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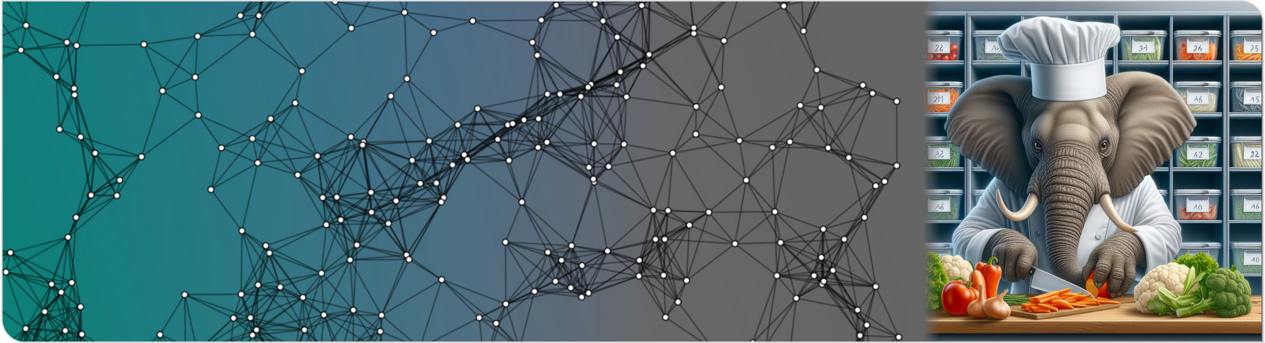
Probability and Computing – Classic Hash Tables

Stefan Walzer, Maximilian Katzmann | WS 2023/2024



Prüfungsanmeldung

- Am einfachsten: Hier angeben, wann ihr Zeit habt:
  `https://www.terminplaner.dfn.de/W4m8QyA9vvp1K19m`
- Alternativ: Email an Stefan und Max.
- Wir bieten euch dann einen Termin per Email an.

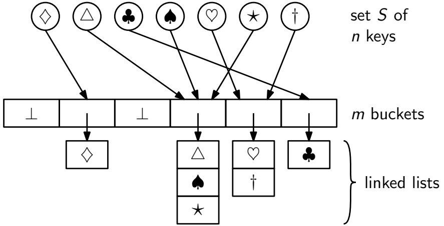
Content

Hash Table with Chaining

e.g. `std::unordered_set`, `java.util.HashMap`

Terminology

D:	*Universe* (or domain) of *keys* (strings, integers, game states in chess)
$S \subseteq D$:	set of n keys (possibly with associated data)
$h : D \to R$:	hash function, range usually $R = [m]$
$\alpha = \frac{n}{m}$:	*load factor*, $\alpha \leq \alpha_{\text{max}} = \mathcal{O}(1)$



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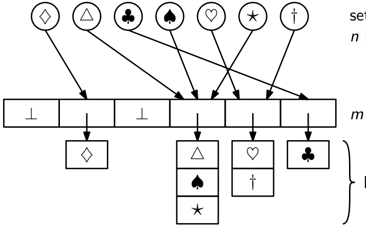

Goal

Operations in time t with $\mathbb{E}[t] = \mathcal{O}(1)$.
Randomness comes from the hash function.

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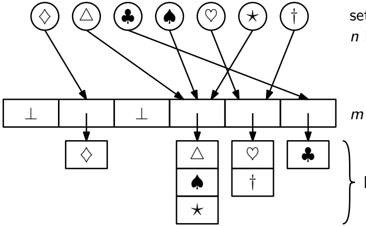

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Ideal Hash Functions

Every function from D to R is equally likely to be h.

Ideal Hash Functions are Impractical

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Naive Idea

- Let R^D denote all functions from D to R. We pick $h \sim \mathcal{U}(R^D)$.
- There are $|R|$ options for the hash of each $x \in D$
- Hence: $|R^D| = |R|^{|D|}$

$x \in D$	x_1	x_2	x_3	$\dots$	$x_{\lvert D \rvert}$
$h(x) \in R$	?	?	?	$\dots$	?

Why $h \sim \mathcal{U}(R^D)$ is desirable

- $h \sim \mathcal{U}(R^D) \Leftrightarrow \forall x_1, \dots, x_n \in D : h(x_1), h(x_2), \dots, h(x_n)$ are *independent* and uniformly random in R.
  $\hookrightarrow$ independence is very useful in an analysis
- In particular: $\forall x_1, \dots, x_n \in D, \forall i_1, \dots, i_n : \Pr_{h \sim \mathcal{U}(R^D)}[h(x_1) = i_1 \wedge \dots \wedge h(x_n) = i_n] = |R|^{-n}$.

Why $h \sim \mathcal{U}(R^D)$ is unwieldy

$\log_2(|R|^{|D|}) = |D| \cdot \log_2(|R|)$ bits to store $h \sim \mathcal{U}(R^D)$ $\rightsquigarrow$ for $D = \{0,1\}^{64}$: more than 2^{64} bits.

Content

What is a Hash Function?
(it depends on who you ask)

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Cryptographic Hash Function

A **collision resistant** function such as $h =$ `sha256sum`

```
$ sha256sum myfile.txt
018a7eaee8a...3e79043e21ab4  myfile.txt
```

Range $R = \{0, 1\}^{256}$. It is hard to find x, y with $h(x) = h(y)$.
$\hookrightarrow$ Files with equal hashes are likely the same.

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Cryptographic Pseudorandom Function

A function $f : \text{Seeds} \times D \to R$ where $\log_2 |\text{Seeds}|$ is small and no efficient algorithm can distinguish

- $f(s, \cdot)$ for $s \sim \mathcal{U}(\text{Seeds})$ and
- $h(\cdot)$ for $h \sim \mathcal{U}(R^D)$,

except with negligible probability.

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Hash Function in Algorithm Engineering

- typically small range $|R| = \mathcal{O}(n)$
  $\hookrightarrow$ cannot be collision resistant
- should **behave like** $h \sim \mathcal{U}(R^D)$ **in my application**
- should be **fast** to evaluate

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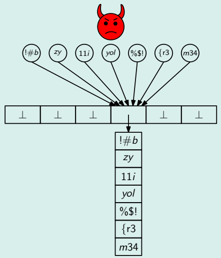
⚠ HashDoS is a thing.



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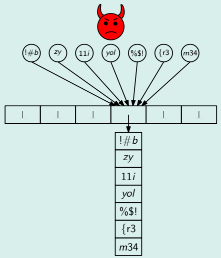
⚠ HashDoS is a thing.
However: Hash function and hash values need not be public.



Hashing in Practice

Black Magic, do not touch!

MurmurHash

Bitshifts, Magic Constants, . . .

```
uint32_t murmur3_32(const uint8_t* key,
            size_t len, uint32_t seed) {
    uint32_t h = seed;
    uint32_t k;
    for (size_t i = len >> 2; i; i--) {
        memcpy(&k, key, sizeof(uint32_t));
        key += sizeof(uint32_t);
        h ^= murmur_32_scramble(k);
        h = (h << 13) | (h >> 19);
        h = h * 5 + 0xe6546b64;
    }
    [...]
    return h;
}
static inline uint32_t murmur_32_scramble(uint32_t k) {
    k *= 0xcc9e2d51;
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Usage

For $R = [m]$, pick seed $\sim \mathcal{U}(\{0,1\}^{32})$ and use

$$h(x) = \texttt{murmur3\_32}(x, \text{seed}) \bmod m.$$

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- **MAYBE,** for your favourite application.

Content

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Approach 1: Ignore the Problem

Simple Uniform Hashing Assumption (SUHA)

- We have access to $h \sim \mathcal{U}(R^D)$ for any R and D.
- h takes $\mathcal{O}(1)$ time to evaluate.
- h takes no space to store.

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1. *Assume* SUHA holds.
2. *Analyse* algorithm under SUHA.
3. *Hope* that algorithm still works with real hash functions.

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- *Modelling* assumption.
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- Excellent track record in non-adversarial settings.

What should a Theorist do?
Approach 2: Bring your own Hash Functions

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Analyse Algorithm using Universal Hashing

1. Define family $\mathcal{H} \subseteq R^D$ of hash functions with $\log(|\mathcal{H}|)$ not too large.
   $\hookrightarrow$ sampling and storing $h \in \mathcal{H}$ is cheap
2. Proof that algorithm with $h \sim \mathcal{U}(\mathcal{H})$ has good expected behaviour.

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- Mathematical structure of $\mathcal{H}$ must be amenable to analysis.
- *Rigorously* covers non-adversarial settings.
- Proofs often difficult.
  $\hookrightarrow$ Wider theory practice gap than with SUHA.

What should a Theorist do?

Approach 3: Let the Cryptographers do the Work

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How to Analyse your Algorithm using Cryptographic Assumptions

1. Analyse algorithm under *SUHA*.
2. Actually use *cryptographic pseudorandom function* f.
   - **Case 1:** Everything still works. Great! :-)
   - **Case 2:** Something fails.

     $\Rightarrow$ Your use case can tell the difference between f and true randomness.

     $\hookrightarrow$ The cryptographers said this is impossible. $\lightning$

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- **YES.** Algorithms become robust even in some adversarial settings.
  $\hookrightarrow$ e.g. Python, Haskell, Ruby, Rust use **SipHash** by default
  `https://en.wikipedia.org/wiki/SipHash`

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- **NO.** Too slow in high-performance settings.

Hash Function	MiB / sec
SipHash	944
Murmur3F	7623
xxHash64	12109

(source: `https://github.com/rurban/smhasher`)

Content

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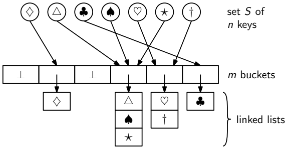
Hash Table with Chaining

Search Time under Chaining

$$\max_{\substack{S \subseteq D \\ |S|=n}} \max_{x \in D} \quad 1 + |\{y \in S \mid h(y) = h(x)\}|$$

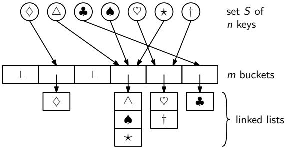


Hash Table with Chaining

Search Time under Chaining

For $n, m \in \mathbb{N}$ and a family $\mathcal{H} \subseteq [m]^D$ of hash functions the *maximum expected search time* is at most

$$T_{\text{chaining}}(n, m, \mathcal{H}) = \max_{\substack{S \subseteq D \\ |S| = n}} \max_{x \in D} \mathbb{E}_{h \sim \mathcal{U}(\mathcal{H})}\Big[1 + |\{y \in S \mid h(y) = h(x)\}|\Big]$$



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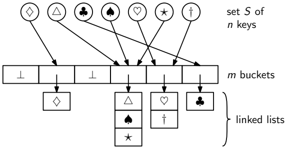
Key set is *worst case*. Only $h \in \mathcal{H}$ is random. Key set is fixed *before* h is chosen.



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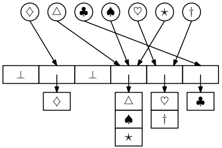

Theorem: Hash Table with Chaining under SUHA

If $\mathcal{H} = [m]^D$ then $T_{\text{chaining}}(n, m, \mathcal{H}) \leq 2 + \alpha = \mathcal{O}(1)$ if $\alpha \in \mathcal{O}(1)$.

Analysis of Hash Table with Chaining under SUHA

Karlsruhe Institute of Technology

Theorem: Hash Table with Chaining under SUHA

Let $\mathcal{H} = [m]^D$, $S \subseteq D$ with $|S| = n$ and $x \in D$ then

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$$\begin{aligned} &\mathbb{E}_{h \sim \mathcal{U}(\mathcal{H})}\Big[1 + |\{y \in S \mid h(y) = h(x)\}|\Big] \\ = &\mathbb{E}_{h \sim \mathcal{U}(\mathcal{H})}\Big[1 + \sum_{y \in S} \mathbb{1}_{\{h(y)=h(x)\}}\Big] \end{aligned}$$

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&= 1 + 1 + \sum_{y\in S\setminus\{x\}} \Pr_{h\sim\mathcal{U}(\mathcal{H})}[h(y) = h(x)] \\
&= 2 + \sum_{y\in S\setminus\{x\}} \frac{1}{m} \leq 2 + \frac{n}{m} = 2 + \alpha. \quad \square
\end{aligned}$$

Content

A Universal Hash Family

Karlsruhe Institute of Technology

Definition: c-universal hash family

A class $\mathcal{H} \subseteq [m]^D$ is called *c-universal* if: $\forall x \neq y \in D : \Pr_{h \sim \mathcal{U}(\mathcal{H})}[h(x) = h(y)] \leq \frac{c}{m}$.

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Note: $\mathcal{H} = [m]^D$ is 1-universal.

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Let $\mathbb{F}_p = \{0, \ldots, p-1\}$ for a prime number p. Then $(\mathbb{F}_p, \times, \oplus)$ is a field where

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The class of Linear Hash Functions

Assume $D \subseteq \mathbb{F}_p$ for prime p. Then the following class is 1-universal:

$$\mathcal{H}_{p,m}^{\text{lin}} := \{x \mapsto ((a \times x) \oplus b) \bmod m \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\}.$$

Proof that $\mathcal{H}^{\text{lin}}_{p,m} := \{x \mapsto ((a \times x) \oplus b) \bmod m \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\}$ is 1-universal.

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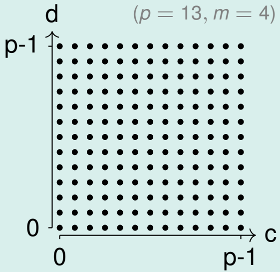

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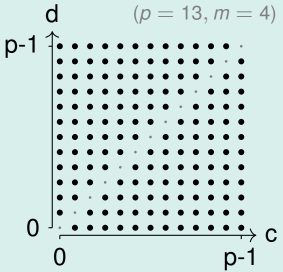
$(p = 13, m = 4)$



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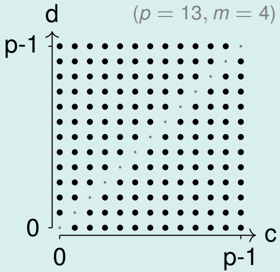

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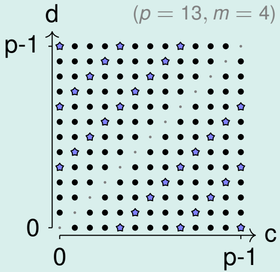

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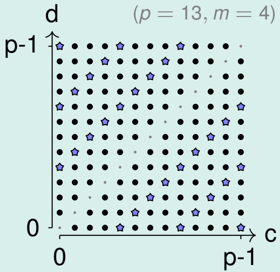

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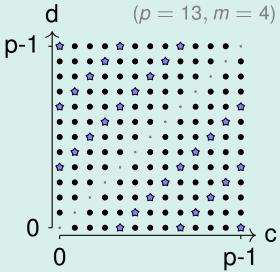

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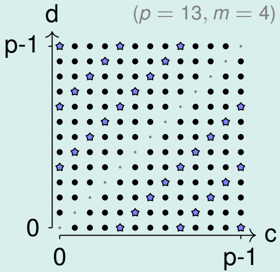

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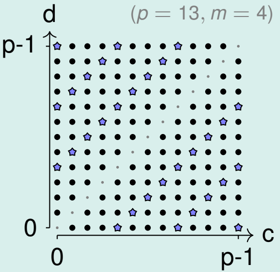

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Proof that $\mathcal{H}^{\text{lin}}_{p,m} := \{x \mapsto ((a \times x) \oplus b) \bmod m \mid a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\}$ is 1-universal.

Let $x \neq y \in \mathbb{F}_p$. (To show: $\Pr_{h\sim\mathcal{H}^{\text{lin}}_{p,m}}[h(x) = h(y)] \leq 1/m$.)

- Define $\begin{aligned} c &= (a \times x) \oplus b \\ d &= (a \times y) \oplus b \end{aligned} \Leftrightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \underbrace{\begin{pmatrix} x & 1 \\ y & 1 \end{pmatrix}}_{\text{regular!}} \begin{pmatrix} a \\ b \end{pmatrix}.$
- The mapping $(a, b) \mapsto (c, d)$ is a bijection (for every $x \neq y$) from $\mathbb{F}_p^* \times \mathbb{F}_p \to P$.
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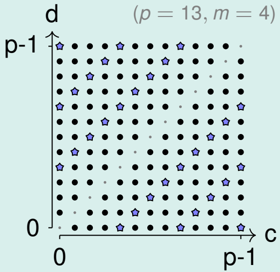

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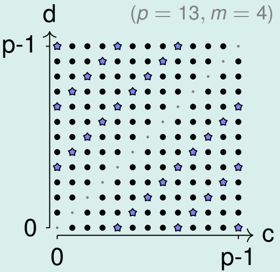

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Analysis of Hash Table with Chaining

… using a Universal Hash Family

Theorem

If $\mathcal{H} \subseteq [m]^D$ is a c-universal hash family then $T_{\text{chaining}}(n, m, \mathcal{H}) \leq 2 + c\alpha = \mathcal{O}(1)$ if $\alpha \in \mathcal{O}(1)$ and $c \in \mathcal{O}(1)$.

Proof: Mostly the same.

$$\forall S \subseteq [D], \forall x \in D : \qquad \mathbb{E}_{h \sim \mathcal{U}(\mathcal{H})}\Big[1 + |\{y \in S \mid h(y) = h(x)\}|\Big]$$

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$$= 2 + \sum_{y \in S \setminus \{x\}} \frac{c}{m} \leq 2 + \frac{cn}{m} = 2 + c\alpha. \quad \square$$

More Universal Families

Examples for Universal Hash Families

- "$\boldsymbol{((ax+b) \bmod p) \bmod m}$" is 1-universal

as discussed: $D = \mathbb{F}_p, \qquad R = [m],$

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- **Multiply-Shift** is 2-universal:

  $$D = \{0, \dots, 2^w - 1\}, \qquad R = \{0, \dots, 2^\ell - 1\}$$

  $$\mathcal{H} = \{x \mapsto \lfloor ((a \cdot x + b) \bmod 2^w)/2^{w-\ell} \rfloor \mid$$
  $$\textbf{odd } a \in \{1, \dots, 2^w - 1\}, b \in \{0, \dots, 2^w - 1\}.\}$$

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  odd $a \in \{1, \ldots, 2^w - 1\}, b \in \{0, \ldots, 2^w - 1\}.\}$

Selling point of multiply shift:

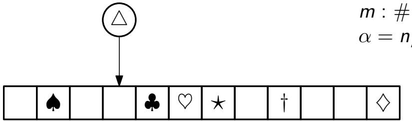
- “$x \bmod 2^w$” drops some higher order bits
- “$\lfloor x/2^{w-\ell} \rfloor$ drops some lower order bits
- No division or modulo operation needed!

For $w = 32$ (taken from Thorup 2015):

```
uint32_t hash(uint32_t x, uint32_t l, uint64_t a) {
        return (a * x + b) >> (64-l);
}
```

Content

Hash Table with Linear Probing



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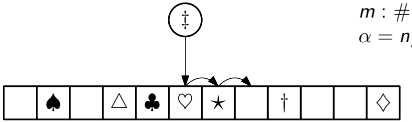

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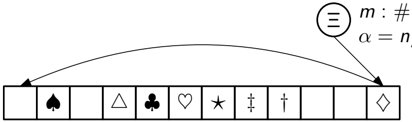

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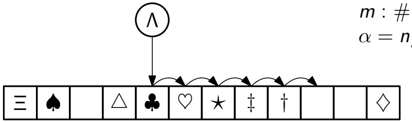

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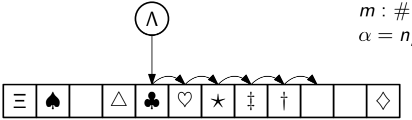

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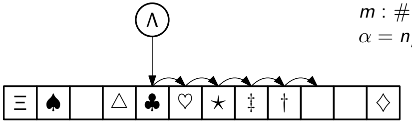

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↪ For details see `https://en.wikipedia.org/wiki/Linear_probing`

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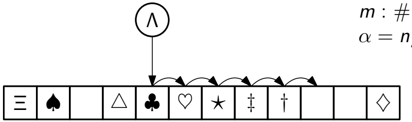

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Hash Table with Linear Probing

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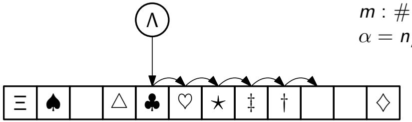

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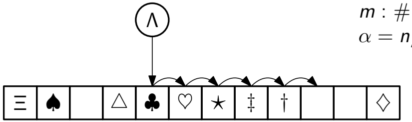

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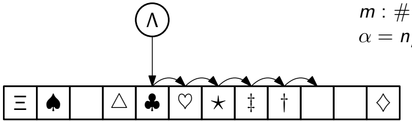

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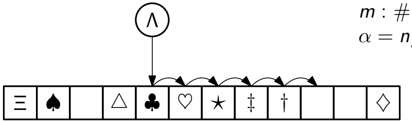

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Theorem: Linear Probing under SUHA

Let $T_{n,m}$ be the random insertion time into a linear probing hash table. If $\frac{1}{2} \leq \alpha = \frac{n}{m} < \alpha_{\max}$ for some $\alpha_{\max} < 1$ then under SUHA we have

$$\mathbb{E}[T_{n,m}] = \mathcal{O}(1).$$

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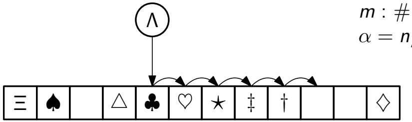

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$$\mathbb{E}[T_{n,m}] = \mathcal{O}\left(\frac{1}{(1-\alpha_{\max})^2}\right) = \mathcal{O}(1). \quad \text{(not here)}$$

Content

Preparation: A concentration bound

Chernoff

For $X \sim \mathit{Bin}(n, p)$ and $\varepsilon \in [0, 1]$ we have $\Pr[X \geq (1 + \varepsilon)\mathbb{E}[X]] \leq \exp(-\varepsilon^2\mathbb{E}[X]/3)$.

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Karlsruhe Institute of Technology

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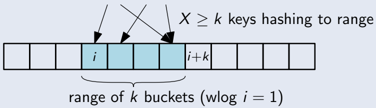

Let $k \in \mathbb{N}$ and $X = |\{y \in S \mid h(y) \in \{1,\dots,k\}\}|$.

Then $\Pr_{h \sim \mathcal{U}(R^D)}[X \geq k] \leq \exp(-(1-\alpha)^2 k/3)$.

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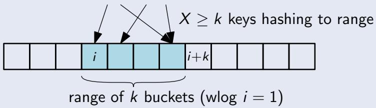

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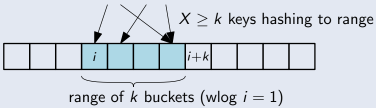

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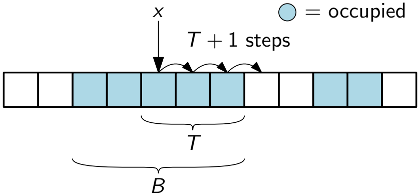
$$\begin{aligned}
\Pr[X \geq k] &= \Pr[X \geq \tfrac{1}{\alpha}\mathbb{E}[X]] \\
&= \Pr[X \geq (1 + \tfrac{1-\alpha}{\alpha})\mathbb{E}[X]] \\
&\leq \exp(-(\tfrac{1-\alpha}{\alpha})^2 \alpha k/3) \\
&\leq \exp(-(1-\alpha)^2 k/3). \quad (\text{using } \tfrac{1}{2} \leq \alpha \leq 1)
\end{aligned}$$

Proof: Expected LP-Insertion Time under SUHA is $\mathcal{O}(1)$

$\mathbb{E}[T]$



$A_{u,v}$: $\{u, v\}$ is maximal occupied block:

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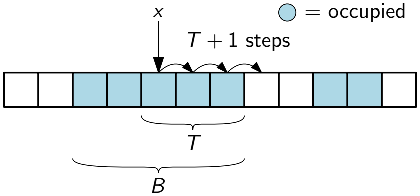
$\mathbb{E}[T] \leq \mathbb{E}[B]$



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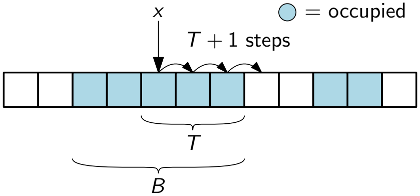

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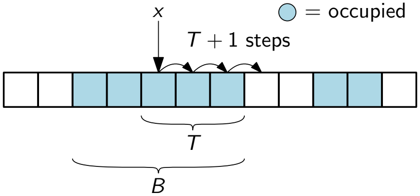

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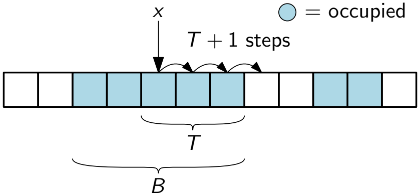

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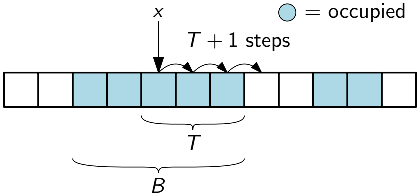

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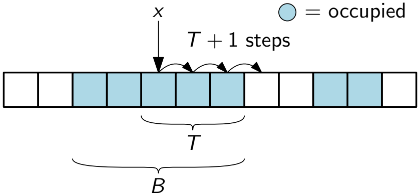

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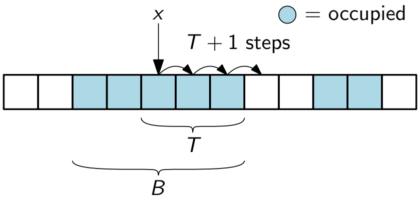

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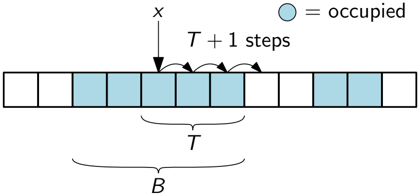

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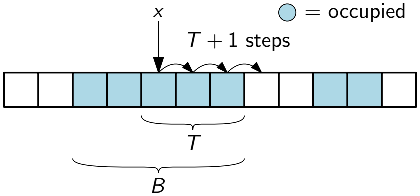
$$\le \sum_{k\ge 1} k^2 \cdot \exp(-(1-\alpha_{\max})^2 k/3) = \mathcal{O}(1).$$

Wolfram Alpha gives: $\int_0^\infty k^2 \exp(-(1-\alpha_{\max})^2 k/3) = \dfrac{54}{(1-\alpha_{\max})^6}$.



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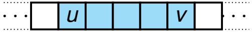

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Content

Degrees of Independence

(Mutual / Collective) Independence

A family $\mathcal{E}$ of **events** is **independent** if $\forall k \in \mathbb{N}$ and distinct $E_1, \dots, E_k \in \mathcal{E}$ we have

$$\Pr\left[\bigcap_{i=1}^{k} E_i\right] = \prod_{i=1}^{k} \Pr[E_i].$$

A family $\mathcal{X}$ of discrete **random variables** is **independent** if $\forall k \in \mathbb{N}$, distinct $X_1, \dots, X_k \in \mathcal{X}$ and all $x_1, \dots, x_k \in \mathbb{R}$ we have

$$\Pr\left[\bigwedge_{i=1}^{k} X_i = x_i\right] = \prod_{i=1}^{k} \Pr[X_i = x_i].$$

Degrees of Independence

Pairwise Independence

A family $\mathcal{E}$ of **events** is **pairwise independent** if for distinct $E_1, E_2 \in \mathcal{E}$ we have

$$\Pr\left[E_1 \cap E_2\right] = \Pr[E_1] \cdot \Pr[E_2].$$

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Degrees of Independence

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d-wise Independence

A family $\mathcal{E}$ of **events** is d**-wise independent** if $\forall k \in \{2, \ldots, d\}$ and distinct $E_1, \ldots, E_k \in \mathcal{E}$ we have

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d-Independent Hash Family

Definition: d-Independent Hash Family

A family $\mathcal{H} \subseteq [R]^D$ of hash functions is *d-independent* if for distinct $x_1, \dots, x_d \in D$ and any $i_1, \dots, i_d \in R$: (grey is implied by black)

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$\mathcal{H}$ is d-independent if for $h \sim \mathcal{U}(\mathcal{H})$

- the family $(h(x))_{x \in D}$ of random variables is d-independent *and*
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Theorem

Let $D = R = \mathbb{F}$ be a finite field. Then

$$\mathcal{H} := \{x \mapsto \sum_{i=0}^{d-1} a_i x^i \mid a_0, \ldots, a_{d-1} \in \mathbb{F}\}$$

is a d-independent family.

Note: $\mathcal{H} \subseteq \mathbb{F}^{\mathbb{F}} \rightsquigarrow$ not yet useful.

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Corollary: Smaller Ranges (proof omitted)

- If m divides $|\mathbb{F}|$, then adding "$\bmod m$" gives a d-independent family $\mathcal{H}' \subseteq [m]^{\mathbb{F}}$.
- If m does not divide $|\mathbb{F}|$, then adding "$\bmod m$" gives a family $\mathcal{H}' \subseteq [m]^{\mathbb{F}}$ such that for $h \sim \mathcal{U}(\mathcal{H}')$ the family $(h(x))_{x \in \mathbb{F}}$ is d-independent but only *approximately* uniformly distributed in $[m]$.

Proof: $\mathcal{H} := \{x \mapsto \sum_{i=0}^{d-1} a_i x^i \mid a_0, \ldots, a_{d-1} \in \mathbb{F}\}$ is d-independent

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Exactly one vector $\vec{a} = M^{-1} \cdot \vec{i}$ solves the equation.

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$$\Rightarrow \Pr_{h \sim \mathcal{U}(\mathcal{H})}[\forall j : h(x_j) = i_j] = \Pr_{a_0, \ldots, a_{d-1} \sim \mathcal{U}(\mathbb{F})}[\vec{a} = M^{-1} \cdot \vec{i}\,] = \mathbb{F}^{-d}. \quad \square$$

Concentration Bound for d-Independent Variables

(Tricky) Exercise

Let d be even and $X_1, \ldots, X_n \sim Ber(p)$ a d-independent family of random variables with $p = \Omega(1/n)$.
Let $X = \sum_{i=1}^{n} X_i$. Then for any $\varepsilon > 0$ we have

$$\Pr[X - \mathbb{E}[X] \geq \varepsilon \mathbb{E}[X]] = \mathcal{O}(\varepsilon^{-d} \mathbb{E}[X]^{-d/2}).$$

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Remark: Weaker than Chernoff, stronger than Chebyshev

Chebycheff gives $\Pr[X - \mathbb{E}[X] \geq \varepsilon \mathbb{E}[X]] \leq \frac{1-p}{\varepsilon^2 \mathbb{E}[X]}$. (requires $d = 2$)
Chernoff gave $\Pr[X - \mathbb{E}[X] \geq \varepsilon \mathbb{E}[X]] \leq \exp(-\varepsilon^2 \mathbb{E}[X]/3)$. (requires $d = n$).

Preparation: A Concentration Bound

again for *d*-independence

Lemma (last slide)

For d-independent $X_1, \ldots, X_n \sim Ber(p)$ and $X = \sum_{i \in [n]} X_i$ we have $\Pr[X \geq (1+\varepsilon)\mathbb{E}[X]] = \mathcal{O}(\varepsilon^{-d}\mathbb{E}[X]^{-d/2})$.

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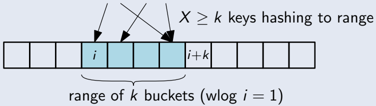
Lemma: $\geq k$ hits in segment of length k



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Let $k \in \mathbb{N}$ and $X = |\{y \in S \mid h(y) \in \{1, \ldots, k\}\}|$.

Then $\Pr[X \geq k] \leq \mathcal{O}((1-\alpha)^{-d} k^{-d/2})$.

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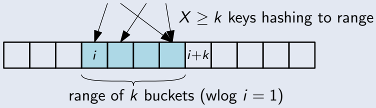

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Let $S = \{x_1, \ldots, x_n\}$ and $X_i = \mathbb{1}_{\{h(x_i) \in \{1,\ldots,k\}\}} \sim Ber(\frac{k}{m})$.
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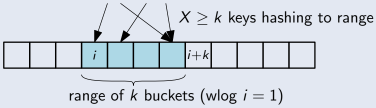

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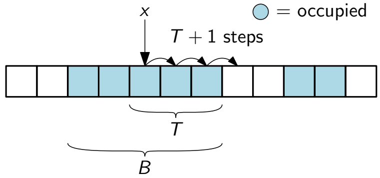
$$\begin{aligned}
\Pr[X \geq k] &= \Pr[X \geq \tfrac{1}{\alpha}\mathbb{E}[X]] \\
&= \Pr[X \geq (1 + \tfrac{1-\alpha}{\alpha})\mathbb{E}[X]] \\
&= \mathcal{O}\left(\left(\frac{1-\alpha}{\alpha}\right)^{-d}(\alpha k)^{-d/2}\right) \\
&\leq \mathcal{O}((1-\alpha)^{-d}k^{-d/2}). \quad \text{(using } \alpha \leq 1\text{)}
\end{aligned}$$

Theorem: Linear Probing with d-independence

Under the same conditions as before, except with 9-independent hash functions, the insertion time $T_{n,m}$ for linear probing satisfies:

$$\mathbb{E}[T_{n,m}] = \mathcal{O}(1)$$



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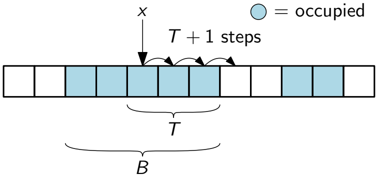
Proof Sketch

$\mathbb{E}[T]$



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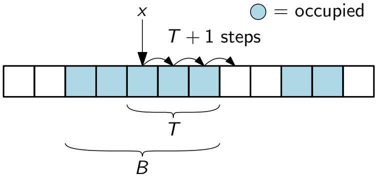

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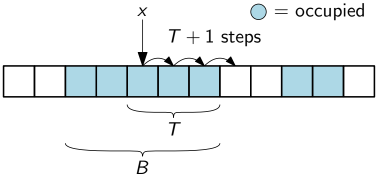

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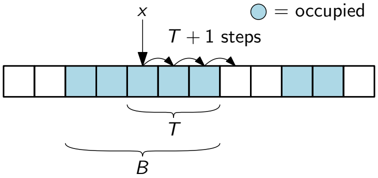

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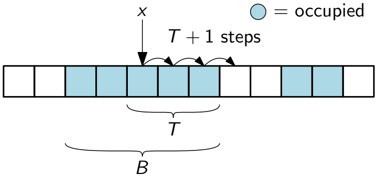

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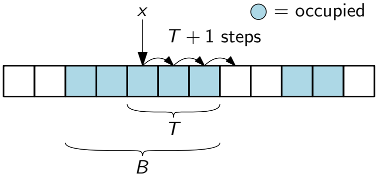
\end{aligned}$$



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(3) If interested, see 3Blue1Brown video: `https://www.youtube.com/watch?v=d-o3eB9sfls`

Final Remarks on Linear Probing + Universal Hashing

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Much more is known about insertion times of linear probing:

- Any 5-independent family gives $\mathcal{O}(\frac{1}{(1-\alpha)^2})$.
  $\hookrightarrow$ A. Pagh, R. Pagh, and Ruzic 2011
- An (artificially bad) 4-independent family gives $\Omega(\log n)$.
  $\hookrightarrow$ Pătraşcu and Thorup 2016
- A (well-designed) 4-independent family gives $\mathcal{O}(\frac{1}{(1-\alpha)^2})$.
  $\hookrightarrow$ Puatracscu and Thorup 2013

Conclusion

Technical Takeaway: Performance of Hash Tables

For both an **ideal hash function** (SUHA) and a random hash function from a suitable **universal class**, a hash table using **linear probing** or **chaining** provably has an expected running time of $\mathcal{O}(1)$ per operation.

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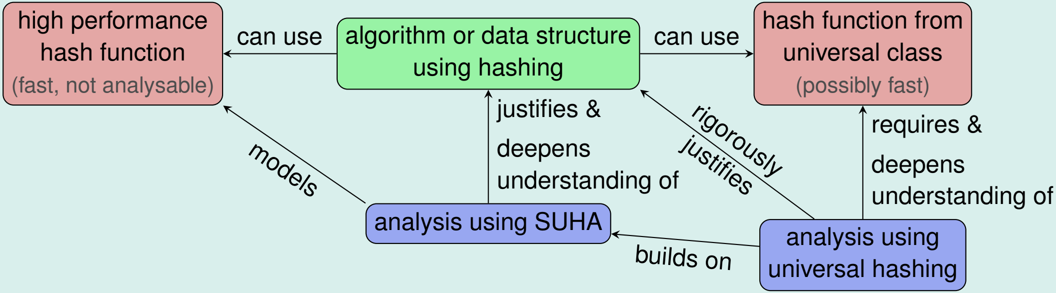
Non-Technical Takeaway: Approaches to analyse hashing based algorithms



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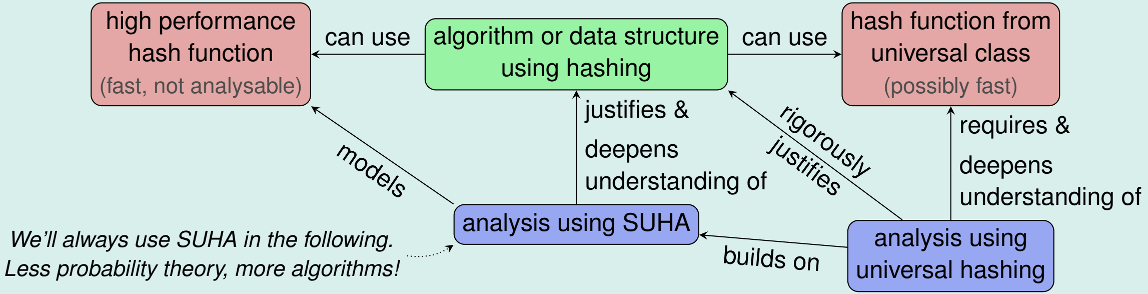

Anhang: Mögliche Prüfungsfragen I

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- Was könnte eine Idealvorstellung einer Hashfunktion sein? Inwiefern wäre eine ideale Hashfunktion nützlich? Was ist das Problem an dieser Vorstellung?
- Was ist die Simple Uniform Hashing Assumption (SUHA)? Was spricht dafür diese Annahme zu treffen? Welche Alternativen gibt es?
- Inwiefern ist eine pseudozufällige Funktion mit kryptographischen Ununterscheidbarkeitsgarantien nützlich für uns? Wie ist der Zusammenhang zur SUHA?*
- Universelles Hashing:
  - Wie ist c-Universalität definiert?
  - Welche c-universellen Hashklasse haben wir kennengelernt? Wie haben wir die c-Universalität bewiesen?
  - Wie ist d-Unabhängigkeit für eine Hashklasse definiert?
  - Welche d-universelle Hashklasse haben wir kennengelernt?
  - Welcher Zusammenhang besteht zwischen d-Unabhängigkeit und c-Universalität? (Übungsaufgabe)
  - Chernoff Schranken sind für Summen unabhängiger Zufallsvariablen gedacht. Was kann man machen, wenn die Zufallsvariablen nur d-unabhängig sind?*

Anhang: Mögliche Prüfungsfragen II

- Betrachten wir Hashing mit verketteten Listen:
  - Welche Schranke an die erwartete Einfügezeit haben wir bewiesen? Wie?
  - An welcher Stelle spielt die Verteilung der Hashfunktion eine Rolle?
  - Nenne eine hinreichende Eigenschaft, die eine universelle Hashklasse haben sollte, damit der Beweis funktioniert.
- Betrachten wir Hashing mit linearem Sondieren:
  - Welche Schranke an die erwartete Laufzeit haben wir bewiesen? Wie?
  - An welcher Stelle spielt die Verteilung der Hashfunktion eine Rolle?
  - Nenne eine hinreichende Eigenschaft, die eine universelle Hashklasse haben sollte, damit der Beweis funktioniert.
  - Wie wir diese Eigenschaft ausgenutzt?*

References I

[1] Anna Pagh, Rasmus Pagh, and Milan Ruzic. "Linear Probing with 5-wise Independence". In: *SIAM Rev.* 53.3 (2011), pp. 547–558. DOI: 10.1137/110827831. URL: https://doi.org/10.1137/110827831.

[2] Mihai Pătraşcu and Mikkel Thorup. "On the *k*-Independence Required by Linear Probing and Minwise Independence". In: *ACM Trans. Algorithms* 12.1 (2016), 8:1–8:27. DOI: 10.1145/2716317. URL: https://doi.org/10.1145/2716317.

[3] Mihai Puatracscu and Mikkel Thorup. "Twisted Tabulation Hashing". In: *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2013, New Orleans, Louisiana, USA, January 6-8, 2013*. Ed. by Sanjeev Khanna. SIAM, 2013, pp. 209–228. DOI: 10.1137/1.9781611973105.16. URL: https://doi.org/10.1137/1.9781611973105.16.

[4] Mikkel Thorup. "High Speed Hashing for Integers and Strings". In: *CoRR* abs/1504.06804 (2015). arXiv: 1504.06804. URL: http://arxiv.org/abs/1504.06804.