Karlsruhe Institute of Technology

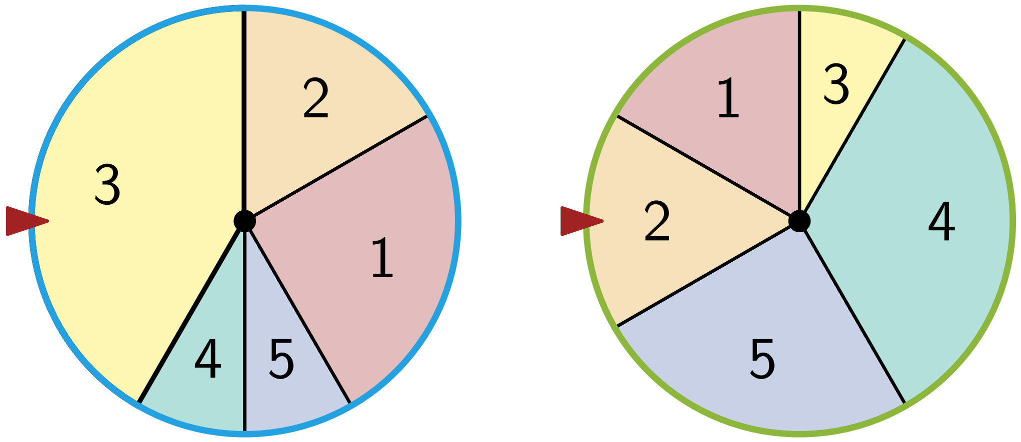
Probability & Computing

Coupling & Erdős-Rényi Random Graphs

Wheels of Fortune

The Problem

- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin?

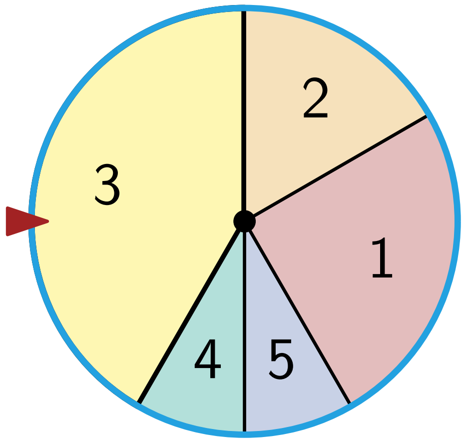


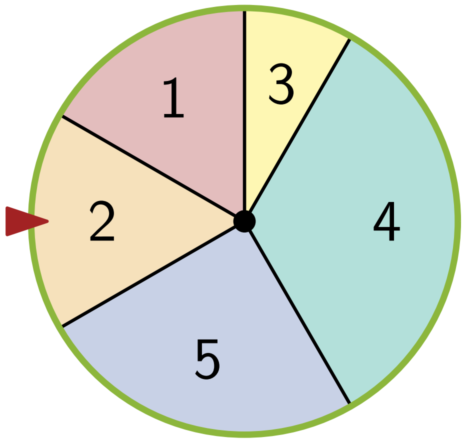
Wheels of Fortune

Karlsruhe Institute of Technology

The Problem

- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?





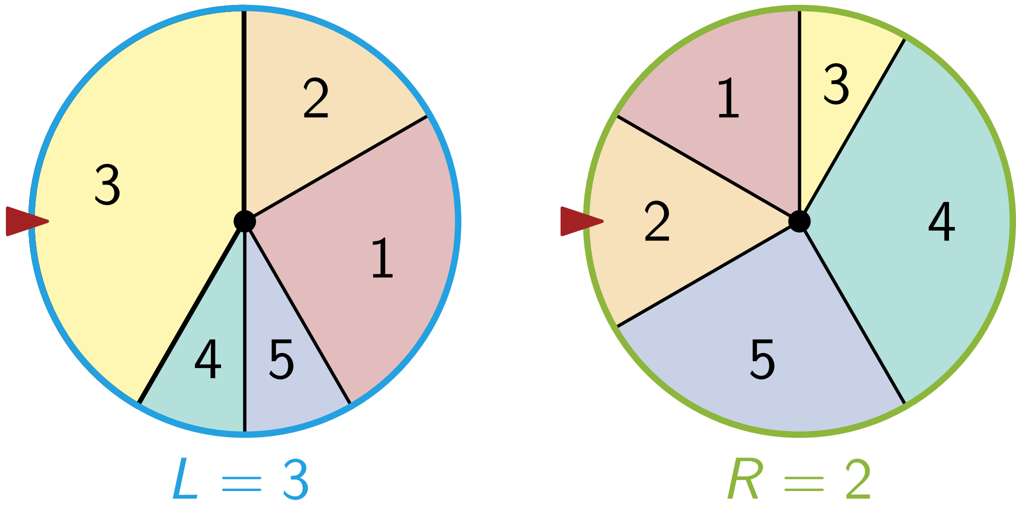
Wheels of Fortune

The Problem

- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel



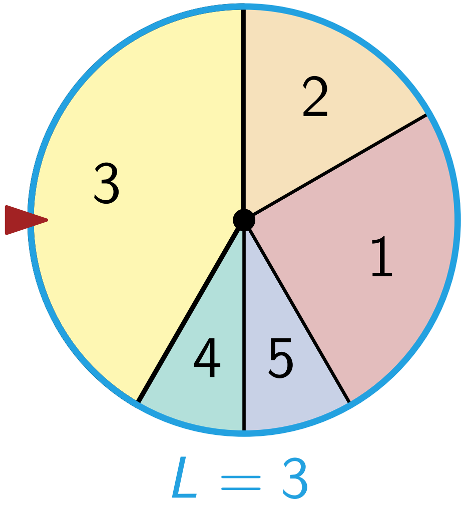
Wheels of Fortune

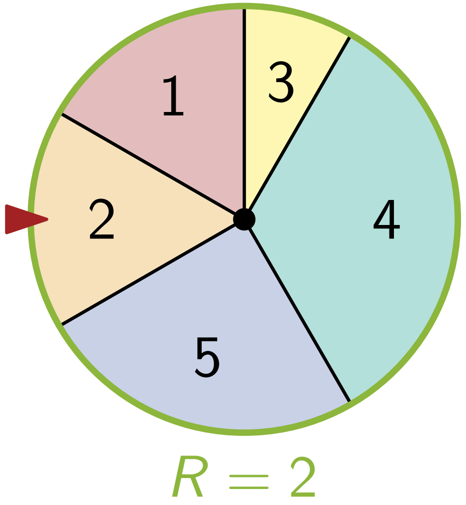
The Problem

- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$





Wheels of Fortune

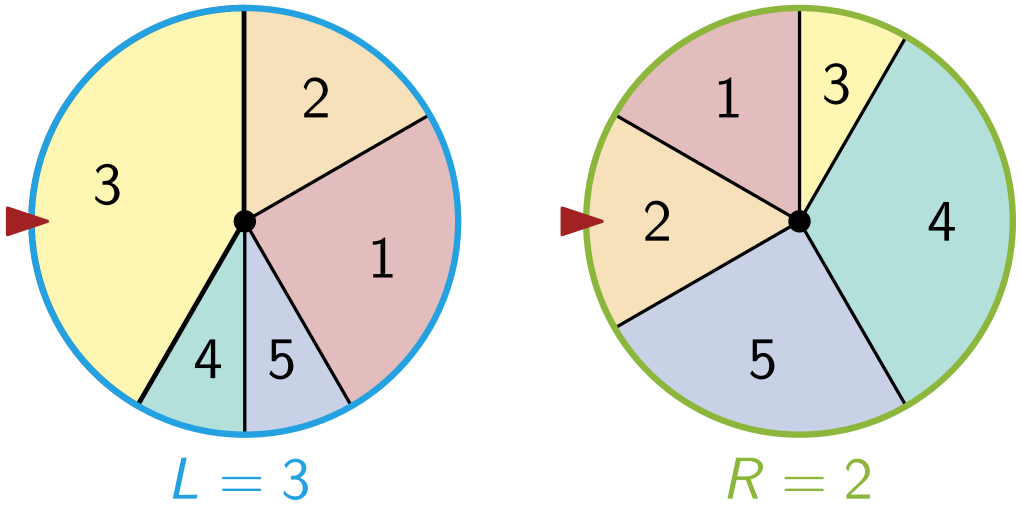
The Problem

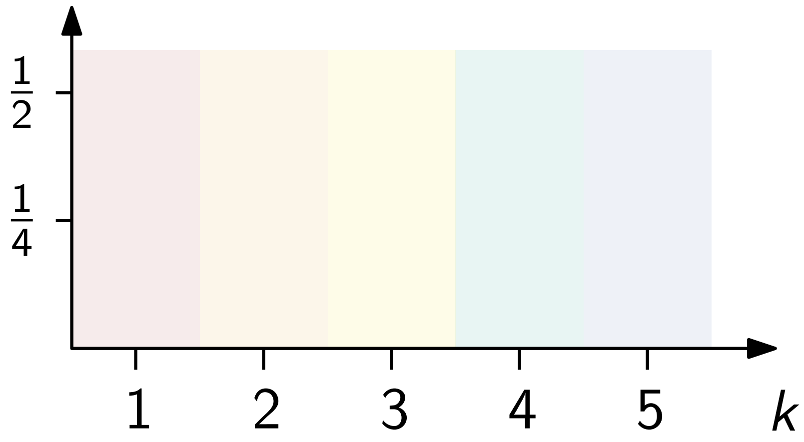
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof



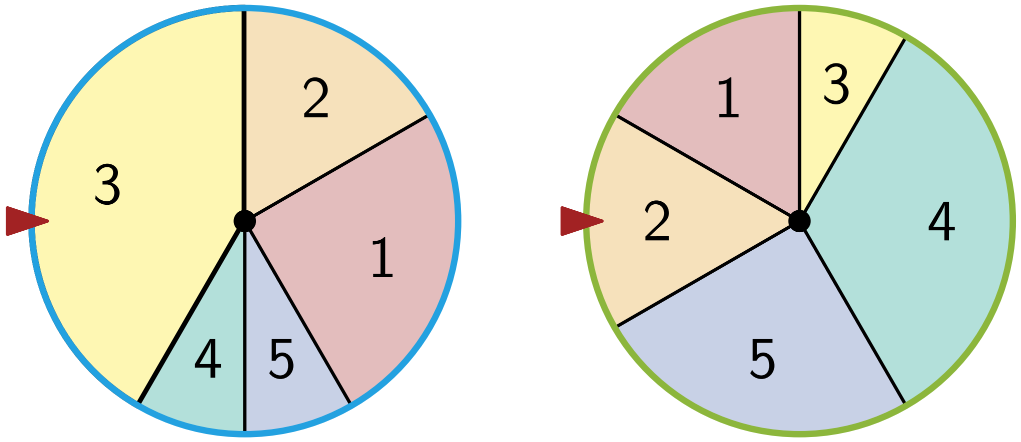


Wheels of Fortune

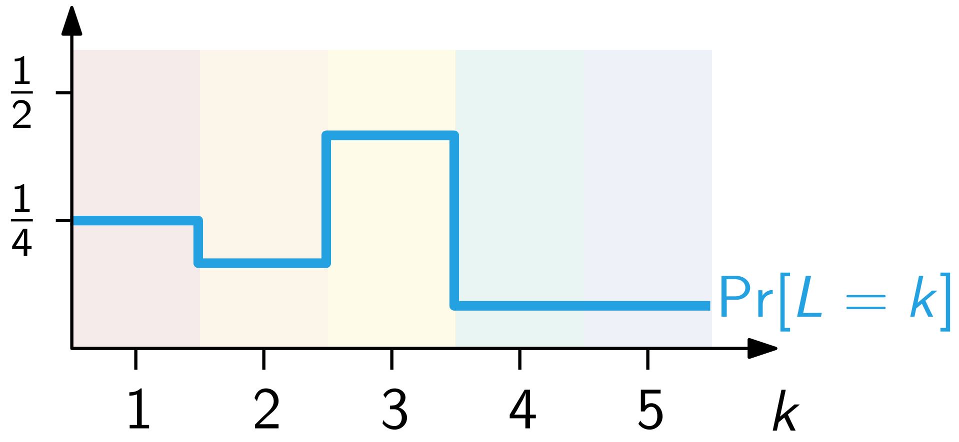
The Problem
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof



$L = 3$ $R = 2$

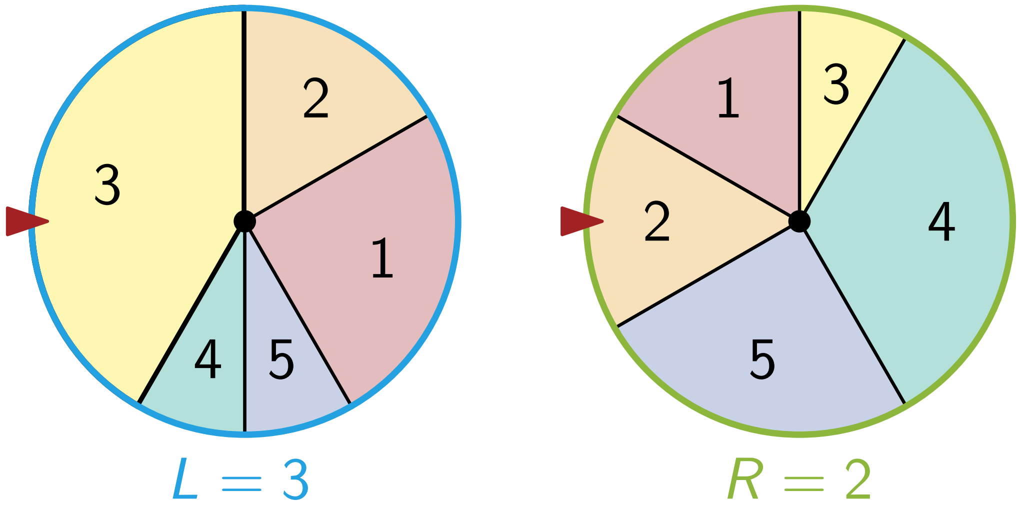


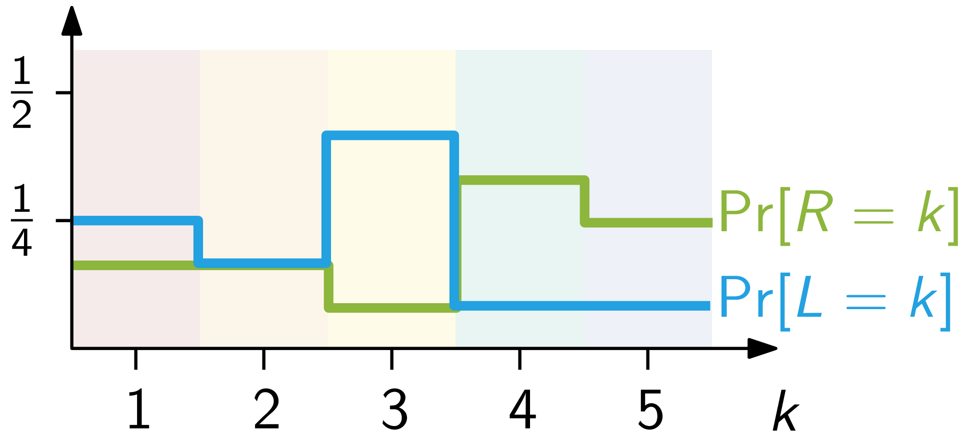
Wheels of Fortune

The Problem
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof





Wheels of Fortune

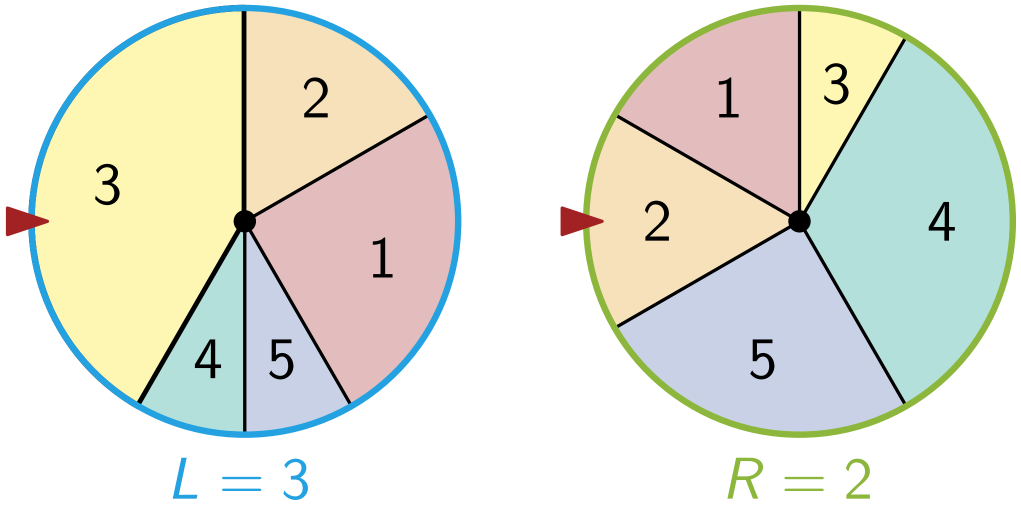
The Problem

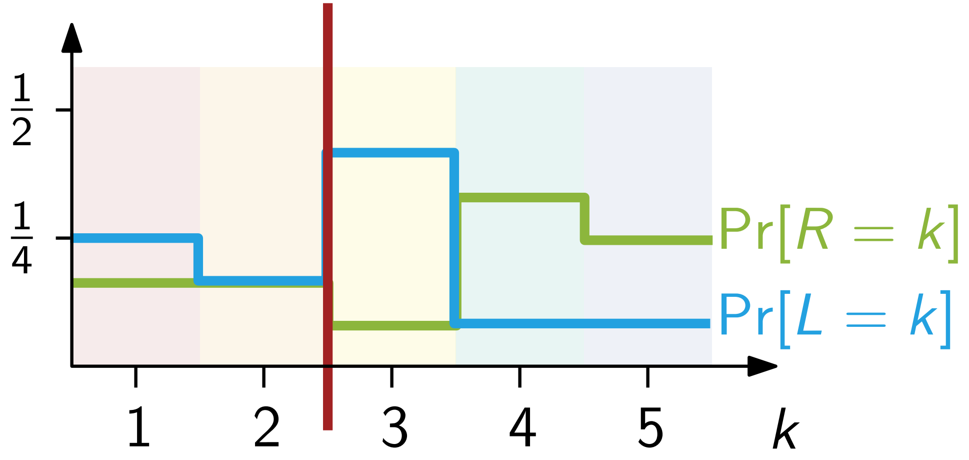
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof





Wheels of Fortune

Karlsruhe Institute of Technology

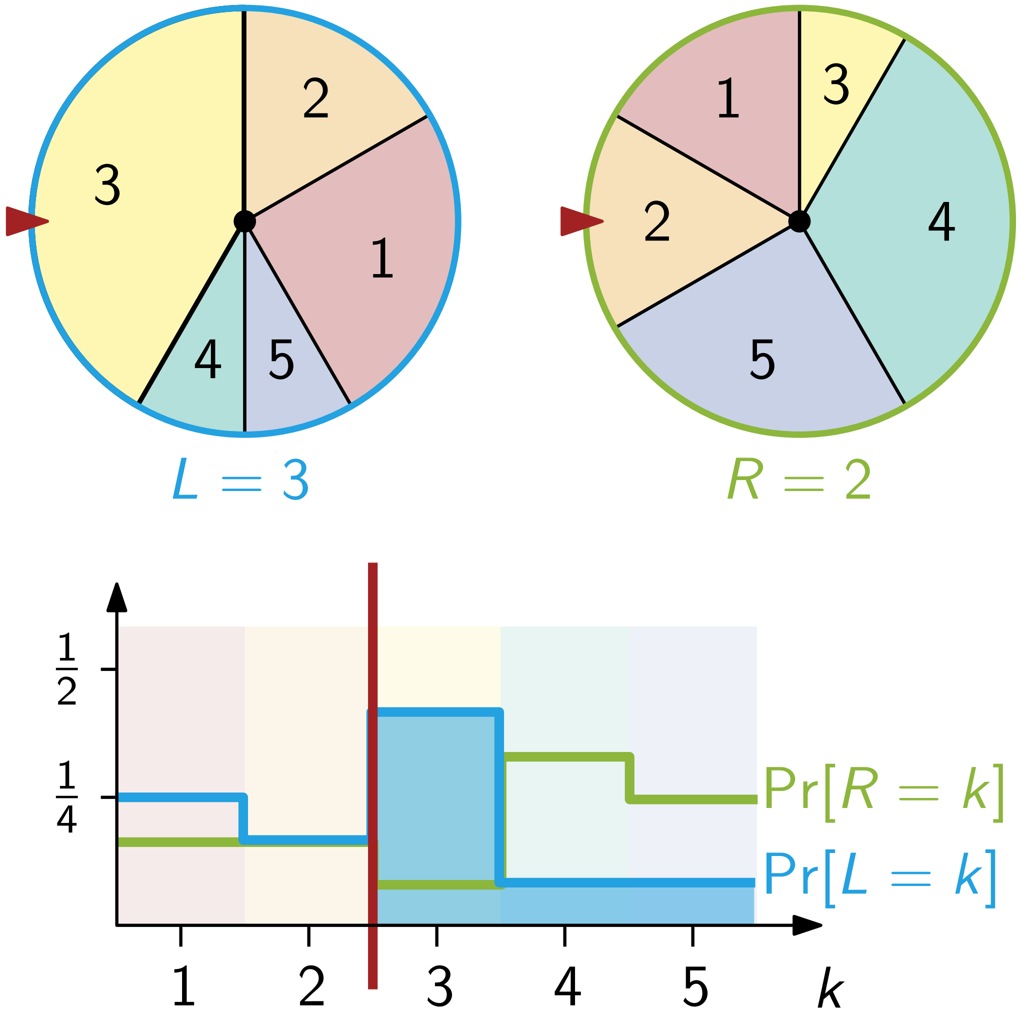
The Problem

- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof



Wheels of Fortune

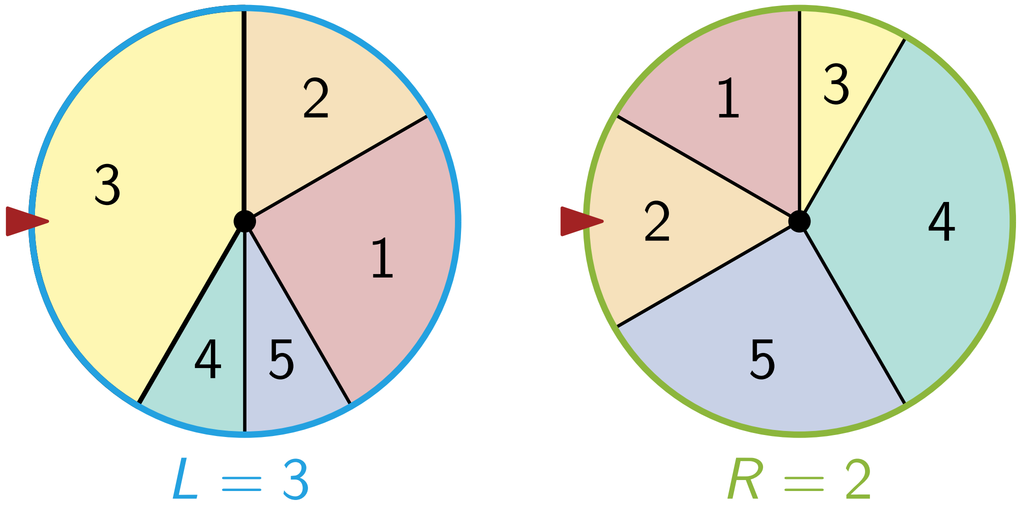
The Problem

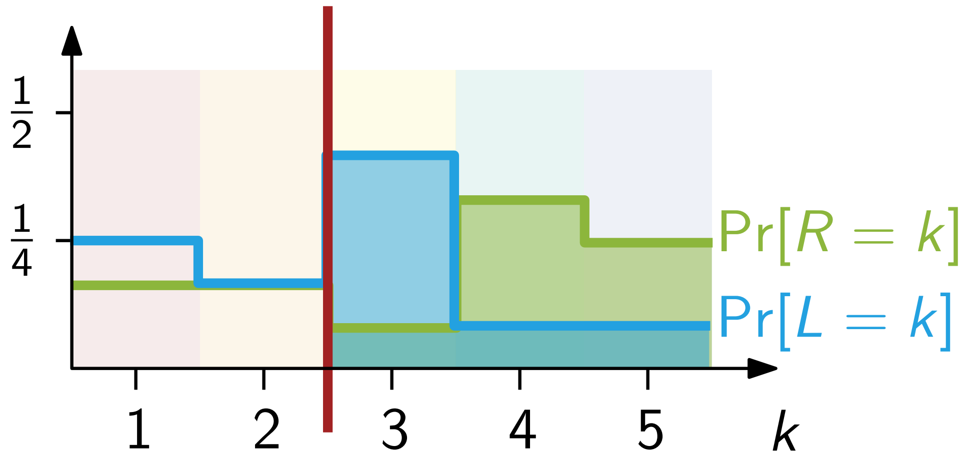
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof



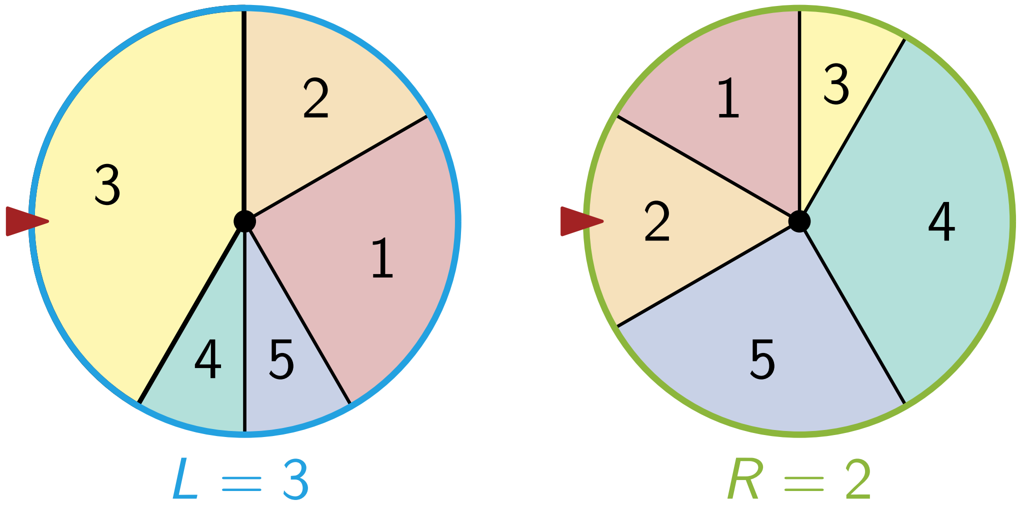


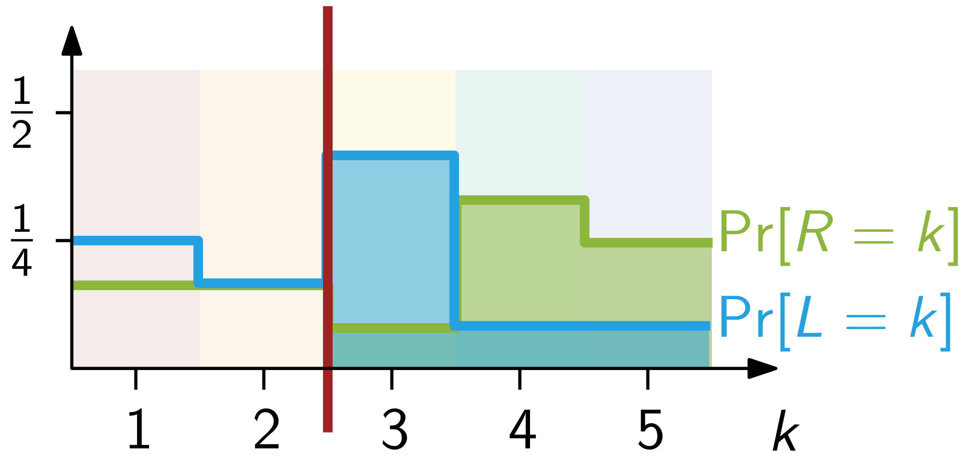
Wheels of Fortune

The Problem
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof
- For each k
  - Compute the sums of the probabilities



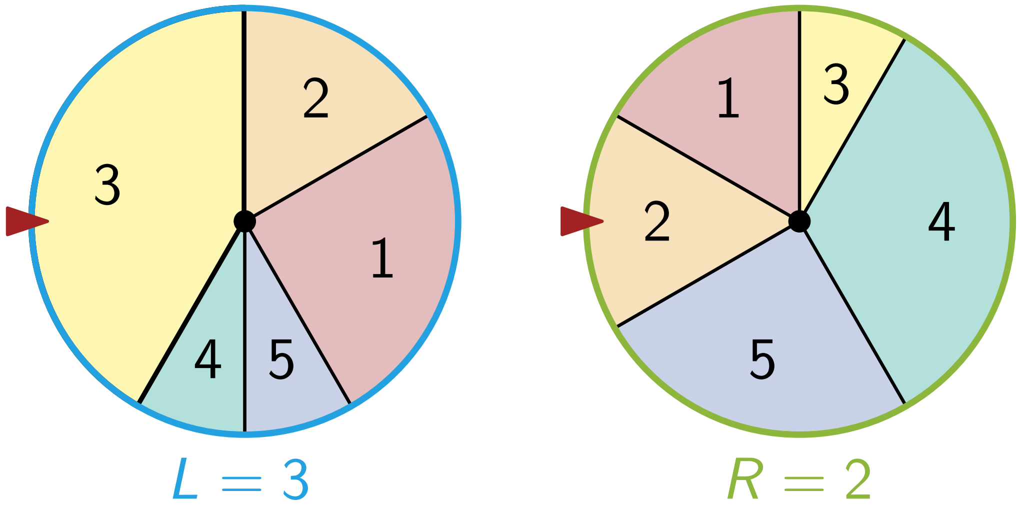


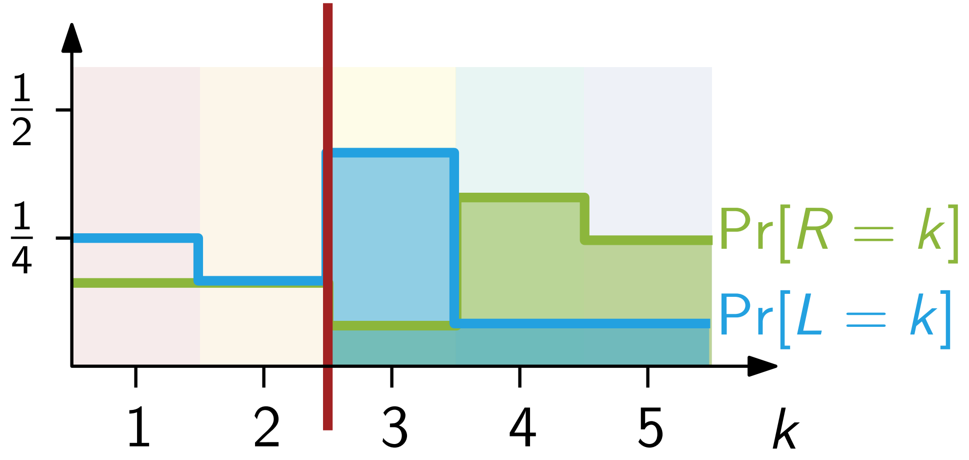
Wheels of Fortune

The Problem
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof
- For each k
  - Compute the sums of the probabilities
  - Compare





Wheels of Fortune

The Problem

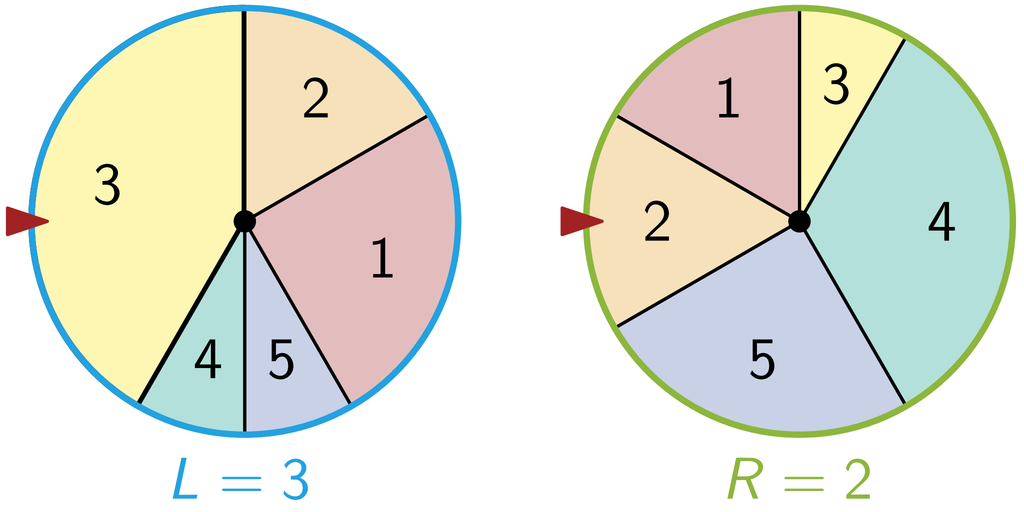
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

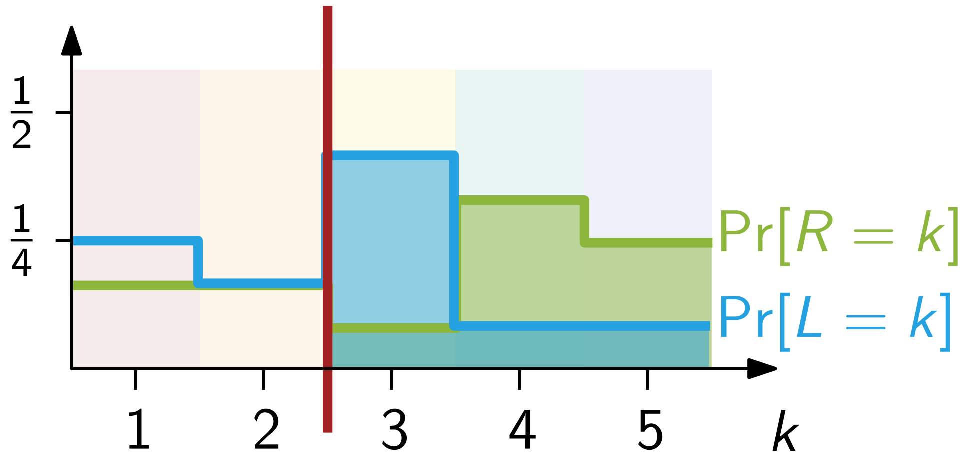
The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof

- For each k
  - Compute the sums of the probabilities
  - Compare
  - Tedious...





Wheels of Fortune

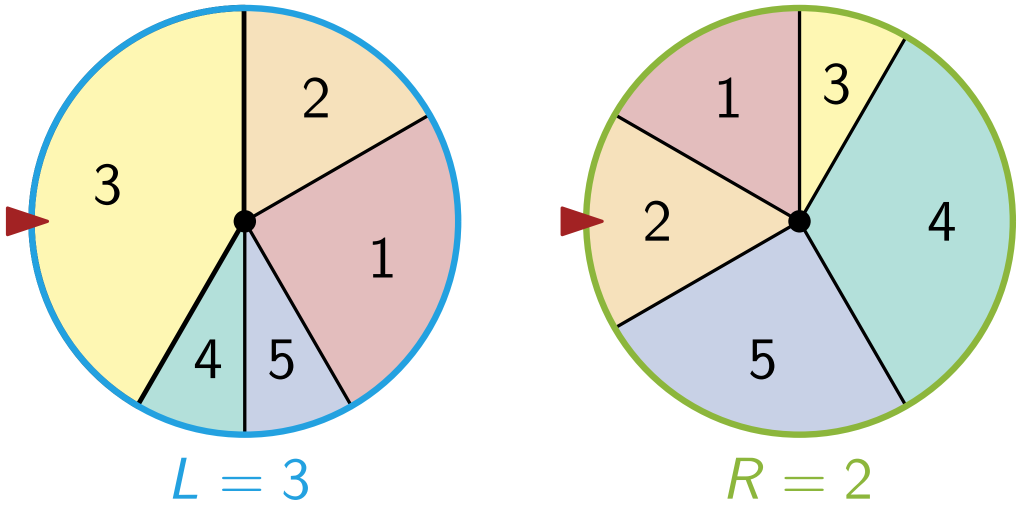
The Problem

- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!



Wheels of Fortune

The Problem

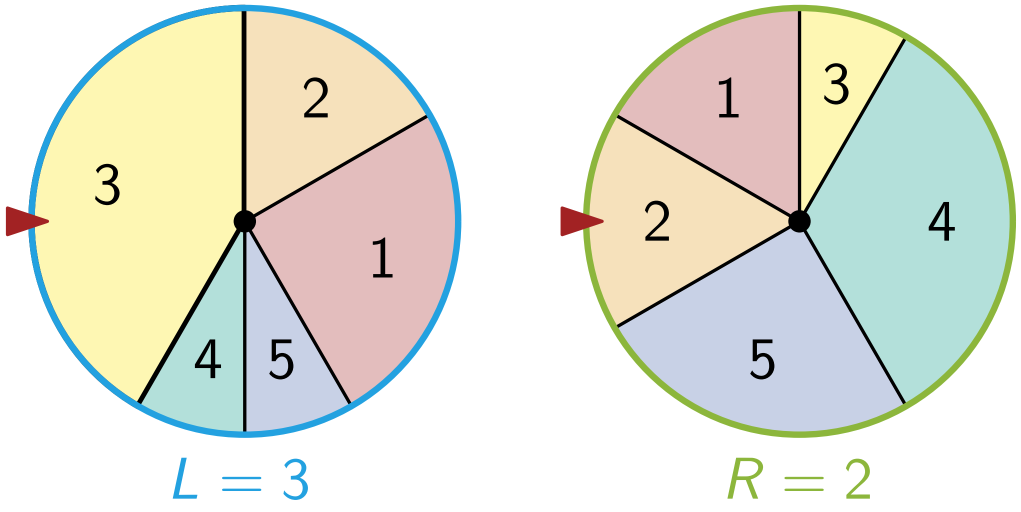
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels



Wheels of Fortune

The Problem

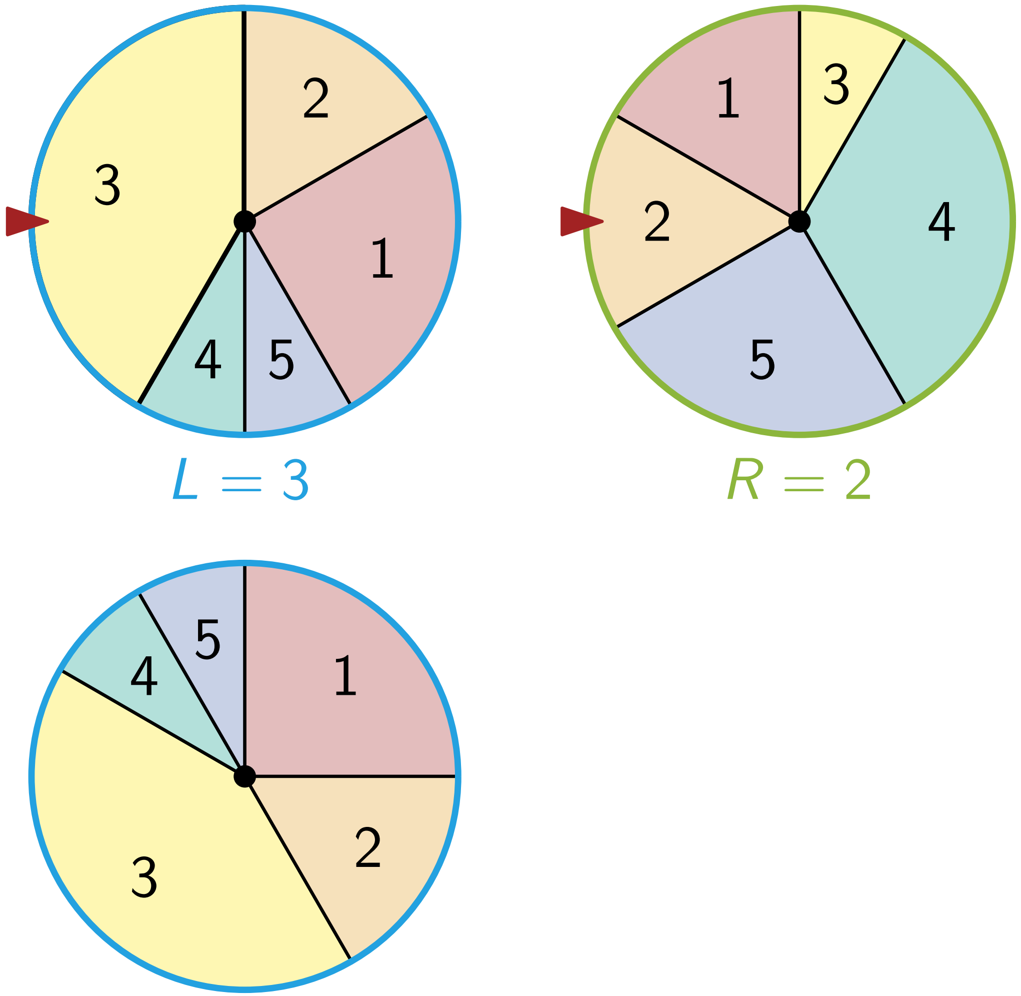
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels



Wheels of Fortune

The Problem

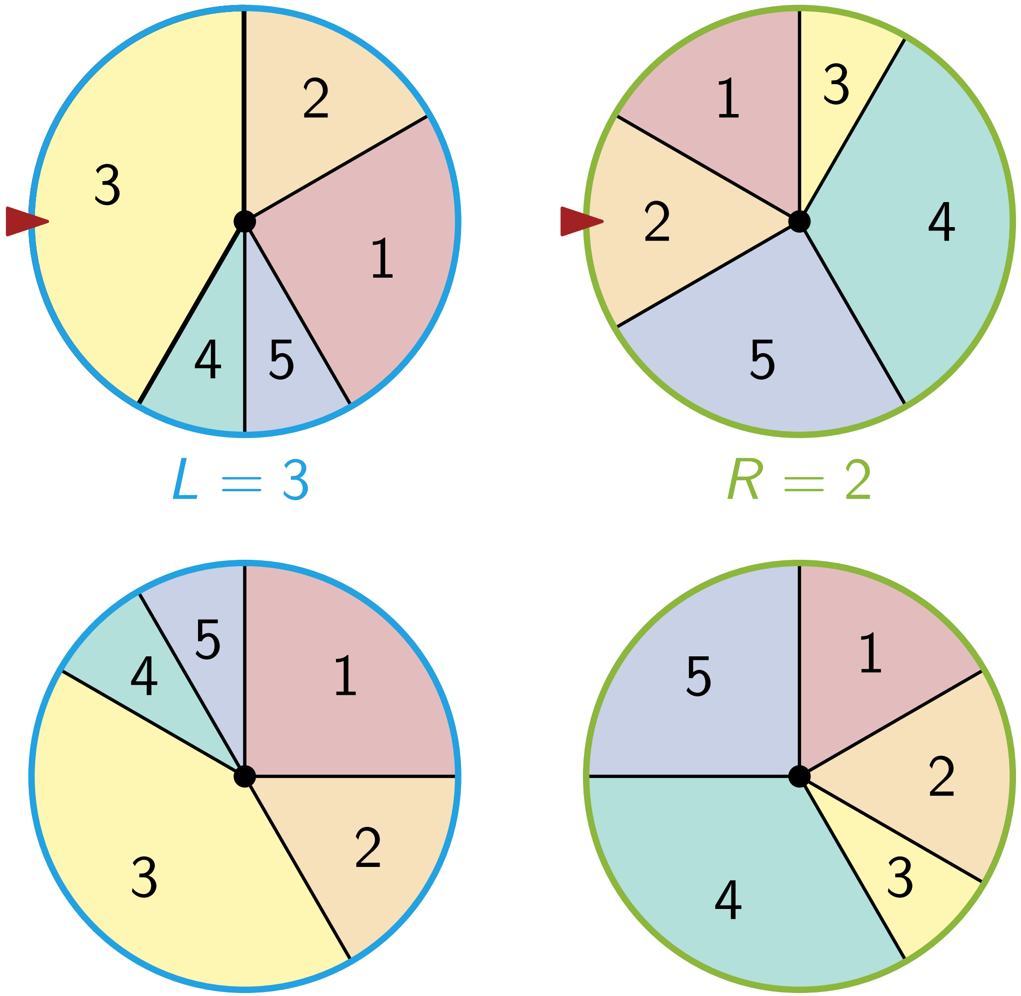
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels



Wheels of Fortune

The Problem

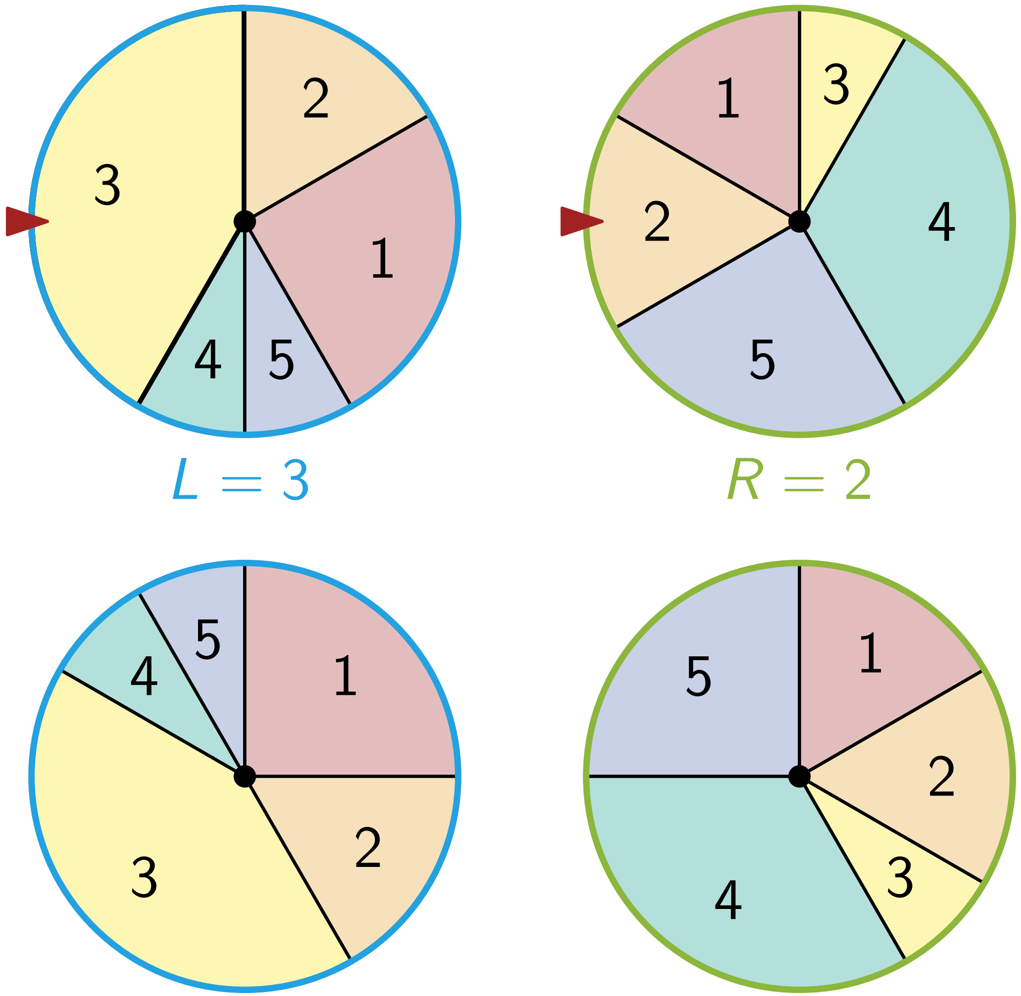
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels   (does not change their distributions)

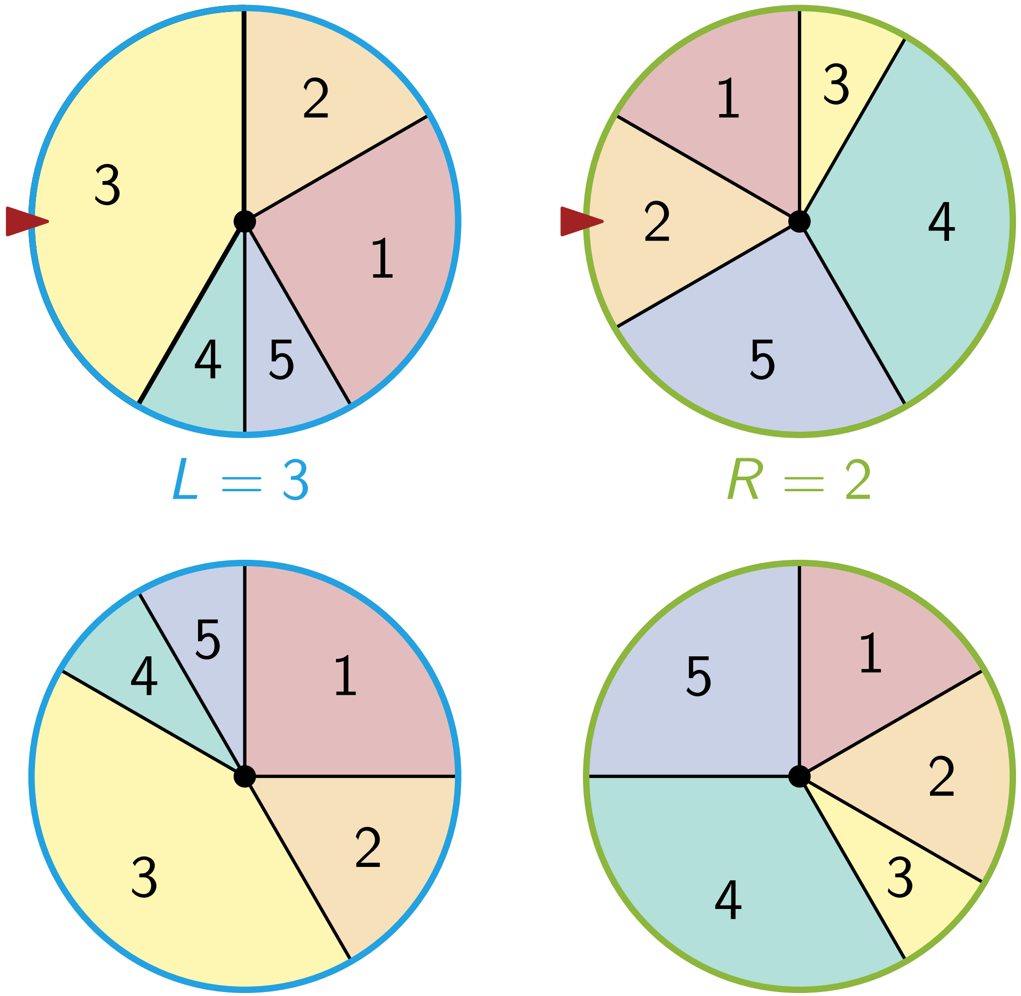


Wheels of Fortune

The Problem
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!
- Sort the wheels (does not change their distributions)
- Adjust sizes



Wheels of Fortune

The Problem

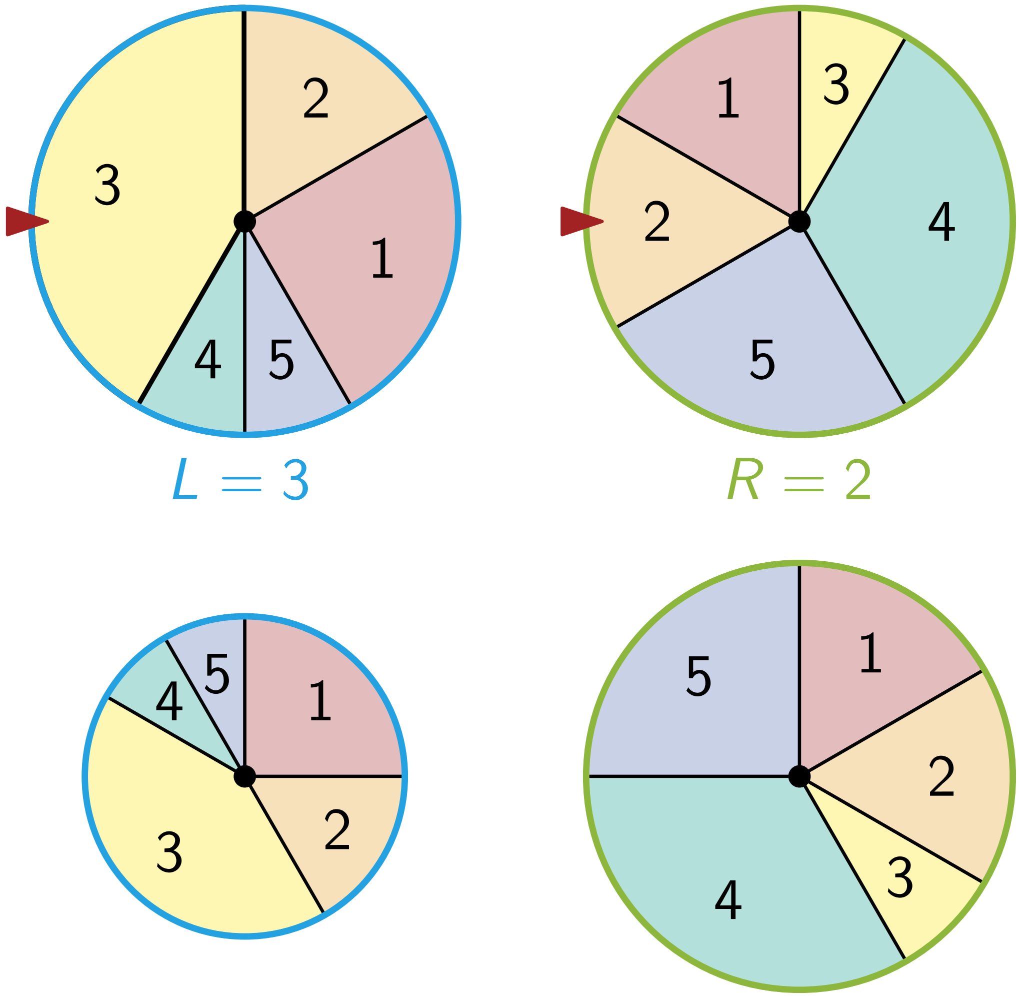
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes



Wheels of Fortune

The Problem

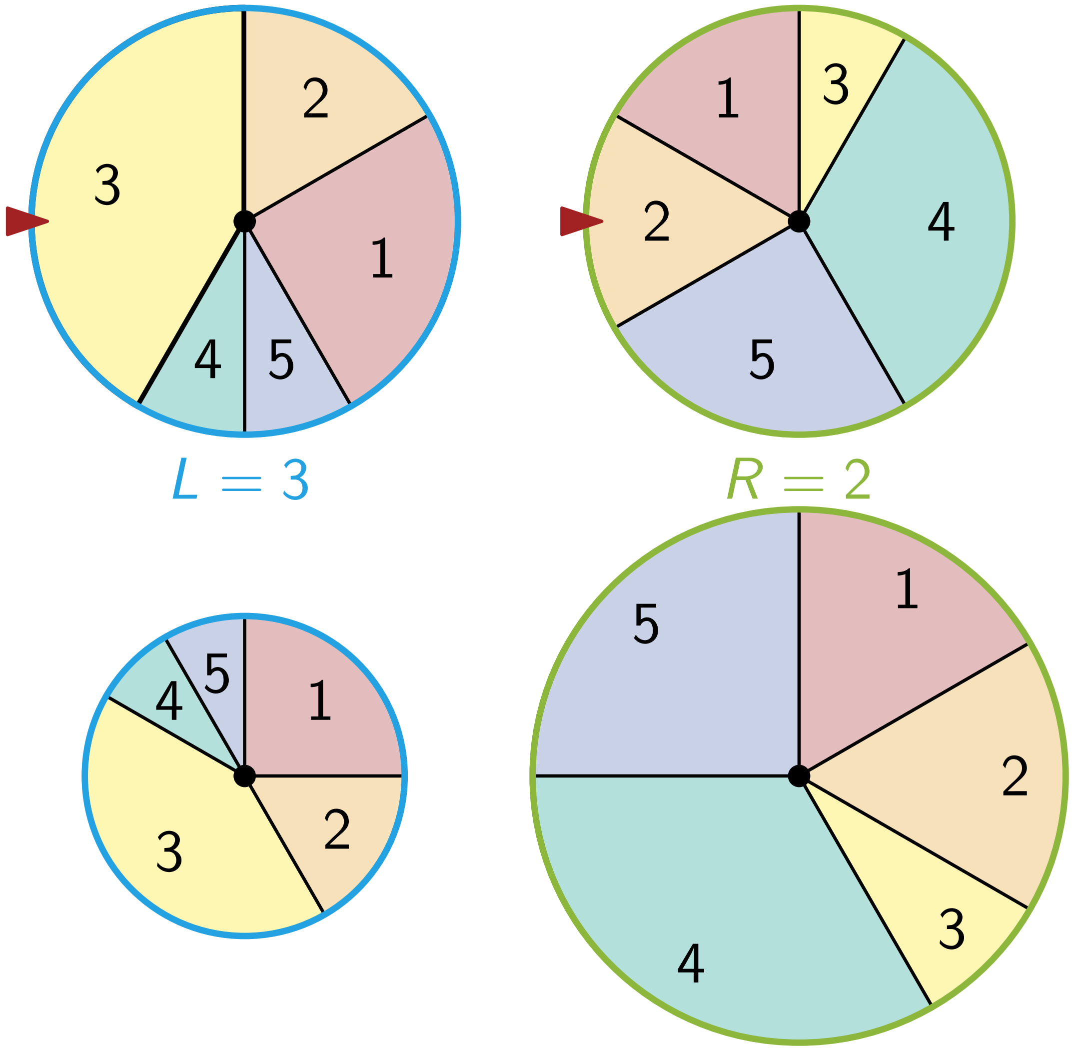
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels   (does not change their distributions)
- Adjust sizes



Wheels of Fortune

The Problem

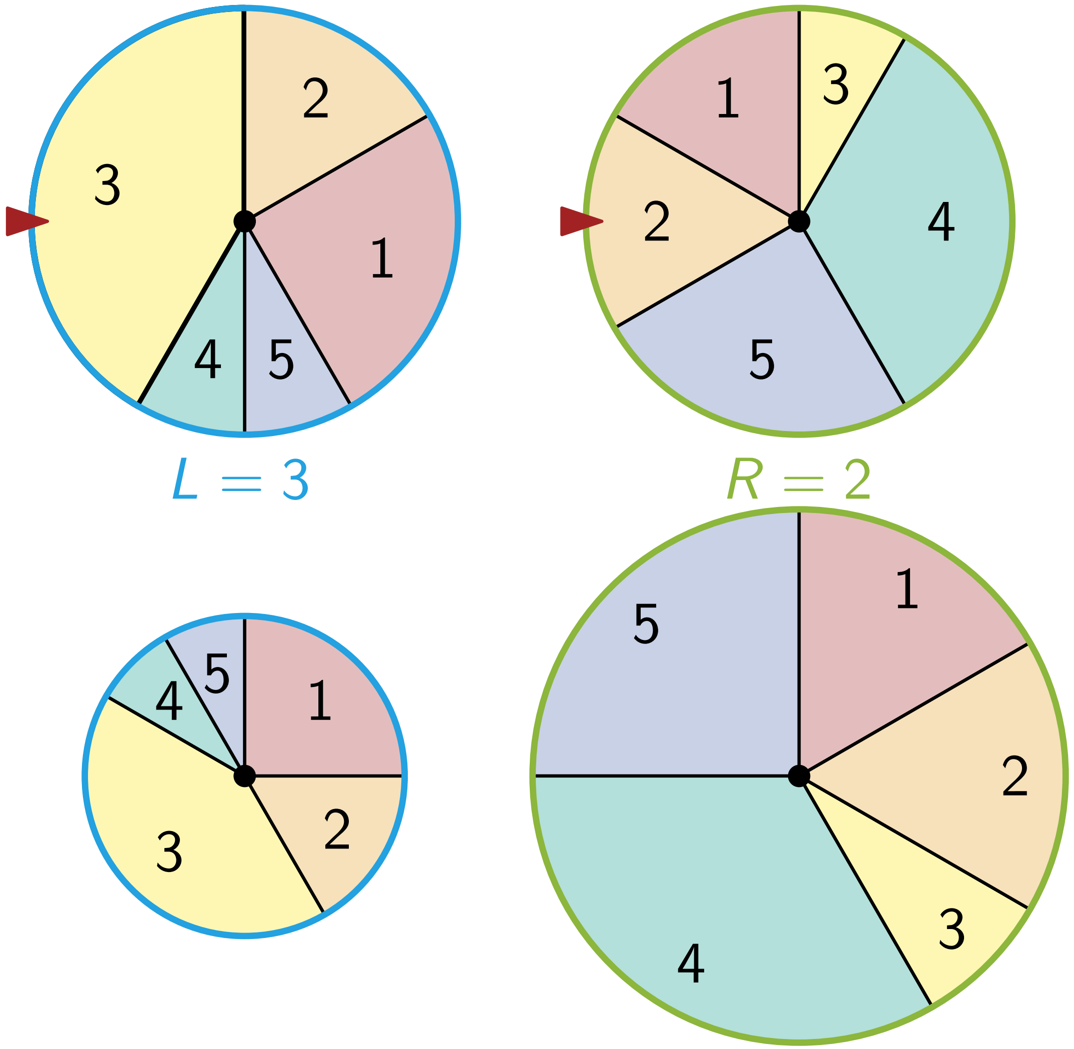
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels   (does not change their distributions)
- Adjust sizes and glue together



Wheels of Fortune

The Problem

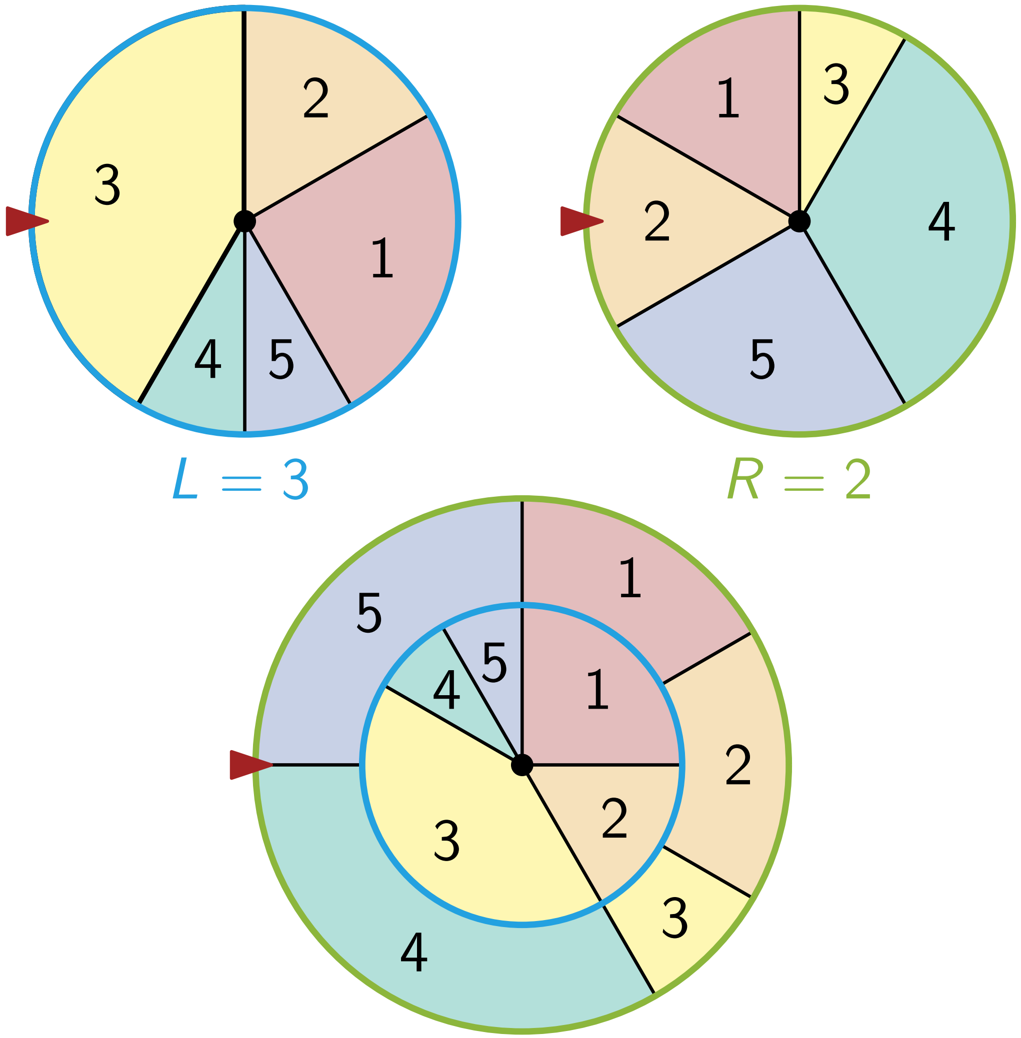
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together



Wheels of Fortune

The Problem

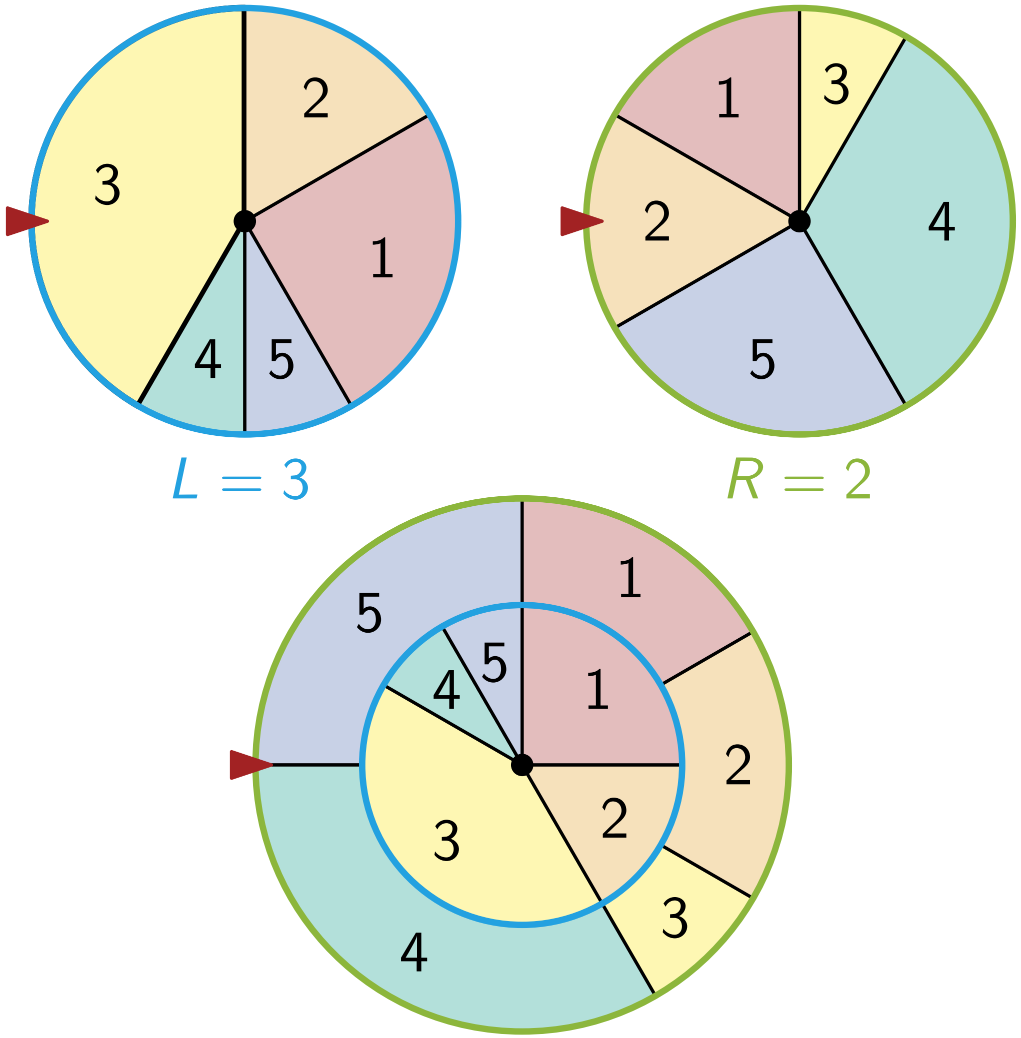
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel



Wheels of Fortune

The Problem

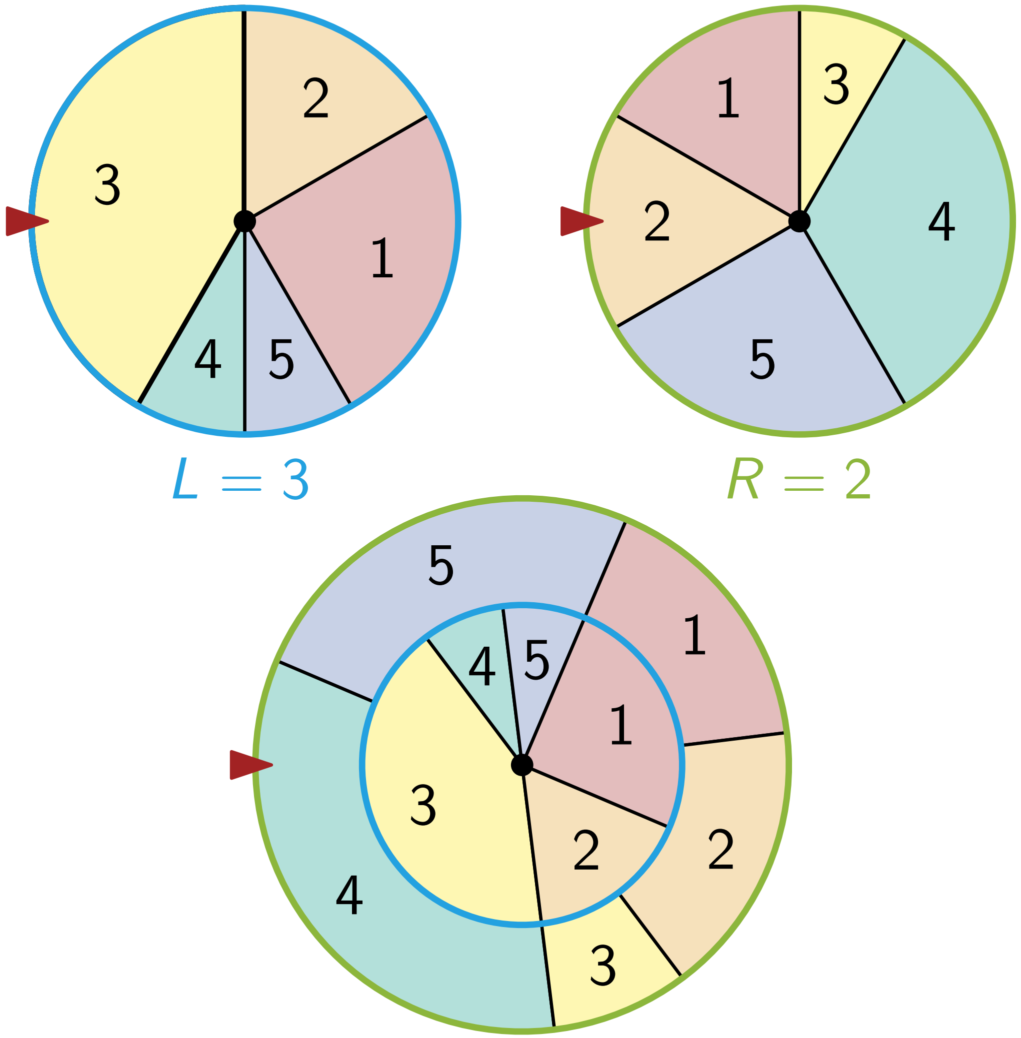
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel



Wheels of Fortune

The Problem

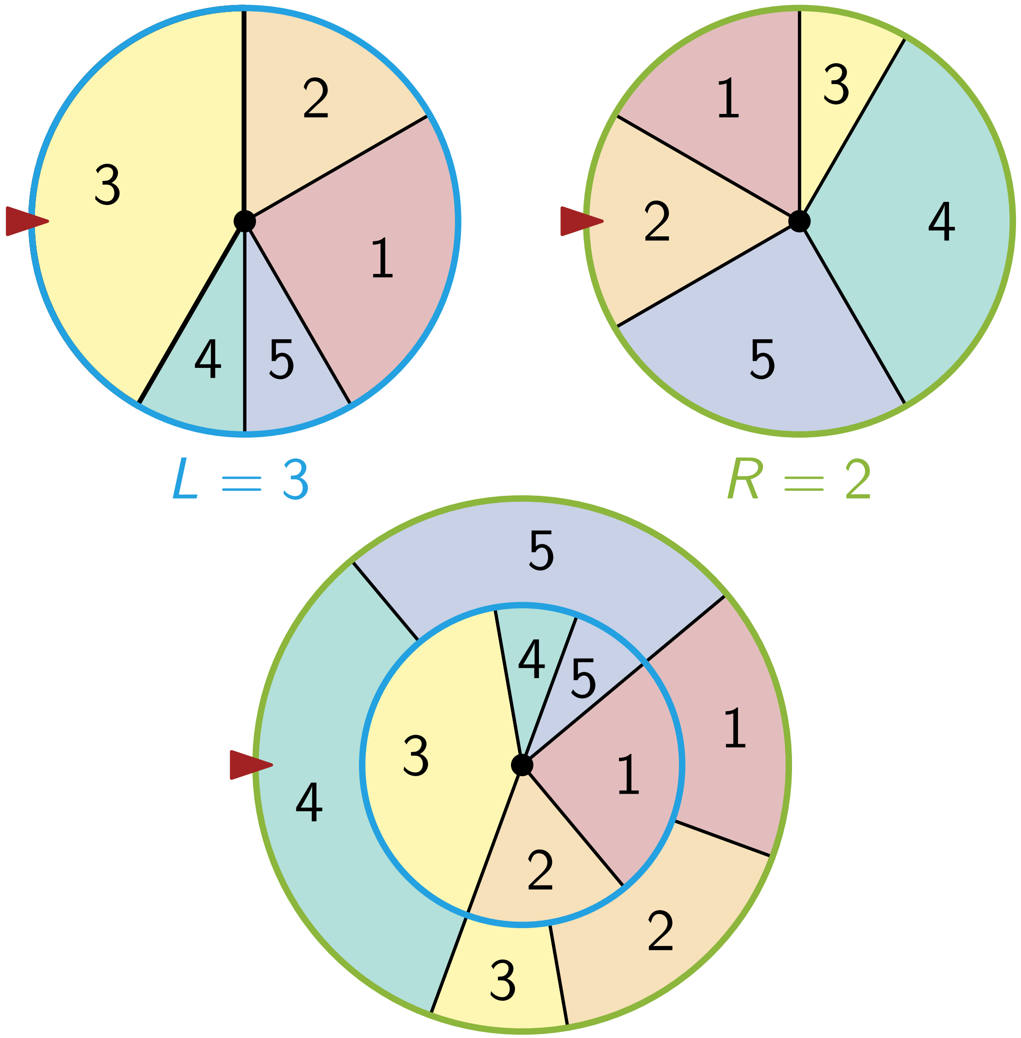
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel



Wheels of Fortune

The Problem

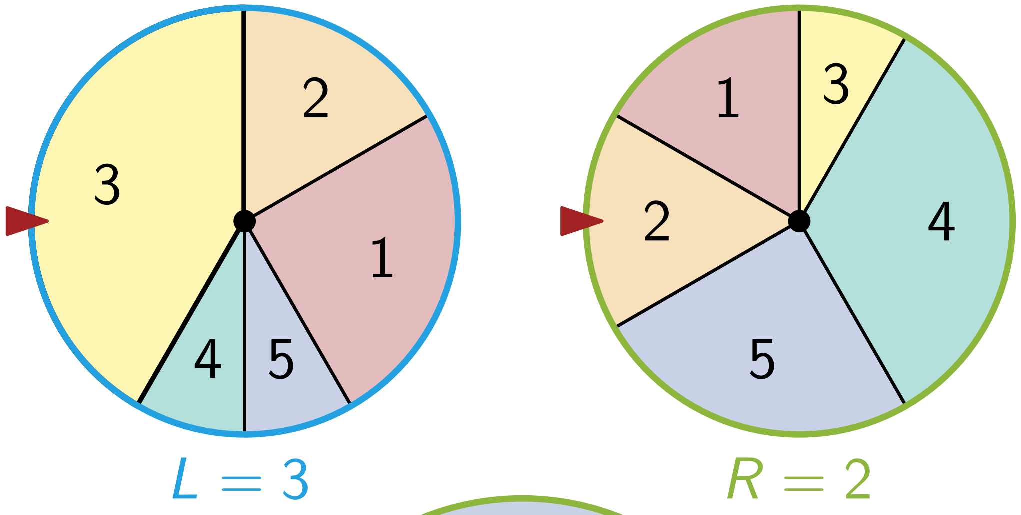
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

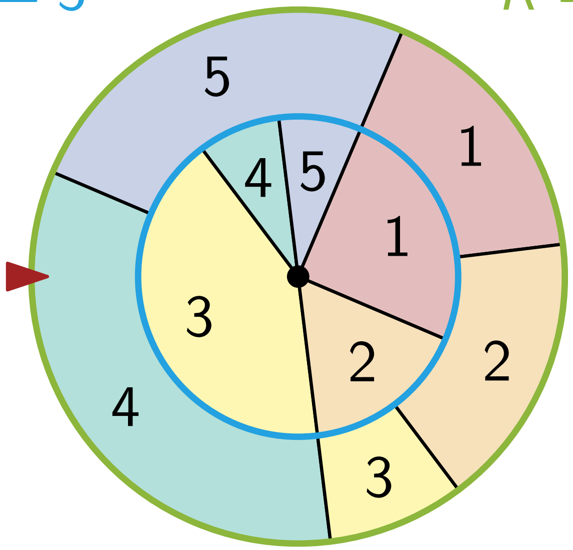
The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel



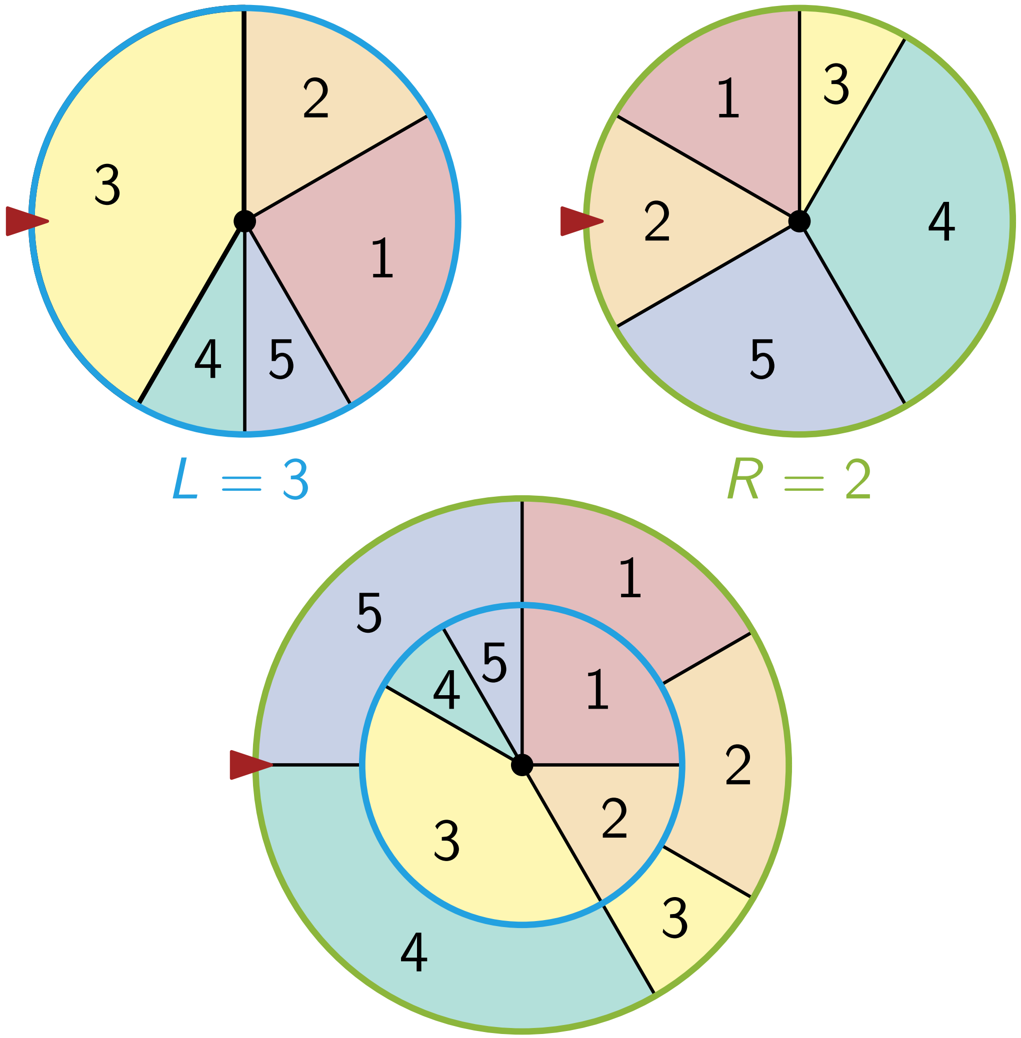


Wheels of Fortune

The Problem
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!
- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel



Wheels of Fortune

The Problem

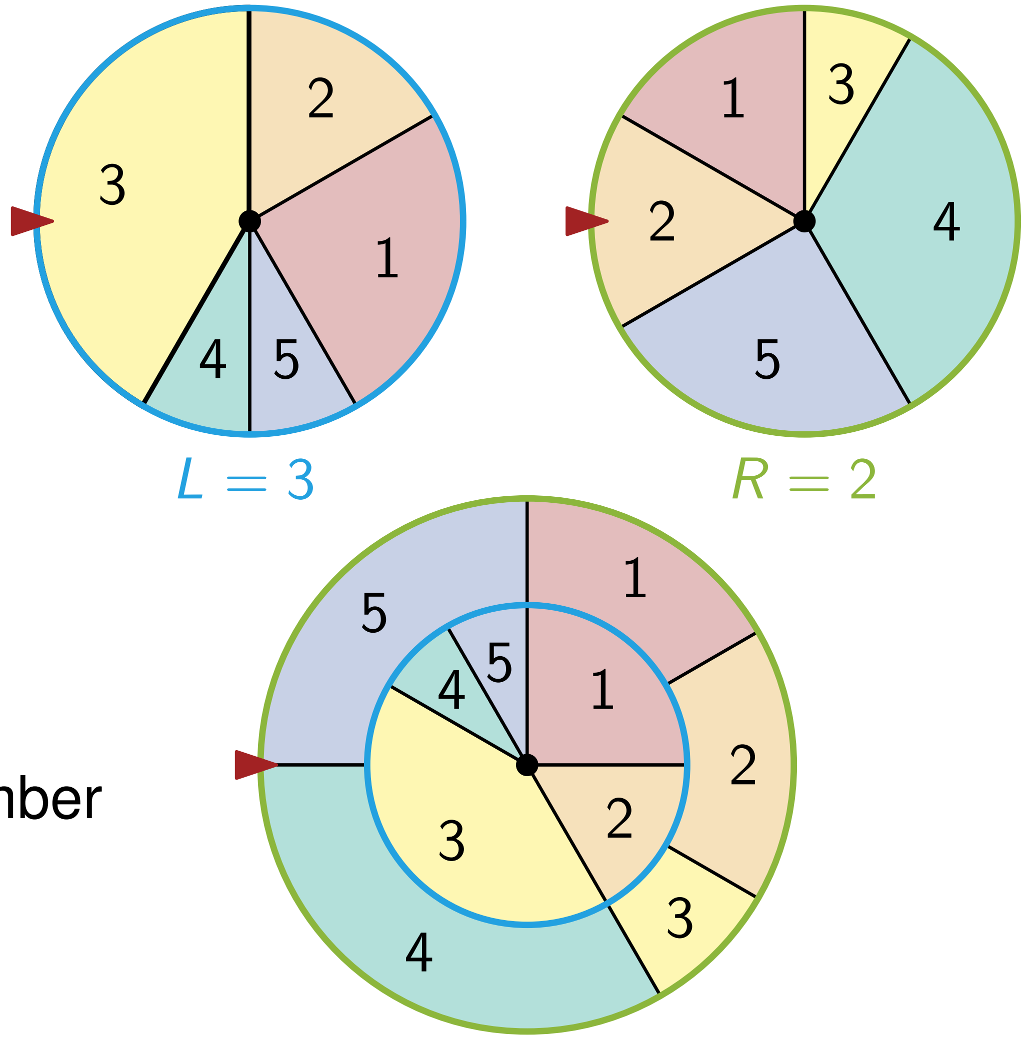
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number



Wheels of Fortune

The Problem

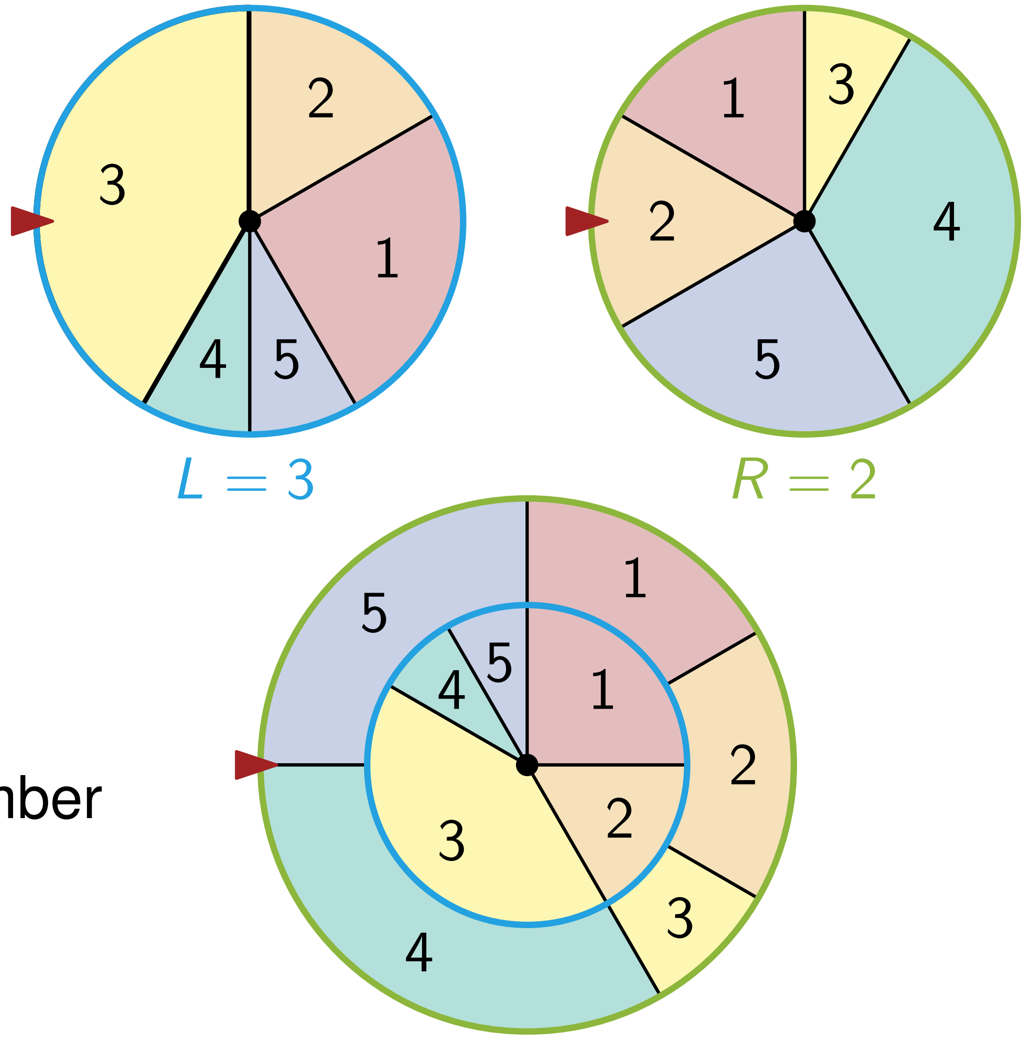
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \overset{d}{=} L'$ and $R \overset{d}{=} R'$ equal distributions



Wheels of Fortune

The Problem

- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

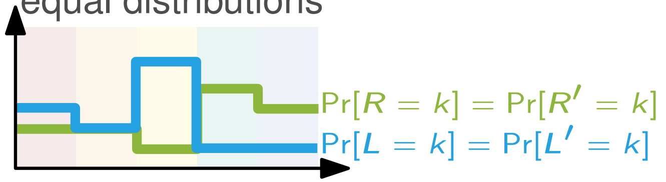
The Maths

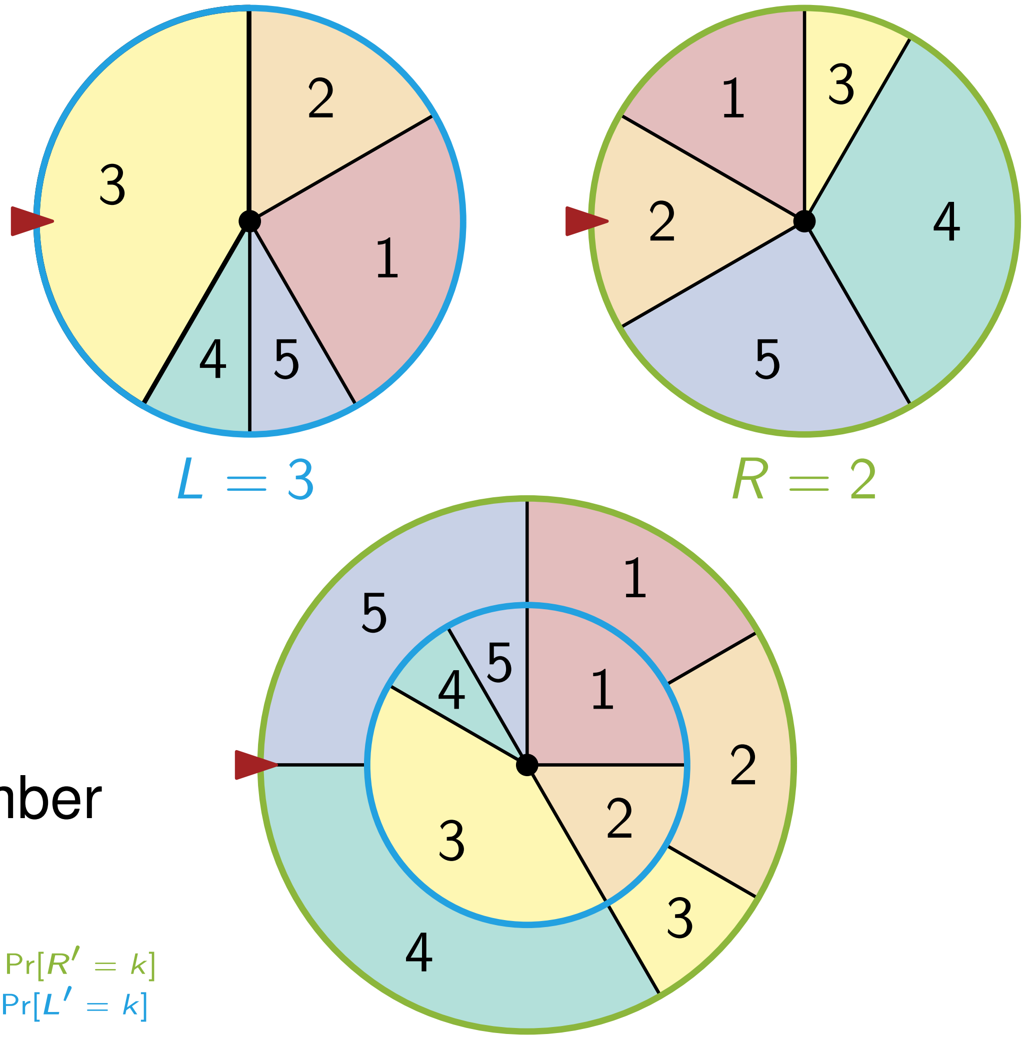
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels   (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \overset{d}{=} L'$ and $R \overset{d}{=} R'$

equal distributions



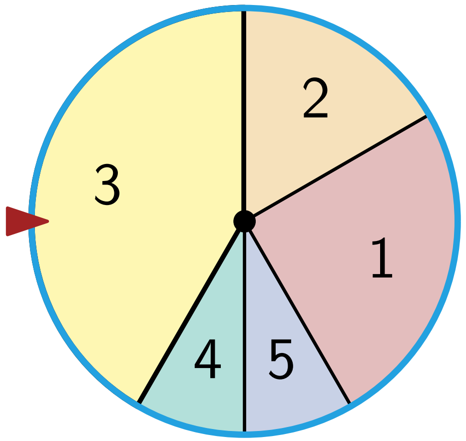


Wheels of Fortune

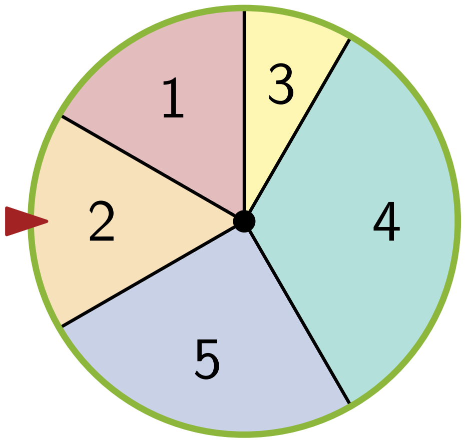
The Problem
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

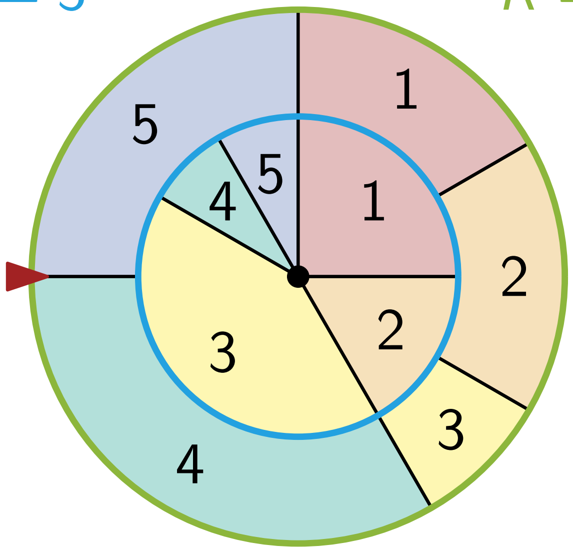
Proof: Frankenstein's Wheel of Fortune!
- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \overset{d}{=} L'$ and $R \overset{d}{=} R'$ equal distributions
- But L' and R' are dependent and *always* $R' \geq L'$



$L = 3$



$R = 2$



Wheels of Fortune

The Problem

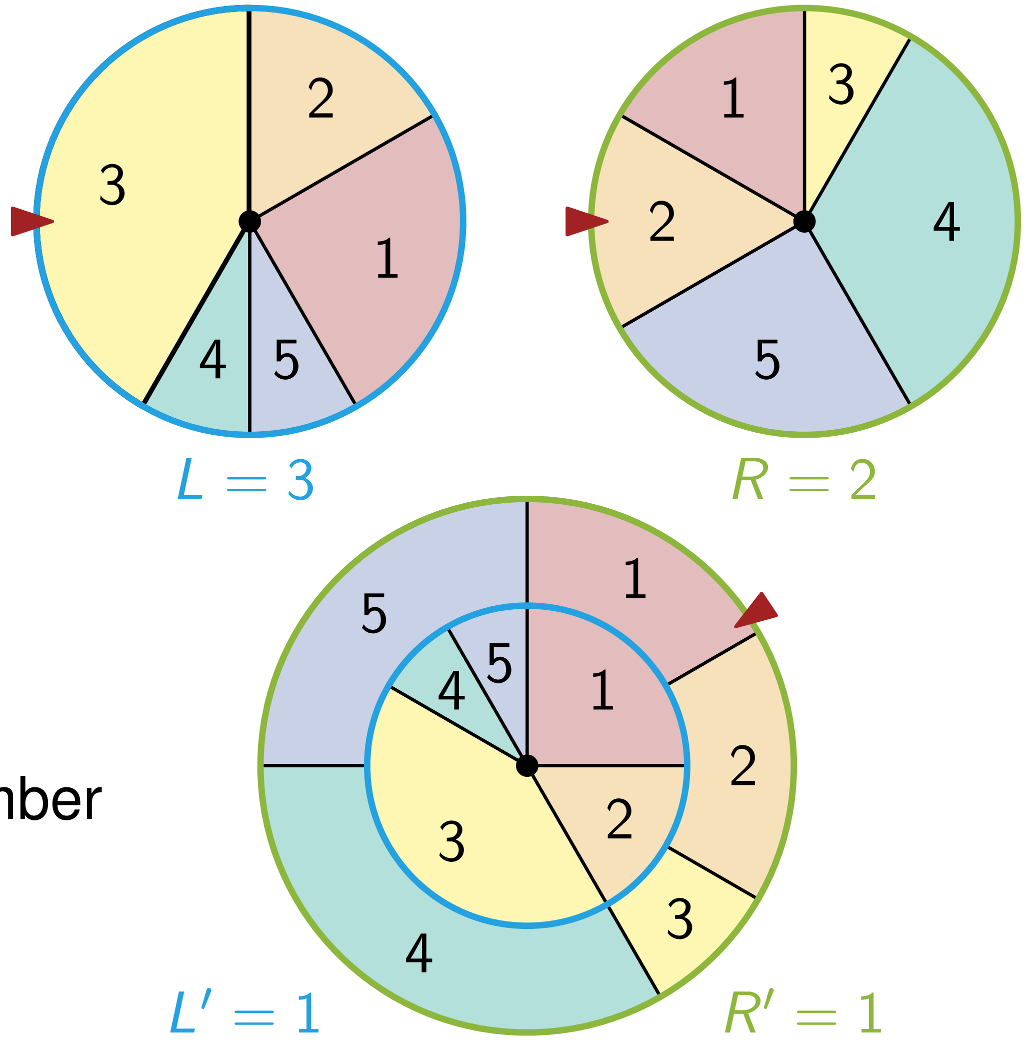
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels   (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \overset{d}{=} L'$ and $R \overset{d}{=} R'$ equal distributions
- But L' and R' are dependent and *always* $R' \geq L'$

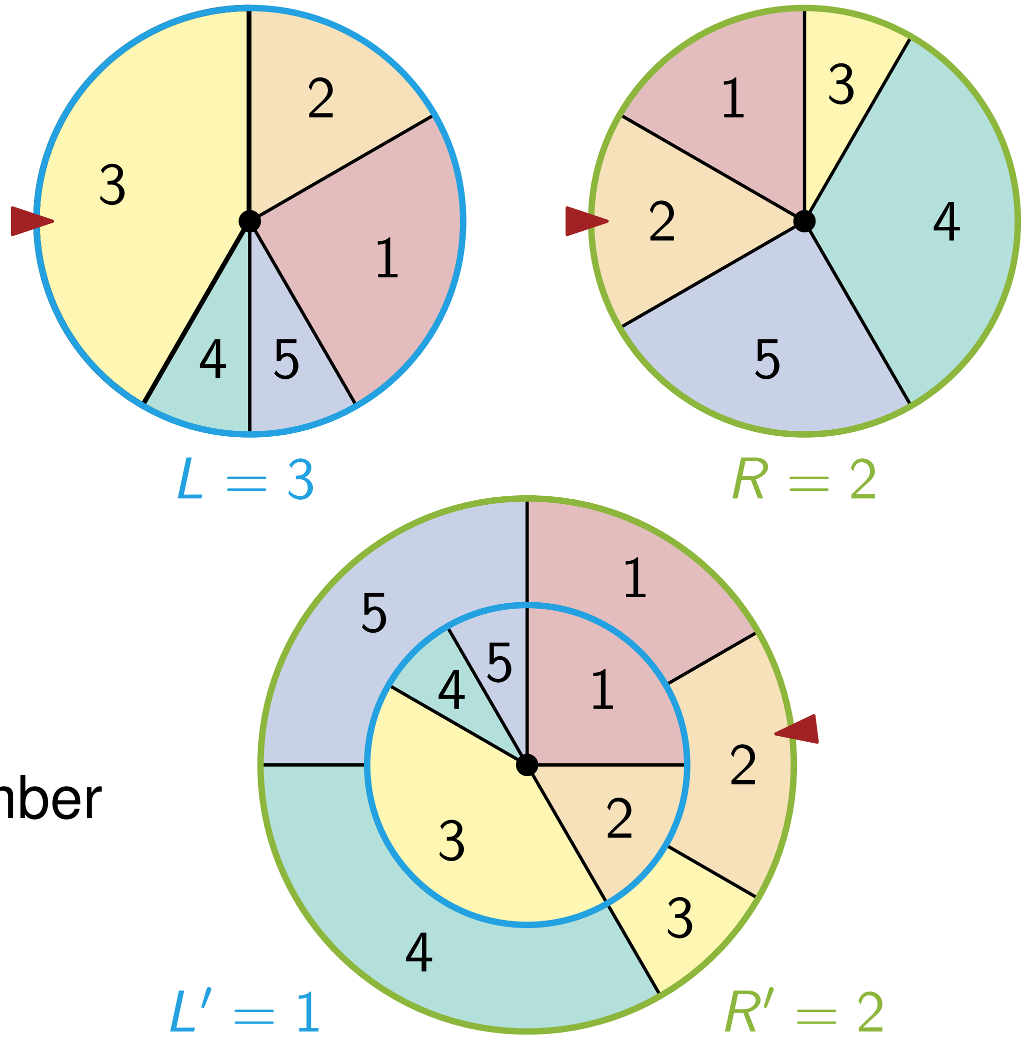


Wheels of Fortune

The Problem
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!
- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \stackrel{d}{=} L'$ and $R \stackrel{d}{=} R'$ equal distributions
- But L' and R' are dependent and *always* $R' \geq L'$

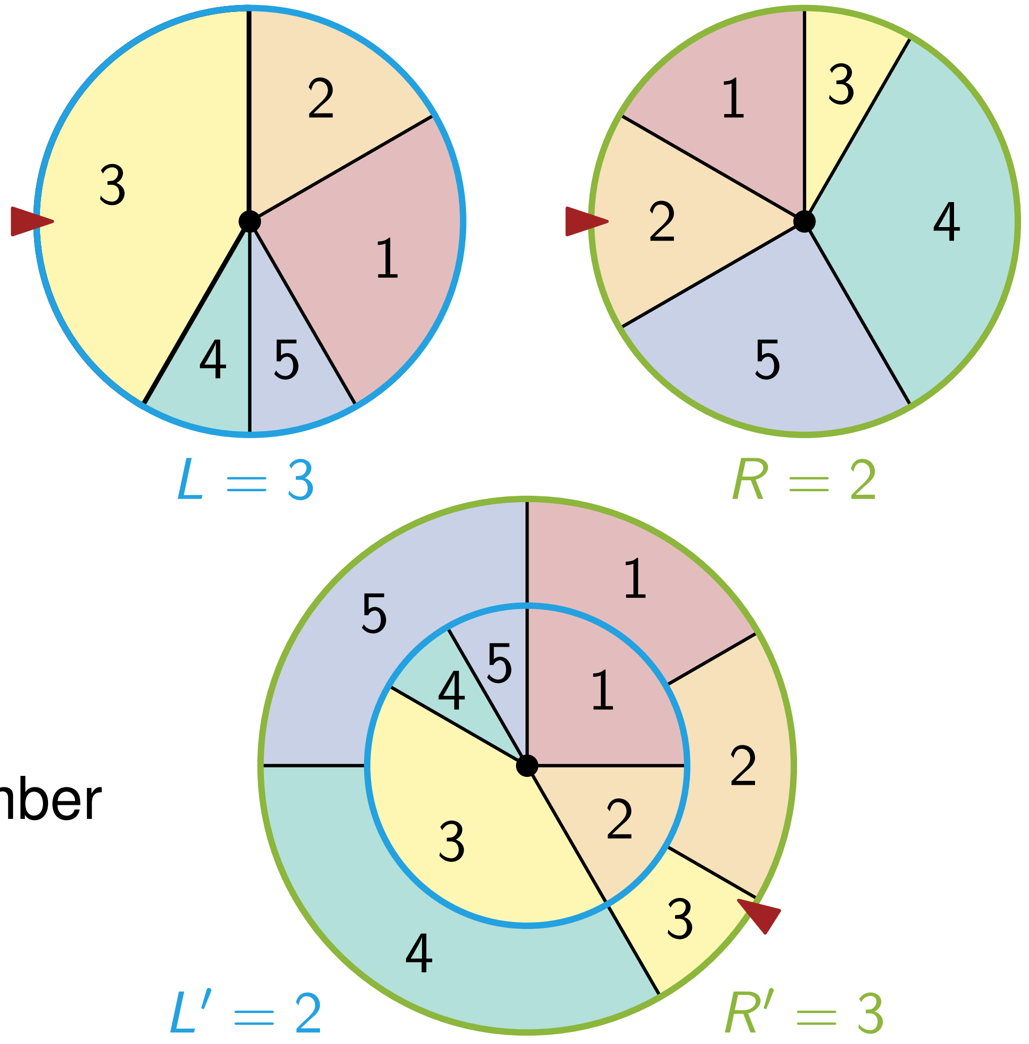


Wheels of Fortune

The Problem
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!
- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \overset{d}{=} L'$ and $R \overset{d}{=} R'$ equal distributions
- But L' and R' are dependent and *always* $R' \geq L'$



Wheels of Fortune

The Problem

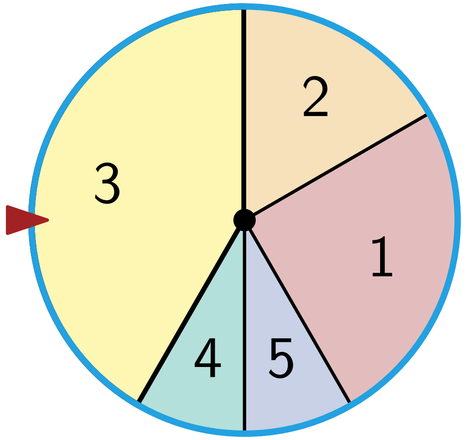
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

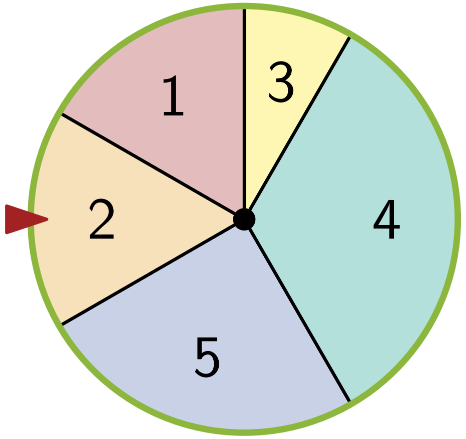
- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

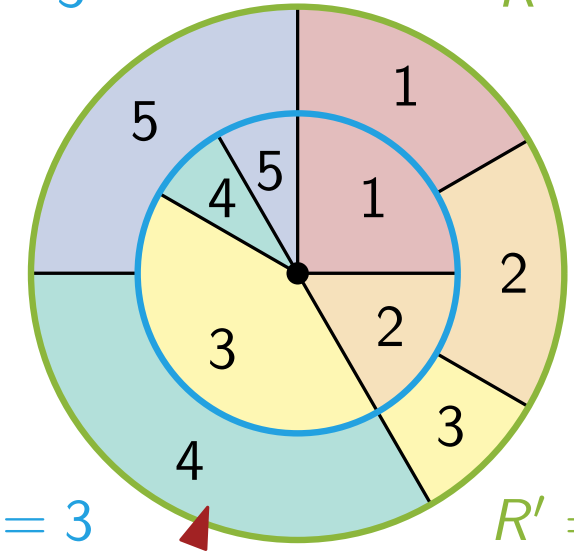
- Sort the wheels   (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \overset{d}{=} L'$ and $R \overset{d}{=} R'$ equal distributions
- But L' and R' are dependent and *always* $R' \geq L'$



$L = 3$



$R = 2$



$L' = 3$ $R' = 4$

Wheels of Fortune

The Problem

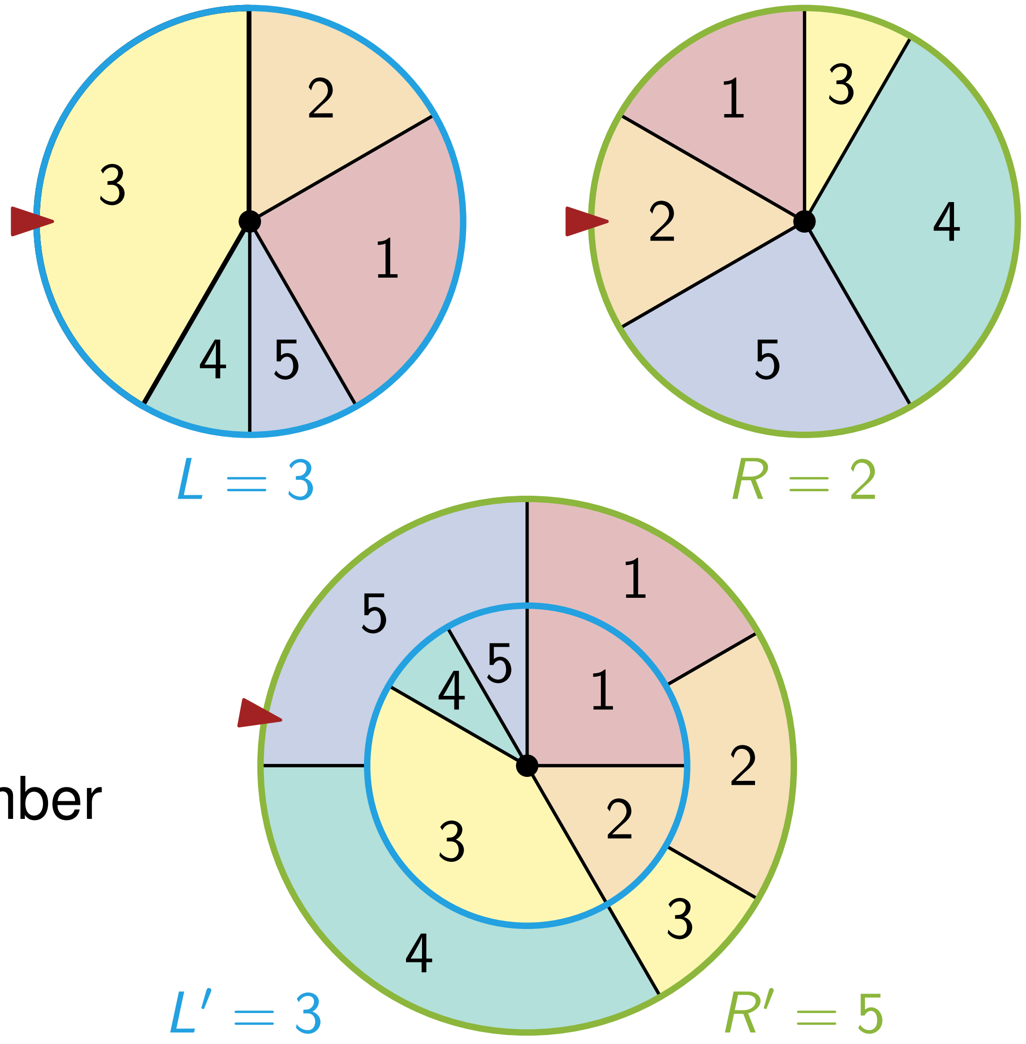
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \stackrel{d}{=} L'$ and $R \stackrel{d}{=} R'$ equal distributions
- But L' and R' are dependent and *always* $R' \geq L'$



Wheels of Fortune

The Problem

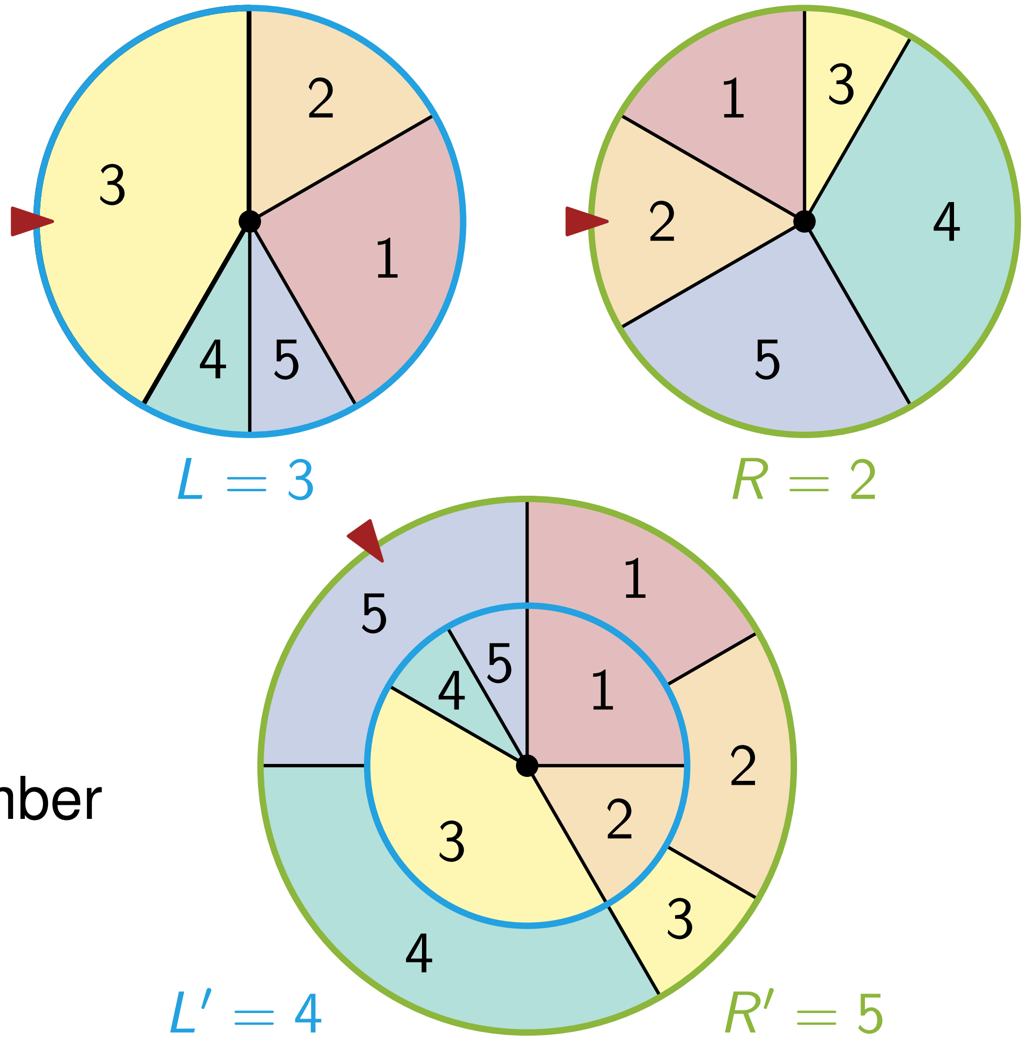
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \overset{d}{=} L'$ and $R \overset{d}{=} R'$ equal distributions
- But L' and R' are dependent and *always* $R' \geq L'$



Wheels of Fortune

The Problem

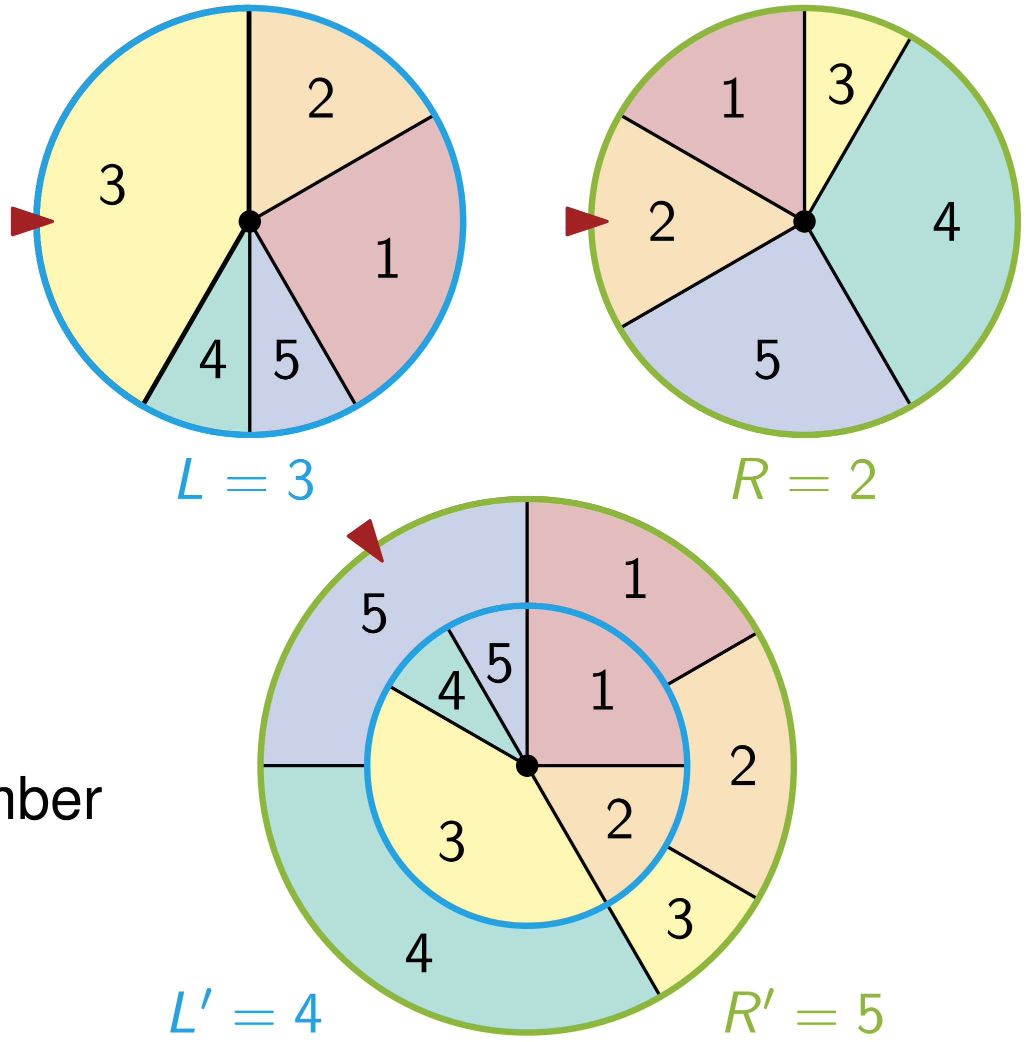
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels   (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \overset{d}{=} L'$ and $R \overset{d}{=} R'$ equal distributions
- But L' and R' are dependent and *always* $R' \geq L'$
  $\Rightarrow \Pr[R' \geq k] \geq \Pr[L' \geq k]$



Wheels of Fortune

The Problem

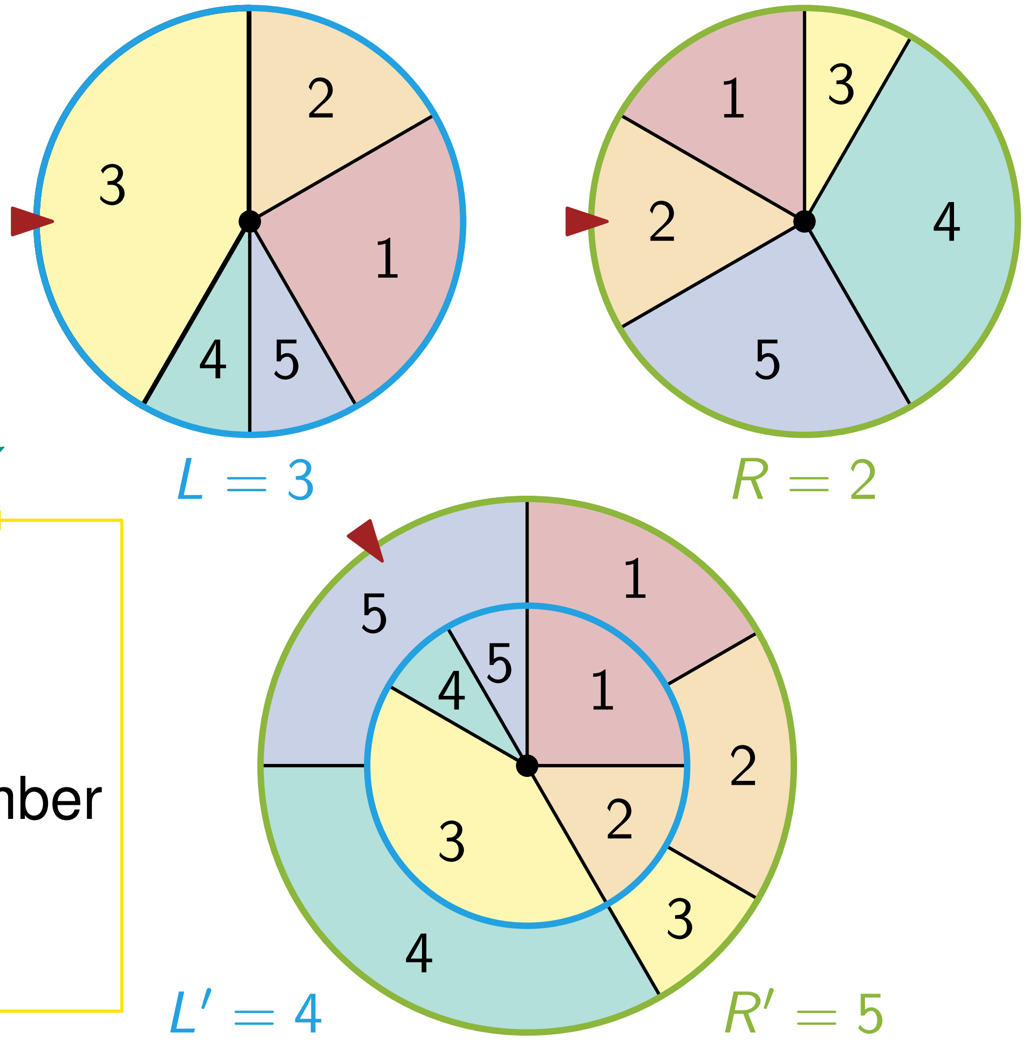
- Consider the two wheels of fortune
- The higher the value the larger the price
- Which do you spin? Why? Can we prove that?

The Maths

- Let L be the value of the left wheel
- Let R be the value of the right wheel
- To show: For all values k: $\Pr[R \geq k] \geq \Pr[L \geq k]$ ✓

Proof: Frankenstein's Wheel of Fortune!

- Sort the wheels (does not change their distributions)
- Adjust sizes and glue together
- Spin as one wheel: L' inner number, R' outer number
- Note that $L \stackrel{d}{=} L'$ and $R \stackrel{d}{=} R'$ equal distributions
- But L' and R' are dependent and *always* $R' \geq L'$
  $\Rightarrow \Pr[R' \geq k] \geq \Pr[L' \geq k]$



What just happened?

Setup & Method

- Random variable L on the left wheel and R on right wheel

What just happened?

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

What just happened?

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel
- Random variable L' on inner wheel and R' on outer wheel

What just happened?

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

  $\overset{d}{=}$ $\overset{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

What just happened?

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

$\overset{d}{=}$ $\overset{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

What just happened?

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

$\overset{d}{=}$ $\overset{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

- Define a relation between random variables to make statements about one using the other

What just happened?

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

$\overset{d}{=}$ $\overset{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

What just happened?

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

$\overset{d}{=}$ $\overset{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X_1', X_2') defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

What just happened?

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

$\overset{d}{=}$ $\overset{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X_1', X_2') defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

What just happened?

Karlsruhe Institute of Technology

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

$L \overset{d}{=} L'$, $R \overset{d}{=} R'$

- Random variable L' on inner wheel and R' on outer wheel

dependent

- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X_1', X_2') defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

What just happened?

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

$\overset{d}{=}$ $\overset{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X_1', X_2') defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

What just happened?

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

$\stackrel{d}{=}$ $\stackrel{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X_1', X_2') defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \stackrel{d}{=} X_1'$ and $X_2 \stackrel{d}{=} X_2'$.

What just happened?

Karlsruhe Institute of Technology

Setup & Method

independent

- Random variable L on the left wheel and R on right wheel

$\overset{d}{=}$ $\overset{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

The coupling defines how L' and R' are related

- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X_1', X_2') defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

What just happened?

Setup & Method

independent

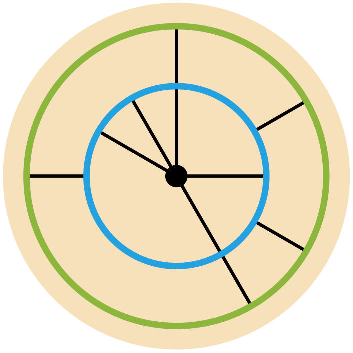
- Random variable L on the left wheel and R on right wheel

$L \overset{d}{=} L'$ and $R \overset{d}{=} R'$

- Random variable L' on inner wheel and R' on outer wheel

dependent

The coupling defines how L' and R' are related



- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X'_1, X'_2) defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \overset{d}{=} X'_1$ and $X_2 \overset{d}{=} X'_2$.

What just happened?

Setup & Method

independent

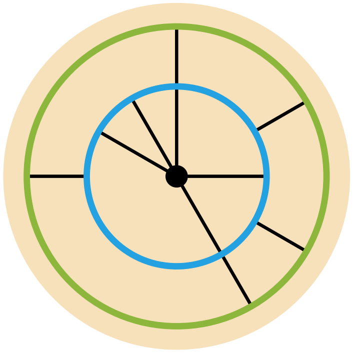
- Random variable L on the left wheel and R on right wheel

$\stackrel{d}{=}$ $\stackrel{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

The coupling defines how L' and R' are related



- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X_1', X_2') defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \stackrel{d}{=} X_1'$ and $X_2 \stackrel{d}{=} X_2'$.

- X_1' and X_2' live in the *same* space

What just happened?

Setup & Method

independent

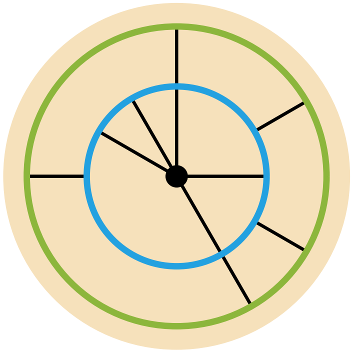
- Random variable L on the left wheel and R on right wheel

$\overset{d}{=}$ $\overset{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

The coupling defines how L' and R' are related



- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X_1', X_2') defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

- X_1' and X_2' live in the *same* space
- Typically we define X_1' and X_2' to be dependent

What just happened?

Setup & Method

independent

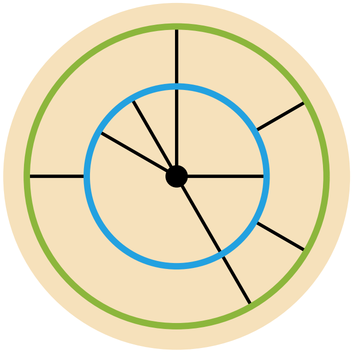
- Random variable L on the left wheel and R on right wheel

$L \overset{d}{=} L'$ $\qquad$ $R \overset{d}{=} R'$

- Random variable L' on inner wheel and R' on outer wheel

dependent

The coupling defines how L' and R' are related



- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X_1', X_2') defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

- X_1' and X_2' live in the *same* space
- Typically we define X_1' and X_2' to be dependent
- Typically we do not talk about the probability spaces explicitly

What just happened?

Setup & Method

independent

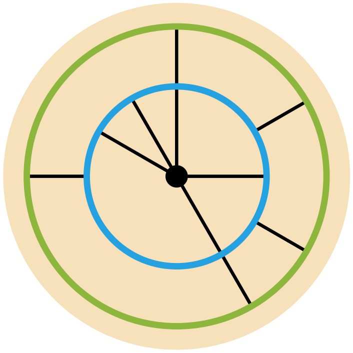
- Random variable L on the left wheel and R on right wheel

$\overset{d}{=}$ $\qquad$ $\overset{d}{=}$

- Random variable L' on inner wheel and R' on outer wheel

dependent

The coupling defines how L' and R' are related



- Define a relation between random variables to make statements about one using the other Here: $\Pr[R' \geq k] \geq \Pr[L' \geq k] \Rightarrow \Pr[R \geq k] \geq \Pr[L \geq k]$

Definition: Let X_1, X_2 be random variables defined on probability spaces $(\Omega_1, \Sigma_1, \Pr_1)$ and $(\Omega_2, \Sigma_2, \Pr_2)$, respectively. A **coupling** of X_1 and X_2 is a pair of random variables (X_1', X_2') defined on a new probability space $(\Omega, \Sigma, \Pr)$ such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

- X_1' and X_2' live in the *same* space
- Typically we define X_1' and X_2' to be dependent
- Typically we do not talk about the probability spaces explicitly

- Abstracting away technicalities, people just "couple" X_1 and X_2 "directly", without introducing X_1' and X_2'

Application: Biased Coins

The Problem

- We have a fair $\{0, 1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0, 1\}$-coin that yields 1 with probability $\frac{2}{3}$

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0, 1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0, 1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

0

Application: Biased Coins

The Problem

- We have a fair $\{0, 1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0, 1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

0 1

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

0 1 0

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

0 1 0 0

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

0 1 0 0 0

Application: Biased Coins

KIT Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0, 1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0, 1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

0 1 0 0 0 1

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$$

$$0$$

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$$

$$0 \; 0$$

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$0\ 0\ 1$$

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0, 1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0, 1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$$

$$0 \; 0 \; 1 \; 1$$

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0, 1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0, 1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$$

$$0 \; 0 \; 1 \; 1 \; 1$$

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$$

$$0 \; 0 \; 1 \; 1 \; 1 \; 1$$

Application: Biased Coins

The Problem

- We have a fair $\{0, 1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0, 1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$ $\quad F = \sum$ 0 1 0 0 0 1 $= 2$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$ $\quad U = \sum$ 0 0 1 1 1 1 $= 4$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

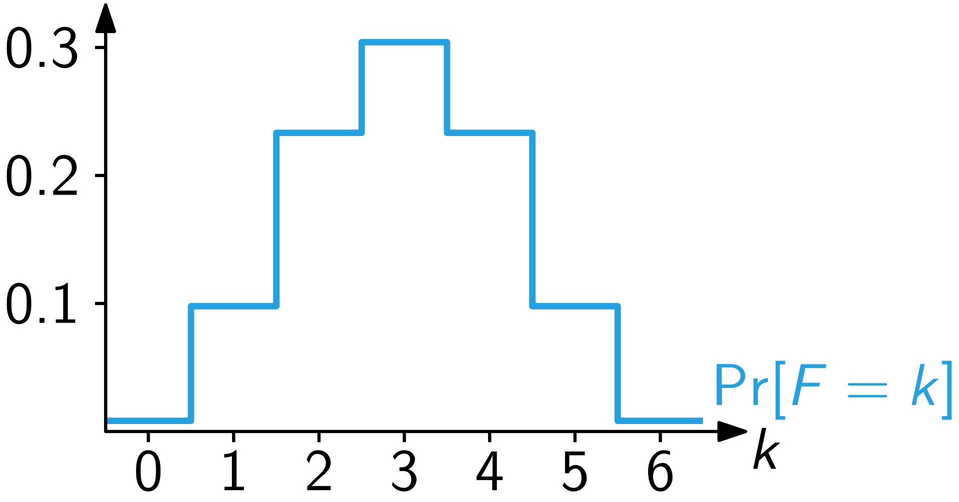
$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof



Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

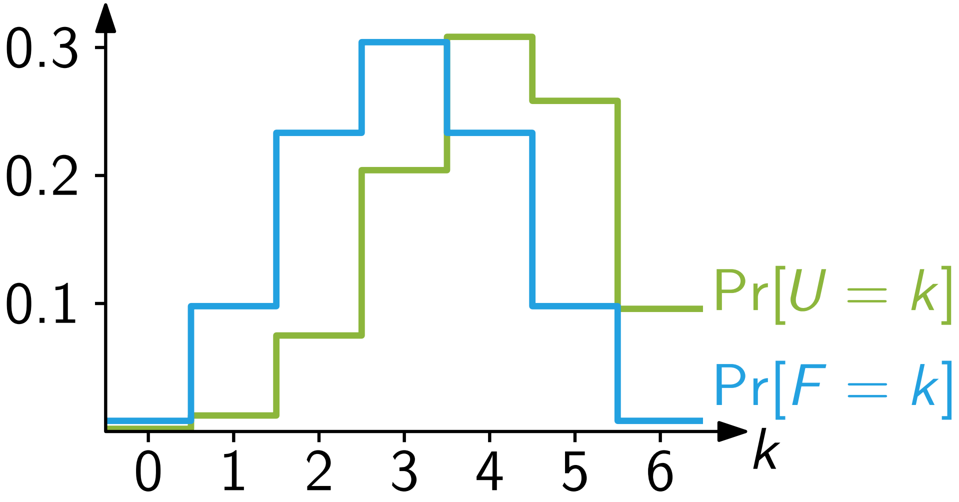
$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof



Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

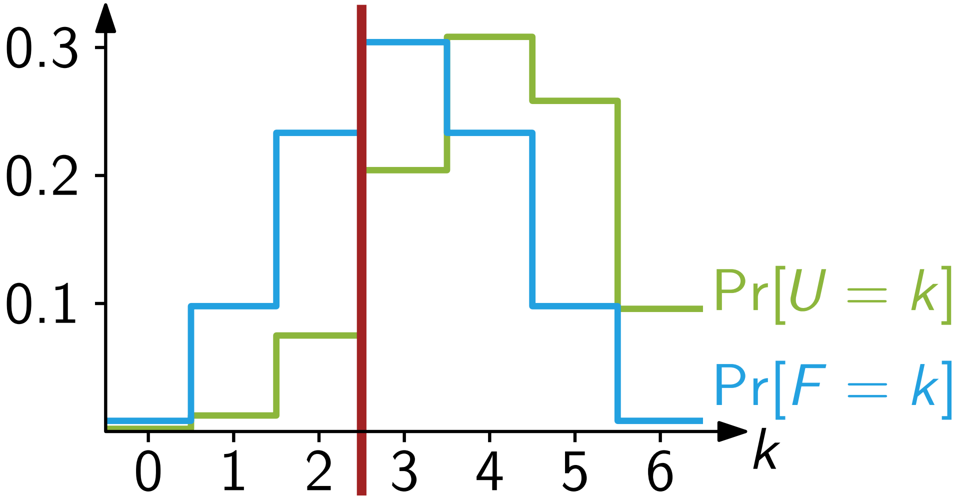
- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$$
$$U = \sum \; 0 \; 0 \; 1 \; 1 \; 1 \; 1 = 4$$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof



Application: Biased Coins

The Problem

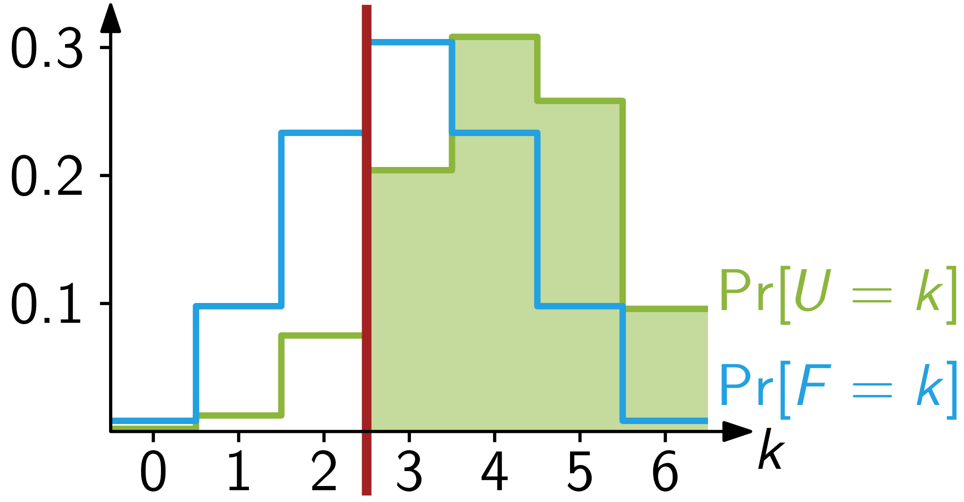
- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$F = \sum$ 0 1 0 0 0 1 $= 2$

$U = \sum$ 0 0 1 1 1 1 $= 4$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof



Application: Biased Coins

The Problem

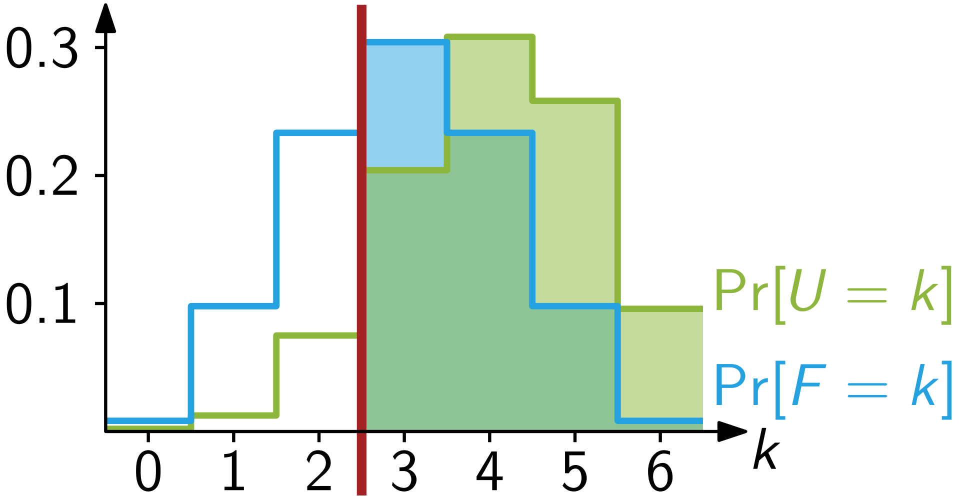
- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof



Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

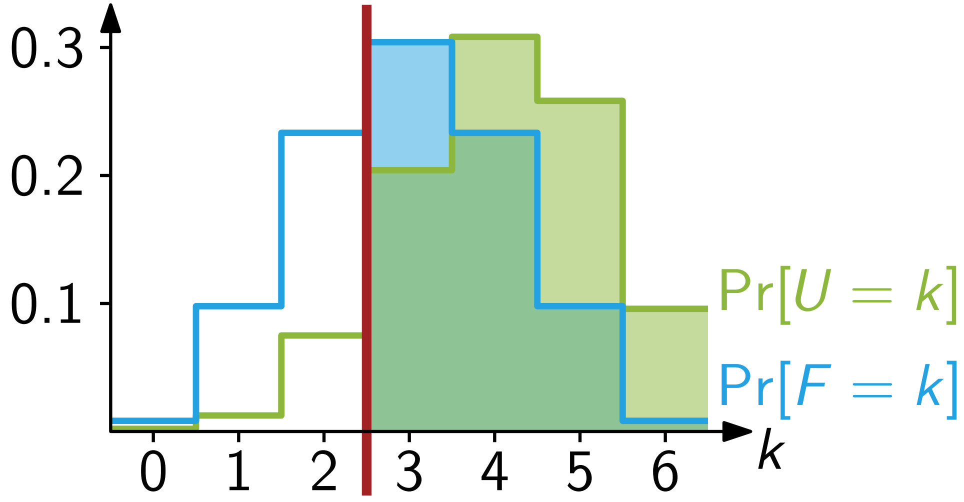
- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof *Compare sums for all* $k \leq 6$



Application: Biased Coins

The Problem

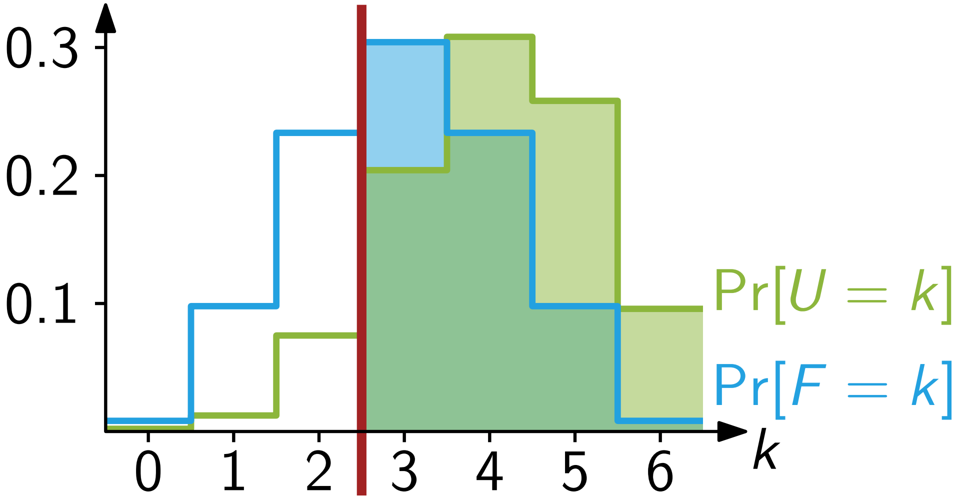
- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

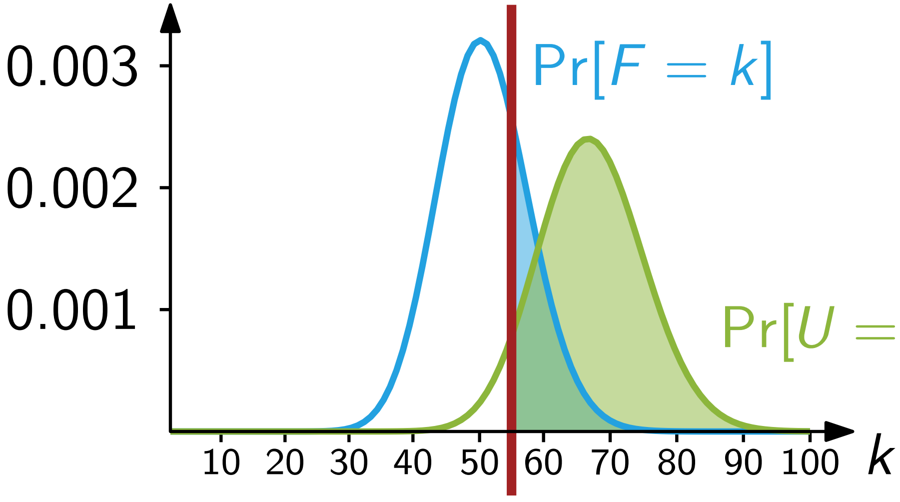
$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$
$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof *Compare sums for all* $k \leq 6$

And if $n = 100$? *so many sums...*





Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

$$F_i \xleftrightarrow{\text{independent}} U_i$$

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$F = \sum$ 0 1 0 0 0 1 $= 2$

$U = \sum$ 0 0 1 1 1 1 $= 4$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \stackrel{d}{=} X_1'$ and $X_2 \stackrel{d}{=} X_2'$.

$F_i \xleftrightarrow{\text{independent}} U_i$

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$$

$$U = \sum \; 0 \; 0 \; 1 \; 1 \; 1 \; 1 = 4$$

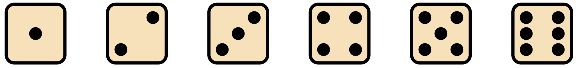
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \stackrel{d}{=} X_1'$ and $X_2 \stackrel{d}{=} X_2'$.



$\Pr \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6}$

$F_i \longleftrightarrow U_i$ independent

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$F = \sum$ 0 1 0 0 0 1 $= 2$

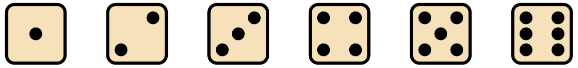
$U = \sum$ 0 0 1 1 1 1 $= 4$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.



Pr $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

$F_i \xleftrightarrow{\text{independent}} U_i$

F_i'

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$$

$$U = \sum \; 0 \; 0 \; 1 \; 1 \; 1 \; 1 = 4$$

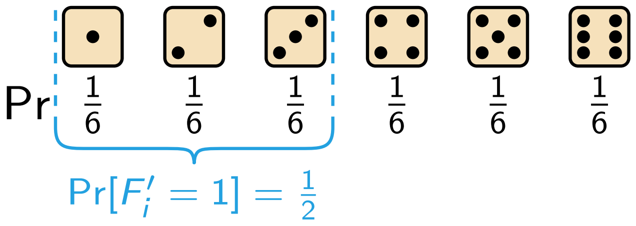
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

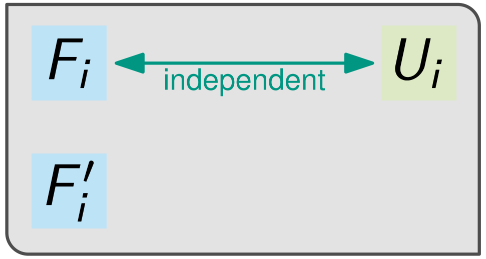
Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.





Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

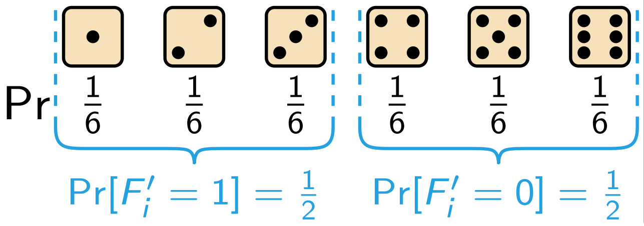
$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

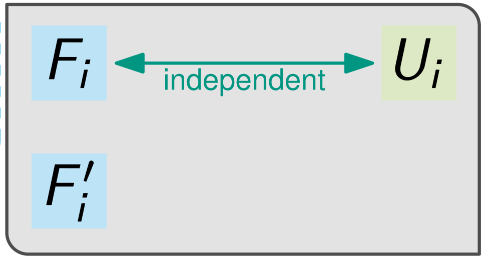
Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.





Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

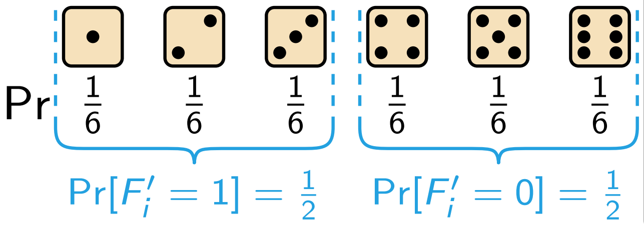
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \stackrel{d}{=} F_i'$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \stackrel{d}{=} X_1'$ and $X_2 \stackrel{d}{=} X_2'$.



$$F_i \xleftrightarrow{\text{independent}} U_i$$

$$F_i \stackrel{d}{=} F_i'$$

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

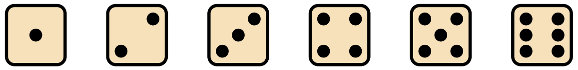
$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \stackrel{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \stackrel{d}{=} X_1'$ and $X_2 \stackrel{d}{=} X_2'$.



Pr $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

$F_i \xleftrightarrow{\text{independent}} U_i$

$F_i \stackrel{d}{=} F_i'$

F_i' $\quad U_i'$

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

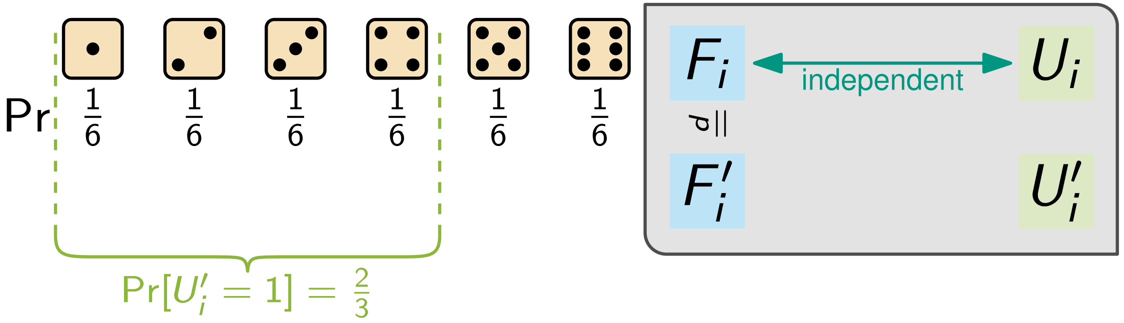
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \overset{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.



Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

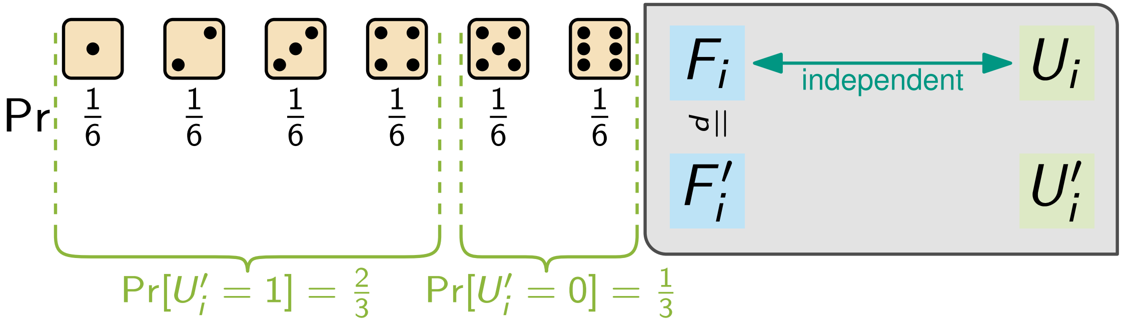
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \stackrel{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \stackrel{d}{=} X_1'$ and $X_2 \stackrel{d}{=} X_2'$.



Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

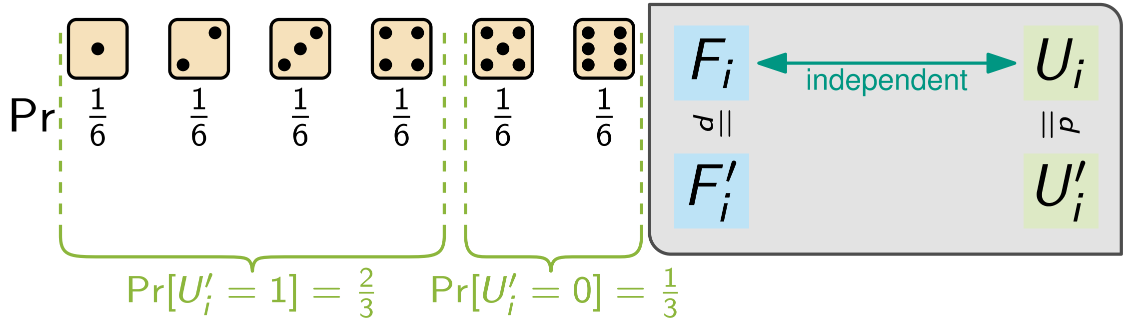
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \stackrel{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4 \Rightarrow U_i \stackrel{d}{=} U_i'$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \stackrel{d}{=} X_1'$ and $X_2 \stackrel{d}{=} X_2'$.



Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$$

$$U = \sum \; 0 \; 0 \; 1 \; 1 \; 1 \; 1 = 4$$

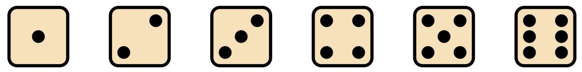
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

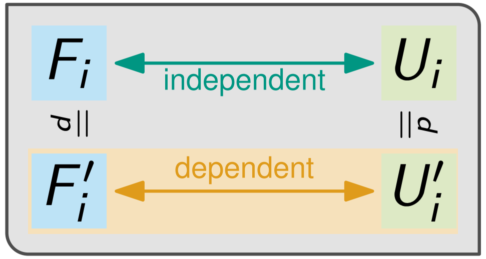
Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F'_i = 1$ iff $W_i \leq 3 \Rightarrow F_i \overset{d}{=} F'_i$
  - Define $U'_i = 1$ iff $W_i \leq 4 \Rightarrow U_i \overset{d}{=} U'_i$
- F'_i and U'_i are dependent and *always* $U'_i \geq F'_i$

Coupling: Random variables X_1, X_2. Define random variables X'_1, X'_2 in a shared probability space such that $X_1 \overset{d}{=} X'_1$ and $X_2 \overset{d}{=} X'_2$.



Pr $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$



Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \overset{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4 \Rightarrow U_i \overset{d}{=} U_i'$
- F_i' and U_i' are dependent and *always* $U_i' \geq F_i'$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

W_i	⚀	⚁	⚂	⚃	⚄	⚅
Pr	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
F_i'	1	1	1	0	0	0
	$\wedge\!\mid$	$\wedge\!\mid$	$\wedge\!\mid$	$\wedge\!\mid$	$\wedge\!\mid$	$\wedge\!\mid$
U_i'	1	1	1	1	0	0

$F_i \xleftrightarrow{\text{independent}} U_i$

$F_i \overset{d}{=} F_i'$, $U_i \overset{d}{=} U_i'$

$F_i' \xleftrightarrow{\text{dependent}} U_i'$

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$
$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \overset{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4 \Rightarrow U_i \overset{d}{=} U_i'$
- F_i' and U_i' are dependent and *always* $U_i' \geq F_i'$

$$F = \sum_{i=1}^{n} F_i$$
$$U = \sum_{i=1}^{n} U_i$$

$$\begin{tikzcd} F_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}"] & U_i \arrow[d, phantom, "\overset{d}{=}"] \\ F_i' \arrow[r, leftrightarrow, "\text{dependent}"] & U_i' \end{tikzcd}$$

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \stackrel{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4 \Rightarrow U_i \stackrel{d}{=} U_i'$
- F_i' and U_i' are dependent and *always* $U_i' \geq F_i'$

$$F = \sum_{i=1}^{n} F_i \qquad U = \sum_{i=1}^{n} U_i$$

$$F' = \sum_{i=1}^{n} F_i' \qquad U' = \sum_{i=1}^{n} U_i'$$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \stackrel{d}{=} X_1'$ and $X_2 \stackrel{d}{=} X_2'$.

$$\begin{array}{ccc} F_i & \xleftrightarrow{\text{independent}} & U_i \\ \| d & & \| d \\ F_i' & \xleftrightarrow{\text{dependent}} & U_i' \end{array}$$

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F'_i = 1$ iff $W_i \leq 3 \Rightarrow F_i \overset{d}{=} F'_i$
  - Define $U'_i = 1$ iff $W_i \leq 4 \Rightarrow U_i \overset{d}{=} U'_i$
- F'_i and U'_i are dependent and *always* $U'_i \geq F'_i$

$$F = \sum_{i=1}^{n} F_i \qquad U = \sum_{i=1}^{n} U_i$$

$$F' = \sum_{i=1}^{n} F'_i \qquad U' = \sum_{i=1}^{n} U'_i$$

Coupling: Random variables X_1, X_2. Define random variables X'_1, X'_2 in a shared probability space such that $X_1 \overset{d}{=} X'_1$ and $X_2 \overset{d}{=} X'_2$.

$$\begin{tikzcd} F_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}"] & U_i \arrow[d, phantom, "\overset{d}{=}"] \\ F'_i \arrow[r, leftrightarrow, "\text{dependent}"] & U'_i \end{tikzcd}$$

Observation: Independent rand. var. X_i, Y_i for $i \in [n]$ with couplings (X'_i, Y'_i) for $i \in [n]$.
Then, for any function f: $(f(X'_1, \dots, X'_n), f(Y'_1, \dots, Y'_n))$ is a coupling of $f(X_1, \dots, X_n)$ and $f(Y_1, \dots, Y_n)$.

Application: Biased Coins

Karlsruhe Institute of Technology

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$$

$$U = \sum \; 0 \; 0 \; 1 \; 1 \; 1 \; 1 = 4$$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \overset{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4 \Rightarrow U_i \overset{d}{=} U_i'$
- F_i' and U_i' are dependent and *always* $U_i' \geq F_i'$

$$F = \sum_{i=1}^{n} F_i \qquad U = \sum_{i=1}^{n} U_i$$

$$F' = \sum_{i=1}^{n} F_i' \qquad U' = \sum_{i=1}^{n} U_i'$$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

$$\begin{tikzcd} F \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}"] & U \arrow[d, phantom, "\overset{d}{=}"] \\ F' \arrow[r, leftrightarrow, "\text{dependent}"] & U' \end{tikzcd} \qquad \begin{tikzcd} F_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}"] & U_i \arrow[d, phantom, "\overset{d}{=}"] \\ F_i' \arrow[r, leftrightarrow, "\text{dependent}"] & U_i' \end{tikzcd}$$

Observation: Independent rand. var. X_i, Y_i for $i \in [n]$ with couplings (X_i', Y_i') for $i \in [n]$.
Then, for any function f: $(f(X_1', \dots, X_n'), f(Y_1', \dots, Y_n'))$ is a coupling of $f(X_1, \dots, X_n)$ and $f(Y_1, \dots, Y_n)$.

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U
- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$

$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \stackrel{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4 \Rightarrow U_i \stackrel{d}{=} U_i'$
- F_i' and U_i' are dependent and *always* $U_i' \geq F_i'$

$$F = \sum_{i=1}^n F_i \qquad U = \sum_{i=1}^n U_i$$

$$F' = \sum_{i=1}^n F_i' \qquad U' = \sum_{i=1}^n U_i'$$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \stackrel{d}{=} X_1'$ and $X_2 \stackrel{d}{=} X_2'$.

$$\begin{tikzcd} F \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\stackrel{d}{=}"] & U \arrow[d, phantom, "\stackrel{d}{=}"] \\ F' \arrow[r, leftrightarrow, "\text{dependent}"] & U' \end{tikzcd} \qquad \begin{tikzcd} F_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\stackrel{d}{=}"] & U_i \arrow[d, phantom, "\stackrel{d}{=}"] \\ F_i' \arrow[r, leftrightarrow, "\text{dependent}"] & U_i' \end{tikzcd}$$

Observation: Independent rand. var. X_i, Y_i for $i \in [n]$ with couplings (X_i', Y_i') for $i \in [n]$.
Then, for any function f: $(f(X_1', \dots, X_n'), f(Y_1', \dots, Y_n'))$ is a coupling of $f(X_1, \dots, X_n)$ and $f(Y_1, \dots, Y_n)$.

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum \text{(0)(1)(0)(0)(0)(1)} = 2$$
$$U = \sum \text{(0)(0)(1)(1)(1)(1)} = 4$$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F'_i = 1$ iff $W_i \leq 3 \Rightarrow F_i \stackrel{d}{=} F'_i$
  - Define $U'_i = 1$ iff $W_i \leq 4 \Rightarrow U_i \stackrel{d}{=} U'_i$
- F'_i and U'_i are dependent and *always* $U'_i \geq F'_i$
  $\Rightarrow U' \geq F'$

$$F = \sum_{i=1}^n F_i \qquad U = \sum_{i=1}^n U_i$$
$$F' = \sum_{i=1}^n F'_i \qquad U' = \sum_{i=1}^n U'_i$$

Coupling: Random variables X_1, X_2. Define random variables X'_1, X'_2 in a shared probability space such that $X_1 \stackrel{d}{=} X'_1$ and $X_2 \stackrel{d}{=} X'_2$.

$$\begin{tikzcd} F \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\stackrel{d}{=}"] & U \arrow[d, phantom, "\stackrel{d}{=}"] \\ F' \arrow[r, leftrightarrow, "\text{dependent}"] & U' \end{tikzcd} \qquad \begin{tikzcd} F_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\stackrel{d}{=}"] & U_i \arrow[d, phantom, "\stackrel{d}{=}"] \\ F'_i \arrow[r, leftrightarrow, "\text{dependent}"] & U'_i \end{tikzcd}$$

Observation: Independent rand. var. X_i, Y_i for $i \in [n]$ with couplings (X'_i, Y'_i) for $i \in [n]$.
Then, for any function f: $(f(X'_1, \ldots, X'_n), f(Y'_1, \ldots, Y'_n))$ is a coupling of $f(X_1, \ldots, X_n)$ and $f(Y_1, \ldots, Y_n)$.

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$F = \sum \; 0 \; 1 \; 0 \; 0 \; 0 \; 1 = 2$

$U = \sum \; 0 \; 0 \; 1 \; 1 \; 1 \; 1 = 4$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \overset{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4 \Rightarrow U_i \overset{d}{=} U_i'$
- F_i' and U_i' are dependent and *always* $U_i' \geq F_i'$

  $\Rightarrow U' \geq F' \Rightarrow \Pr[U' \geq k] \geq \Pr[F' \geq k]$

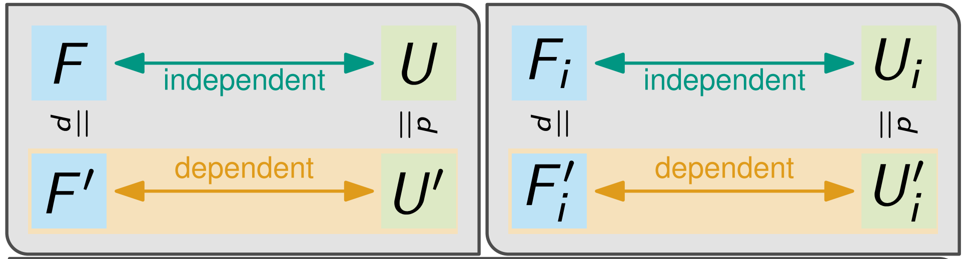
$F = \sum_{i=1}^{n} F_i$

$U = \sum_{i=1}^{n} U_i$

$F' = \sum_{i=1}^{n} F_i'$

$U' = \sum_{i=1}^{n} U_i'$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.



Observation: Independent rand. var. X_i, Y_i for $i \in [n]$ with couplings (X_i', Y_i') for $i \in [n]$.
Then, for any function f: $(f(X_1', \dots, X_n'), f(Y_1', \dots, Y_n'))$ is a coupling of $f(X_1, \dots, X_n)$ and $f(Y_1, \dots, Y_n)$.

Application: Biased Coins

The Problem

- We have a fair $\{0,1\}$-coin that yields 1 with probability $\frac{1}{2}$
- And an unfair $\{0,1\}$-coin that yields 1 with probability $\frac{2}{3}$
- Throw each coin n times, count the 1s, yielding F and U

$$F = \sum 0\ 1\ 0\ 0\ 0\ 1 = 2$$
$$U = \sum 0\ 0\ 1\ 1\ 1\ 1 = 4$$

- You pick a coin. You win if your coin gets more 1s than the other. Which do you pick?

Claim $\Pr[U \geq k] \geq \Pr[F \geq k]$ ✓

Proof

- Let F_i be indicator for ith fair coin
- Let U_i be indicator for ith unfair coin
- Let W_i be the result of a fair die-roll
  - Define $F_i' = 1$ iff $W_i \leq 3 \Rightarrow F_i \overset{d}{=} F_i'$
  - Define $U_i' = 1$ iff $W_i \leq 4 \Rightarrow U_i \overset{d}{=} U_i'$
- F_i' and U_i' are dependent and *always* $U_i' \geq F_i'$
  $\Rightarrow U' \geq F' \Rightarrow \Pr[U' \geq k] \geq \Pr[F' \geq k]$

$$F = \sum_{i=1}^n F_i \qquad U = \sum_{i=1}^n U_i$$
$$F' = \sum_{i=1}^n F_i' \qquad U' = \sum_{i=1}^n U_i'$$

Coupling: Random variables X_1, X_2. Define random variables X_1', X_2' in a shared probability space such that $X_1 \overset{d}{=} X_1'$ and $X_2 \overset{d}{=} X_2'$.

$$\begin{array}{ccc} F & \xleftrightarrow{\text{independent}} & U \\ \overset{d}{=} & & \overset{d}{=} \\ F' & \xleftrightarrow{\text{dependent}} & U' \end{array} \qquad \begin{array}{ccc} F_i & \xleftrightarrow{\text{independent}} & U_i \\ \overset{d}{=} & & \overset{d}{=} \\ F_i' & \xleftrightarrow{\text{dependent}} & U_i' \end{array}$$

Observation: Independent rand. var. X_i, Y_i for $i \in [n]$ with couplings (X_i', Y_i') for $i \in [n]$.
Then, for any function f: $(f(X_1', \ldots, X_n'), f(Y_1', \ldots, Y_n'))$ is a coupling of $f(X_1, \ldots, X_n)$ and $f(Y_1, \ldots, Y_n)$.

The Binomial-Poisson-Approximation or “How I Lied To You”

Setup

- Fair $\{0, 1\}$-coin X with $\Pr[X = 1] = p = \frac{1}{2}$

The Binomial-Poisson-Approximation or “How I Lied To You”

Setup

- Fair $\{0, 1\}$-coin X with $\Pr[X = 1] = p = \frac{1}{2}$

This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$

The Binomial-Poisson-Approximation or "How I Lied To You"

Karlsruhe Institute of Technology

Setup

- Fair $\{0, 1\}$-coin X with $\Pr[X = 1] = p = \frac{1}{2}$

This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$

- Sum of n ind. coins $F = \sum_{i=1}^{n} X_i$, $X_i \sim \mathrm{Ber}(p)$

The Binomial-Poisson-Approximation or "How I Lied To You"

Setup

- Fair $\{0,1\}$-coin X with $\Pr[X=1]=p=\frac{1}{2}$

This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$

- Sum of n ind. coins $F = \sum_{i=1}^{n} X_i$, $X_i \sim \mathrm{Ber}(p)$

This is a Binomial rand. var. $F \sim \mathrm{Bin}(n,p)$

$$\Pr[F=k] = \binom{n}{k} p^k (1-p)^{n-k}$$

The Binomial-Poisson-Approximation or “How I Lied To You”

Setup

- Fair $\{0,1\}$-coin X with $\Pr[X = 1] = p = \frac{1}{2}$ This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$
- Sum of n ind. coins $F = \sum_{i=1}^{n} X_i$, $X_i \sim \mathrm{Ber}(p)$ This is a Binomial rand. var. $F \sim \mathrm{Bin}(n, p)$
  $\Pr[F = k] = \binom{n}{k} p^k (1-p)^{n-k}$
- ... which we have seen today already

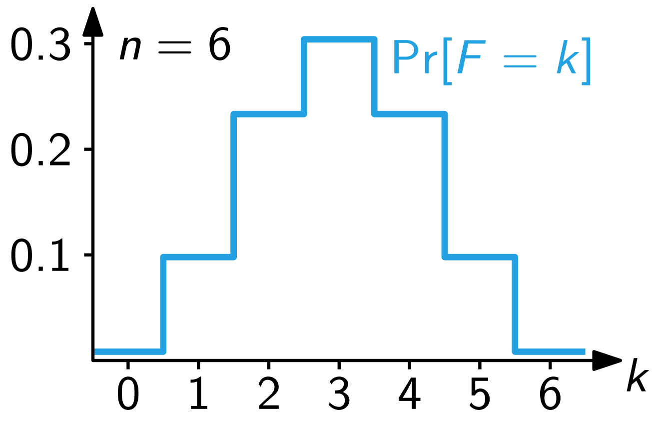
The Binomial-Poisson-Approximation or “How I Lied To You”

Setup

- Fair $\{0,1\}$-coin X with $\Pr[X=1]=p=\frac{1}{2}$ — This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$
- Sum of n ind. coins $F=\sum_{i=1}^{n} X_i$, $X_i \sim \mathrm{Ber}(p)$ — This is a Binomial rand. var. $F \sim \mathrm{Bin}(n,p)$

$$\Pr[F=k]=\binom{n}{k}p^k(1-p)^{n-k}$$

- ... which we have seen today already



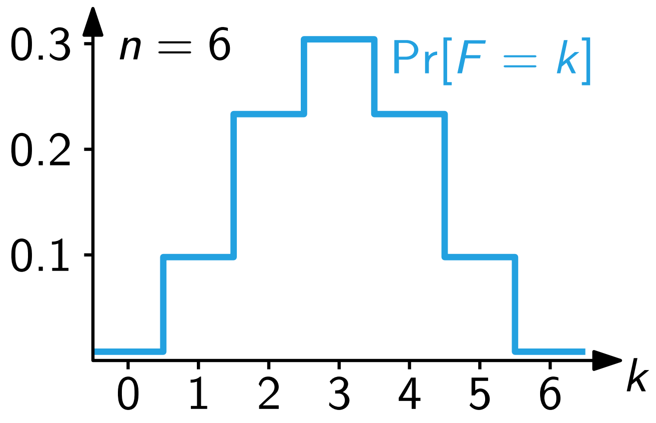
The Binomial-Poisson-Approximation or "How I Lied To You"

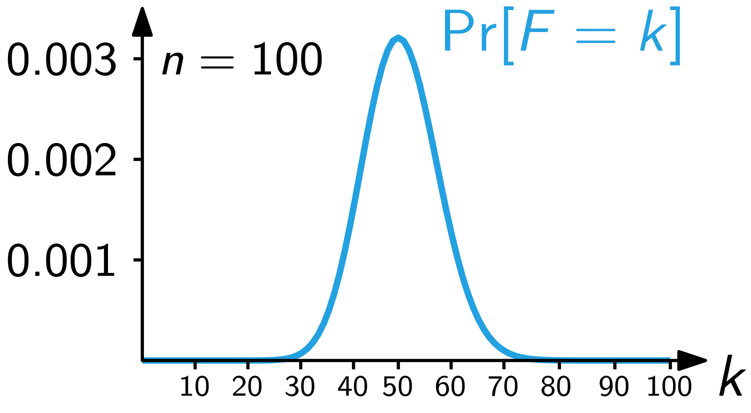
Karlsruhe Institute of Technology

Setup

- Fair $\{0,1\}$-coin X with $\Pr[X=1] = p = \frac{1}{2}$ — This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$
- Sum of n ind. coins $F = \sum_{i=1}^{n} X_i$, $X_i \sim \mathrm{Ber}(p)$ — This is a Binomial rand. var. $F \sim \mathrm{Bin}(n,p)$

  $\Pr[F=k] = \binom{n}{k} p^k (1-p)^{n-k}$
- ... which we have seen today already





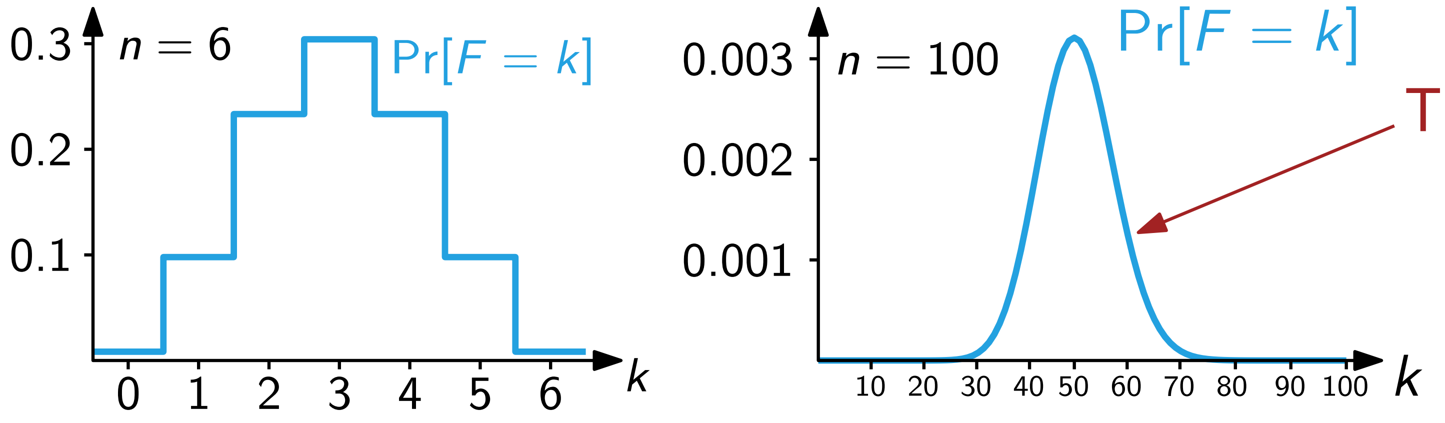
The Binomial-Poisson-Approximation or "How I Lied To You"

Setup

- Fair $\{0,1\}$-coin X with $\Pr[X=1] = p = \frac{1}{2}$ — This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$
- Sum of n ind. coins $F = \sum_{i=1}^{n} X_i$, $X_i \sim \mathrm{Ber}(p)$ — This is a Binomial rand. var. $F \sim \mathrm{Bin}(n,p)$

$$\Pr[F=k] = \binom{n}{k} p^k (1-p)^{n-k}$$

- ... which we have seen today already



This is not a binomial distribution!

The Binomial-Poisson-Approximation or "How I Lied To You"

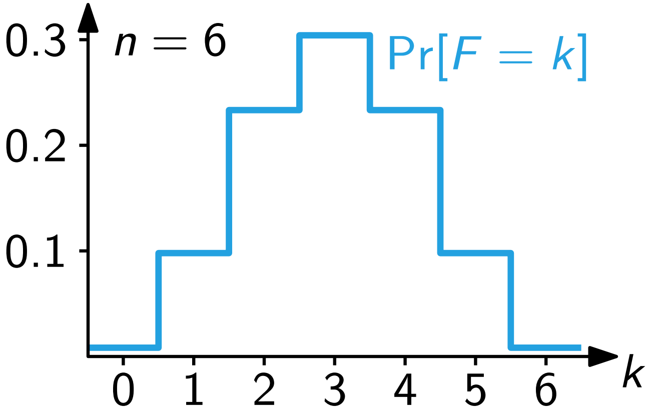
Karlsruhe Institute of Technology

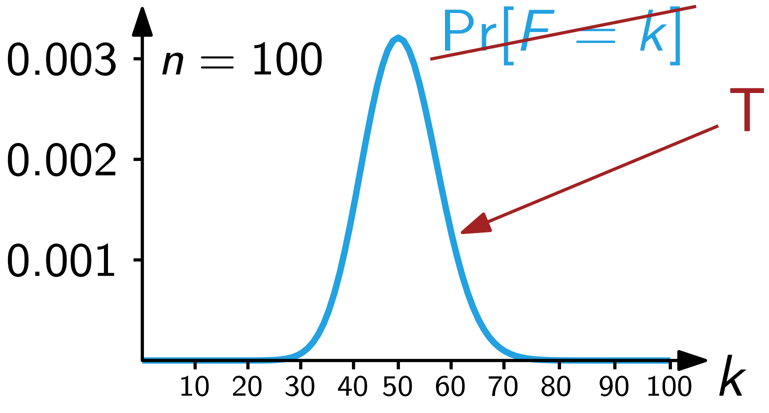
Setup

- Fair $\{0,1\}$-coin X with $\Pr[X=1]=p=\frac{1}{2}$ — This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$
- Sum of n ind. coins $F=\sum_{i=1}^{n} X_i$, $X_i \sim \mathrm{Ber}(p)$ — This is a Binomial rand. var. $F \sim \mathrm{Bin}(n,p)$

$$\Pr[F=k]=\binom{n}{k}p^k(1-p)^{n-k}$$

- ... which we have seen today already





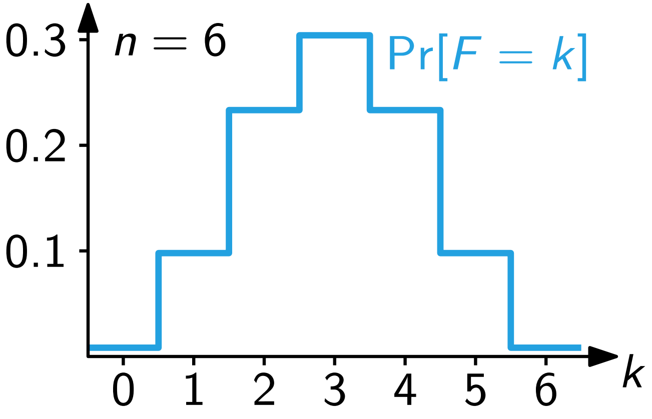
This is not a binomial distribution!

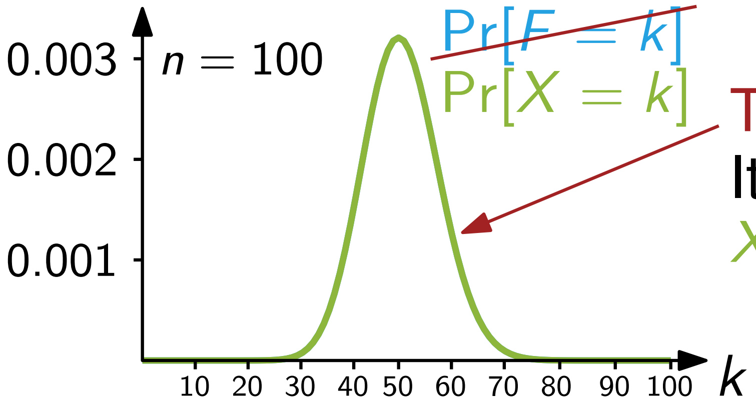
The Binomial-Poisson-Approximation or “How I Lied To You”

Setup

- Fair $\{0,1\}$-coin X with $\Pr[X=1]=p=\frac{1}{2}$ — This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$
- Sum of n ind. coins ${\color{blue}F} = \sum_{i=1}^{n} X_i$, $X_i \sim \mathrm{Ber}(p)$ — This is a Binomial rand. var. $F \sim \mathrm{Bin}(n,p)$

  $\Pr[F=k] = \binom{n}{k} p^k (1-p)^{n-k}$
- ... which we have seen today already





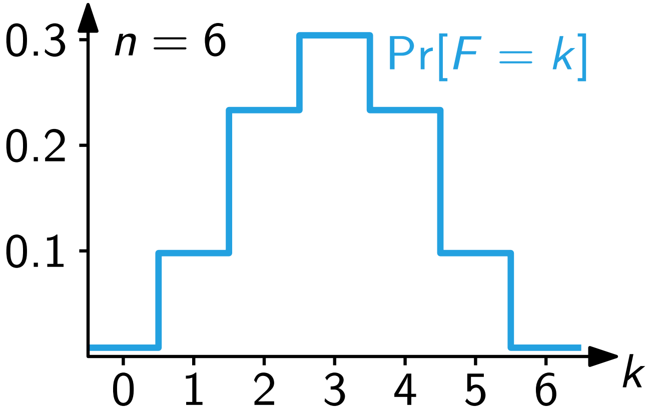
This is not a binomial distribution!
It’s a Poisson distribution with $\lambda = 50$
$X \sim \mathrm{Pois}(\lambda)$: $\Pr[X=k] = \lambda^k e^{-\lambda}/k!$

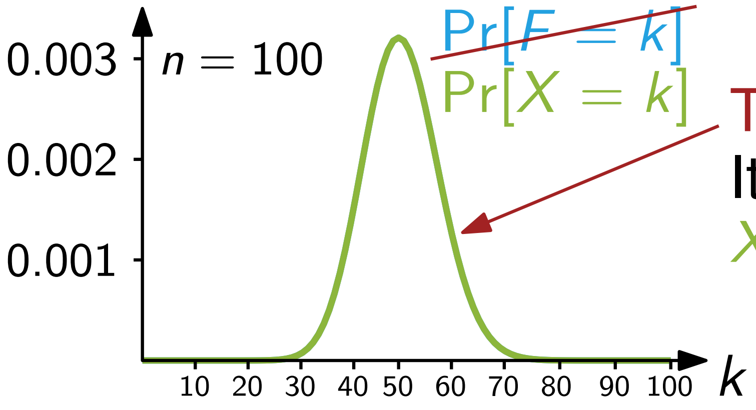
The Binomial-Poisson-Approximation or "How I Lied To You"

Setup

- Fair $\{0,1\}$-coin X with $\Pr[X=1]=p=\frac{1}{2}$ — This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$
- Sum of n ind. coins $F=\sum_{i=1}^{n} X_i$, $X_i \sim \mathrm{Ber}(p)$ — This is a Binomial rand. var. $F \sim \mathrm{Bin}(n,p)$

  $\Pr[F=k]=\binom{n}{k}p^k(1-p)^{n-k}$
- ... which we have seen today already





This is not a binomial distribution!

It's a Poisson distribution with $\lambda=50$

$X \sim \mathrm{Pois}(\lambda)$: $\Pr[X=k]=\lambda^k e^{-\lambda}/k!$

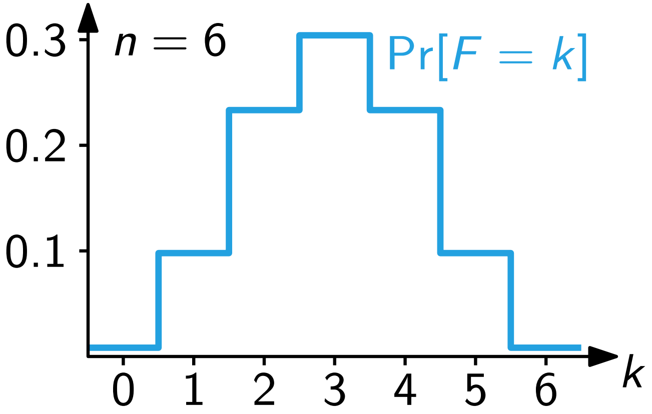
- Why lie? It was easier to plot that way and I thought you wouldn't notice...

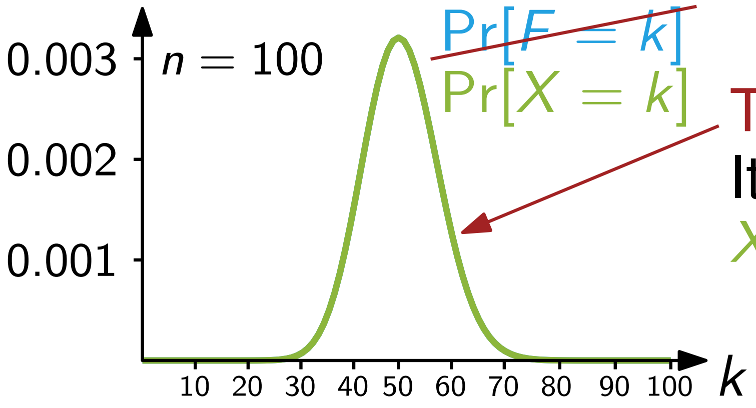
The Binomial-Poisson-Approximation or "How I Lied To You"

Setup

- Fair $\{0,1\}$-coin X with $\Pr[X=1] = p = \frac{1}{2}$ — This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$
- Sum of n ind. coins $F = \sum_{i=1}^{n} X_i,\ X_i \sim \mathrm{Ber}(p)$ — This is a Binomial rand. var. $F \sim \mathrm{Bin}(n,p)$

  $\Pr[F=k] = \binom{n}{k} p^k (1-p)^{n-k}$
- ... which we have seen today already





This is not a binomial distribution!
It's a Poisson distribution with $\lambda = 50$
$X \sim \mathrm{Pois}(\lambda)$: $\Pr[X=k] = \lambda^k e^{-\lambda}/k!$

- Why lie? It was easier to plot that way and I thought you wouldn't notice...
- How dare I? As n increases, the two distributions are very close...

The Binomial-Poisson-Approximation or “How I Lied To You”

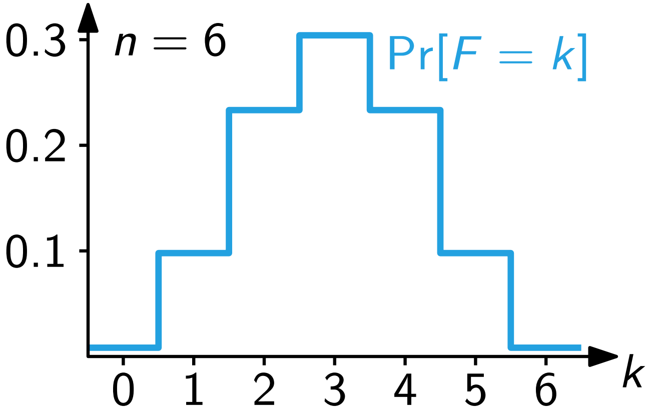
Karlsruhe Institute of Technology

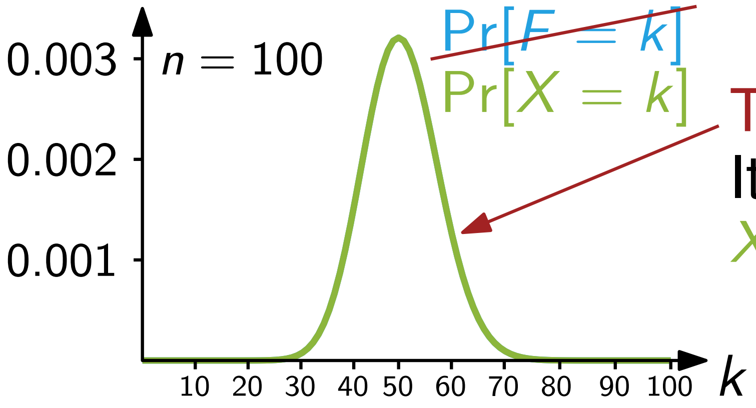
Setup

- Fair $\{0,1\}$-coin X with $\Pr[X=1]=p=\frac{1}{2}$ This is a Bernoulli rand. var. $X \sim \mathrm{Ber}(p)$
- Sum of n ind. coins $F=\sum_{i=1}^{n} X_i$, $X_i \sim \mathrm{Ber}(p)$ This is a Binomial rand. var. $F \sim \mathrm{Bin}(n,p)$

$$\Pr[F=k]=\binom{n}{k}p^k(1-p)^{n-k}$$

- ... which we have seen today already





This is not a binomial distribution!
It's a Poisson distribution with $\lambda=50$
$X \sim \mathrm{Pois}(\lambda)$: $\Pr[X=k]=\lambda^k e^{-\lambda}/k!$

- Why lie? It was easier to plot that way and I thought you wouldn't notice...
- How dare I? As n increases, the two distributions are very close...

What does that mean?

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Total Variation Distance

Karlsruhe Institute of Technology

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X, Y) = \frac{1}{2} \sum_{x \in S} \left| \Pr[X = x] - \Pr[Y = x] \right|$.

- Intuition: Sum over the differences in the probabilities

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X, Y) = \frac{1}{2}\sum_{x \in S} |\Pr[X = x] - \Pr[Y = x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X, Y) = \frac{1}{2}\sum_{x \in S} |\Pr[X = x] - \Pr[Y = x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

$2d_{TV}(X, Y) = \sum_{x \in S} |\Pr[X = x] - \Pr[Y = x]|$

Total Variation Distance

Karlsruhe Institute of Technology

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X, Y) = \frac{1}{2} \sum_{x \in S} |\Pr[X = x] - \Pr[Y = x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

$2d_{TV}(X, Y) = \sum_{x \in S} |\Pr[X = x] - \Pr[Y = x]|$

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

$$2d_{TV}(X,Y) = \sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$$

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

Total Variation Distance

Karlsruhe Institute of Technology

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}2d_{TV}(X,Y) &= \sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]| \\ &= \sum_{x\in S_X}|\Pr[X=x]-\Pr[Y=x]| + \sum_{x\in S_Y}|\Pr[X=x]-\Pr[Y=x]|\end{aligned}$$

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

$S_X = \{x \in S \mid \Pr[X = x] \geq Pr[Y = x]\}$ $\quad S_Y = S \setminus S_X$

$$2d_{TV}(X,Y) = \sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$$
$$= \sum_{x\in S_X} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|}$$

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad$ $S_Y = S \setminus S_X$

$$\begin{aligned}2d_{TV}(X,Y) &= \sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]| \\ &= \sum_{x\in S_X}\cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|}+\sum_{x\in S_Y}\cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} \\ &= \sum_{x\in S_X}\Pr[X=x]-\Pr[Y=x]+\sum_{x\in S_Y}\Pr[Y=x]-\Pr[X=x]\end{aligned}$$

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]| \\
&= \sum_{x\in S_X} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X} \Pr[X=x] - \Pr[Y=x] + \sum_{x\in S_Y} \Pr[Y=x] - \Pr[X=x]
\end{aligned}$$

Total Variation Distance

Karlsruhe Institute of Technology

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X, Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X = x] - \Pr[Y = x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X = x] \geq Pr[Y = x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S} |\Pr[X=x]-\Pr[Y=x]| \\
&= \sum_{x\in S_X} \cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X} \Pr[X=x]-\Pr[Y=x] + \sum_{x\in S_Y} \Pr[Y=x]-\Pr[X=x] \\
&\leq \sum_{x\in S_X} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S_Y} \Pr[Y=x \wedge X\neq x]
\end{aligned}$$

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]| \\
&= \sum_{x\in S_X}\cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|}+\sum_{x\in S_Y}\cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X}\Pr[X=x]-\Pr[Y=x]+\sum_{x\in S_Y}\Pr[Y=x]-\Pr[X=x] \\
&\leq \sum_{x\in S_X}\Pr[X=x \wedge Y\neq x]+\sum_{x\in S_Y}\Pr[Y=x \wedge X\neq x] \\
&\leq \sum_{x\in S}\Pr[X=x \wedge Y\neq x]+\sum_{x\in S}\Pr[Y=x \wedge X\neq x]
\end{aligned}$$

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]| \\
&= \sum_{x\in S_X} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X} \Pr[X=x] - \Pr[Y=x] + \sum_{x\in S_Y} \Pr[Y=x] - \Pr[X=x] \\
&\leq \sum_{x\in S_X} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S_Y} \Pr[Y=x \wedge X\neq x] \\
&\leq \sum_{x\in S} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S} \Pr[Y=x \wedge X\neq x]
\end{aligned}$$

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

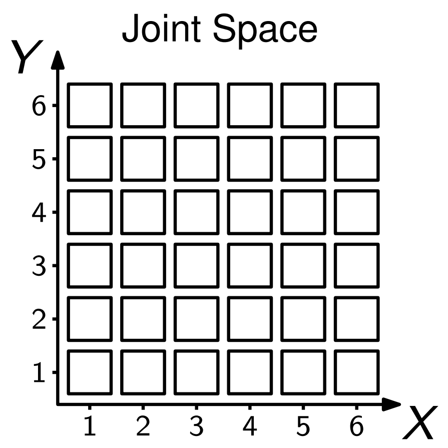
Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \le \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \ge Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S} |\Pr[X=x]-\Pr[Y=x]| \\
&= \sum_{x\in S_X} \cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X} \Pr[X=x]-\Pr[Y=x] + \sum_{x\in S_Y} \Pr[Y=x]-\Pr[X=x] \\
&\le \sum_{x\in S_X} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S_Y} \Pr[Y=x \wedge X \neq x] \\
&\le \sum_{x\in S} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S} \Pr[Y=x \wedge X \neq x]
\end{aligned}$$



Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

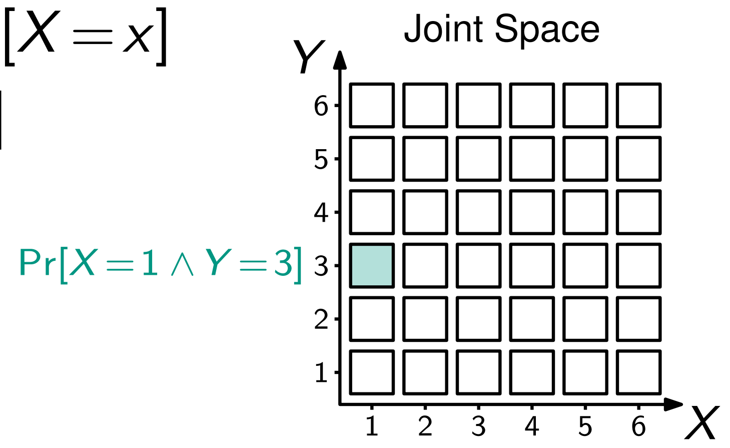
> **Definition**: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x \in S} |\Pr[X = x] - \Pr[Y = x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

> Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X = x] \geq Pr[Y = x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x \in S} |\Pr[X=x] - \Pr[Y=x]| \\
&= \sum_{x \in S_X} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} + \sum_{x \in S_Y} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} \\
&= \sum_{x \in S_X} \Pr[X=x] - \Pr[Y=x] + \sum_{x \in S_Y} \Pr[Y=x] - \Pr[X=x] \\
&\leq \sum_{x \in S_X} \Pr[X=x \wedge Y \neq x] + \sum_{x \in S_Y} \Pr[Y=x \wedge X \neq x] \\
&\leq \sum_{x \in S} \Pr[X=x \wedge Y \neq x] + \sum_{x \in S} \Pr[Y=x \wedge X \neq x]
\end{aligned}$$



Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

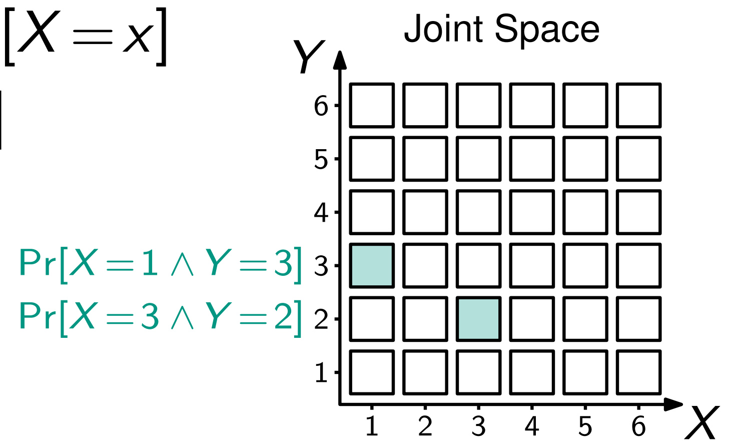
Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x \in S} |\Pr[X = x] - \Pr[Y = x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X = x] \geq Pr[Y = x]\}$ $\quad S_Y = S \setminus S_X$

$$
\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x \in S} |\Pr[X=x] - \Pr[Y=x]| \\
&= \sum_{x \in S_X} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} + \sum_{x \in S_Y} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} \\
&= \sum_{x \in S_X} \Pr[X=x] - \Pr[Y=x] + \sum_{x \in S_Y} \Pr[Y=x] - \Pr[X=x] \\
&\leq \sum_{x \in S_X} \Pr[X=x \wedge Y \neq x] + \sum_{x \in S_Y} \Pr[Y=x \wedge X \neq x] \\
&\leq \sum_{x \in S} \Pr[X=x \wedge Y \neq x] + \sum_{x \in S} \Pr[Y=x \wedge X \neq x]
\end{aligned}
$$



Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

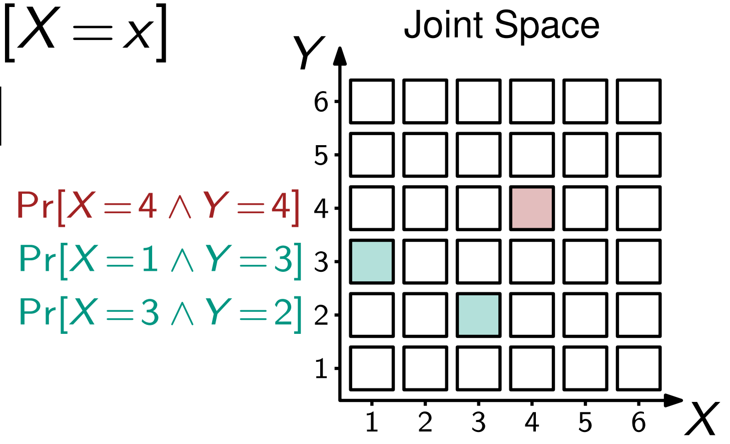
Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A]-\Pr[B]\leq \Pr[A\wedge \bar{B}]$

$S_X = \{x\in S \mid \Pr[X=x]\geq Pr[Y=x]\}$ $\quad S_Y = S\setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]| \\
&= \sum_{x\in S_X}\cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} + \sum_{x\in S_Y}\cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X}\Pr[X=x]-\Pr[Y=x] + \sum_{x\in S_Y}\Pr[Y=x]-\Pr[X=x] \\
&\leq \sum_{x\in S_X}\Pr[X=x\wedge Y\neq x] + \sum_{x\in S_Y}\Pr[Y=x\wedge X\neq x] \\
&\leq \sum_{x\in S}\Pr[X=x\wedge Y\neq x] + \sum_{x\in S}\Pr[Y=x\wedge X\neq x]
\end{aligned}$$



Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

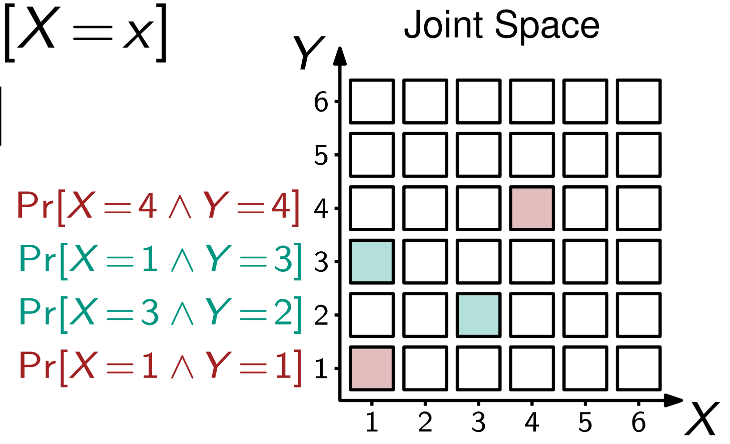
> **Definition**: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

> Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]| \\
&= \sum_{x\in S_X} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X} \Pr[X=x] - \Pr[Y=x] + \sum_{x\in S_Y} \Pr[Y=x] - \Pr[X=x] \\
&\leq \sum_{x\in S_X} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S_Y} \Pr[Y=x \wedge X\neq x] \\
&\leq \sum_{x\in S} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S} \Pr[Y=x \wedge X\neq x]
\end{aligned}$$



Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

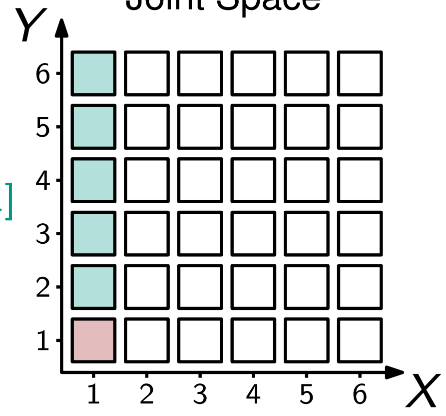
Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S} |\Pr[X=x]-\Pr[Y=x]| \\
&= \sum_{x\in S_X} \cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X} \Pr[X=x]-\Pr[Y=x] + \sum_{x\in S_Y} \Pr[Y=x]-\Pr[X=x] \\
&\leq \sum_{x\in S_X} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S_Y} \Pr[Y=x \wedge X\neq x] \\
&\leq \sum_{x\in S} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S} \Pr[Y=x \wedge X\neq x]
\end{aligned}$$

Joint Space

$\Pr[X=1 \wedge Y\neq 1]$



Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X = x] - \Pr[Y = x]|$.

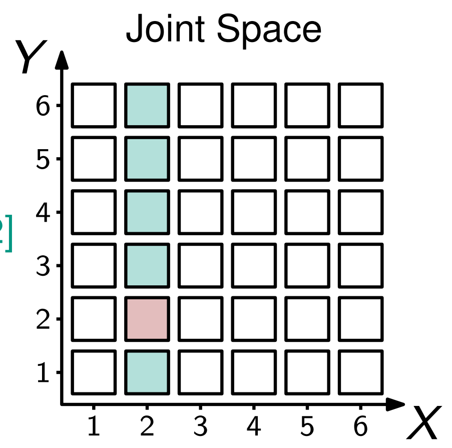
- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X = x] \geq Pr[Y = x]\}$ $\quad S_Y = S \setminus S_X$

$$
\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]| \\
&= \sum_{x\in S_X} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X} \Pr[X=x] - \Pr[Y=x] + \sum_{x\in S_Y} \Pr[Y=x] - \Pr[X=x] \\
&\leq \sum_{x\in S_X} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S_Y} \Pr[Y=x \wedge X\neq x] \\
&\leq \sum_{x\in S} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S} \Pr[Y=x \wedge X\neq x]
\end{aligned}
$$

$\Pr[X=2 \wedge Y\neq 2]$



Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$.

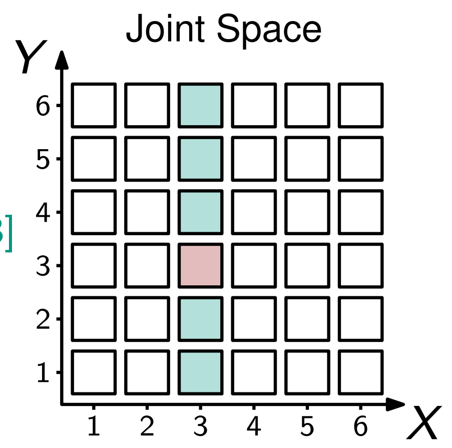
- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$
\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]| \\
&= \sum_{x\in S_X} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X} \Pr[X=x] - \Pr[Y=x] + \sum_{x\in S_Y} \Pr[Y=x] - \Pr[X=x] \\
&\leq \sum_{x\in S_X} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S_Y} \Pr[Y=x \wedge X \neq x] \\
&\leq \sum_{x\in S} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S} \Pr[Y=x \wedge X \neq x]
\end{aligned}
$$

$\Pr[X=3 \wedge Y\neq 3]$



Total Variation Distance

Karlsruhe Institute of Technology

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X, Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X = x] - \Pr[Y = x]|$.

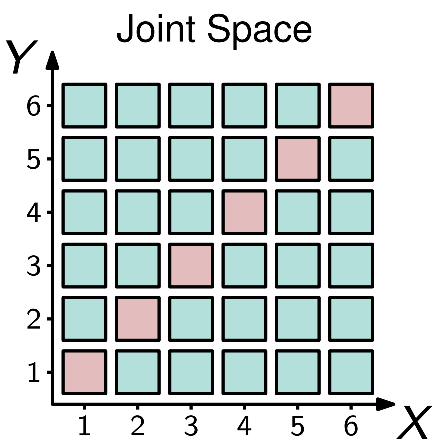
- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X = x] \geq Pr[Y = x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S} |\Pr[X=x]-\Pr[Y=x]| \\
&= \sum_{x\in S_X} \cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X} \Pr[X=x]-\Pr[Y=x] + \sum_{x\in S_Y} \Pr[Y=x]-\Pr[X=x] \\
&\leq \sum_{x\in S_X} \Pr[X=x \wedge Y \neq x] + \sum_{x\in S_Y} \Pr[Y=x \wedge X \neq x] \\
&\leq \sum_{x\in S} \Pr[X=x \wedge Y \neq x] + \sum_{x\in S} \Pr[Y=x \wedge X \neq x]
\end{aligned}$$

$\Pr[X \neq Y]$



Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

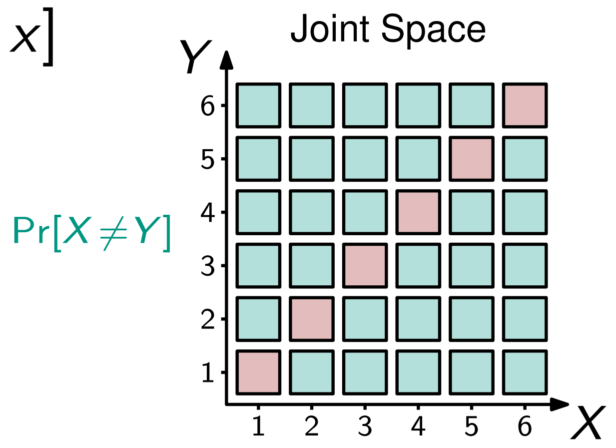
Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S}|\Pr[X=x] - \Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$
\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]| \\
&= \sum_{x\in S_X}\cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|}+\sum_{x\in S_Y}\cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X}\Pr[X=x]-\Pr[Y=x]+\sum_{x\in S_Y}\Pr[Y=x]-\Pr[X=x] \\
&\leq \sum_{x\in S_X}\Pr[X=x \wedge Y\neq x]+\sum_{x\in S_Y}\Pr[Y=x \wedge X\neq x] \\
&\leq \sum_{x\in S}\Pr[X=x \wedge Y\neq x] + \sum_{x\in S}\Pr[Y=x \wedge X\neq x] \\
&= \Pr[X\neq Y] + \Pr[Y\neq X]
\end{aligned}
$$



Total Variation Distance

Karlsruhe Institute of Technology

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

> **Definition**: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]|$.

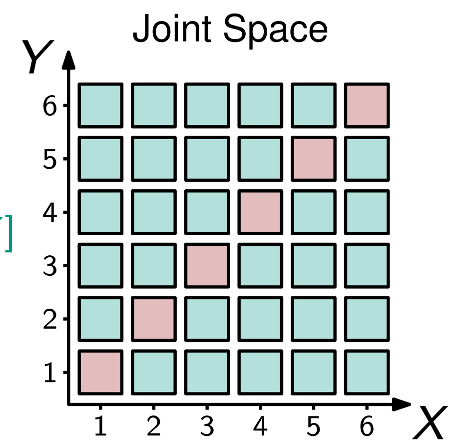
- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

> Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
2d_{TV}(X,Y) &= \sum_{x\in S} |\Pr[X=x] - \Pr[Y=x]| \\
&= \sum_{x\in S_X} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} + \sum_{x\in S_Y} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} \\
&= \sum_{x\in S_X} \Pr[X=x] - \Pr[Y=x] + \sum_{x\in S_Y} \Pr[Y=x] - \Pr[X=x] \\
&\leq \sum_{x\in S_X} \Pr[X=x \wedge Y\neq x] + \sum_{x\in S_Y} \Pr[Y=x \wedge X\neq x] \\
&\leq \boxed{\sum_{x\in S} \Pr[X=x \wedge Y\neq x]} + \sum_{x\in S} \Pr[Y=x \wedge X\neq x] \\
&= \Pr[X\neq Y] + \Pr[Y\neq X] = 2\Pr[X\neq Y]
\end{aligned}$$

$\Pr[X\neq Y]$



Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x \in S} |\Pr[X=x] - \Pr[Y=x]|$.

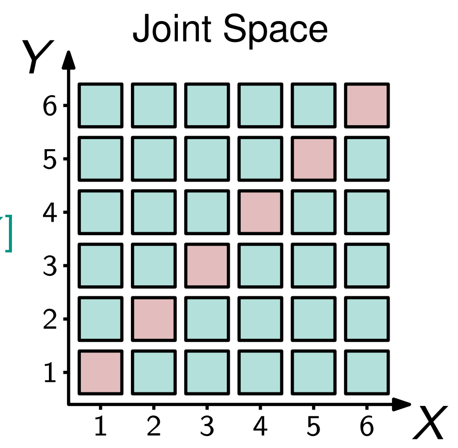
- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \geq Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$
\begin{aligned}
\cancel{2}d_{TV}(X,Y) &= \sum_{x \in S} |\Pr[X=x] - \Pr[Y=x]| \\
&= \sum_{x \in S_X} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} + \sum_{x \in S_Y} \cancel{|}\Pr[X=x] - \Pr[Y=x]\cancel{|} \\
&= \sum_{x \in S_X} \Pr[X=x] - \Pr[Y=x] + \sum_{x \in S_Y} \Pr[Y=x] - \Pr[X=x] \\
&\leq \sum_{x \in S_X} \Pr[X=x \wedge Y \neq x] + \sum_{x \in S_Y} \Pr[Y=x \wedge X \neq x] \\
&\leq \sum_{x \in S} \Pr[X=x \wedge Y \neq x] + \sum_{x \in S} \Pr[Y=x \wedge X \neq x] \\
&= \Pr[X \neq Y] + \Pr[Y \neq X] = \cancel{2}\Pr[X \neq Y]
\end{aligned}
$$

$\Pr[X \neq Y]$



Total Variation Distance

Karlsruhe Institute of Technology

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S}|\Pr[X=x] - \Pr[Y=x]|$.

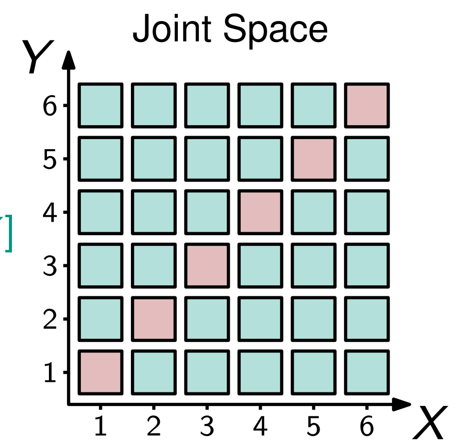
- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \le \Pr[A \wedge \bar{B}]$

$S_X = \{x \in S \mid \Pr[X=x] \ge Pr[Y=x]\}$ $\quad S_Y = S \setminus S_X$

$$\begin{aligned}
\cancel{2}d_{TV}(X,Y) &= \textstyle\sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]| \\
&= \textstyle\sum_{x\in S_X}\cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|}+\sum_{x\in S_Y}\cancel{|}\Pr[X=x]-\Pr[Y=x]\cancel{|} \\
&= \textstyle\sum_{x\in S_X}\Pr[X=x]-\Pr[Y=x]+\sum_{x\in S_Y}\Pr[Y=x]-\Pr[X=x] \\
&\le \textstyle\sum_{x\in S_X}\Pr[X=x \wedge Y\neq x]+\sum_{x\in S_Y}\Pr[Y=x \wedge X\neq x] \\
&\le \textstyle\sum_{x\in S}\Pr[X=x \wedge Y\neq x] + \sum_{x\in S}\Pr[Y=x \wedge X\neq x] \\
&= \Pr[X\neq Y] + \Pr[Y\neq X] = \cancel{2}\Pr[X\neq Y]
\end{aligned}$$

$\Pr[X\neq Y]$



Lemma: $d_{TV}(X,Y) \le \Pr[X \neq Y]$.

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S}|\Pr[X=x]-\Pr[Y=x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

Lemma: $d_{TV}(X,Y) \leq \Pr[X \neq Y]$.

Total Variation Distance

Karlsruhe Institute of Technology

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X, Y) = \frac{1}{2}\sum_{x \in S} |\Pr[X = x] - \Pr[Y = x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

Lemma: $d_{TV}(X, Y) \leq \Pr[X \neq Y]$.

- Note that d_{TV} is defined via the distributions of X and Y

Total Variation Distance

Karlsruhe Institute of Technology

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X, Y) = \frac{1}{2} \sum_{x \in S} |\Pr[X = x] - \Pr[Y = x]|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

Lemma: $d_{TV}(X, Y) \leq \Pr[X \neq Y]$.

- Note that d_{TV} is defined via the distributions of X and Y
- For any coupling (X', Y') of X, Y we have $X' \overset{d}{=} X$ and $Y' \overset{d}{=} Y$. Thus, $d_{TV}(X, Y) = d_{TV}(X', Y')$

Total Variation Distance

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X, Y) = \frac{1}{2} \sum_{x \in S} \left| \Pr[X = x] - \Pr[Y = x] \right|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

Lemma: $d_{TV}(X, Y) \leq \Pr[X \neq Y]$.

- Note that d_{TV} is defined via the distributions of X and Y
- For any coupling (X', Y') of X, Y we have $X' \stackrel{d}{=} X$ and $Y' \stackrel{d}{=} Y$. Thus, $d_{TV}(X, Y) = d_{TV}(X', Y')$

Lemma (coupling inequality): Let X, Y be random variables. Then for any coupling (X', Y') of X and Y it holds that $d_{TV}(X, Y) \leq \Pr[X' \neq Y']$.

Total Variation Distance

Karlsruhe Institute of Technology

- A measure of distance between the distributions of random variables

(Disclaimer: In the following we use a very simplified notation that abstracts away a lot of details!)

Definition: Let X, Y be random variables taking values in a set S. The **total variation distance** of X and Y is $d_{TV}(X,Y) = \frac{1}{2}\sum_{x\in S}\left|\Pr[X=x] - \Pr[Y=x]\right|$.

- Intuition: Sum over the differences in the probabilities
- Maybe a bit tedious to work with, simple bound:

Fréchet: $\Pr[A] - \Pr[B] \leq \Pr[A \wedge \bar{B}]$

Lemma: $d_{TV}(X,Y) \leq \Pr[X \neq Y]$.

- Note that d_{TV} is defined via the distributions of X and Y
- For any coupling (X', Y') of X, Y we have $X' \overset{d}{=} X$ and $Y' \overset{d}{=} Y$. Thus, $d_{TV}(X,Y) = d_{TV}(X',Y')$

Lemma (coupling inequality): Let X, Y be random variables. Then for any coupling (X', Y') of X and Y it holds that $d_{TV}(X,Y) \leq \Pr[X' \neq Y']$.

Lemma (triangle inequality): For rand. var. X, Y, Z: $d_{TV}(X,Z) \leq d_{TV}(X,Y) + d_{TV}(Y,Z)$.

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \text{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \text{Bin}(n, p)$
- Ind. $Y_i \sim \text{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\xrightarrow{\textit{Why?}}$ $Y \sim \mathrm{Pois}(n\lambda)$

$\boxed{\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!}$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n \log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2} \log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \text{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \text{Bin}(n, p)$
- Ind. $Y_i \sim \text{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \text{Pois}(n\lambda)$

Lemma: $X \sim \text{Bin}(n, p)$, $Y \sim \text{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

$X_i \xleftrightarrow{\text{independent}} Y_i$

$\overset{d}{=} \qquad \overset{d}{=}$

Proof

- For each i we couple Y_i and X_i:

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n \log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2} \log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$

$$\begin{tikzcd} X_i \arrow[rr, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}"] & & Y_i \arrow[d, phantom, "\overset{d}{=}"] \\ & & Y_i' \end{tikzcd}$$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \text{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \text{Bin}(n, p)$
- Ind. $Y_i \sim \text{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \text{Pois}(n\lambda)$

Lemma: $X \sim \text{Bin}(n, p)$, $Y \sim \text{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

$$\begin{tikzcd} X_i \arrow[rr, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}"] & & Y_i \arrow[d, phantom, "\overset{d}{=}"] \\ X_i' \arrow[rr, leftrightarrow, "\text{dependent}"] & & Y_i' \end{tikzcd}$$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2} \log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

$$\begin{array}{ccc} X_i & \xleftrightarrow{\text{independent}} & Y_i \\ \overset{d}{\|} & & \overset{d}{\|}\ ✓ \\ X_i' & \xleftrightarrow{\text{dependent}} & Y_i' \end{array}$$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$

The Binomial-Poisson-Approximation

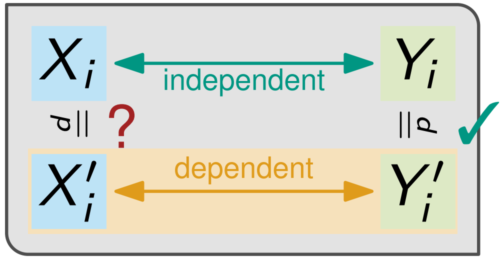
- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n,p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$



The Binomial-Poisson-Approximation

Karlsruhe Institute of Technology

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \le \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\begin{tikzcd}
X_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\;?"] & Y_i \arrow[d, phantom, "\overset{d}{=}\;\checkmark"] \\
X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i'
\end{tikzcd}$$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \text{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \text{Bin}(n, p)$
- Ind. $Y_i \sim \text{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \text{Pois}(n\lambda)$

Lemma: $X \sim \text{Bin}(n, p)$, $Y \sim \text{Pois}(-n \log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2} \log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$\Pr[X_i' = 0]$

$$\begin{tikzcd}
X_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\ ?"] & Y_i \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] \\
X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i'
\end{tikzcd}$$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

$$\begin{tikzcd} X_i \arrow[r, leftrightarrow, "\text{independent}"'] \arrow[d, phantom, "\overset{d}{=}\,?"] & Y_i \arrow[d, phantom, "\overset{d}{=}\,\checkmark"] \\ X'_i \arrow[r, leftrightarrow, "\text{dependent}"] & Y'_i \end{tikzcd}$$

Proof

- For each i we couple Y_i and X_i: $Y'_i = Y_i$, $X'_i = \min\{Y'_i, 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X'_i$

$\Pr[X'_i = 0]$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n \log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2} \log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

$$
\begin{tikzcd}
X_i \arrow[r, leftrightarrow, "\text{independent}"'] \arrow[d, phantom, "\overset{d}{=}\,?"] & Y_i \arrow[d, phantom, "\overset{d}{=}\,\checkmark"] \\
X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i'
\end{tikzcd}
$$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

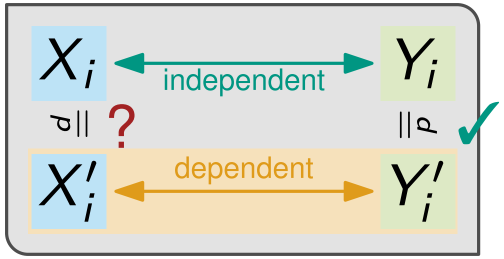
$\Pr[X_i' = 0] = \Pr[Y_i = 0]$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$



Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$\Pr[X_i' = 0] = \Pr[Y_i = 0]$

The Binomial-Poisson-Approximation

Karlsruhe Institute of Technology

- Ind. $X_i \sim \text{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \text{Bin}(n,p)$
- Ind. $Y_i \sim \text{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \text{Pois}(n\lambda)$

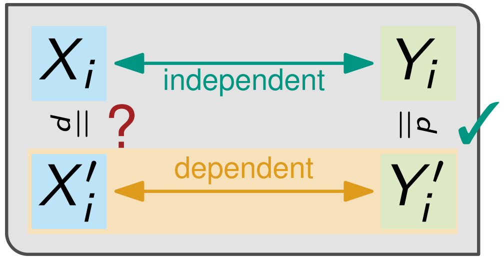
Lemma: $X \sim \text{Bin}(n,p)$, $Y \sim \text{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda}$

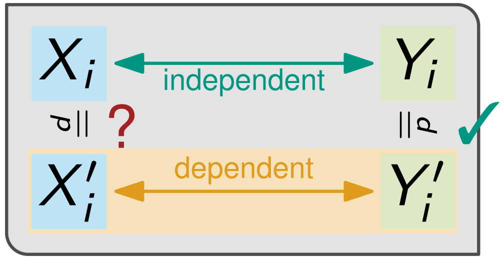


The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n,p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \le \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$



Proof

- For each i we couple Y_i and X_i: $Y'_i = Y_i$, $X'_i = \min\{Y'_i, 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X'_i$

$\Pr[X'_i = 0] = \Pr[Y_i = 0] = e^{-\lambda}$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)}$

$$\begin{array}{ccc} X_i & \xleftrightarrow{\text{independent}} & Y_i \\ \overset{d}{=}\ ? & & \overset{d}{=}\ \checkmark \\ X_i' & \xleftrightarrow{\text{dependent}} & Y_i' \end{array}$$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \checkmark$$

$$\begin{tikzcd}
X_i \arrow[rr, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\,?"] & & Y_i \arrow[d, phantom, "\overset{d}{=}\,\checkmark"] \\
X_i' \arrow[rr, leftrightarrow, "\text{dependent}"] & & Y_i'
\end{tikzcd}$$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n,p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \stackrel{d}{=} X_i'$

$$\begin{array}{ccc} X_i & \xleftrightarrow{\text{independent}} & Y_i \\ \stackrel{d}{=}\ ? & & \stackrel{d}{=}\ \checkmark \\ X_i' & \xleftrightarrow{\text{dependent}} & Y_i' \end{array}$$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0]$ ✓

$\Pr[X_i' = 1]$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

$$\begin{array}{ccc} X_i & \xleftrightarrow{\text{independent}} & Y_i \\ \overset{d}{\parallel}\ ? & & \overset{d}{\parallel}\ \checkmark \\ X_i' & \xleftrightarrow{\text{dependent}} & Y_i' \end{array}$$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \checkmark$$

$$\Pr[X_i' = 1]$$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

$$\begin{tikzcd}
X_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\,?"] & Y_i \arrow[d, phantom, "\overset{d}{=}\,\checkmark"] \\
X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i'
\end{tikzcd}$$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0]$ ✓

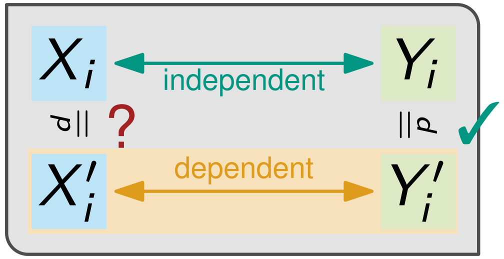
$\Pr[X_i' = 1] = \Pr[Y_i > 0]$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$



Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \stackrel{d}{=} X_i'$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0]$ ✓

$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0]$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

$$\begin{array}{ccc} X_i & \xleftrightarrow{\text{independent}} & Y_i \\ \overset{d}{=}\ ? & & \overset{d}{=}\ \checkmark \\ X_i' & \xleftrightarrow{\text{dependent}} & Y_i' \end{array}$$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \checkmark$$

$$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0]$$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \text{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \text{Bin}(n, p)$
- Ind. $Y_i \sim \text{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \text{Pois}(n\lambda)$

Lemma: $X \sim \text{Bin}(n, p)$, $Y \sim \text{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \checkmark$$

$$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0]$$

$$\begin{tikzcd}
X_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\,?"] & Y_i \arrow[d, phantom, "\overset{d}{=}\,\checkmark"] \\
X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i'
\end{tikzcd}$$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

$$\begin{tikzcd} X_i \arrow[r, leftrightarrow, "\text{independent}"'] \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] & Y_i \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] \\ X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i' \end{tikzcd}$$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0]$ ✓

$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1]$ ✓

The Binomial-Poisson-Approximation

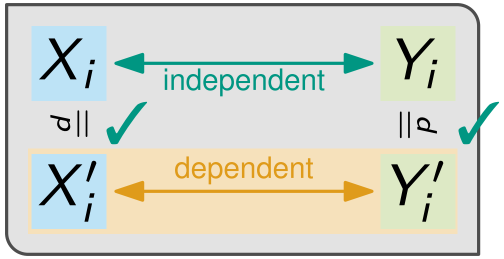
- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$



$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0]$ ✓

$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1]$ ✓

- $X' = \sum_{i=1}^n X_i'$, $Y' = \sum_{i=1}^n Y_i'$

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n \log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2} \log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\begin{tikzcd} X_i \arrow[rr, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] & & Y_i \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] \\ X_i' \arrow[rr, leftrightarrow, "\text{dependent}"] & & Y_i' \end{tikzcd}$$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \ \checkmark$$

$$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1] \ \checkmark$$

- $X' = \sum_{i=1}^{n} X_i'$, $Y' = \sum_{i=1}^{n} Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\begin{tikzcd} X_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] & Y_i \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] \\ X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i' \end{tikzcd}$$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0]$ ✓

$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1]$ ✓

- $X' = \sum_{i=1}^{n} X_i'$, $Y' = \sum_{i=1}^{n} Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$d_{TV}(X, Y) \leq \Pr[X' \neq Y']$

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X, Y) \leq \Pr[X' \neq Y']$.

The Binomial-Poisson-Approximation

Karlsruhe Institute of Technology

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n,p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$
\begin{tikzcd}
X_i \arrow[rr, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] & & Y_i \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] \\
X_i' \arrow[rr, leftrightarrow, "\text{dependent}"] & & Y_i'
\end{tikzcd}
$$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0]$ ✓

$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1]$ ✓

- $X' = \sum_{i=1}^{n} X_i'$, $Y' = \sum_{i=1}^{n} Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$d_{TV}(X,Y) \leq \Pr[X' \neq Y'] \leq \sum_{i=1}^{n} \Pr[X_i' \neq Y_i']$

union bound

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X,Y) \leq \Pr[X' \neq Y']$.

The Binomial-Poisson-Approximation

Karlsruhe Institute of Technology

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

$X_i \xleftrightarrow{\text{independent}} Y_i$

$X_i \overset{d}{=} X_i'$ ✓ $\quad Y_i \overset{d}{=} Y_i'$ ✓

$X_i' \xleftrightarrow{\text{dependent}} Y_i'$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \ ✓$$

$$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1] \ ✓$$

- $X' = \sum_{i=1}^{n} X_i'$, $Y' = \sum_{i=1}^{n} Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$$d_{TV}(X,Y) \leq \Pr[X' \neq Y'] \underset{\text{union bound}}{\leq} \sum_{i=1}^{n} \Pr[X_i' \neq Y_i'] = \sum_{i=1}^{n} \Pr[Y_i' \geq 2]$$

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X,Y) \leq \Pr[X' \neq Y']$.

The Binomial-Poisson-Approximation

Karlsruhe Institute of Technology

- Ind. $X_i \sim \text{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \text{Bin}(n, p)$
- Ind. $Y_i \sim \text{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \text{Pois}(n\lambda)$

Lemma: $X \sim \text{Bin}(n, p)$, $Y \sim \text{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \le \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \checkmark$$

$$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1] \checkmark$$

$$\begin{array}{ccc} X_i & \xleftrightarrow{\text{independent}} & Y_i \\ \overset{d}{\parallel} \checkmark & & \overset{d}{\parallel} \checkmark \\ X_i' & \xleftrightarrow{\text{dependent}} & Y_i' \end{array}$$

- $X' = \sum_{i=1}^{n} X_i'$, $Y' = \sum_{i=1}^{n} Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$$d_{TV}(X, Y) \le \Pr[X' \ne Y'] \underset{\text{union bound}}{\le} \sum_{i=1}^{n} \Pr[X_i' \ne Y_i'] = \sum_{i=1}^{n} \Pr[Y_i' \ge 2]$$

$$= \sum_{i=1}^{n} \sum_{j \ge 2} \Pr[Y_i = j]$$

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X, Y) \le \Pr[X' \ne Y']$.

The Binomial-Poisson-Approximation

Karlsruhe Institute of Technology

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n, p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n, p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X, Y) \leq \frac{n}{2} \log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\begin{tikzcd}
X_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] & Y_i \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] \\
X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i'
\end{tikzcd}$$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \checkmark$$

$$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1] \checkmark$$

- $X' = \sum_{i=1}^{n} X_i'$, $Y' = \sum_{i=1}^{n} Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$$d_{TV}(X, Y) \leq \Pr[X' \neq Y'] \underset{\text{union bound}}{\leq} \sum_{i=1}^{n} \Pr[X_i' \neq Y_i'] = \sum_{i=1}^{n} \Pr[Y_i' \geq 2]$$

$$= \sum_{i=1}^{n} \sum_{j \geq 2} \Pr[Y_i = j]$$

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X, Y) \leq \Pr[X' \neq Y']$.

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \text{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \text{Bin}(n,p)$
- Ind. $Y_i \sim \text{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \text{Pois}(n\lambda)$

Lemma: $X \sim \text{Bin}(n,p)$, $Y \sim \text{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \le \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$
\begin{tikzcd}
X_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] & Y_i \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] \\
X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i'
\end{tikzcd}
$$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0]$ ✓

$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1]$ ✓

- $X' = \sum_{i=1}^n X_i'$, $Y' = \sum_{i=1}^n Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$$d_{TV}(X,Y) \le \Pr[X' \neq Y'] \underset{\text{union bound}}{\le} \sum_{i=1}^n \Pr[X_i' \neq Y_i'] = \sum_{i=1}^n \Pr[Y_i' \ge 2]$$

$$= \sum_{i=1}^n \sum_{j \ge 2} e^{-\lambda}\frac{\lambda^j}{j!}$$

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X,Y) \le \Pr[X' \neq Y']$.

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n,p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \leq \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \checkmark$$
$$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1] \checkmark$$

$X_i \xleftrightarrow{\text{independent}} Y_i$

$X_i \overset{d}{=} X_i'$ ✓, $Y_i \overset{d}{=} Y_i'$ ✓

$X_i' \xleftrightarrow{\text{dependent}} Y_i'$

- $X' = \sum_{i=1}^{n} X_i'$, $Y' = \sum_{i=1}^{n} Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$$d_{TV}(X,Y) \leq \Pr[X' \neq Y'] \underbrace{\leq}_{\text{union bound}} \sum_{i=1}^{n} \Pr[X_i' \neq Y_i'] = \sum_{i=1}^{n} \Pr[Y_i' \geq 2]$$
$$= \sum_{i=1}^{n} e^{-\lambda} \sum_{j \geq 2} \frac{\lambda^j}{j!}$$

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X,Y) \leq \Pr[X' \neq Y']$.

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \text{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \text{Bin}(n,p)$
- Ind. $Y_i \sim \text{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \text{Pois}(n\lambda)$

Lemma: $X \sim \text{Bin}(n,p)$, $Y \sim \text{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \le \frac{n}{2}\log(1-p)^2$.

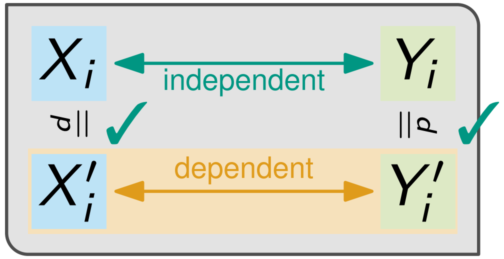
$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0]$ ✓

$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1]$ ✓



- $X' = \sum_{i=1}^n X_i'$, $Y' = \sum_{i=1}^n Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$$d_{TV}(X,Y) \le \Pr[X' \ne Y'] \underset{\text{union bound}}{\le} \sum_{i=1}^n \Pr[X_i' \ne Y_i'] = \sum_{i=1}^n \Pr[Y_i' \ge 2]$$

$$= \sum_{i=1}^n e^{-\lambda} \underbrace{\sum_{j\ge 2} \frac{\lambda^j}{j!}}_{= \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots}$$

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X,Y) \le \Pr[X' \ne Y']$.

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n,p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \le \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k / k!$

$$\begin{tikzcd}
X_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] & Y_i \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] \\
X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i'
\end{tikzcd}$$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \ \checkmark$$
$$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1] \ \checkmark$$

- $X' = \sum_{i=1}^n X_i'$, $Y' = \sum_{i=1}^n Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$$d_{TV}(X,Y) \le \Pr[X' \ne Y'] \underset{\text{union bound}}{\le} \sum_{i=1}^n \Pr[X_i' \ne Y_i'] = \sum_{i=1}^n \Pr[Y_i' \ge 2]$$
$$= \sum_{i=1}^n e^{-\lambda} \underbrace{\sum_{j \ge 2} \frac{\lambda^j}{j!}}_{= \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots \le \frac{\lambda^2}{2}\left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots\right)}$$

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X,Y) \le \Pr[X' \ne Y']$.

The Binomial-Poisson-Approximation

Karlsruhe Institute of Technology

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n,p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \le \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \checkmark$$

$$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1] \checkmark$$

$$\begin{array}{ccc} X_i & \xleftrightarrow{\text{independent}} & Y_i \\ \overset{d}{\|}\ \checkmark & & \overset{d}{\|}\ \checkmark \\ X_i' & \xleftrightarrow{\text{dependent}} & Y_i' \end{array}$$

- $X' = \sum_{i=1}^n X_i'$, $Y' = \sum_{i=1}^n Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$$d_{TV}(X,Y) \le \Pr[X' \ne Y'] \underset{\text{union bound}}{\le} \sum_{i=1}^n \Pr[X_i' \ne Y_i'] = \sum_{i=1}^n \Pr[Y_i' \ge 2]$$

$$= \sum_{i=1}^n e^{-\lambda} \underbrace{\sum_{j\ge 2} \frac{\lambda^j}{j!}}_{= \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots \le \frac{\lambda^2}{2}\left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots\right)} \le \sum_{i=1}^n \frac{\lambda^2}{2} e^{-\lambda} \sum_{j \ge 0} \frac{\lambda^j}{j!}$$

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X,Y) \le \Pr[X' \ne Y']$.

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^{n} X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n,p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^{n} Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \le \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

$$\begin{tikzcd} X_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] & Y_i \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] \\ X'_i \arrow[r, leftrightarrow, "\text{dependent}"] & Y'_i \end{tikzcd}$$

Proof

- For each i we couple Y_i and X_i: $Y'_i = Y_i$, $X'_i = \min\{Y'_i, 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X'_i$

$$\Pr[X'_i = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1-p = \Pr[X_i = 0] \ \checkmark$$
$$\Pr[X'_i = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1] \ \checkmark$$

- $X' = \sum_{i=1}^{n} X'_i$, $Y' = \sum_{i=1}^{n} Y'_i \Rightarrow (X',Y')$ coupling of (X,Y)

$$d_{TV}(X,Y) \le \Pr[X' \neq Y'] \underset{\text{union bound}}{\le} \sum_{i=1}^{n} \Pr[X'_i \neq Y'_i] = \sum_{i=1}^{n} \Pr[Y'_i \ge 2]$$
$$= \sum_{i=1}^{n} e^{-\lambda} \sum_{j \ge 2} \frac{\lambda^j}{j!} \le \sum_{i=1}^{n} \frac{\lambda^2}{2} e^{-\lambda} \underbrace{\sum_{j \ge 0} \frac{\lambda^j}{j!}}_{= e^{\lambda}}$$

Function of Couplings
Couplings (X'_i, Y'_i) of (X_i, Y_i):
$(f(X'_i), f(Y'_i))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X',Y') of X, Y:
$d_{TV}(X,Y) \le \Pr[X' \neq Y']$.

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n,p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \le \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0] \checkmark$$

$$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1] \checkmark$$

$$\begin{array}{ccc} X_i & \xleftrightarrow{\text{independent}} & Y_i \\ \overset{d}{\parallel}\ \checkmark & & \overset{d}{\parallel}\ \checkmark \\ X_i' & \xleftrightarrow{\text{dependent}} & Y_i' \end{array}$$

- $X' = \sum_{i=1}^n X_i'$, $Y' = \sum_{i=1}^n Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

$$d_{TV}(X,Y) \le \Pr[X' \ne Y'] \underset{\text{union bound}}{\le} \sum_{i=1}^n \Pr[X_i' \ne Y_i'] = \sum_{i=1}^n \Pr[Y_i' \ge 2]$$

$$= \sum_{i=1}^n e^{-\lambda} \sum_{j\ge 2} \frac{\lambda^j}{j!} \le \sum_{i=1}^n \frac{\lambda^2}{2} \cancel{e^{-\lambda}} \underbrace{\cancel{\sum_{j\ge 0} \frac{\lambda^j}{j!}}}_{=e^{\lambda}}$$

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X,Y) \le \Pr[X' \ne Y']$.

The Binomial-Poisson-Approximation

- Ind. $X_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\longrightarrow$ $X = \sum_{i=1}^n X_i$ $\longrightarrow$ $X \sim \mathrm{Bin}(n,p)$
- Ind. $Y_i \sim \mathrm{Pois}(\lambda)$ for $i \in [n]$, $\lambda = -\log(1-p)$ $\longrightarrow$ $Y = \sum_{i=1}^n Y_i$ $\longrightarrow$ $Y \sim \mathrm{Pois}(n\lambda)$

Lemma: $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Pois}(-n\log(1-p))$: $d_{TV}(X,Y) \le \frac{n}{2}\log(1-p)^2$.

$\Pr[Y_i = k] = e^{-\lambda}\lambda^k/k!$

$$\begin{tikzcd} X_i \arrow[r, leftrightarrow, "\text{independent}"] \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] & Y_i \arrow[d, phantom, "\overset{d}{=}\ \checkmark"] \\ X_i' \arrow[r, leftrightarrow, "\text{dependent}"] & Y_i' \end{tikzcd}$$

Proof

- For each i we couple Y_i and X_i: $Y_i' = Y_i$, $X_i' = \min\{Y_i', 1\}$
- To show that this is a coupling, we need $X_i \overset{d}{=} X_i'$

$\Pr[X_i' = 0] = \Pr[Y_i = 0] = e^{-\lambda} = e^{\log(1-p)} = 1 - p = \Pr[X_i = 0]$ ✓

$\Pr[X_i' = 1] = \Pr[Y_i > 0] = 1 - \Pr[Y_i = 0] = 1 - \Pr[X_i = 0] = \Pr[X_i = 1]$ ✓

- $X' = \sum_{i=1}^n X_i'$, $Y' = \sum_{i=1}^n Y_i'$ $\Rightarrow$ (X', Y') coupling of (X, Y)

Function of Couplings
Couplings (X_i', Y_i') of (X_i, Y_i):
$(f(X_i'), f(Y_i'))$ coupling of $(f(X_i), f(Y_i))$.

$$d_{TV}(X,Y) \le \Pr[X' \ne Y'] \underset{\text{union bound}}{\le} \sum_{i=1}^n \Pr[X_i' \ne Y_i'] = \sum_{i=1}^n \Pr[Y_i' \ge 2]$$

$$= \sum_{i=1}^n e^{-\lambda} \sum_{j \ge 2} \frac{\lambda^j}{j!} \le \sum_{i=1}^n \frac{\lambda^2}{2} \cancel{e^{-\lambda}} \underbrace{\cancel{\sum_{j\ge 0} \frac{\lambda^j}{j!}}}_{= e^{\lambda}} = \sum_{i=1}^n \frac{\lambda^2}{2} \checkmark$$

Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X,Y) \le \Pr[X' \ne Y']$.

Recap: Theory-Practice Gap

Theory

- Many computational problems are assumed to be hard
- Looks like there are no algorithms that can solve these problems fast

$\underbrace{\text{SAT} \quad \text{Vertex Cover} \quad \text{Independent Set}}_{\text{NP-hard}}$

Recap: Theory-Practice Gap

Theory

- Many computational problems are assumed to be hard
- Looks like there are no algorithms that can solve these problems fast

$\underbrace{\text{SAT} \quad \text{Vertex Cover} \quad \text{Independent Set}}_{\text{NP-hard}}$

Practice

- Many computational problems can be solved extremely fast

Recap: Theory-Practice Gap

Theory

- Many computational problems are assumed to be hard
- Looks like there are no algorithms that can solve these problems fast

$\underbrace{\text{SAT} \quad \text{Vertex Cover} \quad \text{Independent Set}}_{\text{NP-hard}}$

Practice

- Many computational problems can be solved extremely fast
  - "Modern SAT solvers can often handle problems with millions of clauses and hundreds of thousands of variables"

"Propagation = Lazy Clause Generation", Ohrimenko, Stuckey & Codish, CP, 2017

Recap: Theory-Practice Gap

Theory

- Many computational problems are assumed to be hard
- Looks like there are no algorithms that can solve these problems fast

$\underbrace{\text{SAT} \quad \text{Vertex Cover} \quad \text{Independent Set}}_{\text{NP-hard}}$

Practice

- Many computational problems can be solved extremely fast
  - “Modern SAT solvers can often handle problems with millions of clauses and hundreds of thousands of variables”

    “Propagation = Lazy Clause Generation”, Ohrimenko, Stuckey & Codish, CP, 2017
  - For many real-world graphs optimal vertex covers (containing up to millions of nodes) can be found in seconds

    “Branch-and-reduce exponential/FPT algorithms in practice: A case study of vertex cover”, Akiba & Iwata, TCS, 2016

Recap: Theory-Practice Gap

Karlsruhe Institute of Technology

Theory

- Many computational problems are assumed to be hard
- Looks like there are no algorithms that can solve these problems fast

$\underbrace{\text{SAT} \quad \text{Vertex Cover} \quad \text{Independent Set}}_{\text{NP-hard}}$

Practice

- Many computational problems can be solved extremely fast
  - “Modern SAT solvers can often handle problems with millions of clauses and hundreds of thousands of variables”

    “Propagation = Lazy Clause Generation”, Ohrimenko, Stuckey & Codish, CP, 2017
  - For many real-world graphs optimal vertex covers (containing up to millions of nodes) can be found in seconds

    “Branch-and-reduce exponential/FPT algorithms in practice: A case study of vertex cover”, Akiba & Iwata, TCS, 2016

Average-Case Analysis

- Acknowledge difference between theoretical worst-case instances and practical ones
- Represent real world using mathematical models and analyze those theoretically

Random Graph Models

A **graph model** describes a mechanism that can be used to generate a graph.

- Given a set of vertices, how are edges in the graph formed?
- The model consists of *rules* defining which vertices are adjacent
- In *random* graph models these rules involve randomness

Random Graph Models

A **graph model** describes a mechanism that can be used to generate a graph.

- Given a set of vertices, how are edges in the graph formed?
- The model consists of *rules* defining which vertices are adjacent
- In *random* graph models these rules involve randomness

Desirable Properties

- *Simplicity*: We cannot analyze a model that is too complicated

Random Graph Models

A **graph model** describes a mechanism that can be used to generate a graph.

- Given a set of vertices, how are edges in the graph formed?
- The model consists of *rules* defining which vertices are adjacent
- In *random* graph models these rules involve randomness

Desirable Properties

- *Simplicity*: We cannot analyze a model that is too complicated
- *Realism*: We do not want to analyze a model that cannot be used to make predictions about the real world

Random Graph Models

A **graph model** describes a mechanism that can be used to generate a graph.

- Given a set of vertices, how are edges in the graph formed?
- The model consists of *rules* defining which vertices are adjacent
- In *random* graph models these rules involve randomness

Desirable Properties

- *Simplicity*: We cannot analyze a model that is too complicated
- *Realism*: We do not want to analyze a model that cannot be used to make predictions about the real world
- *Fast Generation*: We want to be able to generate many, large benchmark instances to ...
  - analyze structural and algorithmic properties empirically
  - generate hypotheses about asymptotic behavior

Random Graph Models

A **graph model** describes a mechanism that can be used to generate a graph.

- Given a set of vertices, how are edges in the graph formed?
- The model consists of *rules* defining which vertices are adjacent
- In *random* graph models these rules involve randomness

Desirable Properties

- *Simplicity*: We cannot analyze a model that is too complicated
- *Realism*: We do not want to analyze a model that cannot be used to make predictions about the real world
- *Fast Generation*: We want to be able to generate many, large benchmark instances to ...
  - analyze structural and algorithmic properties empirically
  - generate hypotheses about asymptotic behavior

Let's start with a simple model!

Erdős–Rényi Random Graphs

Karlsruhe Institute of Technology

History

- Initially introduced by Edgar Gilbert in 1959

"Random Graphs", Gilbert, Ann. Math. Statist., 1959

Erdős–Rényi Random Graphs

History

- Initially introduced by Edgar Gilbert in 1959

"Random Graphs", Gilbert, Ann. Math. Statist., 1959

- A related version introduced by Paul Erdős and Alfréd Rényi in 1959

"On Random Graphs I", Erdős & Rényi, Publ. Math. Debr., 1959

Erdős–Rényi Random Graphs

History

- Initially introduced by Edgar Gilbert in 1959

"Random Graphs", Gilbert, Ann. Math. Statist., 1959

- A related version introduced by Paul Erdős and Alfréd Rényi in 1959

"On Random Graphs I", Erdős & Rényi, Publ. Math. Debr., 1959

Definitions

$G(n, p)$

- Start with n nodes
- Independently connect any two with fixed probability p

Erdős–Rényi Random Graphs

History

- Initially introduced by Edgar Gilbert in 1959

"Random Graphs", Gilbert, Ann. Math. Statist., 1959

- A related version introduced by Paul Erdős and Alfréd Rényi in 1959

"On Random Graphs I", Erdős & Rényi, Publ. Math. Debr., 1959

Definitions

$G(n, p)$

- Start with n nodes
- Independently connect any two with fixed probability p

$G(n, m)$

- Start with n nodes
- From the $\binom{n}{2}$ possible edges select m uniformly at random

Erdős–Rényi Random Graphs

Karlsruhe Institute of Technology

History

- Initially introduced by Edgar Gilbert in 1959

"Random Graphs", Gilbert, Ann. Math. Statist., 1959

- A related version introduced by Paul Erdős and Alfréd Rényi in 1959

"On Random Graphs I", Erdős & Rényi, Publ. Math. Debr., 1959

Definitions

Gilbert's model, though often meant when talking about Erdős–Rényi graphs

$G(n, p)$

- Start with n nodes
- Independently connect any two with fixed probability p

$G(n, m)$

- Start with n nodes
- From the $\binom{n}{2}$ possible edges select m uniformly at random

Erdős–Rényi Random Graphs

Karlsruhe Institute of Technology

History

- Initially introduced by Edgar Gilbert in 1959
- A related version introduced by Paul Erdős and Alfréd Rényi in 1959

“Random Graphs”, Gilbert, Ann. Math. Statist., 1959

“On Random Graphs I”, Erdős & Rényi, Publ. Math. Debr., 1959

Definitions

Gilbert’s model, though often meant when talking about Erdős–Rényi graphs

$G(n, p)$

- Start with n nodes
- Independently connect any two with fixed probability p

$G(n, m)$

- Start with n nodes
- From the $\binom{n}{2}$ possible edges select m uniformly at random

- For $\tilde{p} = m/\binom{n}{2}$ the *expected* number of edges in $G(n, \tilde{p})$ matches m

Erdős–Rényi Random Graphs

Karlsruhe Institute of Technology

History

- Initially introduced by Edgar Gilbert in 1959

“Random Graphs”, Gilbert, Ann. Math. Statist., 1959

- A related version introduced by Paul Erdős and Alfréd Rényi in 1959

“On Random Graphs I”, Erdős & Rényi, Publ. Math. Debr., 1959

Definitions

Gilbert’s model, though often meant when talking about Erdős–Rényi graphs

$G(n, p)$

- Start with n nodes
- Independently connect any two with fixed probability p

$G(n, m)$

- Start with n nodes
- From the $\binom{n}{2}$ possible edges select m uniformly at random

- For $\tilde{p} = m/\binom{n}{2}$ the *expected* number of edges in $G(n, \tilde{p})$ matches m
- In $G(n, p)$ edges are independent, in $G(n, m)$ they are not

Erdős–Rényi Random Graphs

History

- Initially introduced by Edgar Gilbert in 1959

"Random Graphs", Gilbert, Ann. Math. Statist., 1959

- A related version introduced by Paul Erdős and Alfréd Rényi in 1959

"On Random Graphs I", Erdős & Rényi, Publ. Math. Debr., 1959

Definitions

Gilbert's model, though often meant when talking about Erdős–Rényi graphs

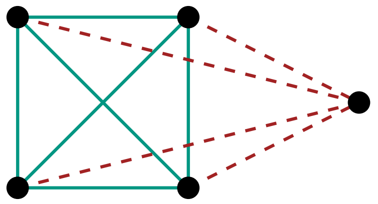
$G(n, p)$

- Start with n nodes
- Independently connect any two with fixed probability p

$G(n, m)$

- Start with n nodes
- From the $\binom{n}{2}$ possible edges select m uniformly at random

- For $\tilde{p} = m/\binom{n}{2}$ the *expected* number of edges in $G(n, \tilde{p})$ matches m
- In $G(n, p)$ edges are independent, in $G(n, m)$ they are not
  - If a $G(5, 6)$ contains a 4-clique, there can be no edge incident to the 5th node



Existence of red edges depends on existence of green ones.

Erdős–Rényi Random Graphs

Karlsruhe Institute of Technology

History

- Initially introduced by Edgar Gilbert in 1959

“Random Graphs”, Gilbert, Ann. Math. Statist., 1959

- A related version introduced by Paul Erdős and Alfréd Rényi in 1959

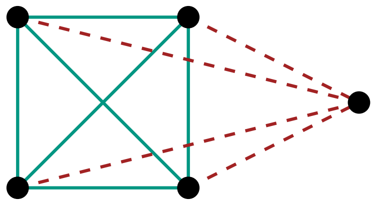
“On Random Graphs I”, Erdős & Rényi, Publ. Math. Debr., 1959

Definitions

Gilbert’s model, though often meant when talking about Erdős–Rényi graphs

$G(n,p)$	$G(n,m)$
■ Start with n nodes ■ Independently connect any two with fixed probability p	■ Start with n nodes ■ From the $\binom{n}{2}$ possible edges select m uniformly at random

- For $\tilde{p} = m/\binom{n}{2}$ the *expected* number of edges in $G(n, \tilde{p})$ matches m
- In $G(n,p)$ edges are independent, in $G(n,m)$ they are not
  - If a $G(5,6)$ contains a 4-clique, there can be no edge incident to the 5th node



Existence of red edges depends on existence of green ones.

- Since many real-world networks are $\overbrace{\textit{sparse}}^{\text{number of edges linear in number of nodes}}$, we focus on $p = \frac{c}{n}$ for $c \in \Theta(1)$

ER – Degree of a Vertex

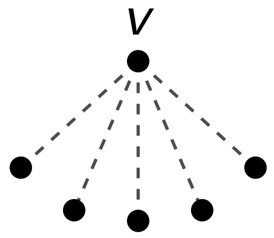
$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges



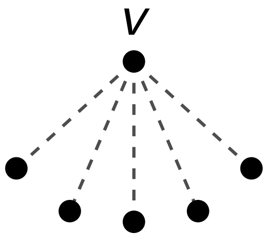
$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p



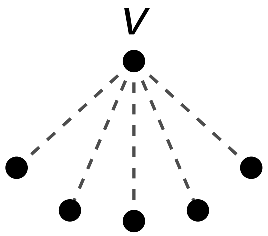
$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p) \longrightarrow \Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$



$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

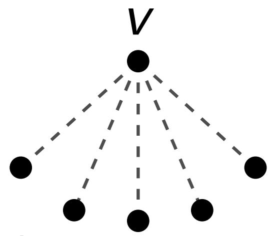
ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$

$$\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$\mathbb{E}[\deg(v)] = (n-1)p \;\&\; \mathrm{Var}[\deg(v)] = (n-1)p(1-p)$$



$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

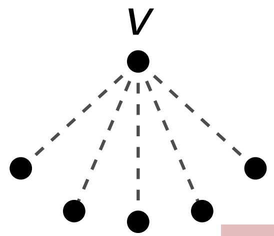
ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$

$$\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$\mathbb{E}[\deg(v)] = (n-1)p \text{ \& } \mathrm{Var}[\deg(v)] = (n-1)p(1-p)$$ *Inconvenient...*



$G(n,p)$
Independently connect any two nodes with fixed probability p.

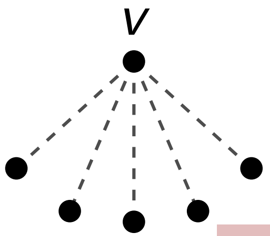
$p = \frac{c}{n}$, $c \in \Theta(1)$

ER – Degree of a Vertex

Karlsruhe Institute of Technology

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ → $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$
  → $\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

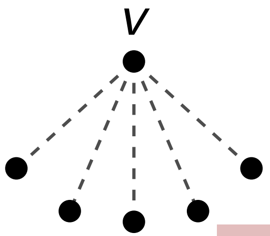
Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$.

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ → $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$
  → $\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$.

Proof

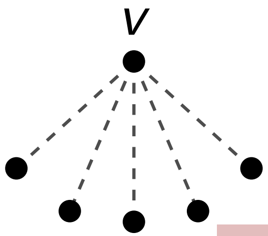
X ←independent→ Y

ER – Degree of a Vertex

Karlsruhe Institute of Technology

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ → $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$
  → $\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$.

Proof

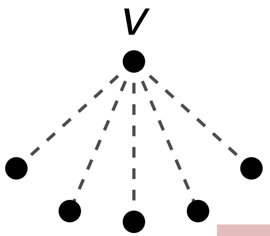
- Independent $Z_i \sim \mathrm{Ber}(p)$ for $i \in [n]$ $\quad Z_1 \quad Z_2 \quad Z_3 \quad \cdots \quad Z_{n-1} \quad Z_n$

$X \xleftrightarrow{\text{independent}} Y$

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \text{Bin}(n-1, p)$



$\rightarrow \Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\rightarrow \mathbb{E}[\deg(v)] = (n-1)p$ & $\text{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*

$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}, c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \text{Bin}(n-1, p)$ and let $Y \sim \text{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$.

Proof

- Independent $Z_i \sim \text{Ber}(p)$ for $i \in [n]$
- $X' = \sum_{i=1}^{n-1} Z_i$

$$\underbrace{Z_1 \quad Z_2 \quad Z_3 \quad \cdots \quad Z_{n-1}}_{X'} \quad Z_n$$

$X \xleftrightarrow{\text{independent}} Y$

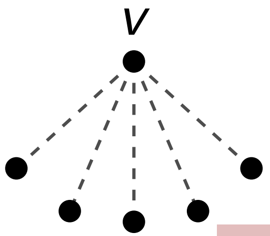
X'

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ $\longrightarrow$ $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

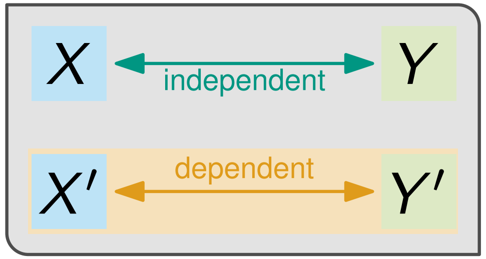
Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$.

Proof

- Independent $Z_i \sim \mathrm{Ber}(p)$ for $i \in [n]$
- $X' = \sum_{i=1}^{n-1} Z_i$, $Y' = X' + Z_n$

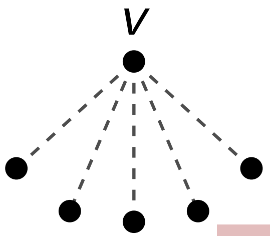
$$\underbrace{\underbrace{Z_1 \quad Z_2 \quad Z_3 \quad \cdots \quad Z_{n-1}}_{X'} \quad Z_n}_{Y'}$$



ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$



$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

$\rightarrow \Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\rightarrow \mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*

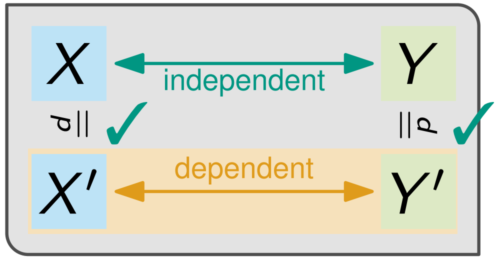
Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$.

Proof

- Independent $Z_i \sim \mathrm{Ber}(p)$ for $i \in [n]$
- $X' = \sum_{i=1}^{n-1} Z_i$, $Y' = X' + Z_n$

$$\underbrace{\underbrace{Z_1 \quad Z_2 \quad Z_3 \quad \cdots \quad Z_{n-1}}_{X'} \quad Z_n}_{Y'}$$

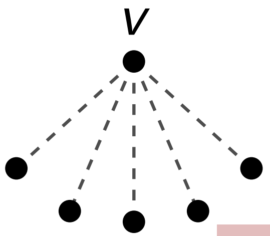


ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ $\longrightarrow$ $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

  $\longrightarrow$ $\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

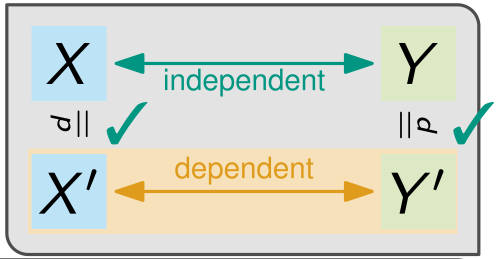
Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$.

Proof

- Independent $Z_i \sim \mathrm{Ber}(p)$ for $i \in [n]$
- $X' = \sum_{i=1}^{n-1} Z_i$, $Y' = X' + Z_n$

  $d_{TV}(X, Y) \leq \Pr[X' \neq Y']$

$$\underbrace{\underbrace{Z_1 \quad Z_2 \quad Z_3 \quad \cdots \quad Z_{n-1}}_{X'} \quad Z_n}_{Y'}$$

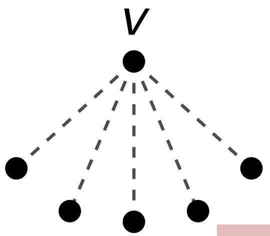


Coupling Inequality
For any coupling (X', Y') of X, Y:
$d_{TV}(X, Y) \leq \Pr[X' \neq Y']$.

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$



$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

$\to \Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\to \mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*

Approximation

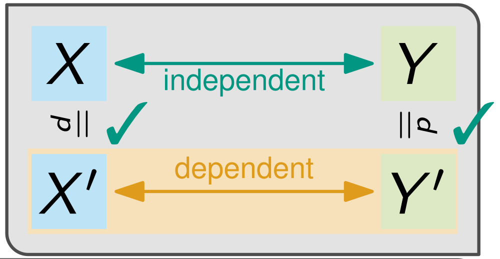
Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1,p)$ and let $Y \sim \mathrm{Bin}(n,p)$. Then, $d_{TV}(X,Y) = o(1)$.

Proof

- Independent $Z_i \sim \mathrm{Ber}(p)$ for $i \in [n]$
- $X' = \sum_{i=1}^{n-1} Z_i$, $Y' = X' + Z_n$

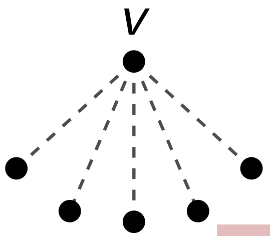
$$d_{TV}(X,Y) \le \Pr[X' \neq Y'] = \Pr[Z_n = 1]$$

$$\underbrace{\underbrace{Z_1 \quad Z_2 \quad Z_3 \quad \cdots \quad Z_{n-1}}_{X'} \quad Z_n}_{Y'}$$



Coupling Inequality
For any coupling (X',Y') of X,Y:
$d_{TV}(X,Y) \le \Pr[X' \neq Y']$.

ER – Degree of a Vertex



Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ $\longrightarrow$ $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$
  $\searrow$ $\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*

$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

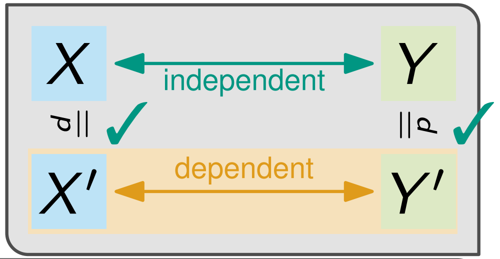
Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1,p)$ and let $Y \sim \mathrm{Bin}(n,p)$. Then, $d_{TV}(X,Y) = o(1)$.

Proof

- Independent $Z_i \sim \mathrm{Ber}(p)$ for $i \in [n]$
- $X' = \sum_{i=1}^{n-1} Z_i$, $Y' = X' + Z_n$

$$d_{TV}(X,Y) \le \Pr[X' \neq Y']$$
$$= \Pr[Z_n = 1] = \frac{c}{n} = o(1) ✓$$

$$\underbrace{\underbrace{Z_1 \quad Z_2 \quad Z_3 \quad \cdots \quad Z_{n-1}}_{X'} \quad Z_n}_{Y'}$$

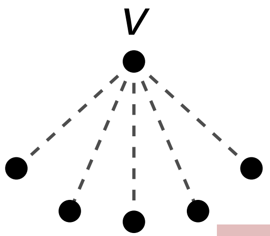


Coupling Inequality
For any coupling (X', Y') of X, Y: $d_{TV}(X,Y) \le \Pr[X' \neq Y']$.

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$



$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

$\to \Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\to \mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$.

Binomial-Poisson-Approximation
$Y \sim \mathrm{Bin}(n, p)$, $Z \sim \mathrm{Pois}(-n \log(1-p))$:
$d_{TV}(Y, Z) \leq \frac{n}{2} \log(1-p)^2$.

Triangle Inequality
$d_{TV}(X, Z) \leq d_{TV}(X, Y) + d_{TV}(Y, Z)$.

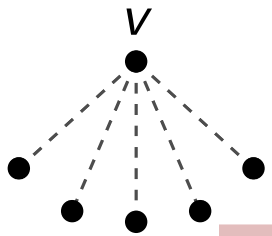
ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$

$\longrightarrow \Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\longrightarrow \mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$. And for $Z \sim \mathrm{Pois}(c + O(\frac{1}{n}))$

$$d_{TV}(X, Z) \leq d_{TV}(X, Y) + d_{TV}(Y, Z)$$

Binomial-Poisson-Approximation
$Y \sim \mathrm{Bin}(n, p)$, $Z \sim \mathrm{Pois}(-n \log(1-p))$:
$d_{TV}(Y, Z) \leq \frac{n}{2} \log(1-p)^2$.

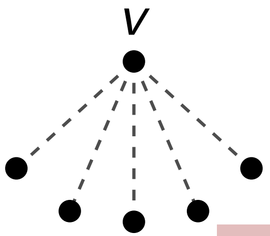
Triangle Inequality
$d_{TV}(X, Z) \leq d_{TV}(X, Y) + d_{TV}(Y, Z)$.

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ → $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

  → $\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$. And for $Z \sim \mathrm{Pois}(c + O(\frac{1}{n}))$

$$\begin{aligned} d_{TV}(X, Z) &\leq d_{TV}(X, Y) + d_{TV}(Y, Z) \\ &= o(1) + \frac{n}{2}\log(1-p)^2 \end{aligned}$$

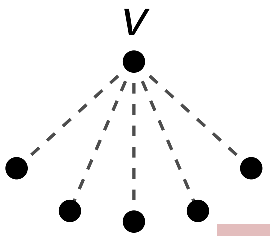
Binomial-Poisson-Approximation
$Y \sim \mathrm{Bin}(n, p)$, $Z \sim \mathrm{Pois}(-n\log(1-p))$:
$d_{TV}(Y, Z) \leq \frac{n}{2}\log(1-p)^2$.

Triangle Inequality
$d_{TV}(X, Z) \leq d_{TV}(X, Y) + d_{TV}(Y, Z)$.

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$



$\longrightarrow \Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\longrightarrow \mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*

$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$. And for $Z \sim \mathrm{Pois}(c + O(\frac{1}{n}))$

$$d_{TV}(X,Z) \leq d_{TV}(X,Y) + d_{TV}(Y,Z)$$
$$= o(1) + \underbrace{\frac{n}{2}\log(1-p)^2}_{\frac{n}{2}(-p-O(p^2)))^2}$$

Binomial-Poisson-Approximation
$Y \sim \mathrm{Bin}(n,p)$, $Z \sim \mathrm{Pois}(-n\log(1-p))$:
$d_{TV}(Y,Z) \leq \frac{n}{2}\log(1-p)^2$.

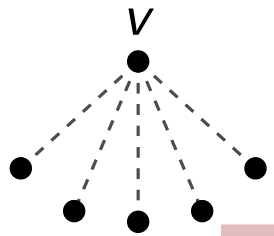
Triangle Inequality
$d_{TV}(X,Z) \leq d_{TV}(X,Y) + d_{TV}(Y,Z)$.

Taylor $p \to 0$: $\log(1-p) = -p - O(p^2)$

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$



$\to \Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\to \mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*

$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$. And for $Z \sim \mathrm{Pois}(c + O(\frac{1}{n}))$

$$d_{TV}(X, Z) \le d_{TV}(X, Y) + d_{TV}(Y, Z)$$
$$= o(1) + \underbrace{\frac{n}{2} \log(1-p)^2}_{\frac{n}{2}(-p - O(p^2)))^2} = \frac{n}{2}(p^2 + O(p^3) + O(p^4))$$

Binomial-Poisson-Approximation
$Y \sim \mathrm{Bin}(n, p)$, $Z \sim \mathrm{Pois}(-n \log(1-p))$:
$d_{TV}(Y, Z) \le \frac{n}{2} \log(1-p)^2$.

Triangle Inequality
$d_{TV}(X, Z) \le d_{TV}(X, Y) + d_{TV}(Y, Z)$.

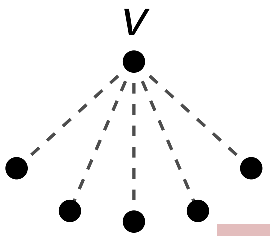
Taylor $p \to 0$: $\log(1-p) = -p - O(p^2)$

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ $\longrightarrow$ $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$. And for $Z \sim \mathrm{Pois}(c + O(\frac{1}{n}))$

$$
\begin{aligned}
d_{TV}(X, Z) &\le d_{TV}(X, Y) + d_{TV}(Y, Z) \\
&= o(1) + \underbrace{\frac{n}{2}\log(1-p)^2}_{\frac{n}{2}(-p - O(p^2)))^2 = \frac{n}{2}(p^2 + O(p^3) + O(p^4))}
\end{aligned}
$$

$\supseteq$

Binomial-Poisson-Approximation
$Y \sim \mathrm{Bin}(n, p)$, $Z \sim \mathrm{Pois}(-n\log(1-p))$:
$d_{TV}(Y, Z) \le \frac{n}{2}\log(1-p)^2$.

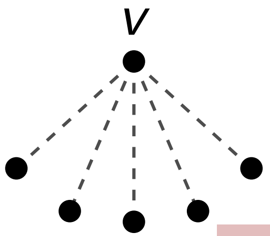
Triangle Inequality
$d_{TV}(X, Z) \le d_{TV}(X, Y) + d_{TV}(Y, Z)$.

Taylor $p \to 0$: $\log(1-p) = -p - O(p^2)$

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$



$\longrightarrow \Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\longrightarrow \mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*

$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$. And for $Z \sim \mathrm{Pois}(c + O(\frac{1}{n}))$

$$d_{TV}(X, Z) \le d_{TV}(X, Y) + d_{TV}(Y, Z)$$
$$= o(1) + \underbrace{\frac{n}{2} \log(1-p)^2}_{\frac{n}{2}(-p - O(p^2)))^2 = \frac{n}{2}(p^2 + O(p^3))}$$

Binomial-Poisson-Approximation
$Y \sim \mathrm{Bin}(n, p)$, $Z \sim \mathrm{Pois}(-n \log(1-p))$:
$d_{TV}(Y, Z) \le \frac{n}{2} \log(1-p)^2$.

Triangle Inequality
$d_{TV}(X, Z) \le d_{TV}(X, Y) + d_{TV}(Y, Z)$.

Taylor $p \to 0$: $\log(1-p) = -p - O(p^2)$

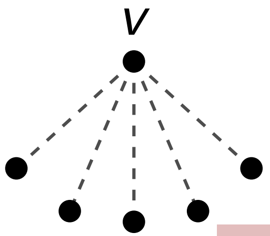
ER – Degree of a Vertex

Karlsruhe Institute of Technology

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \text{Bin}(n-1, p)$ $\longrightarrow$ $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

  $\searrow$ $\mathbb{E}[\deg(v)] = (n-1)p$ & $\text{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \text{Bin}(n-1, p)$ and let $Y \sim \text{Bin}(n,p)$. Then, $d_{TV}(X,Y) = o(1)$. And for $Z \sim \text{Pois}(c + O(\frac{1}{n}))$

$$d_{TV}(X,Z) \le d_{TV}(X,Y) + d_{TV}(Y,Z)$$
$$= o(1) + \underbrace{\tfrac{n}{2}\log(1-p)^2}_{\frac{n}{2}(-p-O(p^2)))^2}$$
$$= \tfrac{n}{2}(p^2 + O(p^3))$$
$$= \tfrac{n}{2}((\tfrac{c}{n})^2 + O((\tfrac{c}{n})^3))$$

Binomial-Poisson-Approximation
$Y \sim \text{Bin}(n,p)$, $Z \sim \text{Pois}(-n\log(1-p))$:
$d_{TV}(Y,Z) \le \frac{n}{2}\log(1-p)^2$.

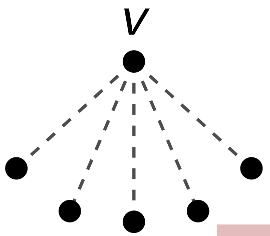
Triangle Inequality
$d_{TV}(X,Z) \le d_{TV}(X,Y) + d_{TV}(Y,Z)$.

Taylor $p \to 0$: $\log(1-p) = -p - O(p^2)$

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ → $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$
  → $\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n,p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$. And for $Z \sim \mathrm{Pois}(c + O(\frac{1}{n}))$

$$d_{TV}(X, Z) \le d_{TV}(X, Y) + d_{TV}(Y, Z)$$
$$= o(1) + \underbrace{\frac{n}{2}\log(1-p)^2}_{\frac{n}{2}(-p-O(p^2)))^2}$$
$$= \frac{n}{2}(p^2 + O(p^3))$$
$$= \frac{n}{2}((\frac{c}{n})^2 + O((\frac{c}{n})^3))$$
$$= \frac{c^2}{2n} + O(\frac{c^3}{n^2}) = o(1)$$

Binomial-Poisson-Approximation
$Y \sim \mathrm{Bin}(n, p)$, $Z \sim \mathrm{Pois}(-n\log(1-p))$:
$d_{TV}(Y, Z) \le \frac{n}{2}\log(1-p)^2$.

Triangle Inequality
$d_{TV}(X, Z) \le d_{TV}(X, Y) + d_{TV}(Y, Z)$.

Taylor $p \to 0$: $\log(1-p) = -p - O(p^2)$

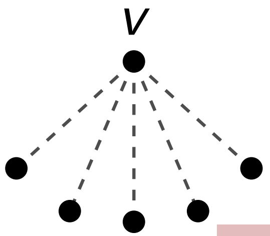
ER – Degree of a Vertex

Karlsruhe Institute of Technology

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ $\longrightarrow$ $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$

$\longrightarrow$ $\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$. And for $Z \sim \mathrm{Pois}(c + O(\frac{1}{n}))$: $d_{TV}(X, Z) = o(1)$.

$$
\begin{aligned}
d_{TV}(X, Z) &\le d_{TV}(X, Y) + d_{TV}(Y, Z) \\
&= o(1) + \underbrace{\tfrac{n}{2}\log(1-p)^2}_{\frac{n}{2}(-p - O(p^2)))^2} = o(1)
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{n}{2}(-p - O(p^2)))^2 &= \tfrac{n}{2}(p^2 + O(p^3)) \\
&= \tfrac{n}{2}((\tfrac{c}{n})^2 + O((\tfrac{c}{n})^3)) \\
&= \tfrac{c^2}{2n} + O(\tfrac{c^3}{n^2}) = o(1)
\end{aligned}
$$

Binomial-Poisson-Approximation
$Y \sim \mathrm{Bin}(n, p)$, $Z \sim \mathrm{Pois}(-n\log(1-p))$:
$d_{TV}(Y, Z) \le \frac{n}{2}\log(1-p)^2$.

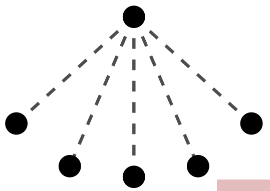
Triangle Inequality
$d_{TV}(X, Z) \le d_{TV}(X, Y) + d_{TV}(Y, Z)$.

Taylor $p \to 0$: $\log(1-p) = -p - O(p^2)$

ER – Degree of a Vertex

Vertex Degree

- Number of neighbors, number of incident edges
- each of $n-1$ potential edges exists with prob. p
- $\deg(v) \sim \mathrm{Bin}(n-1, p)$ → $\Pr[\deg(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$
  → $\mathbb{E}[\deg(v)] = (n-1)p$ & $\mathrm{Var}[\deg(v)] = (n-1)p(1-p)$ *Inconvenient...*



$G(n, p)$
Independently connect any two nodes with fixed probability p.

$p = \frac{c}{n}$, $c \in \Theta(1)$

Approximation

Lemma: Let $p = \frac{c}{n}$ for $c \in \Theta(1)$, let $X \sim \mathrm{Bin}(n-1, p)$ and let $Y \sim \mathrm{Bin}(n, p)$. Then, $d_{TV}(X, Y) = o(1)$. And for $Z \sim \mathrm{Pois}(c + O(\frac{1}{n}))$: $d_{TV}(X, Z) = o(1)$.

$$\begin{aligned} d_{TV}(X, Z) &\leq d_{TV}(X, Y) + d_{TV}(Y, Z) \\ &= o(1) + \underbrace{\tfrac{n}{2} \log(1-p)^2}_{} = o(1) \end{aligned}$$

$$\begin{aligned} \tfrac{n}{2}(-p - O(p^2)))^2 &= \tfrac{n}{2}(p^2 + O(p^3)) \\ &= \tfrac{n}{2}((\tfrac{c}{n})^2 + O((\tfrac{c}{n})^3)) \\ &= \tfrac{c^2}{2n} + O(\tfrac{c^3}{n^2}) = o(1) \end{aligned}$$

- $\mathbb{E}[Z] = \mathrm{Var}[Z] \approx c$, much simpler than the above!

Binomial-Poisson-Approximation
$Y \sim \mathrm{Bin}(n, p)$, $Z \sim \mathrm{Pois}(-n \log(1-p))$: $d_{TV}(Y, Z) \leq \frac{n}{2} \log(1-p)^2$.

Triangle Inequality
$d_{TV}(X, Z) \leq d_{TV}(X, Y) + d_{TV}(Y, Z)$.

Taylor $p \to 0$: $\log(1-p) = -p - O(p^2)$

Conclusion

Karlsruhe Institute of Technology

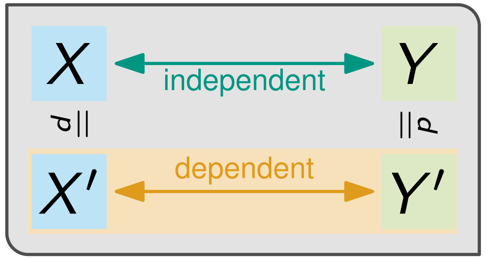
Coupling

- Define relation between rand. var. to make statements about one using the other
- A coupling of (X, Y) is a pair (X', Y') of random variables in a shared probability space such that $X \overset{d}{=} X'$ and $Y \overset{d}{=} Y'$

Conclusion

Coupling

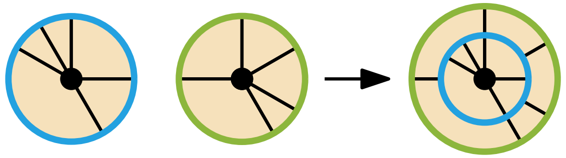
- Define relation between rand. var. to make statements about one using the other
- A coupling of (X, Y) is a pair (X', Y') of random variables in a shared probability space such that $X \overset{d}{=} X'$ and $Y \overset{d}{=} Y'$
- Often X' and Y' dependent

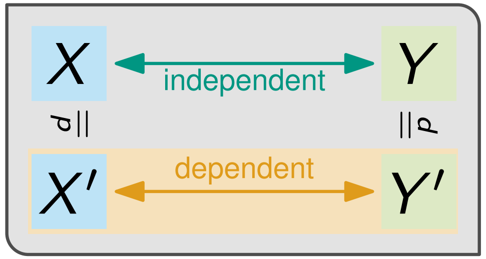


Conclusion

Coupling

- Define relation between rand. var. to make statements about one using the other
- A coupling of (X, Y) is a pair (X', Y') of random variables in a shared probability space such that $X \overset{d}{=} X'$ and $Y \overset{d}{=} Y'$
- Often X' and Y' dependent
- Examples: Wheel of fortune & Unfair dice

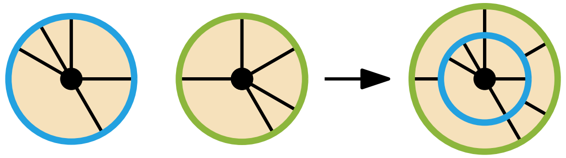


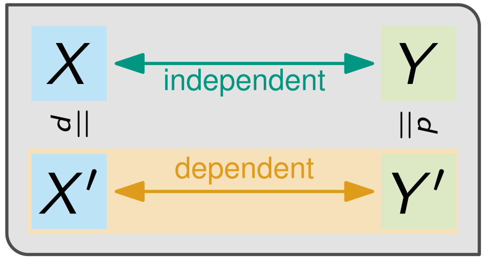


Conclusion

Coupling

- Define relation between rand. var. to make statements about one using the other
- A coupling of (X, Y) is a pair (X', Y') of random variables in a shared probability space such that $X \overset{d}{=} X'$ and $Y \overset{d}{=} Y'$
- Often X' and Y' dependent
- Examples: Wheel of fortune & Unfair dice
- Coupling inequality to bound *total variation distance*

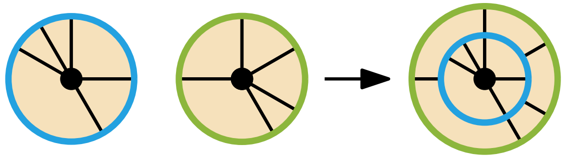


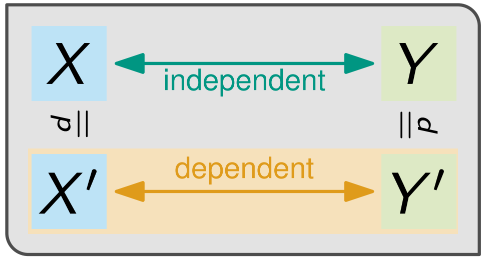


Conclusion

Coupling

- Define relation between rand. var. to make statements about one using the other
- A coupling of (X, Y) is a pair (X', Y') of random variables in a shared probability space such that $X \overset{d}{=} X'$ and $Y \overset{d}{=} Y'$
- Often X' and Y' dependent
- Examples: Wheel of fortune & Unfair dice
- Coupling inequality to bound *total variation distance*





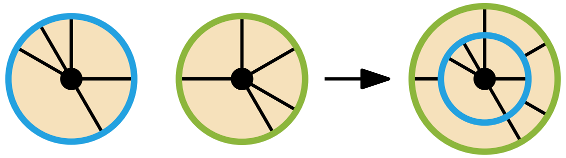
Random Graph Models

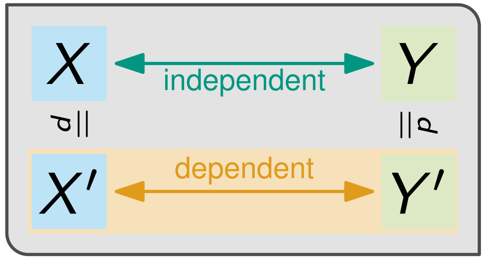
- Mathematical models represent real-world networks and allow for theoretical analysis
- Desirable properties: simple, realistic, fast to generate

Conclusion

Coupling

- Define relation between rand. var. to make statements about one using the other
- A coupling of (X, Y) is a pair (X', Y') of random variables in a shared probability space such that $X \overset{d}{=} X'$ and $Y \overset{d}{=} Y'$
- Often X' and Y' dependent
- Examples: Wheel of fortune & Unfair dice
- Coupling inequality to bound *total variation distance*





Random Graph Models

- Mathematical models represent real-world networks and allow for theoretical analysis
- Desirable properties: simple, realistic, fast to generate

Erdős-Rényi Random Graphs

- $G(n, p)$: Start with n nodes, connect any two with fixed probability p, independently
- In sparse $G(n, p)$ the degree of a vertex is approximately Poisson-distributed

Outlook: Degree Distribution vs. Degree Distribution

Distributions

- Probability distribution of the degree of a *given* vertex in a $G(n, \frac{c}{n})$ approaches $\mathrm{Pois}(c)$

Outlook: Degree Distribution vs. Degree Distribution

Karlsruhe Institute of Technology

Distributions

- Probability distribution of the degree of a *given* vertex in a $G(n, \frac{c}{n})$ approaches $\mathrm{Pois}(c)$
- Empirical distribution of the degrees of *all* vertices in a graph $G = (V, E)$

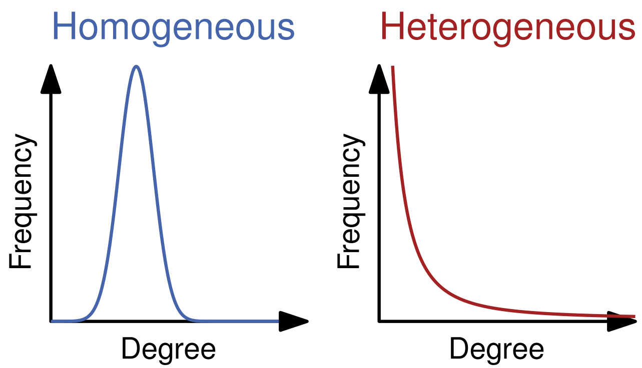
  $N_d = \sum_{v \in V} \mathbb{1}_{\{\deg(v) = d\}}$ (normalized: $\frac{1}{n} N_d$, for $n = |V|$)

Outlook: Degree Distribution vs. Degree Distribution

Distributions

- Probability distribution of the degree of a *given* vertex in a $G(n, \frac{c}{n})$ approaches $\mathrm{Pois}(c)$
- Empirical distribution of the degrees of *all* vertices in a graph $G = (V, E)$

  $N_d = \sum_{v \in V} \mathbb{1}_{\{\deg(v)=d\}}$ (normalized: $\frac{1}{n} N_d$, for $n = |V|$)



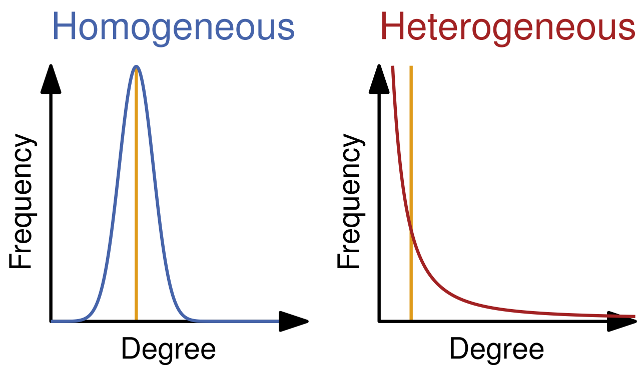
Characterizing a Distribution

Outlook: Degree Distribution vs. Degree Distribution

Distributions

- Probability distribution of the degree of a *given* vertex in a $G(n, \frac{c}{n})$ approaches $\mathrm{Pois}(c)$
- Empirical distribution of the degrees of *all* vertices in a graph $G = (V, E)$

$N_d = \sum_{v \in V} \mathbb{1}_{\{\deg(v) = d\}}$ (normalized: $\frac{1}{n} N_d$, for $n = |V|$)



Characterizing a Distribution

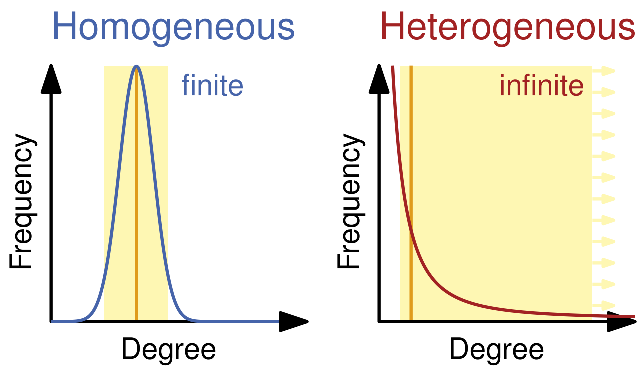
- *Mean*: What degree would we expect for a vertex?

Outlook: Degree Distribution vs. Degree Distribution

Distributions

- Probability distribution of the degree of a *given* vertex in a $G(n, \frac{c}{n})$ approaches $\mathrm{Pois}(c)$
- Empirical distribution of the degrees of *all* vertices in a graph $G = (V, E)$

  $N_d = \sum_{v \in V} \mathbb{1}_{\{\deg(v)=d\}}$ (normalized: $\frac{1}{n} N_d$, for $n = |V|$)



Characterizing a Distribution

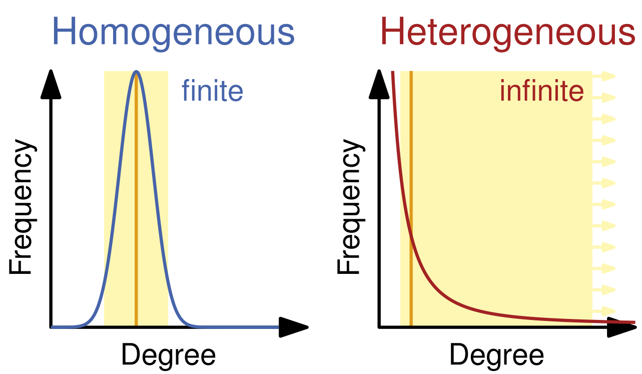
- *Mean*: What degree would we expect for a vertex?
- *Variance*: (very rough intuition) How far would we expect the degree of a vertex to deviate from the mean?

Outlook: Degree Distribution vs. Degree Distribution

Distributions

- Probability distribution of the degree of a *given* vertex in a $G(n, \frac{c}{n})$ approaches $\mathrm{Pois}(c)$
- Empirical distribution of the degrees of *all* vertices in a graph $G = (V, E)$

$N_d = \sum_{v \in V} \mathbb{1}_{\{\deg(v) = d\}}$ (normalized: $\frac{1}{n} N_d$, for $n = |V|$)



Characterizing a Distribution

- *Mean*: What degree would we expect for a vertex?
- *Variance*: (very rough intuition) How far would we expect the degree of a vertex to deviate from the mean?

Empirical Distribution of $G(n, \frac{c}{n})$ → homogeneous