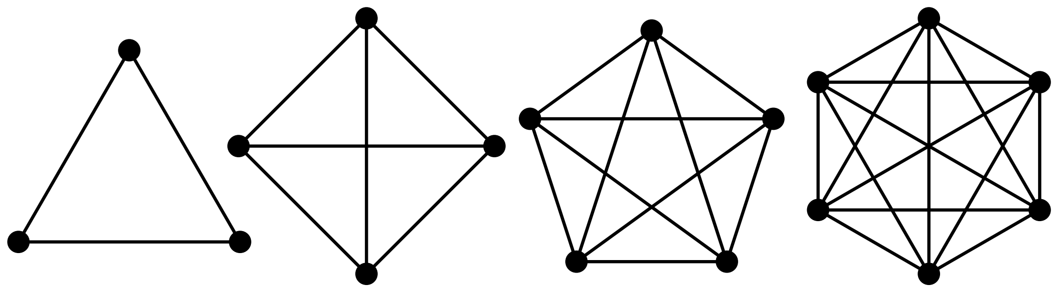
Probability & Computing

Probabilistic Method

Complete Coloring

The Problem

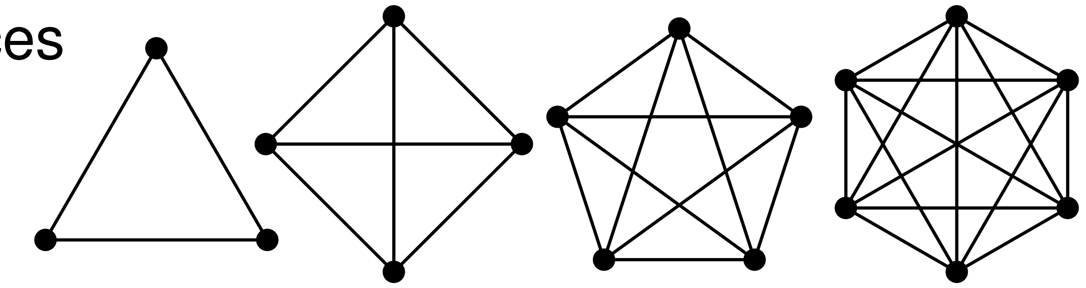
- Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)



Complete Coloring

The Problem

- Let G be the complete graph on n vertices (every vertex is adjacent to every other vertex)
- A k-clique is a complete subgraph with k vertices



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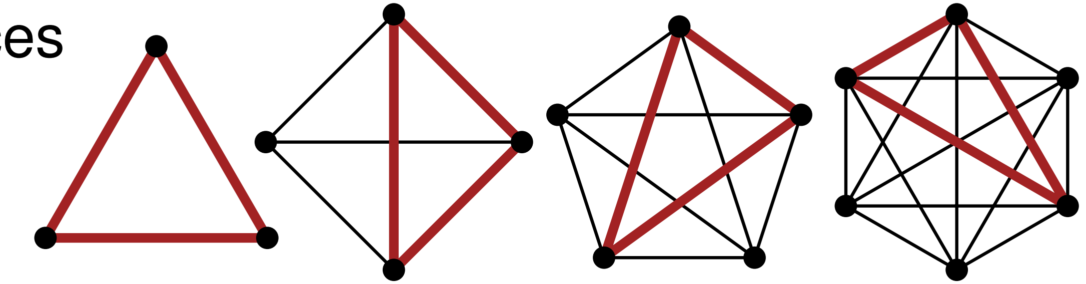

3-cliques

Complete Coloring

Karlsruhe Institute of Technology

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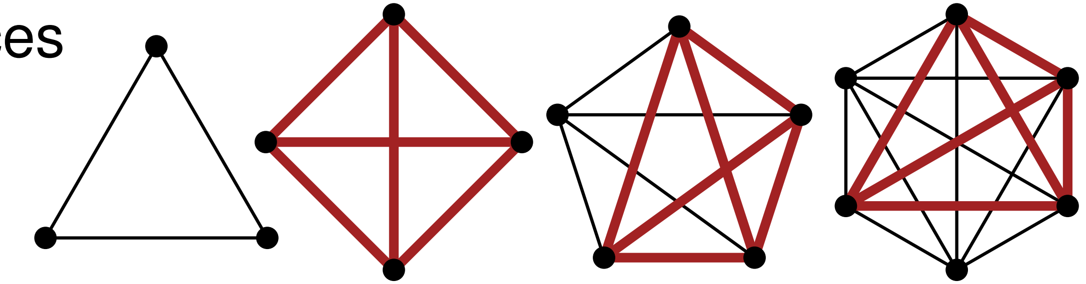

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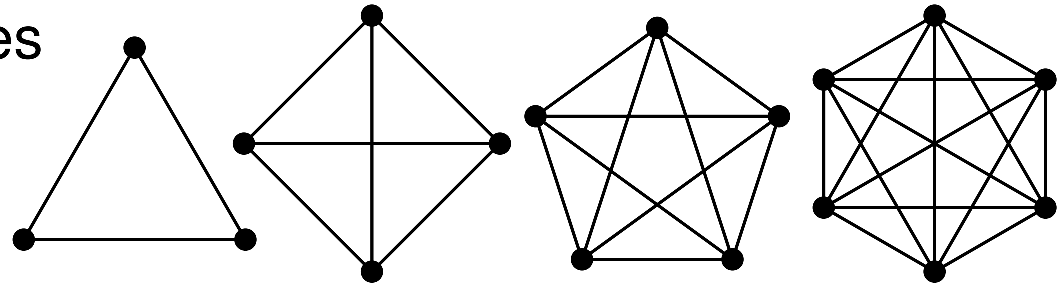

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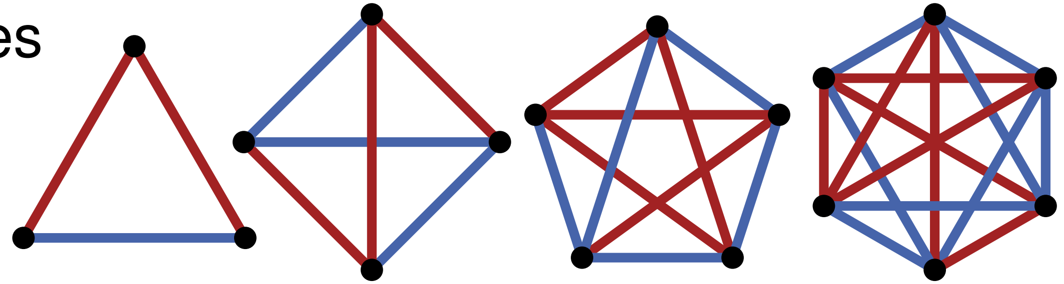

colorings

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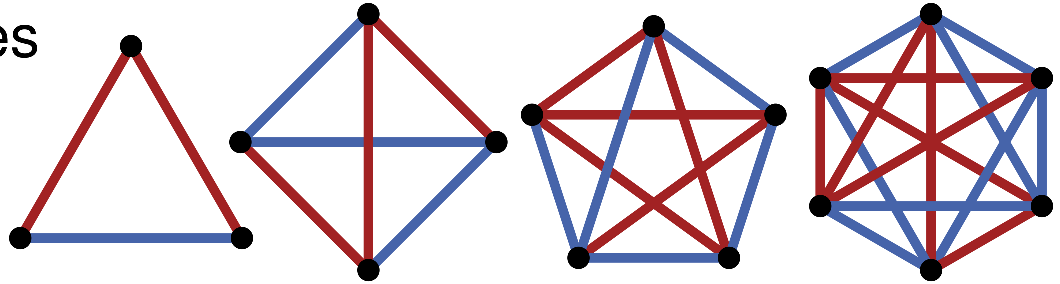

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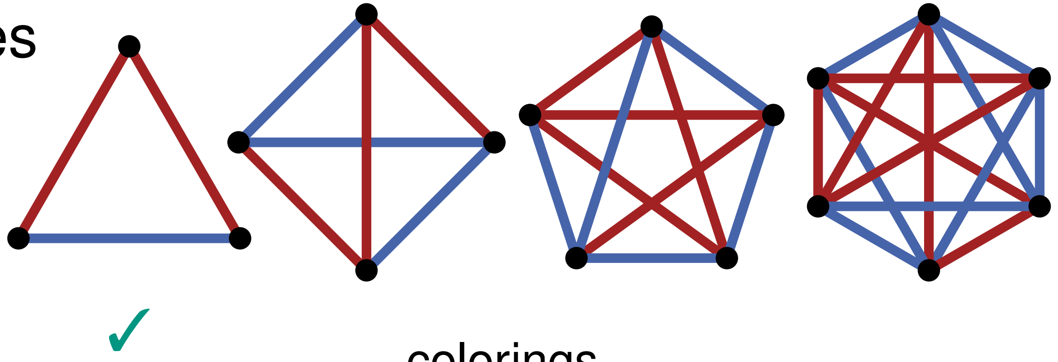

colorings
$k = 3$

Complete Coloring

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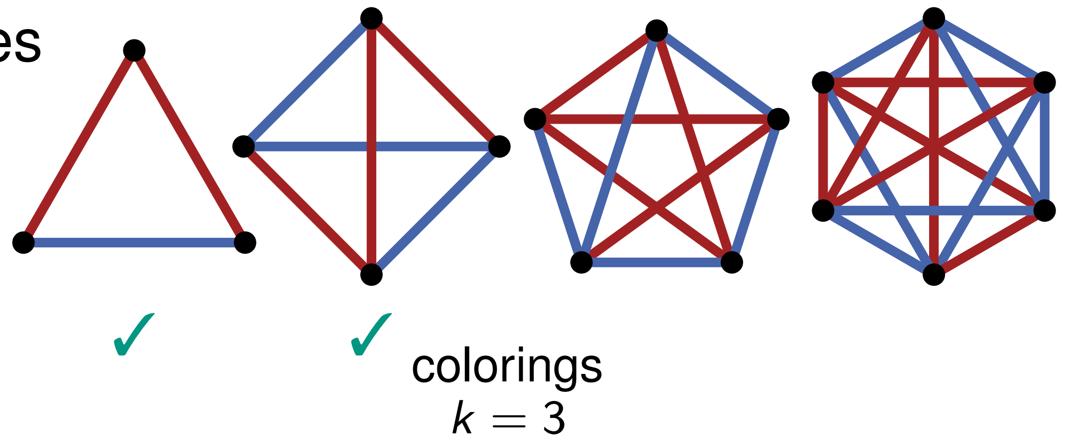

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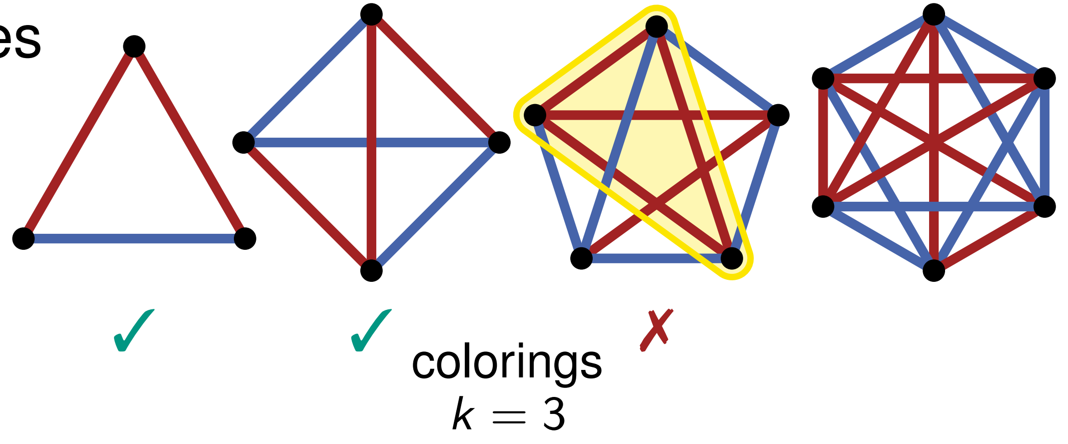

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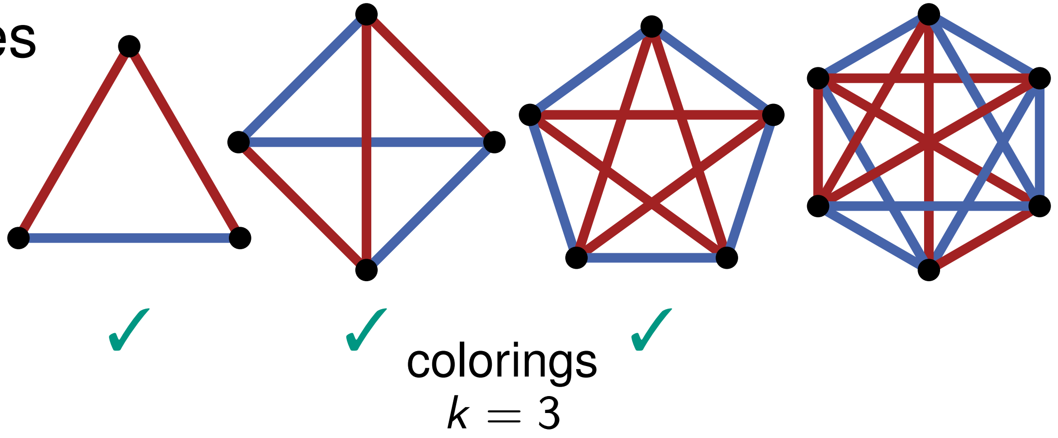

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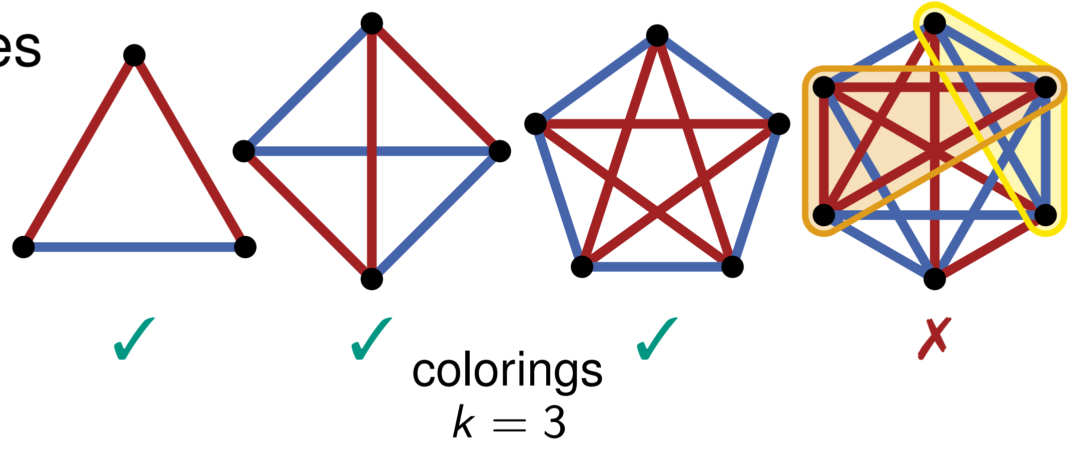

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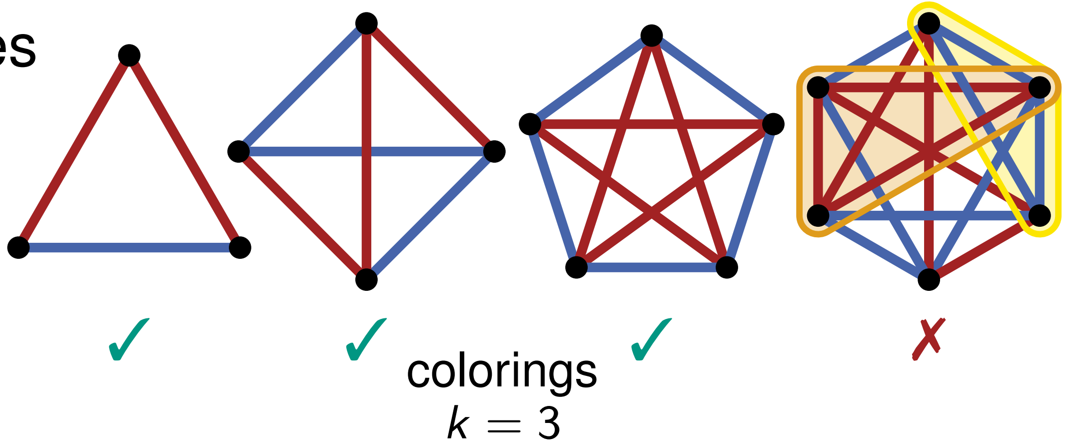

The Solution?

- Brute-force algorithm?

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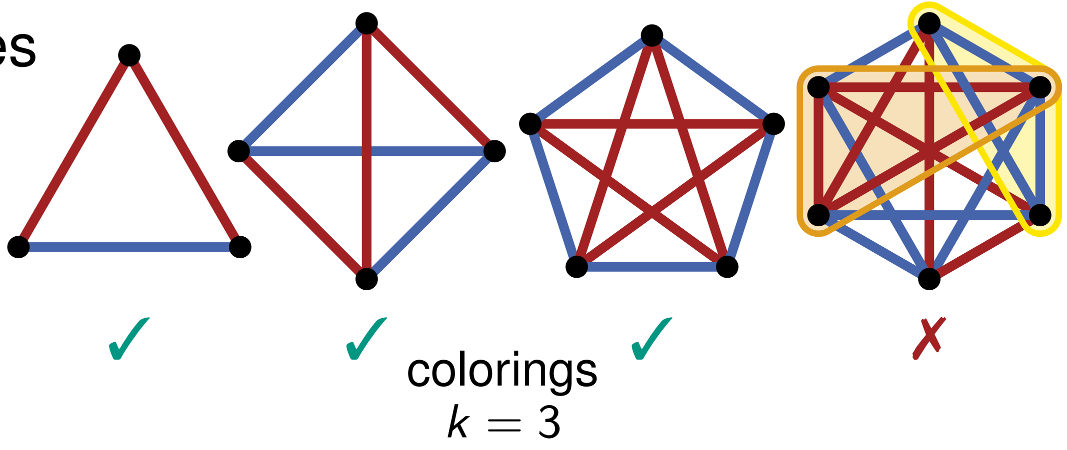

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  - $n = 6 \Rightarrow 2^{n(n-1)} = 2^{30} = 1{,}073{,}741{,}824$ possible colorings

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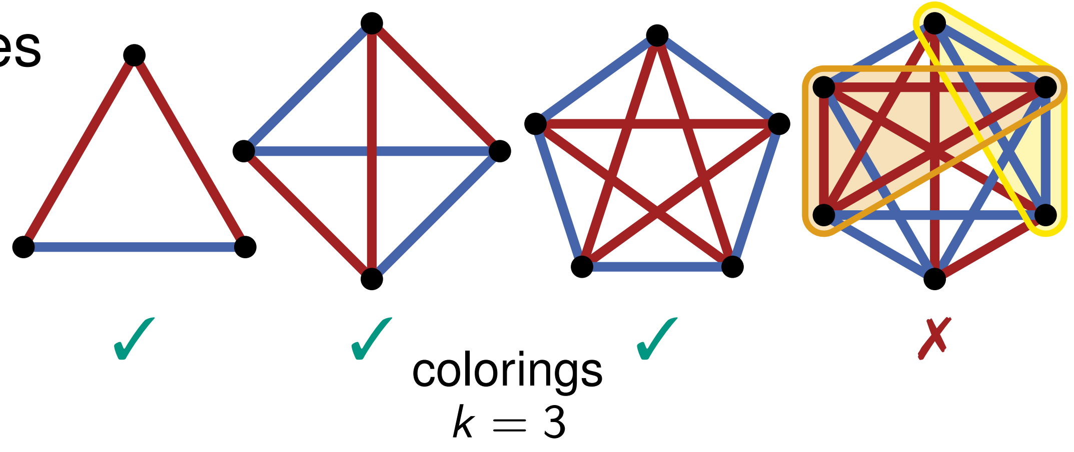

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  - $n = 6 \Rightarrow 2^{n(n-1)} = 2^{30} = 1{,}073{,}741{,}824$ possible colorings
  - $k = 3 \Rightarrow \binom{6}{3} = 20$ triangles to check $\Rightarrow$ 60 edges per coloring

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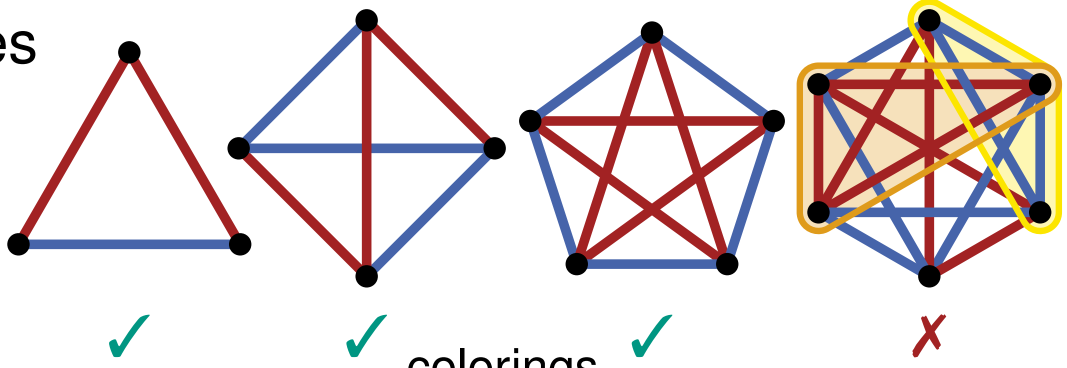

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naive implementation: 20min
no coloring exists

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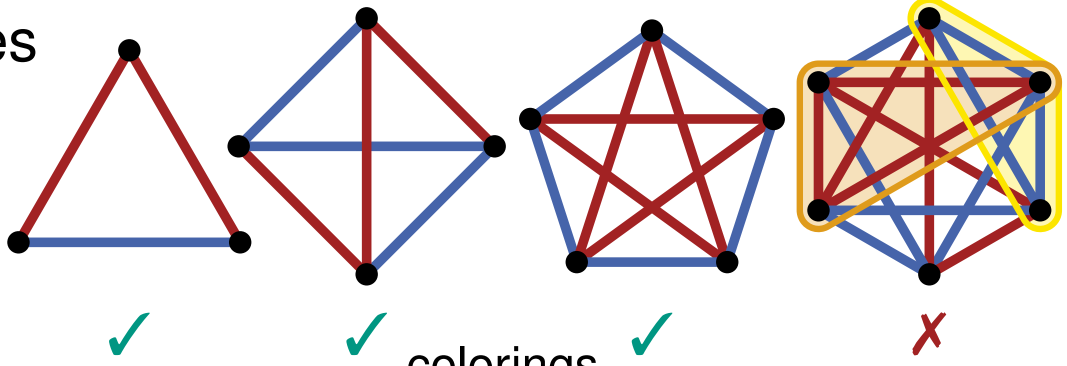

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  - What about $n = 1000$ and $k = 20$?

naive implementation: 20min
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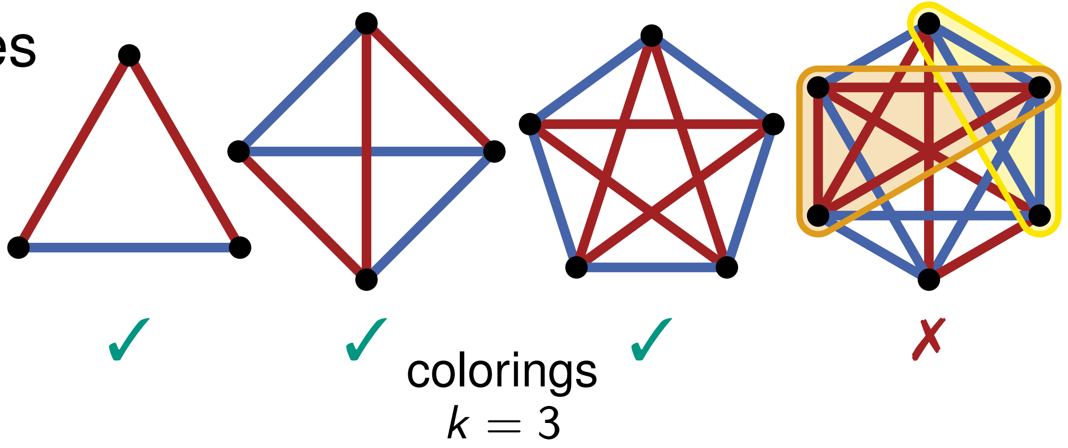

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- Randomized algorithm?

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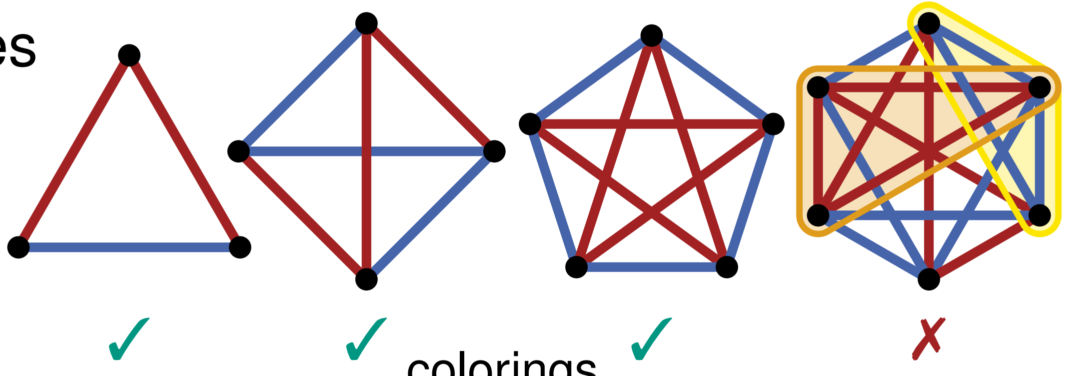

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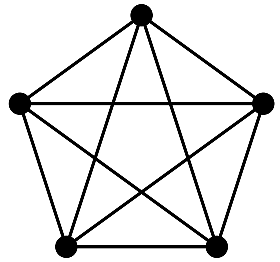
  naive implementation: 20min
  no coloring exists
- Randomized algorithm?
  - How often shall we try before assuming that no coloring exists?

Randomized Coloring

Algorithm

- For each edge independently, choose one of the colors with probability $1/2$



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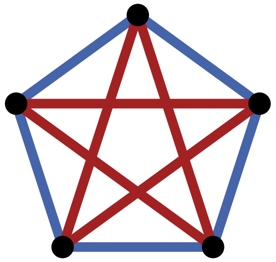

Randomized Coloring

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Algorithm

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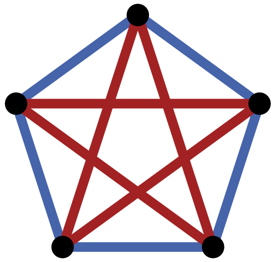

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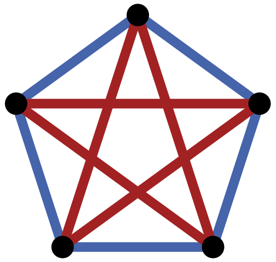

$X = 0$

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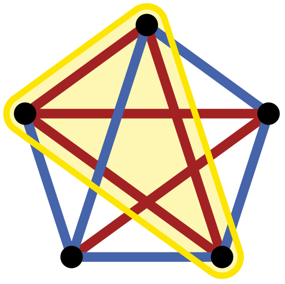

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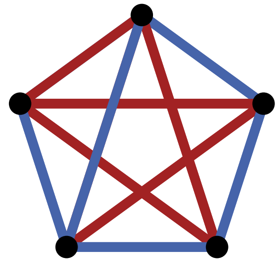
- Let $H_1, \ldots, H_{\binom{n}{k}}$ be all the different k-cliques



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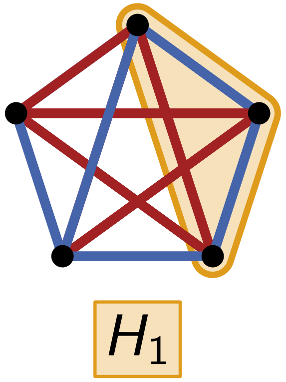

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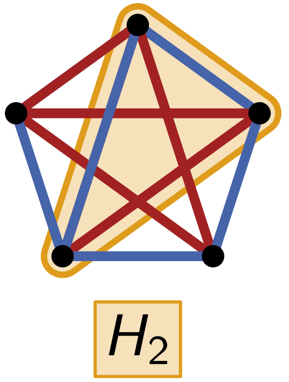

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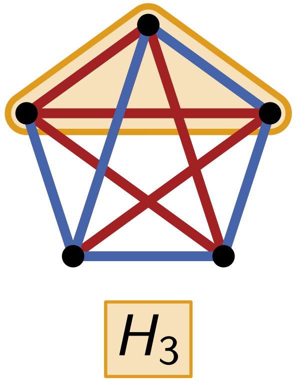

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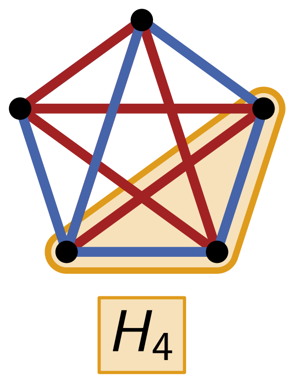

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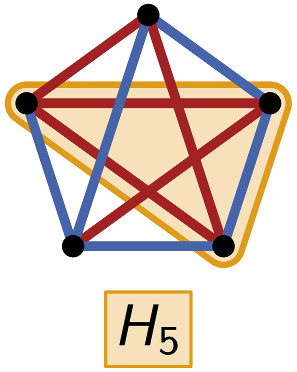

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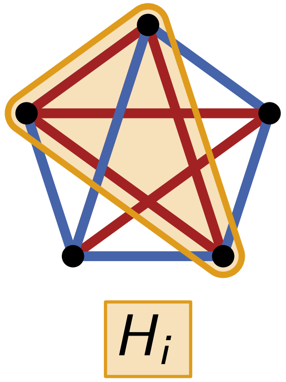

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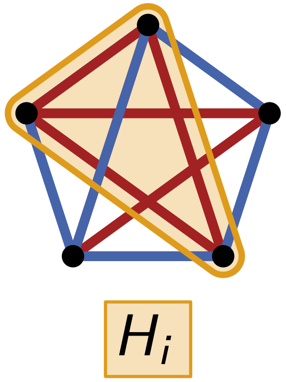

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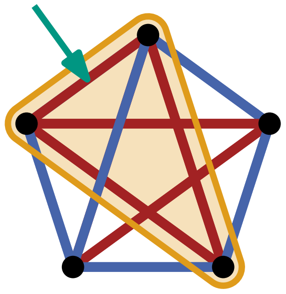

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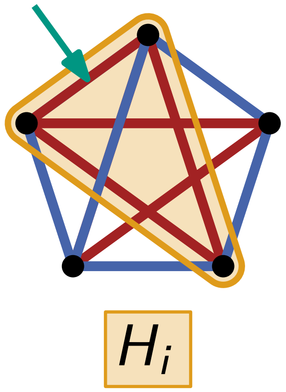

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Karlsruhe Institute of Technology

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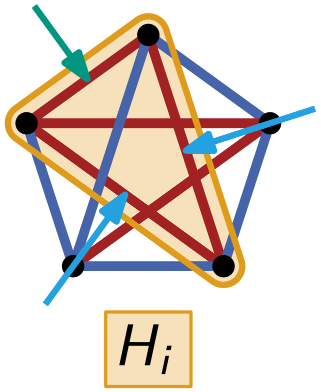
  - The $\binom{k}{2} - 1$ remaining edges need to get the same color



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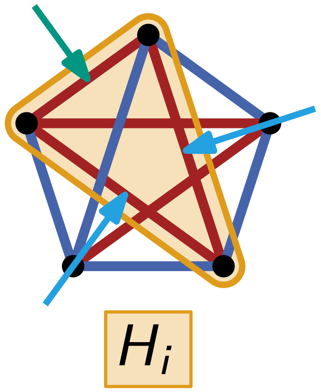

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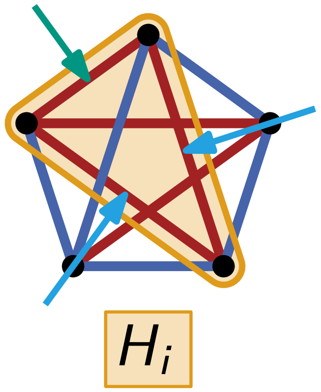
$$\Pr[X_i = 1] = \left(\tfrac{1}{2}\right)^{\binom{k}{2}-1}$$
$$= 2^{-\binom{k}{2}+1}$$



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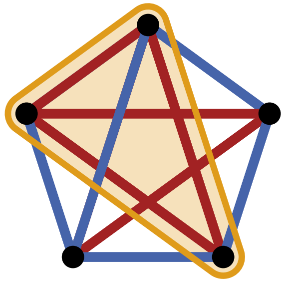

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Randomized Coloring

Algorithm

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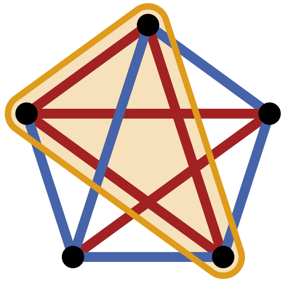

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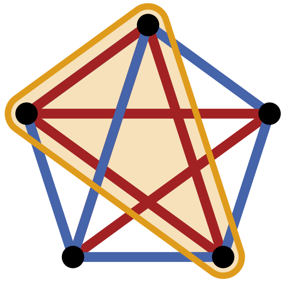

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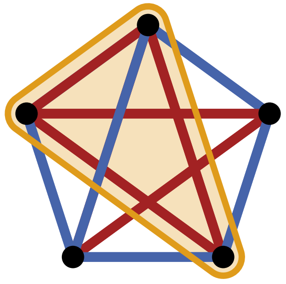

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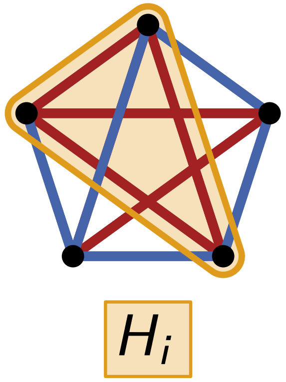

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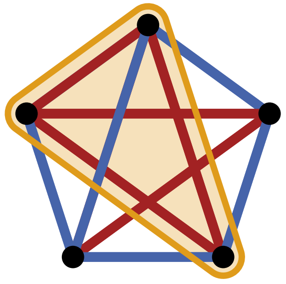

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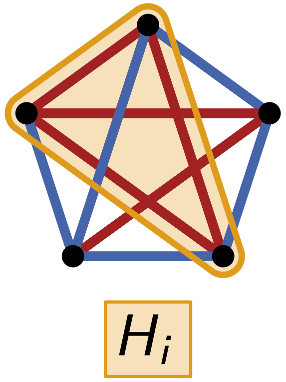
$=$

Randomized Coloring

Karlsruhe Institute of Technology

Algorithm



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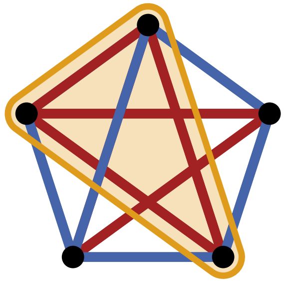

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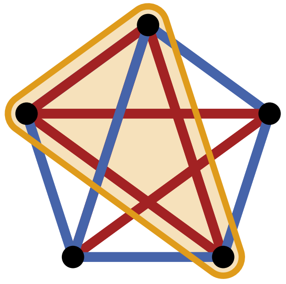

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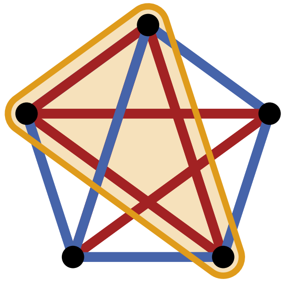

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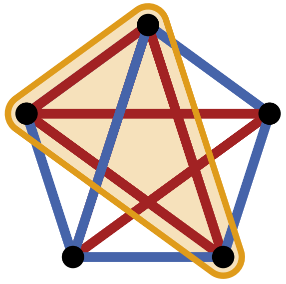

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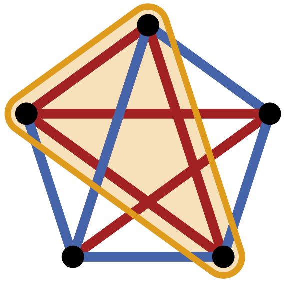

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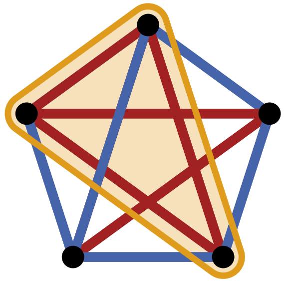

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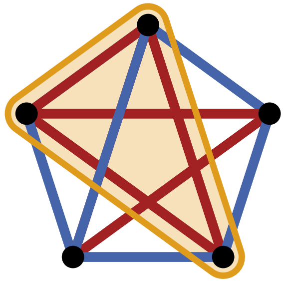

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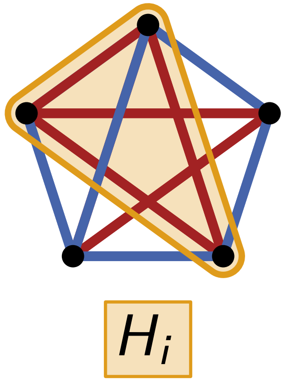

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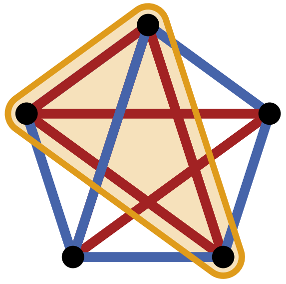

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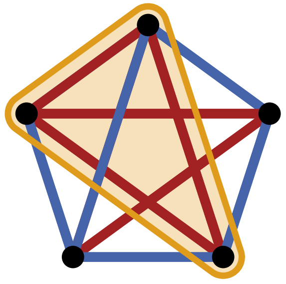

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Randomized Coloring

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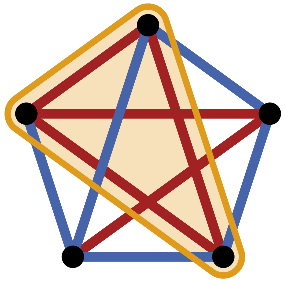

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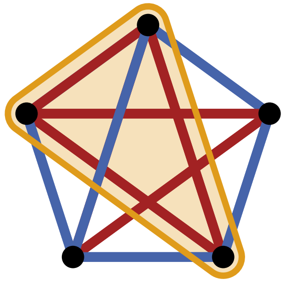

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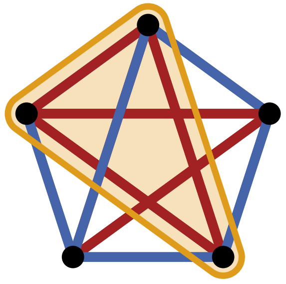

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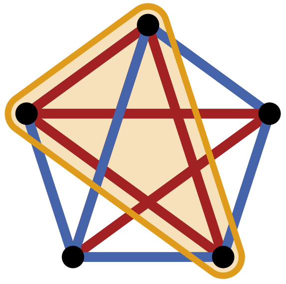

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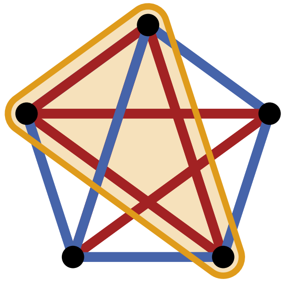

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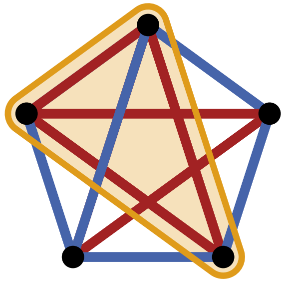

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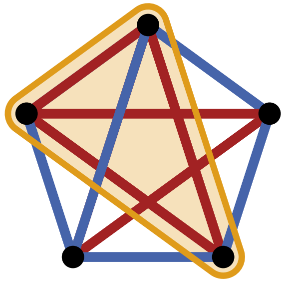
Randomized Coloring

Karlsruhe Institute of Technology

Algorithm



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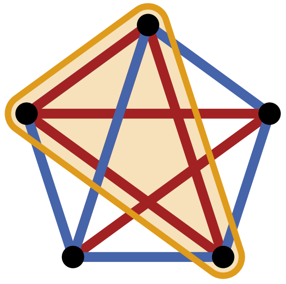

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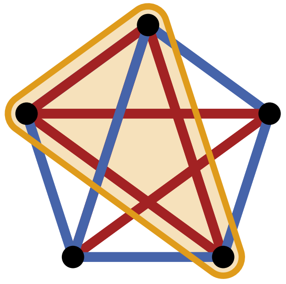

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Karlsruhe Institute of Technology

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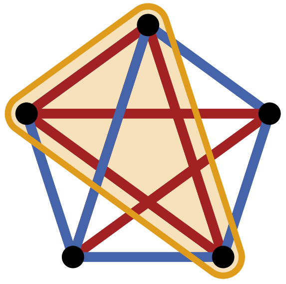

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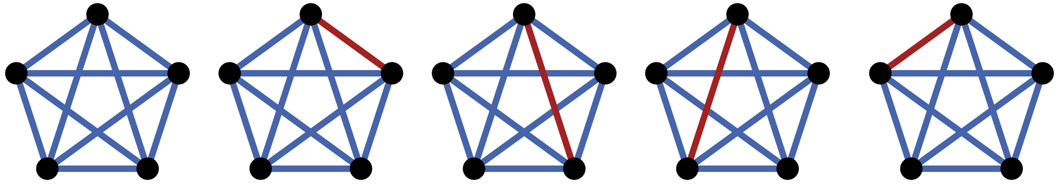

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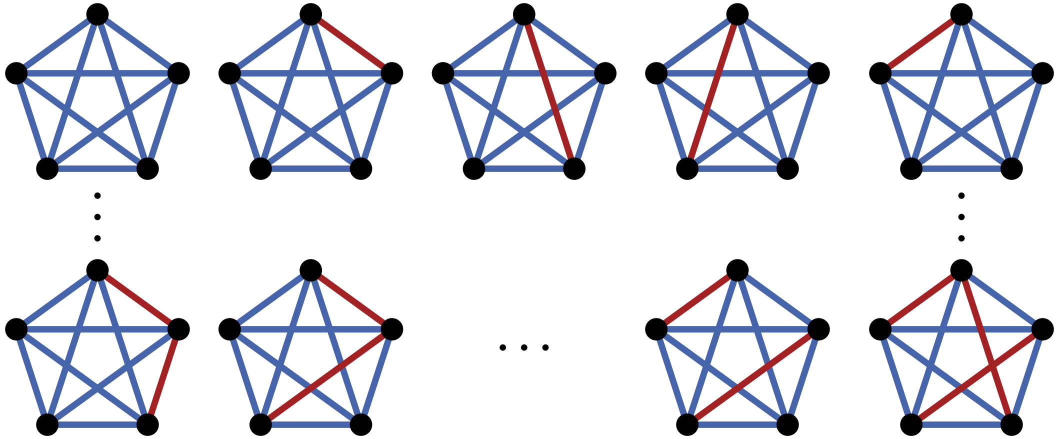

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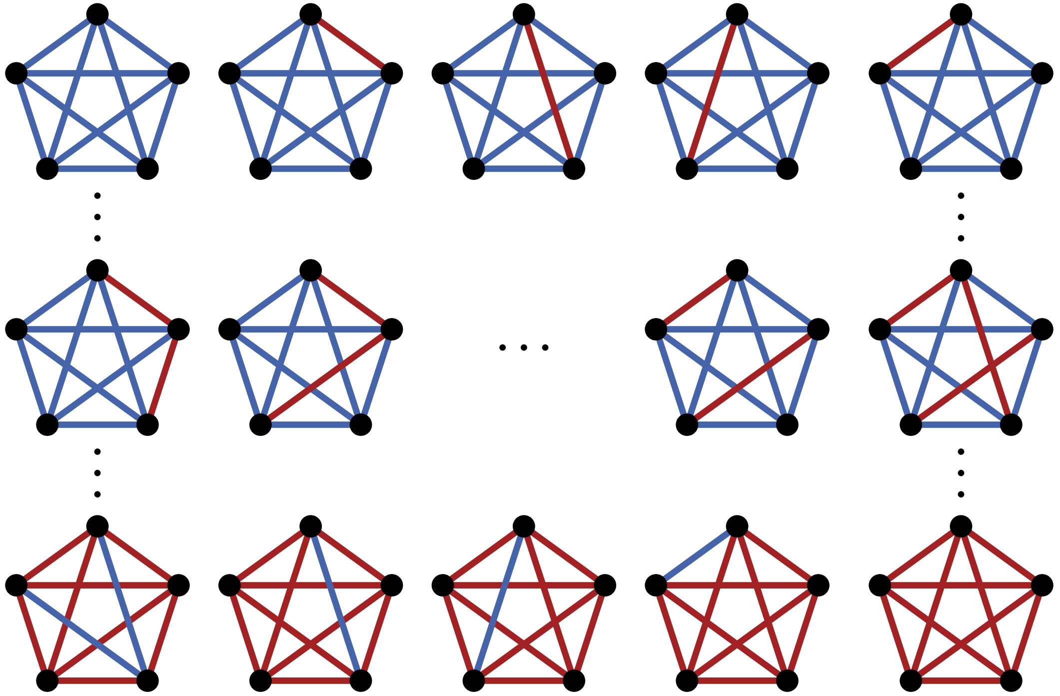

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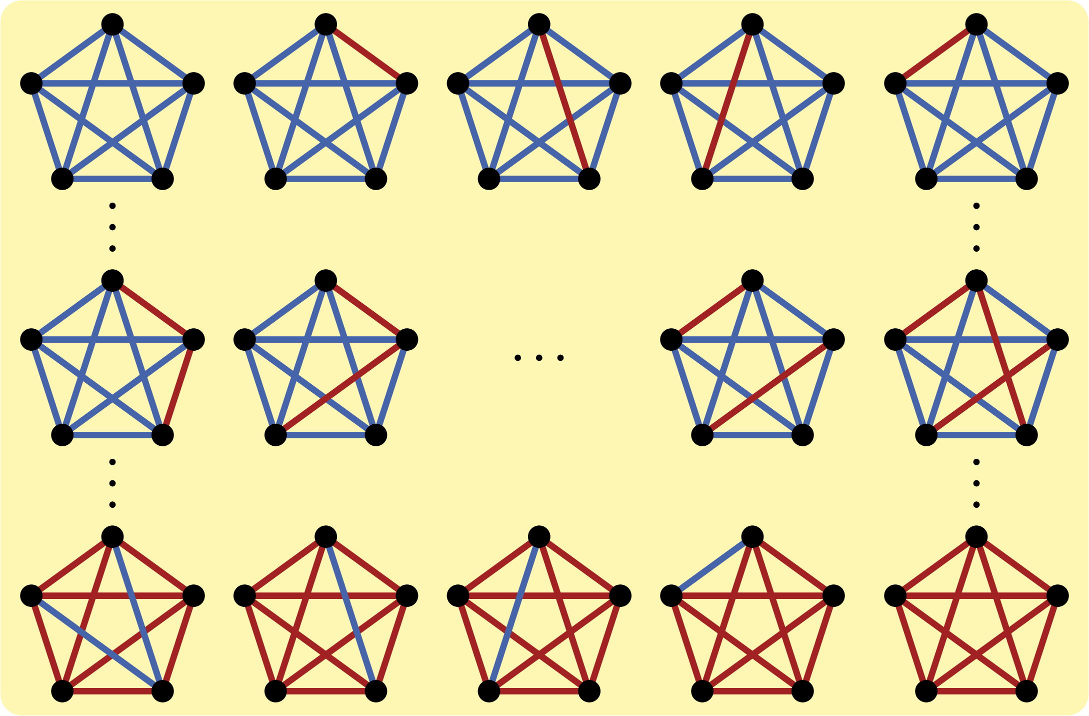

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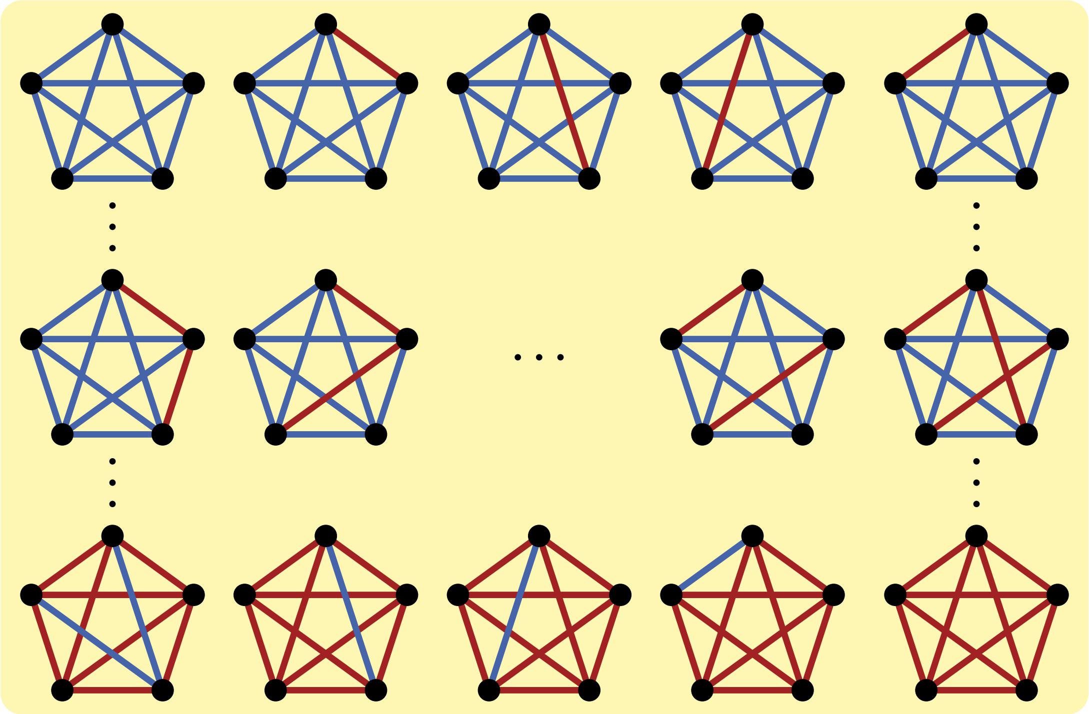

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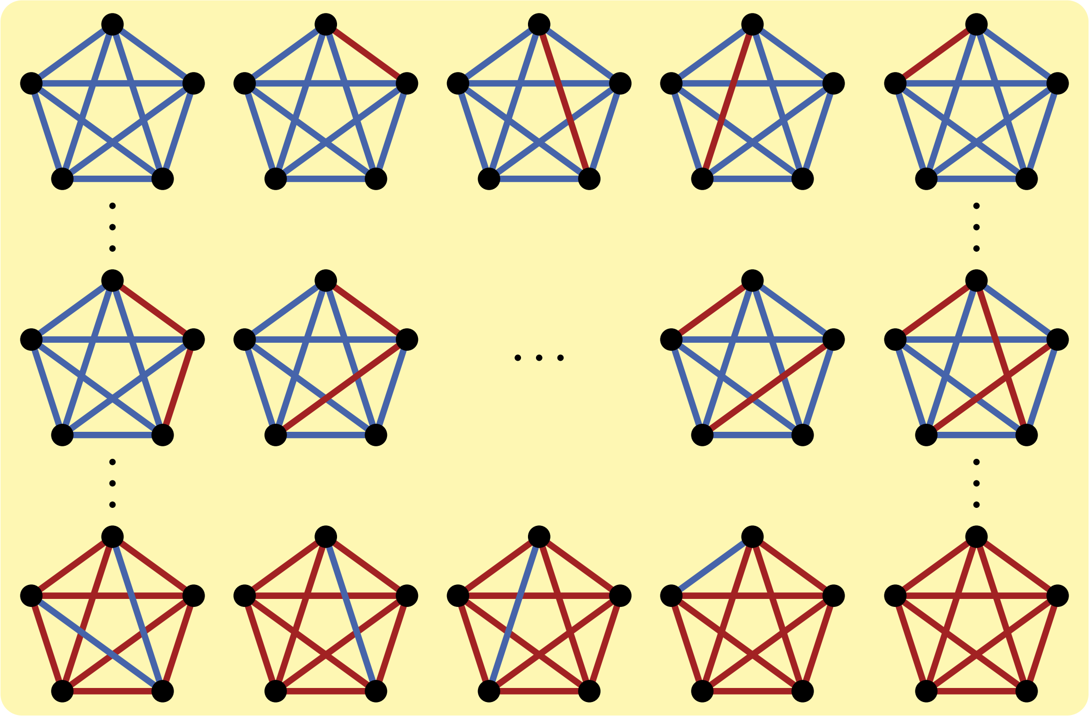
Just Shown

- $X = 0 \Rightarrow$ coloring returned by algorithm contains *no* monochromatic k-clique
- $\Pr[X = 0] > 0$



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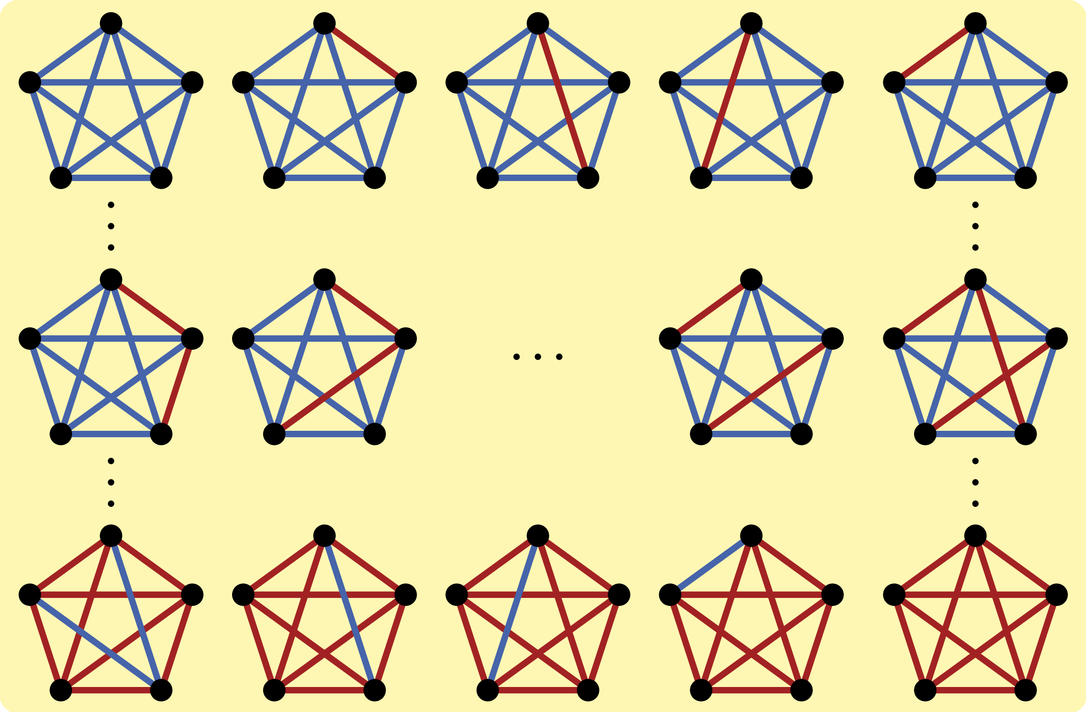

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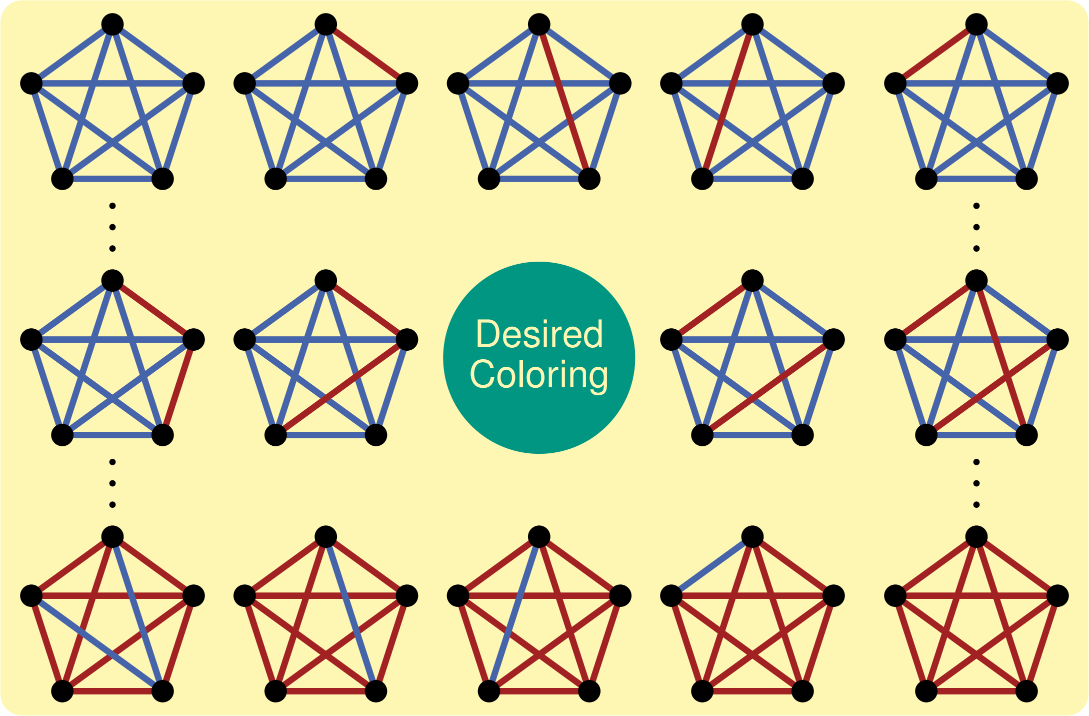

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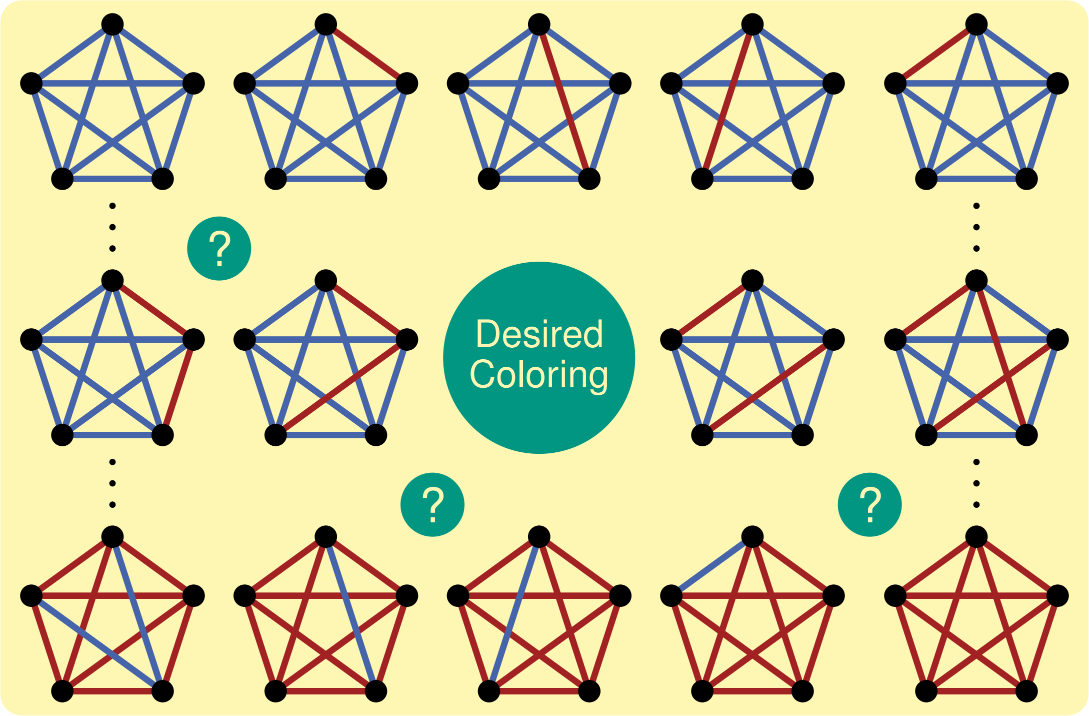

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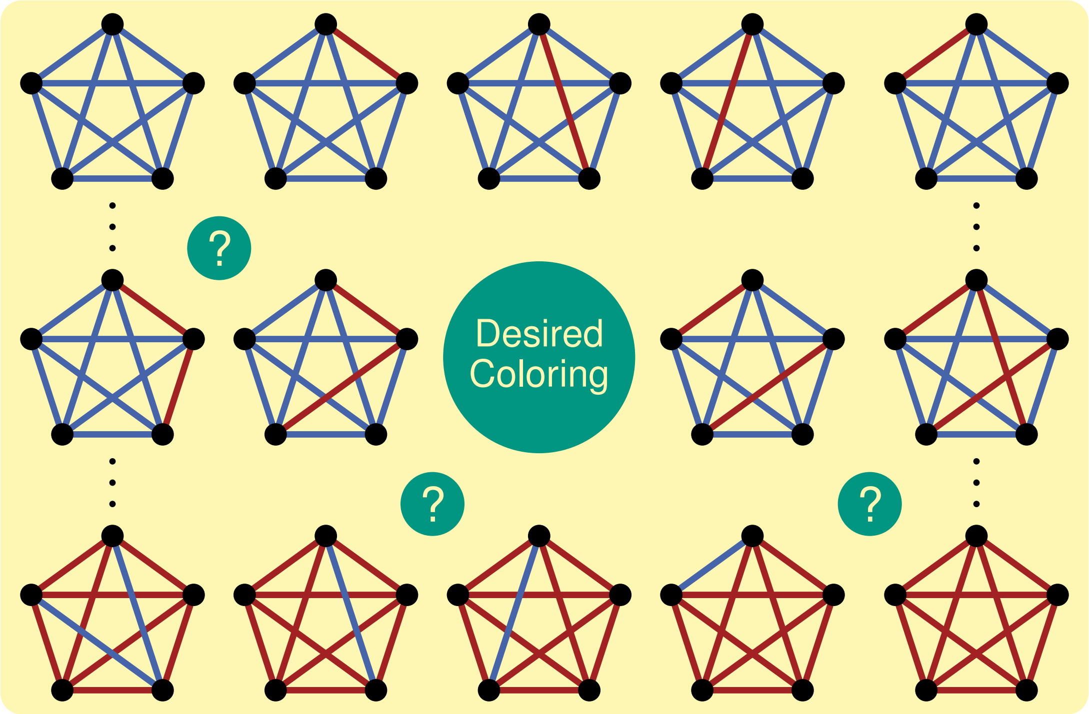

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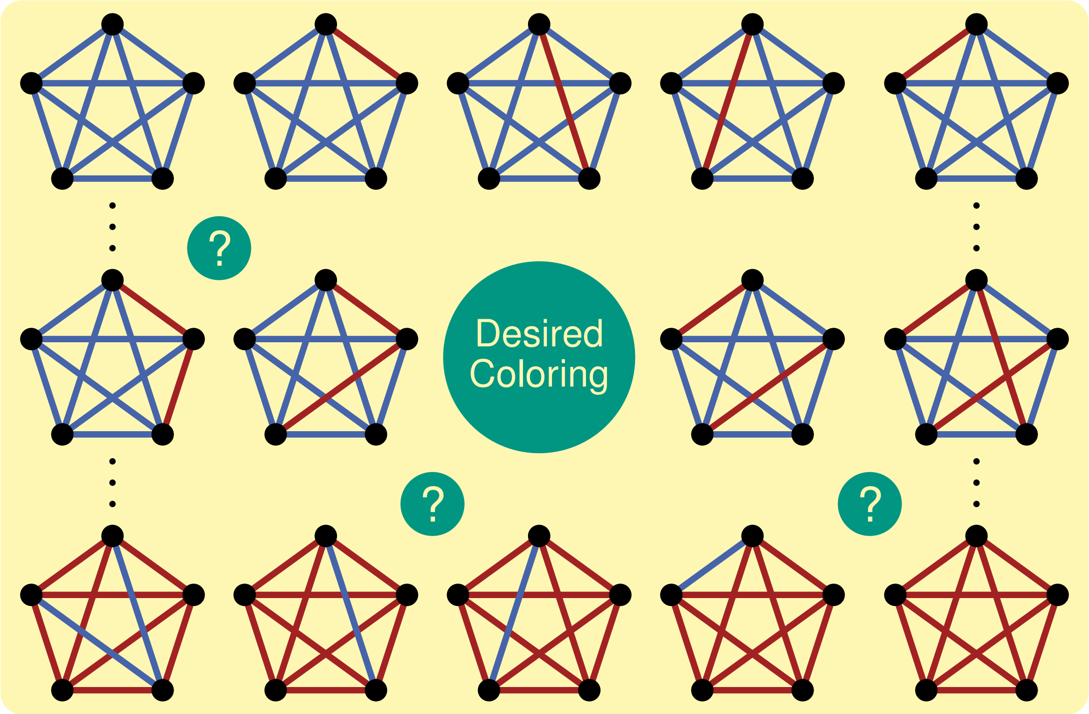

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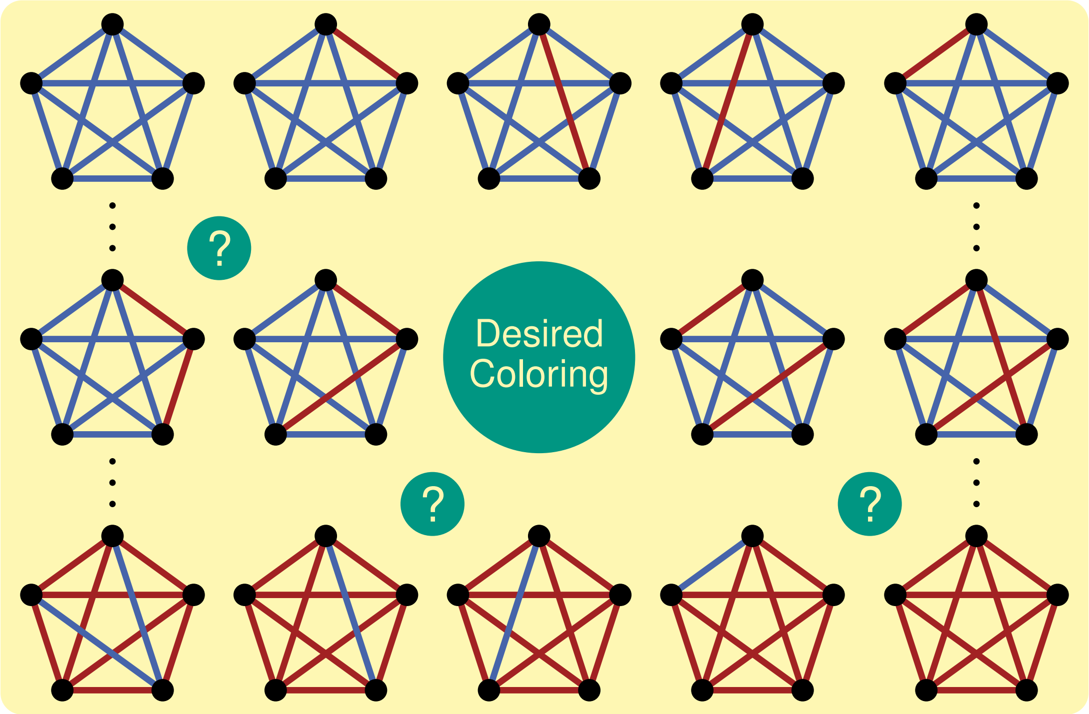

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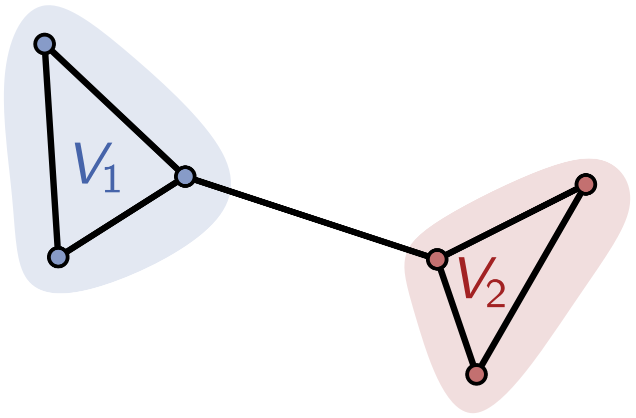

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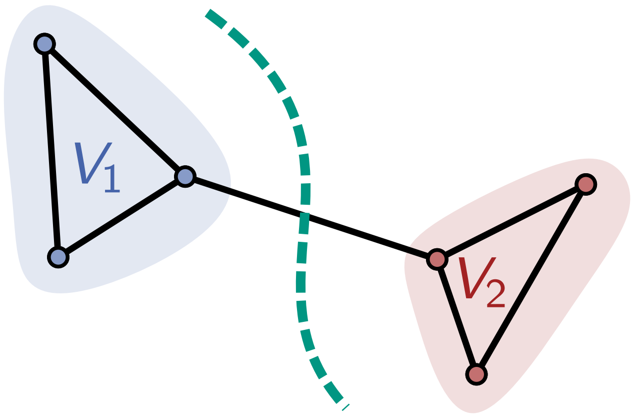

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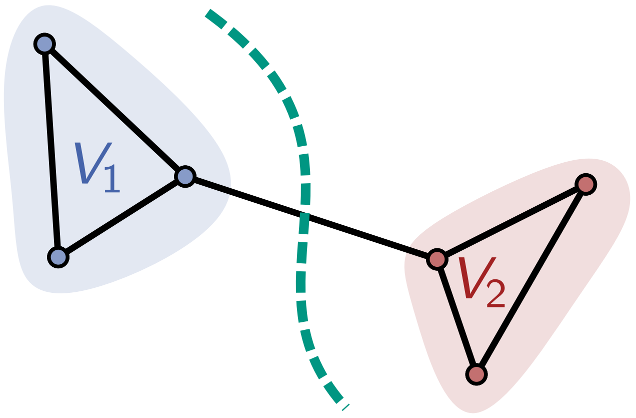

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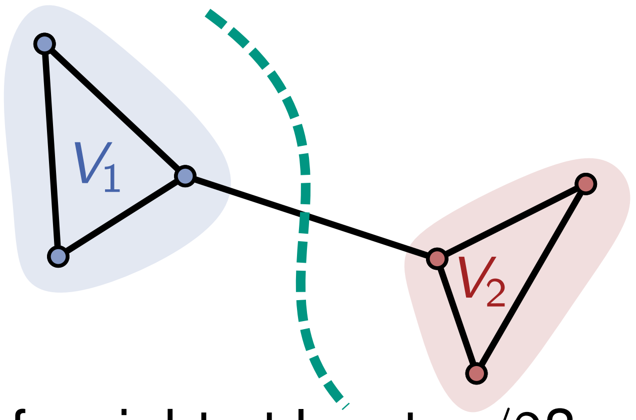

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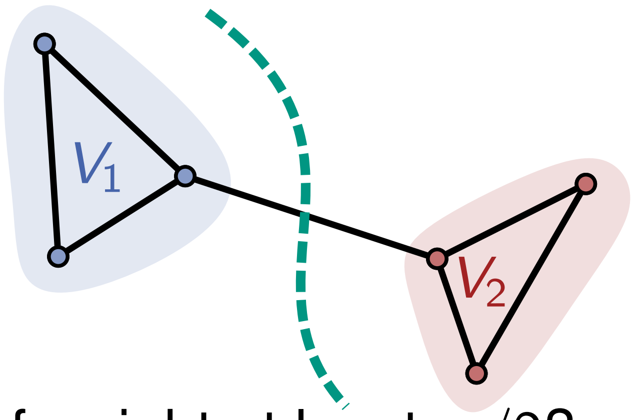

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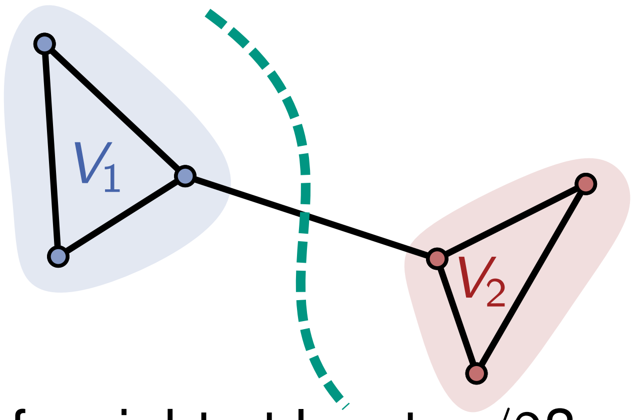

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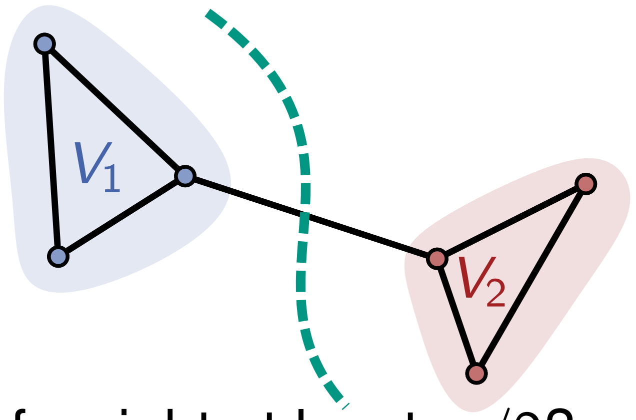

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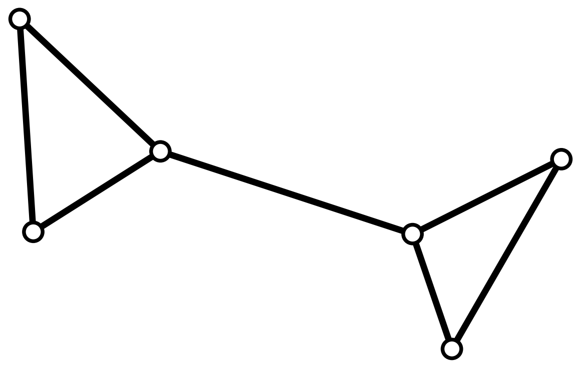

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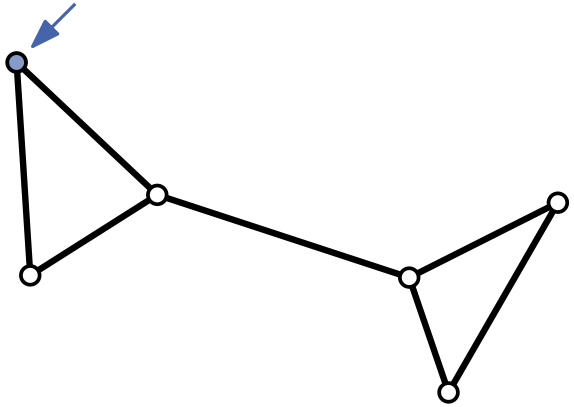

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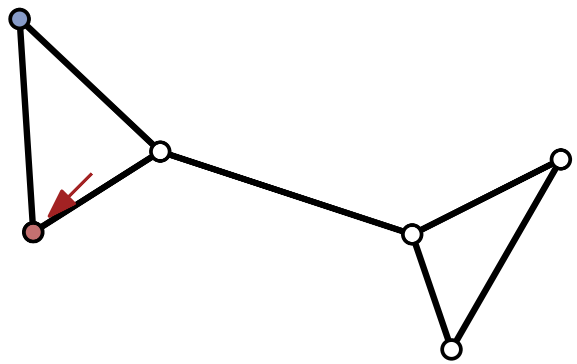

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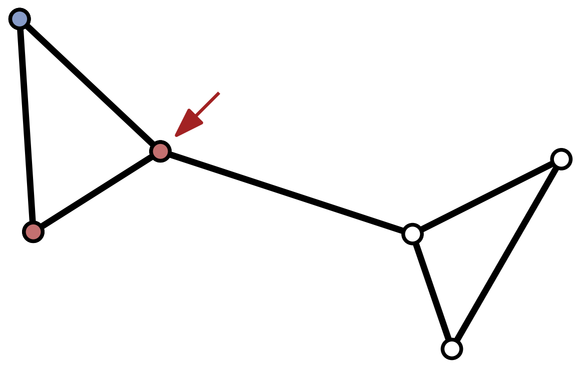

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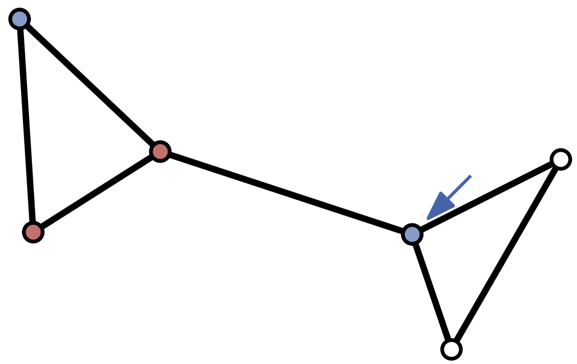

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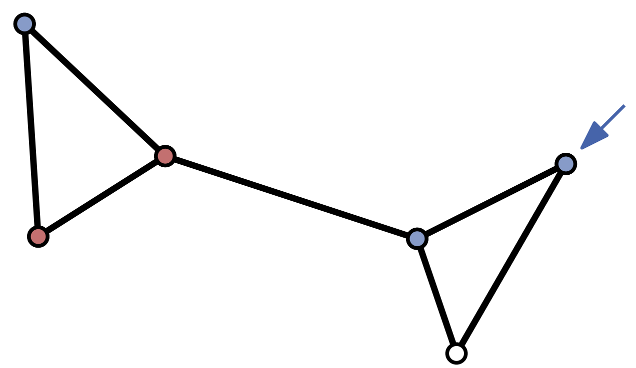

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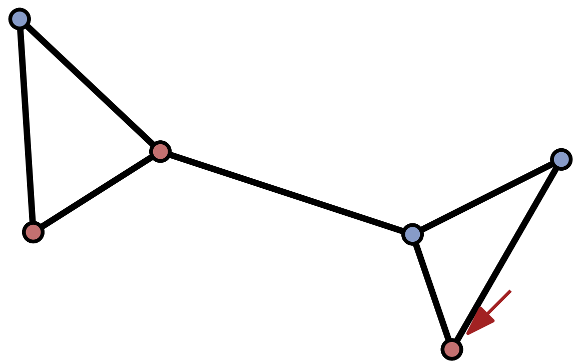

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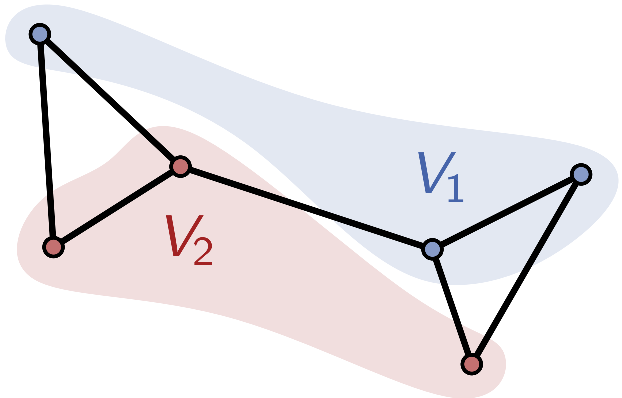

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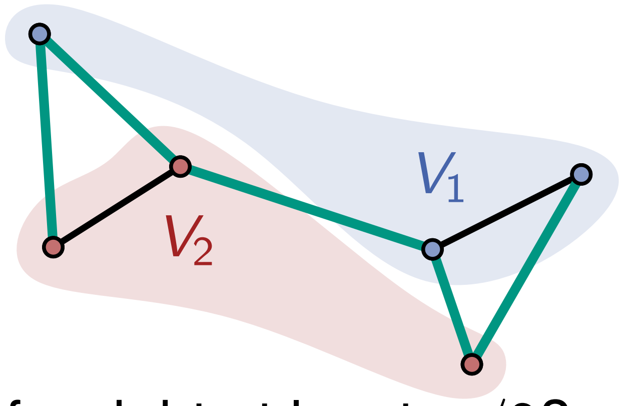

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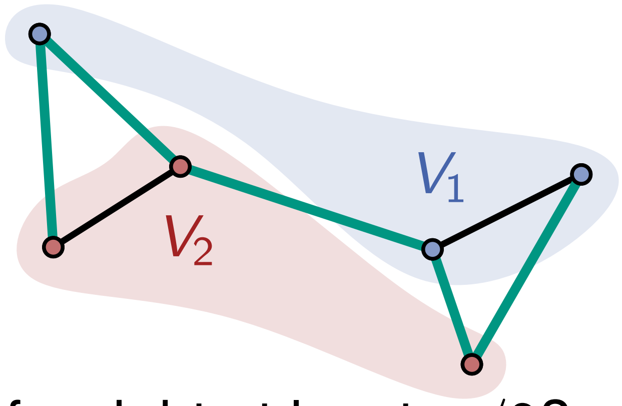

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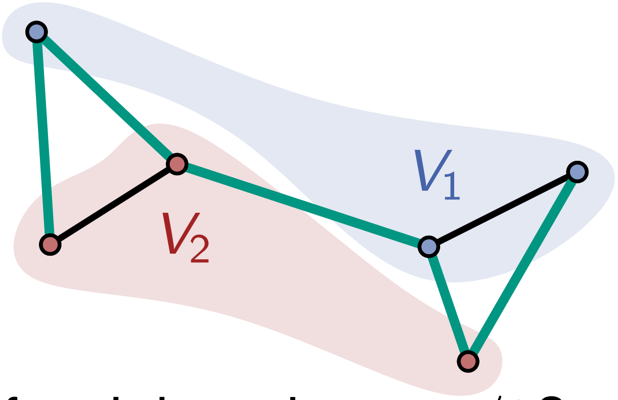

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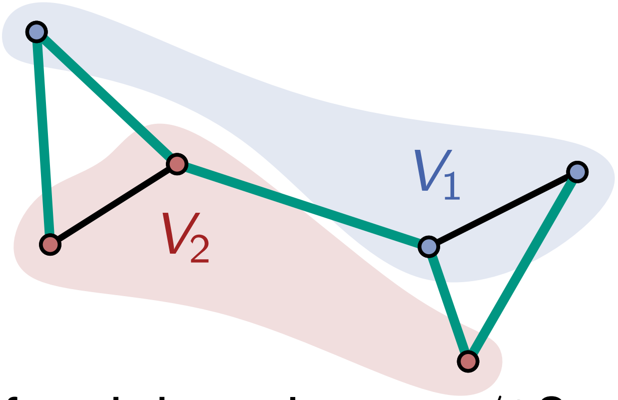

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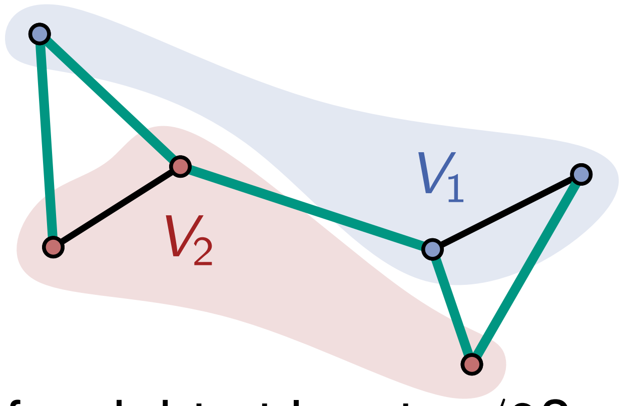

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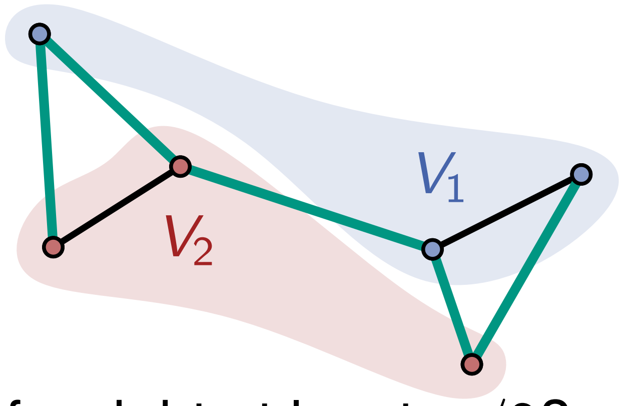

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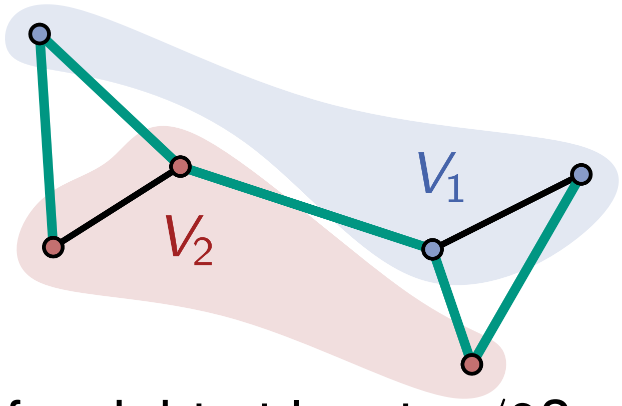

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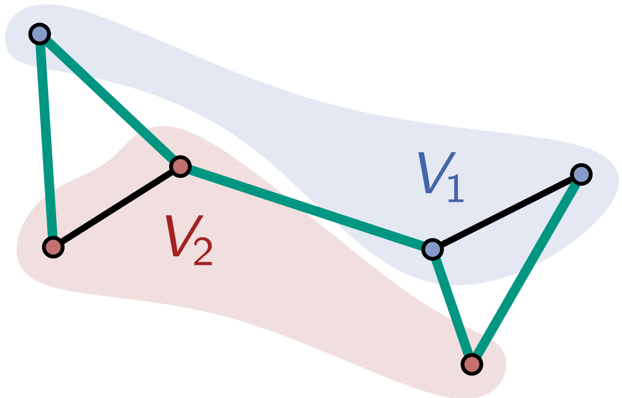
Application: Cuts

Karlsruhe Institute of Technology



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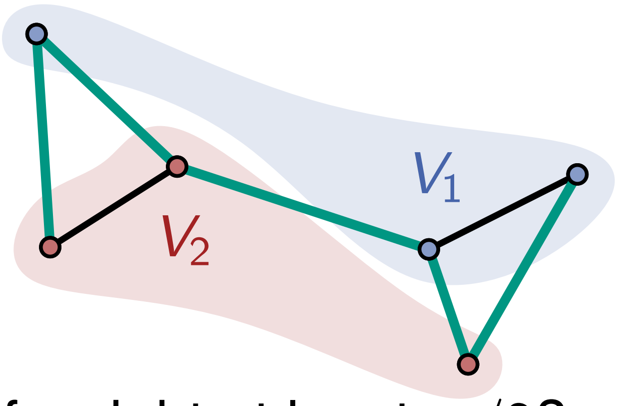

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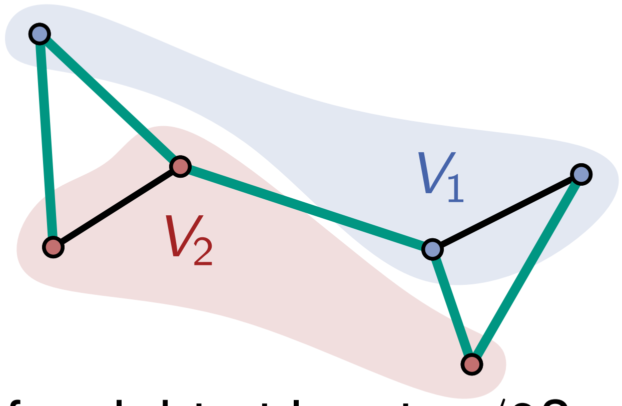

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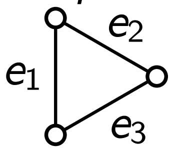

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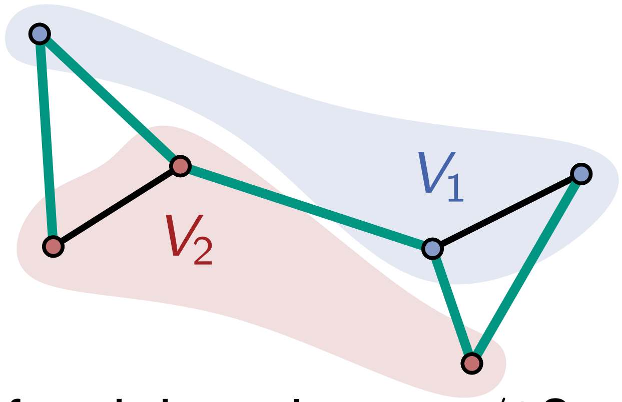

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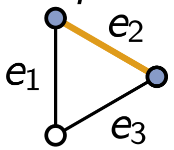

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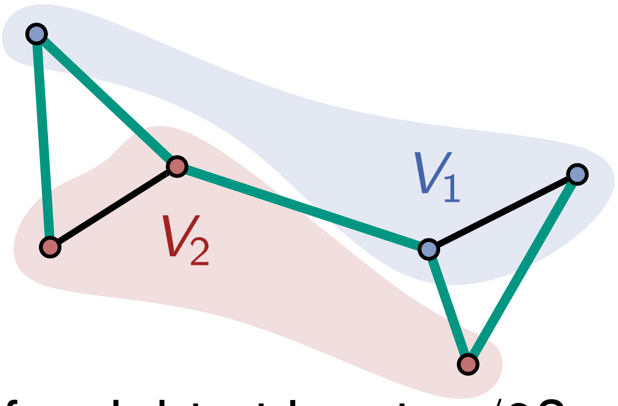

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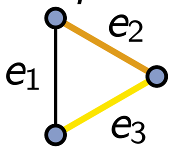

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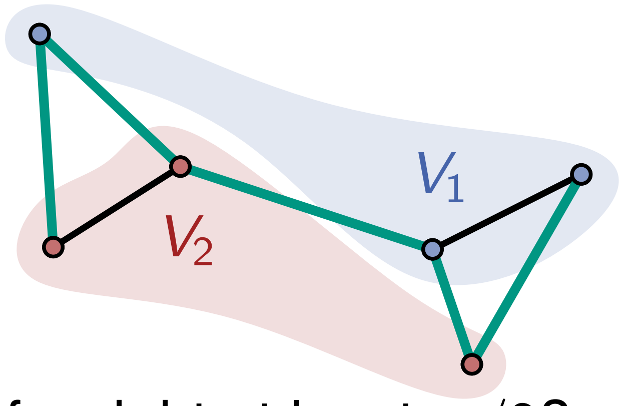

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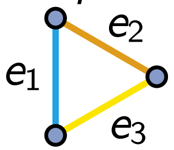

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Theorem: Let X be a random variable taking values in a set S. Then, $\Pr[X \geq \mathbb{E}[X]] > 0$ and $\Pr[X \leq \mathbb{E}[X]] > 0$.

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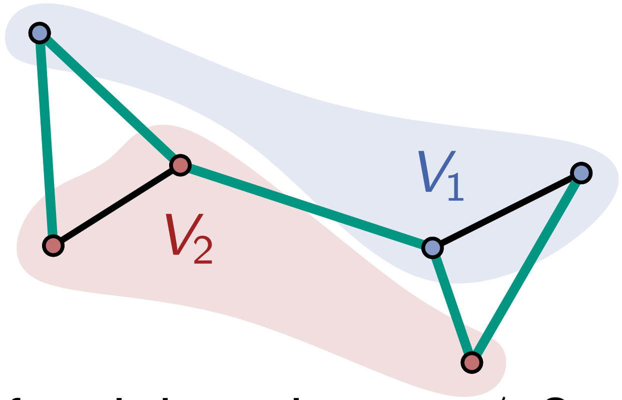

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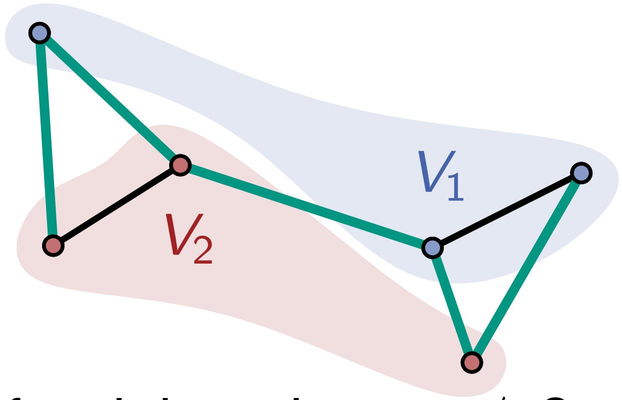

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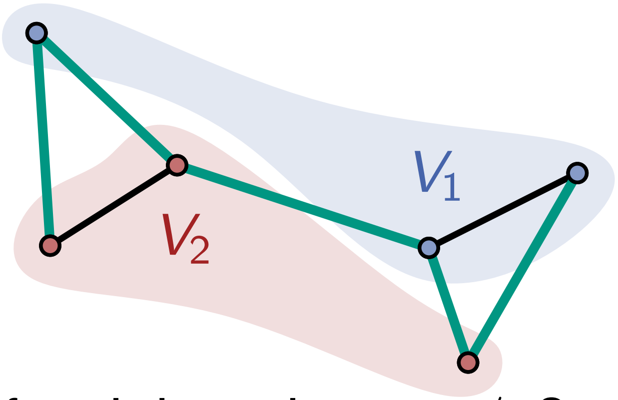

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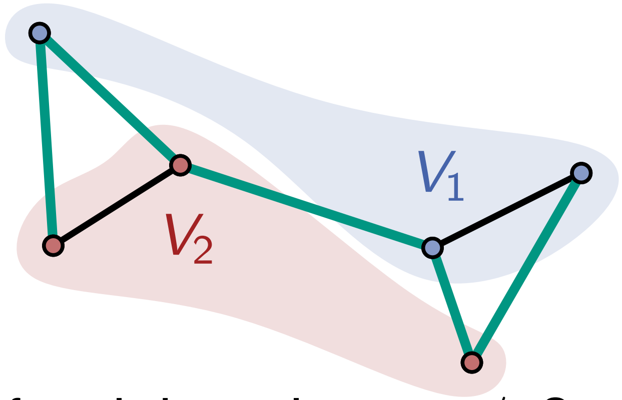

Recap

- $G = (V, E)$ an unweighted, undirected, connected graph
- *Cut*: partition of V into V_1, V_2 s.t. $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$
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- Question now: In a graph with m edges, does there *exist* a cut of weight at least $m/2$?

Random Process

- Add each vertex to one of the two sets with equal prob. $\frac{1}{2}$

Probabilistic Method: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

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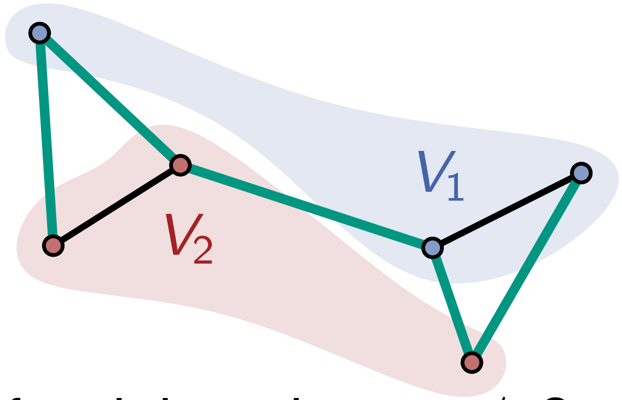

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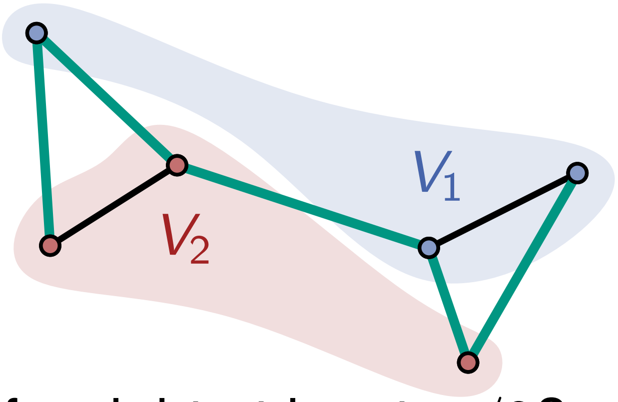

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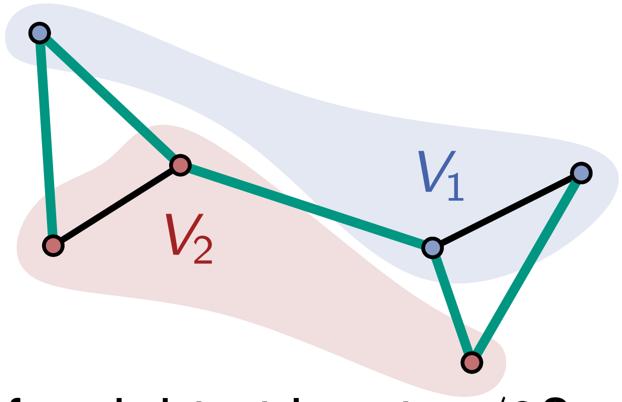

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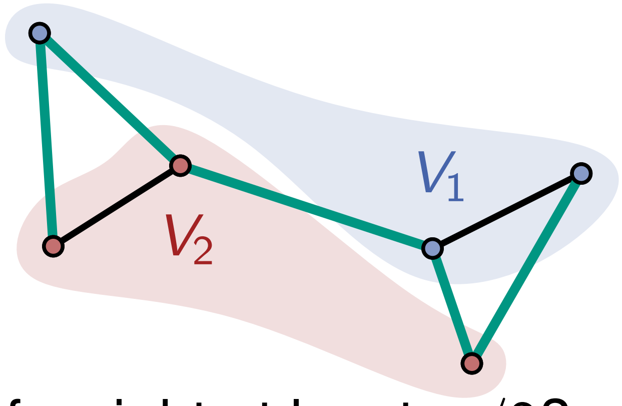

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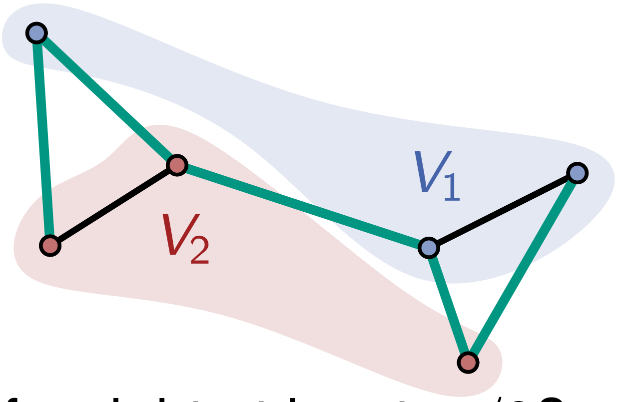

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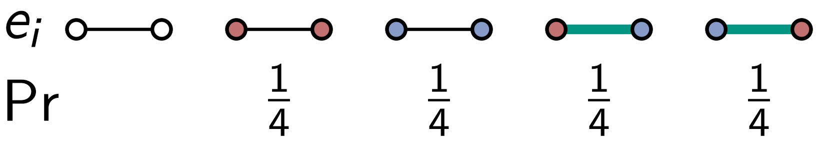

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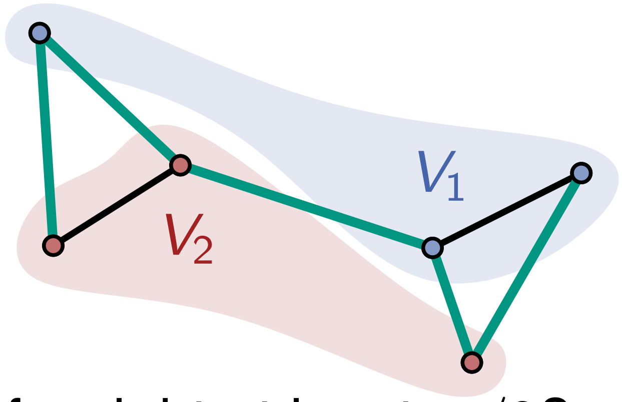

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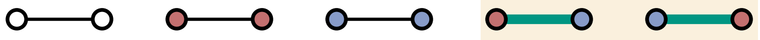

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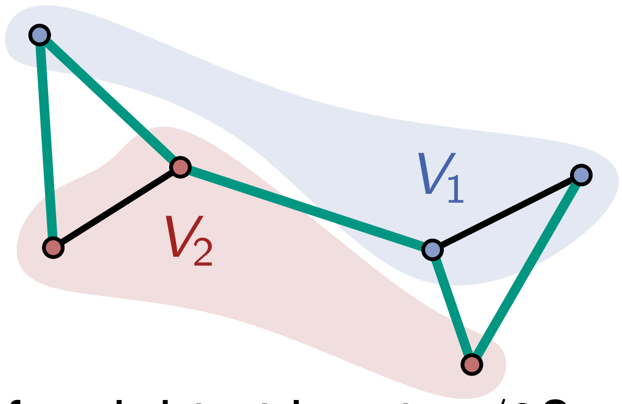

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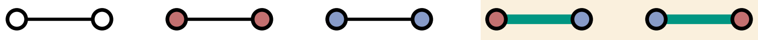
e_i 

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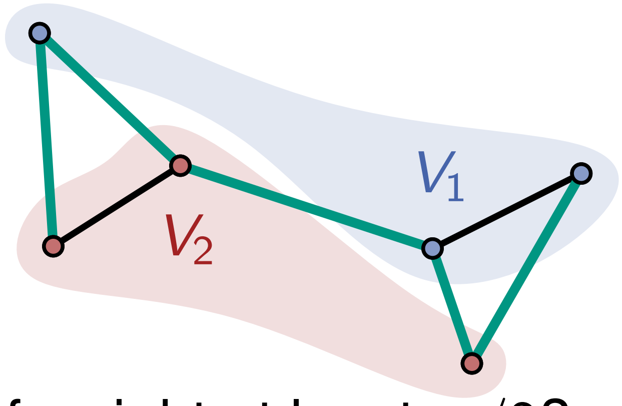

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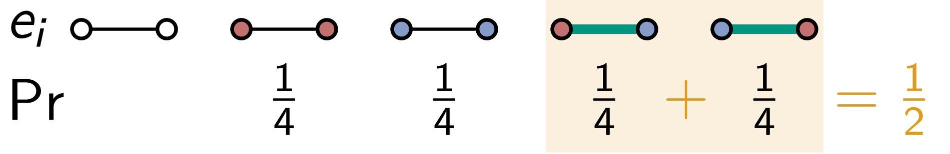
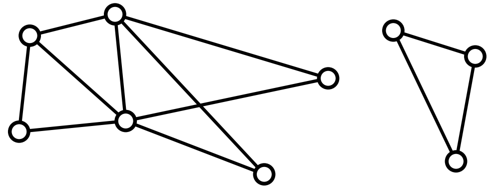


Application: Independent Sets

The Problem

- Two vertices in a graph are *independent*, if they are not adjacent



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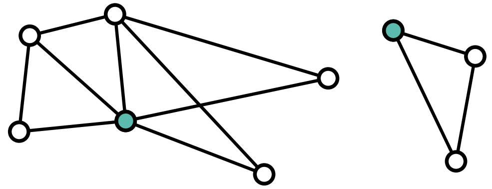

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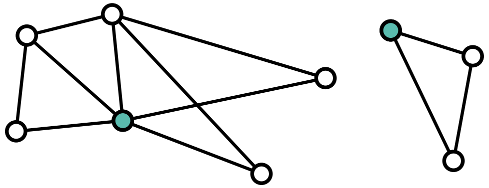

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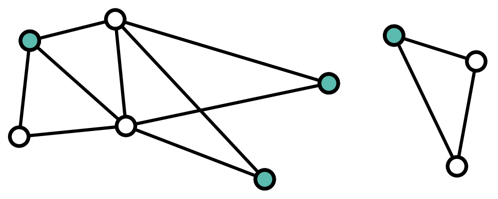

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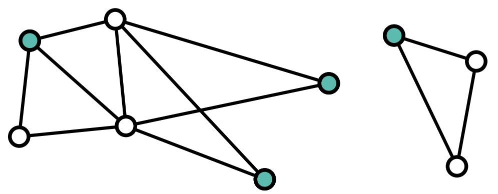

$\alpha(G) = 4$

Application: Independent Sets

Karlsruhe Institute of Technology

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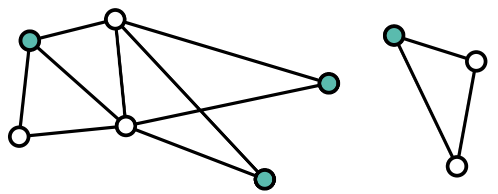

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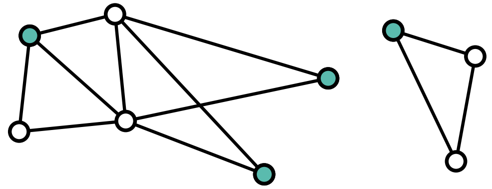

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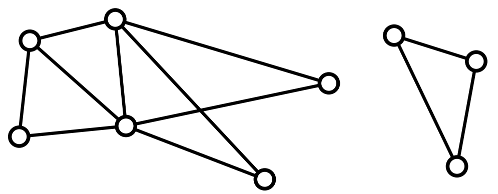
Theorem: Let G be a graph with n vertices and $m \geq n/2$ edges. Then $\alpha(G) \geq n^2/(4m)$.

Application: Independent Sets

Karlsruhe Institute of Technology

The Problem



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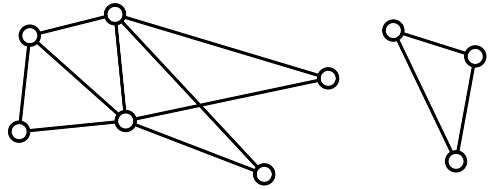

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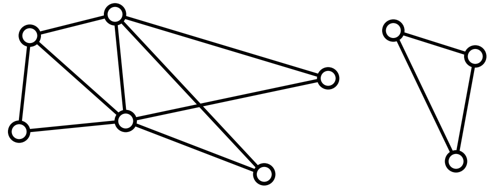

$d = \frac{24}{9}$

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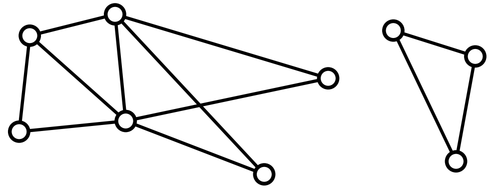

$d = \frac{24}{9} \Rightarrow$ Survival rate: $\frac{3}{8}$

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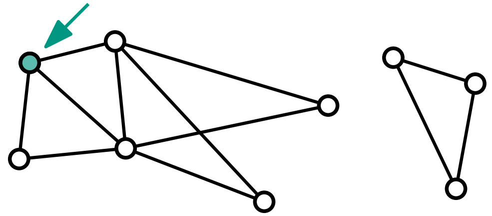

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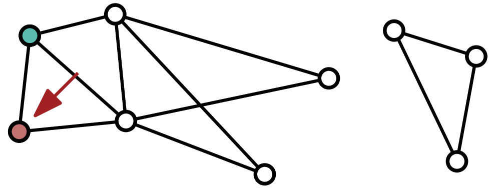

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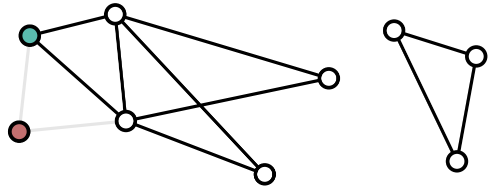

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Theorem: Let G be a graph with n vertices and $m \geq n/2$ edges. Then $\alpha(G) \geq n^2/(4m)$.

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Probabilistic Method: Show that something exists by proving that it has a *positive* probability of occuring from a random process.

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- Let $d = 2m/n$ be the average degree of G
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Application: Independent Sets

Karlsruhe Institute of Technology

The Problem

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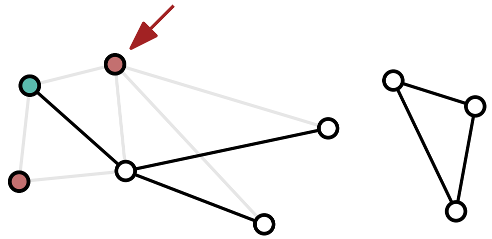

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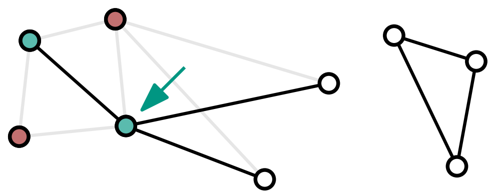

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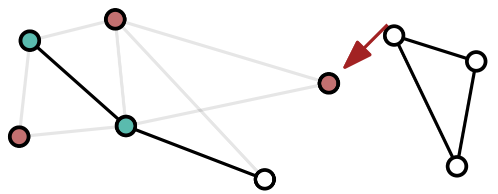

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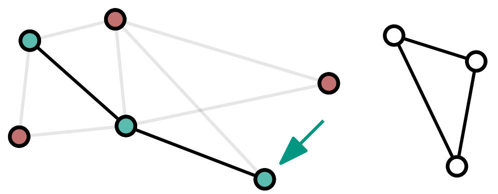

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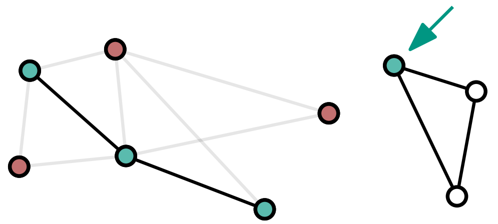

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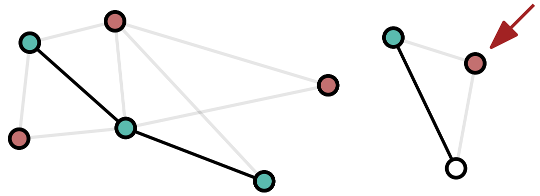

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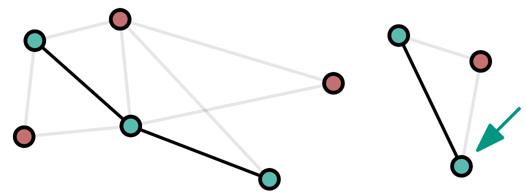

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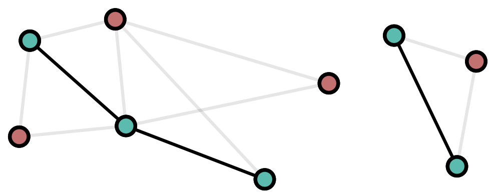

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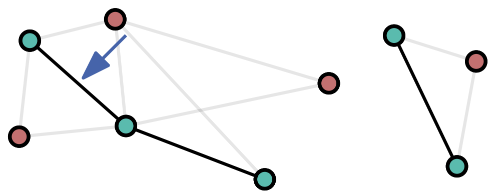

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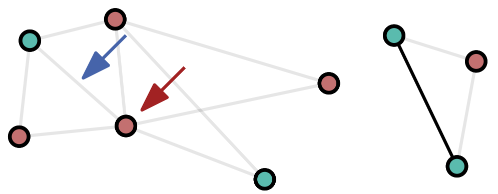

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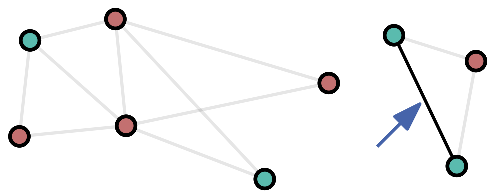

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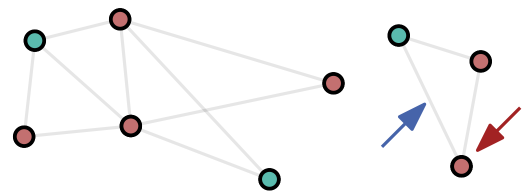

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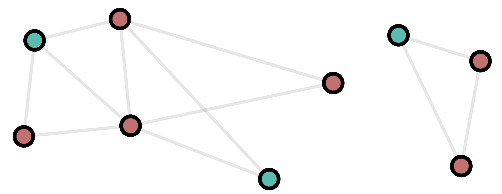

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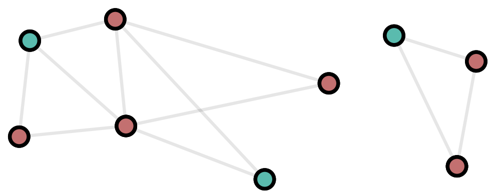

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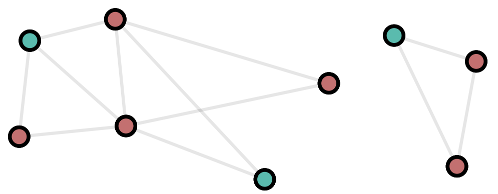

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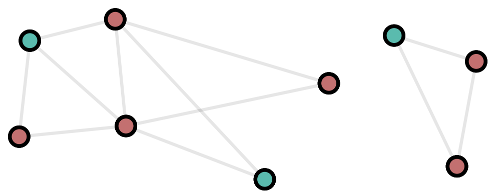

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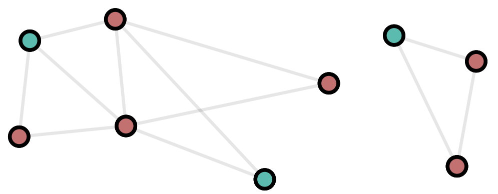

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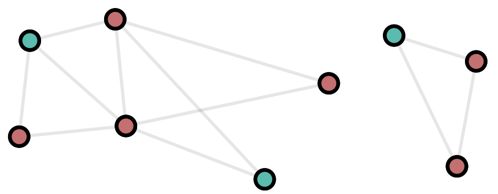

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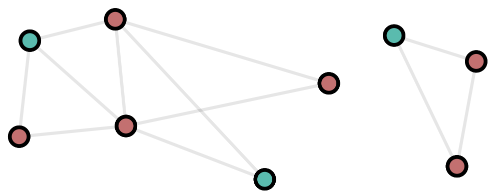

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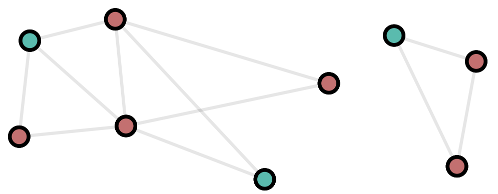

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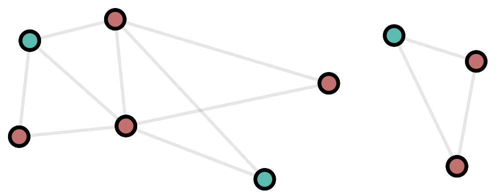

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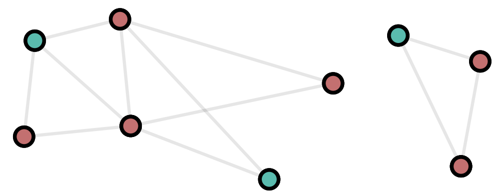

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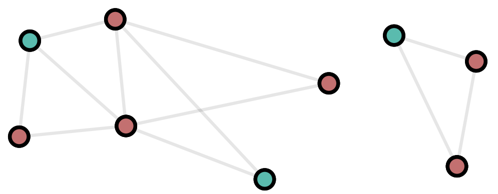

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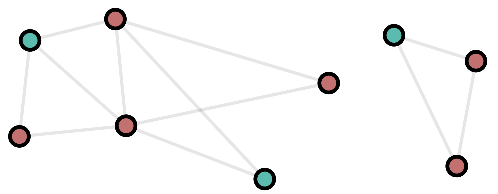

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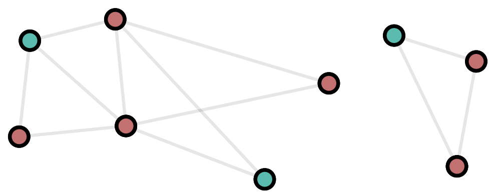

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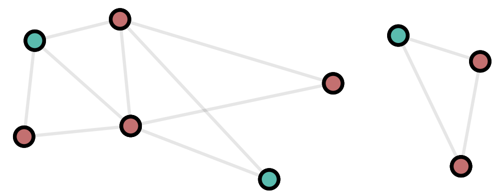

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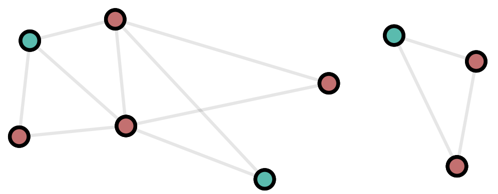

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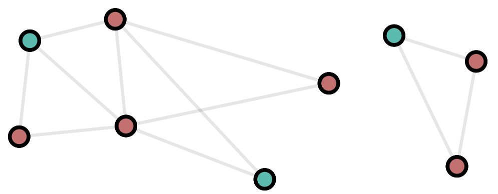

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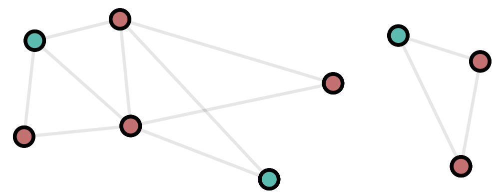

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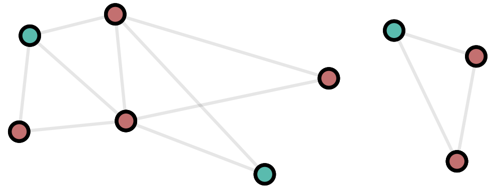

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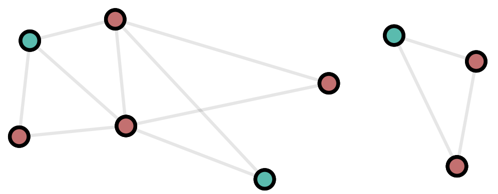

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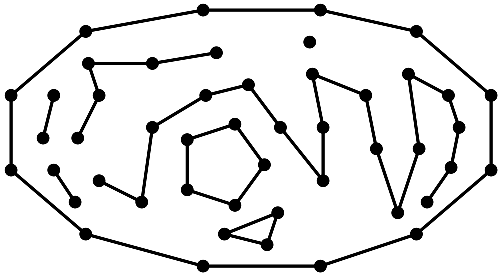

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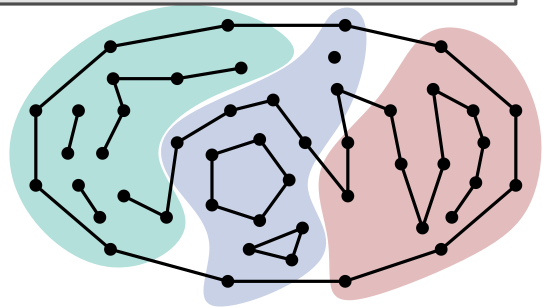

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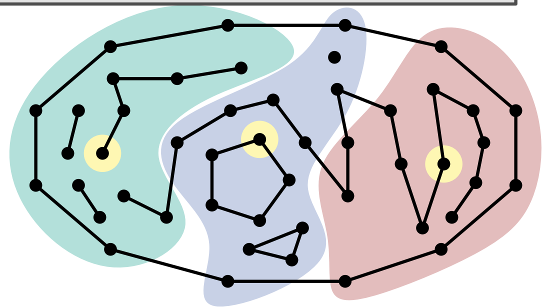

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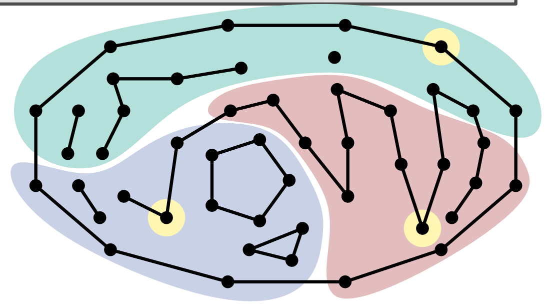

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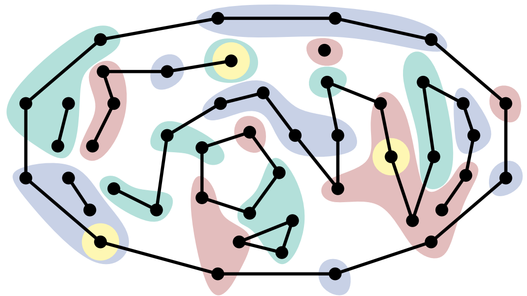

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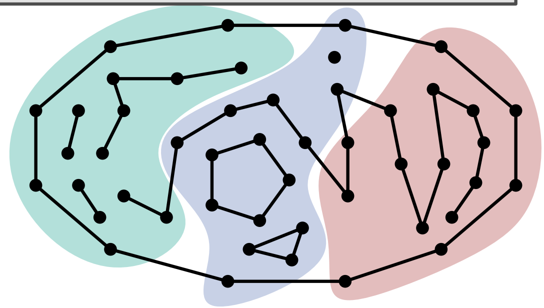

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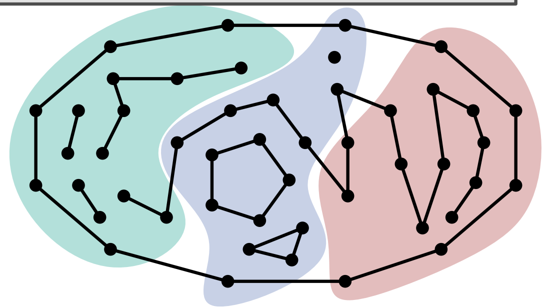

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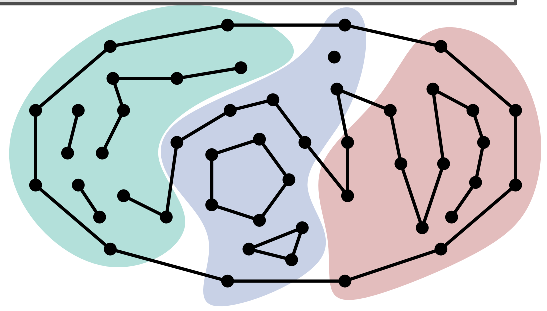

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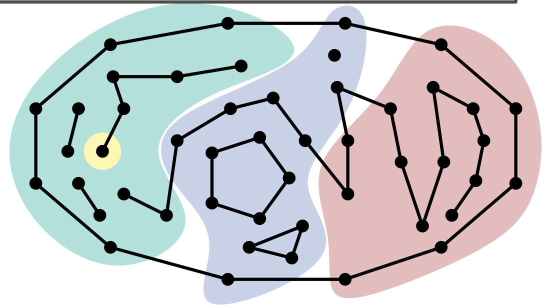

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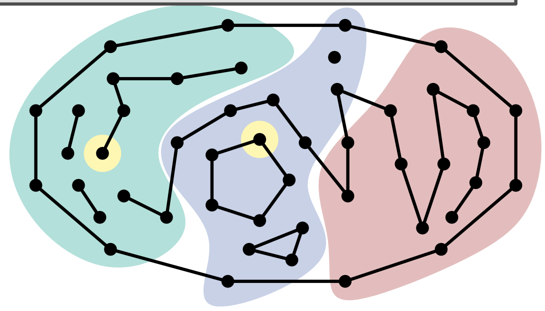

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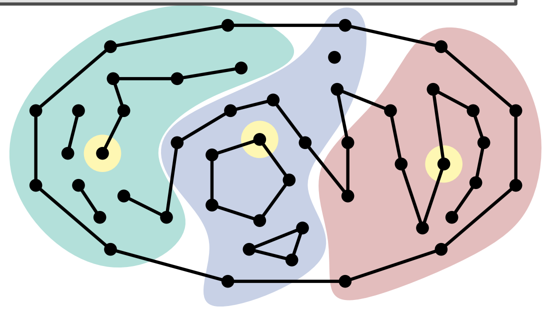

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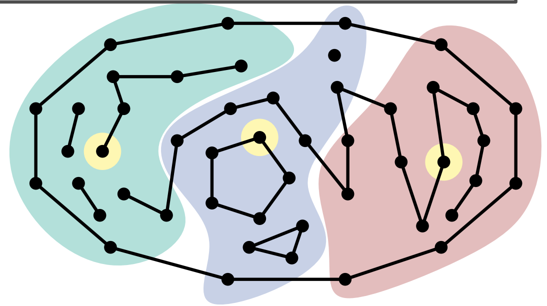

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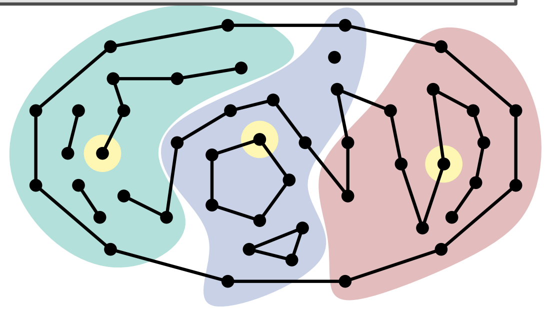

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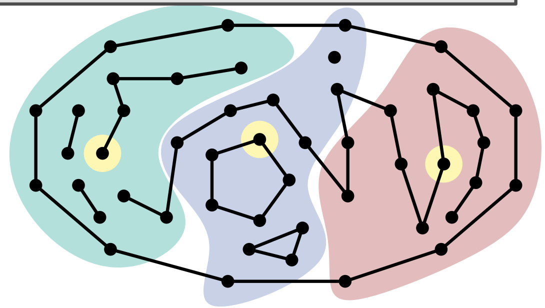

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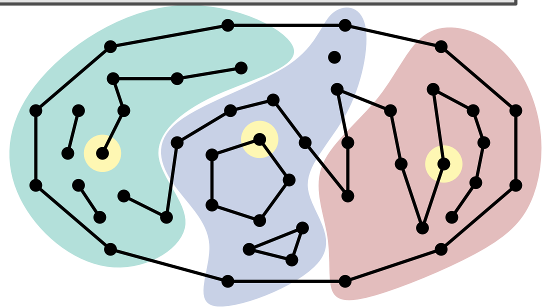

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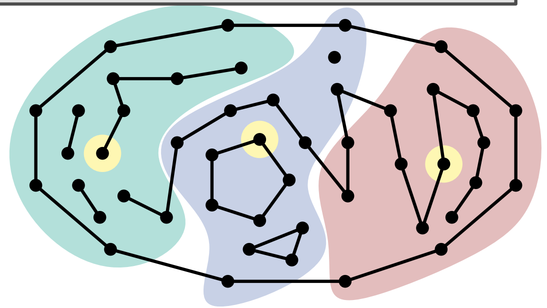

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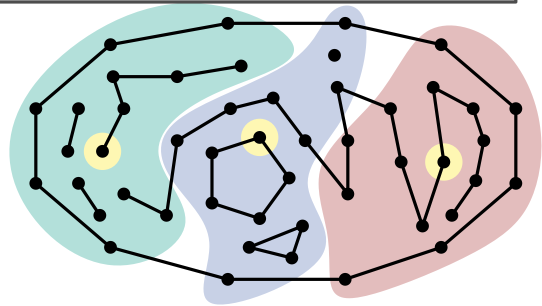

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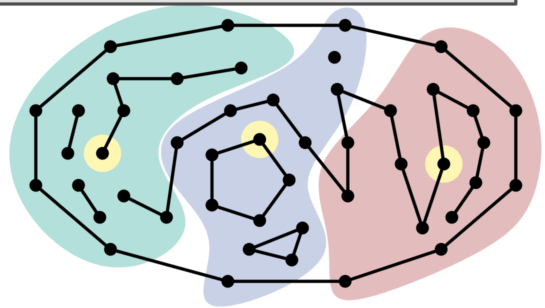

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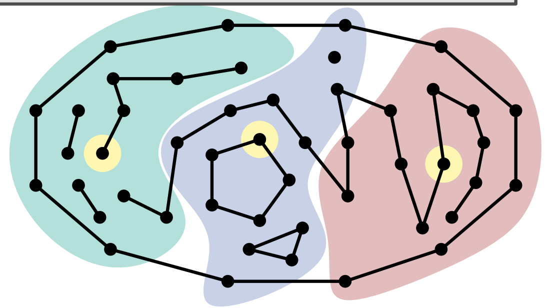

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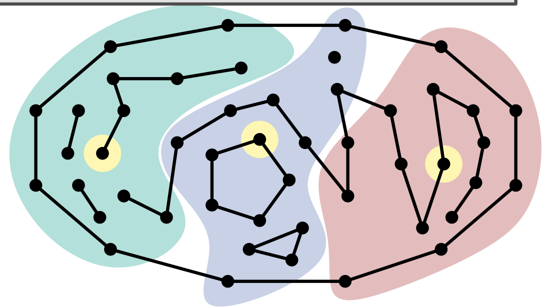

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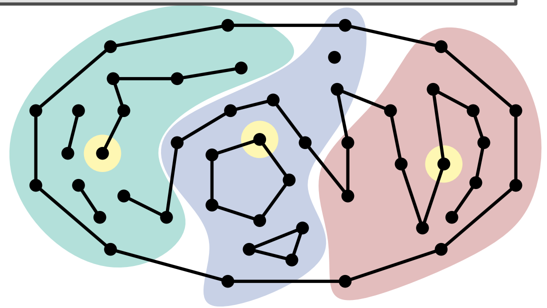

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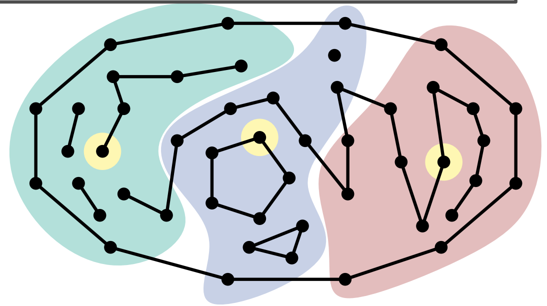

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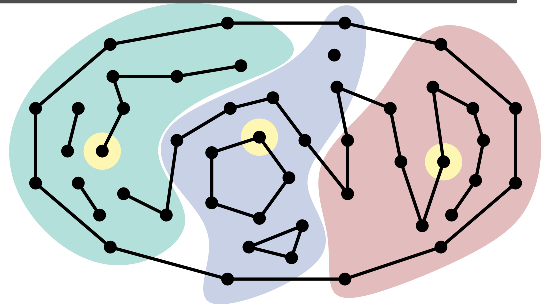

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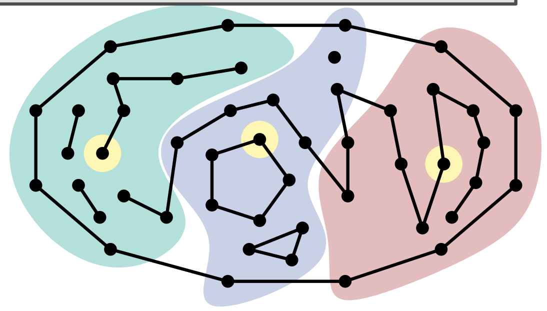

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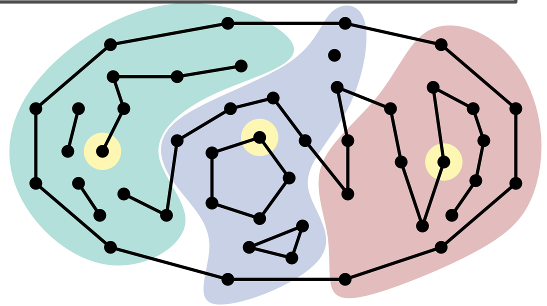

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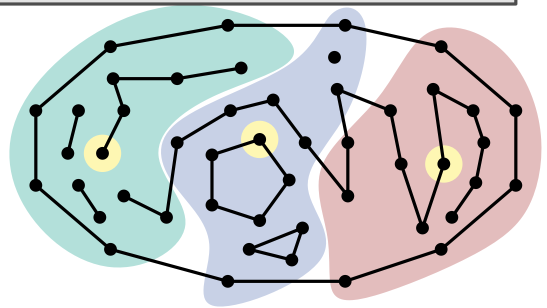

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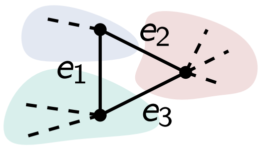

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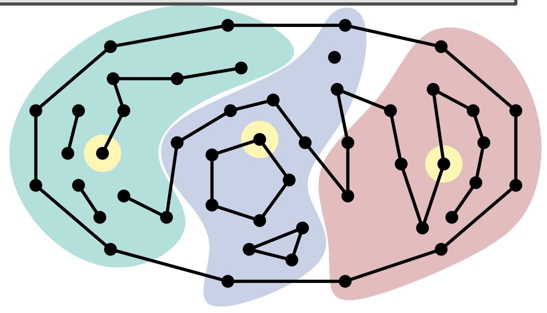

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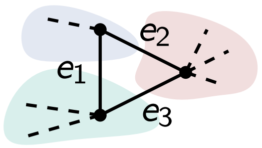

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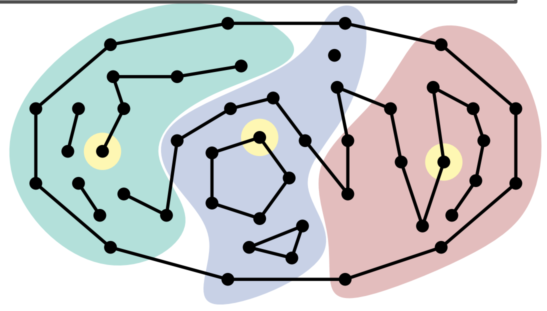

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- Assume $|V_i| = k = 8\Delta$ for all i (otherwise remove vertices from too large V_i)
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- To show: $\Pr[\text{“}S \text{ independent”}] > 0$
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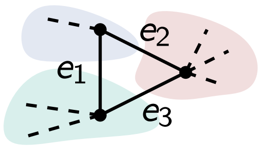
$k > 1$

The events are not independent!



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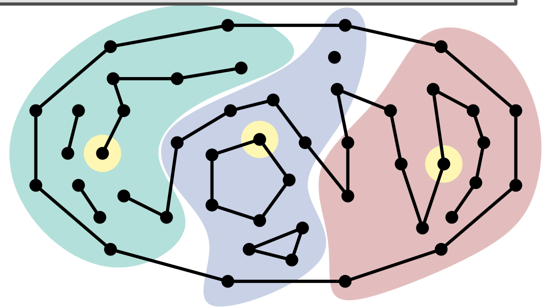
Application: Dependent Independent Set

Theorem: Let $G = (V, E)$ be a graph with max-degree Δ. For any partition $V_1 \cup \ldots \cup V_t = V$ such that $|V_i| \geq 8\Delta$, there exists an independent set containing one vertex from each V_i.

Proof

Probabilistic Method: Show that something exists by proving that it has a *positive* probability of occuring from a random process.



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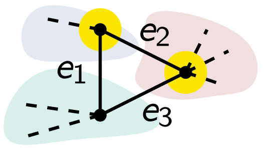

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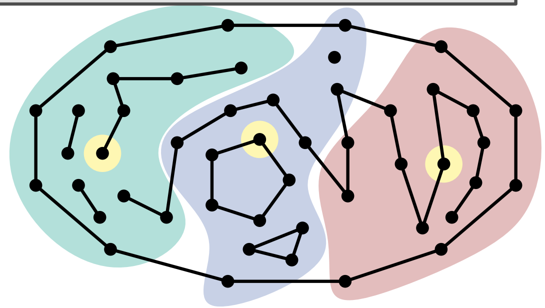

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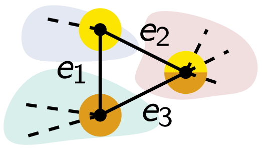

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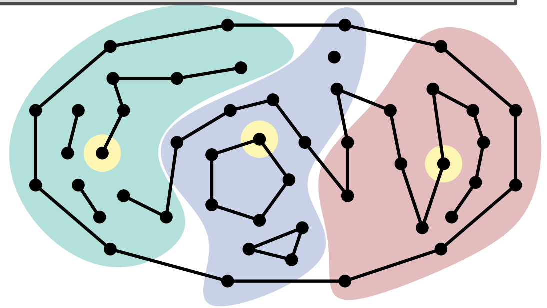

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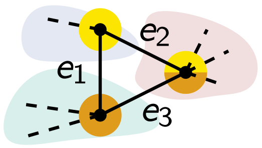

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To be or not to be... independent

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Definition: Event A is **independent of an event** B if $\Pr[A \mid B] = \Pr[A]$. $(\Pr[A \cap B] = \Pr[A]\Pr[B])$

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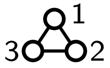
Graph



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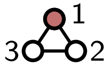

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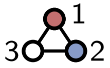

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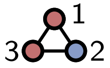

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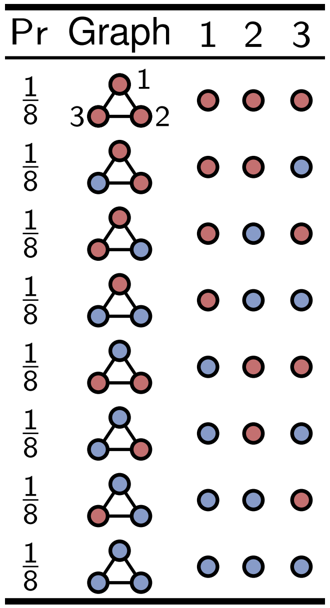

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- Let A_{ij} for $i < j$ be the event that i and j have the same color

Pr	Graph	1	2	3	A_{12}	A_{13}	A_{23}
$\frac{1}{8}$	1, 2, 3	red	red	red			
$\frac{1}{8}$		red	red	blue			
$\frac{1}{8}$		red	blue	red			
$\frac{1}{8}$		red	blue	blue			
$\frac{1}{8}$		blue	red	red			
$\frac{1}{8}$		blue	red	blue			
$\frac{1}{8}$		blue	blue	red			
$\frac{1}{8}$		blue	blue	blue			

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$\frac{1}{8}$		red	red	red	✓	✓	✓
$\frac{1}{8}$		red	red	blue			
$\frac{1}{8}$		red	blue	red			
$\frac{1}{8}$		red	blue	blue			
$\frac{1}{8}$		blue	red	red			
$\frac{1}{8}$		blue	red	blue			
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$\frac{1}{8}$		$\color{red}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	✓	✓	✓
$\frac{1}{8}$		$\color{red}\bullet$	$\color{red}\bullet$	$\color{blue}\bullet$	✓	✗	✗
$\frac{1}{8}$		$\color{red}\bullet$	$\color{blue}\bullet$	$\color{red}\bullet$			
$\frac{1}{8}$		$\color{red}\bullet$	$\color{blue}\bullet$	$\color{blue}\bullet$			
$\frac{1}{8}$		$\color{blue}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$			
$\frac{1}{8}$		$\color{blue}\bullet$	$\color{red}\bullet$	$\color{blue}\bullet$			
$\frac{1}{8}$		$\color{blue}\bullet$	$\color{blue}\bullet$	$\color{red}\bullet$			
$\frac{1}{8}$		$\color{blue}\bullet$	$\color{blue}\bullet$	$\color{blue}\bullet$			

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$\frac{1}{8}$		red	red	blue	✓	✗	✗
$\frac{1}{8}$		red	blue	red	✗	✓	✗
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$\frac{1}{8}$		blue	red	red	✗	✗	✓
$\frac{1}{8}$		blue	red	blue	✗	✓	✗
$\frac{1}{8}$		blue	blue	red	✓	✗	✗
$\frac{1}{8}$		blue	blue	blue	✓	✓	✓

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Karlsruhe Institute of Technology

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$\frac{1}{8}$		red	red	blue	✓	✗	✗
$\frac{1}{8}$		red	blue	red	✗	✓	✗
$\frac{1}{8}$		red	blue	blue	✗	✗	✓
$\frac{1}{8}$		blue	red	red	✗	✗	✓
$\frac{1}{8}$		blue	red	blue	✗	✓	✗
$\frac{1}{8}$		blue	blue	red	✓	✗	✗
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Pr	Graph	1	2	3	A_{12}	A_{13}	A_{23}
$\frac{1}{8}$	1, 2, 3	🔴	🔴	🔴	✓	✓	✓
$\frac{1}{8}$		🔴	🔴	🔵	✓	✗	✗
$\frac{1}{8}$		🔴	🔵	🔴	✗	✓	✗
$\frac{1}{8}$		🔴	🔵	🔵	✗	✗	✓
$\frac{1}{8}$		🔵	🔴	🔴	✗	✗	✓
$\frac{1}{8}$		🔵	🔴	🔵	✗	✓	✗
$\frac{1}{8}$		🔵	🔵	🔴	✓	✗	✗
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$\frac{1}{8}$	1, 2, 3	●	●	●	✓	✓	✓
$\frac{1}{8}$		●	●	●	✓	✗	✗
$\frac{1}{8}$		●	●	●	✗	✓	✗
$\frac{1}{8}$		●	●	●	✗	✗	✓
$\frac{1}{8}$		●	●	●	✗	✗	✓
$\frac{1}{8}$		●	●	●	✗	✓	✗
$\frac{1}{8}$		●	●	●	✓	✗	✗
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Pr	Graph	1	2	3	A_{12}	A_{13}	A_{23}
$\frac{1}{8}$		red	red	red	✓	✓	✓
$\frac{1}{8}$		red	red	blue	✓	✗	✗
$\frac{1}{8}$		red	blue	red	✗	✓	✗
$\frac{1}{8}$		red	blue	blue	✗	✗	✓
$\frac{1}{8}$		blue	red	red	✗	✗	✓
$\frac{1}{8}$		blue	red	blue	✗	✓	✗
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Pr	Graph	1	2	3	A_{12}	A_{13}	A_{23}
$\frac{1}{8}$	1 3 2	$\color{red}{\bullet}$	$\color{red}{\bullet}$	$\color{red}{\bullet}$	✓	✓	✓
$\frac{1}{8}$		$\color{red}{\bullet}$	$\color{red}{\bullet}$	$\color{blue}{\bullet}$	✓	✗	✗
$\frac{1}{8}$		$\color{red}{\bullet}$	$\color{blue}{\bullet}$	$\color{red}{\bullet}$	✗	✓	✗
$\frac{1}{8}$		$\color{red}{\bullet}$	$\color{blue}{\bullet}$	$\color{blue}{\bullet}$	✗	✗	✓
$\frac{1}{8}$		$\color{blue}{\bullet}$	$\color{red}{\bullet}$	$\color{red}{\bullet}$	✗	✗	✓
$\frac{1}{8}$		$\color{blue}{\bullet}$	$\color{red}{\bullet}$	$\color{blue}{\bullet}$	✗	✓	✗
$\frac{1}{8}$		$\color{blue}{\bullet}$	$\color{blue}{\bullet}$	$\color{red}{\bullet}$	✓	✗	✗
$\frac{1}{8}$		$\color{blue}{\bullet}$	$\color{blue}{\bullet}$	$\color{blue}{\bullet}$	✓	✓	✓

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$\frac{1}{8}$		blue	red	red	✗	✗	✓
$\frac{1}{8}$		blue	red	blue	✗	✓	✗
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$\frac{1}{8}$		red	red	blue	✓	✗	✗
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$\frac{1}{8}$		red	blue	blue	✗	✗	✓
$\frac{1}{8}$		blue	red	red	✗	✗	✓
$\frac{1}{8}$		blue	red	blue	✗	✓	✗
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$\frac{1}{8}$		$\color{red}\bullet$	$\color{red}\bullet$	$\color{blue}\bullet$	✓	✗	✗
$\frac{1}{8}$		$\color{red}\bullet$	$\color{blue}\bullet$	$\color{red}\bullet$	✗	✓	✗
$\frac{1}{8}$		$\color{red}\bullet$	$\color{blue}\bullet$	$\color{blue}\bullet$	✗	✗	✓
$\frac{1}{8}$		$\color{blue}\bullet$	$\color{red}\bullet$	$\color{red}\bullet$	✗	✗	✓
$\frac{1}{8}$		$\color{blue}\bullet$	$\color{red}\bullet$	$\color{blue}\bullet$	✗	✓	✗
$\frac{1}{8}$		$\color{blue}\bullet$	$\color{blue}\bullet$	$\color{red}\bullet$	✓	✗	✗
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$\frac{1}{8}$		🔴	🔵	🔴	✗	✓	✗
$\frac{1}{8}$		🔴	🔵	🔵	✗	✗	✓
$\frac{1}{8}$		🔵	🔴	🔴	✗	✗	✓
$\frac{1}{8}$		🔵	🔴	🔵	✗	✓	✗
$\frac{1}{8}$		🔵	🔵	🔴	✓	✗	✗
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Pr	Graph	1	2	3	A_{12}	A_{13}	A_{23}
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$\frac{1}{8}$		🔴	🔵	🔴	✗	✓	✗
$\frac{1}{8}$		🔴	🔵	🔵	✗	✗	✓
$\frac{1}{8}$		🔵	🔴	🔴	✗	✗	✓
$\frac{1}{8}$		🔵	🔴	🔵	✗	✓	✗
$\frac{1}{8}$		🔵	🔵	🔴	✓	✗	✗
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$\frac{1}{8}$		$\textcolor{blue}{\bullet}$	$\textcolor{red}{\bullet}$	$\textcolor{red}{\bullet}$	✗	✗	✓
$\frac{1}{8}$		$\textcolor{blue}{\bullet}$	$\textcolor{red}{\bullet}$	$\textcolor{blue}{\bullet}$	✗	✓	✗
$\frac{1}{8}$		$\textcolor{blue}{\bullet}$	$\textcolor{blue}{\bullet}$	$\textcolor{red}{\bullet}$	✓	✗	✗
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$\frac{1}{8}$		🔴	🔵	🔴	✗	✓	✗
$\frac{1}{8}$		🔴	🔵	🔵	✗	✗	✓
$\frac{1}{8}$		🔵	🔴	🔴	✗	✗	✓
$\frac{1}{8}$		🔵	🔴	🔵	✗	✓	✗
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Pr	Graph	1	2	3	A_{12}	A_{13}	A_{23}
$\frac{1}{8}$	1, 2, 3	red	red	red	✓	✓	✓
$\frac{1}{8}$		red	red	blue	✓	✗	✗
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To be or not to be... independent

Independence

Definition: Event A is **independent of an event** B if $\Pr[A \mid B] = \Pr[A]$. $(\Pr[A \cap B] = \Pr[A]\Pr[B])$

Definition: Event A is **independent of a set of events** $\mathcal{E}$ if for all subsets $\mathcal{E}' = \{B_1, B_2, \dots, B_k\} \subseteq \mathcal{E}$ we have $\Pr[A \mid \bigcap_{i \in [k]} B_i] = \Pr[A]$.

Example

- Triangle, independently color each vertex red/blue with prob. $\frac{1}{2}$
- Let A_{ij} for $i < j$ be the event that i and j have the same color
- $A = A_{12}$, $B = A_{23}$:

  $\Pr[A_{12}] = \boxed{\frac{1}{2}}$

  $\Pr[A_{12} \mid A_{23}] = \frac{\Pr[A_{12} \cap A_{23}]}{\Pr[A_{23}]} = \frac{1/4}{1/2} = \frac{1}{2}$

  (same holds for all choices of A and B)
- All A_{ij} are *pairwise* independent

- $A = A_{12}$, $\mathcal{E} = \{A_{13}, A_{23}\}$:

  $\Pr[A_{12} \mid A_{13} \cap A_{23}]$

  $= \frac{\Pr[A_{12} \cap A_{13} \cap A_{23}]}{\Pr[A_{13} \cap A_{23}]} = \frac{1/4}{1/4} = \boxed{1}$
- A_{ij} not independent of the other events

Pr	Graph	1	2	3	A_{12}	A_{13}	A_{23}
$\frac{1}{8}$	1, 2, 3	red	red	red	✓	✓	✓
$\frac{1}{8}$		red	red	blue	✓	✗	✗
$\frac{1}{8}$		red	blue	red	✗	✓	✗
$\frac{1}{8}$		red	blue	blue	✗	✗	✓
$\frac{1}{8}$		blue	red	red	✗	✗	✓
$\frac{1}{8}$		blue	red	blue	✗	✓	✗
$\frac{1}{8}$		blue	blue	red	✓	✗	✗
$\frac{1}{8}$		blue	blue	blue	✓	✓	✓

Lovász Local Lemma (LLL)

Theorem: Let $E_1, ..., E_n$ be events such that each E_i for $i \in [n]$ is independent of all but at most $d > 0$ of the other events. Let $p = \max_{i \in [n]} \Pr[E_i]$. If $4dp \leq 1$, then $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$.

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$$\Pr[\textstyle\bigcap_{i \in [n]} \neg E_i]$$

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$$\Pr[\overbrace{\textstyle\bigcap_{i\in[n]} \neg E_i}^{\mathcal{I}([n])}]$$

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"Chain Rule"

$$\underbrace{\Pr[\overbrace{\textstyle\bigcap_{i \in [n]} \neg E_i}^{\mathcal{I}([n])}]} \overset{\text{Chain Rule}}{=} \prod_{i \in [n]} \Pr[\neg E_i \mid \mathcal{I}([i-1])]$$

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Proof

$$\Pr[\underbrace{\textstyle\bigcap_{i \in [n]} \neg E_i}_{\mathcal{I}([n])}] \overset{\text{“Chain Rule”}}{=} \prod_{i \in [n]} \Pr[\neg E_i \mid \mathcal{I}([i-1])]$$
$$= \prod_{i \in [n]} (1 - \Pr[E_i \mid \mathcal{I}([i-1])])$$

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- Since $d > 0$ and $4dp \leq 1$, we have $4p \leq 1$ and thus $2p \leq 1/2$

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Proof

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Conditional Probability: $\Pr[A \cap B] = \Pr[A \mid B] \cdot \Pr[B]$

$$\Pr[\overbrace{\textstyle\bigcap_{i\in[n]} \neg E_i}^{\mathcal{I}([n])}] \overset{\text{“Chain Rule”}}{=} \prod_{i\in[n]} \Pr[\neg E_i \mid \mathcal{I}([i-1])]$$

$$= \prod_{i\in[n]} (1 - \underbrace{\Pr[E_i \mid \mathcal{I}([i-1])]}_{\text{Claim: } \le 2p})$$

$$\ge \prod_{i\in[n]} (1 - 2p)$$

$$\ge \prod_{i\in[n]} 1/2$$

- Since $d > 0$ and $4dp \le 1$, we have $4p \le 1$ and thus $2p \le 1/2$

Lovász Local Lemma (LLL)

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Theorem: Let $E_1, \dots, E_n$ be events such that each E_i for $i \in [n]$ is independent of all but at most $d > 0$ of the other events. Let $p = \max_{i \in [n]} \Pr[E_i]$. If $4dp \leq 1$, then $\Pr[\bigcap_{i \in [n]} \neg E_i] > 0$.

- If $d = 0$, everything is independent and we can just compute the probability as the product
- For each $i \in [n]$ let $D_i \subseteq [n]$ be the such that E_i is independent of $\{E_1, \dots, E_n\} \setminus (\bigcup_{j \in \{i\} \cup D_i} E_j)$, then $|D(i)| \leq d$.

(Remove events defined by D_i to make E_i independent of the rest.)

Proof

$$\Pr[\overbrace{\bigcap_{i \in [n]} \neg E_i}^{\mathcal{I}([n])}] \overset{\text{“Chain Rule”}}{=} \prod_{i \in [n]} \Pr[\neg E_i \mid \mathcal{I}([i-1])]$$

$$= \prod_{i \in [n]} (1 - \underbrace{\Pr[E_i \mid \mathcal{I}([i-1])]}_{\text{Claim: } \leq 2p})$$

← *remains to prove this!*

$$\geq \prod_{i \in [n]} (1 - 2p)$$

$$\geq \prod_{i \in [n]} 1/2 \quad > 0 \ ✓$$

Notation: For $S \subseteq [n]$ write $\mathcal{I}(S) = \bigcap_{i \in S} \neg E_i$

Conditional Probability: $\Pr[A \cap B] = \Pr[A \mid B] \cdot \Pr[B]$

- Since $d > 0$ and $4dp \leq 1$, we have $4p \leq 1$ and thus $2p \leq 1/2$

LLL – Proof of Claim

Claim: For all $S_i \subseteq \{1, ..., n\} \setminus \{i\}$, $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$.

Proof

LLL: Events $E_1, ..., E_n$

- $p = \max_{i \in [n]} \Pr[E_i]$
- E_i independent of $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with $|D_i| \leq d$
- $4dp \leq 1$

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Proof (via induction over the size $s = |S_i|$)

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LLL – Proof of Claim

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Claim: For all $S_i \subseteq \{1, ..., n\} \setminus \{i\}$, $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$.

Proof (via induction over the size $s = |S_i|$)

Start: $s = 0$

LLL: Events $E_1, ..., E_n$

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Proof (via induction over the size $s = |S_i|$)
Start: $s = 0$ → $S_i = \emptyset$

LLL: Events $E_1, ..., E_n$
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- E_i independent of $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with $|D_i| \leq d$
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Start: $s = 0$ ➜ $S_i = \emptyset$ ➜ $\Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i]$

LLL: Events $E_1, ..., E_n$

- $p = \max_{i \in [n]} \Pr[E_i]$
- E_i independent of $\{E_1, ..., E_n\} \setminus \bigcup_{j \in \{i\} \cup D_i} E_j$ with $|D_i| \leq d$
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Start: $s = 0$ → $S_i = \emptyset$ → $\Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p$ ✓

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Proof (via induction over the size $s = |S_i|$)

Start: $s = 0$ ➞ $S_i = \emptyset$ ➞ $\Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p$ ✓

Step: $s > 0$

LLL: Events $E_1, ..., E_n$

- $p = \max_{i \in [n]} \Pr[E_i]$
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Start: $s = 0 \rightarrow S_i = \emptyset \rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p$ ✓

Step: $s > 0$

- Case 1: $D_i' = S_i \cap D_i = \emptyset$

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Start: $s = 0 \to S_i = \emptyset \to \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p$ ✓

Step: $s > 0$

- Case 1: $D_i' = S_i \cap D_i = \emptyset$
  - E_i is independent of $\{E_j \mid j \in S_i\}$

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- Case 2: $D_i' = S_i \cap D_i \neq \emptyset$

$\Pr[E_i \mid \mathcal{I}(S_i)]$

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Start: $s = 0 \rightarrow S_i = \emptyset \rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p$ ✓

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$$\Pr[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]}$$

$$\mathcal{I}(S_i) = \bigcap_{j \in S_i} \neg E_j$$

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Proof (via induction over the size $s = |S_i|$)

Start: $s = 0 \rightarrow S_i = \emptyset \rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p$ ✓

Step: $s > 0$

- Case 1: $D'_i = S_i \cap D_i = \emptyset$
  - E_i is independent of $\{E_j \mid j \in S_i\} \rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p$ ✓
- Case 2: $D'_i = S_i \cap D_i \neq \emptyset$

$$\Pr[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]}$$

$$\mathcal{I}(S_i) = \bigcap_{j \in S_i} \neg E_j$$

$$S_i = (S_i \setminus D'_i) \cup D'_i$$

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- Case 2: $D_i' = S_i \cap D_i \neq \emptyset$

$$\Pr[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]}$$

$$\mathcal{I}(S_i) = \bigcap_{j \in S_i} \neg E_j$$

$$S_i = (S_i \setminus D_i') \cup D_i' \quad \rightarrow \quad = \bigcap_{j \in S_i \setminus D_i'} \neg E_j \cap \bigcap_{j \in D_i'} \neg E_j$$

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$$\Pr[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]}$$

$$\mathcal{I}(S_i) = \bigcap_{j \in S_i} \neg E_j$$

$$S_i = (S_i \setminus D_i') \cup D_i' \rightarrow = \bigcap_{j \in S_i \setminus D_i'} \neg E_j \cap \bigcap_{j \in D_i'} \neg E_j$$

$$= \mathcal{I}(S_i \setminus D_i') \cap \mathcal{I}(D_i')$$

LLL: Events $E_1, ..., E_n$
- $p = \max_{i \in [n]} \Pr[E_i]$
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Claim: For all $S_i \subseteq \{1, ..., n\} \setminus \{i\}$, $\Pr[E_i \mid \mathcal{I}(S_i)] \leq 2p$.

Proof (via induction over the size $s = |S_i|$)

Start: $s = 0 \rightarrow S_i = \emptyset \rightarrow \Pr[E_i \mid \mathcal{I}(S_i)] = \Pr[E_i] \leq p \leq 2p$ ✓

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- Case 2: $D_i' = S_i \cap D_i \neq \emptyset$

$$\Pr[E_i \mid \mathcal{I}(S_i)] = \frac{\Pr[E_i \cap \mathcal{I}(S_i)]}{\Pr[\mathcal{I}(S_i)]} = \frac{\Pr[E_i \cap \mathcal{I}(D_i') \cap \mathcal{I}(S_i \setminus D_i')]}{\Pr[\mathcal{I}(S_i)]}$$

$$\mathcal{I}(S_i) = \bigcap_{j \in S_i} \neg E_j$$

$$S_i = (S_i \setminus D_i') \cup D_i' \quad \rightarrow \quad = \bigcap_{j \in S_i \setminus D_i'} \neg E_j \cap \bigcap_{j \in D_i'} \neg E_j$$

$$= \mathcal{I}(S_i \setminus D_i') \cap \mathcal{I}(D_i')$$

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$$S_i = (S_i \setminus D_i') \cup D_i' \quad \rightarrow \quad = \bigcap_{j \in S_i \setminus D_i'} \neg E_j \cap \bigcap_{j \in D_i'} \neg E_j$$

$$= \mathcal{I}(S_i \setminus D_i') \cap \mathcal{I}(D_i')$$

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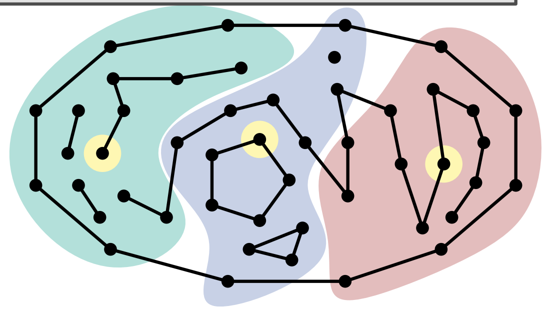
Application: Dependent Independent Set (2nd Try)

Theorem: Let $G=(V,E)$ be a graph with max-degree Δ. For any partition $V_1 \cup \ldots \cup V_t = V$ such that $|V_i| \geq 8\Delta$, there exists an independent set containing one vertex from each V_i.

Probabilistic Method: Show that something exists by proving that it has a *positive* probability of occuring from a random process.



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Random Process

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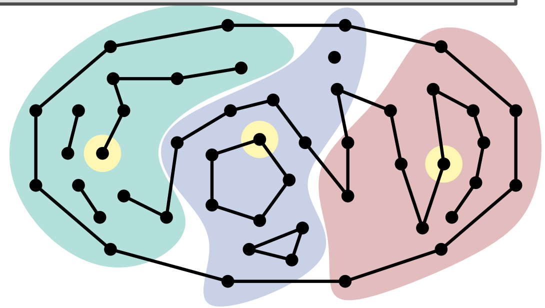

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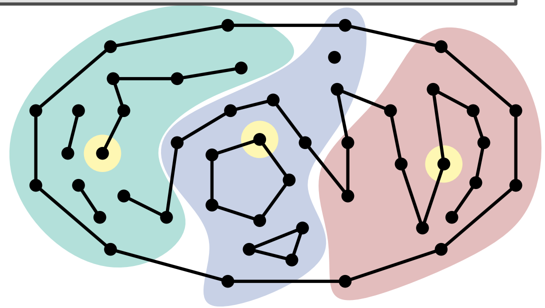

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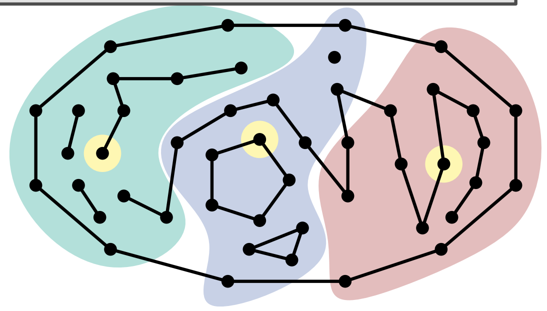

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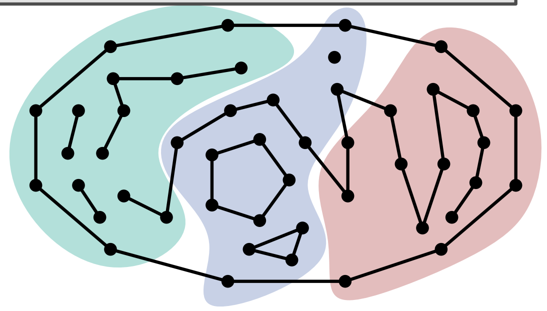

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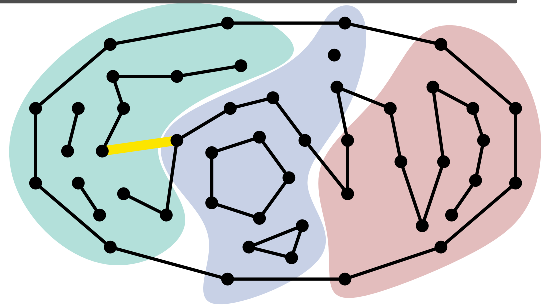

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Theorem: Let $G = (V, E)$ be a graph with max-degree Δ. For any partition $V_1 \cup \ldots \cup V_t = V$ such that $|V_i| \geq 8\Delta$, there exists an independent set containing one vertex from each V_i.

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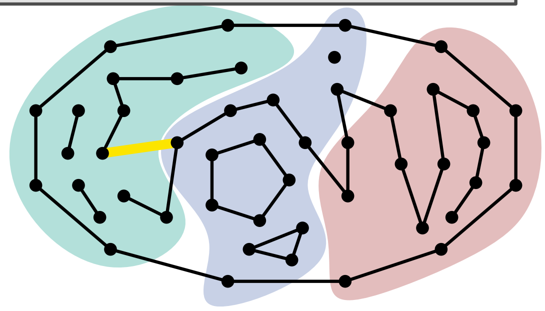

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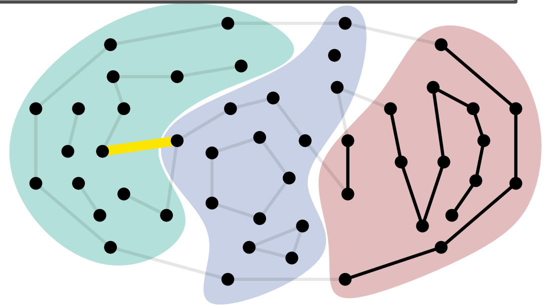

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Karlsruhe Institute of Technology

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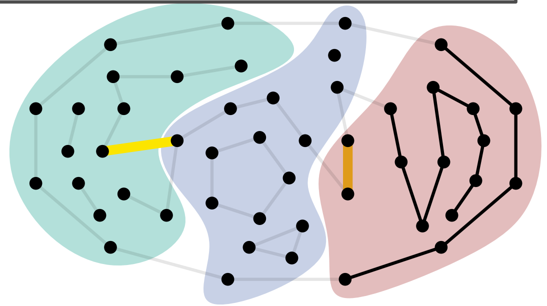

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No matter the outcome of A_f for $f \in E \setminus D_e$, the probability for a node in V_i or V_j to be chosen remains the same $\Rightarrow A_e$ is independent of all events but D_e

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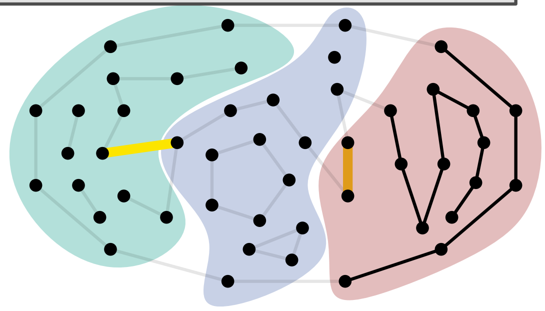

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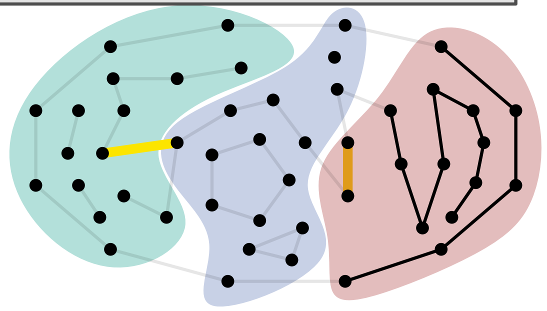

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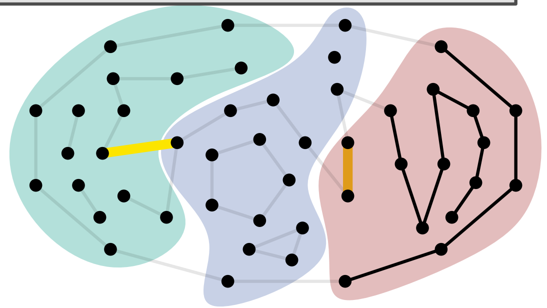

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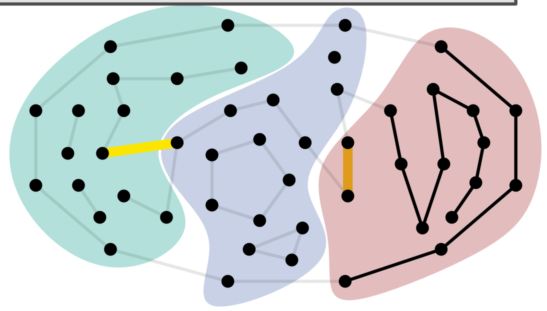

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Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

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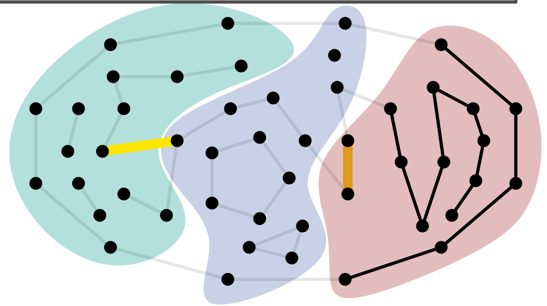

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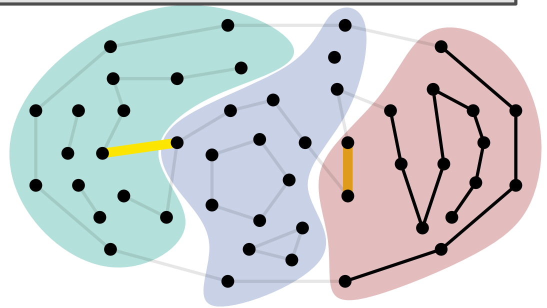

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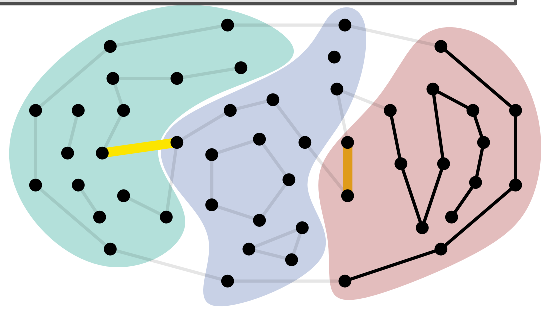

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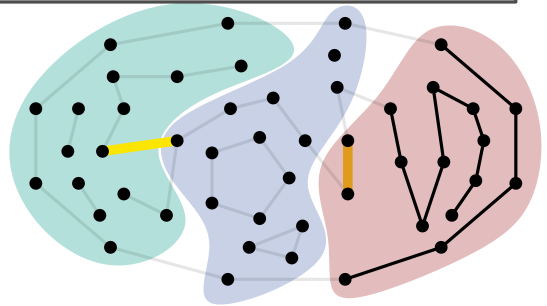

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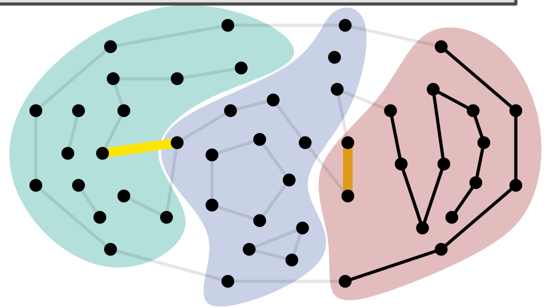

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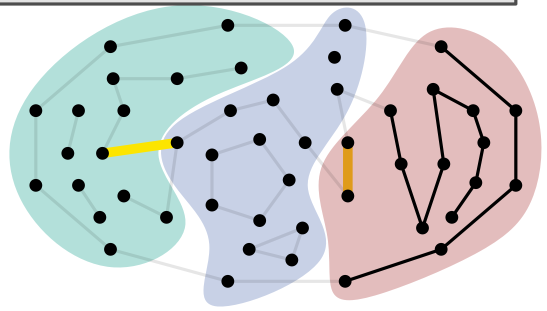

Proof

Random Process

- Assume $|V_i| = k = 8\Delta$ for all i (otherwise remove vertices from too large V_i)
- Obtain S by ind. choosing one vertex unif. at random from each V_i

Positive Probability

- To show: Pr["S independent"] > 0
- S is independent iff no edge $e = \{\underset{V_i}{u}, \underset{V_j}{v}\}$ has $\overbrace{e \subseteq S}^{\text{(both endpoints in } S\text{)}}$,

  Let A_e be the event that $e \subseteq S$

  $\Pr[\text{“}S\text{ independent”}] = \Pr[\bigcap_{e \in E} \neg A_e] > 0$ ✓

  $\Pr[A_e] \leq \frac{1}{k^2} =: p$

  $D_e = \{A_{e'} \mid e' \cap (V_i \cup V_j) \neq \emptyset\} \rightarrow |D_e| \leq \underset{|V_i|}{k}\Delta + \underset{|V_j|}{k}\Delta \leq 2k\Delta =: d$

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- $4dp \leq 1$

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$$4dp = 4 \cdot 2k\Delta \cdot \frac{1}{k^2} = \frac{8\Delta}{k} = 1$$

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

Application: Independent Paths

- Given a network and k vertex pairs that want to communicate

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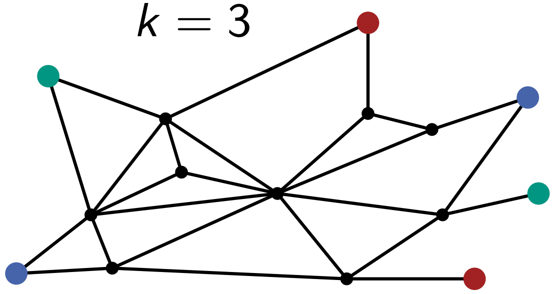

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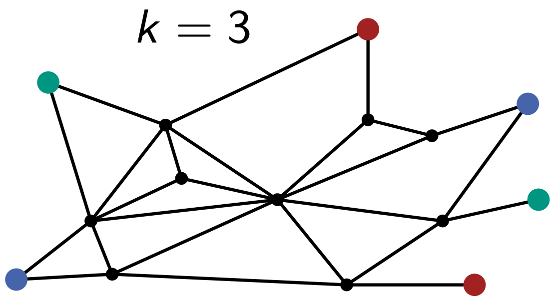

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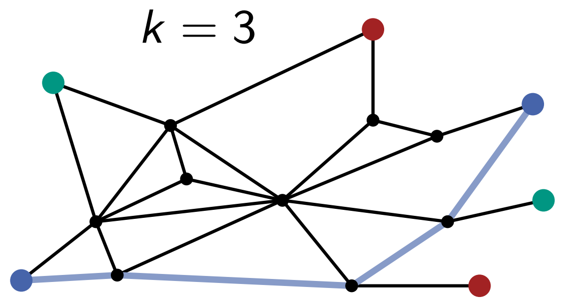

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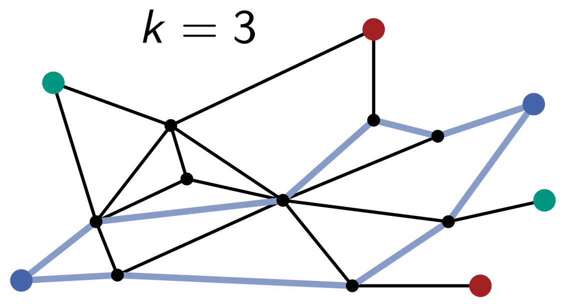

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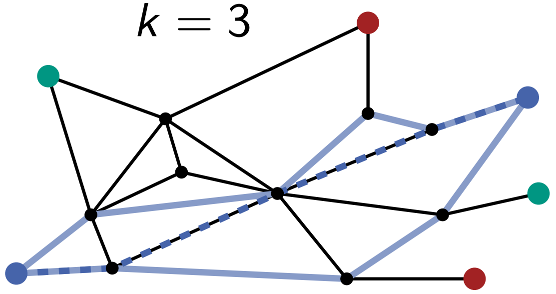

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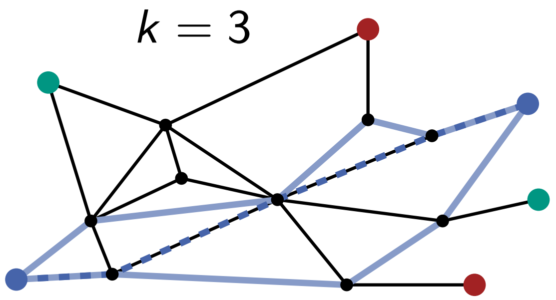

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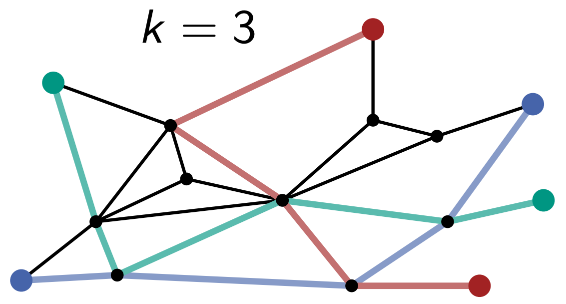

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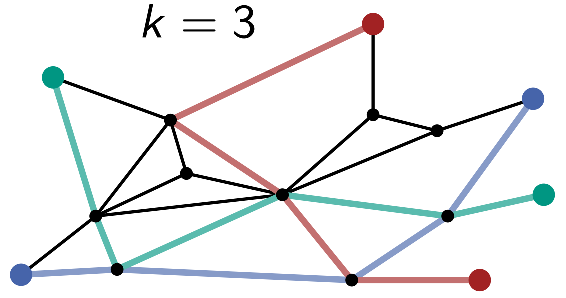

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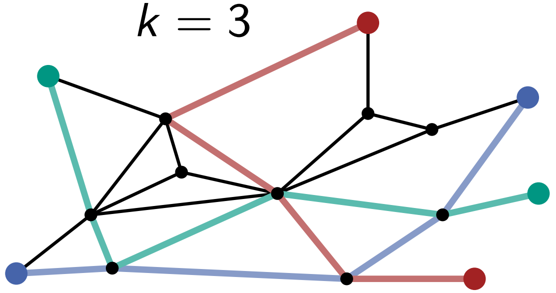

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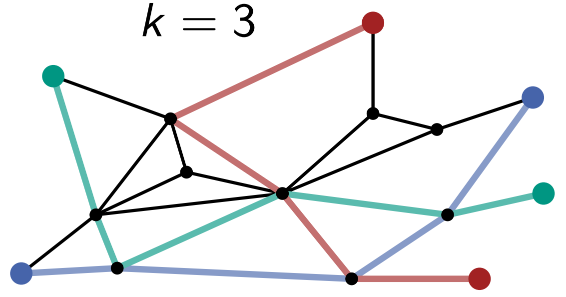

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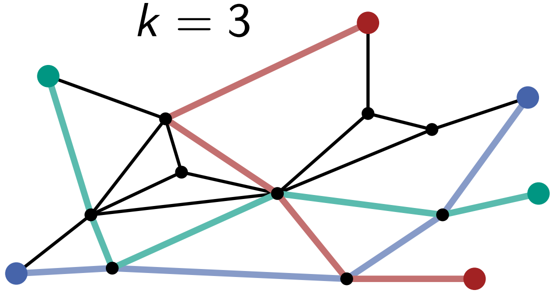

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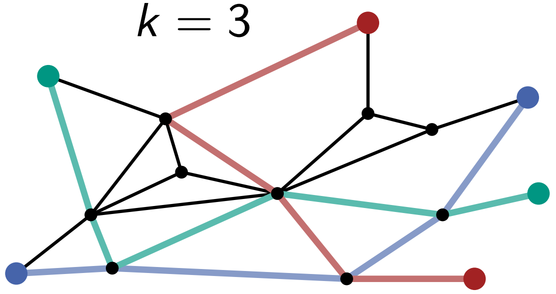

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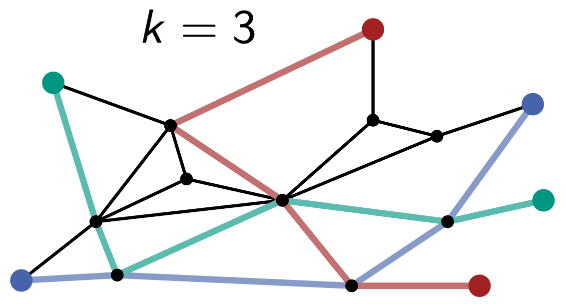

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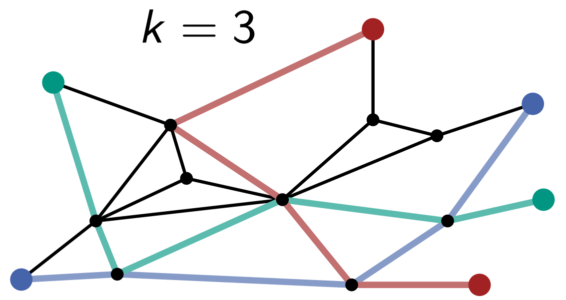

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$\Pr[E_{ij}] \leq \frac{\ell}{m}$ Whatever the choice of P_i, there are $\geq m$ choices for P_j, at most ℓ of which share an edge with P_i

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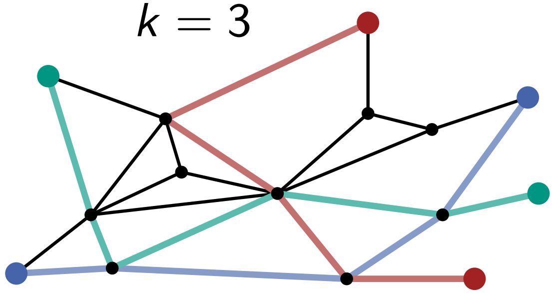

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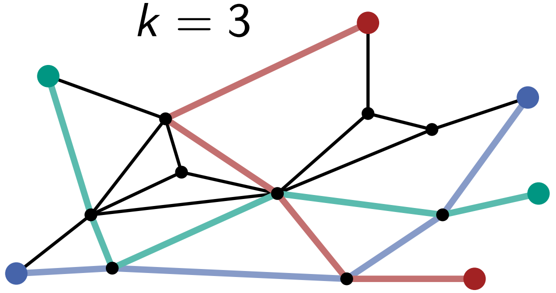

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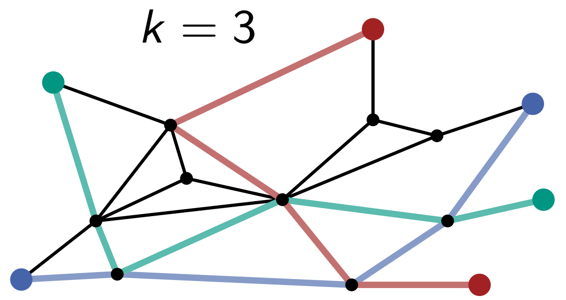

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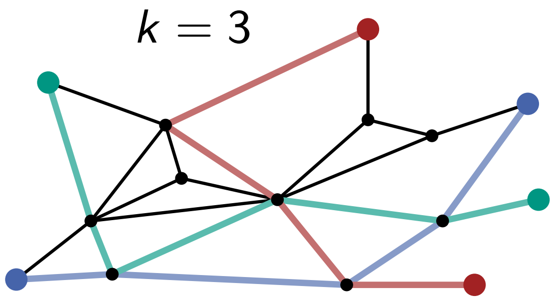

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$D_{ij} = \{E_{st} \mid \{s,t\} \cap \{i,j\} \neq \emptyset\}$

Removing D_{ij} is discarding all events that could tell us something about whether P_i and P_j can intersect

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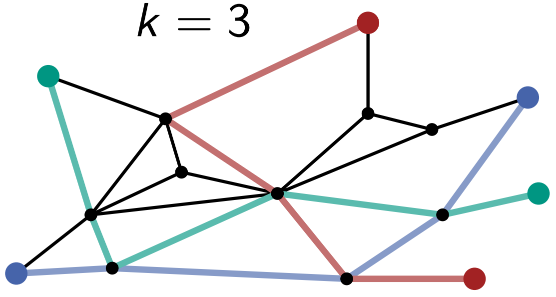

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$$|D_{ij}| = (k-1) + (k-1) - 1$$

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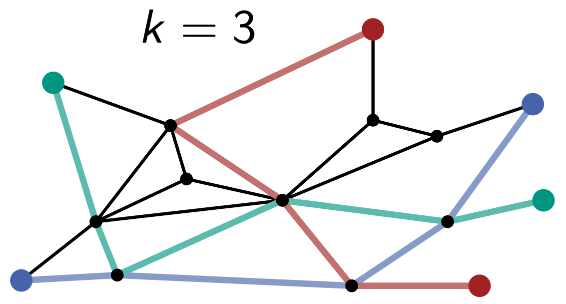

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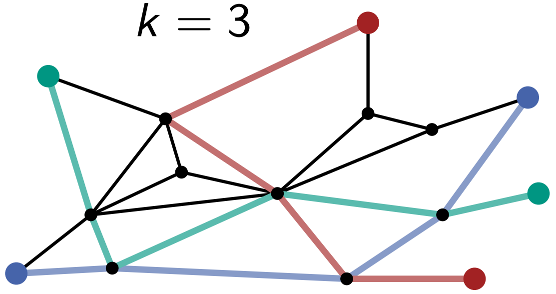

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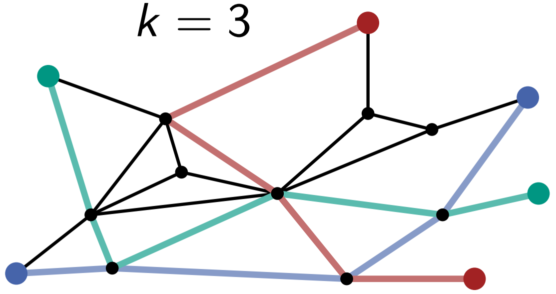
$k = 3$



Theorem: Let $m = \min_{i \in [k]}\{|S_i|\}$. Then, there exists a valid choice if any path in S_i shares edges with at most $\ell \leq m/(8k)$ paths in S_j for $i \neq j$.

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Random Process: Ind., unif. at random choose P_i from S_i

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$$\Pr[\textstyle\bigcap_{i<j} \neg E_{ij}] \overset{?}{>} 0$$

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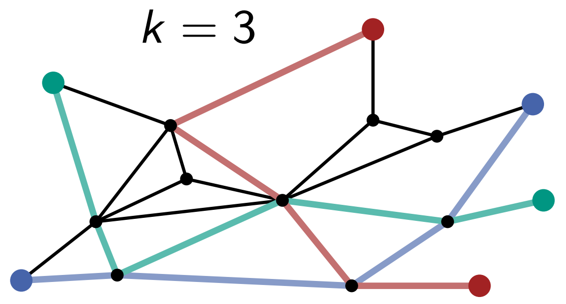

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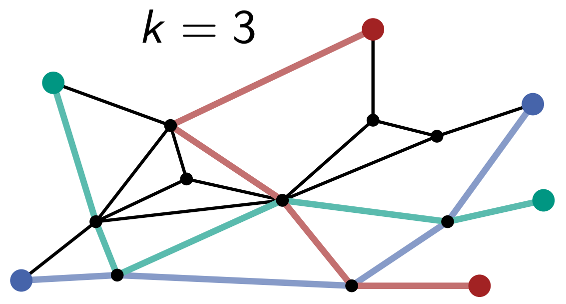

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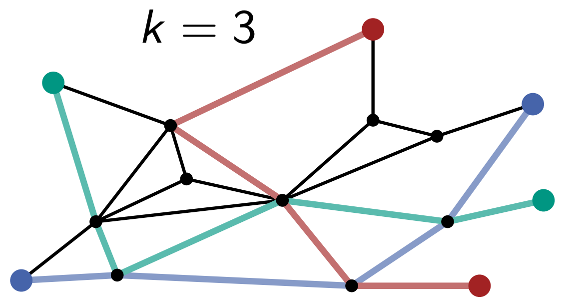

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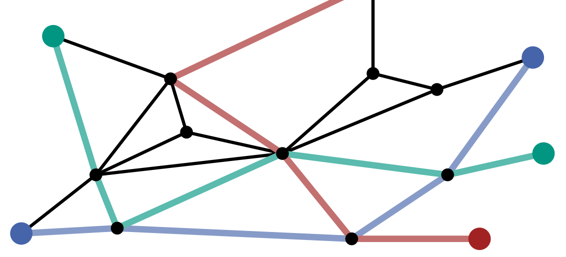

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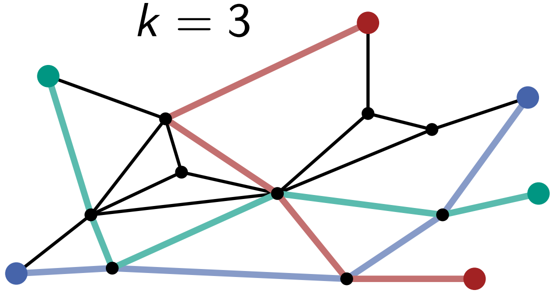

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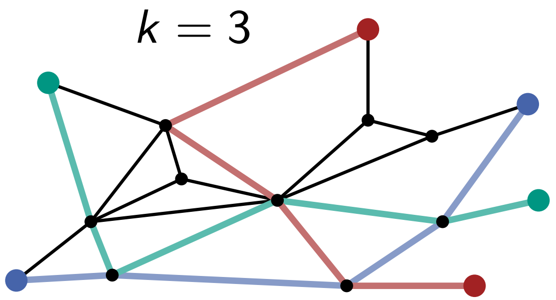

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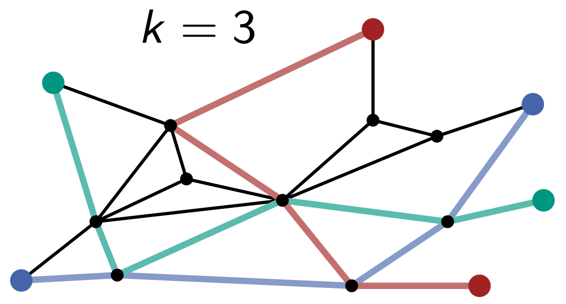

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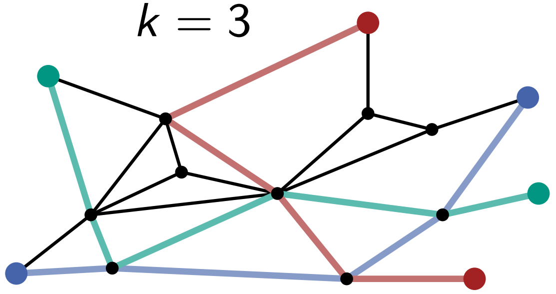

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Conclusion

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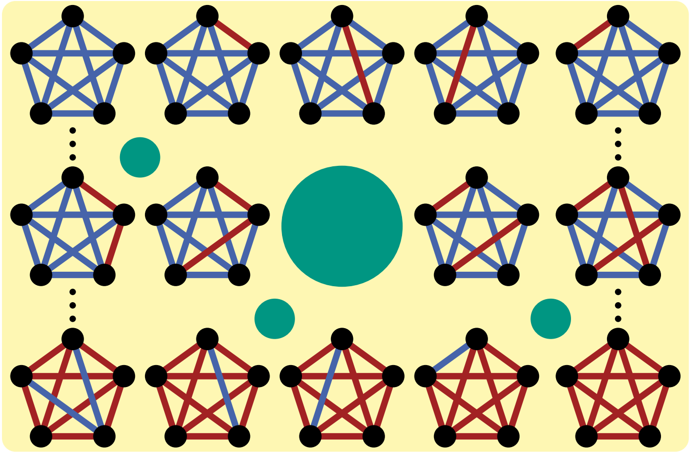
Probabilistic Method

- Show that something exists *deterministically*, by showing that it occurs with positive probability from a random process
- Reasoning: At least one object in the sample space has the desired property



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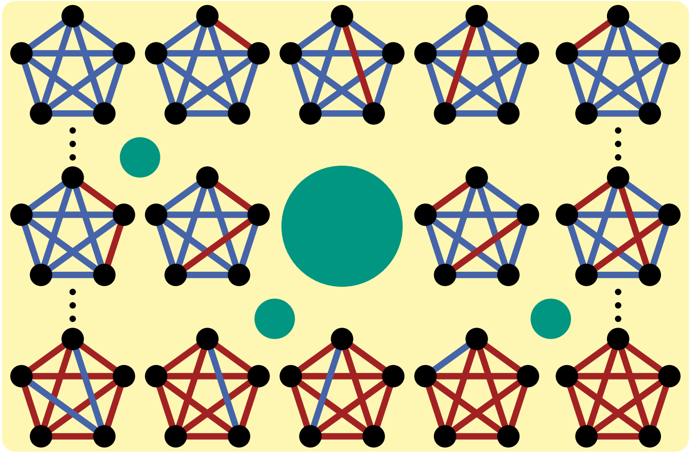

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- Useful tool when applying probabilistic method
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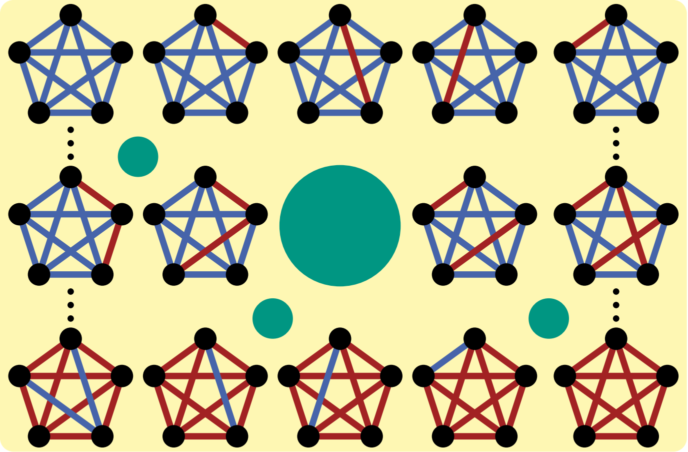

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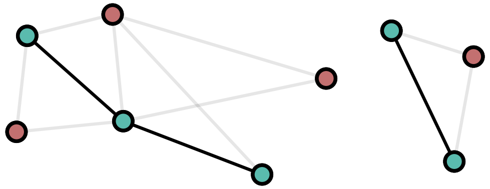
Sample via Modification

- Example Vertex Cover: remove vertices/edges at random



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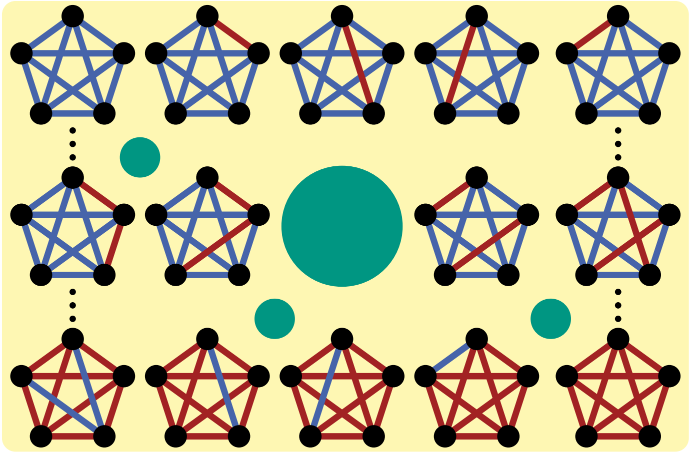

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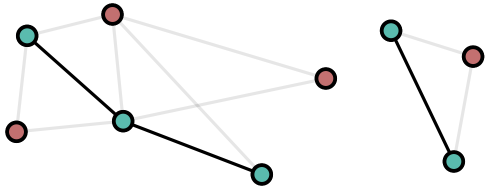

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Lovász Local Lemma

- Show that something exists by showing that all events that prevent its existence do not occur, with positive probability
- Lemma works as long as there are not too many dependencies