Probability & Computing

Overview & The Power of Randomness

Why is randomness useful in computation?

- Randomness facilitates the development of algorithms and data structures.

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- But we have to pay for that . . .
  - Maybe we only *expect* the approach to be fast
  - Maybe we only *expect* the approach to work correctly
- Goal: develop methods that fail only rarely

Why is randomness useful in computation?

- Useful when bridging the theory-practice gap regarding the performance of an appraoch

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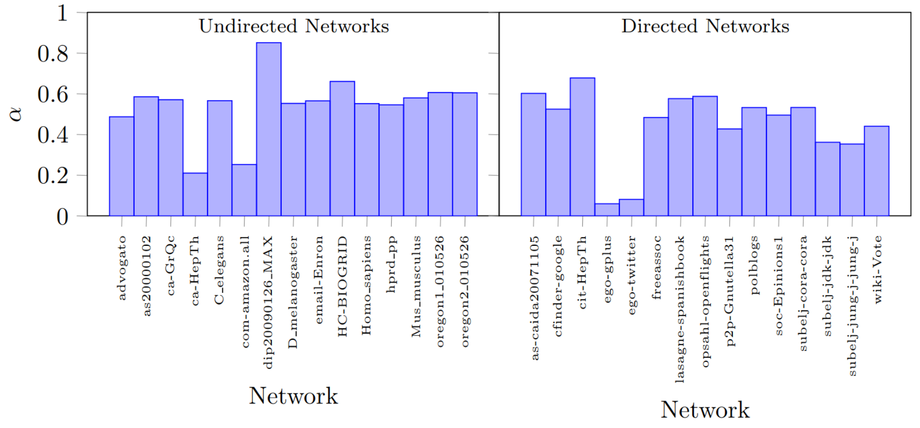

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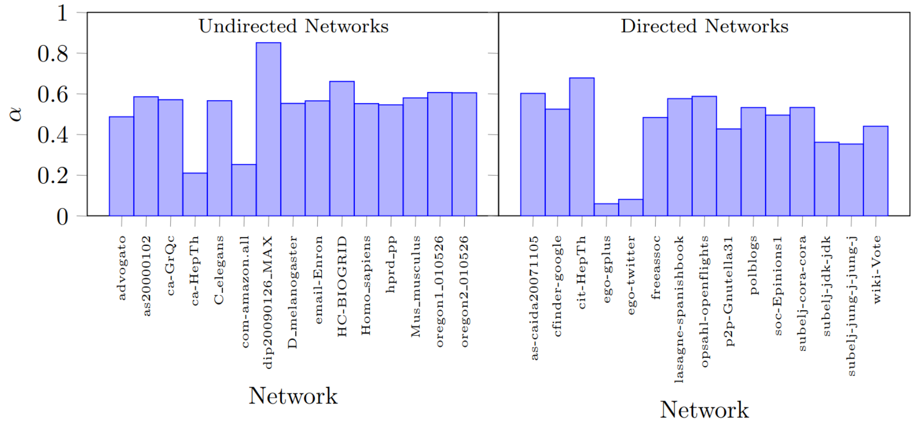

Average-Case Analysis

- Distinguish practical instances from the worst case

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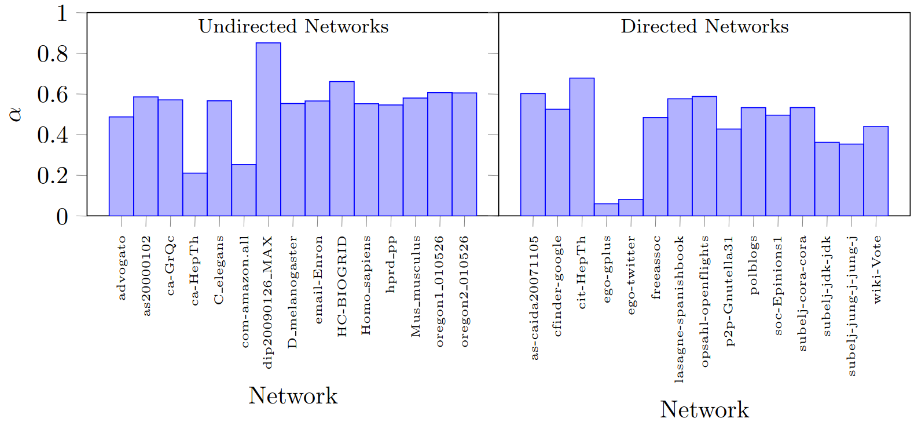

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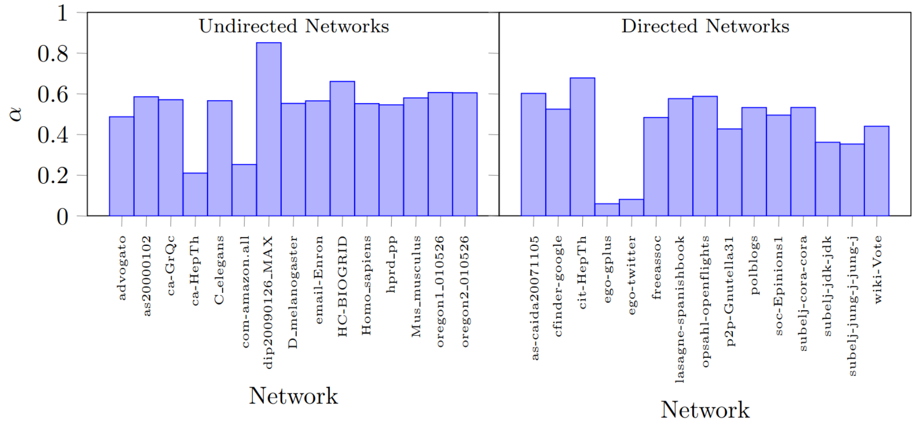

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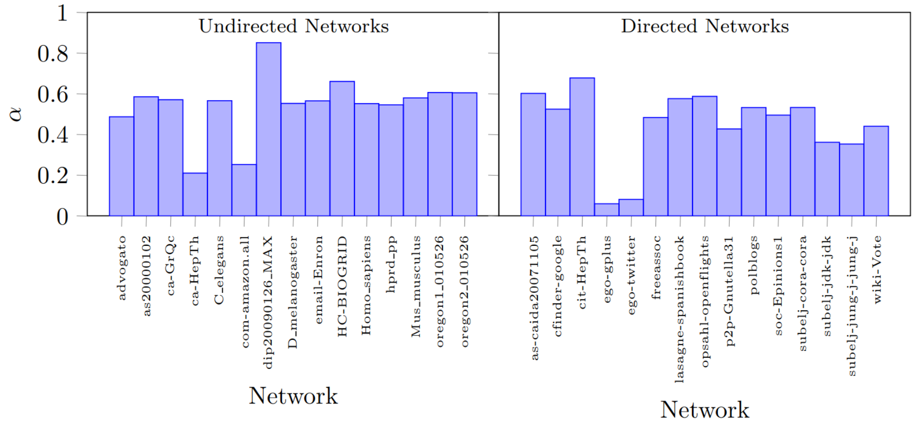

Average-Case Analysis

- Distinguish practical instances from the worst case
- Define probabilistic distributions (over possible inputs) that favor realistic instances
- Analyze performance assuming input is drawn from the distribution
- Expect good performance when hard instances are sufficiently unlikely

Overview

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Randomized Algorithms & Data Structures

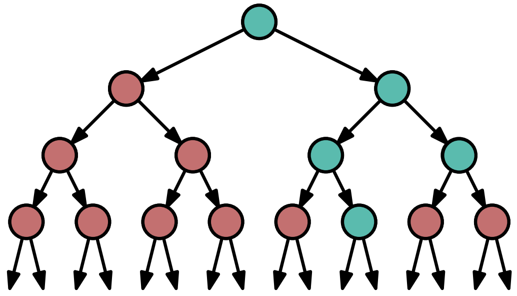
- Probability Amplification
- Streaming / Online-algorithms
- Hashing

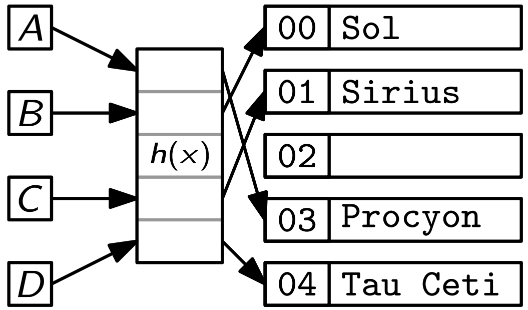
Average-Case Analysis

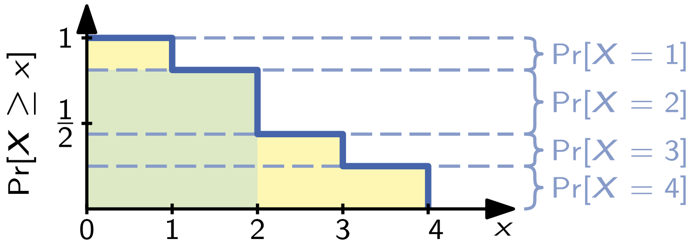
- Random Graphs
- Algorithm Analysis

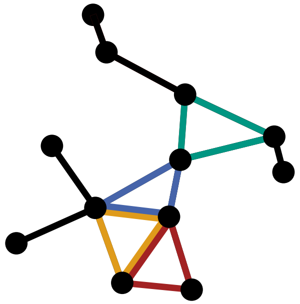
Toolbox

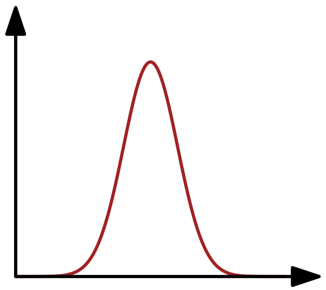
- Probabilistic Method
- Yao’s Principle
- Coupling
- Dealing with stochastic dependencies
- Concentration bounds

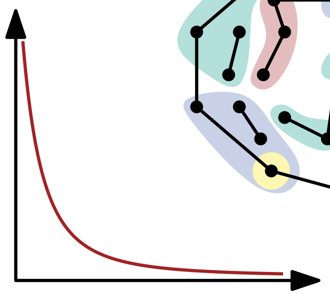


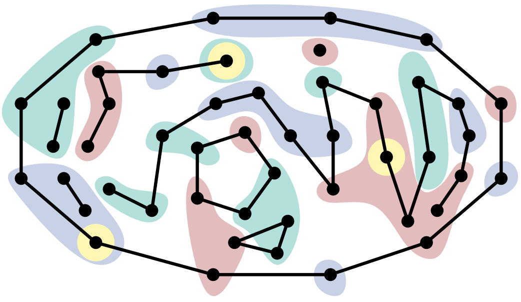












Organization

Team

Max
Lecture
(first part)

Stefan
Lecture
(second part)

Hans-Peter
Exercise

Thursday 11:30

Tuesday 8:00 (every other week)

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Except this week!

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Questions? Ilias, Discord, Matrix?

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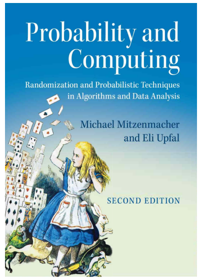
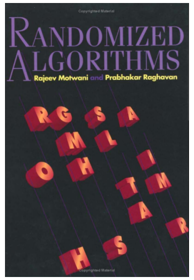
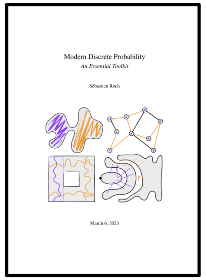
- Slides
- Previous script
- *Probability and Computing*
- *Randomized Algorithms*
- *Modern Discrete Probability*







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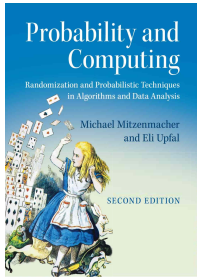

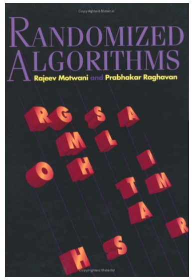

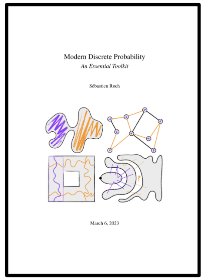

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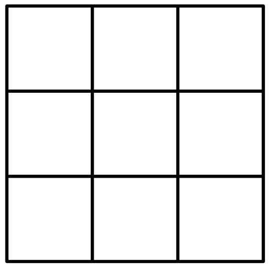
Exam
- Oral
- Requirment: sheets handed in regularly

Power of Randomness: Let's Play a Game

Tic-Tac-Toe

- Players take turns placing ○ and ✕ in 3×3 grid
- First to get three in a line wins

Power of Randomness: Let’s Play a Game



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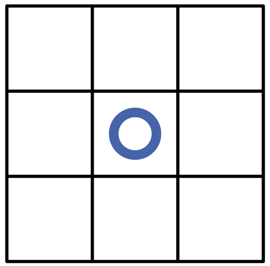

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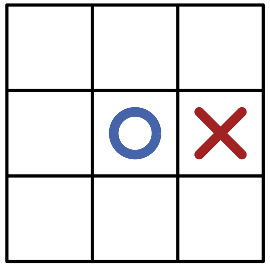

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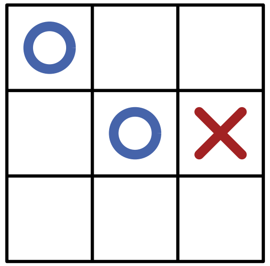

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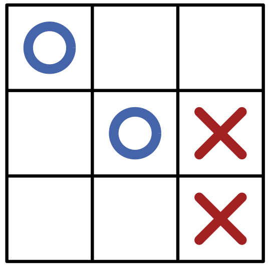

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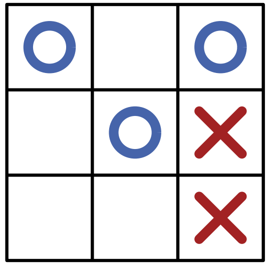

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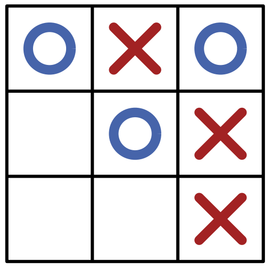

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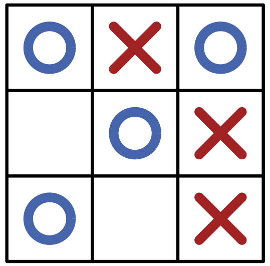

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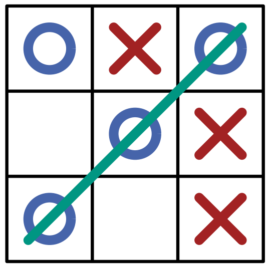

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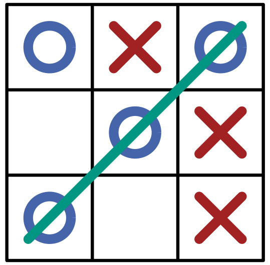

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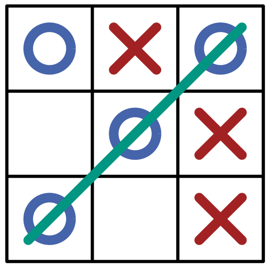

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Tree of Moves

- Each node is a board configuration
- A parent-child relation represents a valid move

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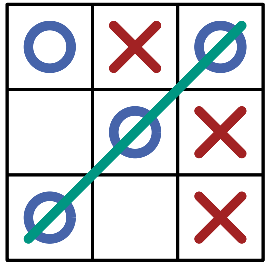

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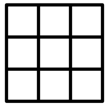

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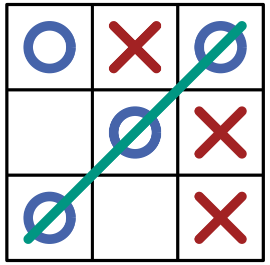

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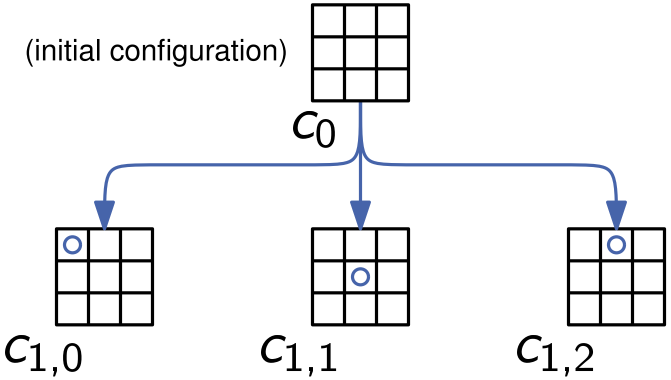

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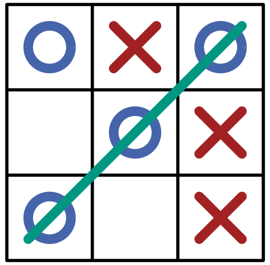

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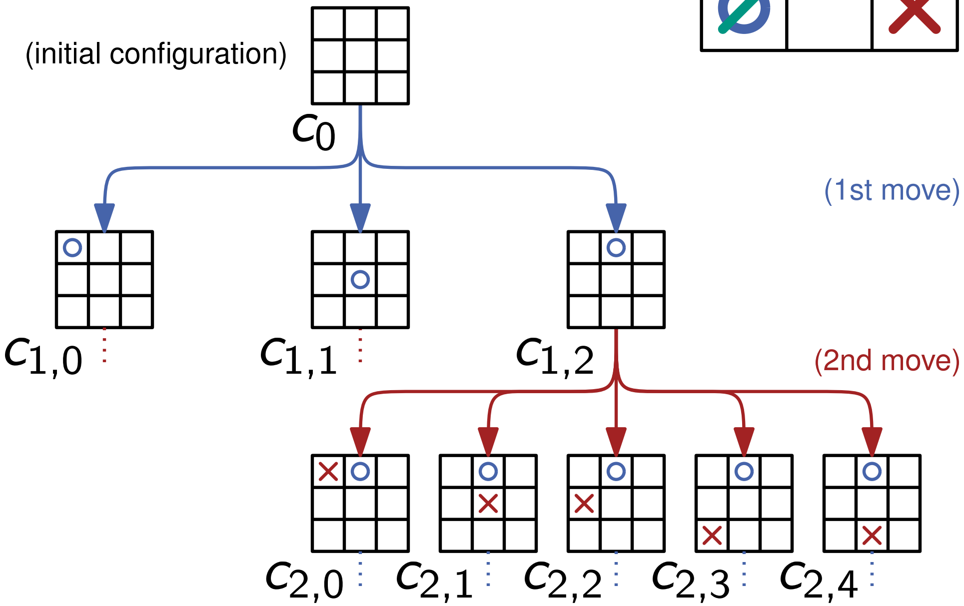

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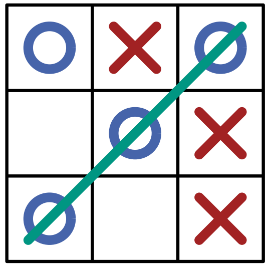

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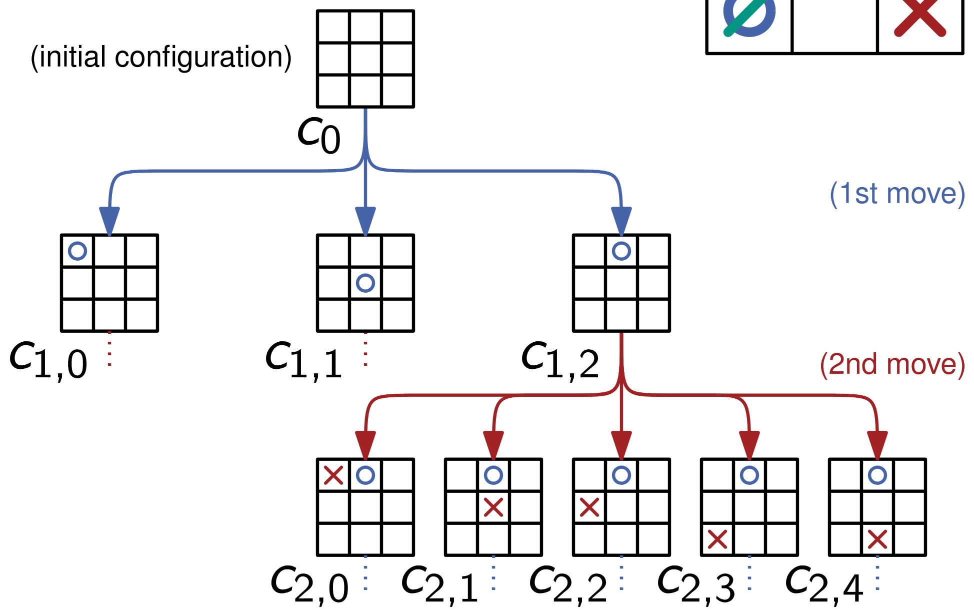
- Label a config 1 if Player 2 can win, 0 o.w.

What label do we put on the root?



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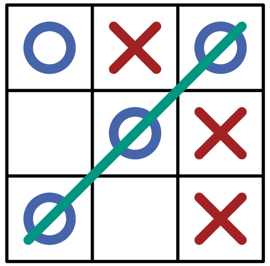

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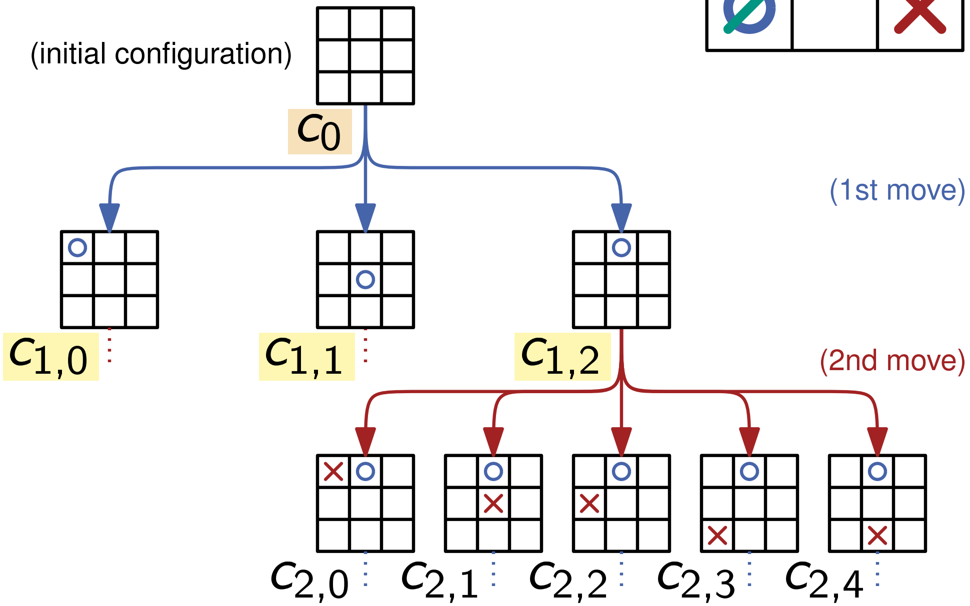
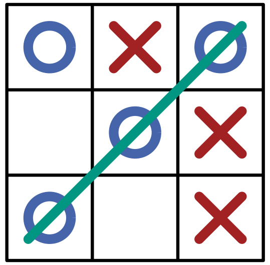


Power of Randomness: Let's Play a Game



Tic-Tac-Toe

- Players take turns placing O and X in 3×3 grid
- First to get three in a line wins

Can Player 2 win the game?

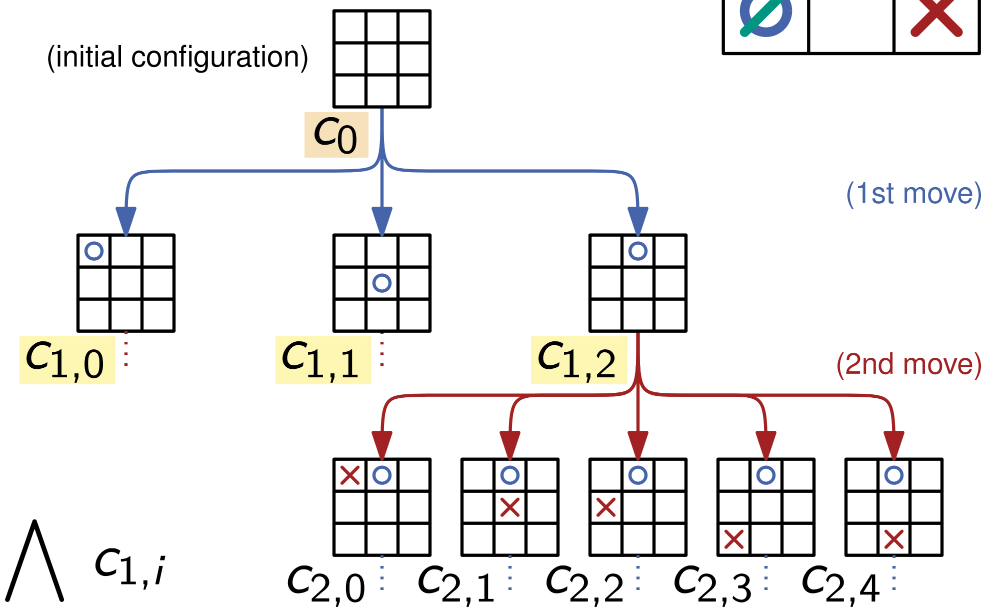
Tree of Moves

- Each node is a board configuration
- A parent-child relation represents a valid move
- Label a config 1 if Player 2 can win, 0 o.w.

What label do we put on the root?

- $c_0 = 1$ if there exists *no* i such that $c_{1,i} = 0$
  or equivalently, if for *all* i we have $c_{1,i} = 1$

$$c_0 = \bigwedge_{i \in [2]} c_{1,i}$$



Power of Randomness: Let's Play a Game

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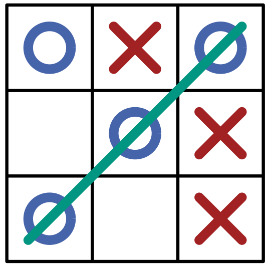

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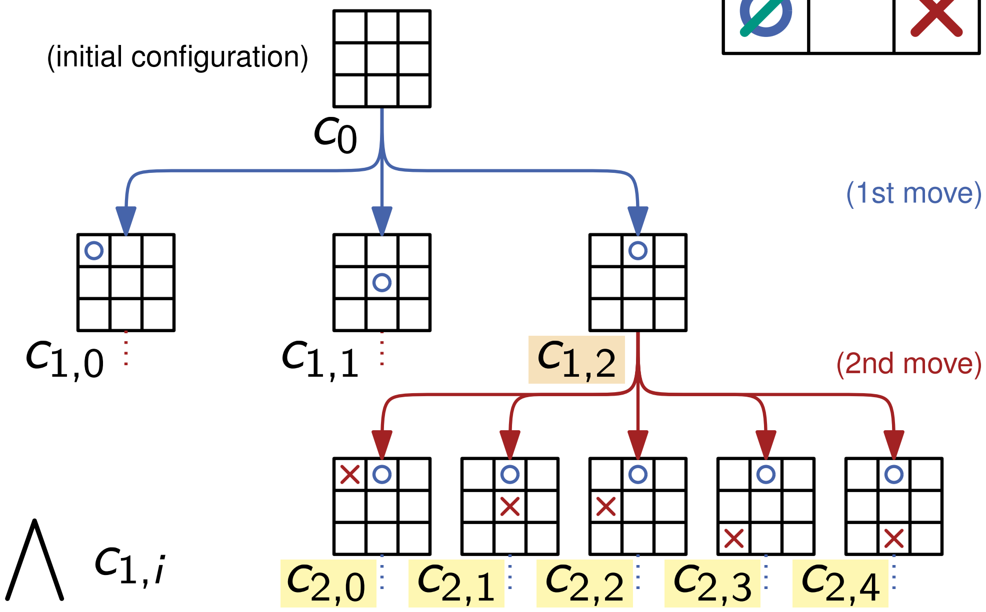

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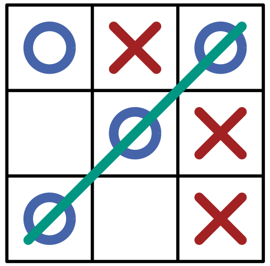

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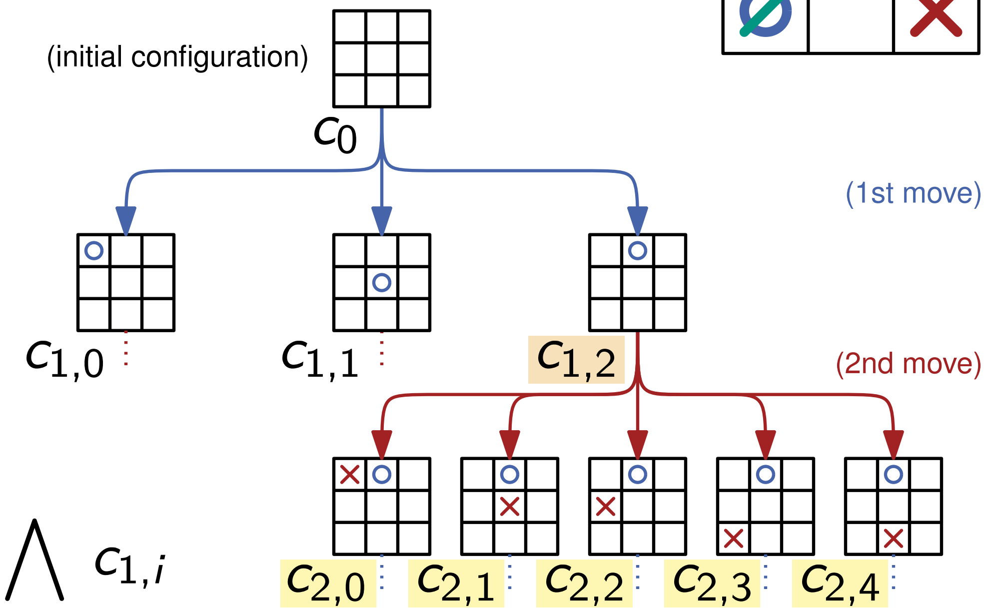


AND/OR-Trees

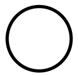
Structure

- Node types: $\wedge$-nodes, $\vee$-nodes, and leaves

AND/OR-Trees

Structure

- Node types: $\wedge$-nodes, $\vee$-nodes, and leaves
- The root is a leaf



AND/OR-Trees

Structure

- Node types: $\wedge$-nodes, $\vee$-nodes, and leaves
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AND/OR-Trees

Structure

- Node types: $\wedge$-nodes, $\vee$-nodes, and leaves
- The root is a leaf or an $\wedge$-node
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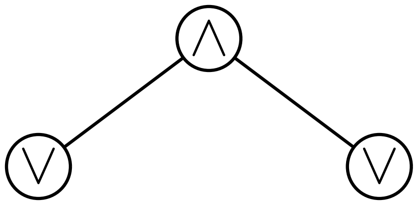

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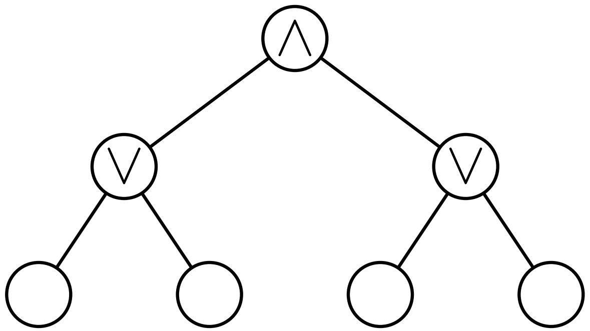

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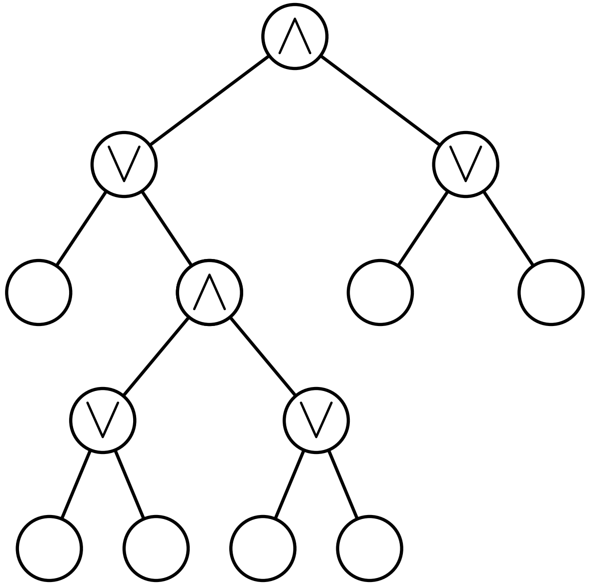

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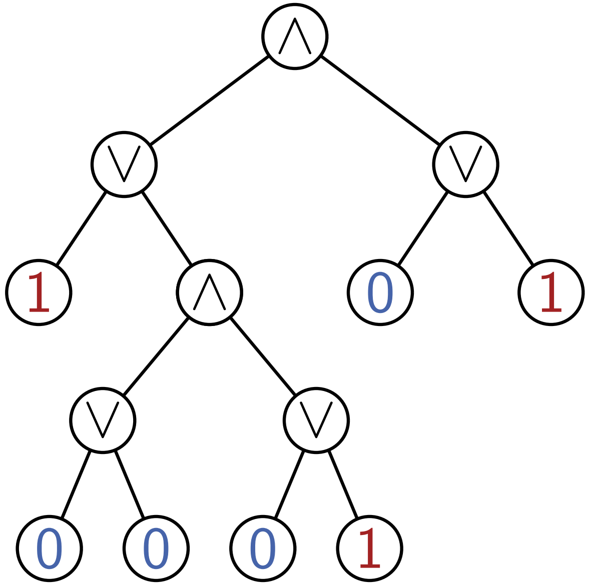
- Leaves contain boolean values



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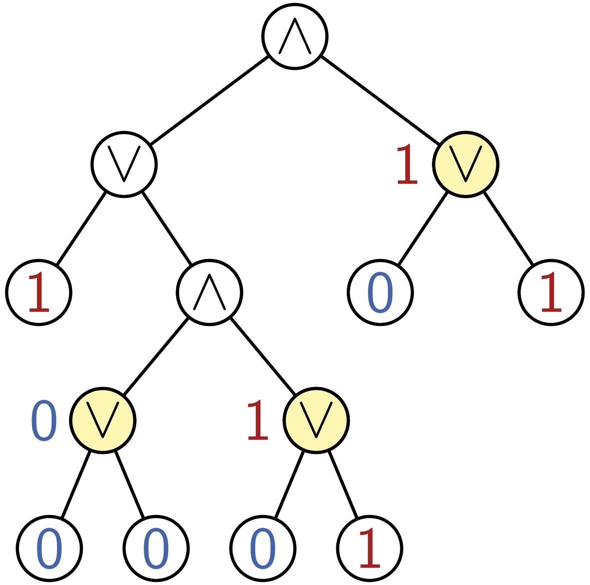

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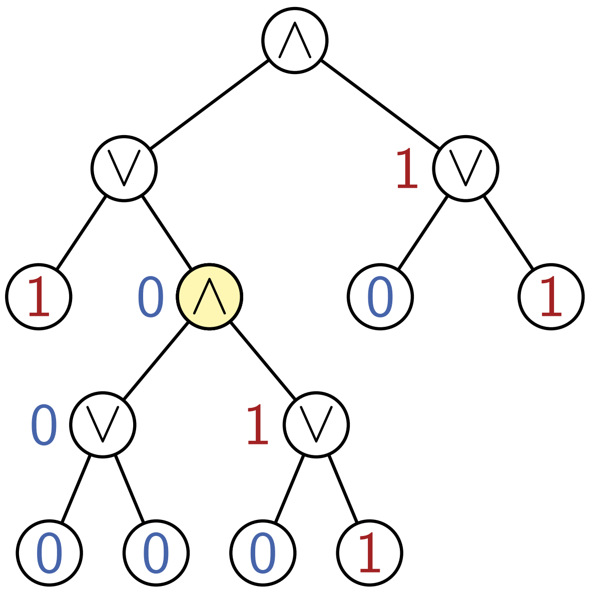

AND/OR-Trees

Structure

- Node types: ∧-nodes, ∨-nodes, and leaves
- The root is a leaf or an ∧-node
- ∧-nodes have only ∨-nodes as children
- ∨-nodes have only AND/OR-trees as children

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- Leaves contain boolean values
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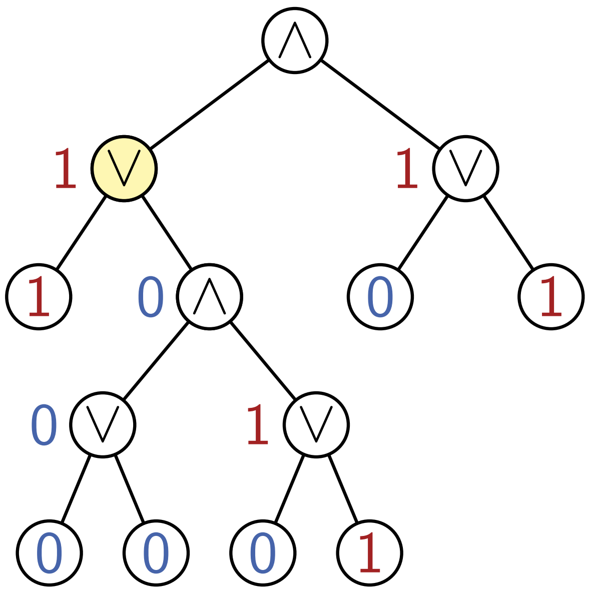

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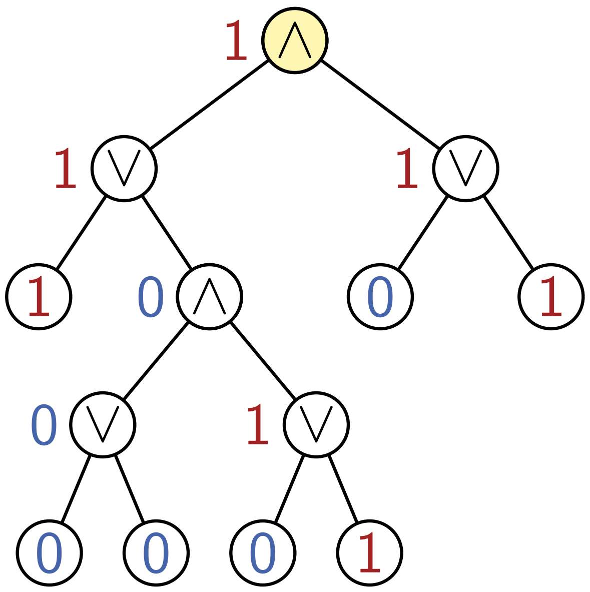

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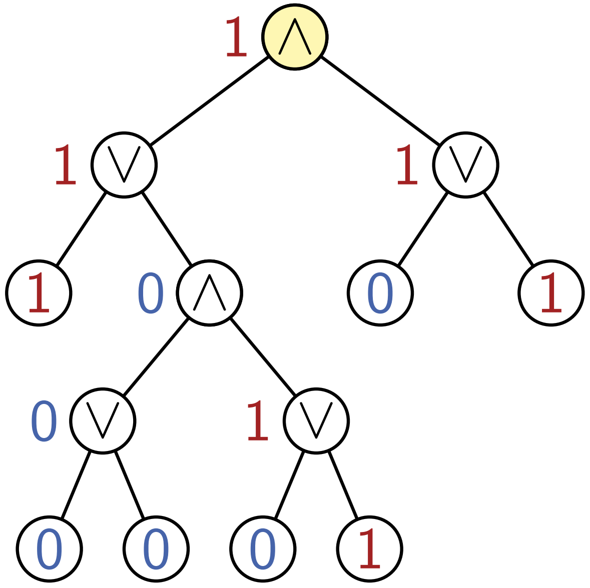

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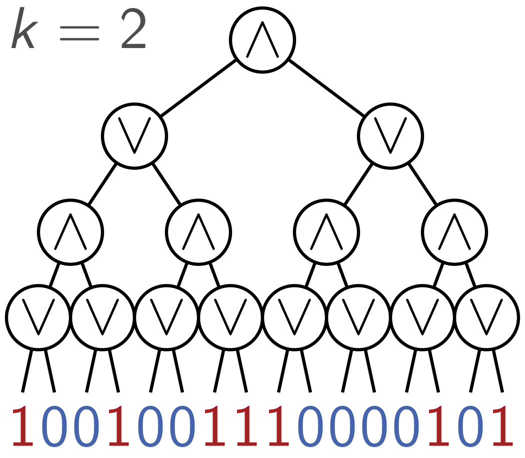
Example Complexities

- Tic-Tac-Toe: 31896 (non-symmetric) games (leaves)
- Checkers: approx. 10^{40} leaves
- Chess: approx. 10^{123} leaves
- Go (19×19): approx. 10^{360} leaves

Deterministic Evaluation

Simplifying Assumption

- Each inner node has two children
- All leaves have the same depth $2k$



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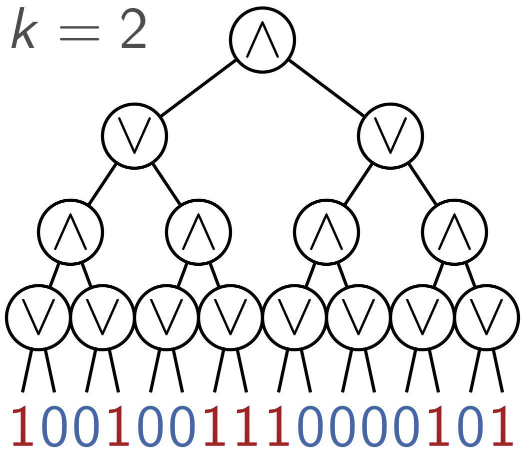
  $\Rightarrow$ A bit-string of length $n = 4^k$ encodes the input completely



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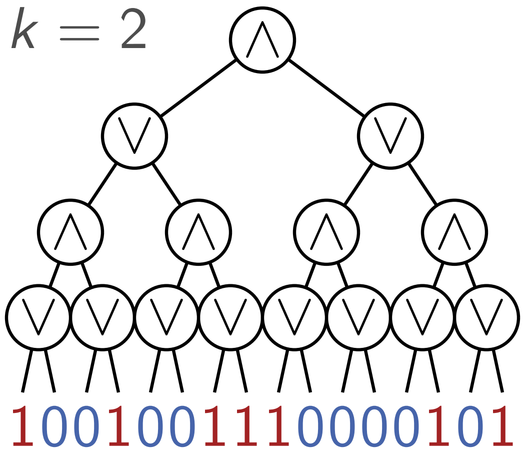

A Simple Deterministic Algorithm

- Compute all nodes bottom up

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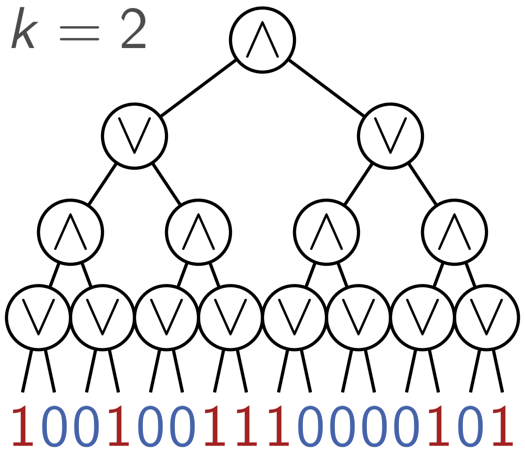

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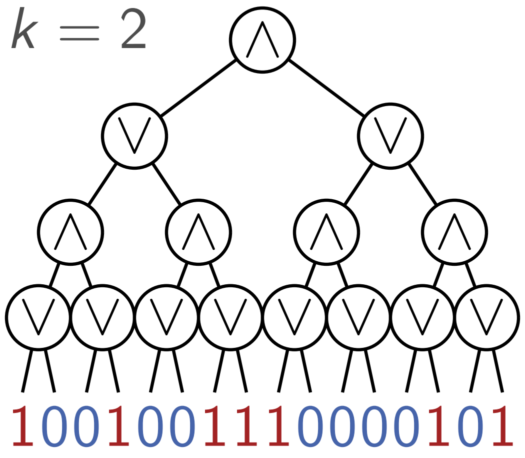

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$$\sum_{\ell=0}^{2k} 2^\ell = 2^{2k+1} - 1 = \Theta(4^k) = \Theta(n)$$

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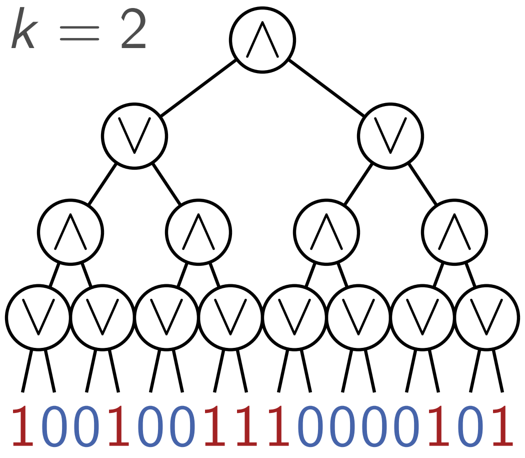

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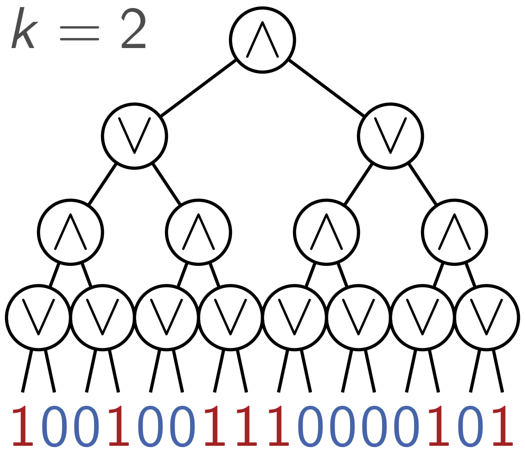

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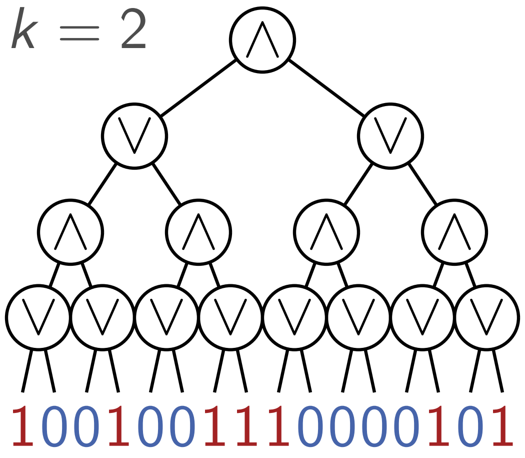

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Karlsruhe Institute of Technology

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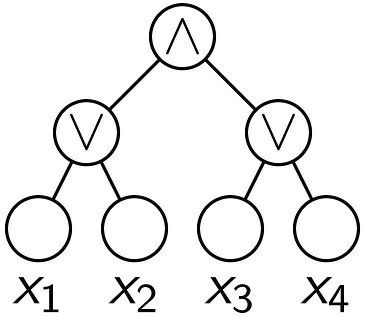

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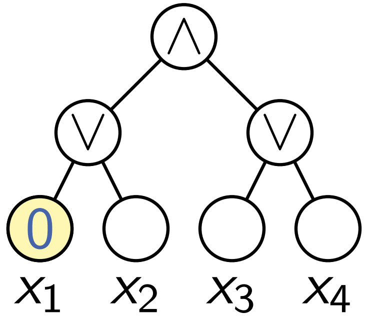

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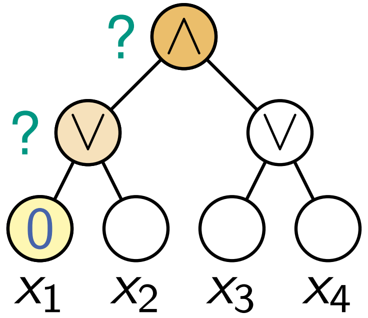

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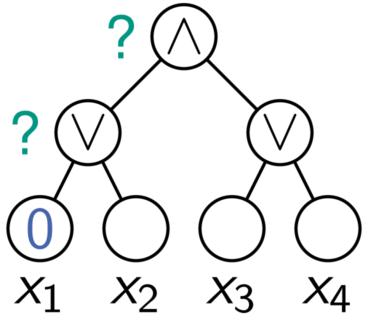

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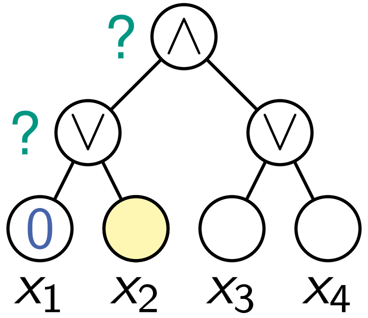

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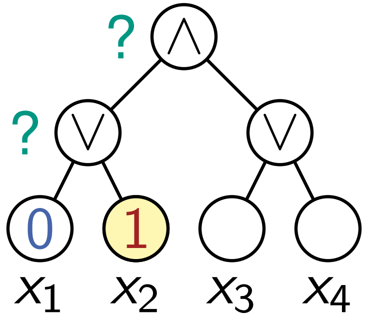

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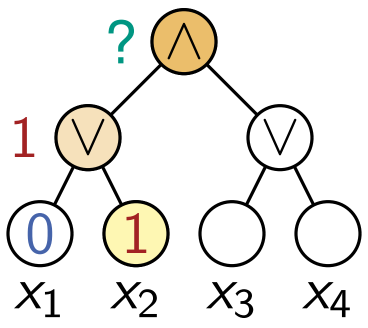

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- Compute all nodes bottom up
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$$\sum_{\ell=0}^{2k} 2^\ell = 2^{2k+1} - 1 = \Theta(4^k) = \Theta(n)$$

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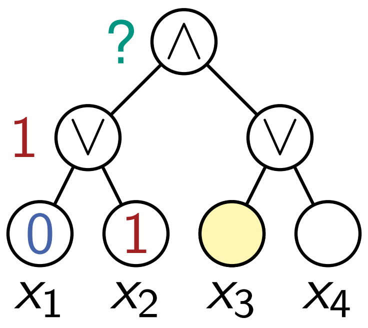

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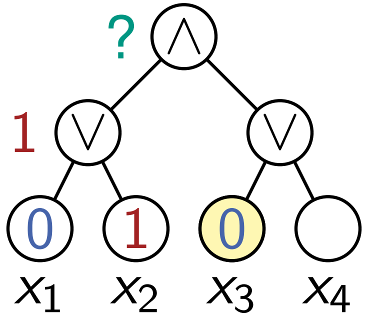

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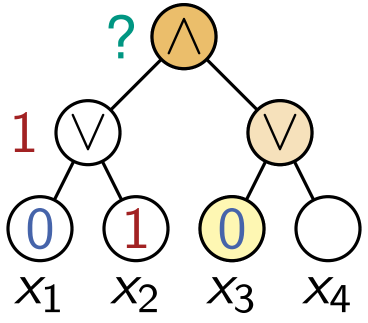

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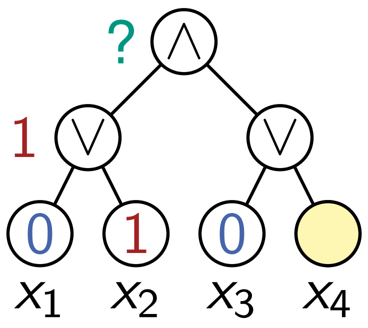

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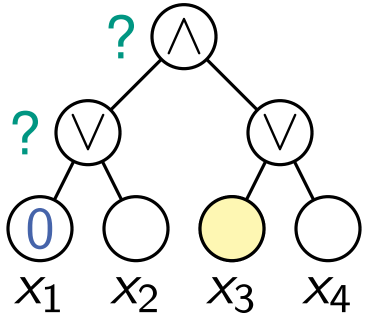

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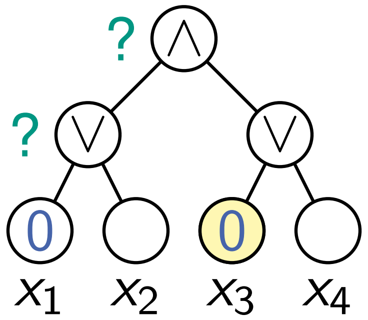

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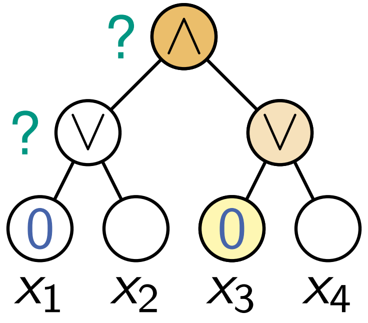

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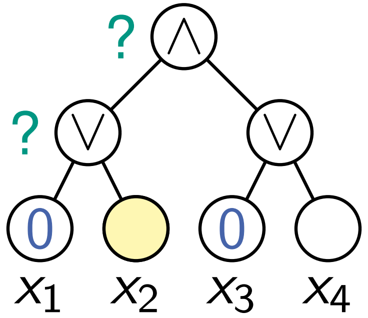

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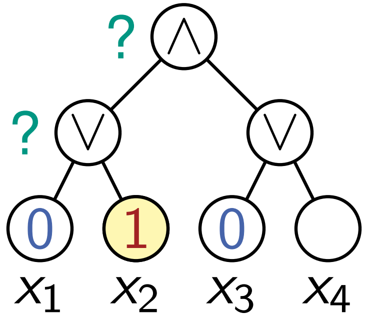

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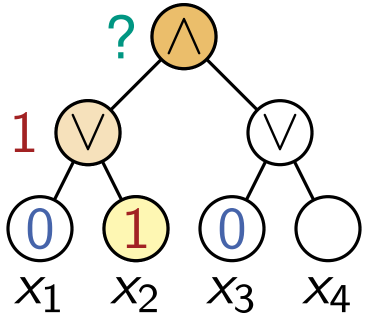

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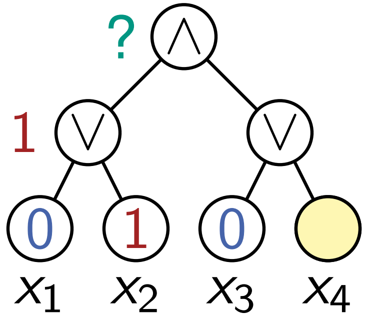

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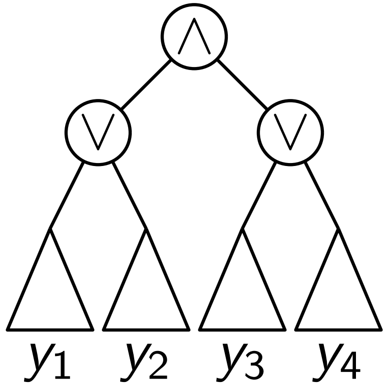

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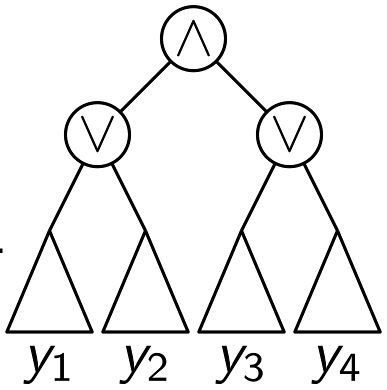

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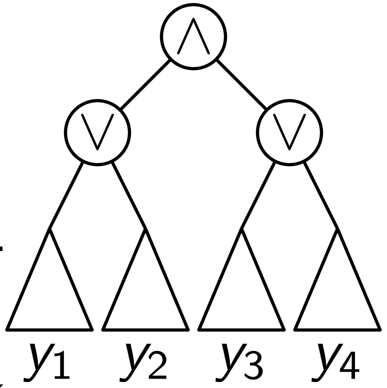

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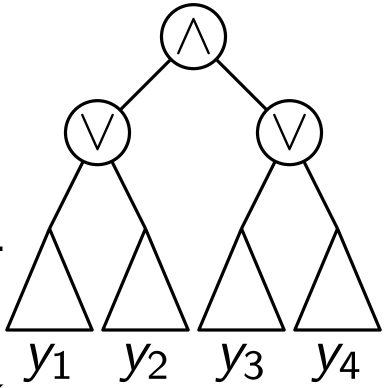

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Randomized Evaluation

Karlsruhe Institute of Technology

Idea

- We can evaluate an $\wedge$-node to 0 if we find *one* 0-child
- We can evaluate an $\vee$-node to 1 if we find *one* 1-child

} while ignoring the other child!

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Algorithm

evalAndNode(v)
- **if** v is leaf **then**
  - **return value**(v)
- $c :=$ **uniformSample**(v.**children**)
- **if evalOrNode**(c) $= 0$ **then**
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- $c' :=$ the other child
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Here each of the two children is selected with equal probability $1/2$.

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- Execute as evalAndNode(r) for root-node r

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Karlsruhe Institute of Technology

Idea

- We can evaluate an $\wedge$-node to 0 if we find *one* 0-child
- We can evaluate an $\vee$-node to 1 if we find *one* 1-child

} while ignoring the other child!

Algorithm

```
evalAndNode(v)
  if v is leaf then
    return value(v)
  c := uniformSample(v.children)
  if evalOrNode(c) = 0 then
    return 0
  c' := the other child
  return evalOrNode(c')
```

Here each of the two children is selected with equal probability $1/2$.

```
evalOrNode(v)
  c := uniformSample(v.children)
  if evalAndNode(c) = 1 then
    return 1
  c' := the other child
  return evalAndNode(c')
```

$\vee$-nodes are not leaves in our setting

- Execute as evalAndNode(r) for root-node r

How long does that take?

Randomized Evaluation – Running Time

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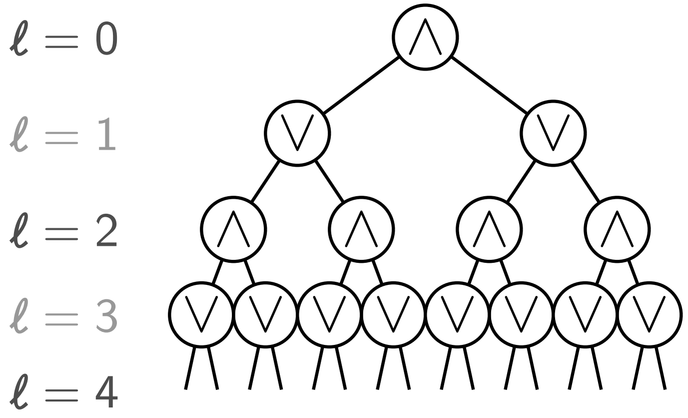
(that the number X of visited leaves at depth $2k$ is $\leq 3^k = 3^{\log_4(n)} = n^{\log_4(3)}$ in expectation)

- Expected number of nodes evaluated on *even* layer $\ell = 2i$ is at most 3^i



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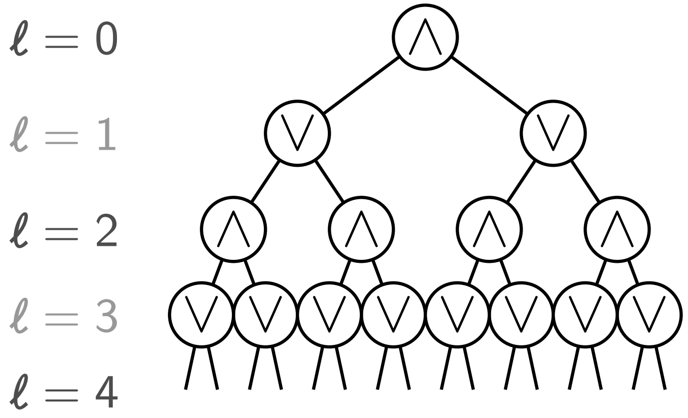

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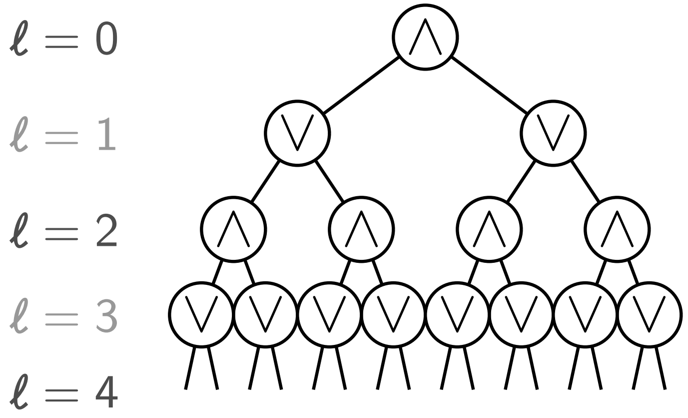

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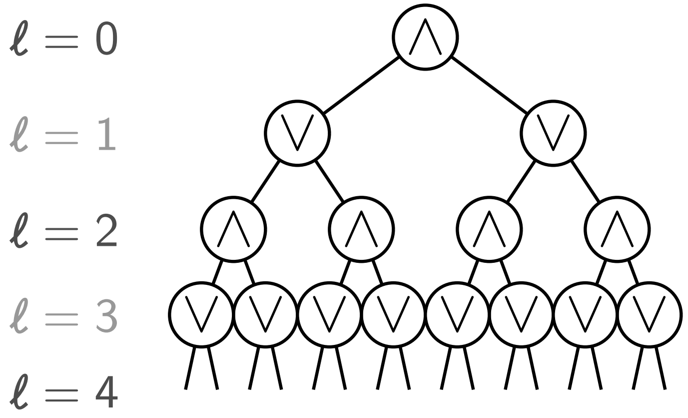

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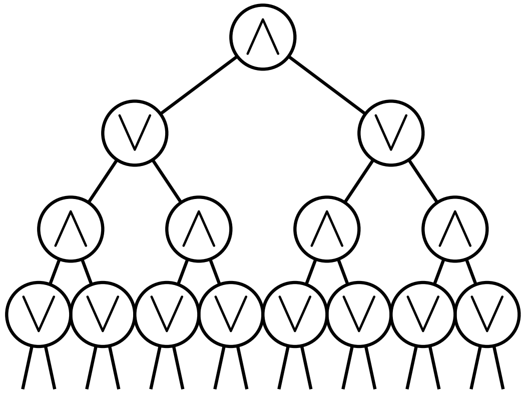
$\ell = 0$

$\ell = 1$

$\ell = 2$

$\ell = 3$

$\ell = 4$



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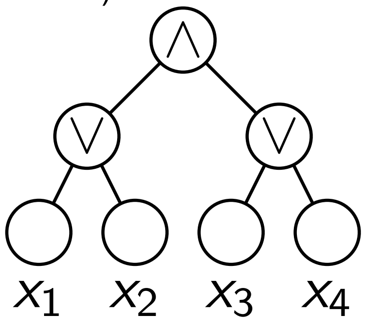

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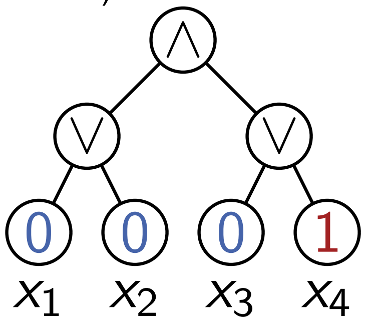

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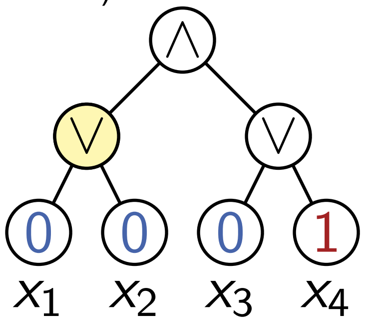

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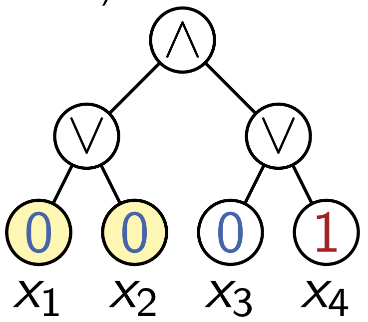

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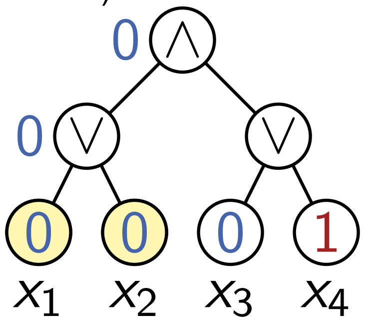

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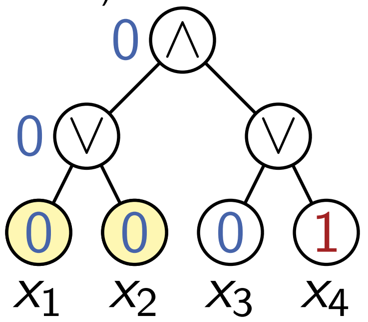

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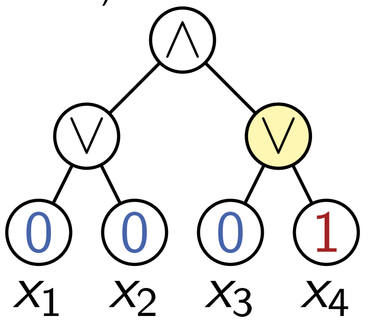

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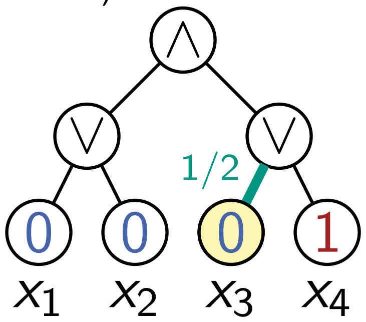
- Let X_R be number of leaves visited when going right first
  - $\Pr[\text{RE} \to x_3] = 1/2$



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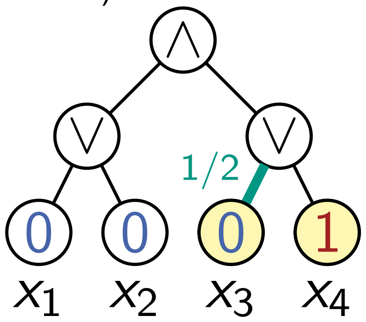

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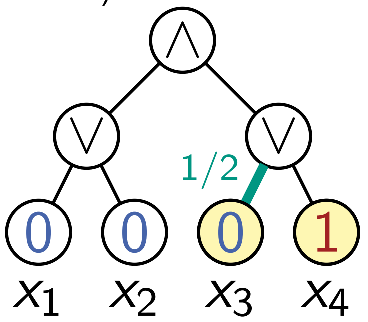

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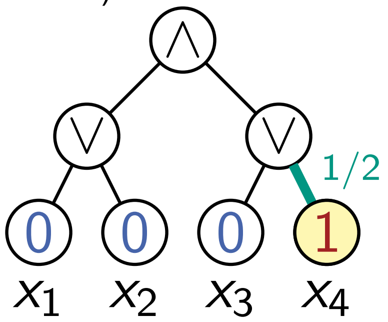

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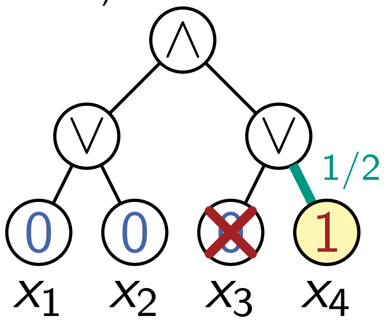

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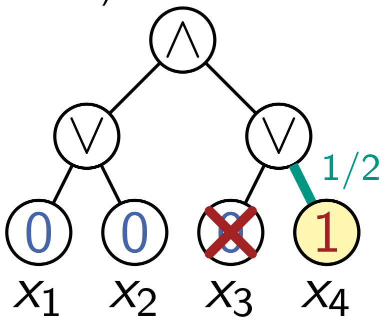

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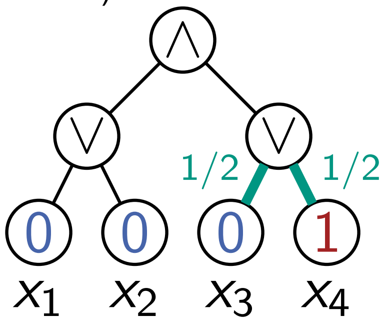

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$\mathbb{E}[X_L] = 2$

$\mathbb{E}[X_R] = 2 + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{7}{2}$

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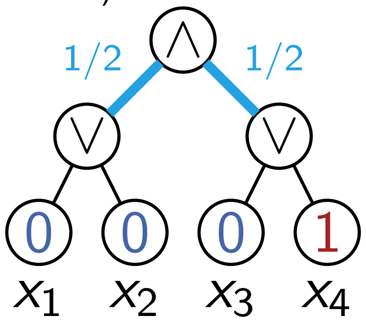
Base: $k = 1$



- Case analysis over all bit-strings x_1, x_2, x_3, x_4, example 0001
- Let X_L be number of leaves visited when going left first
  - Independent of leaf choice, need to look at other too: $X_L = 2$
  - When left $\vee$-node is checked, root value is determined
- Let X_R be number of leaves visited when going right first
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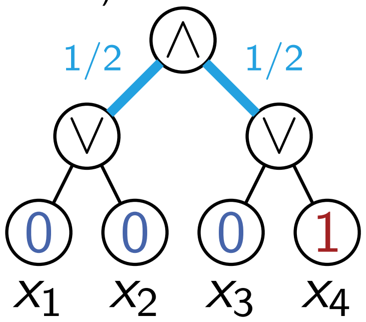

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$\mathbb{E}[X] = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{7}{2} = \frac{11}{4} \leq 3$ ✓

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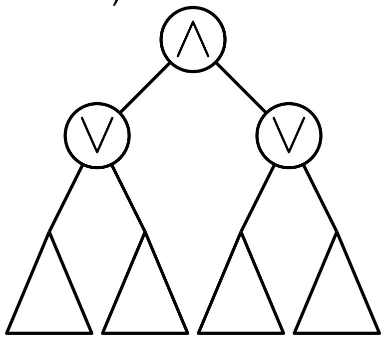

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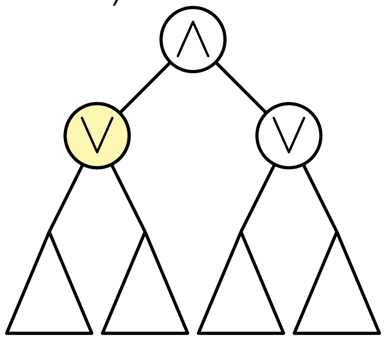

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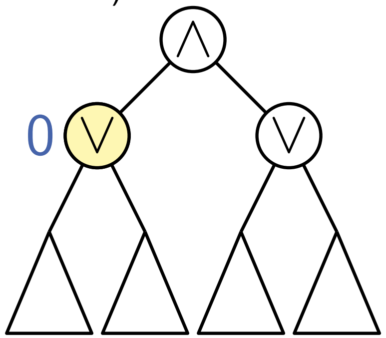

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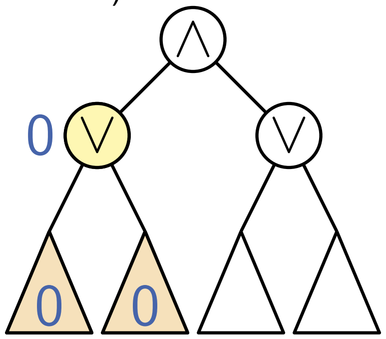

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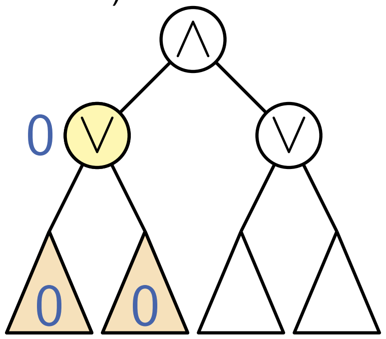

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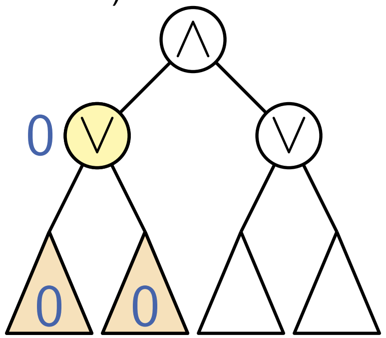

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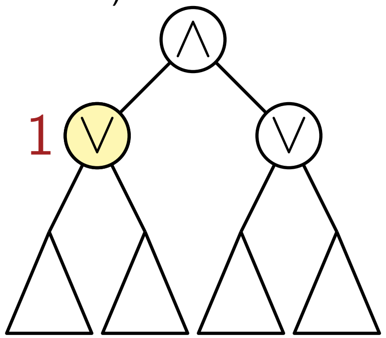

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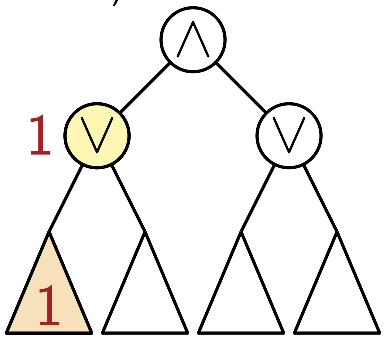

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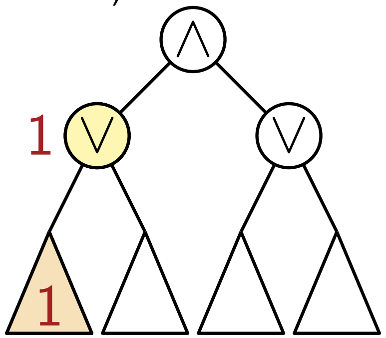

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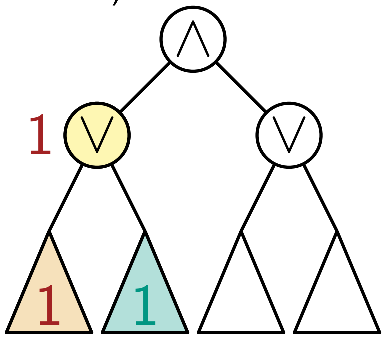

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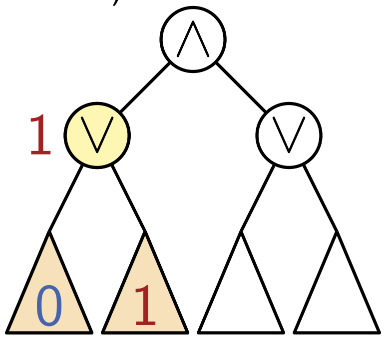

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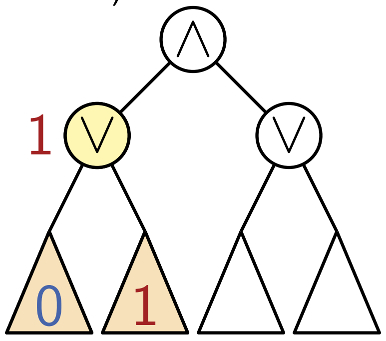

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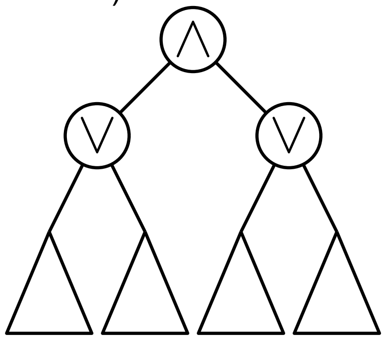

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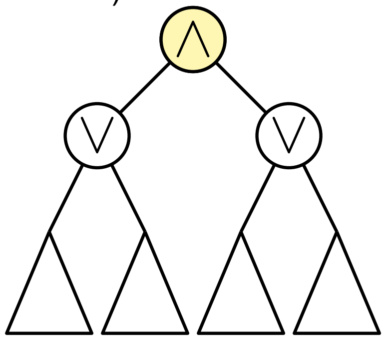

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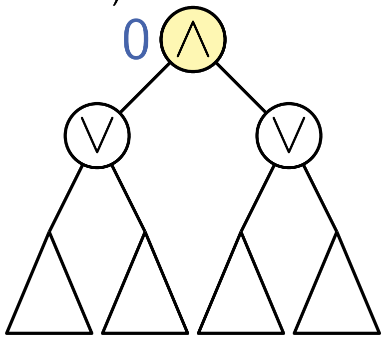

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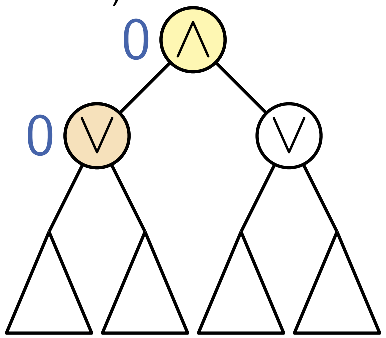

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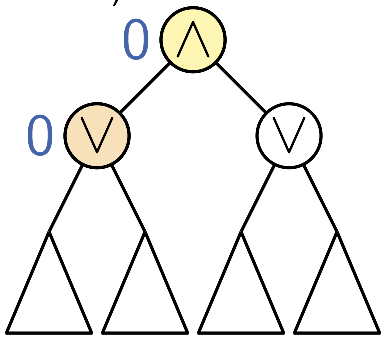

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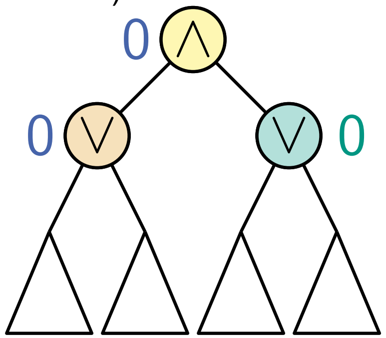

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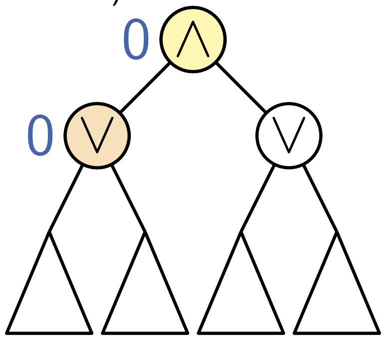

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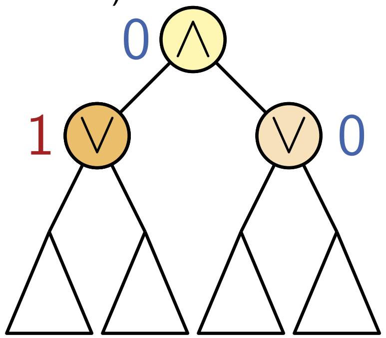

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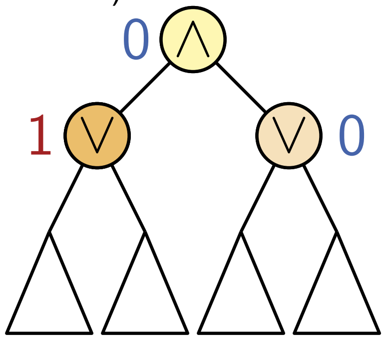

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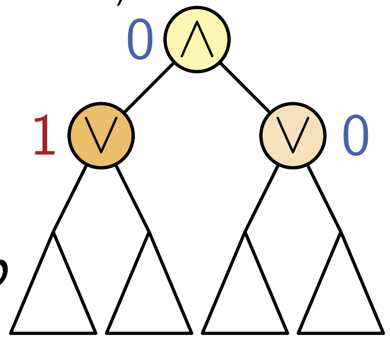

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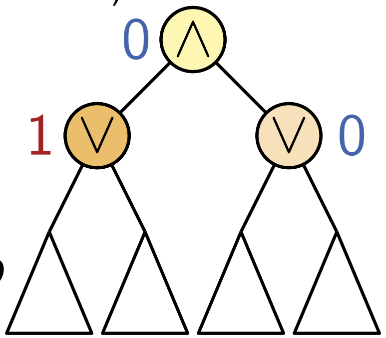

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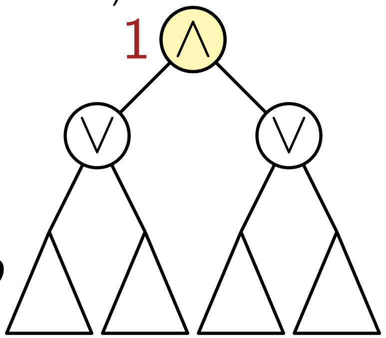

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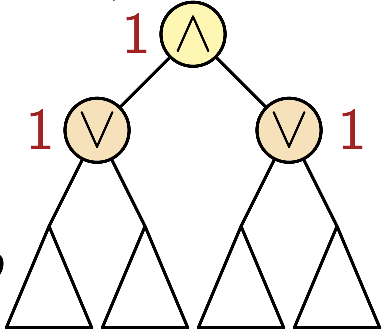

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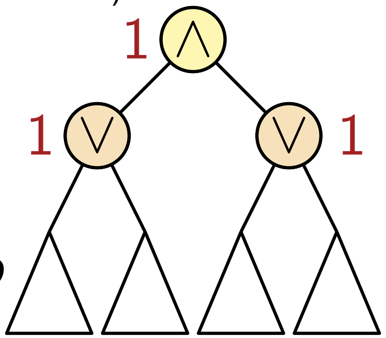

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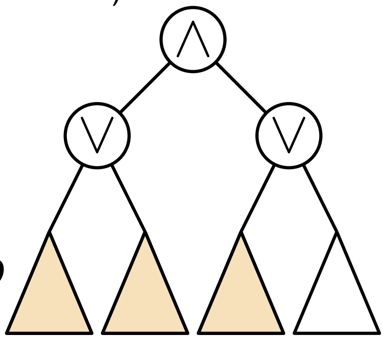

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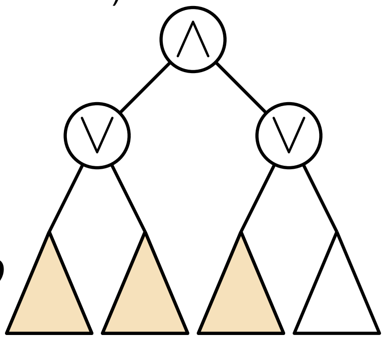

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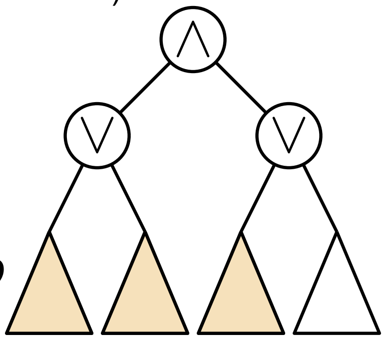

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$\mathbb{E}[X] \leq 3 \cdot 3^{k-1} = 3^k$ ✓

Power of Randomness: Average-Case Analysis

Binary Search Trees

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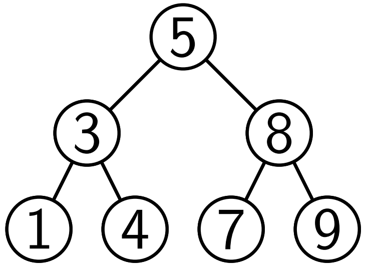

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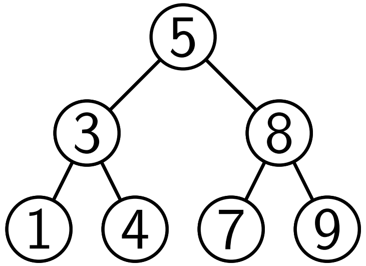

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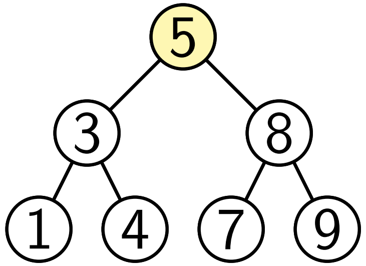

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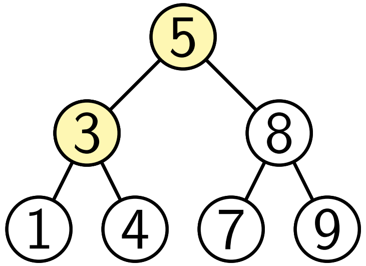

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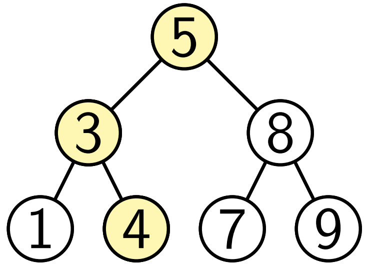

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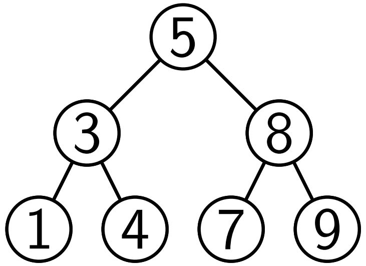

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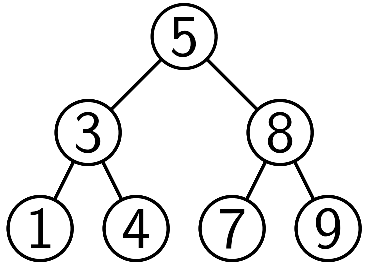

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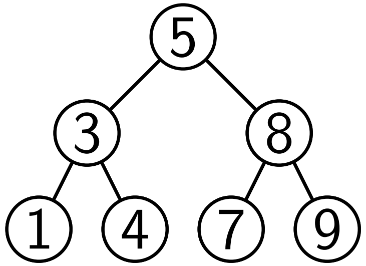

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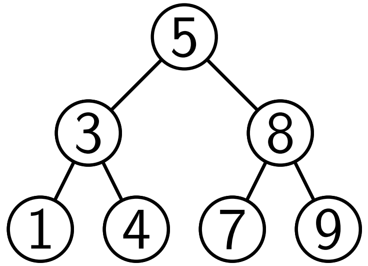

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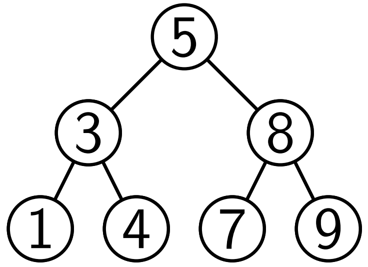

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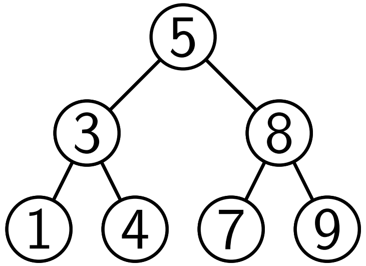

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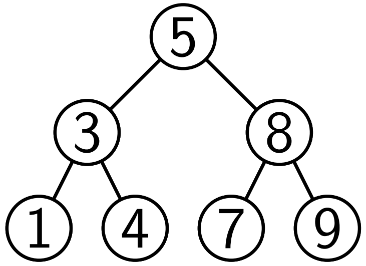
Is all that necessary?

Keep it Simple

Simple Insert Strategy

- Place a new element where it belongs. ✓



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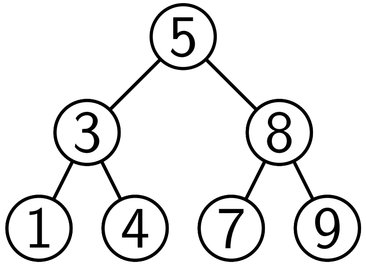
- Example: Insert 2



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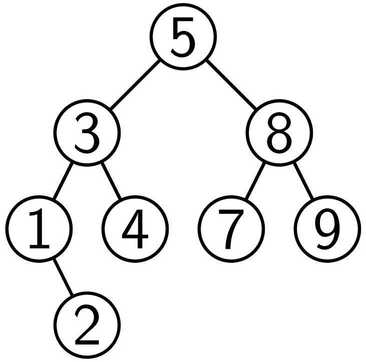

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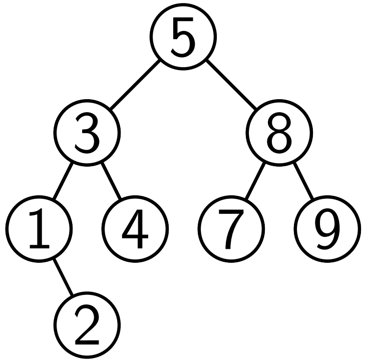

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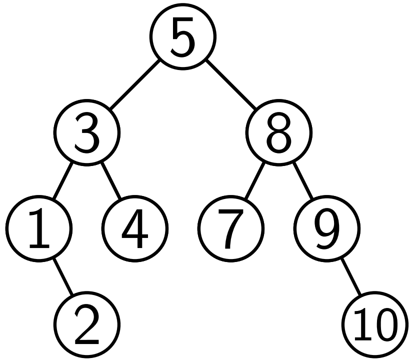

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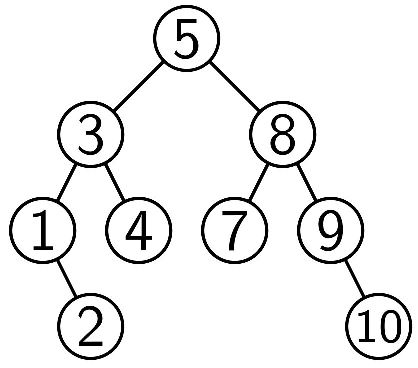

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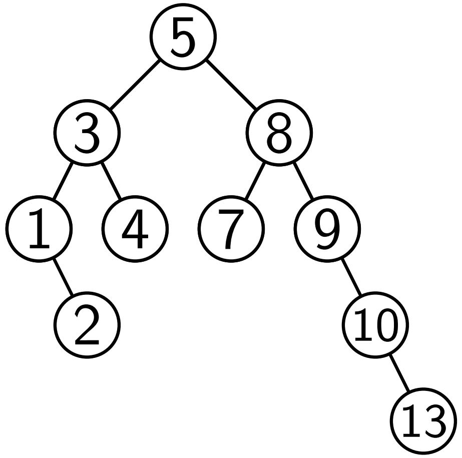

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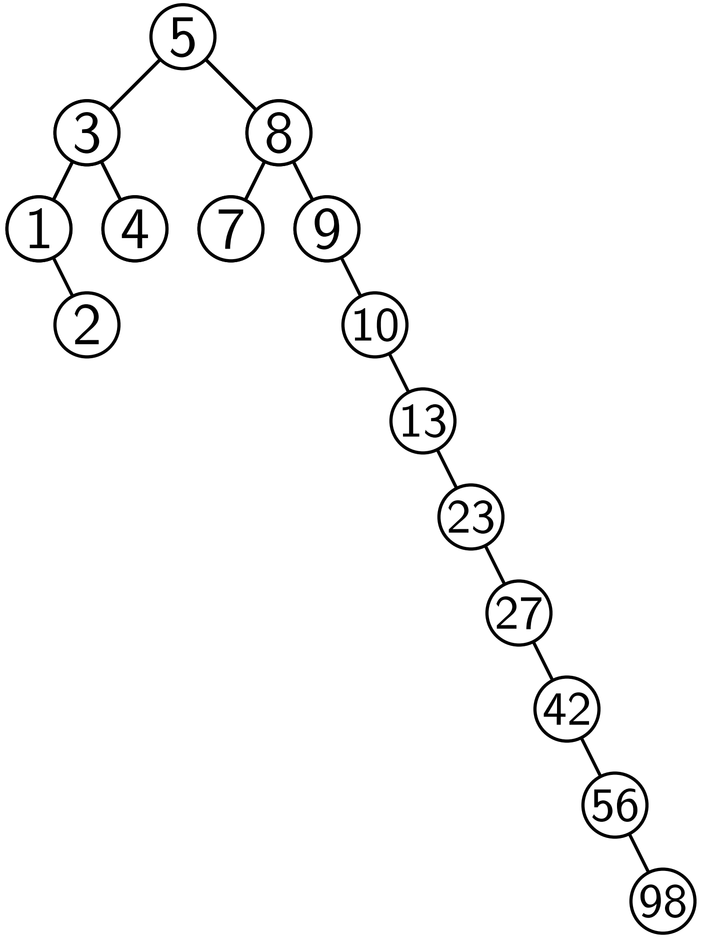

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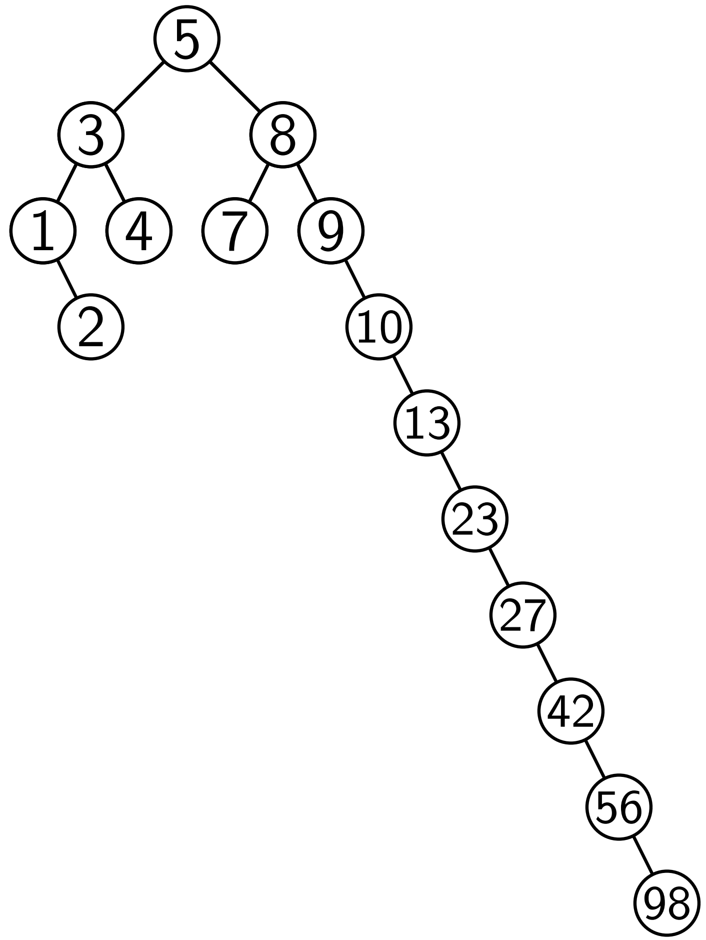

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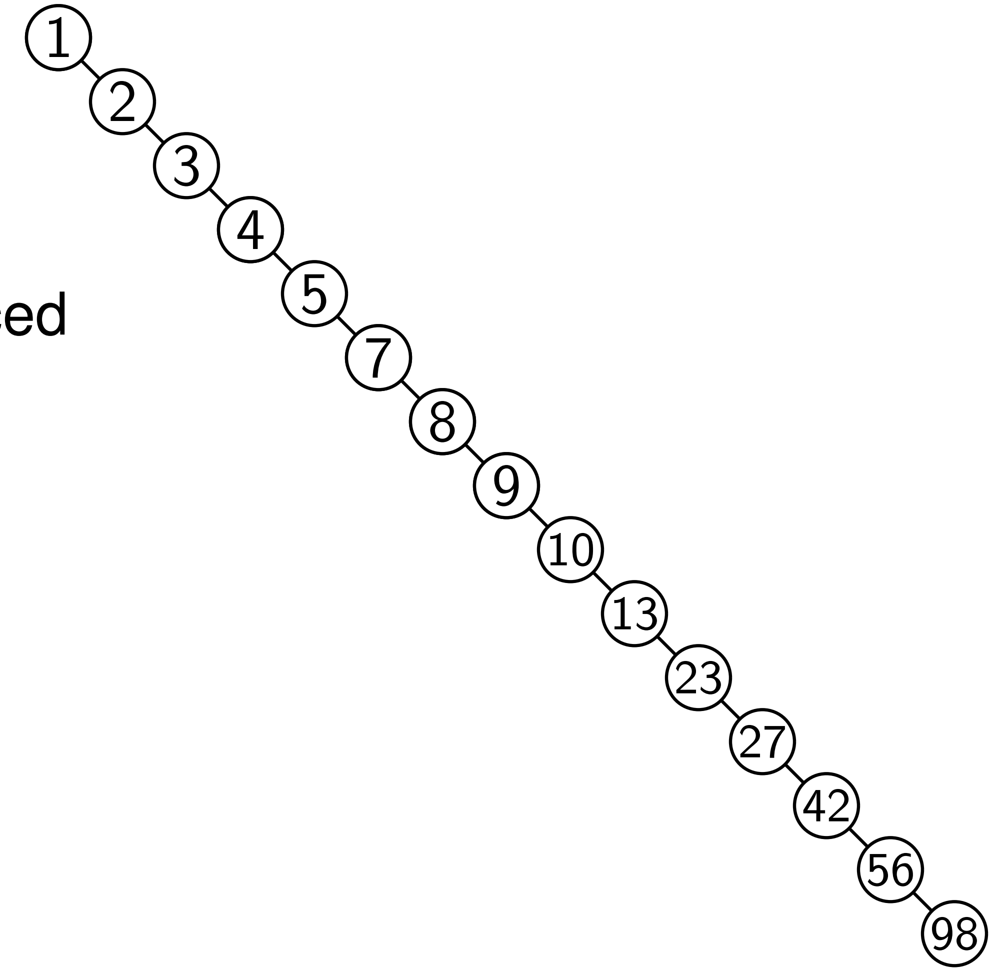

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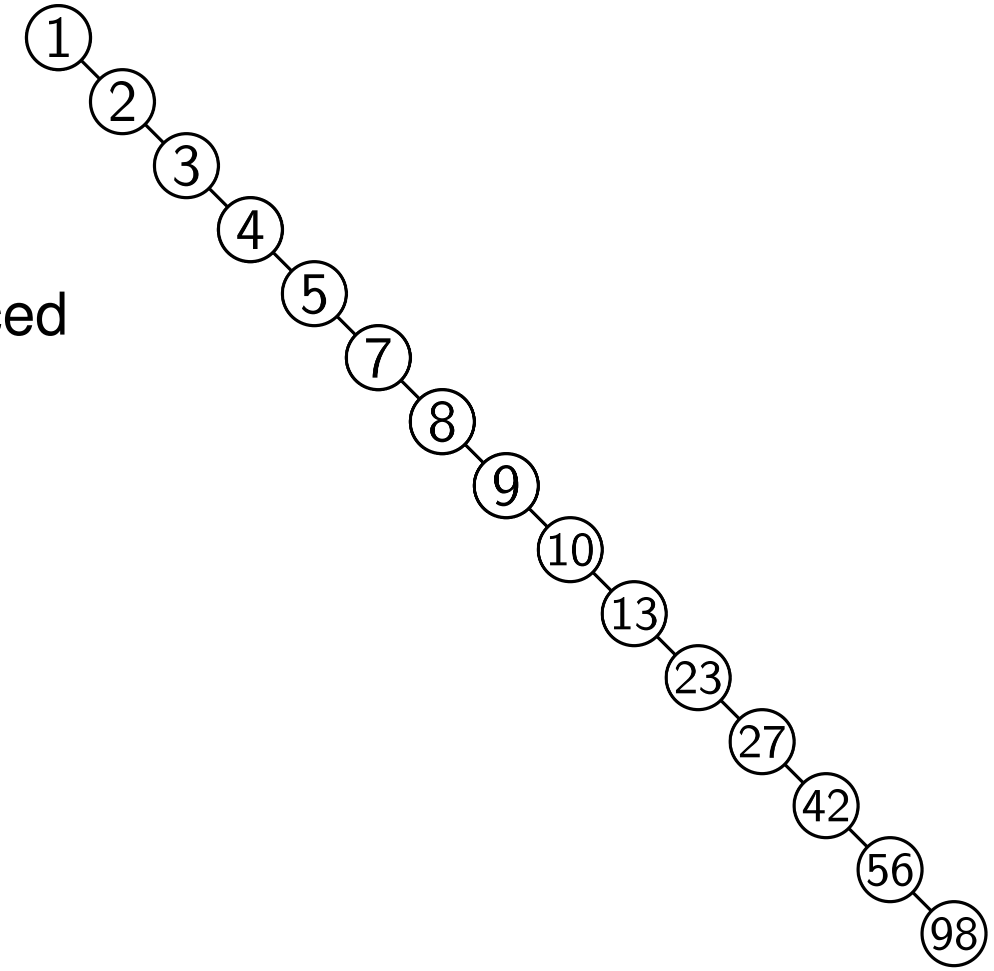

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Karlsruhe Institute of Technology

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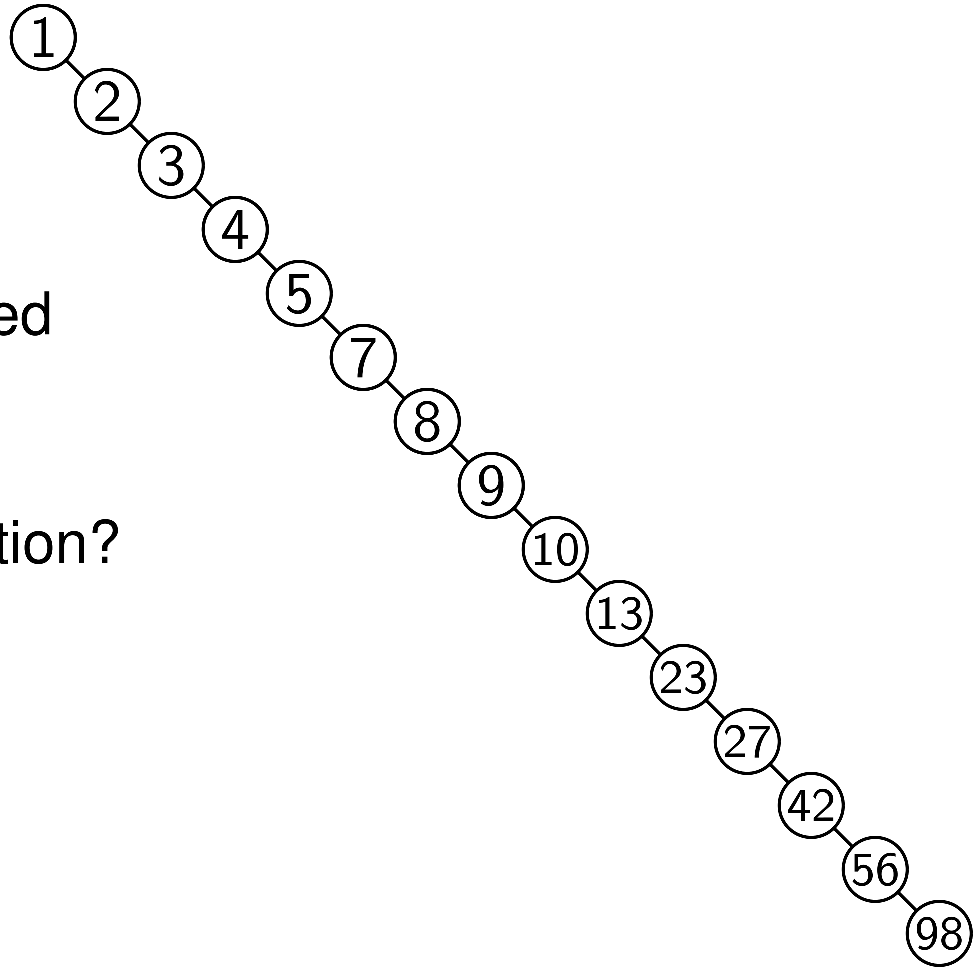

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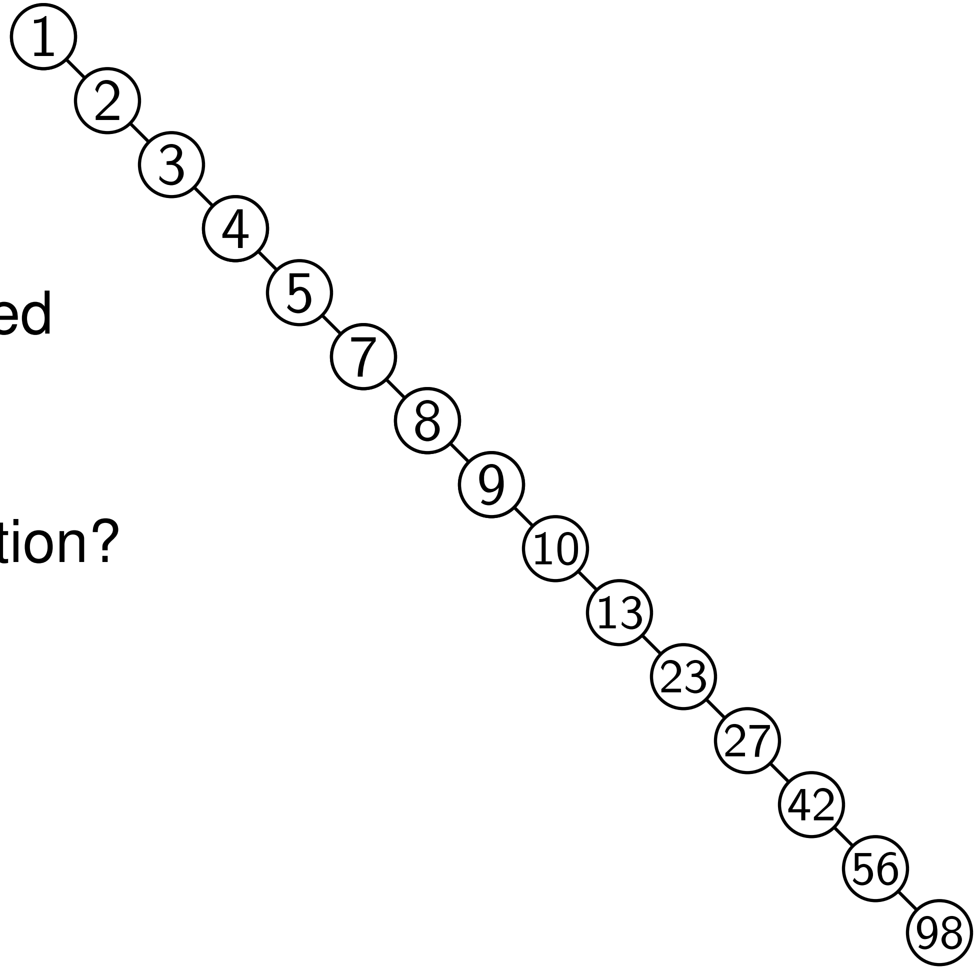

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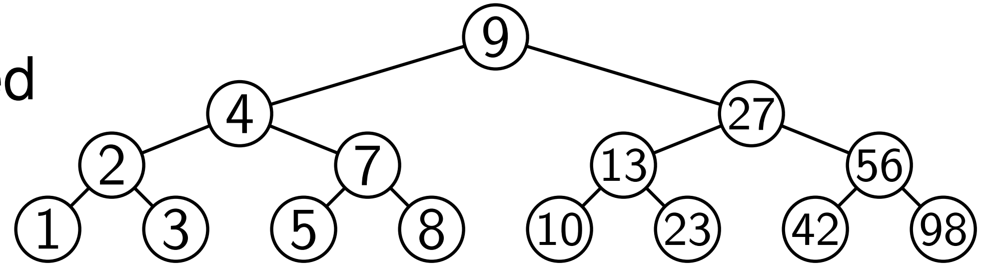

https://oeis.org/A056971

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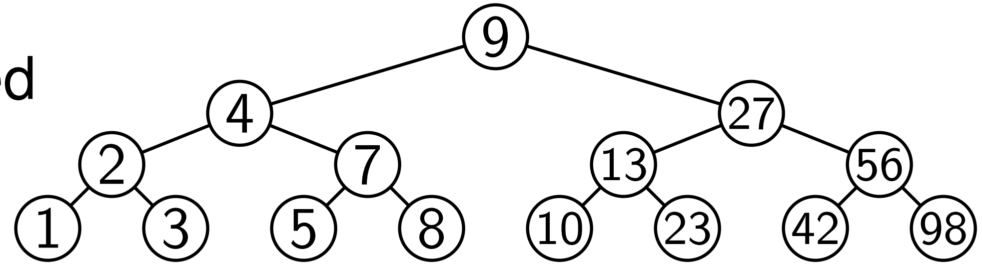

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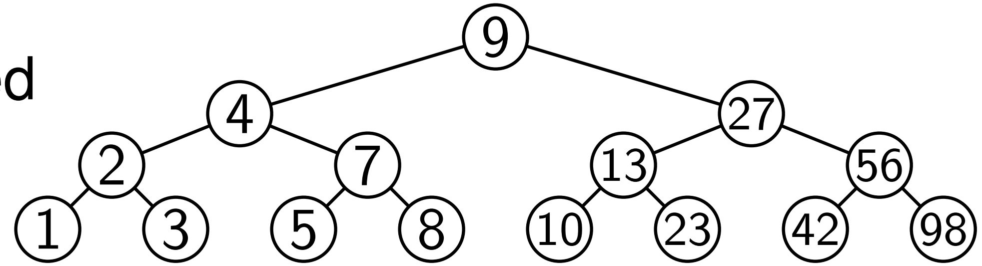

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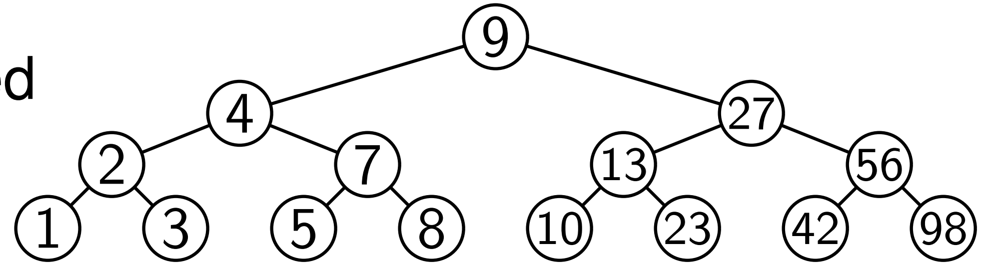

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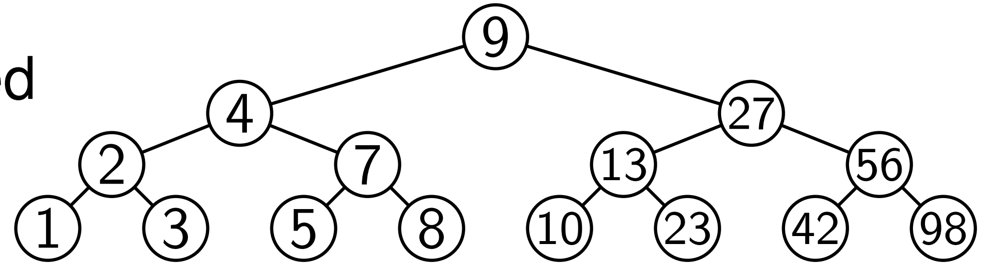

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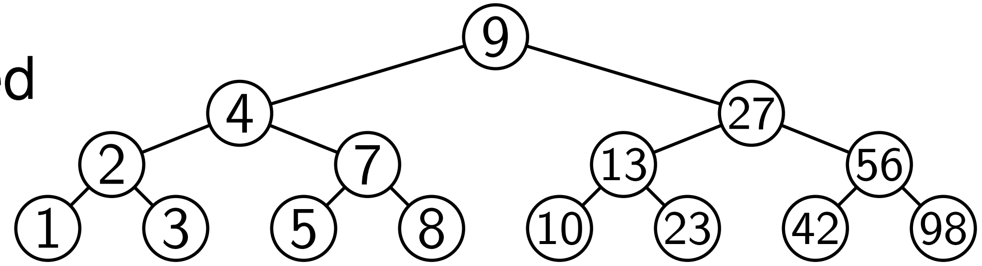

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In the following: uniform random permutation of the numbers

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Theorem: Let S be a permutation of $M = \{1, 2, \ldots, n\}$ chosen uniformly at random. Then, the expected depth of a binary search tree with the Simple Insert Strategy is $O(\log(n))$.

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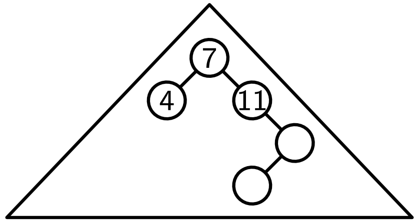

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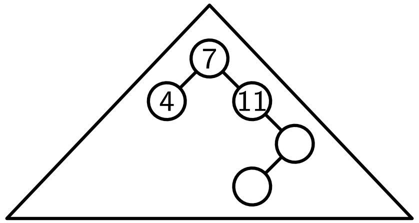

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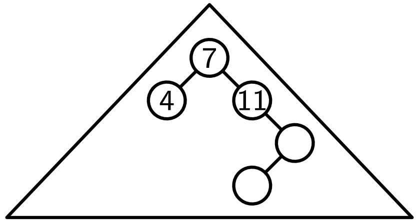

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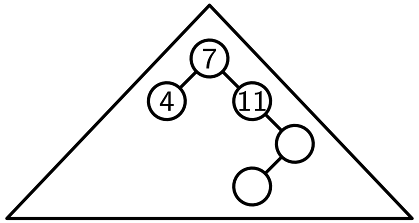

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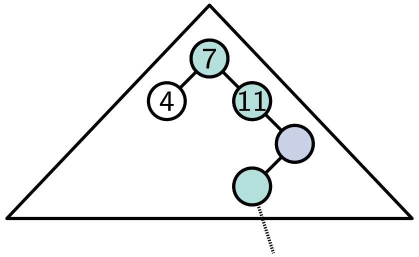

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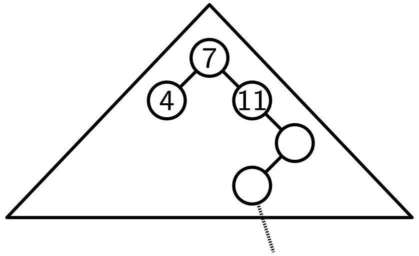

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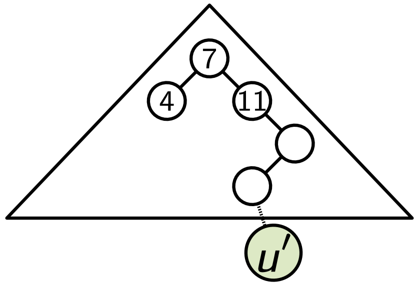

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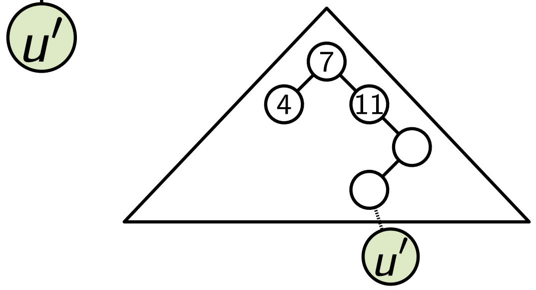

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- From then on, u' is on the path that would lead to v
- Case 1: $u' = u$: u is on path ✓
- Case 2: $u' \neq u$: ($u < u'$) & u is in left sub-tree of u' but v is in right u not on path ✓

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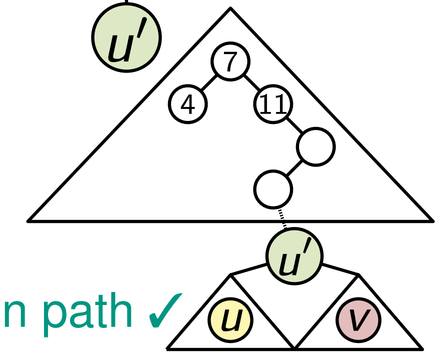

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- w.l.o.g. we can assume the elements to be $1, ..., n$, as we are only interested in the order

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$u > v$

$M_{v,u} = \{v, ..., u\}$ (for symmetry reasons)

- Let $S_{u,v}$ be the subsequence of S containing the elements in $M_{u,v}$
- Then $S_{u,v}$ is a uniform random permutation of $M_{u,v}$
- The probability that u is first in $S_{u,v}$ is
  $\Pr[\text{“}u \text{ first in } S_{u,v}\text{”}] = 1/|M_{u,v}| = 1/(v - u + 1)$

$$M_{u,v} = \{u, u+1, u+2, v\}$$
$$S = (..., u, ..., u+2, ..., v, ..., u+1, ...)$$
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Conclusion

Organizational

- Homepage: scale.iti.kit.edu/teaching/2023ws/randalg
- A place for questions will be linked on the website

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- At the cost of certainty (may be slow, may be wrong)

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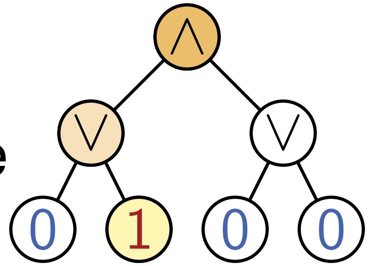

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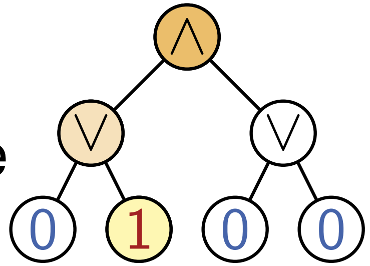

Average-Case Analysis

- Model real world using probability distributions over inputs
- If worst case is unlikely, expect good running times
- Example: Binary search-trees with simple insert strategy have same expected depth as complicated deterministic data structures