub>] =  $\frac{\Pr[A_{12} \cap A_{13} \cap A_{23}]}{\Pr[A_{13} \cap A_{23}]} = \frac{1/4}{1/4} = 1$







**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

• If d = 0, everything is independent and we can just compute the probability as the product



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

 If d = 0, everything is independent and we can just compute the probability as the product
 For each i ∈ [n] let D<sub>i</sub> ⊆ [n] be the such that E<sub>i</sub> is independent of {E<sub>1</sub>, ..., E<sub>n</sub>} \ (U<sub>j∈{i}∪D<sub>i</sub></sub> E<sub>j</sub>), then |D(i)| ≤ d.



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

 If d = 0, everything is independent and we can just compute the probability as the product
 For each i ∈ [n] let D<sub>i</sub> ⊆ [n] be the such that E<sub>i</sub> is independent of {E<sub>1</sub>, ..., E<sub>n</sub>} \ (U<sub>j∈{i}∪D<sub>i</sub></sub> E<sub>j</sub>), then |D(i)| ≤ d.

### Proof

 $\Pr[\bigcap_{i\in[n]}\neg E_i]$ 



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

 If d = 0, everything is independent and we can just compute the probability as the product
 For each i ∈ [n] let D<sub>i</sub> ⊆ [n] be the such that E<sub>i</sub> is independent of {E<sub>1</sub>, ..., E<sub>n</sub>} \ (U<sub>j∈{i}∪D<sub>i</sub></sub> E<sub>j</sub>), then |D(i)| ≤ d. (Remove events defined by D<sub>i</sub> to make E<sub>i</sub> independent of the rest.)

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 

**Proof**  $\mathcal{I}([n])$ 



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

 If d = 0, everything is independent and we can just compute the probability as the product
 For each i ∈ [n] let D<sub>i</sub> ⊆ [n] be the such that E<sub>i</sub> is independent of {E<sub>1</sub>,..., E<sub>n</sub>} \ (U<sub>i∈{i}</sub>, E<sub>i</sub>), then |D(i)| ≤ d. (Remove events defined by D<sub>i</sub> to make E<sub>i</sub> independent of the rest.)

Proof 
$$\mathcal{I}([n])$$
  

$$\underbrace{\Pr[\bigcap_{i \in [n]} \neg E_i]}_{= \Pr[\neg E_n \cap (\neg E_{n-1} \cap \neg E_{n-2} \cap ... \cap \neg E_1)]}$$

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

- If d = 0, everything is independent and we can just compute the probability as the product • For each  $i \in [n]$  let  $D_i \in [n]$  be the such that
- For each  $i \in [n]$  let  $D_i \subseteq [n]$  be the such that  $E_i$  is independent of  $\{E_1, ..., E_n\} \setminus (\bigcup_{j \in \{i\} \cup D_i} E_j)$ , then  $|D(i)| \leq d$ .

(Remove events defined by  $D_i$  to make  $E_i$  independent of the rest.)

Proof 
$$\mathcal{I}([n])$$
  

$$\underbrace{\Pr[\bigcap_{i \in [n]} \neg E_i]}_{= \Pr[\neg E_n \cap (\neg E_{n-1} \cap \neg E_{n-2} \cap ... \cap \neg E_1)]}$$

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 

**Conditional Probability**:  $\Pr[A \cap B] = \Pr[A | B] \cdot \Pr[B]$ 



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

- If d = 0, everything is independent and we can just compute the probability as the product
- For each  $i \in [n]$  let  $D_i \subseteq [n]$  be the such that  $E_i$  is independent of  $\{E_1, ..., E_n\} \setminus (\bigcup_{j \in \{i\} \cup D_i} E_j)$ , then  $|D(i)| \leq d$ .

(Remove events defined by  $D_i$  to make  $E_i$  independent of the rest.)

Notation: For 
$$S \subseteq [n]$$
 write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 

**Conditional Probability**:  $\Pr[A \cap B] = \Pr[A \mid B] \cdot \Pr[B]$ 

 $= \Pr\left[\neg E_n \cap \left(\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1\right)\right]$  $= \Pr\left[\neg E_n \mid \left(\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1\right)\right] \cdot \Pr\left[\left(\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1\right)\right]$ 

Proof

 $\mathcal{I}([n])$ 

 $\Pr[\bigcap_{i\in[n]}\neg E_i]$ 



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

- If d = 0, everything is independent and we can just compute the probability as the product
- For each  $i \in [n]$  let  $D_i \subseteq [n]$  be the such that  $E_i$  is independent of  $\{E_1, ..., E_n\} \setminus (\bigcup_{j \in \{i\} \cup D_i} E_j)$ , then  $|D(i)| \leq d$ .

(Remove events defined by  $D_i$  to make  $E_i$  independent of the rest.)

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 

**Conditional Probability**:  $\Pr[A \cap B] = \Pr[A \mid B] \cdot \Pr[B]$ 

 $\underbrace{\Pr\left[\bigcap_{i\in[n]}\neg E_{i}\right]}_{=\Pr\left[\neg E_{n}\cap\left(\neg E_{n-1}\cap\neg E_{n-2}\cap\ldots\cap\neg E_{1}\right)\right]}_{=\Pr\left[\neg E_{n}\mid\left(\neg E_{n-1}\cap\neg E_{n-2}\cap\ldots\cap\neg E_{1}\right)\right]\cdot\Pr\left[\left(\neg E_{n-1}\cap\neg E_{n-2}\cap\ldots\cap\neg E_{1}\right)\right]}$ 

Proof

 $\mathcal{I}([n])$


**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

- If d = 0, everything is independent and we can just compute the probability as the product
- For each  $i \in [n]$  let  $D_i \subseteq [n]$  be the such that  $E_i$  is independent of  $\{E_1, ..., E_n\} \setminus (\bigcup_{j \in \{i\} \cup D_i} E_j)$ , then  $|D(i)| \leq d$ .

(Remove events defined by  $D_i$  to make  $E_i$  independent of the rest.)

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 

**Conditional Probability**:  $\Pr[A \cap B] = \Pr[A \mid B] \cdot \Pr[B]$ 

 $= \Pr[\neg E_n \cap (\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)]$   $= \Pr[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)] \cdot \Pr[(\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)]$   $= \Pr[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)] \cdot \Pr[(\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)]$ 

 $= \Pr[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)] \cdot \Pr[\neg E_{n-1} \mid (\neg E_{n-2} \cap \dots \cap \neg E_1)] \cdot \Pr[(\neg E_{n-2} \cap \dots \cap \neg E_1)]$ 

Proof

 $\mathcal{I}([n])$ 

 $\Pr[\bigcap_{i\in[n]}\neg E_i]$ 



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

- If d = 0, everything is independent and we can just compute the probability as the product
- For each  $i \in [n]$  let  $D_i \subseteq [n]$  be the such that  $E_i$  is independent of  $\{E_1, ..., E_n\} \setminus (\bigcup_{j \in \{i\} \cup D_i} E_j)$ , then  $|D(i)| \leq d$ .

(Remove events defined by  $D_i$  to make  $E_i$  independent of the rest.)

$$\begin{array}{l} \operatorname{Proof} \quad \mathcal{I}([n]) \\ \underset{i \in [n]}{\operatorname{Pr}[\bigcap_{i \in [n]} \neg E_i]} \\ = \operatorname{Pr}[\neg E_n \cap (\neg E_{n-1} \cap \neg E_{n-2} \cap \ldots \cap \neg E_1)] \\ = \operatorname{Pr}[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \ldots \cap \neg E_1)] \cdot \operatorname{Pr}[(\neg E_{n-1} \cap \neg E_{n-2} \cap \ldots \cap \neg E_1)] \\ = \operatorname{Pr}[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \ldots \cap \neg E_1)] \cdot \operatorname{Pr}[\neg E_{n-1} \mid (\neg E_{n-2} \cap \ldots \cap \neg E_1)] \\ = \operatorname{Pr}[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \ldots \cap \neg E_1)] \cdot \operatorname{Pr}[\neg E_{n-1} \mid (\neg E_{n-2} \cap \ldots \cap \neg E_1)] \cdot \operatorname{Pr}[(\neg E_{n-2} \cap \ldots \cap \neg E_1)] \\ \\ \end{array} \right]$$



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

- If d = 0, everything is independent and we can just compute the probability as the product
- For each  $i \in [n]$  let  $D_i \subseteq [n]$  be the such that  $E_i$  is independent of  $\{E_1, ..., E_n\} \setminus (\bigcup_{j \in \{i\} \cup D_i} E_j)$ , then  $|D(i)| \leq d$ .

(Remove events defined by  $D_i$  to make  $E_i$  independent of the rest.)

$$\begin{array}{l} \text{Proof} \quad \mathcal{I}([n]) \\ \begin{array}{l} \Pr[\bigcap_{i \in [n]} \neg E_i] \\ = \Pr[\neg E_n \cap (\neg E_{n-1} \cap \neg E_{n-2} \cap \ldots \cap \neg E_1)] \\ = \Pr[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \ldots \cap \neg E_1)] \cdot \Pr[(\neg E_{n-1} \cap \neg E_{n-2} \cap \ldots \cap \neg E_1)] \\ = \Pr[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \ldots \cap \neg E_1)] \cdot \Pr[\neg E_{n-1} \mid (\neg E_{n-2} \cap \ldots \cap \neg E_1)] \\ = \Pr[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \ldots \cap \neg E_1)] \cdot \Pr[\neg E_{n-1} \mid (\neg E_{n-2} \cap \ldots \cap \neg E_1)] \cdot \Pr[(\neg E_{n-2} \cap \ldots \cap \neg E_1)] \\ \end{array}$$



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

- If d = 0, everything is independent and we can just compute the probability as the product
- For each  $i \in [n]$  let  $D_i \subseteq [n]$  be the such that  $E_i$  is independent of  $\{E_1, ..., E_n\} \setminus (\bigcup_{j \in \{i\} \cup D_i} E_j)$ , then  $|D(i)| \leq d$ . (Remove events defined by  $D_i$  to make  $E_i$  independent of the rest.)

$$\begin{array}{c} \mathsf{Proof} \quad \mathcal{I}([n]) \\ \mathsf{Pr}[\bigcap_{i \in [n]} \neg E_i] \quad \stackrel{\text{``Chain Rule''}}{=} \quad \prod_{i \in [n]} \mathsf{Pr}[\neg E_i \mid \mathcal{I}([i-1])] \\ \quad = \mathsf{Pr}[\neg E_n \cap (\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)] \\ \quad = \mathsf{Pr}[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)] \cdot \mathsf{Pr}[(\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)] \\ \quad = \mathsf{Pr}[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)] \cdot \mathsf{Pr}[\neg E_{n-1} \mid (\neg E_{n-2} \cap \dots \cap \neg E_1)] \\ \quad = \mathsf{Pr}[\neg E_n \mid (\neg E_{n-1} \cap \neg E_{n-2} \cap \dots \cap \neg E_1)] \cdot \mathsf{Pr}[\neg E_{n-1} \mid (\neg E_{n-2} \cap \dots \cap \neg E_1)] \\ \quad \mathcal{I}([n-1]) \quad \qquad \mathcal{I}([n-2]) \end{array}$$



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

 If d = 0, everything is independent and we can just compute the probability as the product
 For each i ∈ [n] let D<sub>i</sub> ⊆ [n] be the such that E<sub>i</sub> is independent of {E<sub>1</sub>, ..., E<sub>n</sub>} \ (U<sub>j∈{i}∪D<sub>i</sub></sub> E<sub>j</sub>), then |D(i)| ≤ d.

**Proof**  $\mathcal{I}([n])$  $\Pr[\bigcap_{i \in [n]} \neg E_i] \stackrel{\text{"Chain Rule"}}{=} \prod_{i \in [n]} \Pr[\neg E_i \mid \mathcal{I}([i-1])]$ 

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 



**Theorem**: Let  $E_1, \ldots, E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

• If d = 0, everything is independent and we can just compute the probability as the product • For each  $i \in [n]$  let  $D_i \subseteq [n]$  be the such that  $E_i$  is independent of  $\{E_1, ..., E_n\} \setminus (\bigcup_{j \in \{i\} \cup D_i} E_j)$ , then  $|D(i)| \leq d$ .

**Proof**  $\mathcal{I}([n])$   $\Pr[\bigcap_{i \in [n]} \neg E_i] \stackrel{\text{"Chain Rule"}}{=} \prod_{i \in [n]} \Pr[\neg E_i \mid \mathcal{I}([i-1])]$  $= \prod_{i \in [n]} (1 - \Pr[E_i \mid \mathcal{I}([i-1])])$ 

(Remove events defined by  $D_i$  to make  $E_i$  independent of the rest.)

| Notation: For $S \subseteq [n]$ write         | te |
|---|----|
| $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ |    |



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

 If d = 0, everything is independent and we can just compute the probability as the product
 For each i ∈ [n] let D<sub>i</sub> ⊆ [n] be the such that E<sub>i</sub> is independent of {E<sub>1</sub>, ..., E<sub>n</sub>} \ (U<sub>j∈{i}∪D<sub>i</sub></sub> E<sub>j</sub>), then |D(i)| ≤ d.

 $\begin{array}{c} \mathbf{Proof} \quad \mathcal{I}([n]) \\ \Pr[\bigcap_{i \in [n]} \neg E_i] \stackrel{\text{``Chain Rule''}}{=} \prod_{i \in [n]} \Pr[\neg E_i \mid \mathcal{I}([i-1])] \\ = \prod_{i \in [n]} (1 - \Pr[E_i \mid \mathcal{I}([i-1])]) \\ \hline \mathbf{Claim:} \leq 2p \end{array}$ 

| Notation: For $S \subset [r]$                            | ] write |
|--|---------|
| $\mathcal{I}(S) = \bigcap_{i \in S} \neg \overline{E_i}$ | -       |



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

 If d = 0, everything is independent and we can just compute the probability as the product
 For each i ∈ [n] let D<sub>i</sub> ⊆ [n] be the such that E<sub>i</sub> is independent of {E<sub>1</sub>, ..., E<sub>n</sub>} \ (U<sub>j∈{i}∪D<sub>i</sub></sub> E<sub>j</sub>), then |D(i)| ≤ d.

| Notation: For $T(S) = O$ | $S \subseteq [n]$ | write |
|--------------------------|-------------------|-------|



**Theorem**: Let  $E_1, \ldots, E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

• If d = 0, everything is independent and we can just compute the probability as the product • For each  $i \in [n]$  let  $D_i \subseteq [n]$  be the such that  $E_i$  is independent of  $\{E_1, ..., E_n\} \setminus (\bigcup_{i \in \{i\} \cup D_i} E_j)$ , then  $|D(i)| \leq d$ .

**Proof**  $\mathcal{I}([n])$   $\Pr[\bigcap_{i \in [n]} \neg E_i] \stackrel{\text{"Chain Rule"}}{=} \prod_{i \in [n]} \Pr[\neg E_i \mid \mathcal{I}([i-1])]$  $=\prod_{i\in[n]}(1-\Pr[E_i\mid\mathcal{I}([i-1])])$ Claim:  $\leq 2p$  $\geq \prod_{i \in [n]} (1-2p)$ 

(Remove events defined by  $D_i$  to make  $E_i$  independent of the rest.)

> Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$

**Conditional Probability:**  $\Pr[A \cap B] = \Pr[A \mid B] \cdot \Pr[B]$ 

• Since d > 0 and  $4dp \le 1$ , we have  $4p \le 1$  and thus  $2p \le 1/2$ 



Notation: For  $S \subseteq [n]$  write

**Conditional Probability**:  $Pr[A \cap B] = Pr[A | B] \cdot Pr[B]$ 

 $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 

**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

 If d = 0, everything is independent and we can just compute the probability as the product
 For each i ∈ [n] let D<sub>i</sub> ⊆ [n] be the such that E<sub>i</sub> is independent of {E<sub>1</sub>, ..., E<sub>n</sub>} \ (U<sub>i∈{i}∪D<sub>i</sub></sub> E<sub>j</sub>), then |D(i)| ≤ d.

Proof 
$$\mathcal{I}([n])$$
  
 $\Pr[\bigcap_{i \in [n]} \neg E_i] \stackrel{\text{``Chain Rule''}}{=} \prod_{i \in [n]} \Pr[\neg E_i \mid \mathcal{I}([i-1])]$   
 $= \prod_{i \in [n]} (1 - \Pr[E_i \mid \mathcal{I}([i-1])])$   
 $Claim: \leq 2p$   
 $\geq \prod_{i \in [n]} (1 - 2p)$   
 $\geq \prod_{i \in [n]} 1/2$   
Since  $d > 0$  and  $4dp \leq 1$ , we have  $4p \leq 1$  and thus  $2p \leq 1/2$ 

 $p \leq 1/2$ 



**Theorem**: Let  $E_1, ..., E_n$  be events such that each  $E_i$  for  $i \in [n]$  is independent of all but at most d > 0 of the other events. Let  $p = \max_{i \in [n]} \Pr[E_i]$ . If  $4dp \le 1$ , then  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

 If d = 0, everything is independent and we can just compute the probability as the product
 For each i ∈ [n] let D<sub>i</sub> ⊆ [n] be the such that E<sub>i</sub> is independent of {E<sub>1</sub>, ..., E<sub>n</sub>} \ (U<sub>i∈{i}∪D<sub>i</sub></sub> E<sub>j</sub>), then |D(i)| ≤ d.

Proof 
$$\mathcal{I}([n])$$
  
 $\Pr[\bigcap_{i \in [n]} \neg E_i] \stackrel{\text{"Chain Rule"}}{=} \prod_{i \in [n]} \Pr[\neg E_i \mid \mathcal{I}([i-1])]$   
 $= \prod_{i \in [n]} (1 - \Pr[E_i \mid \mathcal{I}([i-1])])$   
Claim:  $\leq 2p$  ← remains to prove this!  
 $\geq \prod_{i \in [n]} (1 - 2p)$   
 $\geq \prod_{i \in [n]} 1/2 > 0 \checkmark$   
Since  $d > 0$  and  $4dp \leq 1$ , we have  $4p \leq 1$  and thus  $2p \leq 1/2$ 



**Claim**: For all  $S_i \subseteq \{1, ..., n\} \setminus \{i\}$ ,  $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$ .

#### Proof

LLL: Events 
$$E_1, ..., E_n$$
  
•  $p = \max_{i \in [n]} \Pr[E_i]$   
•  $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
•  $4dp \le 1$ 

Notation: For 
$$S \subseteq [n]$$
 write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 



#### **Claim**: For all $S_i \subseteq \{1, ..., n\} \setminus \{i\}$ , $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$ .

**Proof** (via induction over the size  $s = |S_i|$ )

LLL: Events 
$$E_1, ..., E_n$$
  
•  $p = \max_{i \in [n]} \Pr[E_i]$   
•  $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
•  $4dp \le 1$ 

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 



**Claim**: For all  $S_i \subseteq \{1, ..., n\} \setminus \{i\}$ ,  $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$ .

**Proof** (via induction over the size  $s = |S_i|$ ) Start: s = 0

LLL: Events  $E_1, ..., E_n$   $p = \max_{i \in [n]} \Pr[E_i]$   $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with  $|D_i| \le d$  $4dp \le 1$ 

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 



**Claim**: For all  $S_i \subseteq \{1, ..., n\} \setminus \{i\}$ ,  $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$ .

**Proof** (via induction over the size  $s = |S_i|$ ) Start:  $s = 0 \rightarrow S_i = \emptyset$ 

LLL: Events  $E_1, ..., E_n$   $p = \max_{i \in [n]} \Pr[E_i]$   $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with  $|D_i| \le d$  $4dp \le 1$ 

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 



**Claim**: For all  $S_i \subseteq \{1, ..., n\} \setminus \{i\}$ ,  $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$ .

**Proof** (via induction over the size  $s = |S_i|$ ) Start:  $s = 0 \rightarrow S_i = \emptyset \rightarrow \Pr[E_i | \mathcal{I}(S_i)] = \Pr[E_i]$ 

LLL: Events  $E_1, ..., E_n$   $p = \max_{i \in [n]} \Pr[E_i]$   $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with  $|D_i| \le d$  $4dp \le 1$ 

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 



Claim: For all  $S_i \subseteq \{1, ..., n\} \setminus \{i\}$ ,  $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$ . Proof (via induction over the size  $s = |S_i|$ ) Start:  $s = 0 \rightarrow S_i = \emptyset \rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark$ LLL: Events  $E_1, ..., E_n$   $p = \max_{i \in [n]} \Pr[E_i]$   $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with  $|D_i| \leq d$   $4dp \leq 1$ Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ Conditional Probability:  $\Pr[A \cap B] = \Pr[A|B] \cdot \Pr[B]$ 



**Claim**: For all  $S_i \subseteq \{1, ..., n\} \setminus \{i\}$ ,  $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$ .

**Proof** (via induction over the size  $s = |S_i|$ ) Start:  $s = 0 \rightarrow S_i = \emptyset \rightarrow \Pr[E_i | \mathcal{I}(S_i)] = \Pr[E_i] \le p \le 2p \checkmark$ LLL: Events  $E_1, ..., E_n$   $p = \max_{i \in [n]} \Pr[E_i]$   $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with  $|D_i| \le d$ 

• 
$$4dp \leq 1$$

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 



Claim: For all  $S_i \subseteq \{1, ..., n\} \setminus \{i\}$ ,  $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$ . Proof (via induction over the size  $s = |S_i|$ ) Start:  $s = 0 \twoheadrightarrow S_i = \emptyset \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark$ Step: s > 0Case 1:  $D'_i = S_i \cap D_i = \emptyset$ LLL: Events  $E_1, ..., E_n$   $p = \max_{i \in [n]} \Pr[E_i]$   $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with  $|D_i| \leq d$  $4dp \leq 1$ 

Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ 











Claim: For all  $S_i \subseteq \{1, ..., n\} \setminus \{i\}$ ,  $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$ . Proof (via induction over the size  $s = |S_i|$ ) Start:  $s = 0 \rightarrow S_i = \emptyset \rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p$ Step: s > 0• Case 1:  $D'_i = S_i \cap D_i = \emptyset$ •  $E_i$  is independent of  $\{E_j \mid j \in S_i\} \rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p$ Notation: For  $S \subseteq [n]$  write  $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$ Conditional Probability:  $\Pr[A \cap B] = \Pr[A \mid B] \cdot \Pr[B]$ 







$$\begin{array}{l} \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \twoheadrightarrow S_i = \emptyset \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \textbf{LLL: Events } E_1, ..., E_n \\ \bullet p = \max_{i \in [n]} \Pr[E_i] \\ \bullet p = \max_{i \in [n]} \Pr[E_i] \\ \bullet E_i \text{ independent of } \{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j \\ \texttt{with } |D_i| \leq d \\ \bullet dap \leq 1 \\ \textbf{Case 1: } D'_i = S_i \cap D_i = \emptyset \\ \bullet E_i \text{ is independent of } \{E_j \mid j \in S_i\} \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \\ \bullet Case 2: D'_i = S_i \cap D_i \neq \emptyset \\ \Pr[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]} \\ \end{array}$$



$$\begin{array}{l} \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \twoheadrightarrow S_i = \emptyset \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \textbf{LLL: Events } E_1, ..., E_n \\ \bullet p = \max_{i \in [n]} \Pr[E_i] \\ \bullet p = \max_{i \in [n]} \Pr[E_i \mid \mathcal{I}(S_i)] \\ \bullet p = \max_{i \in [n]} \Pr[E_i \mid \mathcal{I}(S_i)] \\ \bullet p = \max_{i \in [n]} \Pr[\mathcal{I}(S_i)] \\ \bullet$$



$$\begin{array}{l} \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \twoheadrightarrow S_i = \emptyset \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \textbf{Step: } s > 0 \\ \textbf{Step: } s > 0 \\ \textbf{Case 1: } D'_i = S_i \cap D_i = \emptyset \\ \textbf{Step: } E_i \text{ is independent of } \{E_j \mid j \in S_i\} \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \\ \textbf{Step: } E_i \text{ is independent of } \{E_j \mid j \in S_i\} \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \\ \textbf{Step: } Case 2: D'_i = S_i \cap D_i \neq \emptyset \\ \textbf{Step: } Pr[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]} \\ \textbf{T}(S_i) = \bigcap_{i \in S_i} \neg E_i \end{array}$$













$$\begin{array}{l} \hline \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \hline \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \twoheadrightarrow S_i = \emptyset \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \hline \textbf{LLI: Events } E_1, ..., E_n \\ \bullet p = \max_{i \in [n]} \Pr[E_i] \\ \bullet E_i \text{ independent of } \{E_1, ..., E_n \} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j \\ \bullet Case 1: D'_i = S_i \cap D_i = \emptyset \\ \bullet E_i \text{ is independent of } \{E_j \mid j \in S_i\} \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \\ \bullet \text{ Case 2: } D'_i = S_i \cap D_i \neq \emptyset \\ \hline \textbf{Pr}[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(S_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]} \\ \hline \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)]} \\ \hline \end{array}$$



$$\begin{array}{l} \hline \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \hline \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \Rightarrow S_i = \emptyset \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \hline \textbf{LLL: Events } E_1, ..., E_n \\ \bullet p = \max_{i \in [n]} \Pr[E_i] \\ \bullet E_i \text{ independent of } \{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i \in S_j\} \\ \bullet E_i \text{ is independent of } \{E_j \mid j \in S_i\} \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \\ \bullet \text{ Case 1: } D'_i = S_i \cap D_i = \emptyset \\ \bullet \text{ Case 2: } D'_i = S_i \cap D_i \neq \emptyset \\ \hline \textbf{Pr}[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(S_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]} \\ \hline \textbf{Pr}[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(S_i \setminus D'_i)]} \\ = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) |\mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) |\mathcal{I}(S_i \setminus D'_i)]} \\ \hline \end{array}$$



$$\begin{array}{l} \hline \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \hline \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \Rightarrow S_i = \emptyset \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \begin{array}{l} \textbf{LL: Events } E_1, ..., E_n \\ \bullet p = \max_{i \in [n]} \Pr[E_i] \\ \bullet E_i \text{ independent of } \{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j \\ \bullet E_i \text{ independent of } \{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j \\ \bullet E_i \text{ is independent of } \{E_j \mid j \in S_i\} \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \\ \bullet \text{ Case } 2: D'_i = S_i \cap D_i \neq \emptyset \\ \bullet \text{ Case } 2: D'_i = S_i \cap D_i \neq \emptyset \\ \Pr[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(S_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]} \\ \hline \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)]} \end{array}$$



$$\begin{array}{l} \hline \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \hline \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \Rightarrow S_i = \emptyset \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \hline \textbf{LLI: Events } E_1..., E_n \\ \bullet p = \max_{i \in [n]} \Pr[E_i] \\ \bullet E_i \text{ independent of } \{E_1, ..., E_n \} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j \\ \hline \textbf{Step: } s > 0 \\ \bullet \text{ Case 1: } D'_i = S_i \cap D_i = \emptyset \\ \bullet E_i \text{ is independent of } \{E_j \mid j \in S_i\} \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \\ \bullet \text{ Case 2: } D'_i = S_i \cap D_i \neq \emptyset \\ \hline \textbf{Pr}[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D_i) \cap \mathcal{I}(S_i \setminus D'_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]} \\ \hline \textbf{Conditional Probability:} \\ \hline \textbf{Pr}[E_i \cap \mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{S}_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{I}(S_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{I}(S_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(\mathcal{I}(S_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal{I}(S_i \setminus D'_i)] \\ \hline \textbf{Pr}[\mathcal{I}(S_i \setminus D'_i)] \cdot \Pr[\mathcal$$



$$\begin{array}{l} \hline \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \hline \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \twoheadrightarrow S_i = \emptyset \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \hline \textbf{LL: Events } E_1, ..., E_n \\ \bullet p = \max_{i \in [n]} \Pr[E_i] \\ \bullet E_i \text{ independent of } \{E_1, ..., E_n \} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j \\ \hline \textbf{Step: } s > 0 \\ \bullet \text{ Case 1: } D'_i = S_i \cap D_i = \emptyset \\ \bullet E_i \text{ is independent of } \{E_j \mid j \in S_i\} \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \\ \bullet \text{ Case 2: } D'_i = S_i \cap D_i \neq \emptyset \\ \bullet \text{ Case 2: } D'_i = S_i \cap D_i \neq \emptyset \\ \hline \textbf{Pr}[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(S_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]}{\Pr[\mathcal{I}(D'_i) \cap \mathcal{I}(S_i \setminus D'_i)]} \\ \hline \begin{array}{c} \textbf{Pr}[E_i \cap \mathcal{I}(D'_i) | \mathcal{I}(S_i \setminus D'_i)] \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) | \mathcal{I}(S_i \setminus D'_i)] \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) | \mathcal{I}(S_i \setminus D'_i)] \\ \hline \end{array} \right. \\ \hline \end{array}$$


















$$\begin{array}{l} \hline \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \hline \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \Rightarrow S_i = \emptyset \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \hline \textbf{Stap: } s > 0 \\ \hline \textbf{Case 1: } D'_i = S_i \cap D_i = \emptyset \\ \hline \textbf{E}_i \text{ is independent of } \{E_j \mid j \in S_i\} \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \\ \hline \textbf{Mattion: For } S \subseteq [n] \text{ write } \\ \hline \textbf{Case 2: } D'_i = S_i \cap D_i \neq \emptyset \\ \hline \textbf{Pr}[E_i \mid \mathcal{I}(S_i)] \leq \frac{p}{\Pr[\mathcal{I}(D'_i)|\mathcal{I}(S_i \setminus D'_i)]} \text{ remains to show } \geq \frac{1}{2} \\ \hline \textbf{Conditional Probability: } \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B|B] \cdot \Pr[B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B|B] \cdot \Pr[A \cap B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B|B] \cdot \Pr[A \cap B] \\ \hline \textbf{Pr}[A \cap B] = \Pr[A|B|B| \cdot \Pr[A \cap B] \\ \hline \textbf{Pr}[A \cap B] \\ \hline \textbf{Pr}[A$$

 $\Pr[A \cap B] \leq \Pr[A]$ 



$$\begin{array}{l} \hline \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \hline \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \twoheadrightarrow S_i = \emptyset \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \hline \textbf{LL: Events } E_1, ..., E_n \\ \bullet p = \max_{i \in [n]} \Pr[E_i] \\ \bullet E_i \text{ independent of } \{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j \\ \hline \textbf{Step: } s > 0 \\ \hline \textbf{Case 1: } D'_i = S_i \cap D_i = \emptyset \\ \bullet E_i \text{ is independent of } \{E_j \mid j \in S_i\} \twoheadrightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \\ \hline \textbf{Mataion: For } S \subseteq [n] \text{ write } \\ \hline \textbf{Case 2: } D'_i = S_i \cap D_i \neq \emptyset \\ \hline \textbf{Pr}[\mathcal{I}(D'_i)] \leq \frac{p}{\Pr[\mathcal{I}(D'_i)]\mathcal{I}(S_i \setminus D'_i)]} \text{ premains to show } \geq \frac{1}{2} \\ \Pr[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] = \Pr[\bigcap_{j \in D'_i} \neg E_j \mid \mathcal{I}(S_i \setminus D'_i)] \end{array}$$













$$\begin{array}{l} \hline \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \hline \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \Rightarrow S_i = \emptyset \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \hline \textbf{Start: } s = 0 \Rightarrow S_i = \emptyset \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \hline \textbf{Start: } s > 0 \\ \hline \textbf{Case 1: } D'_i = S_i \cap D_i = \emptyset \\ \hline \textbf{Case 2: } D'_i = S_i \cap D_i \neq \emptyset \\ \hline \textbf{Pr}[\mathcal{I}(S_i)] \leq \frac{p}{\Pr[\mathcal{I}(D'_i)|\mathcal{I}(S_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] = \Pr[\bigcap_{j \in D'_i} \neg E_j \mid \mathcal{I}(S_i \setminus D'_i)] \\ = 1 - \Pr[\bigcup_{j \in D'_i} \Pr[E_j \mid \mathcal{I}(S_i \setminus D'_i)] \\ \hline \textbf{(via union bound)} \geq 1 - \sum_{j \in D'_i} \Pr[E_j \mid \mathcal{I}(S_i \setminus D'_i)] \\ \hline \textbf{D'_i} \leq D_i \leq D_i \leq 1 - \sum_{j \in D'_i} \Pr[E_j \mid \mathcal{I}(S_i \setminus D'_i)] \\ \hline \textbf{D'_i} \leq D_i \leq D_i \leq D_i \leq 1 - \sum_{j \in D'_i} \Pr[E_j \mid \mathcal{I}(S_i \setminus D'_i)] \\ \hline \textbf{D'_i} \leq D_i \leq D_i \leq D_i \leq D_i \leq 2p \end{cases}$$



$$\begin{array}{l} \hline \textbf{Claim: For all } S_i \subseteq \{1, ..., n\} \setminus \{i\}, \Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p. \\ \hline \textbf{Proof (via induction over the size } s = |S_i|) \\ Start: s = 0 \Rightarrow S_i = \emptyset \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \hline \textbf{Start: } s = 0 \Rightarrow S_i = \emptyset \Rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p \checkmark \\ \hline \textbf{Start: } s > 0 \\ \hline \textbf{Case 1: } D'_i = S_i \cap D_i = \emptyset \\ \hline \textbf{Case 2: } D'_i = S_i \cap D_i \neq \emptyset \\ \hline \textbf{Pr}[E_i \mid \mathcal{I}(S_i)] \leq \frac{p}{\Pr[\mathcal{I}(D'_i)|\mathcal{I}(S_i \setminus D'_i)]} \\ \hline \textbf{Pr}[\mathcal{I}(D'_i) \mid \mathcal{I}(S_i \setminus D'_i)] = \Pr[\bigcap_{j \in D'_i} \neg E_j \mid \mathcal{I}(S_i \setminus D'_i)] \\ = 1 - \Pr[\bigcup_{j \in D'_i} \Pr[E_j \mid \mathcal{I}(S_i \setminus D'_i)] \\ \hline \textbf{(via union bound)} \geq 1 - \sum_{j \in D'_i} \Pr[E_j \mid \mathcal{I}(S_i \setminus D'_i)] \\ \hline \textbf{D'_i} \leq D_i \leq D_i \leq 1 \\ \hline \textbf{D'_i} \leq D_i \leq 2p \\ \hline \textbf{D'_i} = \mathbf{D'_i} \\ \hline \textbf{D'_i} \\ \hline \textbf{D'_i} = \mathbf{D'_i} \\ \hline \textbf{D'_i} \\ \hline \textbf{D'_i} = \mathbf{D'_i} \\ \hline \textbf{D'_i} \\ \hline \textbf{D'_i}$$

**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each V<sub>i</sub> Positive Probability
- To show:  $\Pr["S independent"] > 0$  (both endpoints in *S*)
- *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

### Proof

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each V<sub>i</sub> Positive Probability
- To show:  $\Pr["S independent"] > 0$  (both endpoints in *S*)
- *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in F} \neg A_e]$





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a positive

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each V<sub>i</sub> Positive Probability
  LLL: Events

• To show: 
$$\Pr["S independent"] > 0$$
 (both endpoints in S)

• *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e]$  LLL: Events  $E_1, ..., E_n$   $p = \max_{i \in [n]} \Pr[E_i]$   $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with  $|D_i| \le d$   $4dp \le 1$ Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0.$ 





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a positive

#### Proof

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each V<sub>i</sub> Positive Probability
  LLL: Events
- To show:  $\Pr["S independent"] > 0$  (both endpoints in S)
- *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e]$  $\Pr[A_e] \leq \frac{1}{k^2} =: p$







**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

exists by proving that it has a *positive* 

#### Proof

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each  $V_i$ Positive Probability
- To show: Pr["*S* independent"] > 0 (both endpoints in *S*)
- S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ . Let  $A_e$  be the event that  $e \subseteq S$  $\Pr["S independent"] = \Pr[\bigcap_{e \in F} \neg A_e]$  $\Pr[A_e] \leq \frac{1}{k^2} =: p$

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

13

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain *S* by ind. choosing one vertex unif. at random from each  $V_i$ *Positive Probability*

To show: Pr["S independent"] > 0 (both endpoints in S)
S is independent iff no edge 
$$e = \{u, v\}$$
 has  $e \subseteq S$ ,  
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$   
Pr["S independent"] = Pr[ $\bigcap_{e \in E} \neg A_e$ ]  
Pr[ $A_e$ ]  $\leq \frac{1}{k^2} =: p$ 

Maximilian Katzmann, Stefan Walzer - Probability & Computing







**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

#### Proof

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each  $V_i$ Positive Probability

• To show: 
$$\Pr["S \text{ independent"}] > 0$$
 (both endpoints in S)  
• S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ ,  
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$   
 $\Pr["S \text{ independent"}] = \Pr[\bigcap_{e \in E} \neg A_e]$   
 $\Pr[A_e] \le \frac{1}{k^2} \rightleftharpoons p$   
 $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_i) \neq \emptyset\}$ 



**LLL**: Events  $E_1, \ldots, E_n$  $\blacksquare p = \max_{i \in [n]} \Pr[E_i]$ •  $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{i \in \{i\} \cup D_i} E_j$ with  $|D_i| \leq d$ ■ 4*dp* < 1 Then,  $Pr[\bigcap_{i\in[n]} \neg E_i] > 0.$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each  $V_i$ Positive Probability
- To show:  $\Pr["S independent"] > 0$  (both endpoints in *S*) • S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_i$  $\Pr["S independent"] = \Pr[\bigcap_{e \in F} \neg A_e]$  $\Pr[A_e] \leq \frac{1}{k^2} \eqqcolon p$  $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\}$

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

This is like isolating  $V_i$ ,  $V_j$  from the remainder of the graph

```
Probabilistic Method: Show that something
exists by proving that it has a positive
probability of occuring from a random process.
```





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

#### Proof

Random Process

 $\Pr[A_e] \leq \frac{1}{k^2} =: p$ 

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each V<sub>i</sub> Positive Probability
  LLL: Events A

• To show: 
$$Pr["S independent"] > 0$$
 (both endpoints in S)

• *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ , Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$  $\Pr["S independent"] = \Pr[\bigcap_{e \in E} \neg A_e]$ 

Probabilistic Method: Show that something

exists by proving that it has a *positive* probability of occuring from a random process.

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

This is like isolating  $V_i$ ,  $V_j$  from the remainder of the graph No matter the outcome of  $A_f$  for  $f \in E \setminus D_e$ , the probability for a node in  $V_i$  or  $V_j$  to be chosen remains the same  $\Rightarrow A_e$  is independent of all events but  $D_e$ 

 $D_{e} = \{A_{e'} \mid e' \cap (V_i \cup V_i) \neq \emptyset\}$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

exists by proving that it has a *positive* 

### Proof

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each  $V_i$ Positive Probability

• To show: 
$$\Pr["S \text{ independent"}] > 0$$
 (both endpoints in *S*)  
• *S* is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ ,  
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$   
 $\Pr["S \text{ independent"}] = \Pr[\bigcap_{e \in E} \neg A_e]$   
 $\Pr[A_e] \le \frac{1}{k^2} =: p$   
 $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_i) \neq \emptyset\} \rightarrow |D_e| < k\Delta + k\Delta$ 







**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

### Proof

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each V<sub>i</sub> Positive Probability
  LLL: Events

To show: Pr["S independent"] > 0 (both endpoints in S)
S is independent iff no edge 
$$e = \{u, v\}$$
 has  $e \subseteq S$ ,  
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$   
Pr["S independent"] = Pr[ $\bigcap_{e \in E} \neg A_e$ ]  
Pr[ $A_e$ ]  $\leq \frac{1}{k^2} \rightleftharpoons p$   $|V_i| = |V_j|$   
 $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta$ 







**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

### Proof

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each V<sub>i</sub> Positive Probability
  LLL: Events

To show: Pr["S independent"] > 0 (both endpoints in S)
S is independent iff no edge 
$$e = \{u, v\}$$
 has  $e \subseteq S$ ,  
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$   
Pr["S independent"] = Pr[ $\bigcap_{e \in E} \neg A_e$ ]  
Pr[ $A_e$ ]  $\leq \frac{1}{k^2} \rightleftharpoons p$   $|V_i| |V_j|$   
 $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta$ 







**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

### Proof

13

Random Process

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each  $V_i$ Positive Probability

To show: Pr["S independent"] > 0 (both endpoints in S)
S is independent iff no edge e = {u, v} has e ⊆ S,
Let A<sub>e</sub> be the event that e ⊆ S 
$$V_i V_j$$
Pr["S independent"] = Pr[ $\bigcap_{e \in E} \neg A_e$ ]
Pr[A<sub>e</sub>] ≤  $\frac{1}{k^2} =: p$   $|V_i| |V_j|$ 
D<sub>e</sub> = {A<sub>e'</sub> | e' ∩ (V<sub>i</sub> ∪ V<sub>j</sub>) ≠ Ø} → |D<sub>e</sub>| ≤  $\frac{k\Delta}{k\Delta} + \frac{k\Delta}{k\Delta} \le 2k\Delta =: d$ 

Maximilian Katzmann, Stefan Walzer - Probability & Computing







Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

### Proof

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain *S* by ind. choosing one vertex unif. at random from each  $V_i$ *Positive Probability*

• To show: 
$$\Pr["S \text{ independent}"] > 0$$
 (both endpoints in S)  
• S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ ,  
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$   
 $\Pr["S \text{ independent}"] = \Pr[\bigcap_{e \in E} \neg A_e]$   
 $\Pr[A_e] \leq \frac{1}{k^2} =: p$   $|V_i| |V_j|$   
 $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta =: d$   
 $Pr[A_e] \leq L_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta =: d$ 

**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

### Proof

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each V<sub>i</sub> Positive Probability
  LLL: Events B

To show: 
$$\Pr["S \text{ independent"}] > 0$$
 (both endpoints in S)
S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ ,
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$ 
 $\Pr["S \text{ independent"}] = \Pr[\bigcap_{e \in E} \neg A_e]$ 
 $\Pr[A_e] \leq \frac{1}{k^2} = p$ 
 $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta = d$ 
 $Pr[A_e] \leq 2k\Delta = d$ 



LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .  
 $4dp = 4 \cdot 2k\Delta \cdot \frac{1}{k^2}$ 





**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$  such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 

### Proof

- Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each  $V_i$ *Positive Probability*

• To show: 
$$\Pr["S \text{ independent"}] > 0$$
 (both endpoints in S)  
• S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ ,  
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$   
 $\Pr["S \text{ independent"}] = \Pr[\bigcap_{e \in E} \neg A_e]$   
 $\Pr[A_e] \leq \frac{1}{k^2} =: p$   
 $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta =: d$   
•  $p = \max_{i \in [n]} \Pr[E_i]$   
•  $E_i \text{ independent of } \{E_1, \dots, E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \leq d$   
•  $4dp \leq 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0.$   
4 $dp = 4 \cdot 2k\Delta \cdot \frac{1}{k^2}$   
 $= \frac{8\Delta}{k}$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

exists by proving that it has a *positive* 

### Proof

Random Process

probability of occuring from a random process. • Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )



• Obtain S by ind. choosing one vertex unif. at random from each  $V_i$ Positive Probability LLL: Events F1..... Fn

To show: 
$$\Pr["S \text{ independent"}] > 0$$
 (both endpoints in S)
S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ ,
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$ 
 $\Pr["S \text{ independent"}] = \Pr[\bigcap_{e \in E} \neg A_e]$ 
 $\Pr[A_e] \leq \frac{1}{k^2} =: p$ 
 $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta =: d$ 
 $pr[A_e] \leq 2k\Delta =: d$ 
 $Pr[A_e] = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta =: d$ 
 $Pr[A_e] = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta =: d$ 
 $Pr[A_e] = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta =: d$ 
 $Pr[A_e] = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta =: d$ 
 $Pr[A_e] = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\}$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

exists by proving that it has a *positive* 

### Proof

Random Process

probability of occuring from a random process. • Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )



• Obtain S by ind. choosing one vertex unif. at random from each  $V_i$ Positive Probability LLL: Events F1..... Fn

• To show: 
$$\Pr["S \text{ independent"}] > 0$$
 (both endpoints in S)  
• S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ ,  
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$   
 $\Pr["S \text{ independent"}] = \Pr[\bigcap_{e \in E} \neg A_e]$   
 $\Pr[A_e] \leq \frac{1}{k^2} =: p$   
 $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta =: d$   
•  $p = \max_{i \in [n]} \Pr[E_i]$   
•  $E_i \text{ independent of } \{E_1, \dots, E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \leq d$   
•  $4dp \leq 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0.$   
4 $dp = 4 \cdot 2k\Delta \cdot \frac{1}{k^2}$   
 $= \frac{8\Delta}{k} = 1$ 



**Theorem**: Let G = (V, E) be a graph with max-degree  $\Delta$ . For any partition  $V_1 \cup ... \cup V_t = V$ such that  $|V_i| \ge 8\Delta$ , there exists an independent set containing one vertex from each  $V_i$ .

Probabilistic Method: Show that something

exists by proving that it has a *positive* 

### Proof

- probability of occuring from a random process. • Assume  $|V_i| = k = 8\Delta$  for all *i* (otherwise remove vertices from too large  $V_i$ )
- Obtain S by ind. choosing one vertex unif. at random from each  $V_i$ Positive Probability **LLL**: Events  $E_1, \ldots, E_n$

To show: 
$$\Pr["S \text{ independent"}] > 0$$
 (both endpoints in S)
S is independent iff no edge  $e = \{u, v\}$  has  $e \subseteq S$ ,
Let  $A_e$  be the event that  $e \subseteq S$   $\bigvee_i \bigvee_j$ 
 $\Pr["S \text{ independent"}] = \Pr[\bigcap_{e \in E} \neg A_e] > 0 \checkmark$ 
 $\Pr[A_e] \leq \frac{1}{k^2} =: p$ 
 $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq k\Delta + k\Delta \leq 2k\Delta =: d$ 
 $p = \max_{i \in [n]} \Pr[E_i]$ 
 $P = \max_{i \in [n]} \Pr[E_i]$ 
 $E_i \text{ independent of } \{E_1, \dots, E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ 
 $W_i \mid |D_i| \leq d$ 
 $A_{dp} \leq 1$ 
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .
 $A_{dp} = 4 \cdot 2k\Delta \cdot \frac{1}{k^2}$ 
 $P = \max_{i \in [n]} \Pr[E_i]$ 
 $A_{dp} \leq 1$ 
 $A_{dp} = 4 \cdot 2k\Delta \cdot \frac{1}{k^2}$ 
 $A_{e} \mid V_i \mid |V_i|$ 
 $A_{e} \mid V_i \mid V_i \mid V_i$ 
 $A_{e} \mid V_i \mid$ 

Karlsruhe Institute of Technolo

• Given a network and *k* vertex pairs that want to communicate





Given a network and *k* vertex pairs that want to communicate
 Each *i* ∈ [*k*] has a set S<sub>i</sub> of candidate communication paths





Given a network and *k* vertex pairs that want to communicate
 Each *i* ∈ [*k*] has a set S<sub>i</sub> of candidate communication paths





Given a network and *k* vertex pairs that want to communicate
 Each *i* ∈ [*k*] has a set S<sub>i</sub> of candidate communication paths




Given a network and *k* vertex pairs that want to communicate
 Each *i* ∈ [*k*] has a set S<sub>i</sub> of candidate communication paths





- Given a network and k vertex pairs that want to communicate
- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)





- Given a network and k vertex pairs that want to communicate
- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)





- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)



**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \leq m/(8k)$  paths in  $S_j$  for  $i \neq j$ .





- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

Proof







- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \leq m/(8k)$  paths in  $S_j$  for  $i \neq j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ 





- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_i$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge







- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

k = 3

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0$ 



Given a network and k vertex pairs that want to communicate

• Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths

Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0$ 

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0$  $\Pr[E_{ij}] \leq \frac{\ell}{m}$  Whatever the choice of  $P_i$ , there are  $\geq m$  choices for  $P_j$ , **Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

k = 3

LLL: Events 
$$E_1, ..., E_n$$
  
**p** = max<sub>i \in [n]</sub> Pr[ $E_i$ ]  
**E**<sub>i</sub> independent of { $E_1, ..., E_n$ }\ $\bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
**4** $dp \le 1$   
Then,  $Pr[\bigcap_{i \in [n]} \neg E_i] > 0.$ 





- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

k = 3

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0$  $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p$ 

LLL: Events 
$$E_1, ..., E_n$$
  
**p** = max<sub>i∈[n]</sub> Pr[ $E_i$ ]  
E<sub>i</sub> independent of { $E_1, ..., E_n$ }\ $\bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
4 $dp \le 1$   
Then,  $Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .



Given a network and k vertex pairs that want to communicate

• Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths

Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0$   $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p$ • Let  $E_i$  and  $P_j$  share an edge  $E_{ij}$  independent of every other event but not of *all* others

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0$   $\Pr[E_{ij}] \le \frac{\ell}{m} \eqqcolon p$ •  $\lim_{k \to \infty} E_{ij}$  independent of every other event but not of *all* others •  $\lim_{k \to \infty} E_{i1}, \dots, E_{i\ell}$  occur, then  $\Pr[E_{ij}] = 0$  for  $j > \ell$  **Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

k = 3

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .



Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$   $\Pr[E_{ij}] \leq \frac{\ell}{m} \eqqcolon p$ Removing  $D_{ij}$  is discarding all events that could tell us something about whether  $P_i$  and  $P_i$  can intersect

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$  $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \quad |D_{ij}| = (k-1) + (k-1) - 1$  **Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$   $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \quad |D_{ij}| = (k-1) + (k-1) - 1$  $E_{i1}, \dots, E_{ik}$ 

 $P_i$  may intersect any of the other k-1 paths

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$   $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \mid |D_{ij}| = (k-1) + (k-1) - 1$  $E_{j1}, \dots, E_{jk}$ 

 $P_i$  may intersect any of the other k-1 paths

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0.$ 





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$   $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \quad |D_{ij}| = (k-1) + (k-1) - 1$  $(counted E_{ij} twice)$ 

LLL: Events 
$$E_1, ..., E_n$$
  
•  $p = \max_{i \in [n]} \Pr[E_i]$   
•  $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
•  $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .



Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$  $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \quad |D_{ij}| = (k-1) + (k-1) - 1 < 2k =: d$ 

LLL: Events 
$$E_1, ..., E_n$$
  
•  $p = \max_{i \in [n]} \Pr[E_i]$   
•  $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
•  $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$  $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \quad |D_{ij}| = (k-1) + (k-1) - 1 < 2k =: d$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

k = 3

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .



Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$  $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \quad |D_{ij}| = (k-1) + (k-1) - 1 < 2k =: d$ 

4*d p* 



**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

if any path in S, cha

k = 3

Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$   $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \quad |D_{ij}| = (k-1) + (k-1) - 1 < 2k =: d$  $4dp = 4 \cdot 2k \cdot \frac{\ell}{m}$ 

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$   $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \mid |D_{ij}| = (k-1) + (k-1) - 1 < 2k =: d$  $4dp = 4 \cdot 2k \cdot \frac{\ell}{m} = \ell \cdot \frac{8k}{m}$ 

#### LLL: Events $E_1, ..., E_n$ $p = \max_{i \in [n]} \Pr[E_i]$ $E_i$ independent of $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with $|D_i| \le d$ $4dp \le 1$ Then, $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

**Probabilistic Method:** Show that something

probability of occuring from a random process.

exists by proving that it has a *positive* 





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \leq m/(8k)$  paths in  $S_j$  for  $i \neq j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$   $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \mid |D_{ij}| = (k-1) + (k-1) - 1 < 2k =: d$  $4dp = 4 \cdot 2k \cdot \frac{\ell}{m} = \ell \cdot \frac{8k}{m}$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \leq m/(8k)$  paths in  $S_j$  for  $i \neq j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] \stackrel{?}{>} 0 \mid D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$   $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p \mid |D_{ij}| = (k-1) + (k-1) - 1 < 2k =: d$  $4dp = 4 \cdot 2k \cdot \frac{\ell}{m} = \ell \cdot \frac{8k}{m} \leq 1$ 

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .





Given a network and k vertex pairs that want to communicate

- Each  $i \in [k]$  has a set  $S_i$  of candidate communication paths
- Does there exist a choice of paths (one P<sub>i</sub> from each S<sub>i</sub>) that are pairwise edge-disjoint? (NP-complete to decide)

**Theorem**: Let  $m = \min_{i \in [k]} \{|S_i|\}$ . Then, there exists a valid choice if any path in  $S_i$  shares edges with at most  $\ell \le m/(8k)$  paths in  $S_j$  for  $i \ne j$ .

#### Proof

Random Process : Ind., unif. at random choose  $P_i$  from  $S_i$ Positive Probability

• Let  $E_{ij}$  be the event that  $P_i$  and  $P_j$  share an edge  $\Pr[\bigcap_{i < j} \neg E_{ij}] > 0$   $D_{ij} = \{E_{st} \mid \{s, t\} \cap \{i, j\} \neq \emptyset\}$   $\Pr[E_{ij}] \leq \frac{\ell}{m} =: p$   $|D_{ij}| = (k-1) + (k-1) - 1 < 2k =: d$  $4dp = 4 \cdot 2k \cdot \frac{\ell}{m} = \ell \cdot \frac{8k}{m} \leq 1$ 

**Probabilistic Method**: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

k = 3

LLL: Events 
$$E_1, ..., E_n$$
  
 $p = \max_{i \in [n]} \Pr[E_i]$   
 $E_i$  independent of  $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$   
with  $|D_i| \le d$   
 $4dp \le 1$   
Then,  $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$ .

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms



# Conclusion

### **Probabilistic Method**

- Show that something exists *deterministically*, by showing that it occurs with positive probability from a random process
- Reasoning: At least one object in the sample space has the desired property





#### Show that something exists *deterministically*, by showing that it occurs with positive probability from a random process

Probabilistic Method

Reasoning: At least one object in the sample space has the desired property

### **Expectation Argument**

• Useful tool when applying probabilistic method •  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

### Conclusion



#### 15 Maximilian Katzmann, Stefan Walzer – Probability & Computing

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

### Conclusion

### **Probabilistic Method**

- Show that something exists *deterministically*, by showing that it occurs with positive probability from a random process
- Reasoning: At least one object in the sample space has the desired property

### **Expectation Argument**

Useful tool when applying probabilistic method  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

### **Sample via Modification**

Example Vertex Cover: remove vertices/edges at random







# Conclusion

### **Probabilistic Method**

- Show that something exists *deterministically*, by showing that it occurs with positive probability from a random process
- Reasoning: At least one object in the sample space has the desired property

### **Expectation Argument**

Useful tool when applying probabilistic method  $\Pr[X \ge \mathbb{E}[X]] > 0$  and  $\Pr[X \le \mathbb{E}[X]] > 0$ .

### **Sample via Modification**

Example Vertex Cover: remove vertices/edges at random

# Lovász Local Lemma

- Show that something exists by showing that all events that prevent its existence do not occur, with positive probability
- Lemma works as long as there are not too many dependencies





