Karlsruhe Institute of Technology

Probability & Computing

Continuous Probability Spaces & Random Geometric Graphs

Motivation – Radioactive Decay

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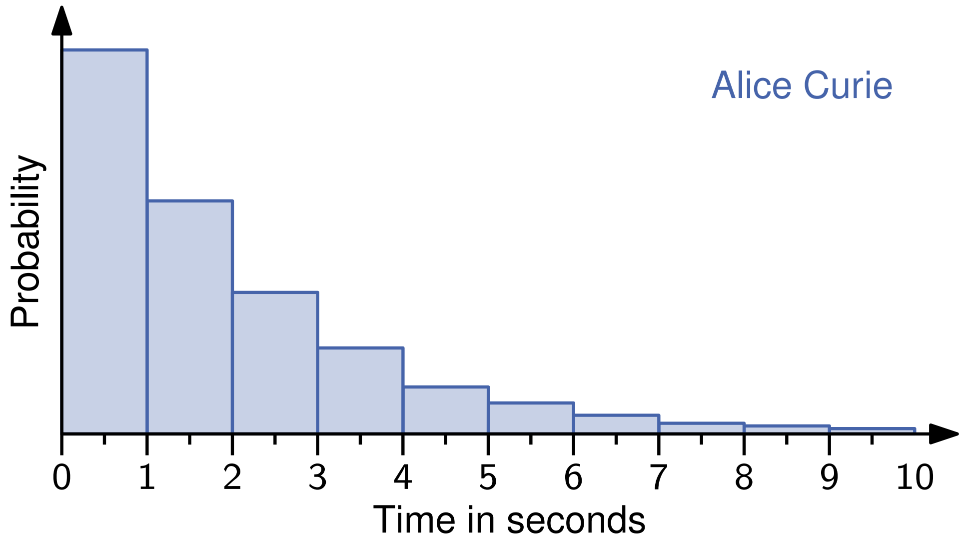

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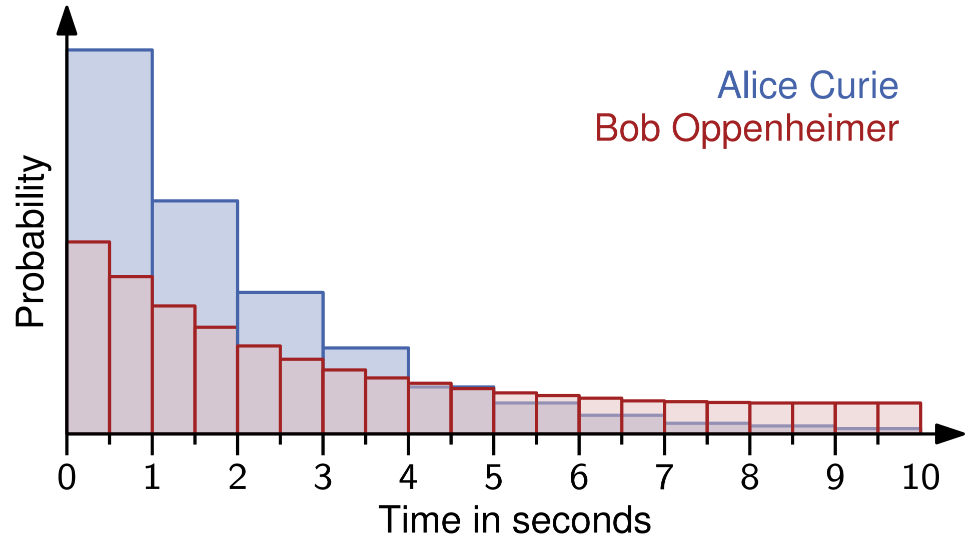

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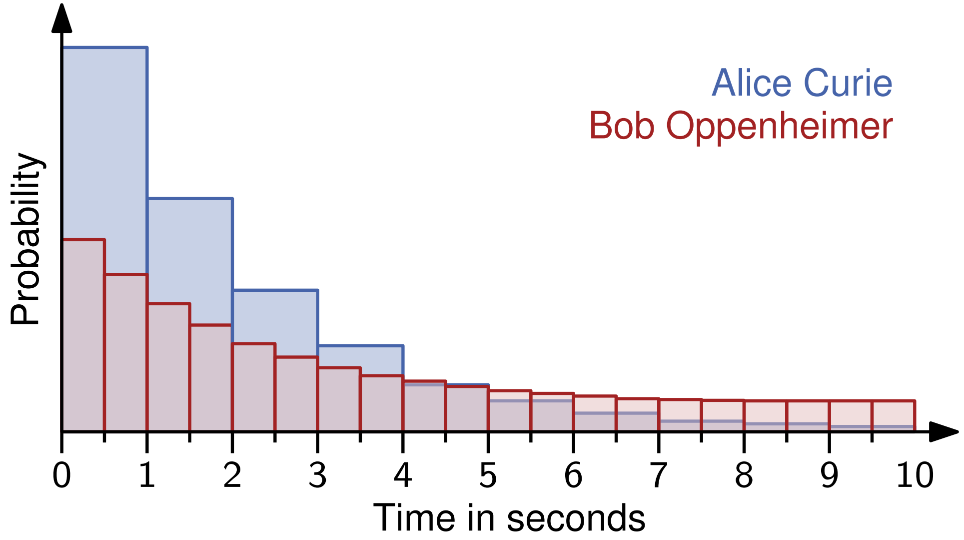

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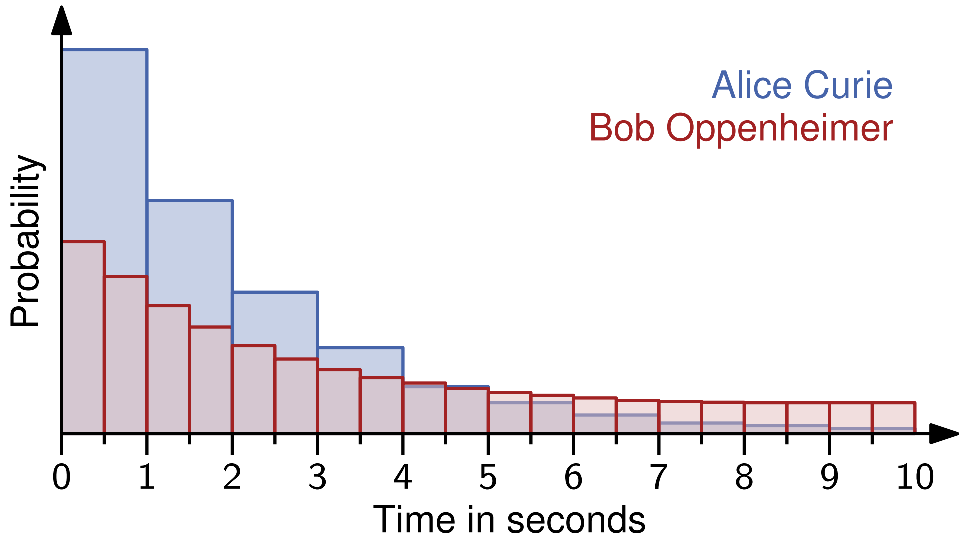

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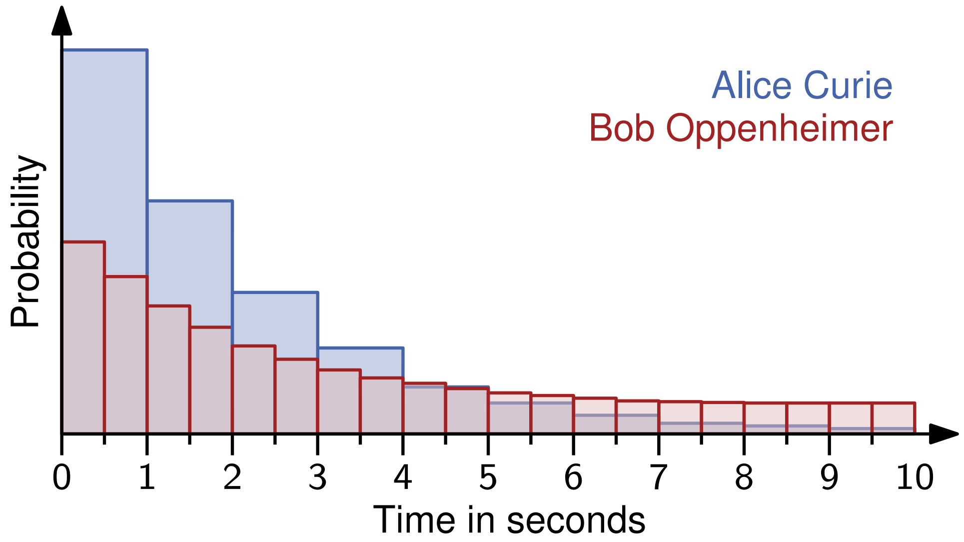

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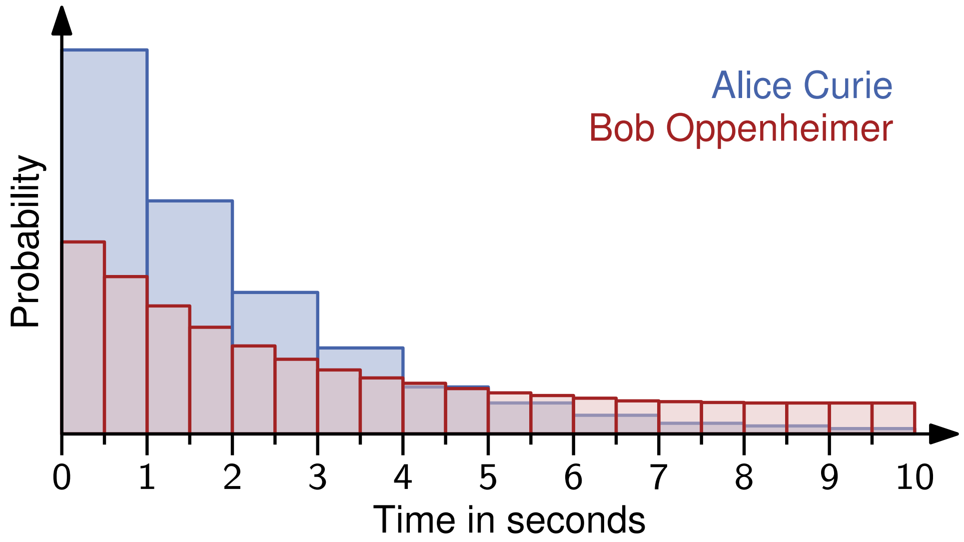

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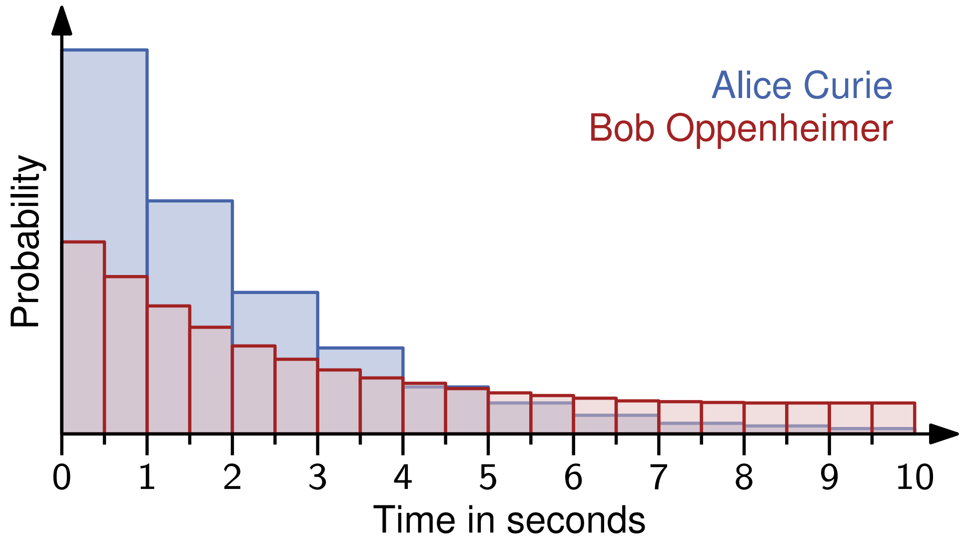

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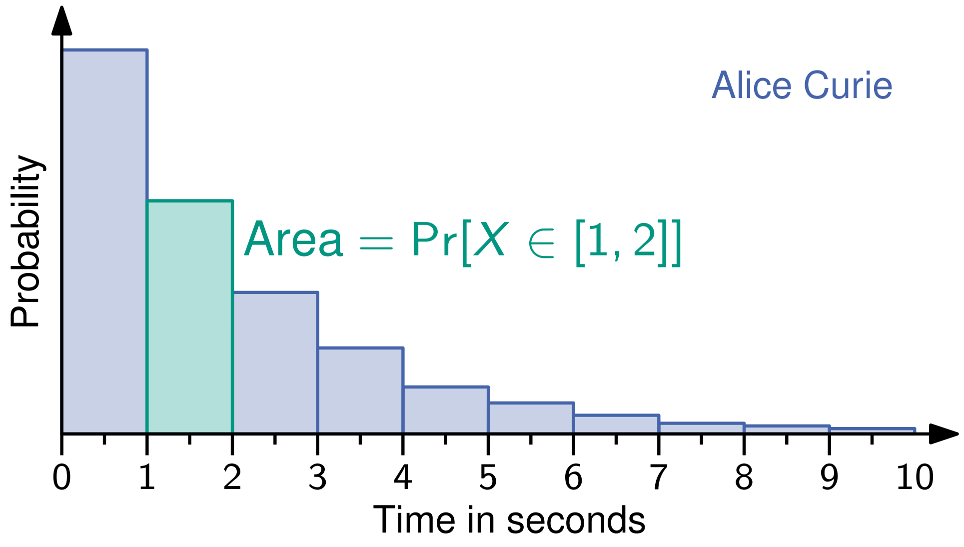

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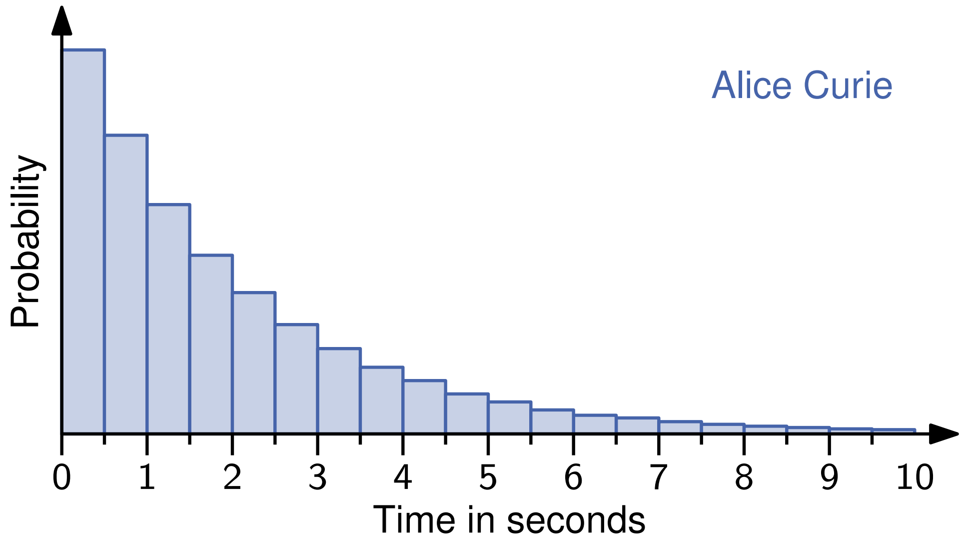

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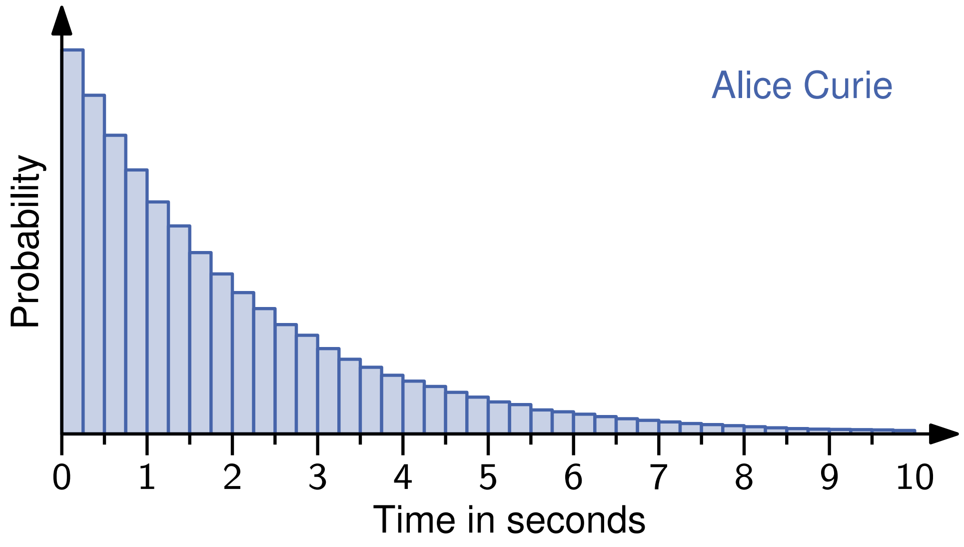

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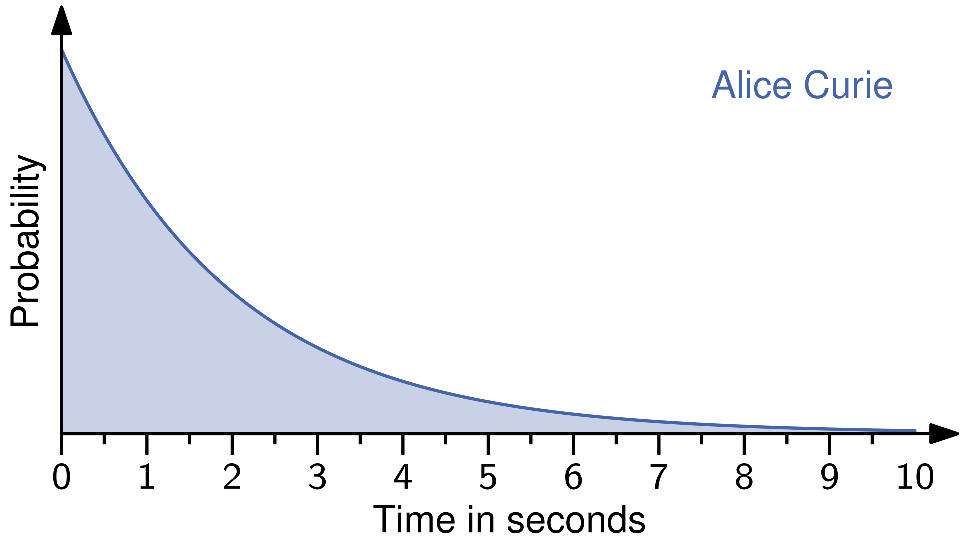

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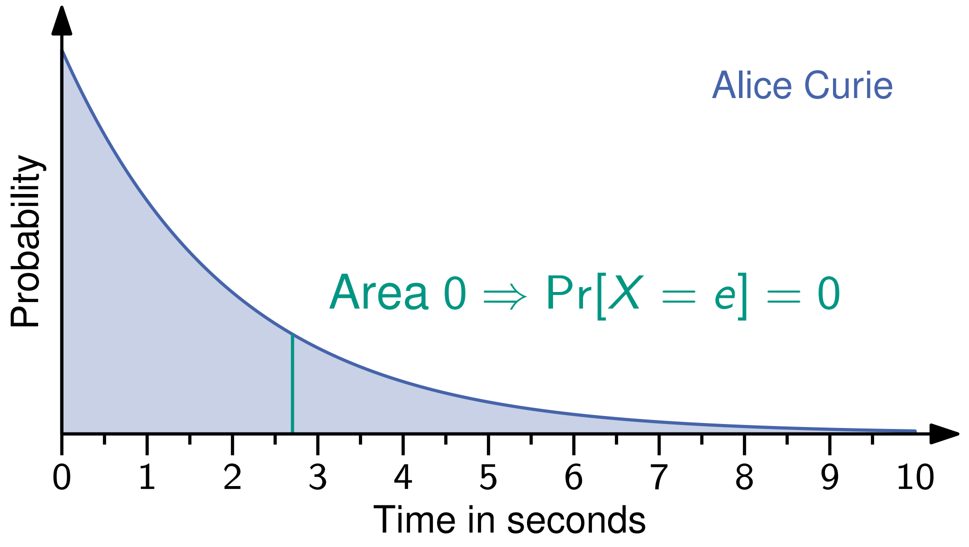

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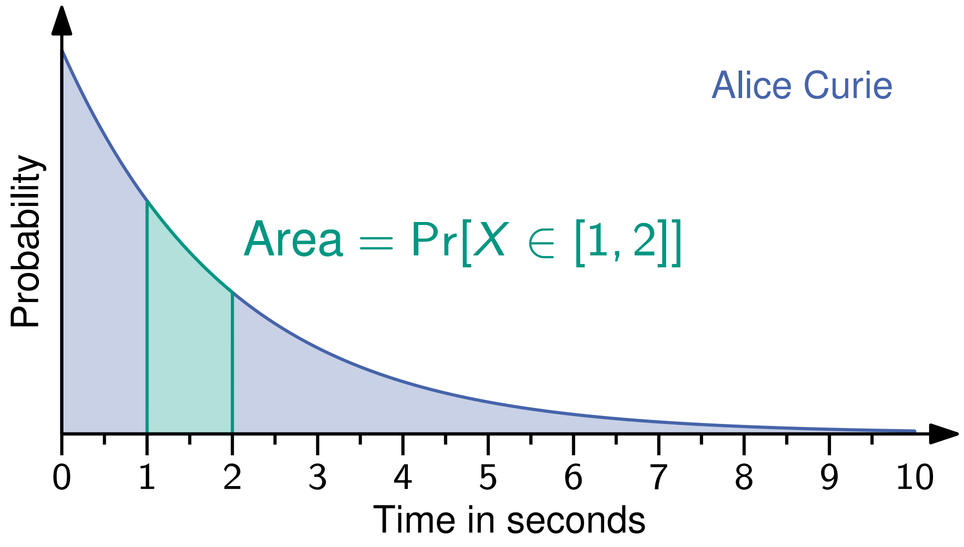

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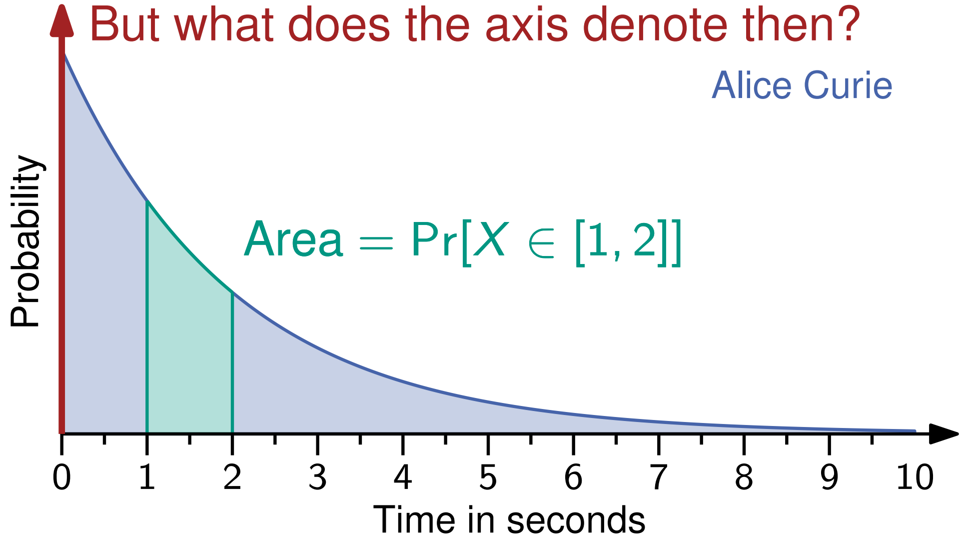

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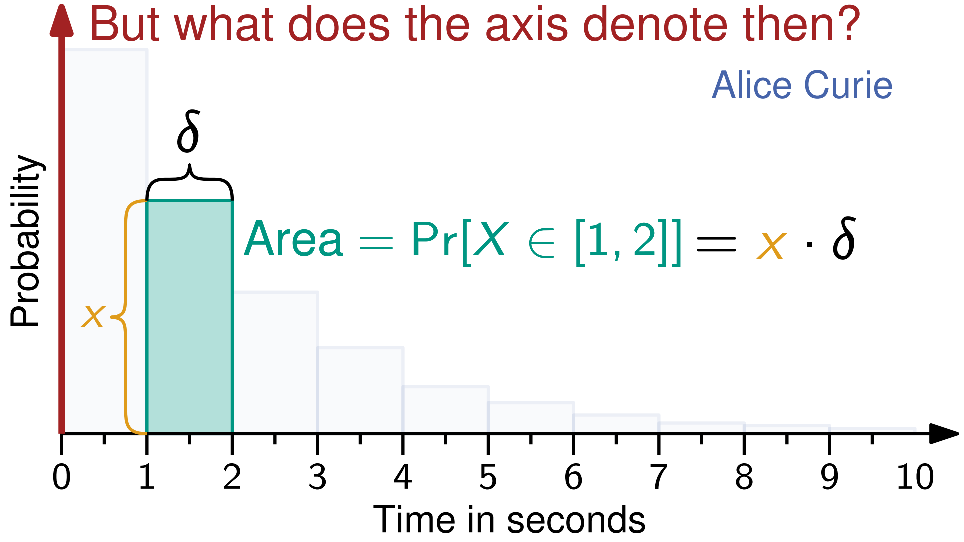

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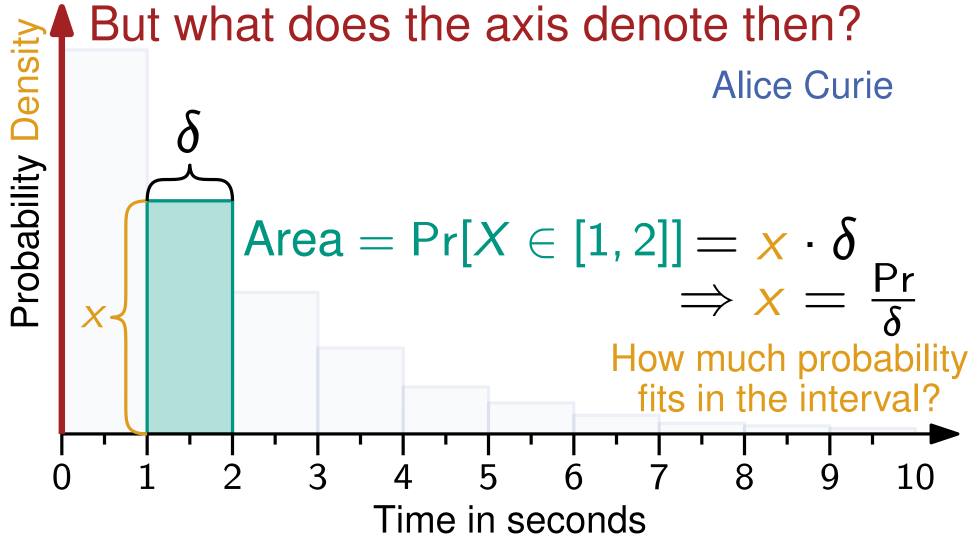

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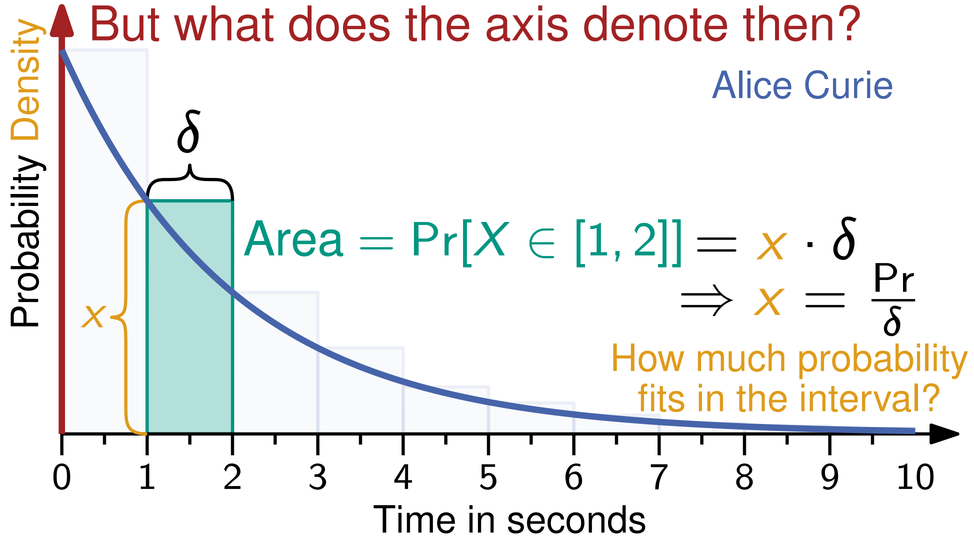
- We describe distributions using probability density functions



youtube.com/watch?v=ZA4JkHKZM50

Working in Continuous Probability Spaces

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Karlsruhe Institute of Technology

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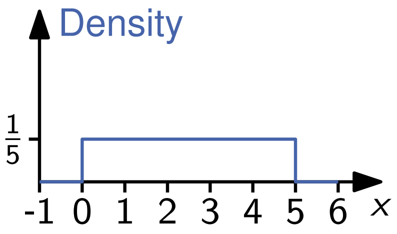

Working in Continuous Probability Spaces

Karlsruhe Institute of Technology

Discrete Random Variable X

- Cumulative distribution function

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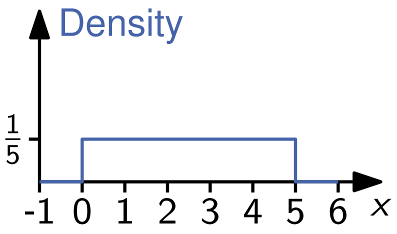

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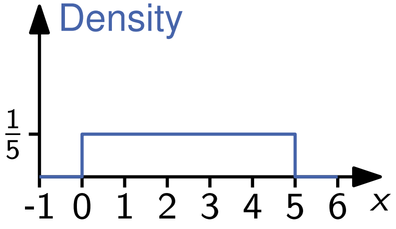

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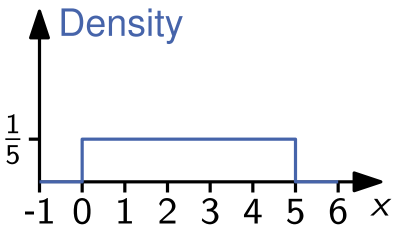

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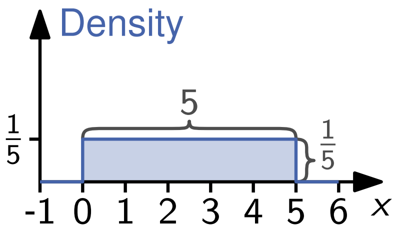

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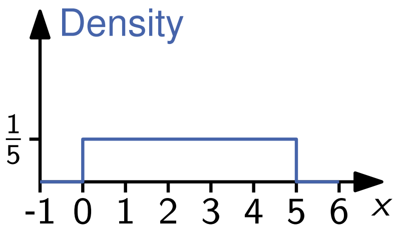

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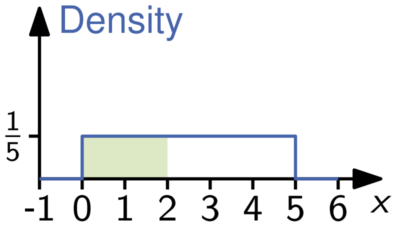

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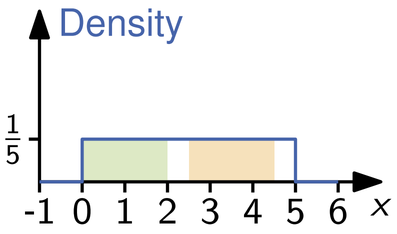

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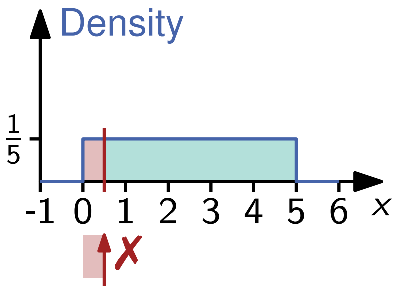

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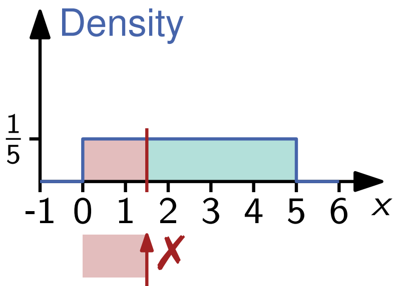

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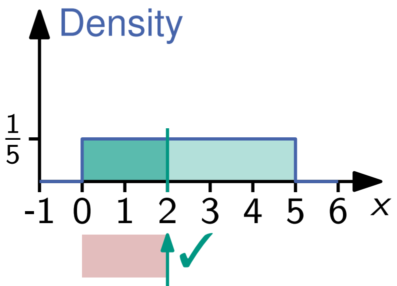

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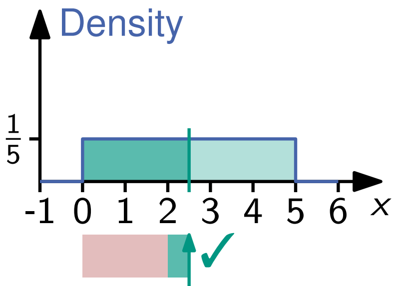

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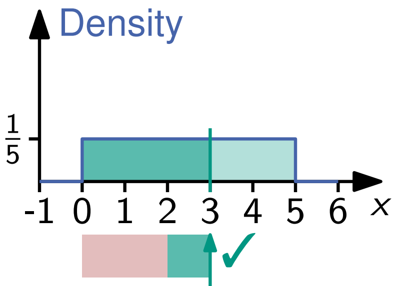

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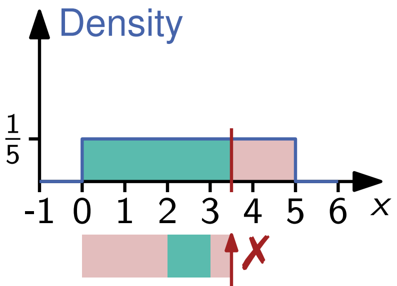

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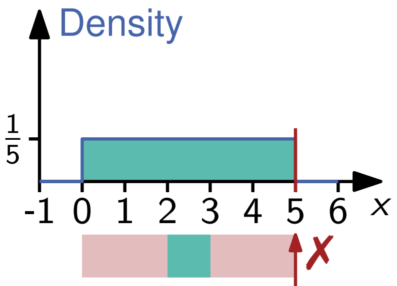

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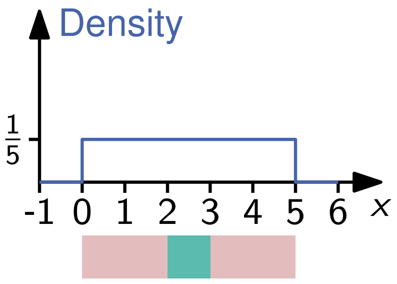

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Working in Continuous Probability Spaces

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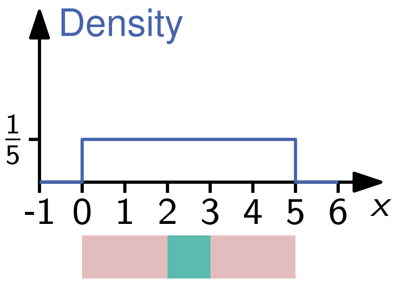

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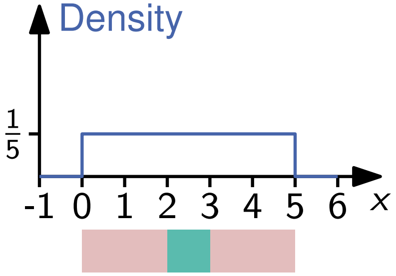

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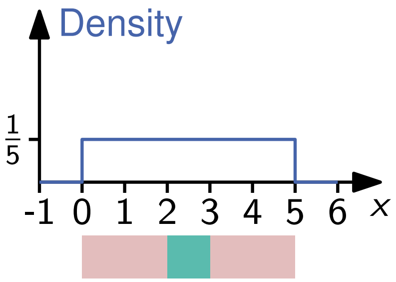

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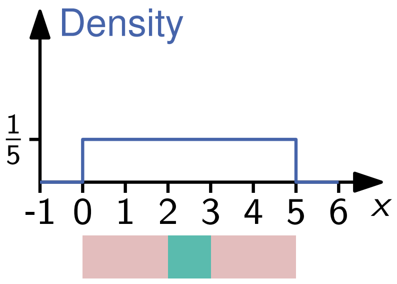

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Karlsruhe Institute of Technology

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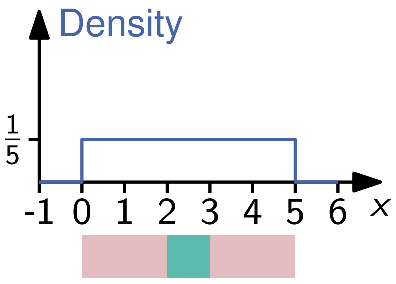

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$$= \left[\tfrac{x}{5}\right]_0^3 - \left[\tfrac{x}{5}\right]_0^2 = \tfrac{3}{5} - \tfrac{2}{5} = \tfrac{1}{5} ✓$$

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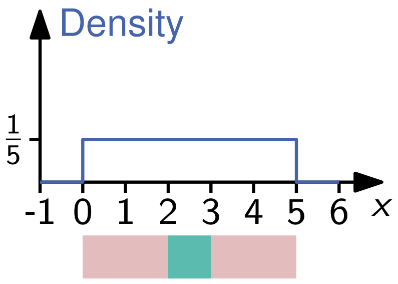

- In general: $X \sim \mathcal{U}([a, b])$
  $\Pr[X \in [c, d] \subseteq [a, b]] = \frac{d-c}{b-a}$

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Exponential Distribution $X \sim \mathrm{Exp}(\lambda)$

- “Rate” parameter $\lambda > 0$

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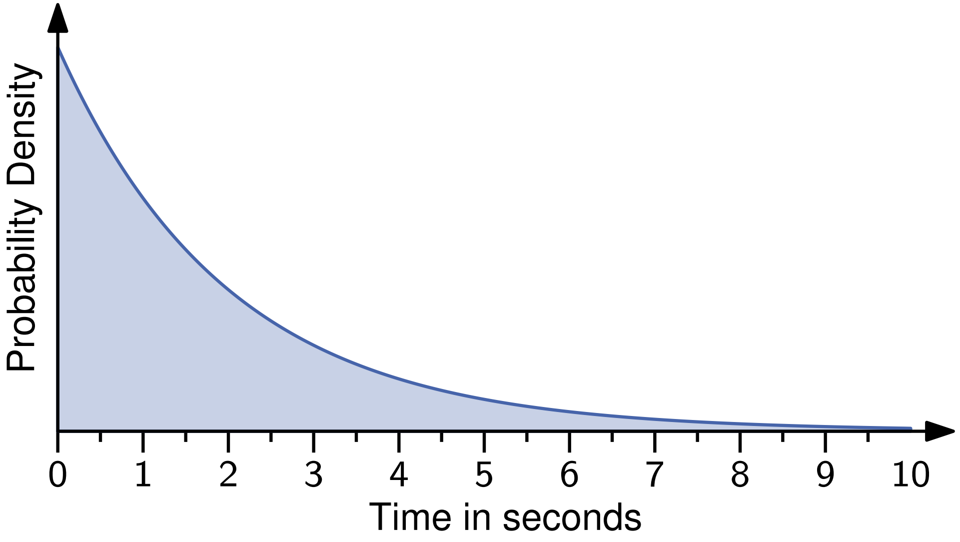

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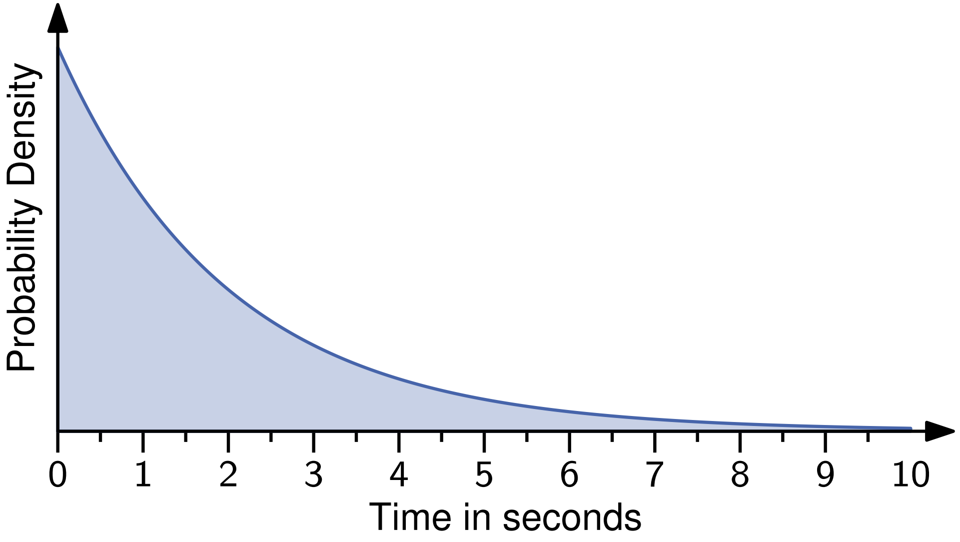

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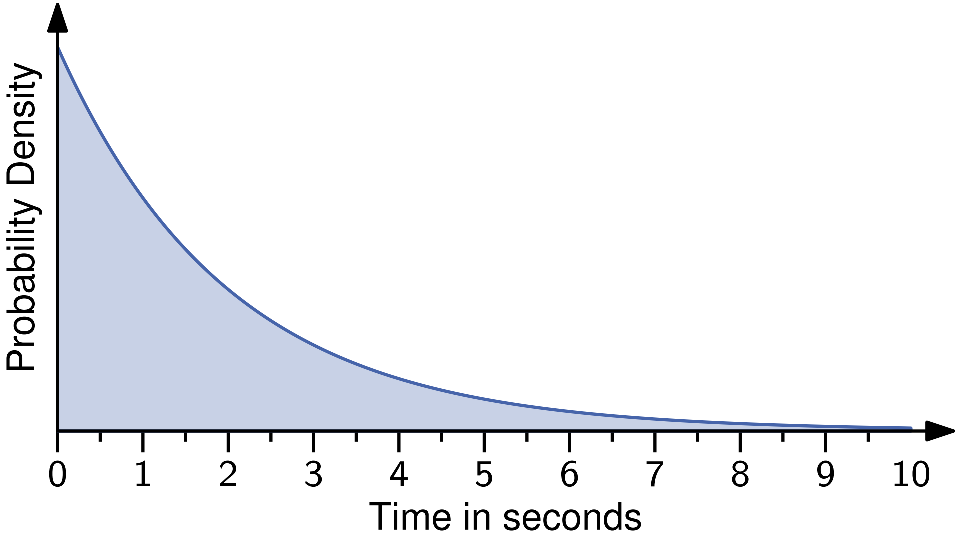

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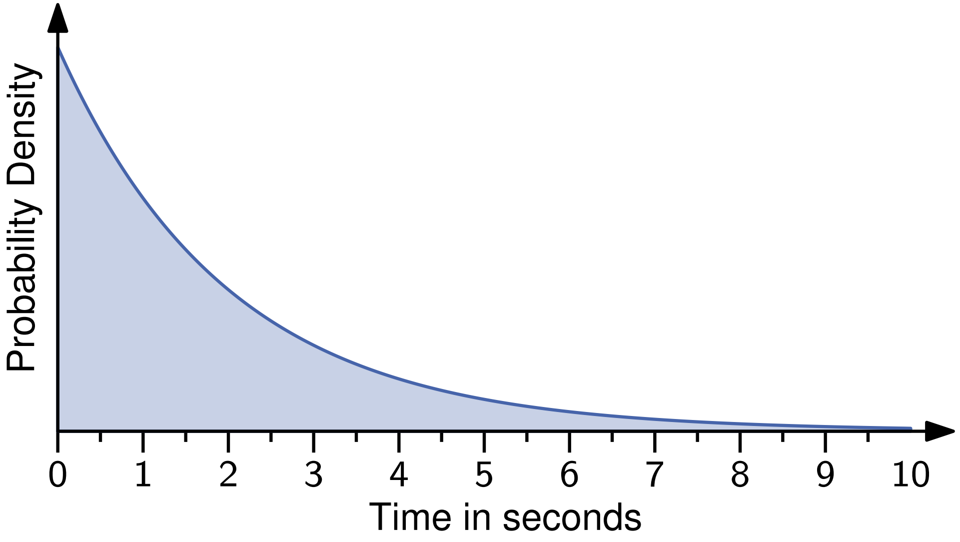

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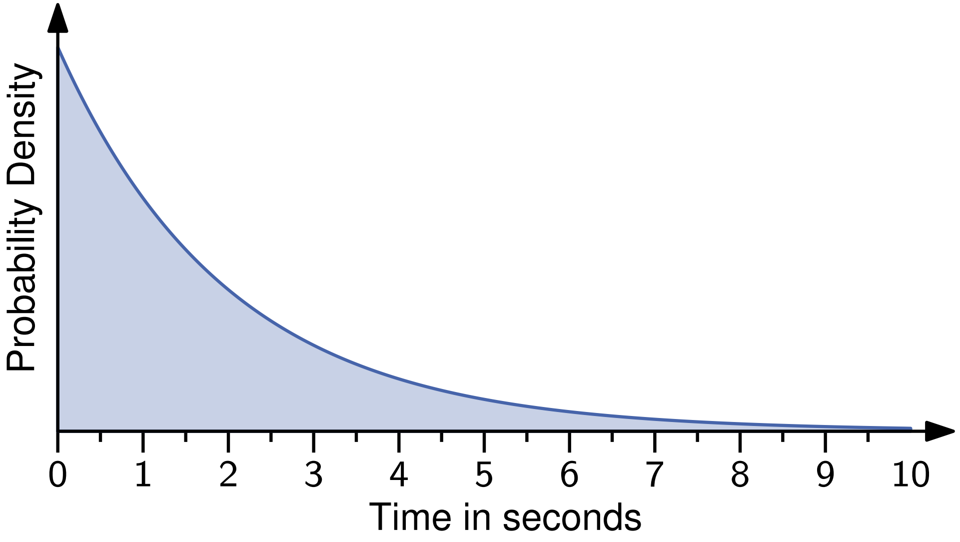

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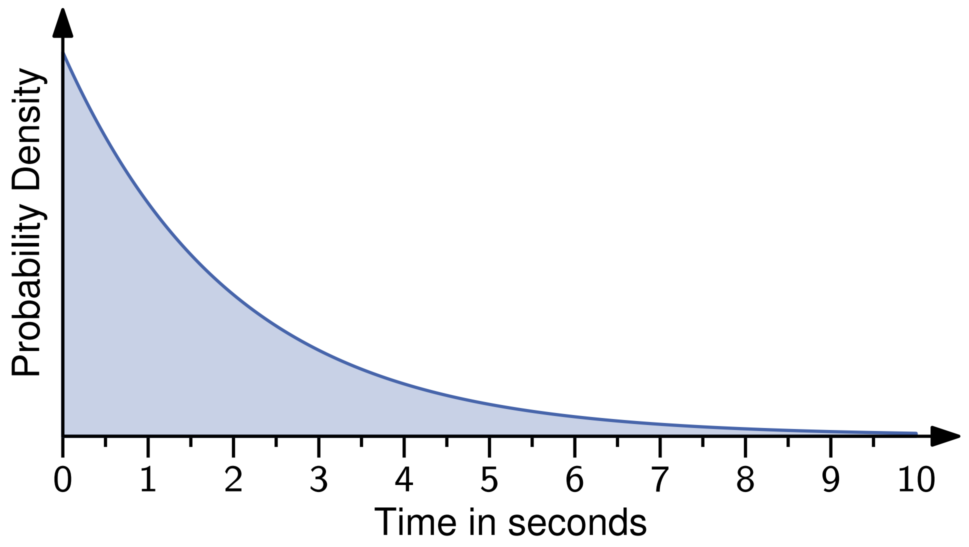

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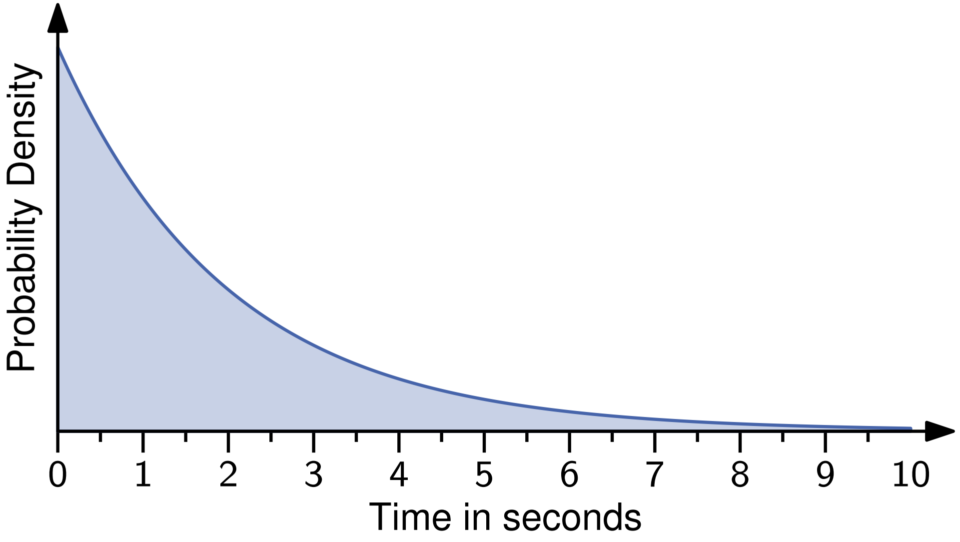

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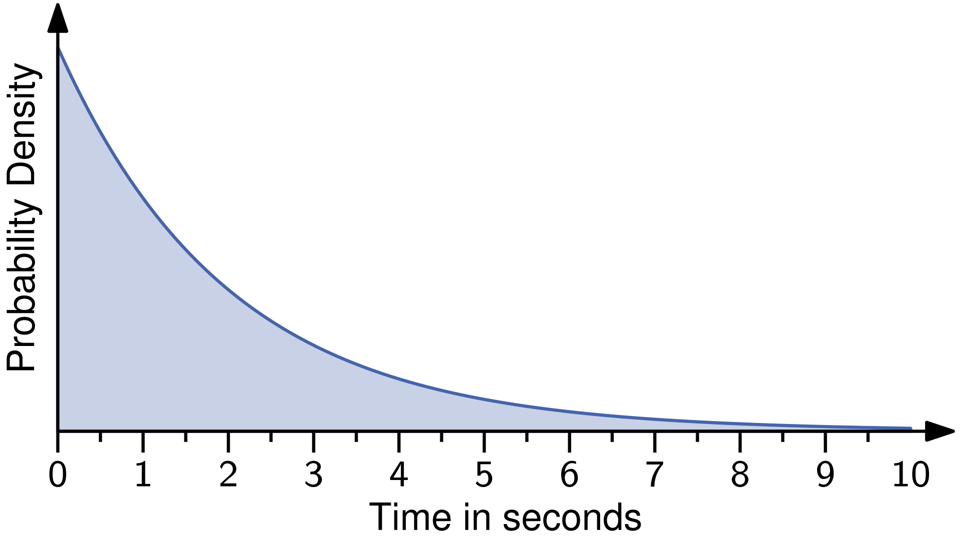

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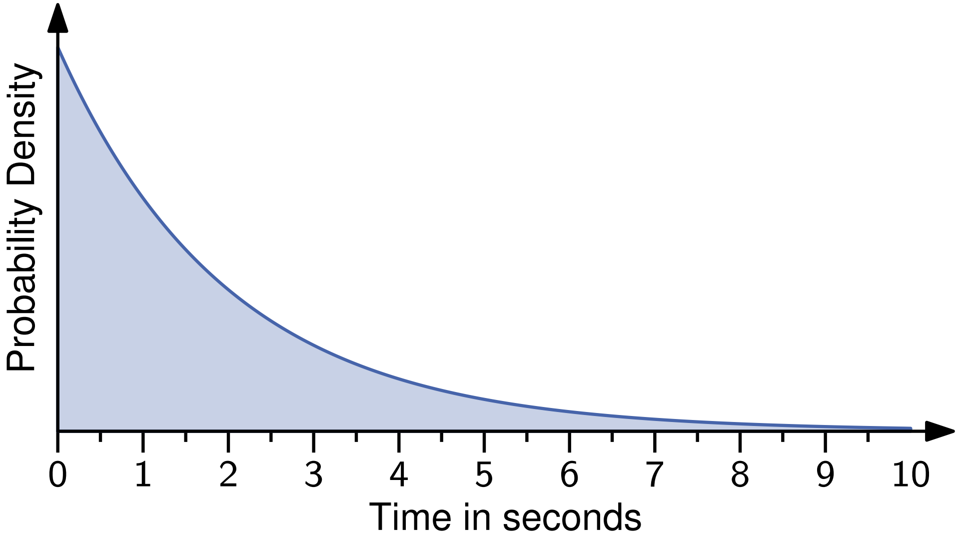

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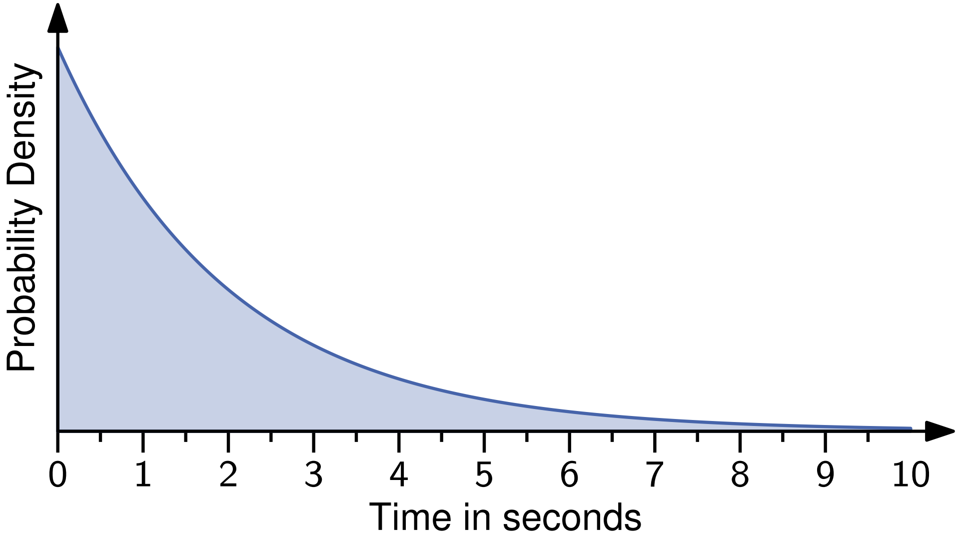

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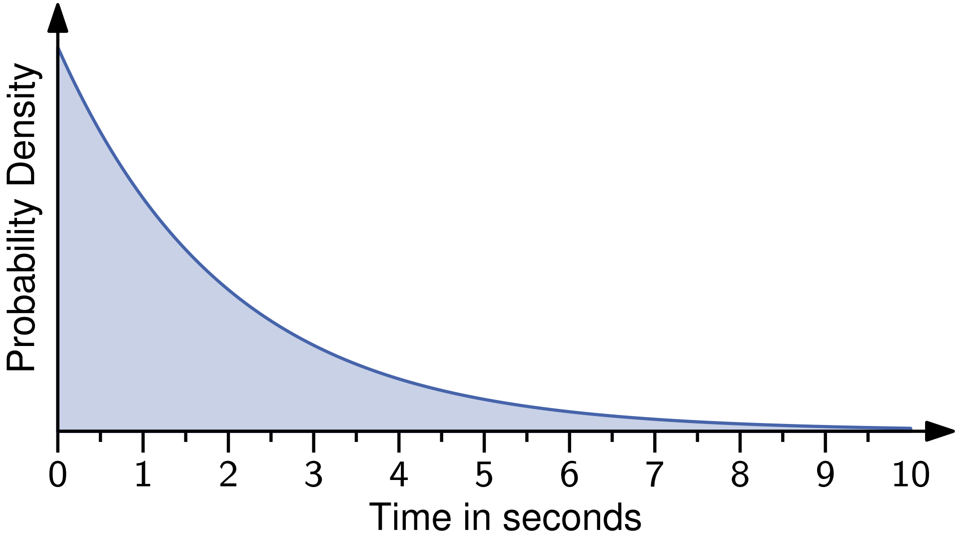

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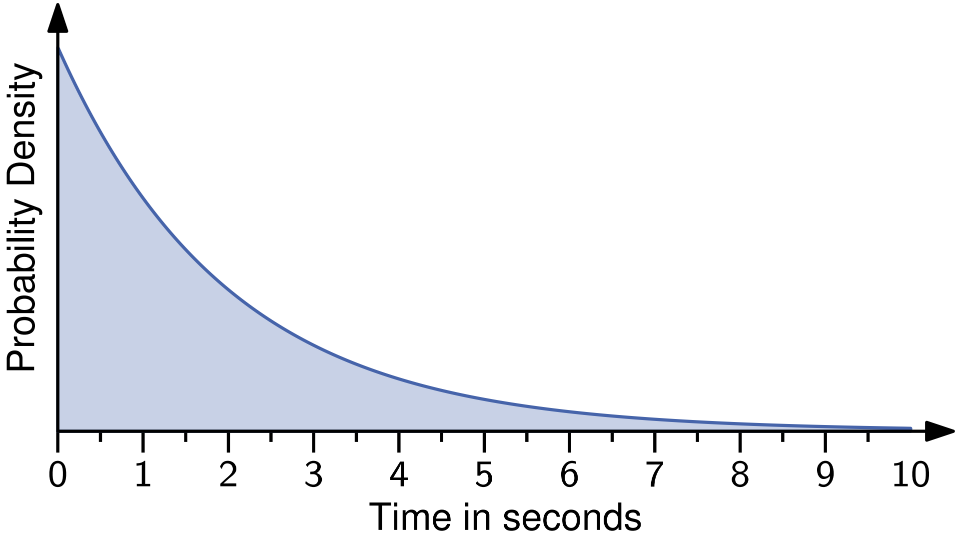

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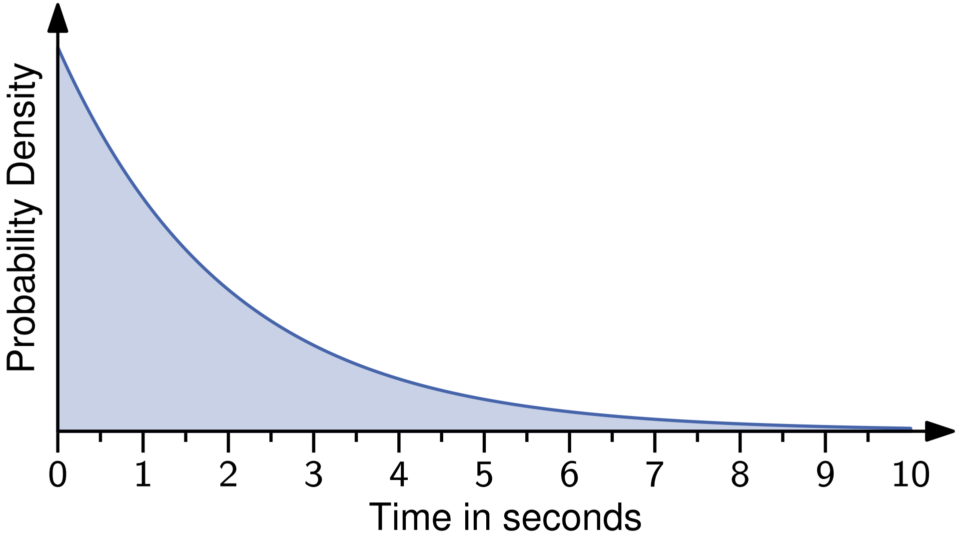

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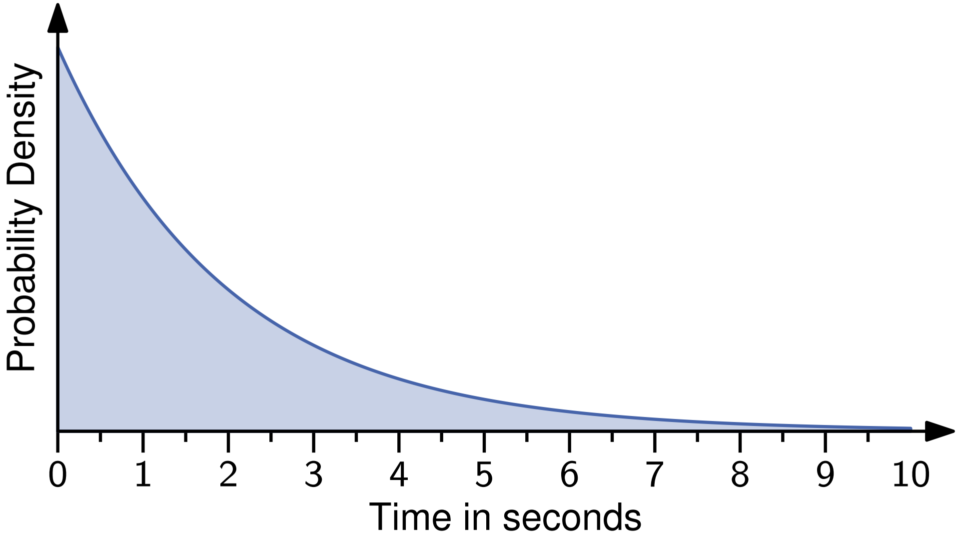

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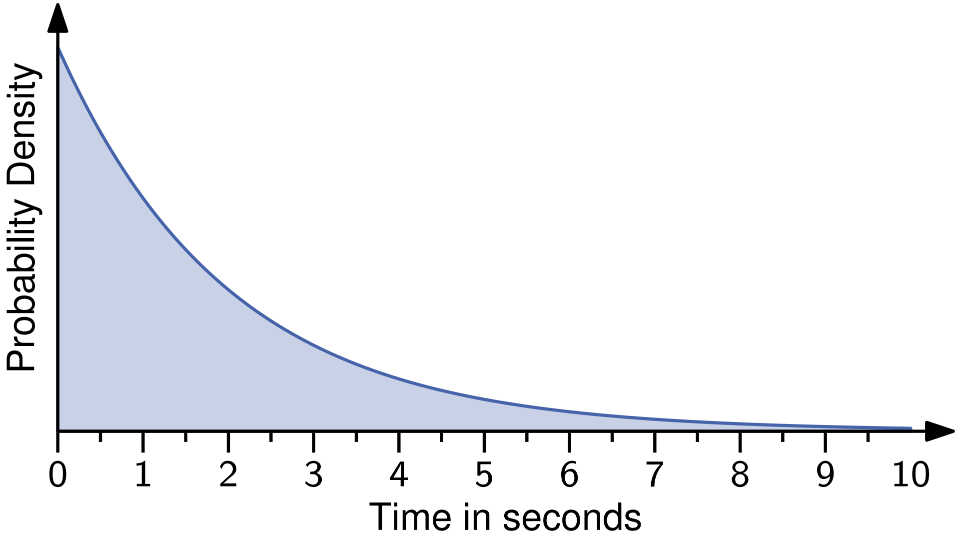

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Example: Radioactive Decay

Exponential Distribution $X \sim \mathrm{Exp}(\lambda)$

- "Rate" parameter $\lambda > 0$
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- "Time until first success"
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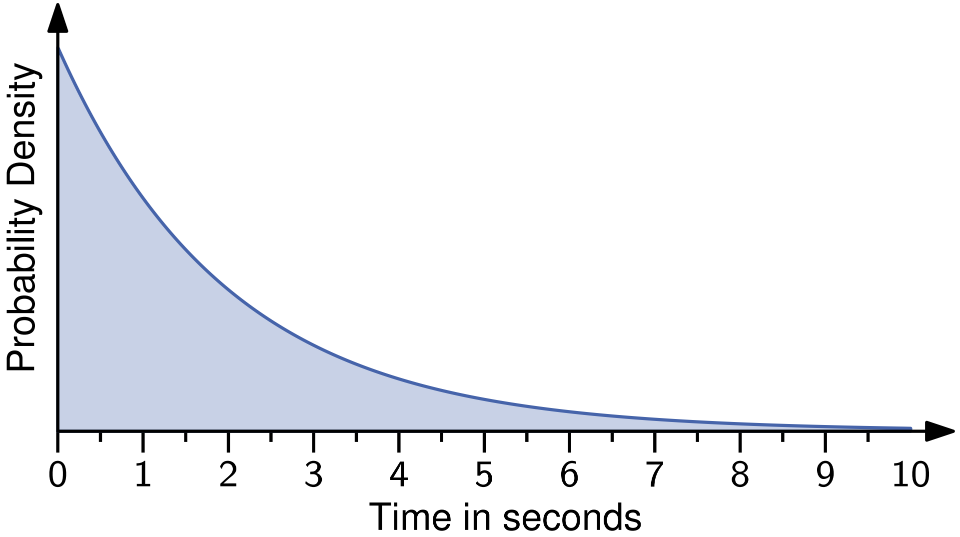

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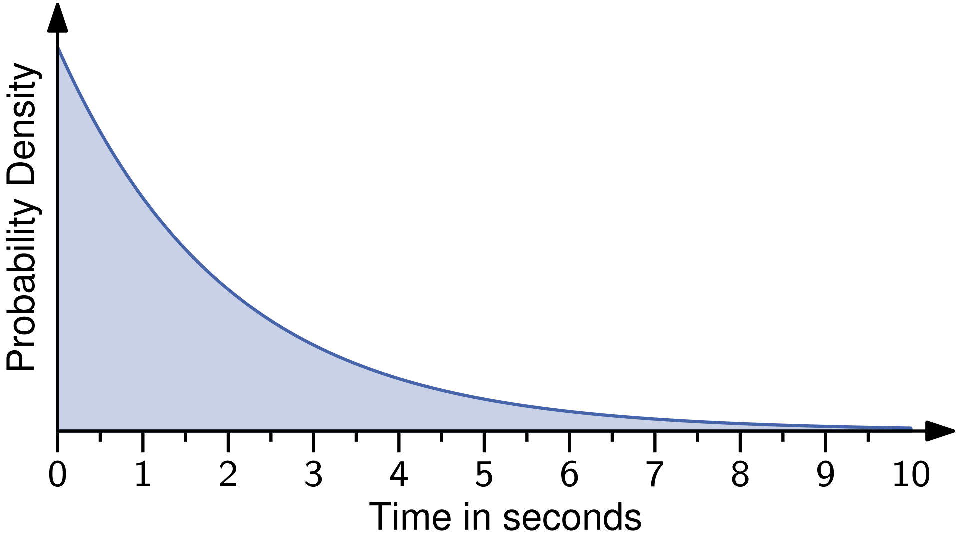

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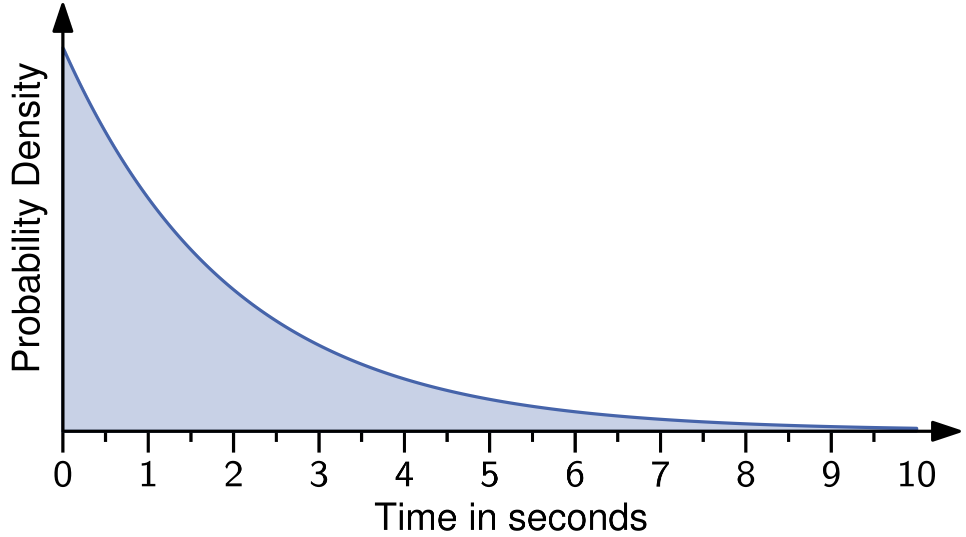

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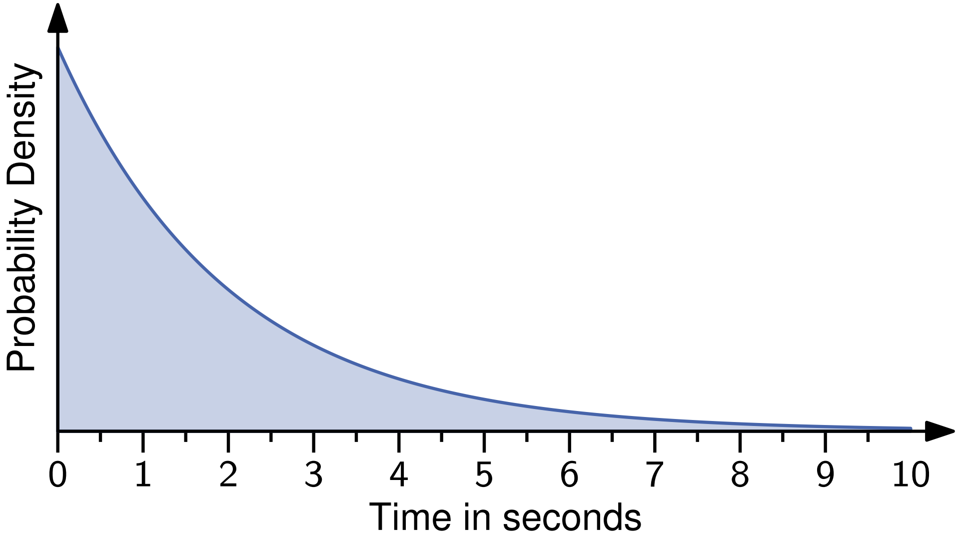

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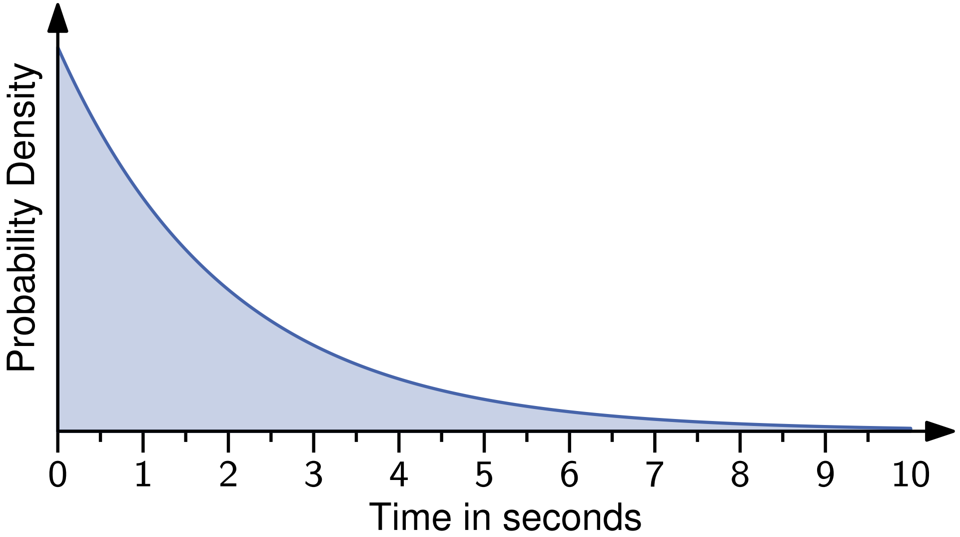

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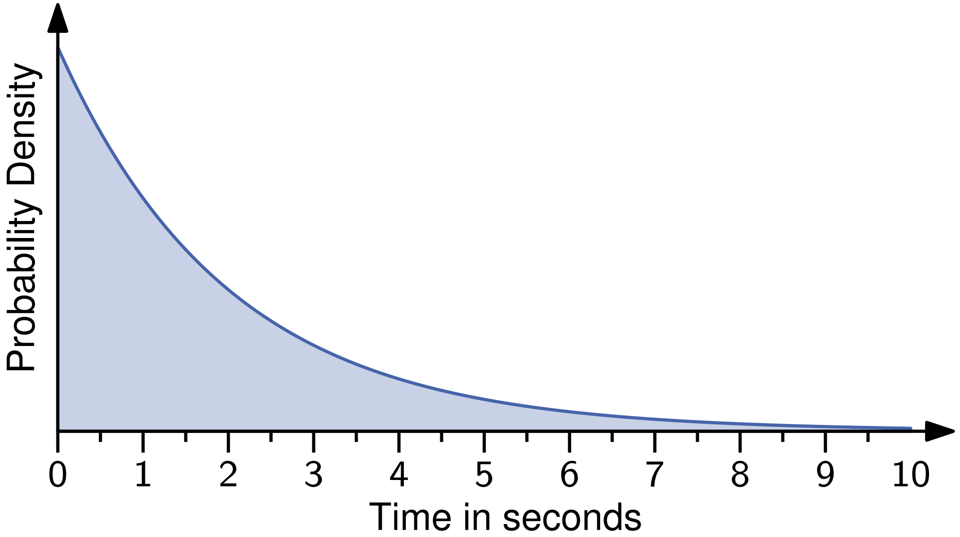

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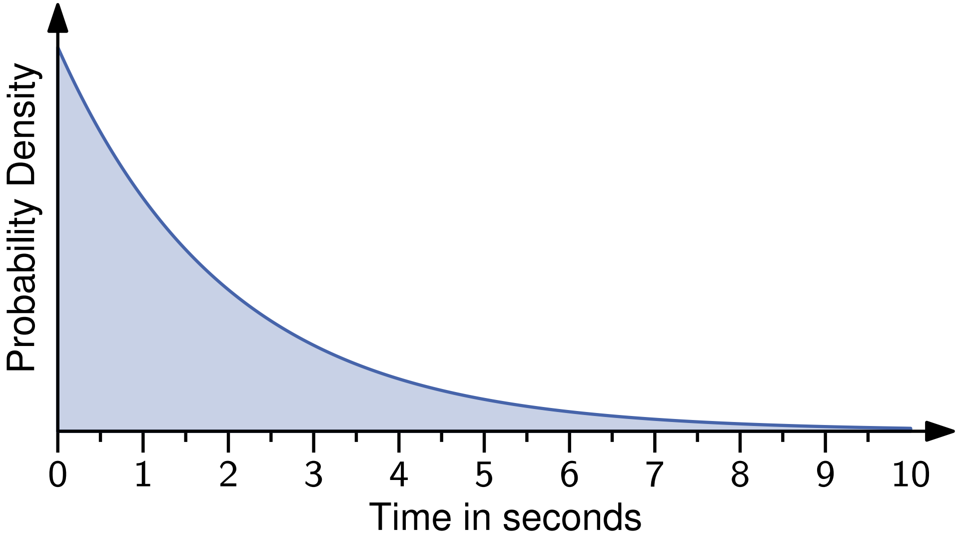

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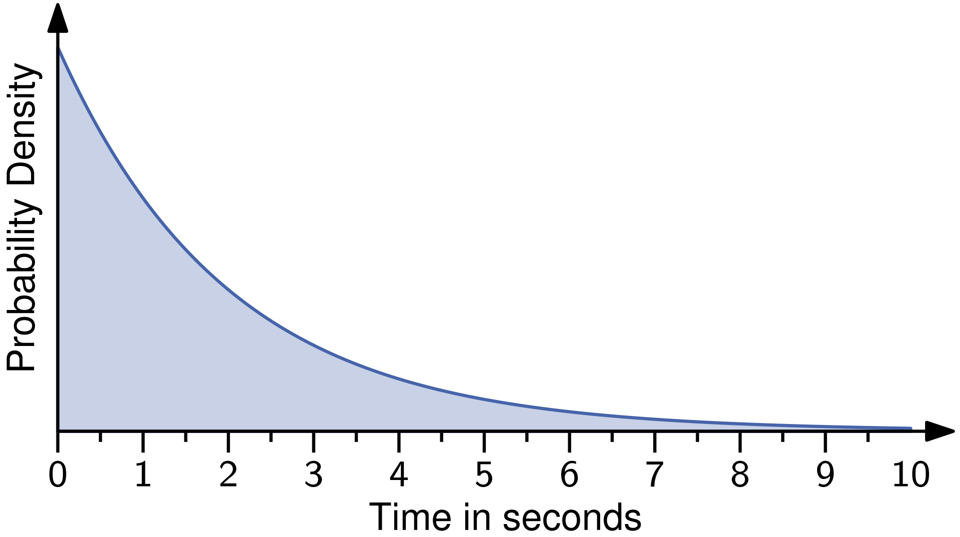

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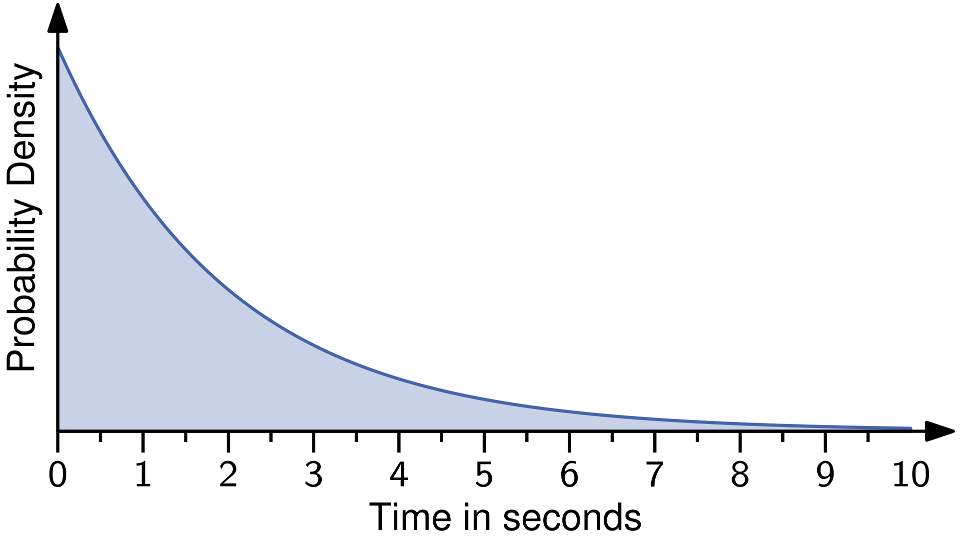

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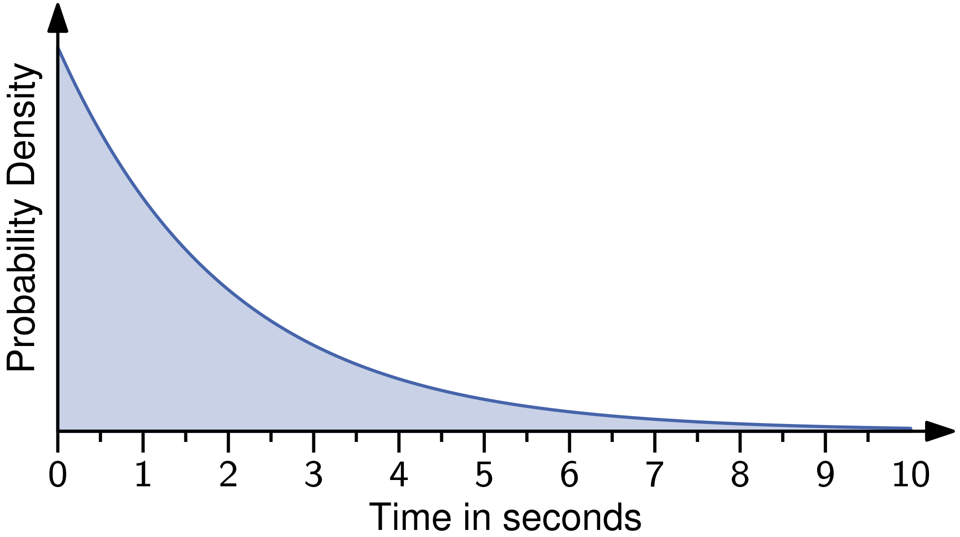

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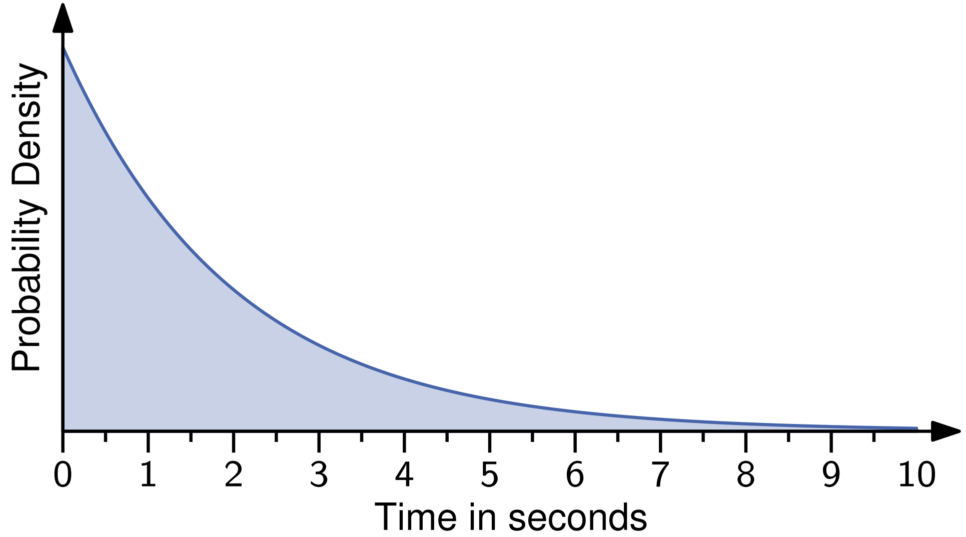

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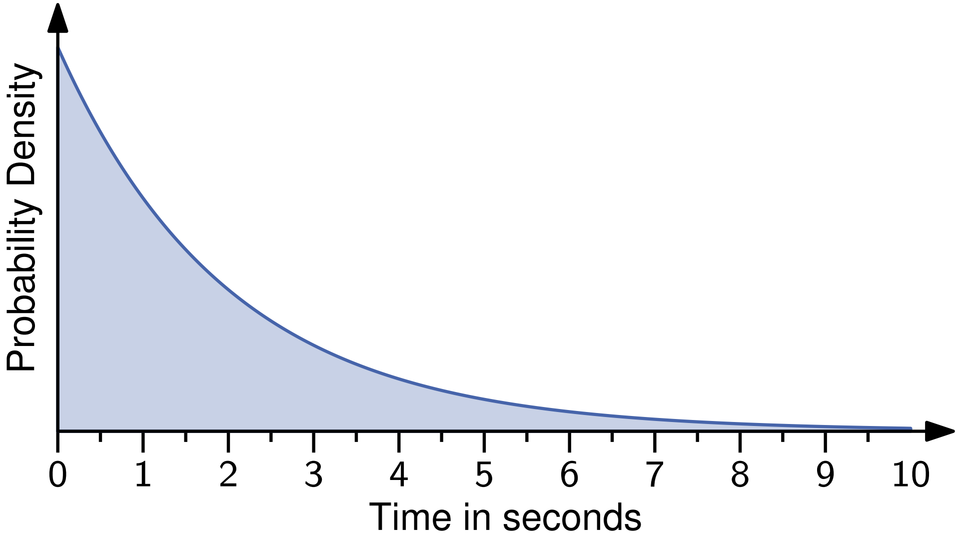

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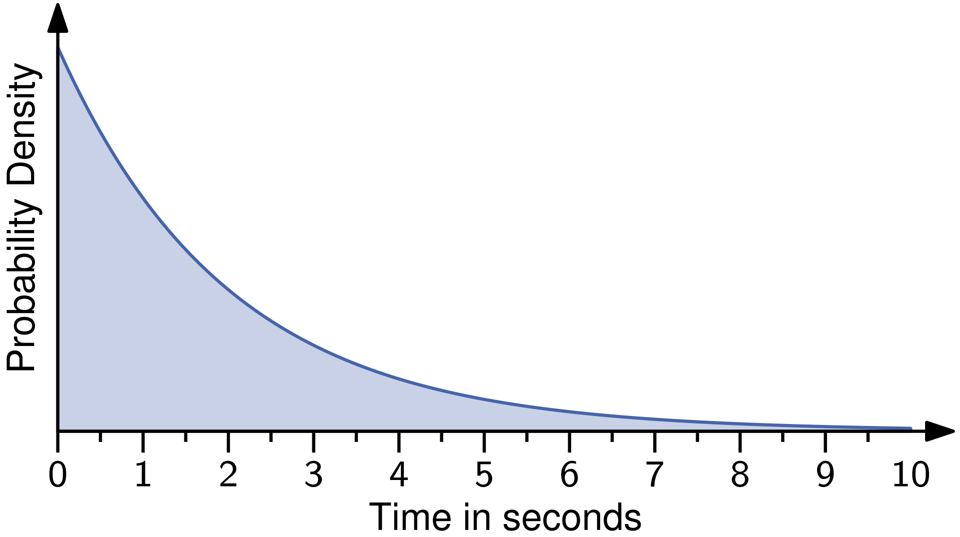

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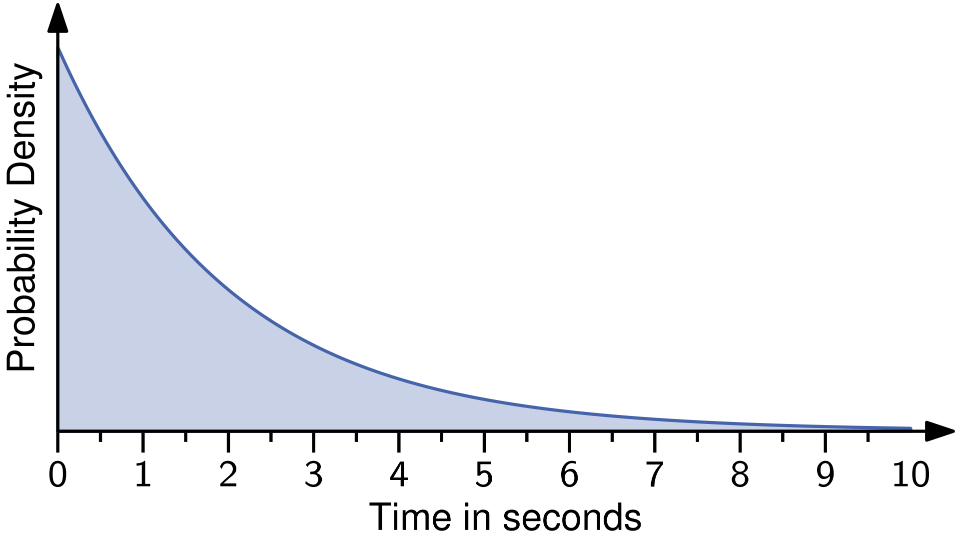

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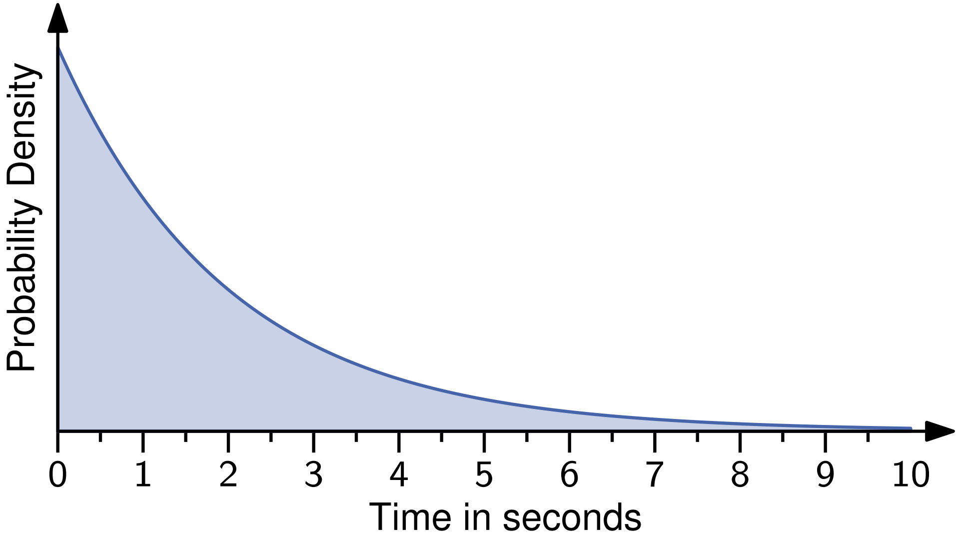

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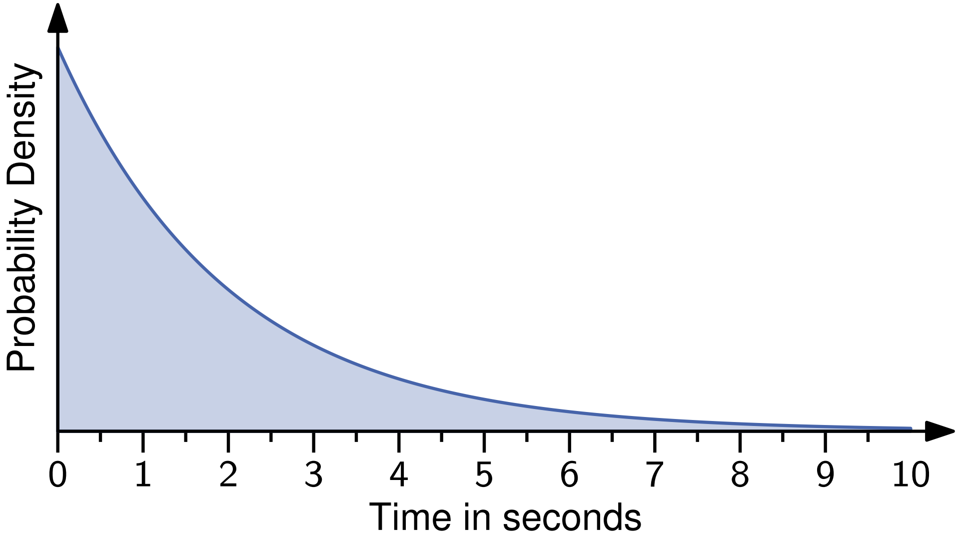

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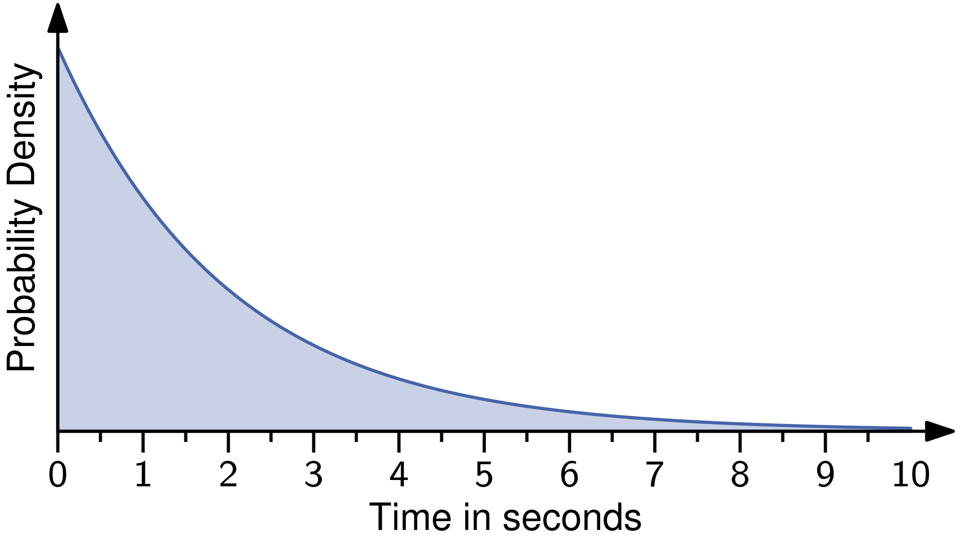

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Example: Radioactive Decay

Exponential Distribution $X \sim \mathrm{Exp}(\lambda)$

- "Rate" parameter $\lambda > 0$
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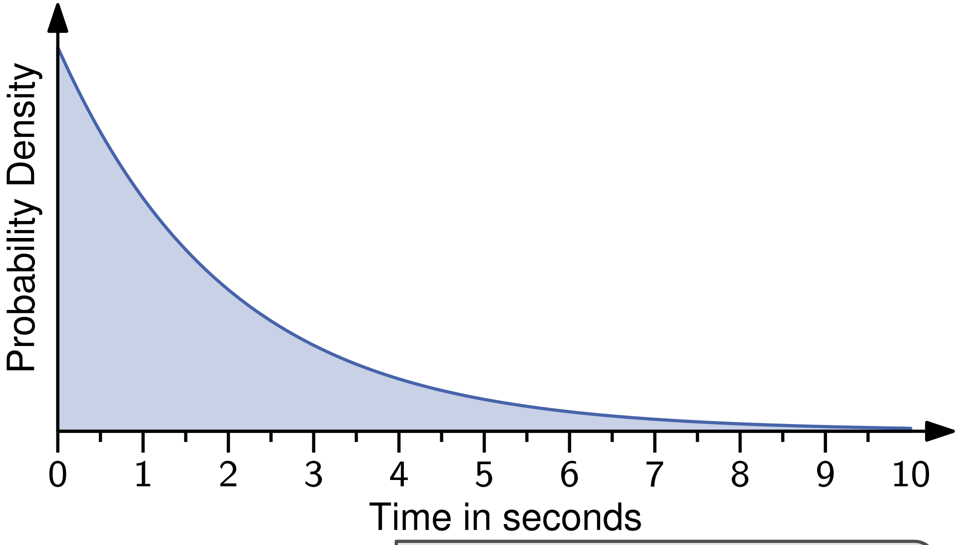

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Karlsruhe Institute of Technology

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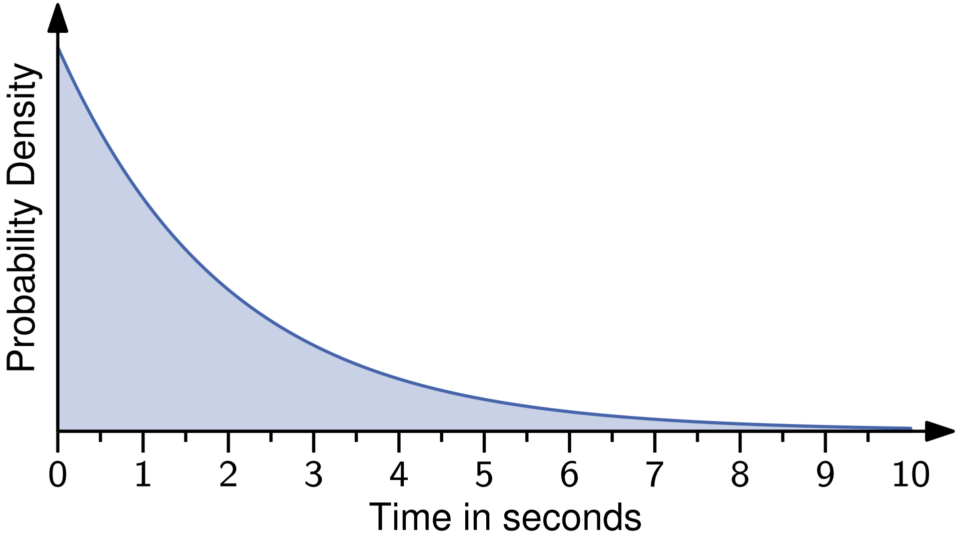

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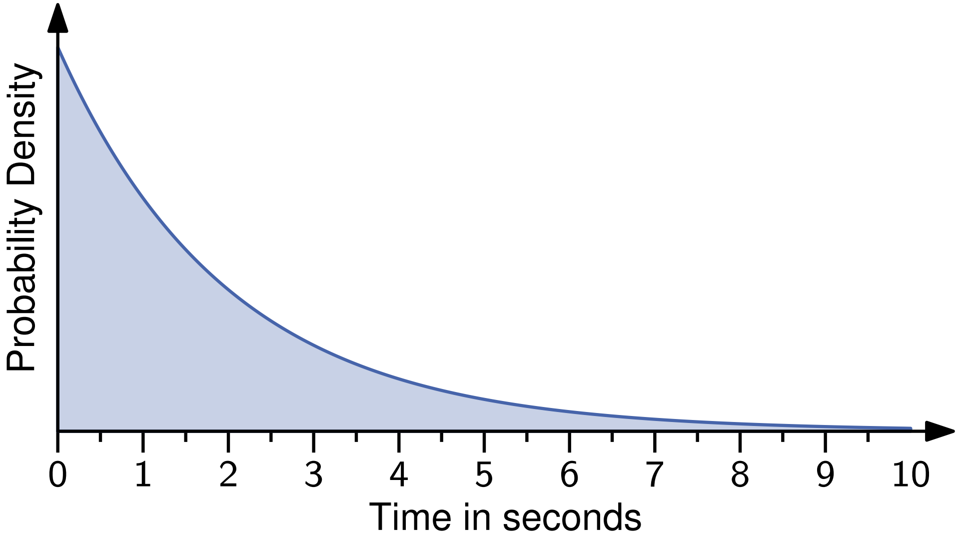

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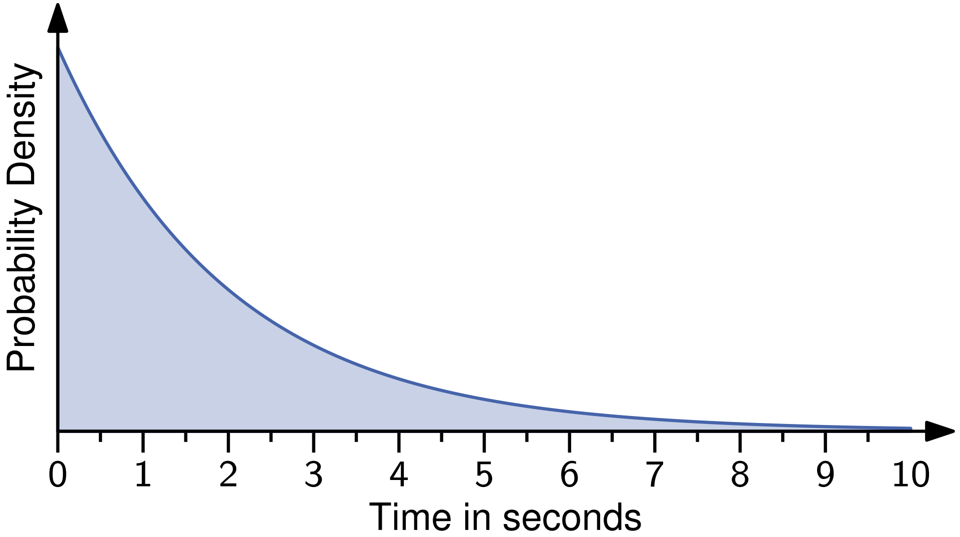

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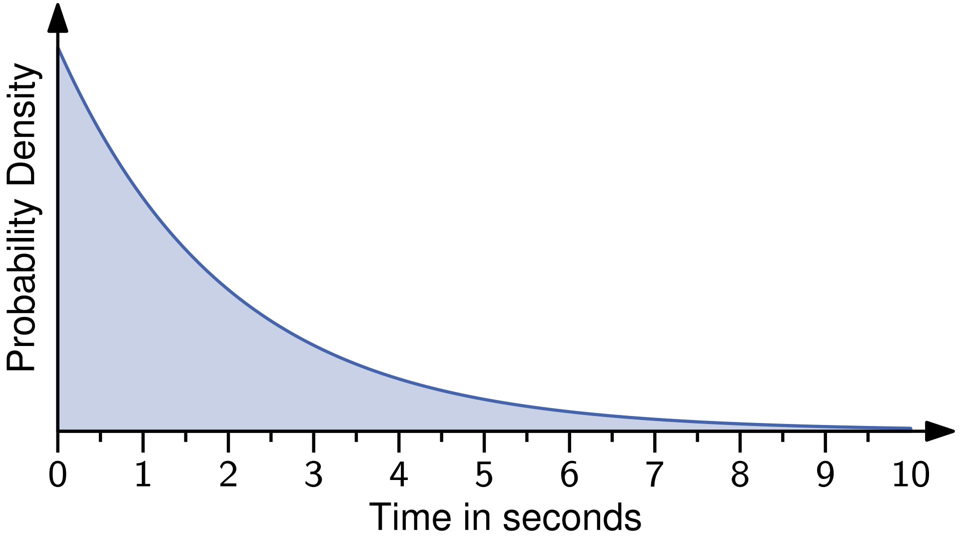

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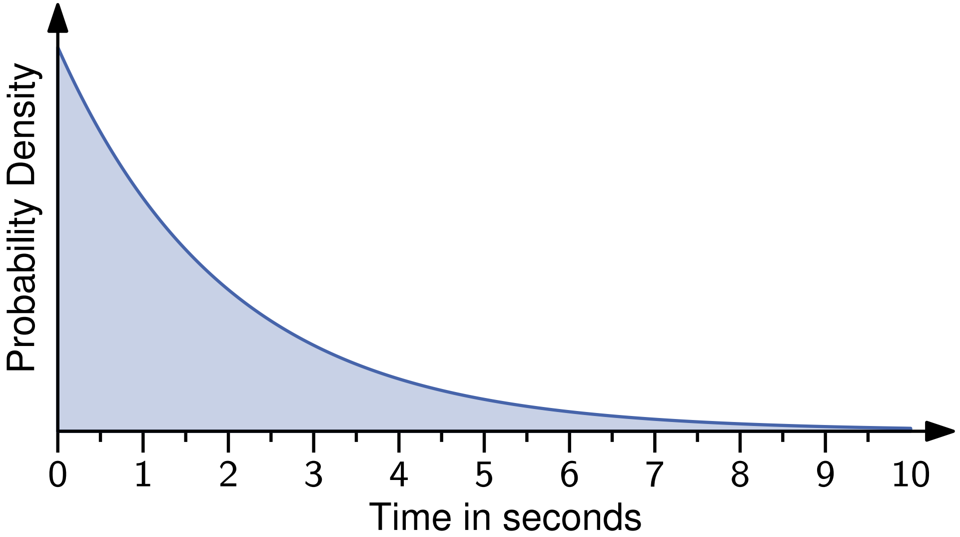

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Exponential Distribution: Memorylessness

Motivation

- What is the probability of having to wait longer than an additional time $s > 0$ after already having waited time $t > 0$?

$X \sim \mathrm{Exp}(\lambda)$

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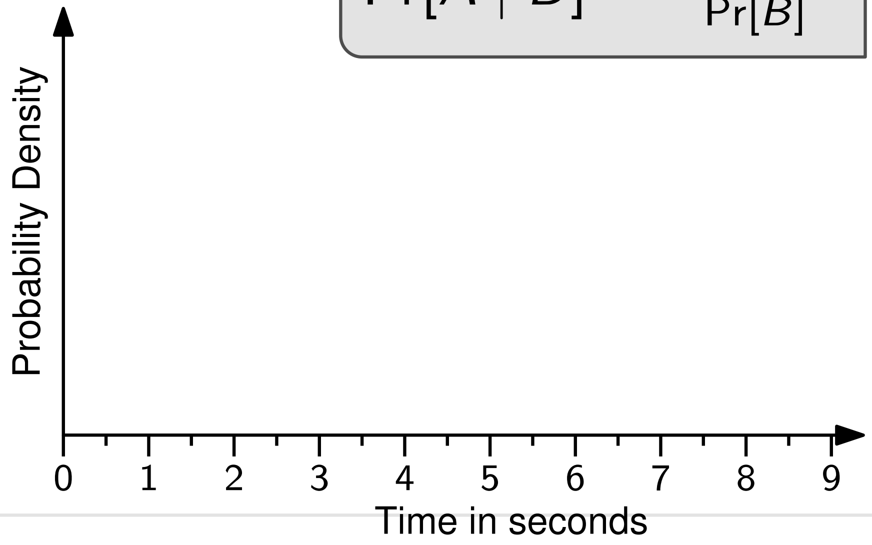

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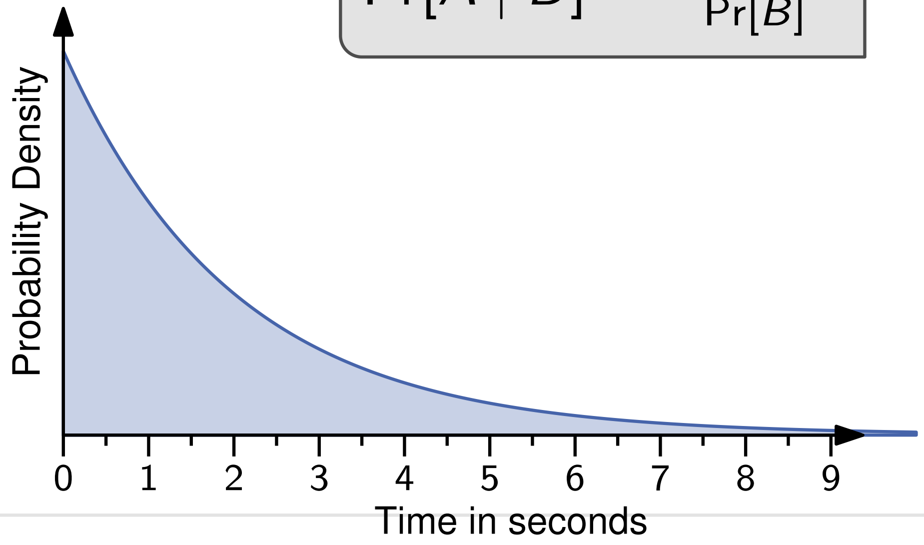

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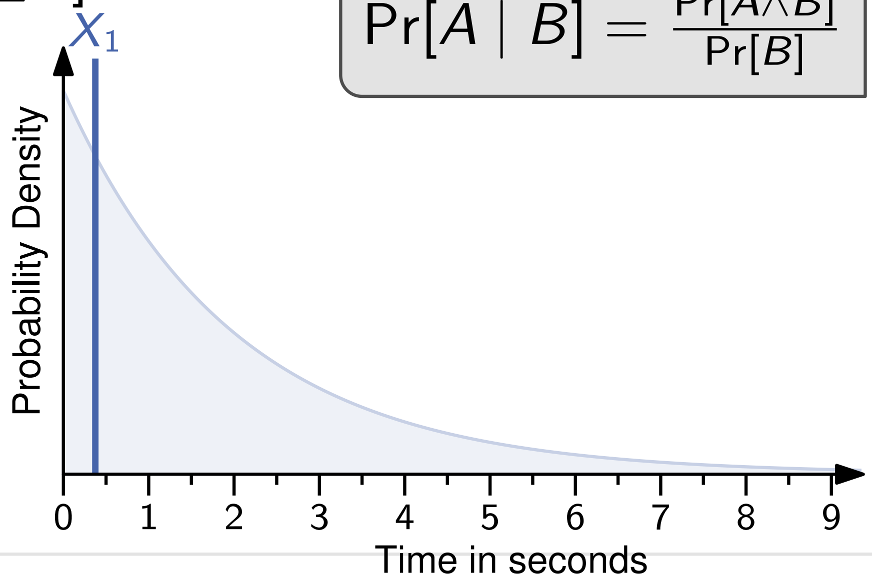

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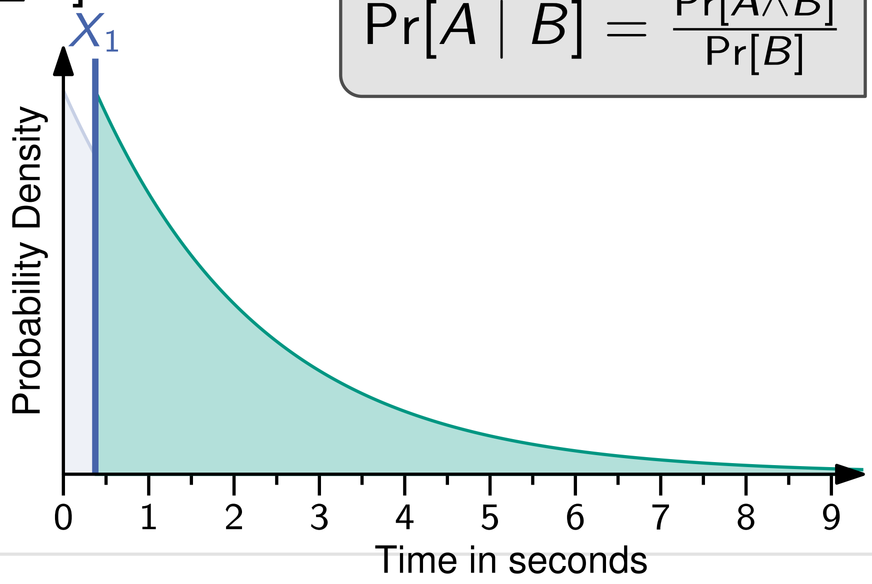

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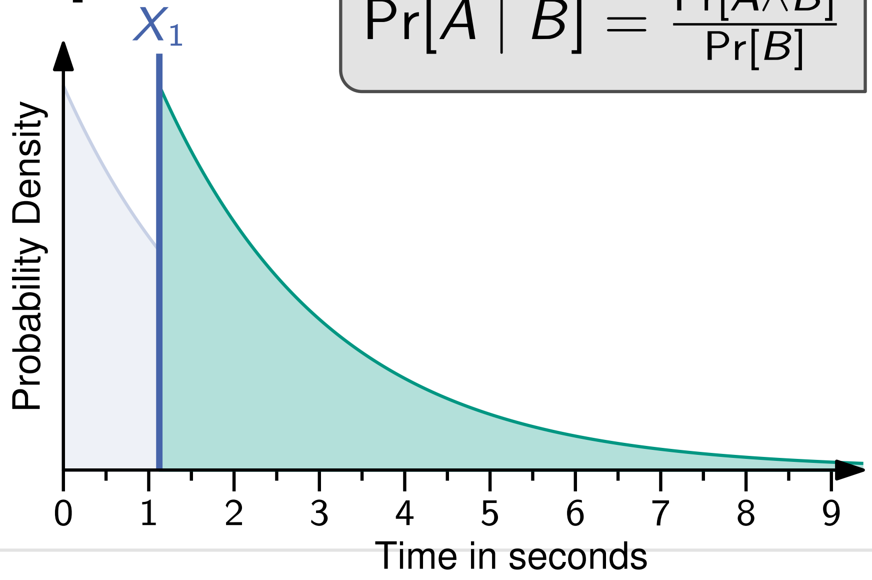

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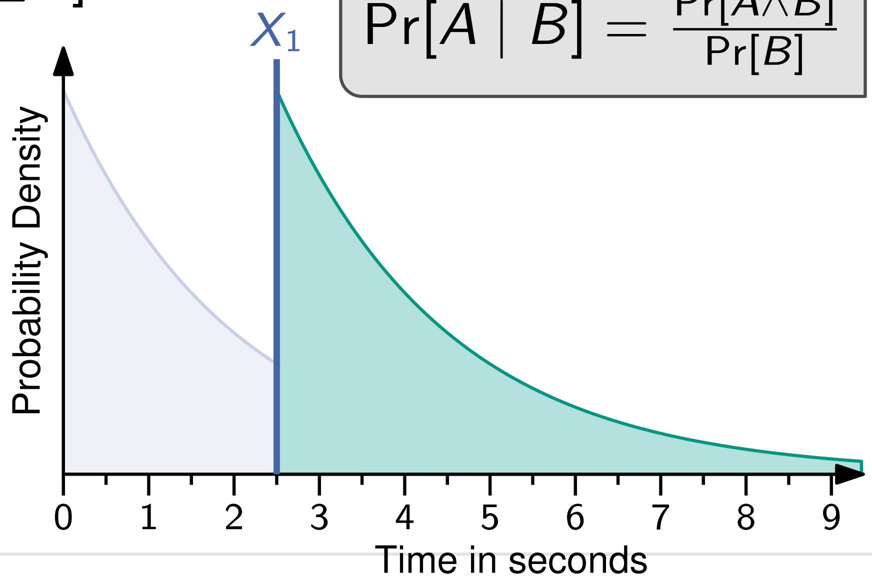
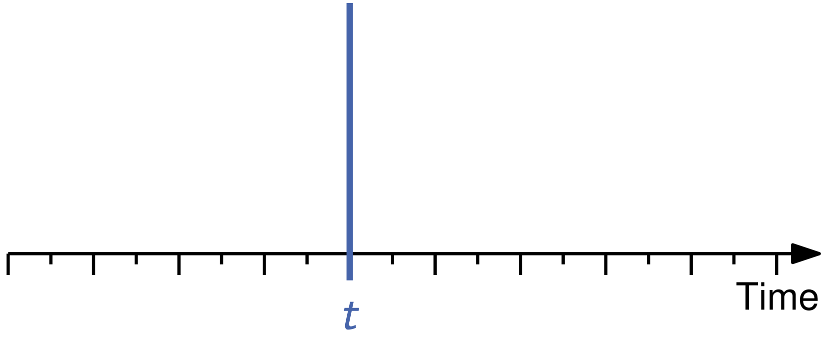


Counting Decays

Motivation

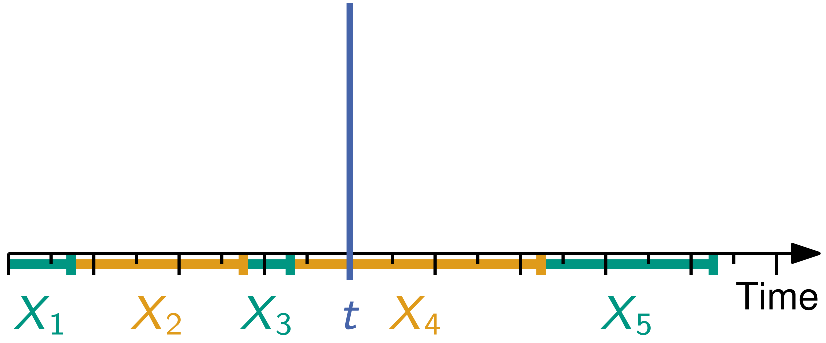
- Count number of particles emitted within a given time t



Counting Decays

Motivation

- Count number of particles emitted within a given time t
- Let $X_1, X_2, X_3, \ldots \sim \mathrm{Exp}(\lambda)$ be independent waiting times



Counting Decays

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- Let $N(a, b)$ be the number of emissions in $[a, b]$
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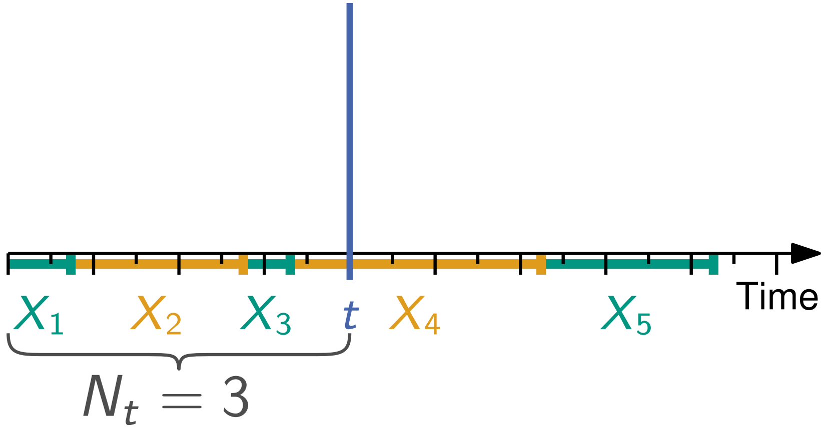

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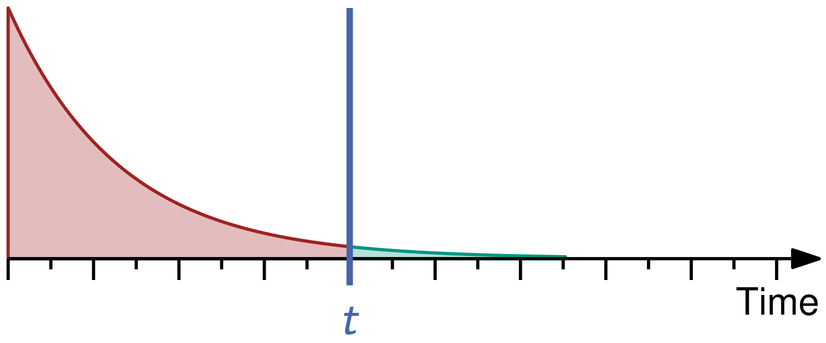

Specific Values

$$\Pr[N_t = 0]$$

$X \sim \mathrm{Exp}(\lambda)$
$f_X(x) = \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$
$F_X(x) = 1 - e^{-\lambda x}$

Counting Decays

Karlsruhe Institute of Technology

Motivation

- Count number of particles emitted within a given time t
- Let $X_1, X_2, X_3, \ldots \sim \mathrm{Exp}(\lambda)$ be independent waiting times
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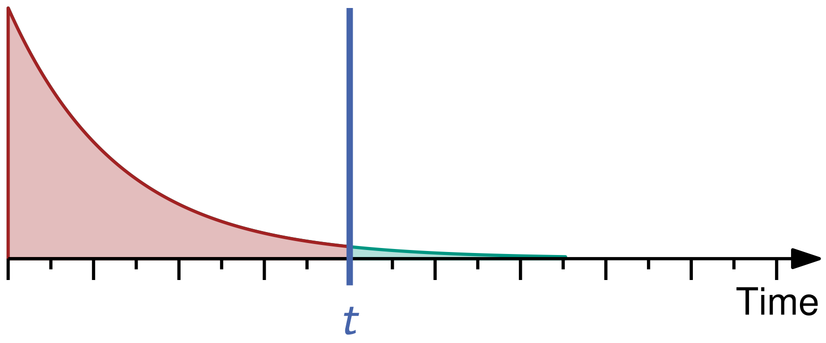

$\boldsymbol{X \sim \mathrm{Exp}(\lambda)}$
$f_X(x) = \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$
$F_X(x) = 1 - e^{-\lambda x}$

Specific Values

$$\Pr[N_t = 0] = \Pr[X_1 > t]$$

Counting Decays

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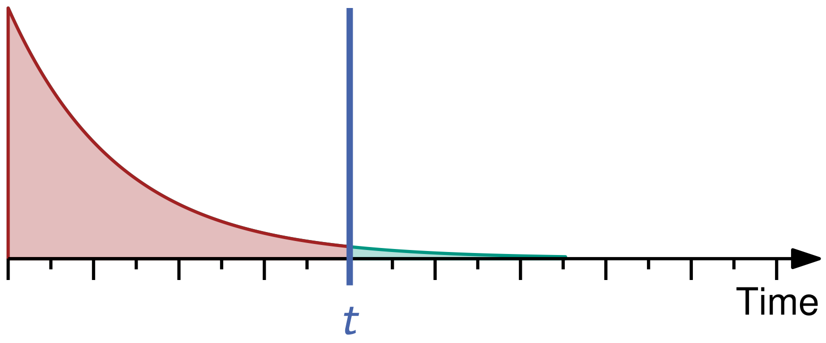
Specific Values

$$\Pr[N_t = 0] = \Pr[X_1 > t] = 1 - \Pr[X_1 \leq t] = 1 - F_{X_1}(t)$$



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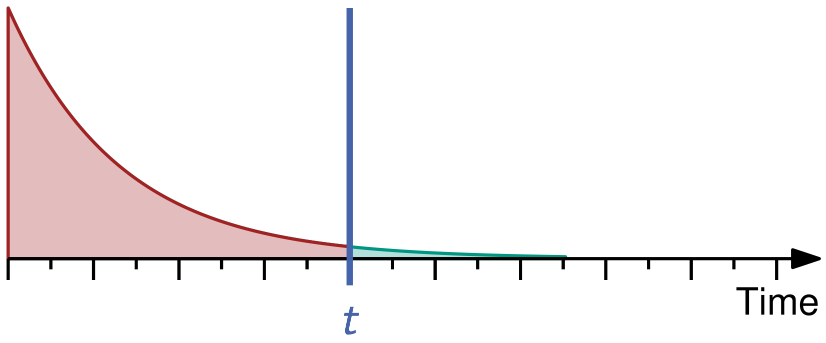

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Specific Values

$$\Pr[N_t = 0] = \Pr[X_1 > t] = 1 - \Pr[X_1 \leq t] = 1 - F_{X_1}(t) = e^{-\lambda t}$$

Counting Decays

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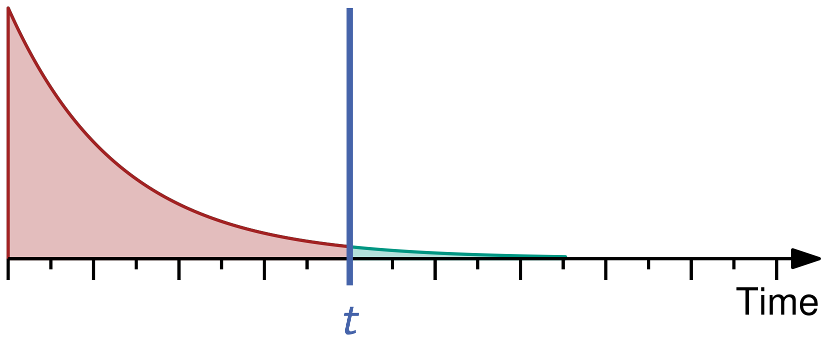
Specific Values

$$\Pr[N_t = 0] = e^{-\lambda t}$$



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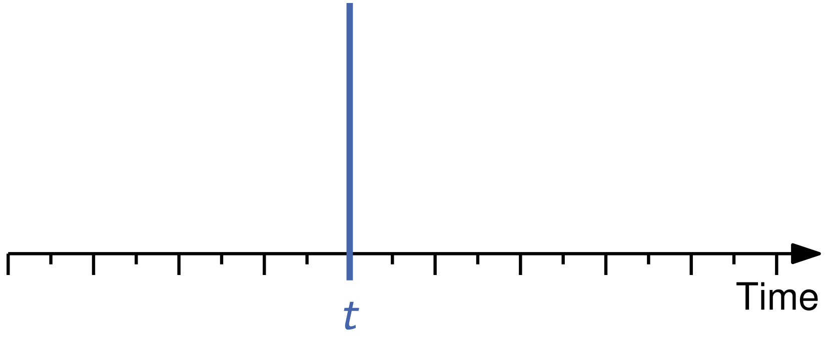

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Specific Values

$$\Pr[N_t = 0] = e^{-\lambda t}$$
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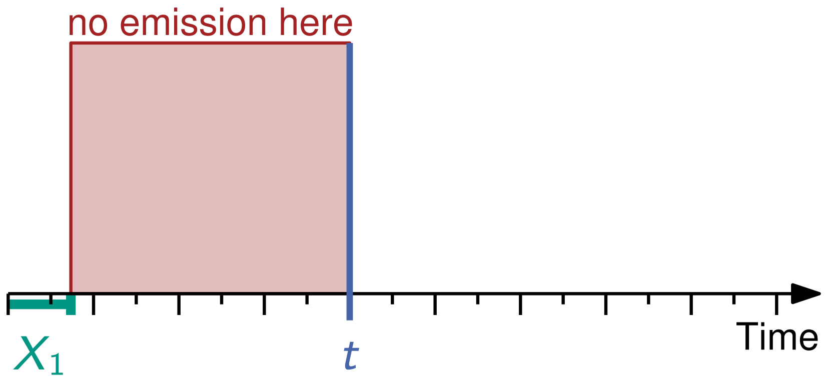

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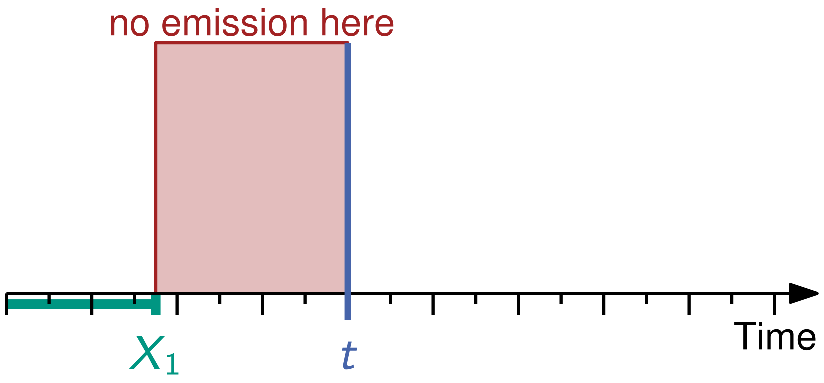

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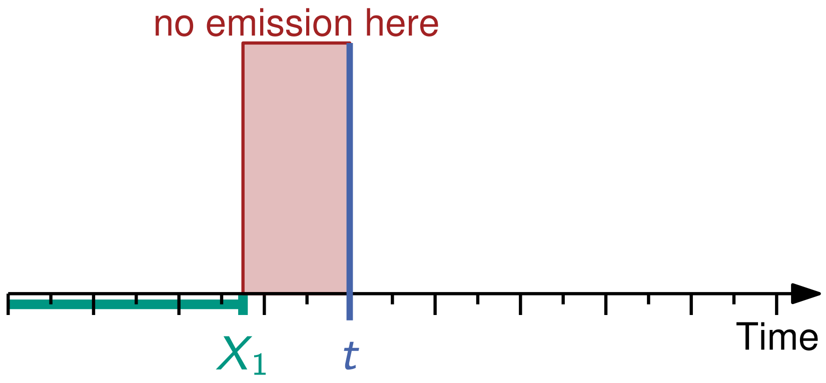

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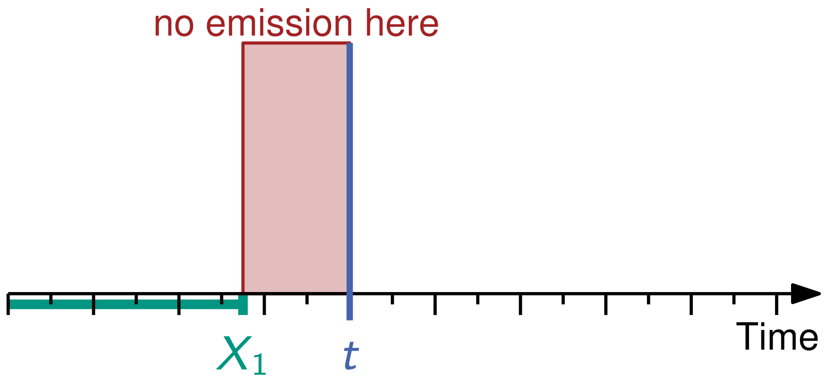

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Law of Total Probability: $\Pr[A] = \int_{-\infty}^{\infty} \Pr[A \mid X = x] \cdot f_X(x) \mathrm{d}x$

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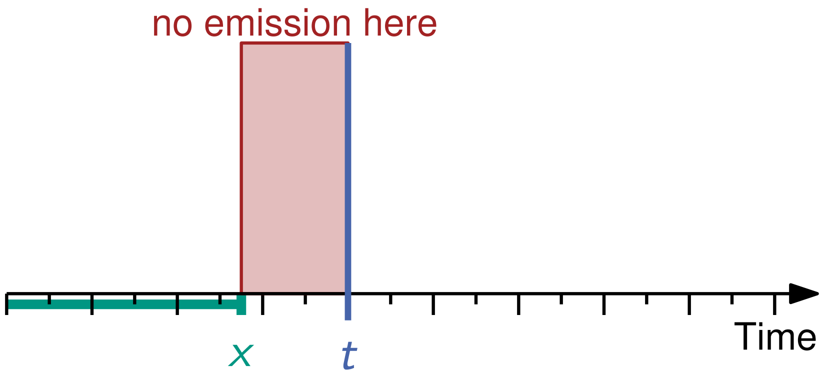

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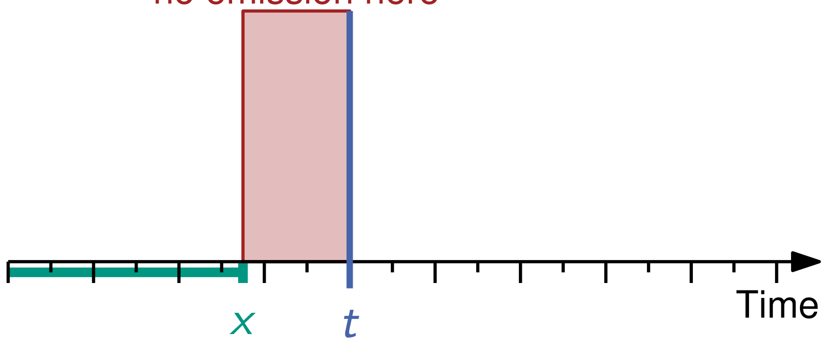

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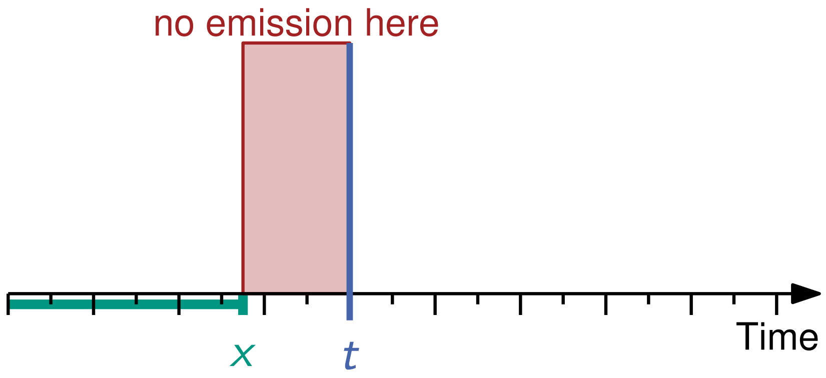

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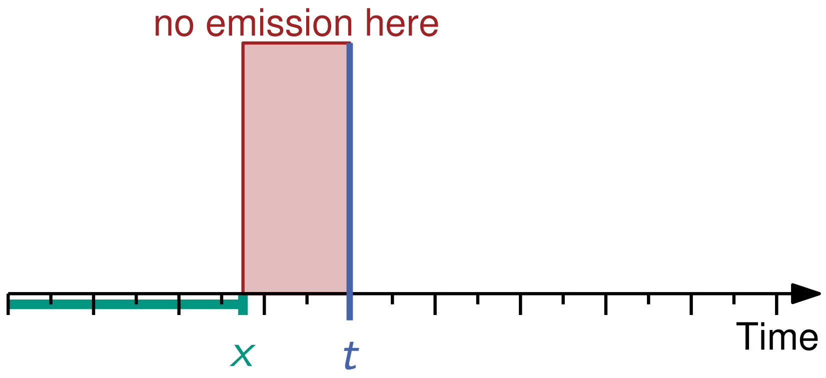

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Counting Decays

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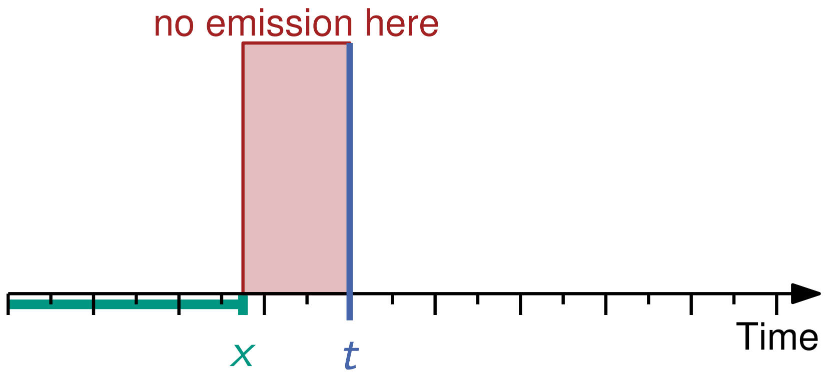

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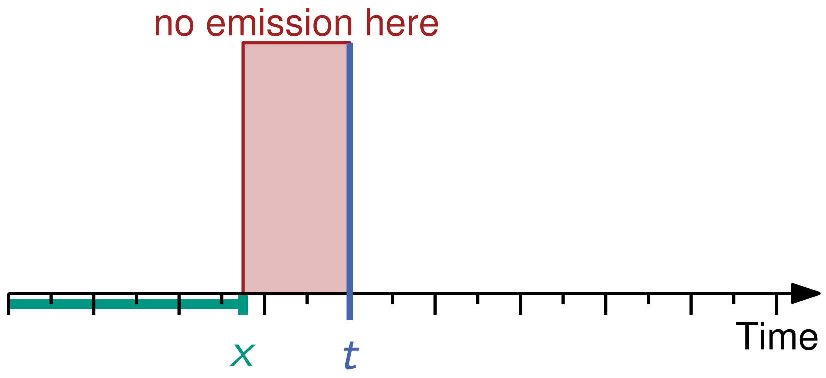

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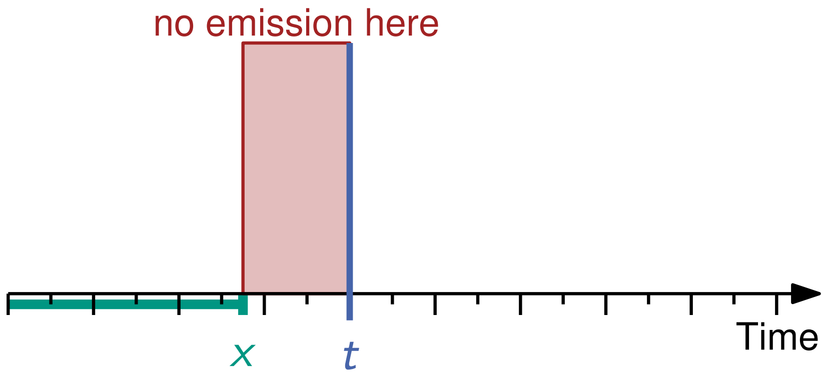
Specific Values

$$\Pr[N_t = 0] = e^{-\lambda t}$$

$$\begin{aligned}\Pr[N_t = 1] &= \int_{-\infty}^{\infty} \Pr[X_1 \leq t \land N(x, t) = 0 \mid X_1 = x] f_{X_1}(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \Pr[X_1 \leq t \land N(x, t) = 0 \mid X_1 = x]\lambda e^{-\lambda x}\mathbb{1}_{x\geq 0}\mathrm{d}x \\ &= \int_{0}^{t} \Pr[X_1 \leq t \land N(x, t) = 0 \mid X_1 = x]\lambda e^{-\lambda x}\mathbb{1}_{x\geq 0}\mathrm{d}x\end{aligned}$$

Counting Decays



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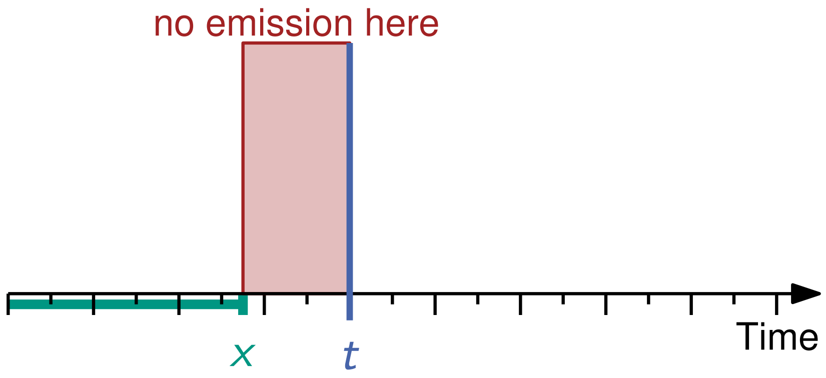
Counting Decays



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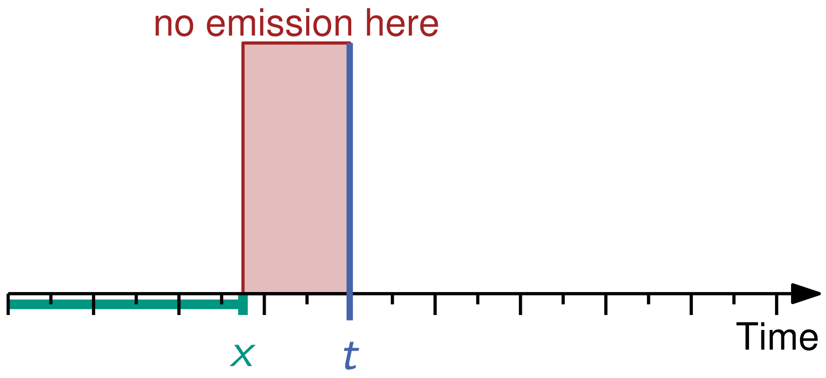
Counting Decays



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$$
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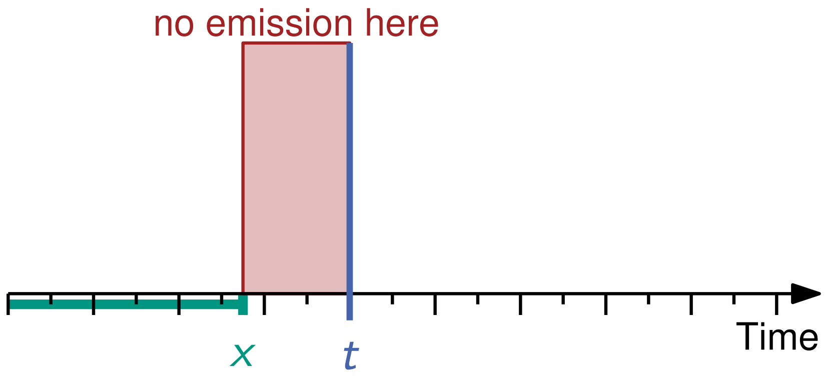

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&= \int_{0}^{t} \Pr[N(x, t) = 0 \mid X_1 = x] \lambda e^{-\lambda x}\mathrm{d}x
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independent

Counting Decays

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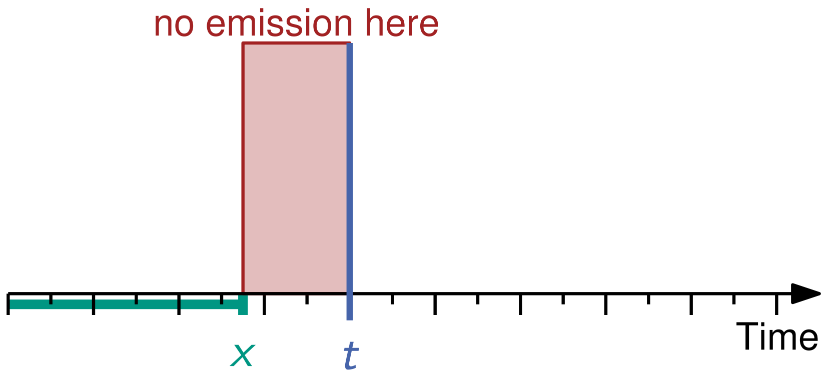

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$$F_X(x) = 1 - e^{-\lambda x}$$

$$\Pr[N_t = 0] = e^{-\lambda t}$$

$$\begin{aligned}
\Pr[N_t = 1] &= \int_{-\infty}^{\infty} \Pr[X_1 \leq t \wedge N(x, t) = 0 \mid X_1 = x] f_{X_1}(x)\mathrm{d}x \\
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&= \int_0^t \Pr[N(x, t) = 0 \mid \cancel{X_1 = x}] \lambda e^{-\lambda x}\mathrm{d}x
\end{aligned}$$

independent

Counting Decays

Motivation

- Count number of particles emitted within a given time t
- Let $X_1, X_2, X_3, \ldots \sim \mathrm{Exp}(\lambda)$ be independent waiting times
- Let $N(a, b)$ be the number of emissions in $[a, b]$
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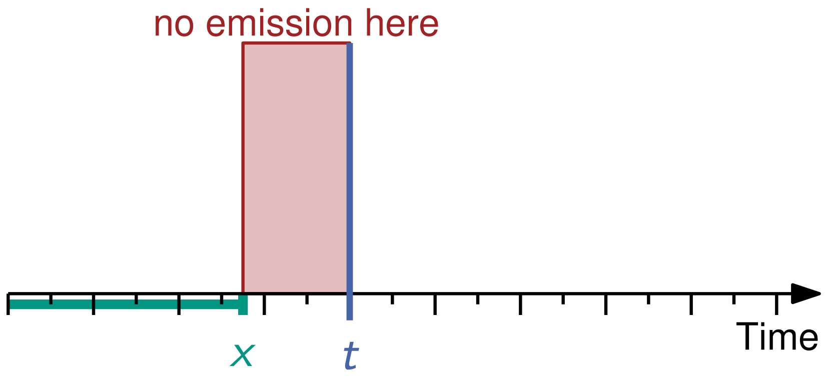

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Specific Values

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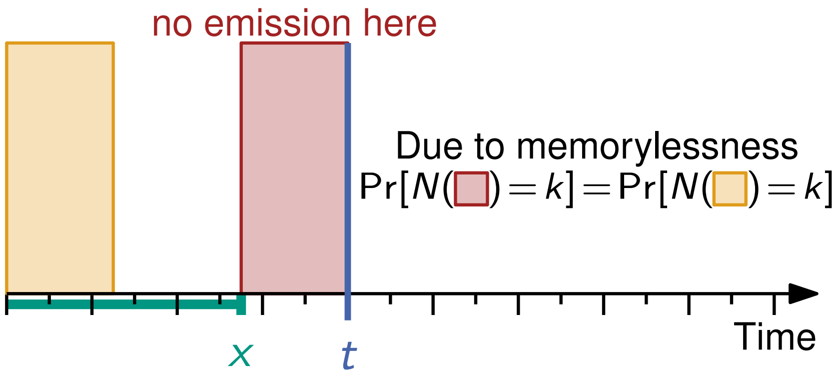
Counting Decays



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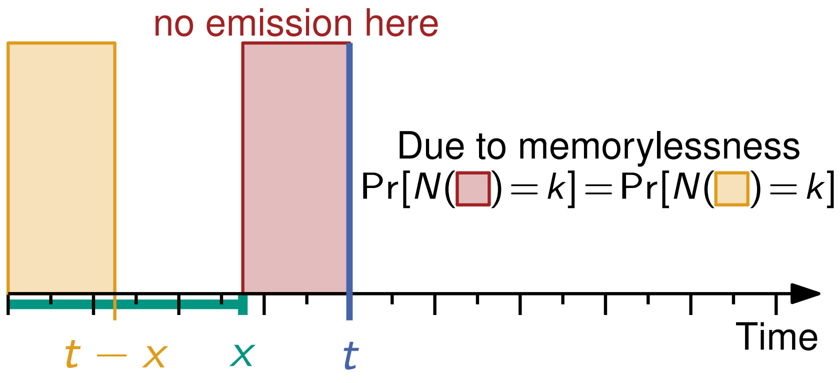

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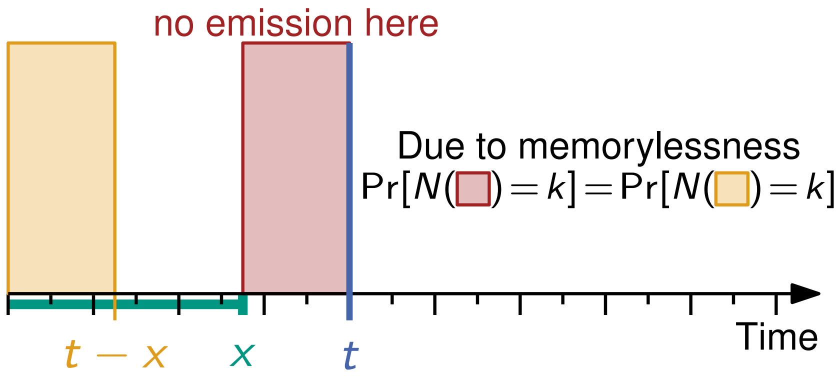
Counting Decays



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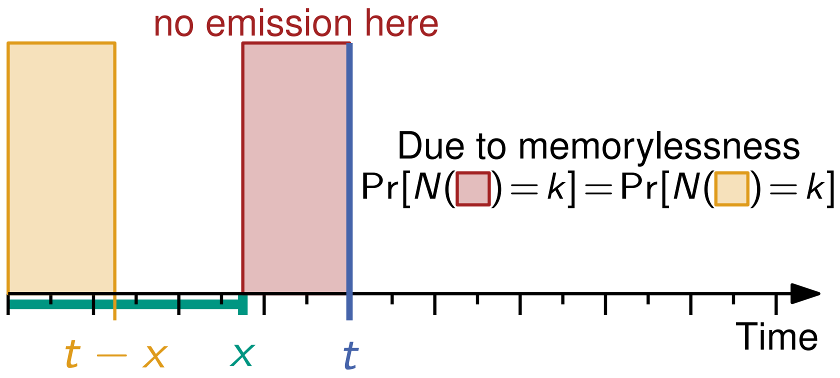

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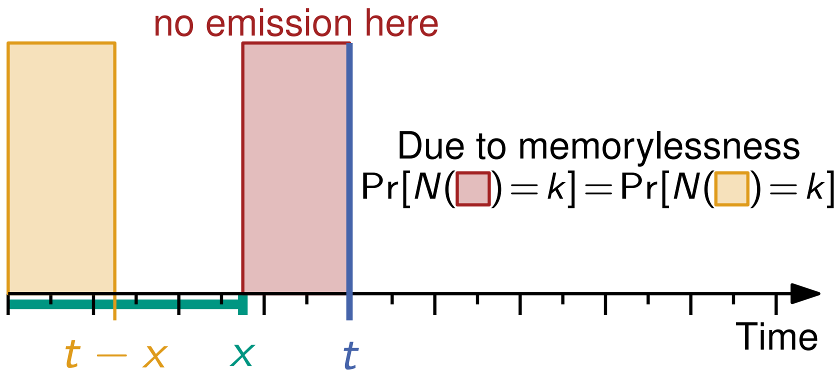

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Counting Decays

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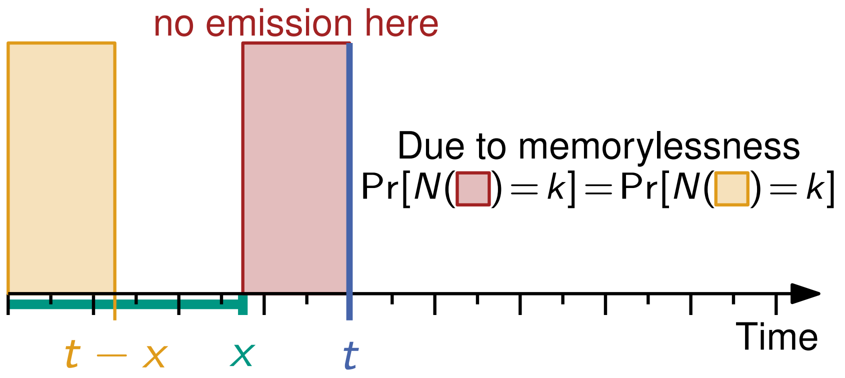

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&= \lambda e^{-\lambda t} \int_0^t 1\mathrm{d}x
\end{aligned}$$

Counting Decays

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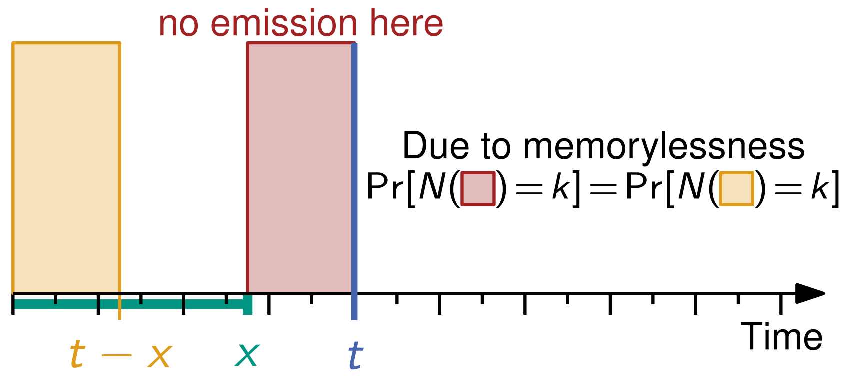

Specific Values

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&= \lambda e^{-\lambda t} \int_0^t 1\mathrm{d}x = \lambda t e^{-\lambda t}
\end{aligned}$$

Counting Decays

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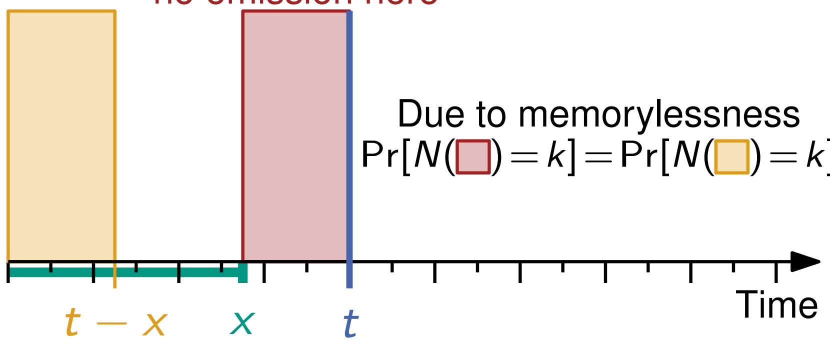

$\Pr[N(\square) = k] = \Pr[N(\square) = k]$

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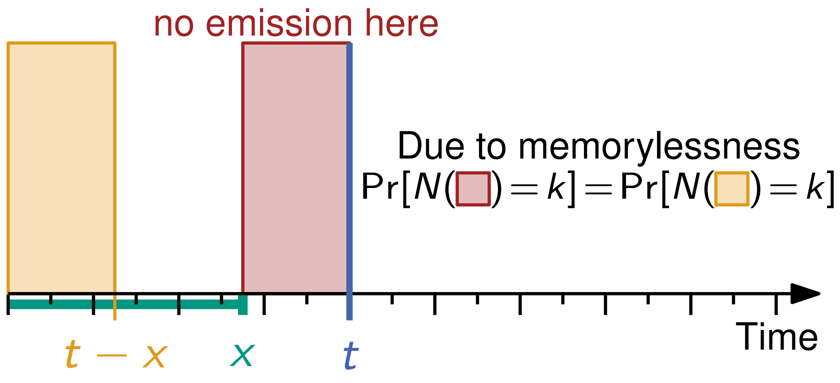

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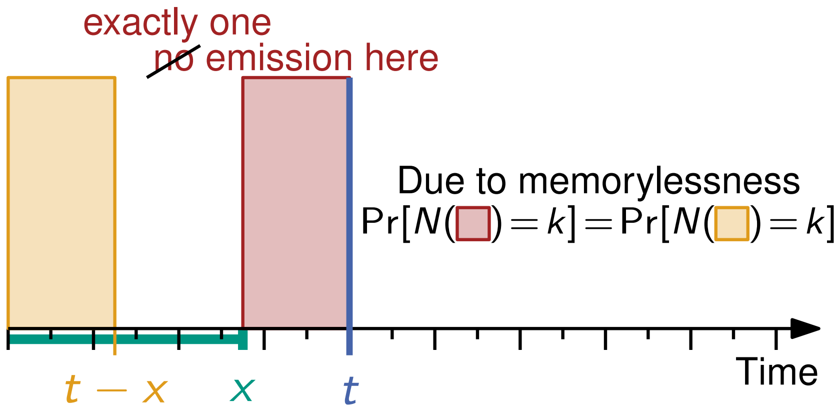
$\Pr[N_t = 2]$

Counting Decays



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Counting Decays

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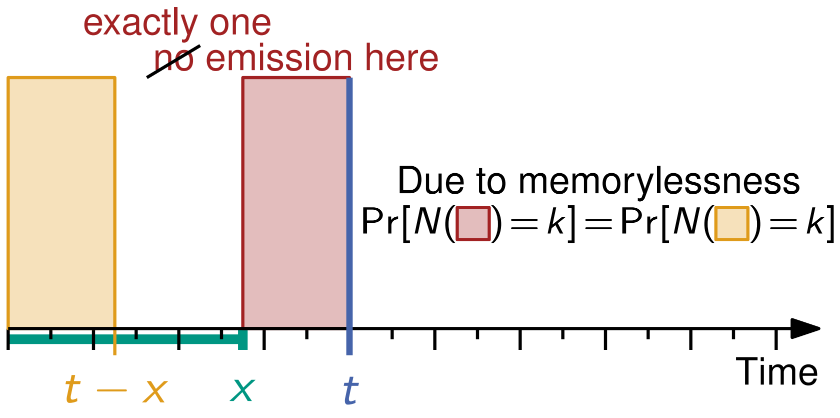

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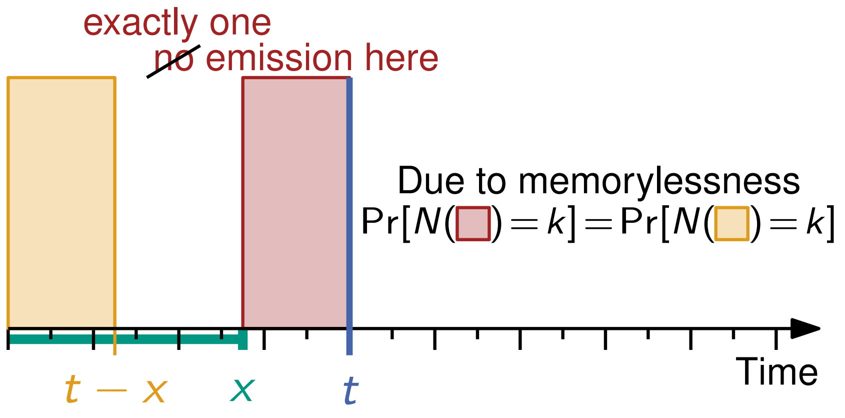

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independent

Counting Decays

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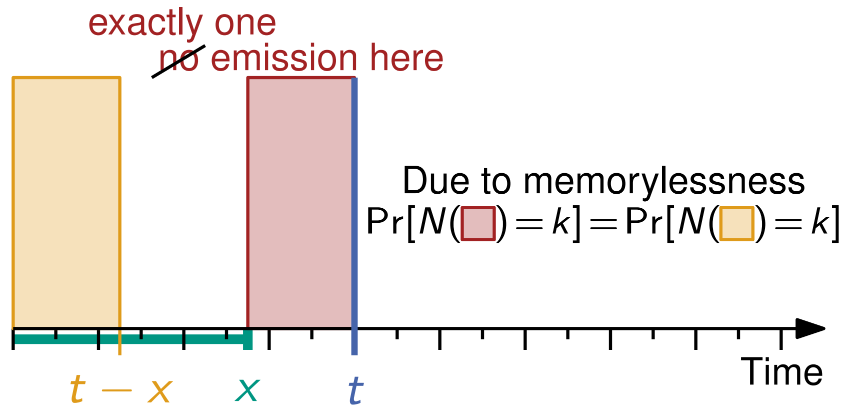

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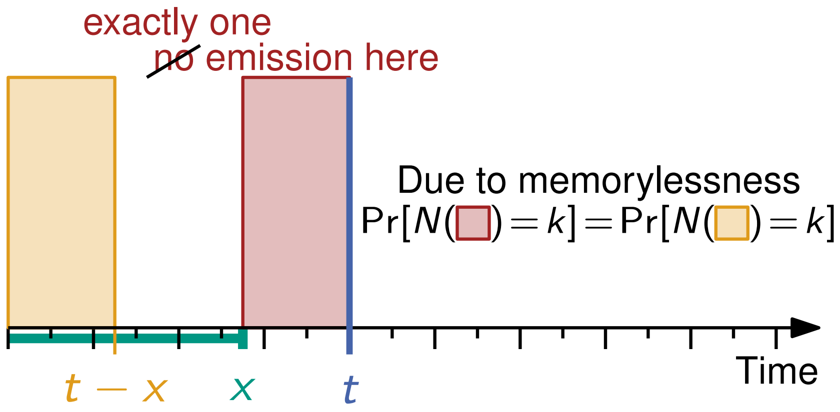
Counting Decays



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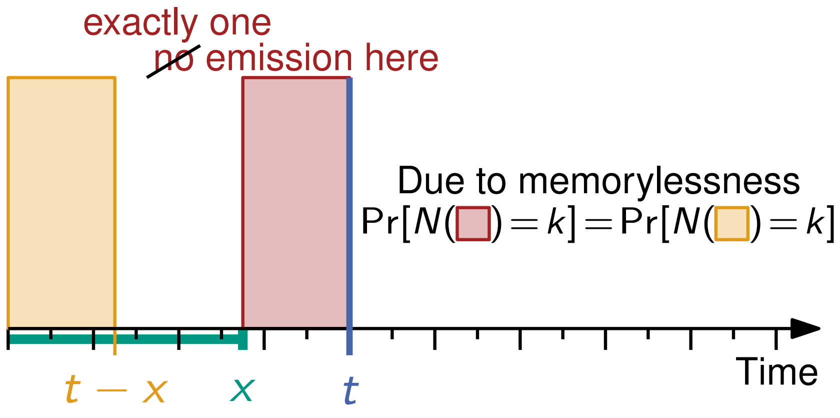

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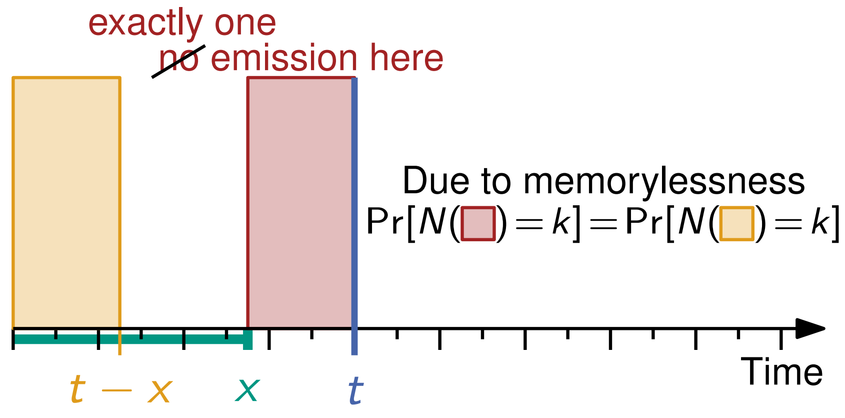

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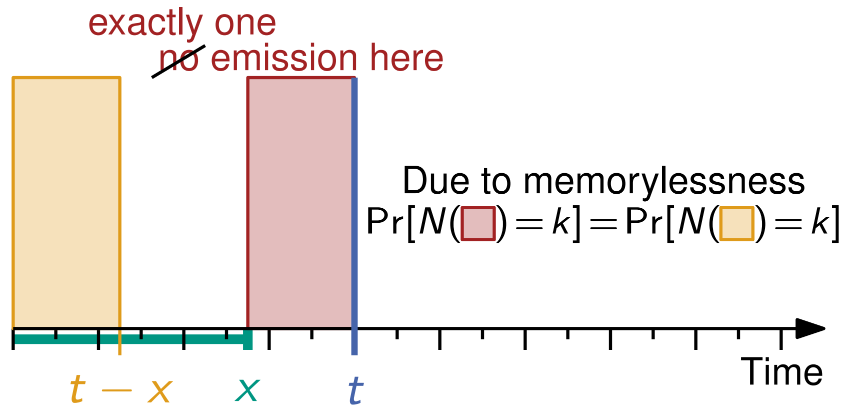

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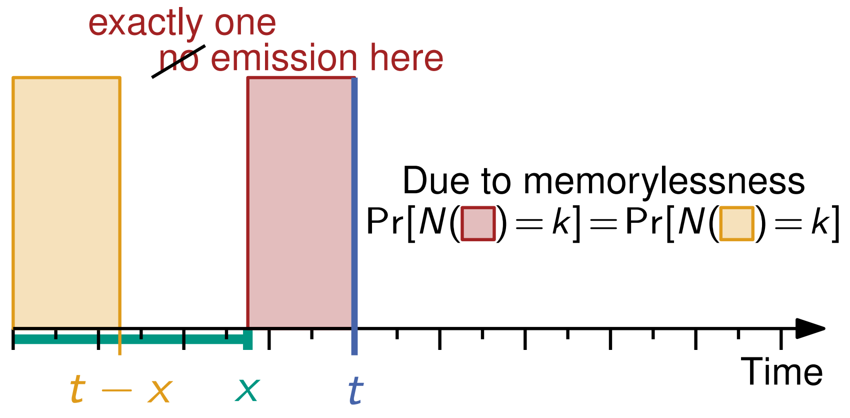

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Karlsruhe Institute of Technology

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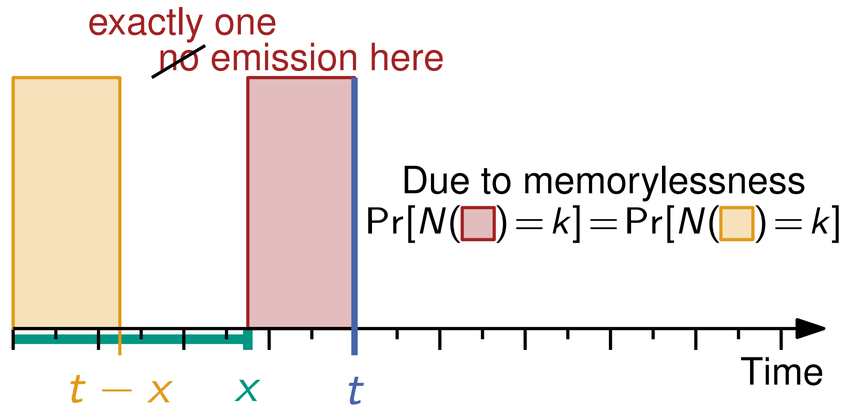

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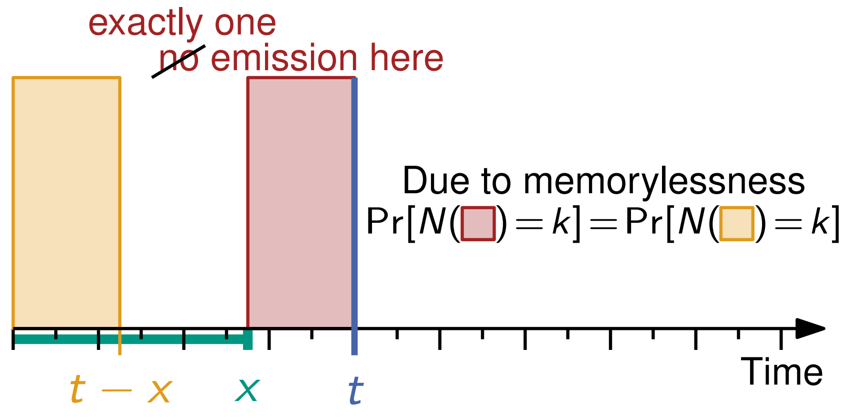

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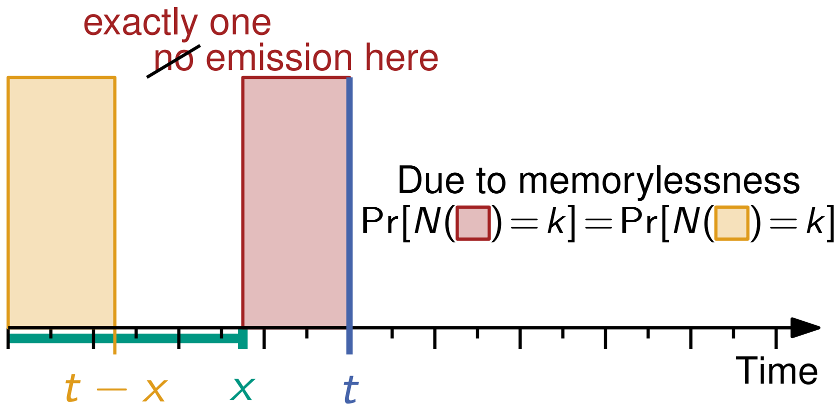

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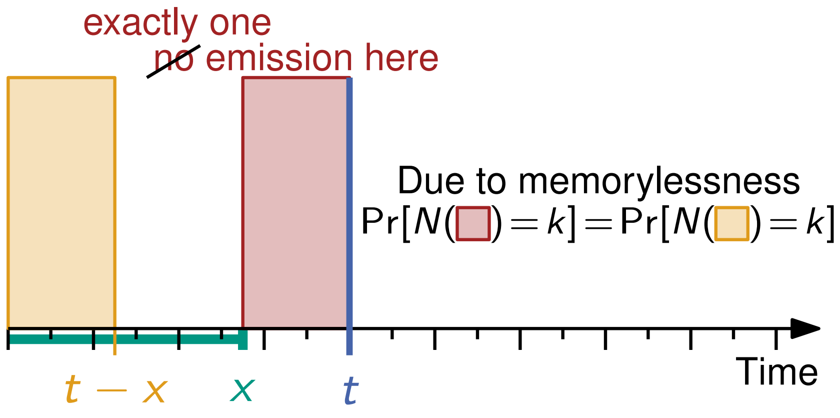

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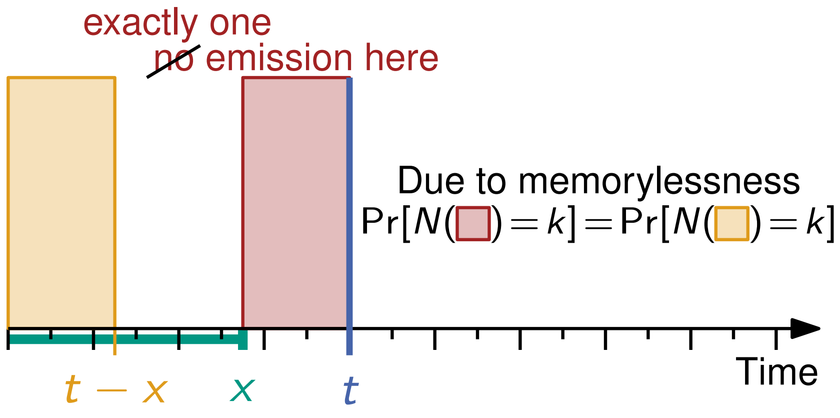

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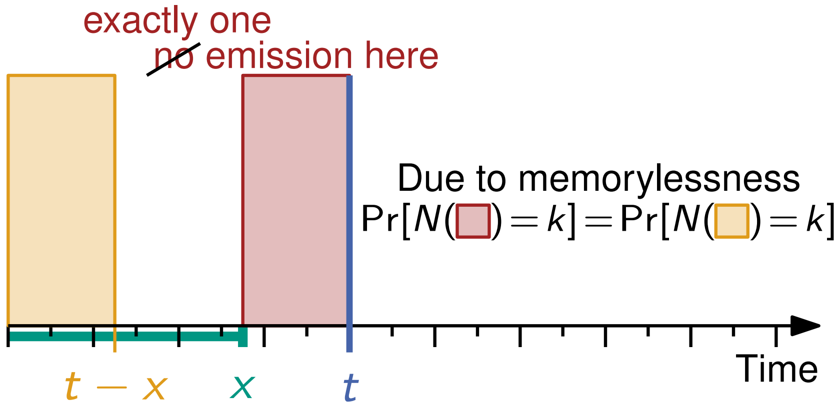

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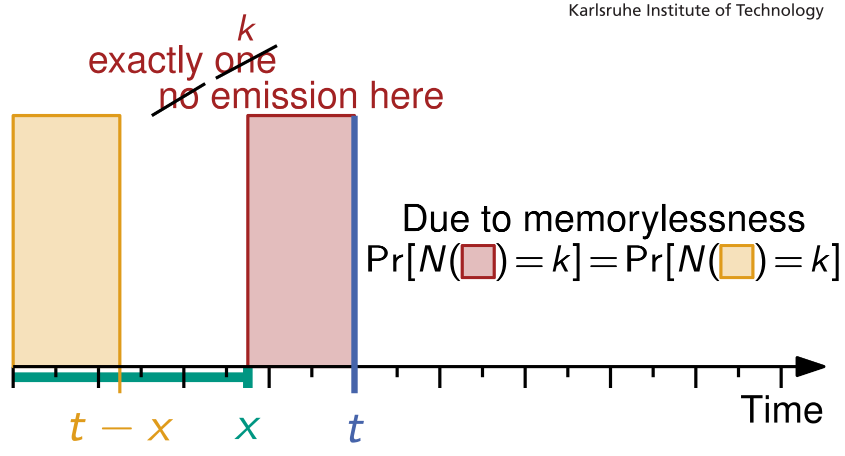

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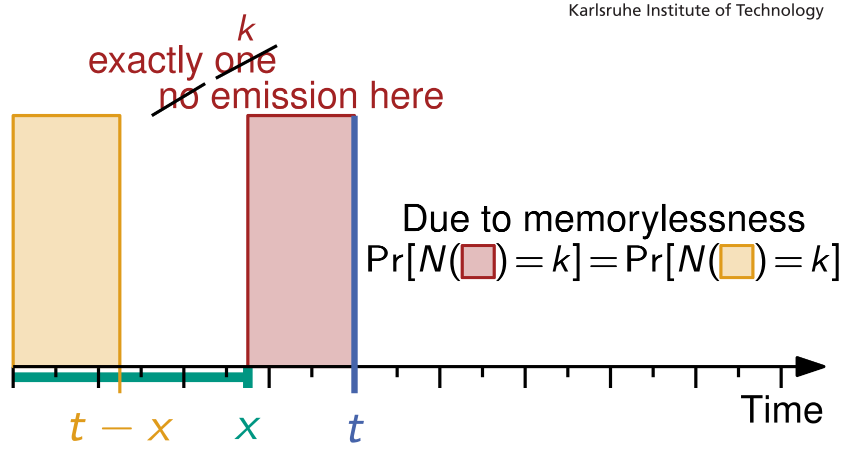

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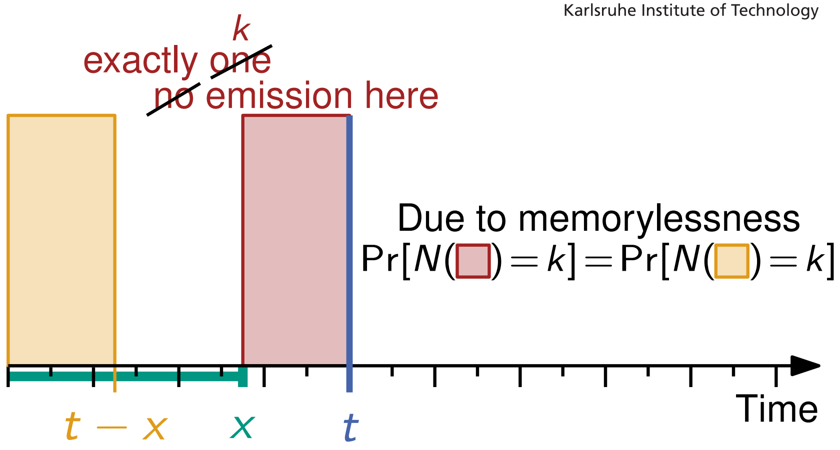

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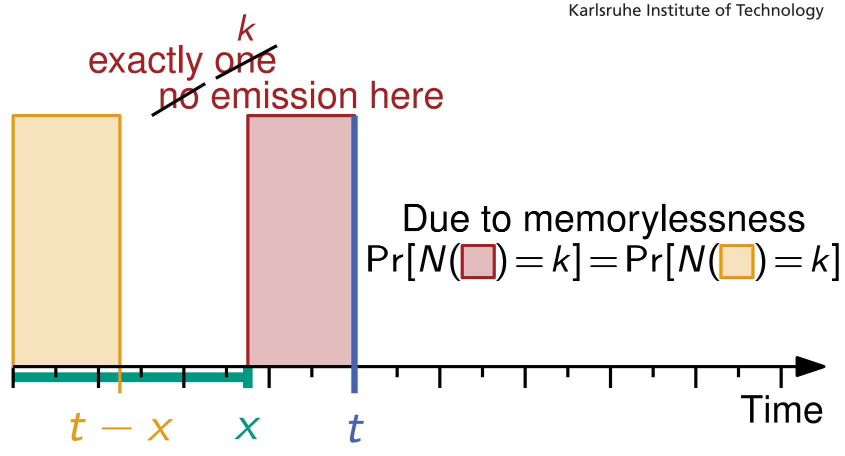

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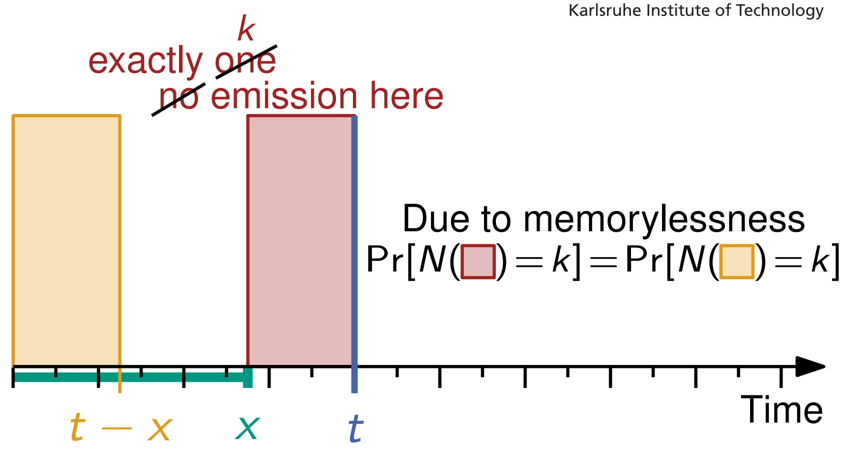

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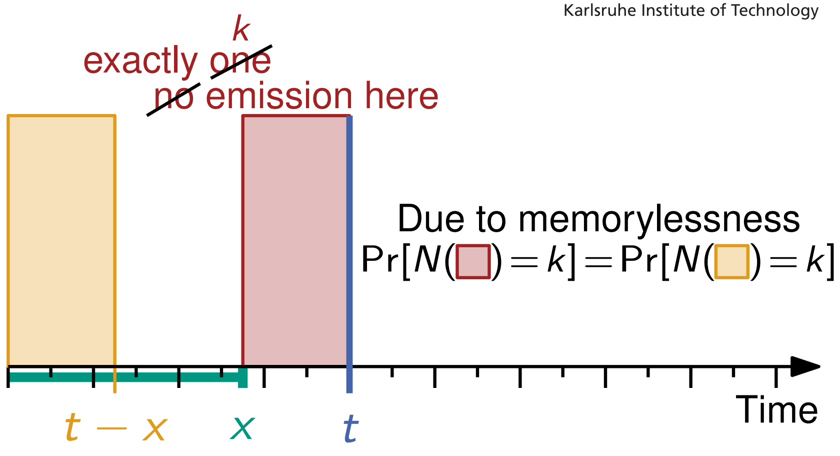

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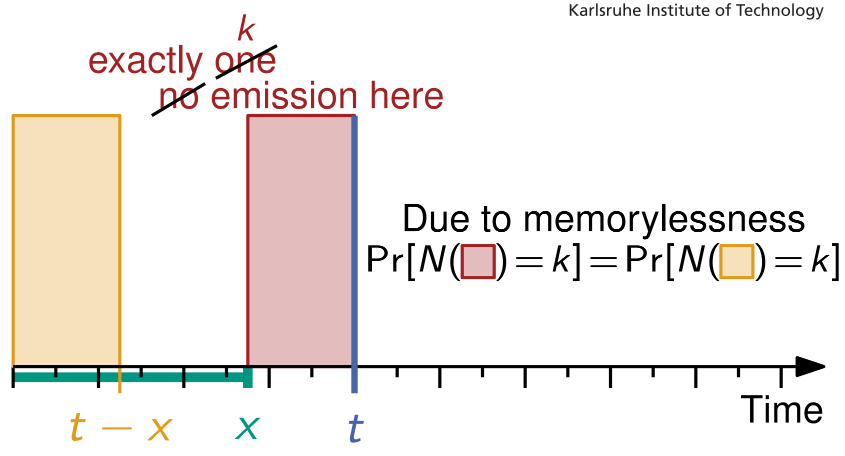

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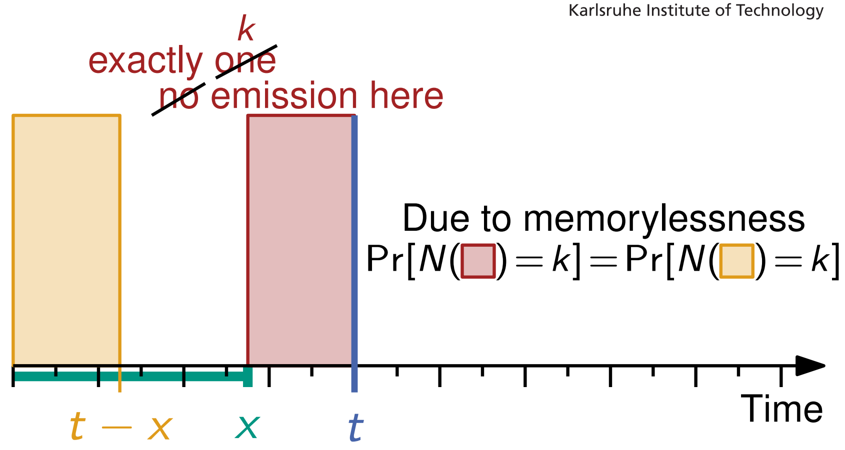

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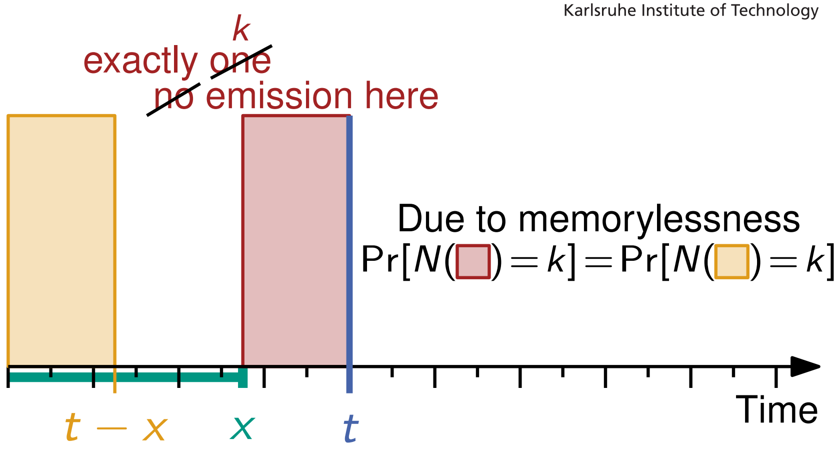

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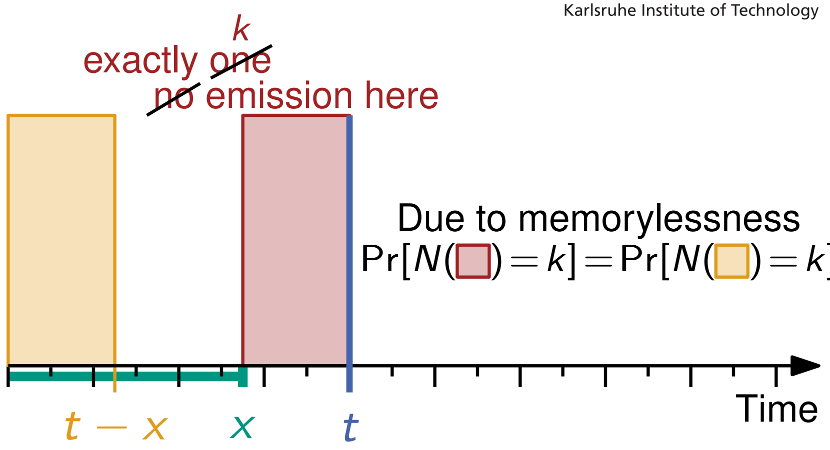

Due to memorylessness
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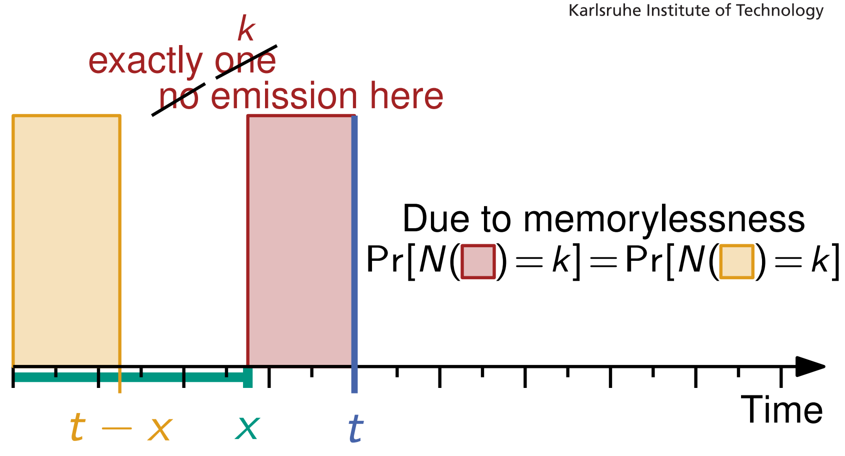

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Definition: A **Poisson process** with *intensity* λ is a collection of random variables $X_1, X_2, \ldots \in \mathbb{R}$ such that, if $N(a, b) = |\{i \mid X_i \in [a, b]\}|$, then

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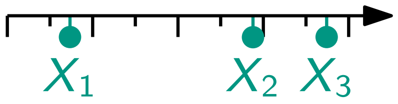

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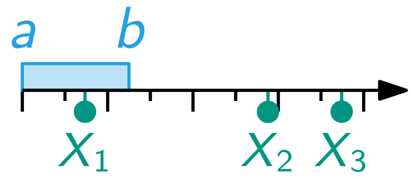

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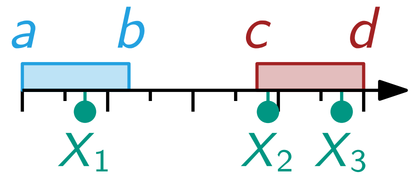

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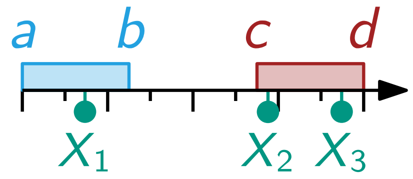

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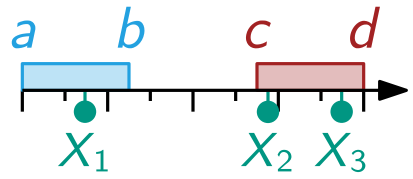

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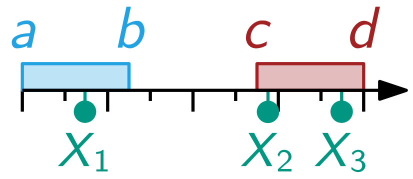

- Simple case: $N(0, b) = 1$, where is X_1?

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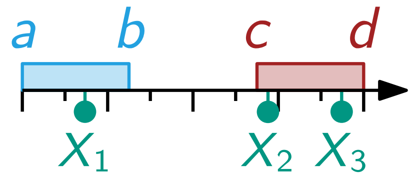

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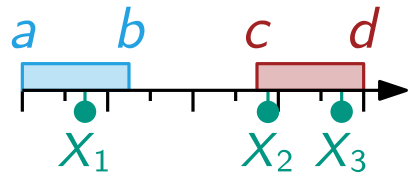

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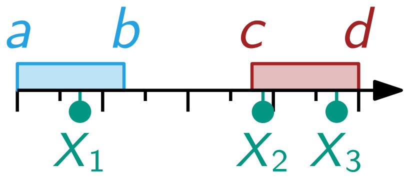

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independence of disjoint intervals

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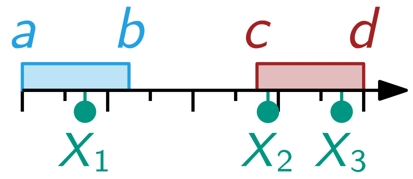

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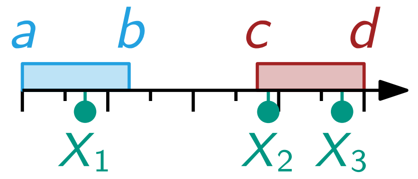

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independence of disjoint intervals

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Definition: A **Poisson process** with *intensity* λ is a collection of random variables $X_1, X_2, \ldots \in \mathbb{R}$ such that, if $N(a,b) = |\{i \mid X_i \in [a,b]\}|$, then

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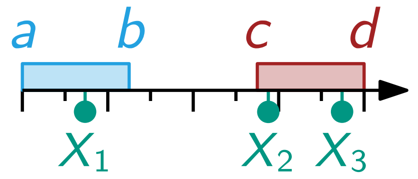

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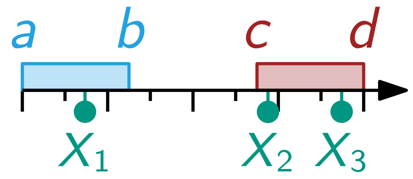

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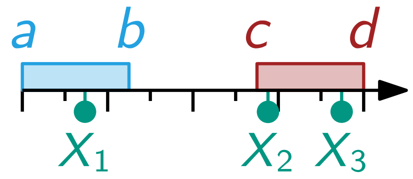

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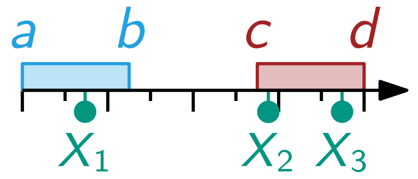

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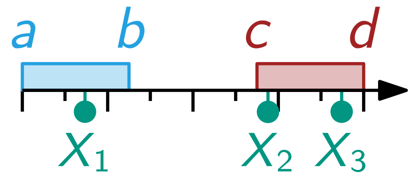

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- In general: the positions of the points are distributed uniformly in an interval

Continuous Spaces: Joint Distributions

Definition: For two random variables X, Y the **joint cumulative distribution function** is

$$F_{X,Y}(a, b) = \Pr[X \leq a \wedge Y \leq b].$$

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Karlsruhe Institute of Technology

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Definition: The **conditional density** of X with respect to an event A is

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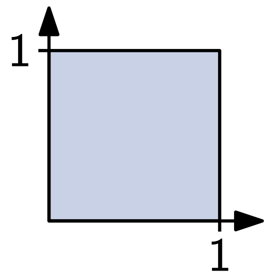
Example: $\mathcal{U}([0,1]^2)$

Uniform Distribution on the Unit Square

- We want to draw a point P uniformly at random from $[0,1]^2$



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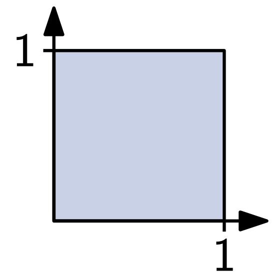

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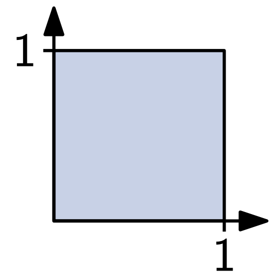

Marginal Distributions

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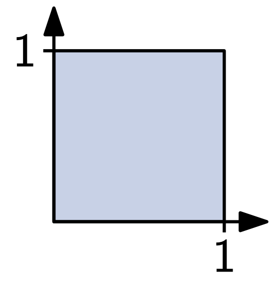

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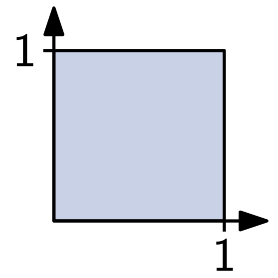

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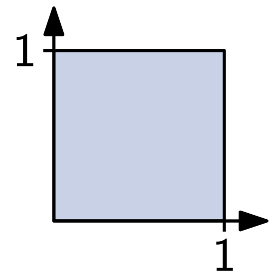

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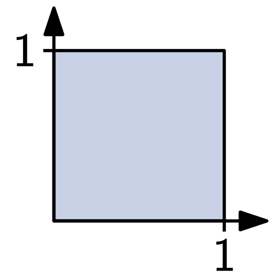

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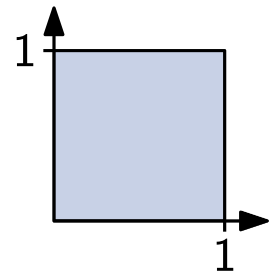

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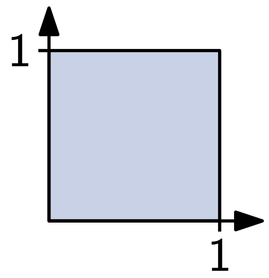

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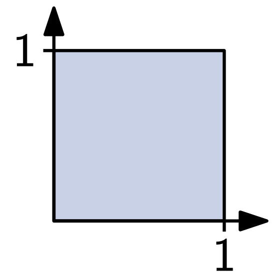

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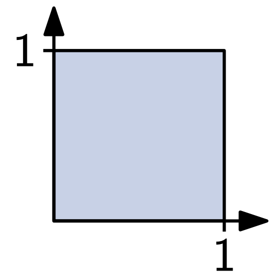

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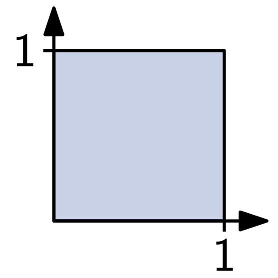

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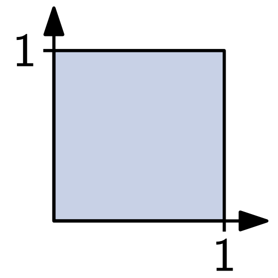
Example: $\mathcal{U}([0,1]^2)$

Uniform Distribution on the Unit Square



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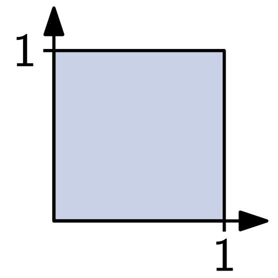

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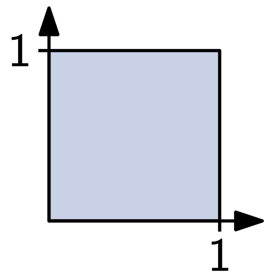

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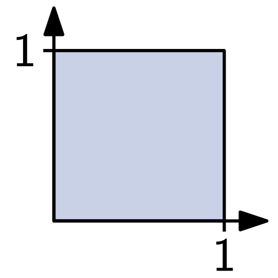

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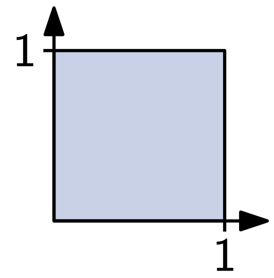

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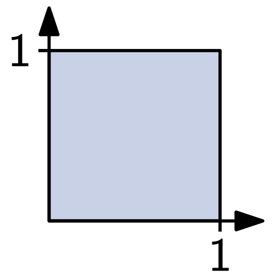

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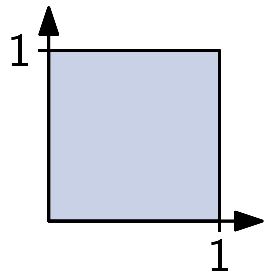

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- Sample $P = (X,Y) \sim \mathcal{U}([0,1]^2)$ by independently sampling $X, Y \sim \mathcal{U}([0,1])$!

Application: Random Geometric Graphs

Motivation

- Average-case analysis: analyze models that represent the real world
- So far: Erdős-Rényi random graphs (connect two vertices independently with equal prob)

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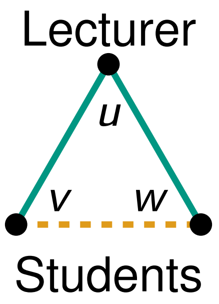

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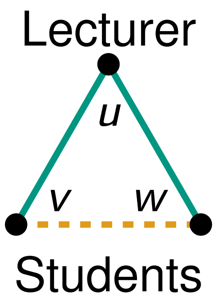

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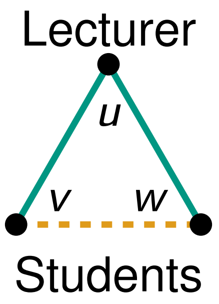

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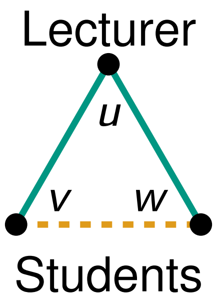

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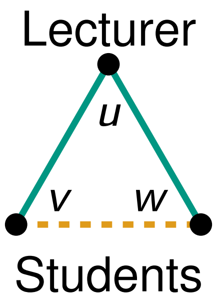

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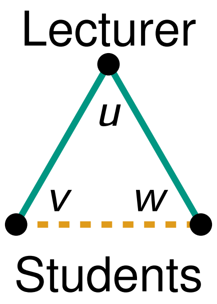

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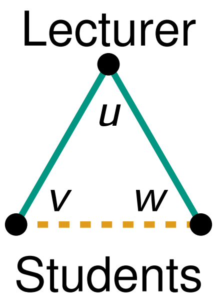

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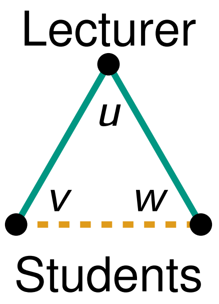

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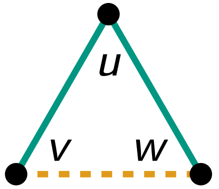

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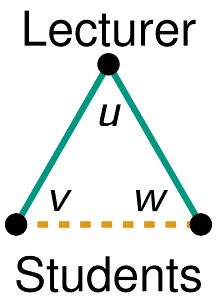

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Simple & Realistic!

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Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

Application: Simple Random Geometric Graphs

- Number: n vertices

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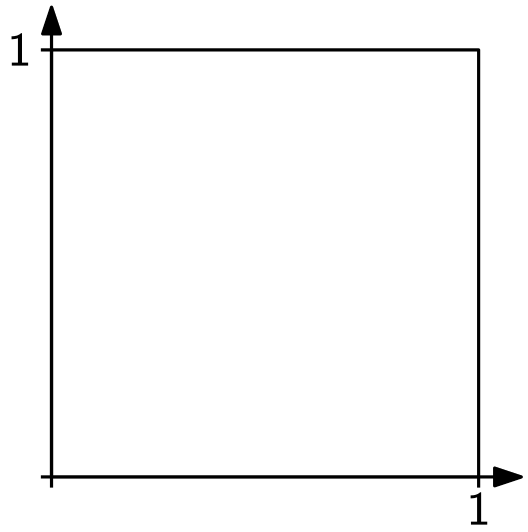

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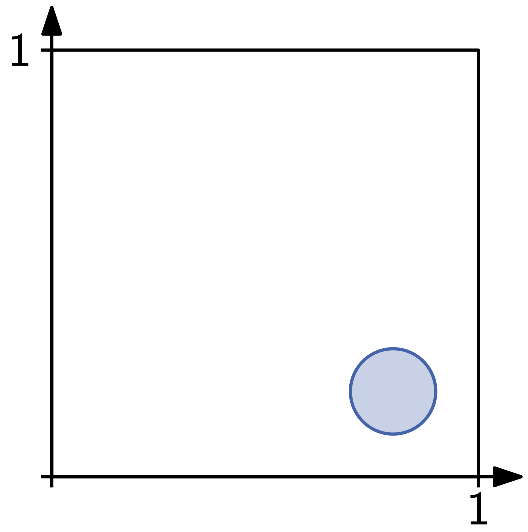

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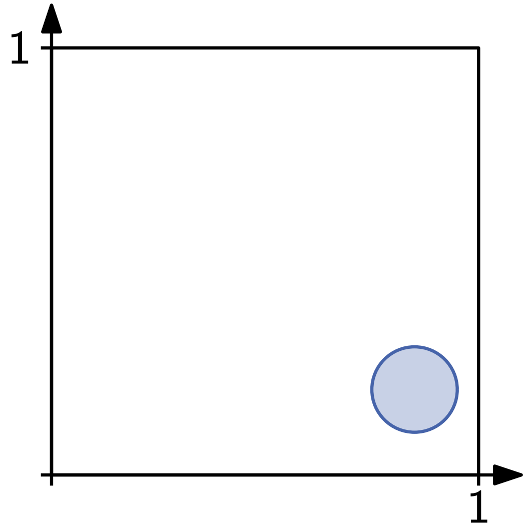

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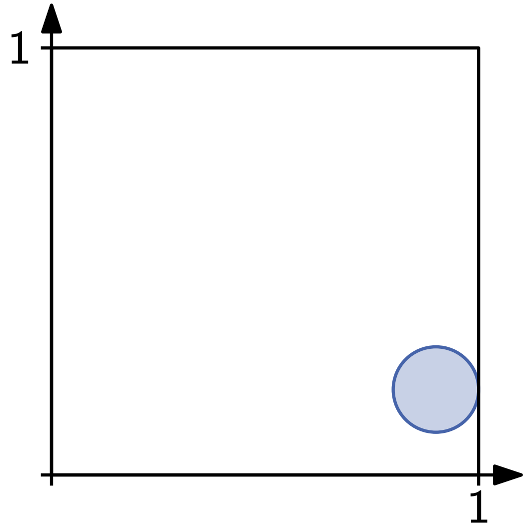
Connection probability depends on distance



Application: Simple Random Geometric Graphs

- Number: n vertices
- Space: 2-dimensional torus $\mathbb{T}^2$ (unit square with opposite sides identified)

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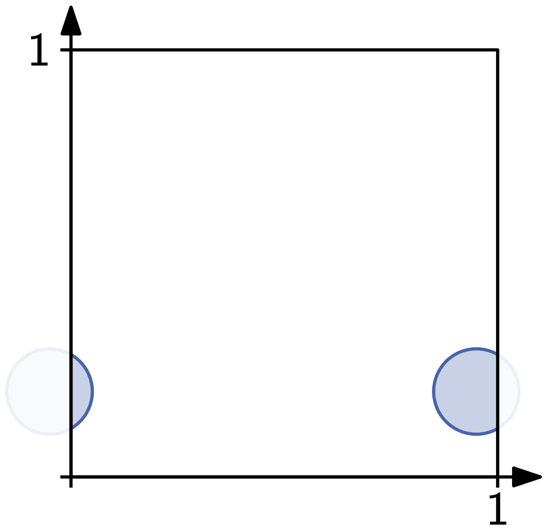

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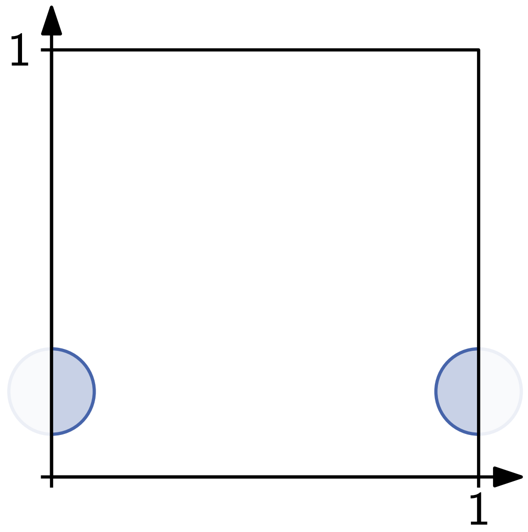

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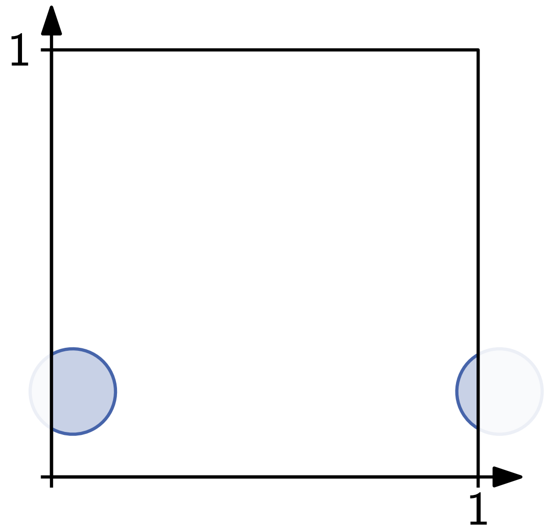

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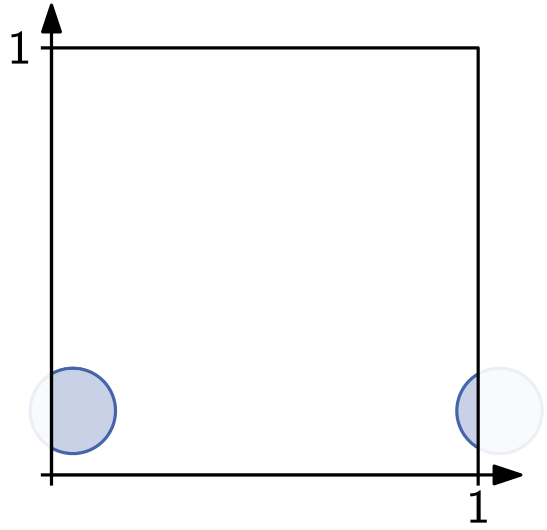

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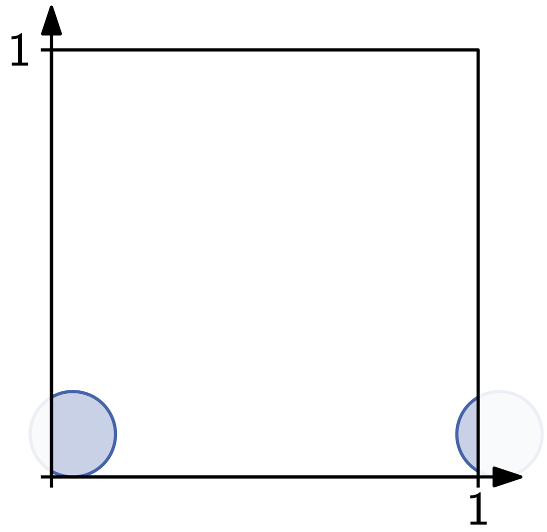

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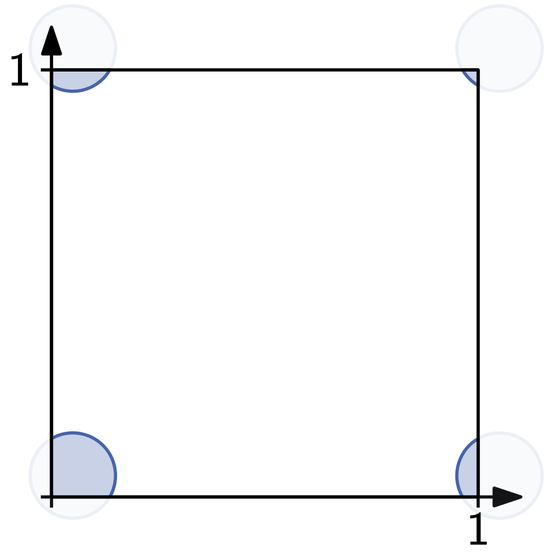

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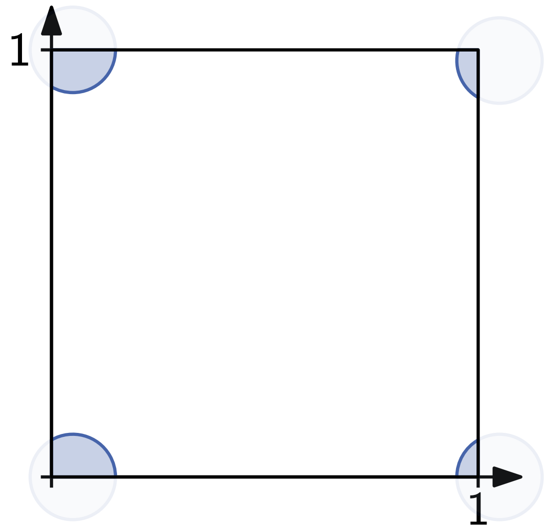

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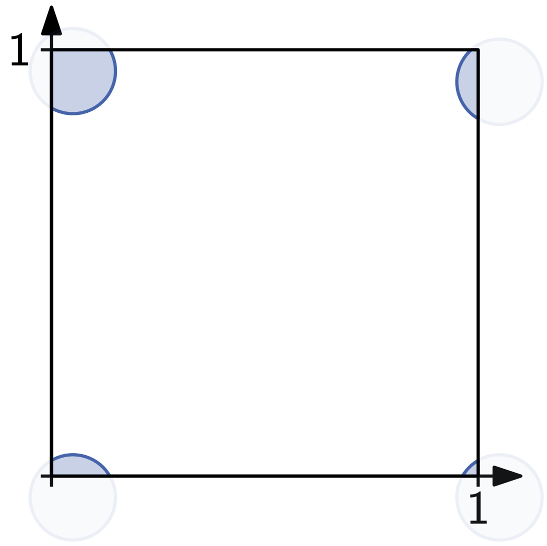

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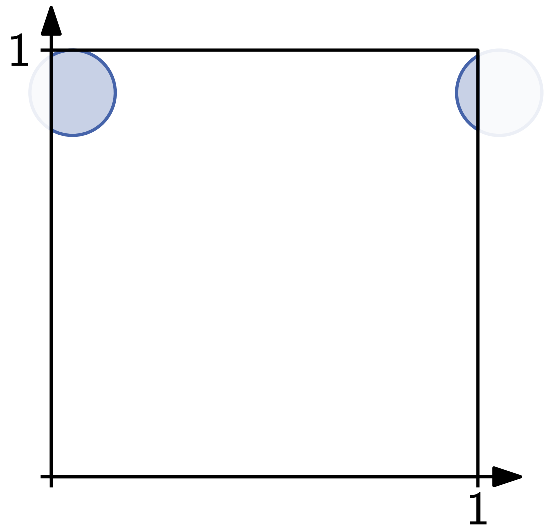

Application: Simple Random Geometric Graphs

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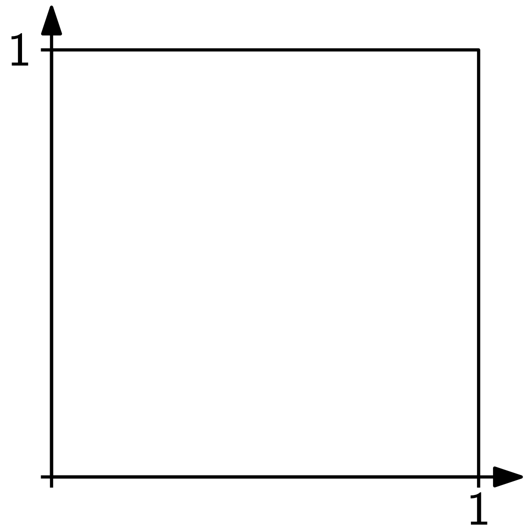

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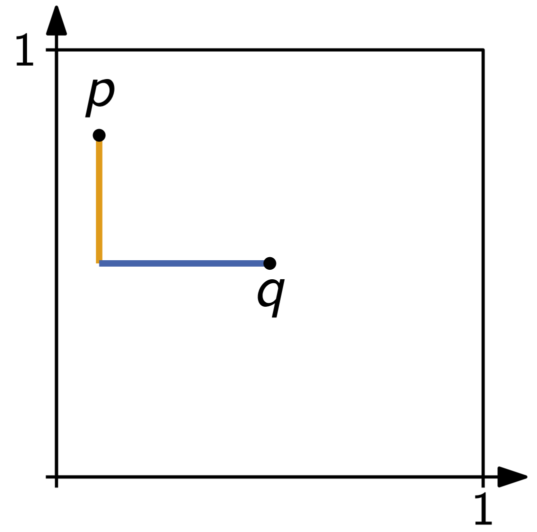

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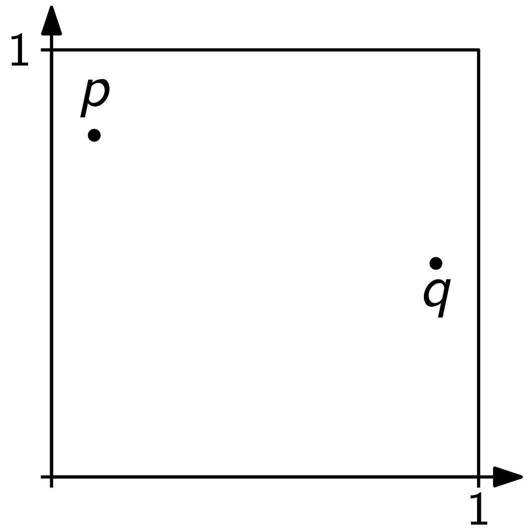

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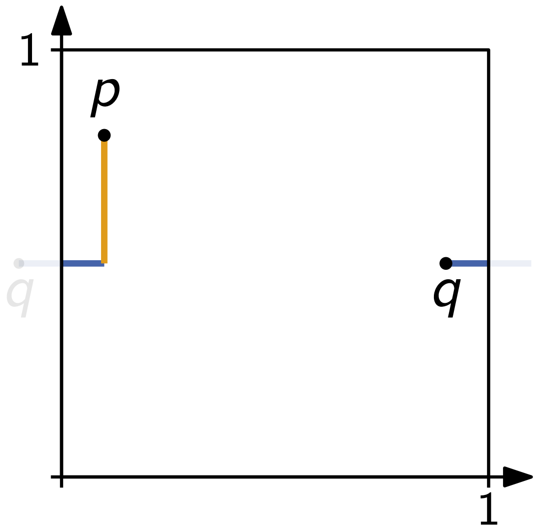

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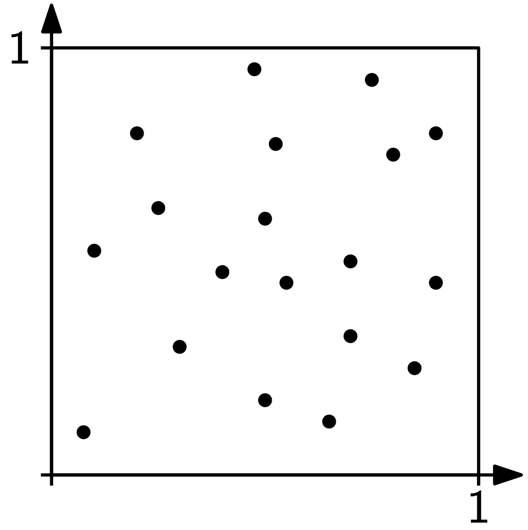

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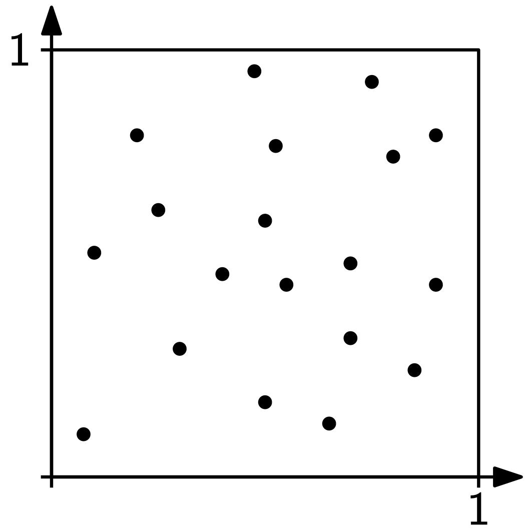
Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance



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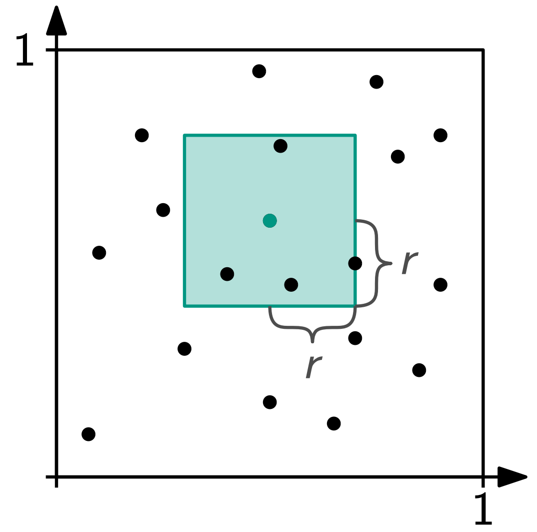

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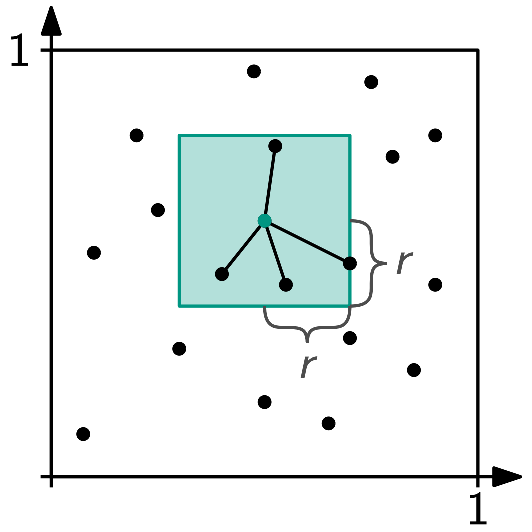

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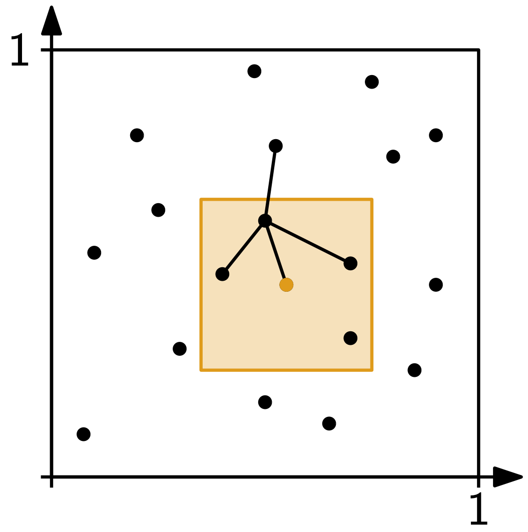

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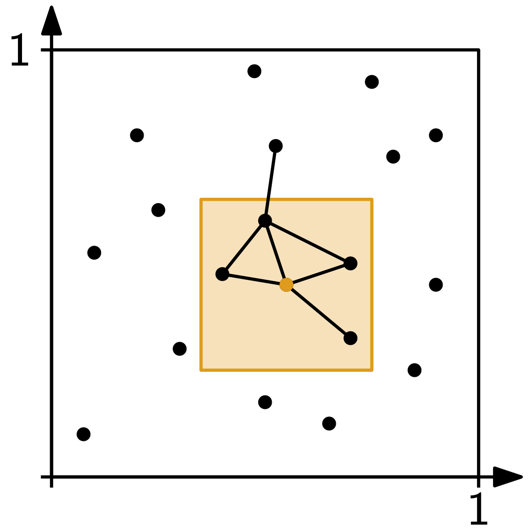

Application: Simple Random Geometric Graphs

Karlsruhe Institute of Technology

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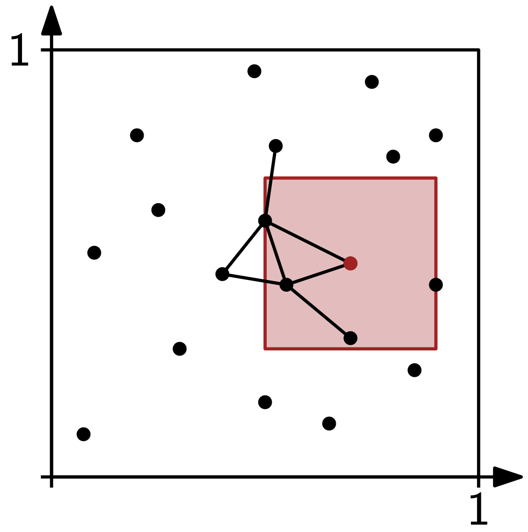

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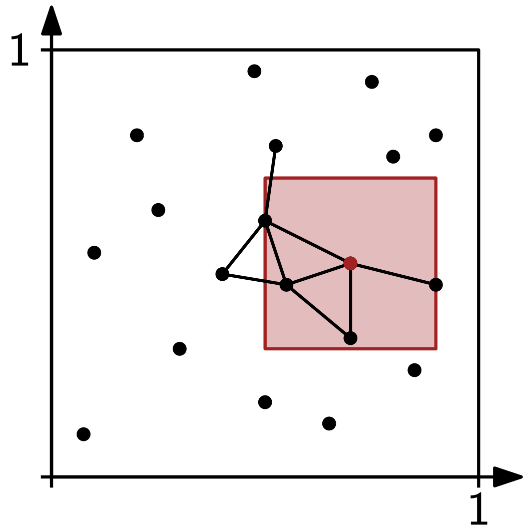

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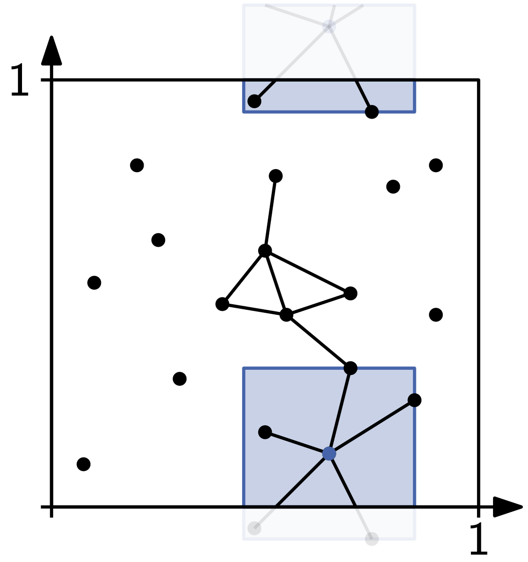

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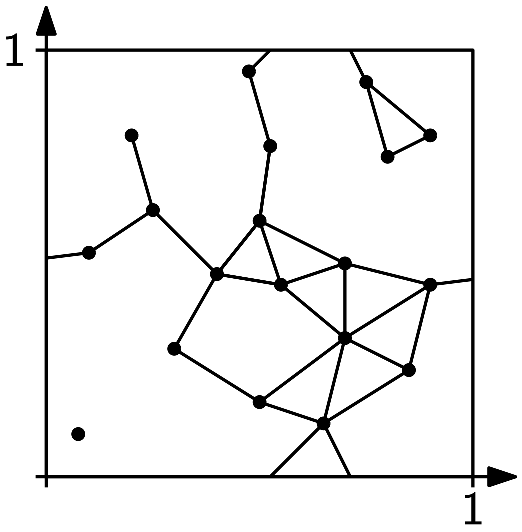

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threshold parameter

Expected Degree of v

Random Geometric Graph
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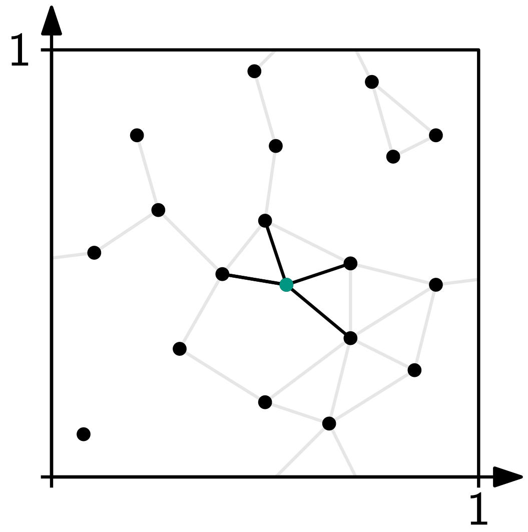

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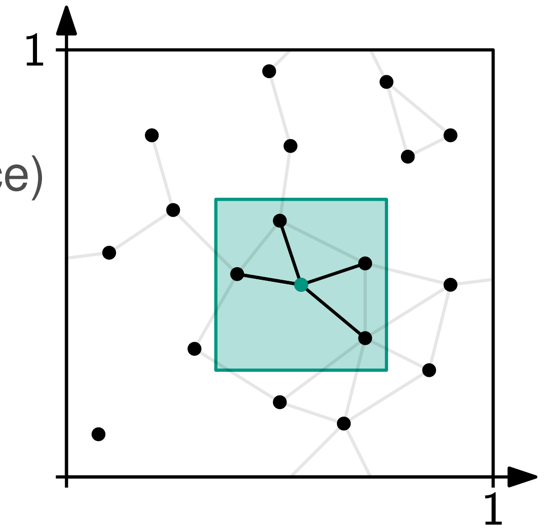
threshold parameter

Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

Expected Degree of v

- Neighbors of v are in $\boxed{N(v)}$ (here $N(v)$ denotes the *region* in the ground space)



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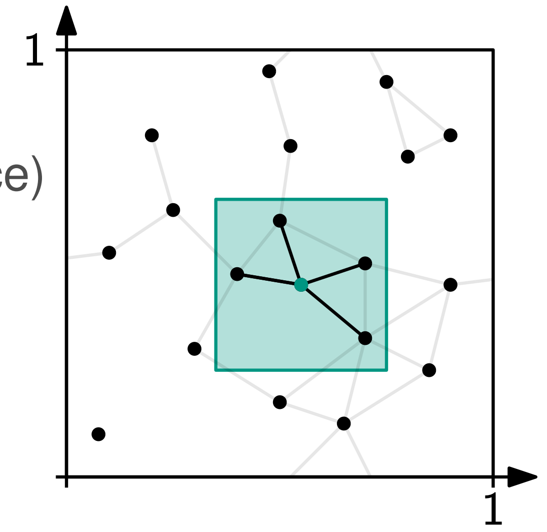
Expected Degree of *v*

- Neighbors of v are in $\boxed{N(v)}$ (here $N(v)$ denotes the *region* in the ground space)
- $\mathbb{E}[\deg(v)] = \mathbb{E}[\sum_{u \in V \setminus \{v\}} \mathbb{1}_{\{P_u \in N(v)\}}] = \sum_{u \in V \setminus \{v\}} \Pr[d(P_u, P_v) \le r]$



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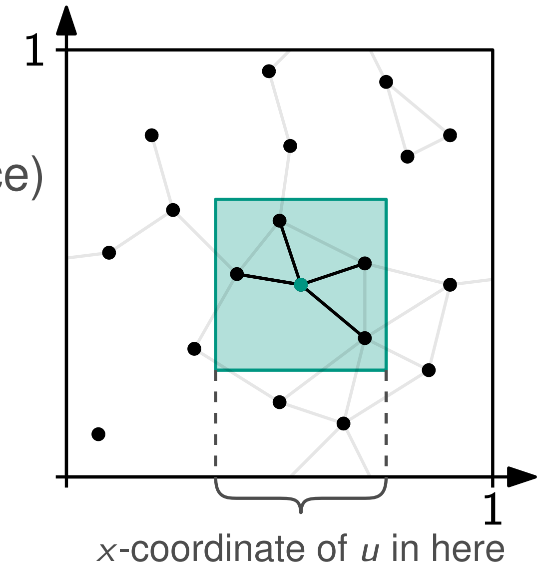

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Random Geometric Graph
Nodes distributed in metric space
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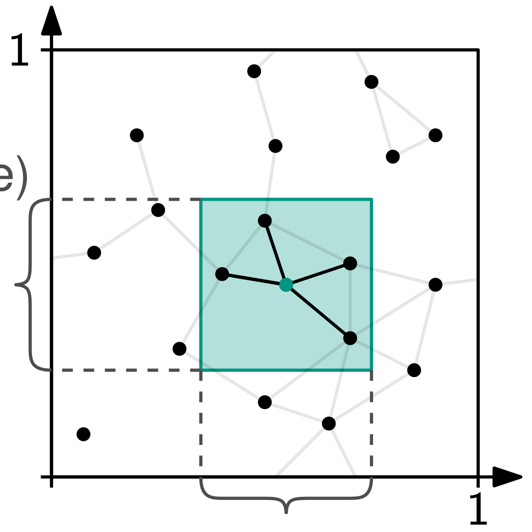

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Random Geometric Graph
Nodes distributed in metric space
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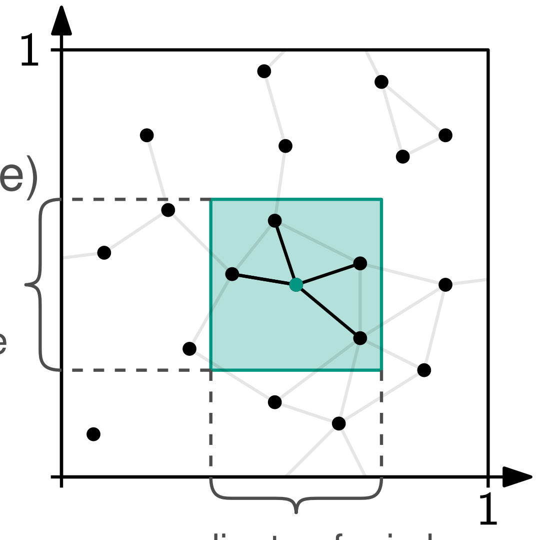

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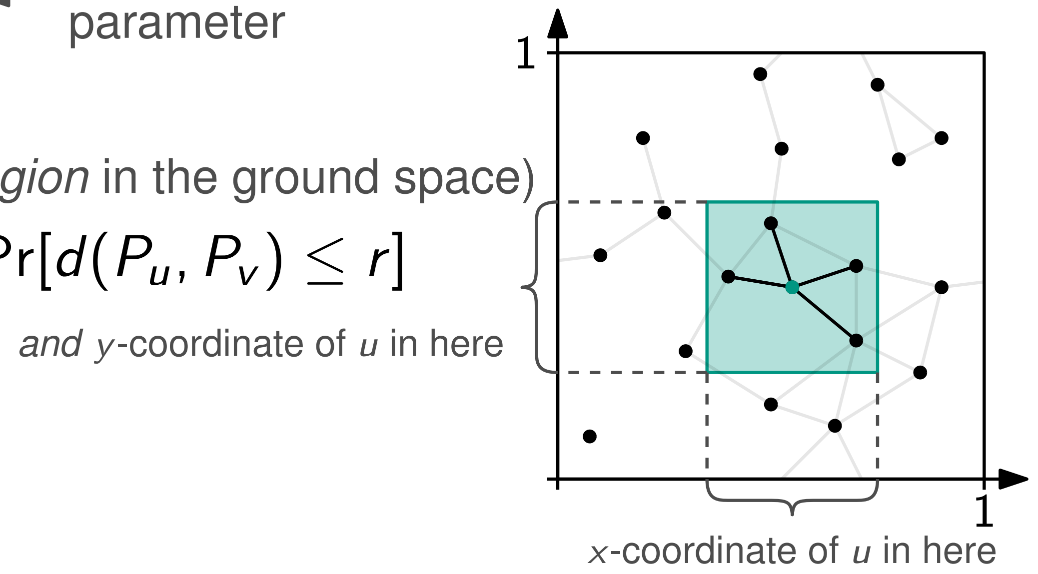

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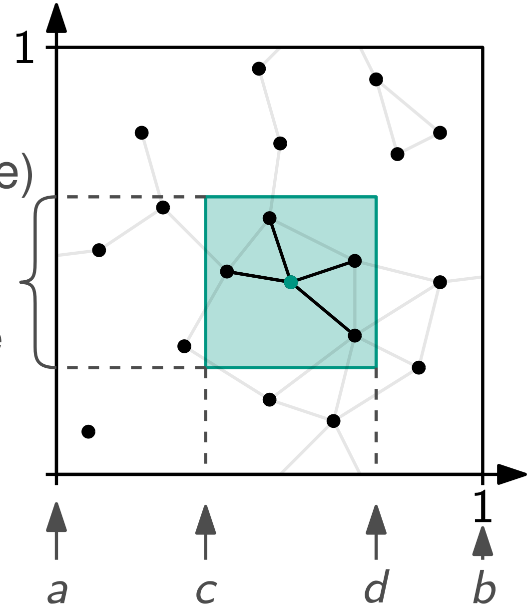
and y-coordinate of u in here



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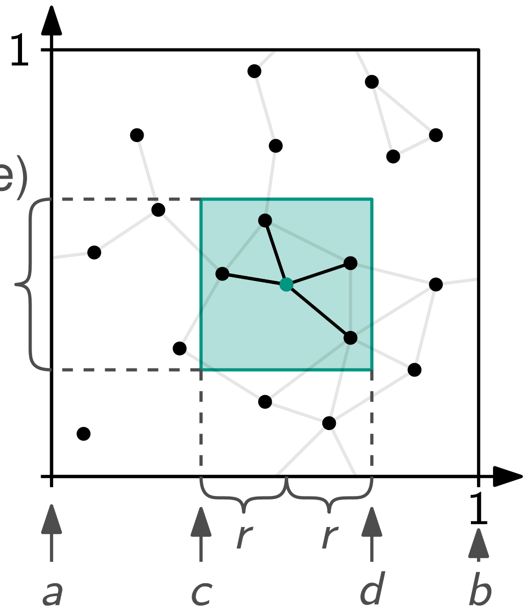

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$$\downarrow$$

$$= \sum_{u \in V \setminus \{v\}} \frac{2r}{1-0} \cdot \frac{2r}{1-0}$$

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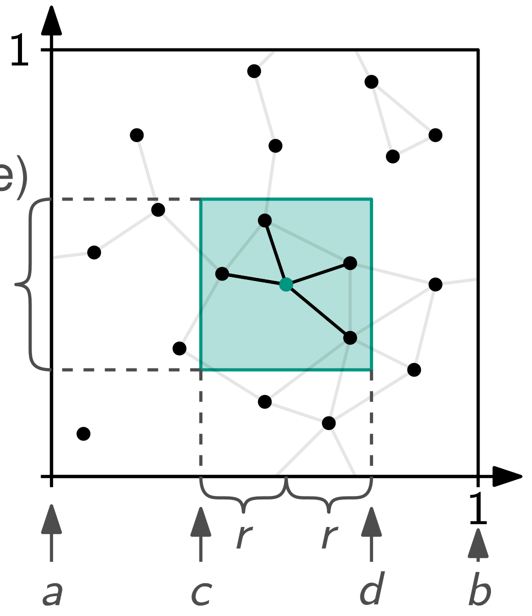

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Karlsruhe Institute of Technology

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$$= \sum_{u\in V\setminus\{v\}} \frac{2r}{1-0} \cdot \frac{2r}{1-0}$$
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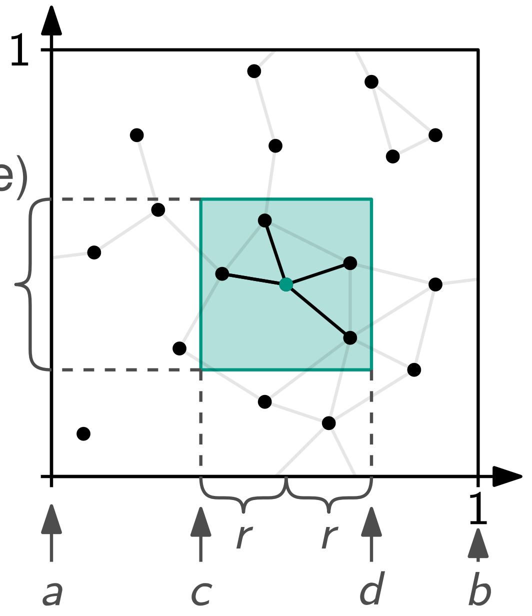

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← threshold parameter

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(area of the region $N(v)$)

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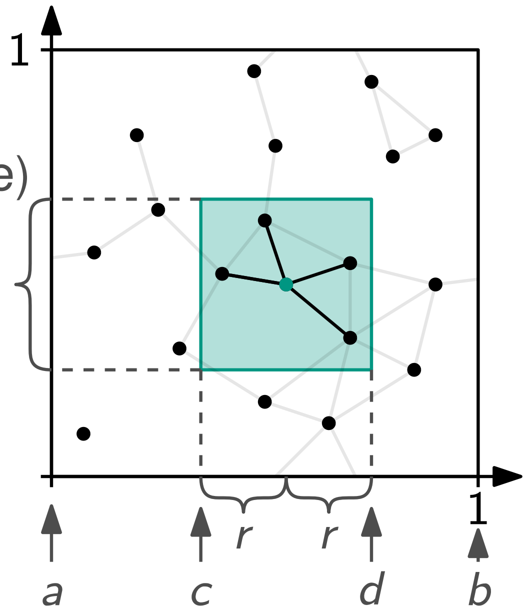

Simple Random Geometric Graphs – Locality

Locality

- Two vertices v and w are likelier to connect if they have a common neighbor u

Simple Random Geometric Graphs – Locality

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Realistic assumption: $r = \Theta(n^{-1/2})$ such that $\mathbb{E}[\deg(v)] = \Theta(1)$

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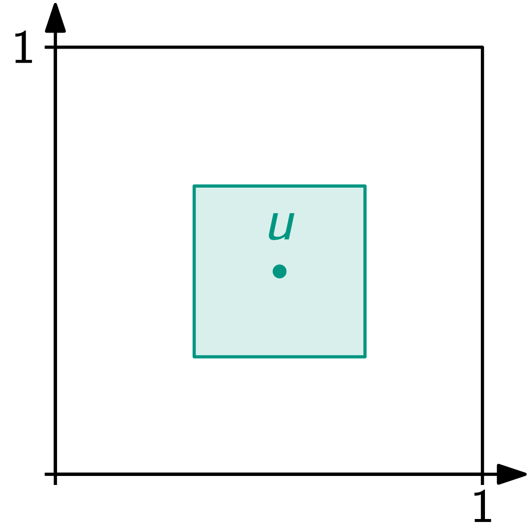

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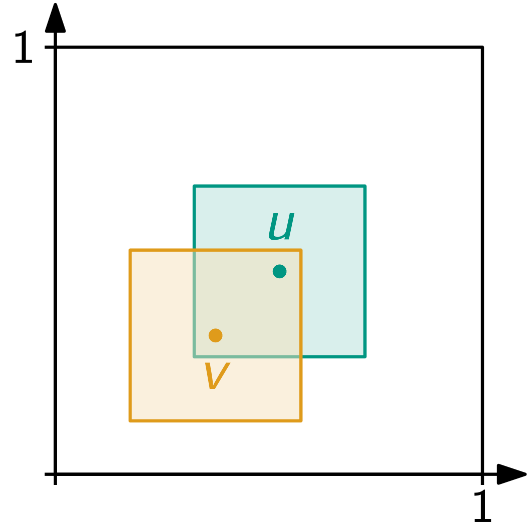

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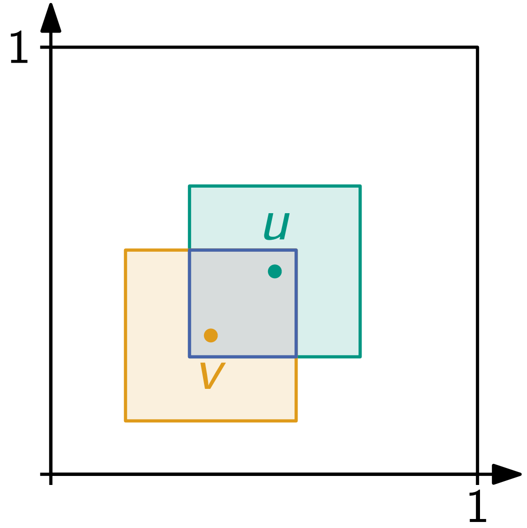

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

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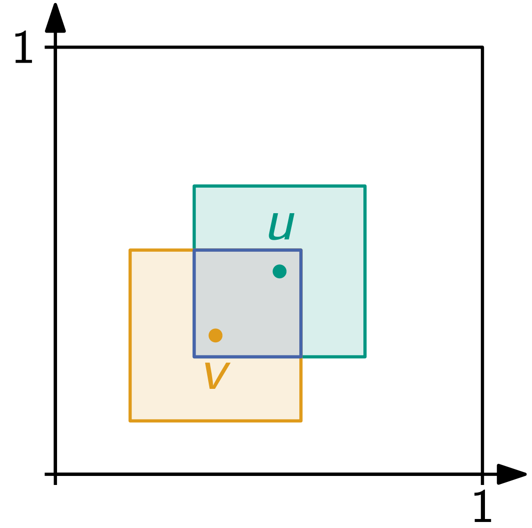
Numerator $\Pr[w \in N(v) \wedge v \in N(u) \wedge w \in N(u)]$



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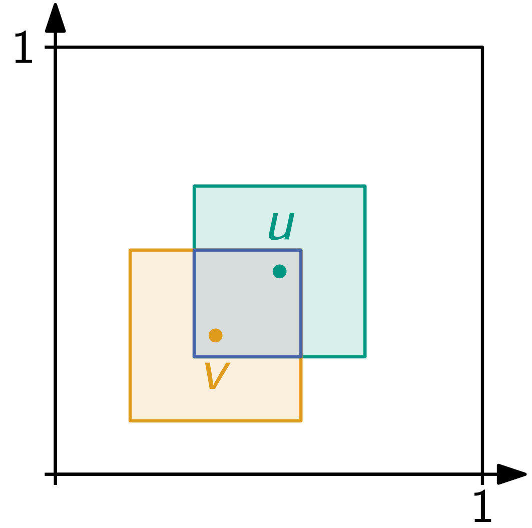

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Law of Total Probability
$\Pr[A] = \int_{-\infty}^{\infty} \Pr[A \mid X = x] f_X(x) \mathrm{d}x$

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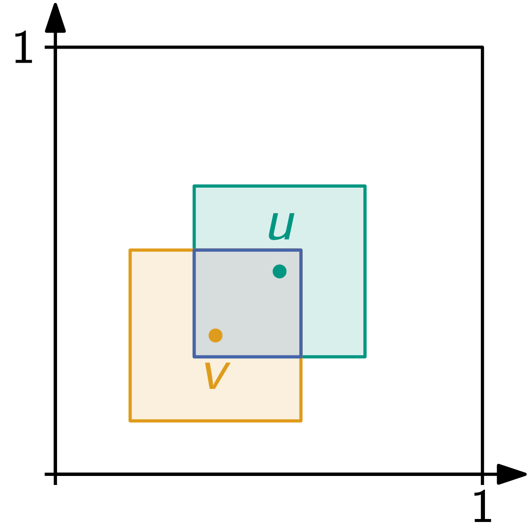

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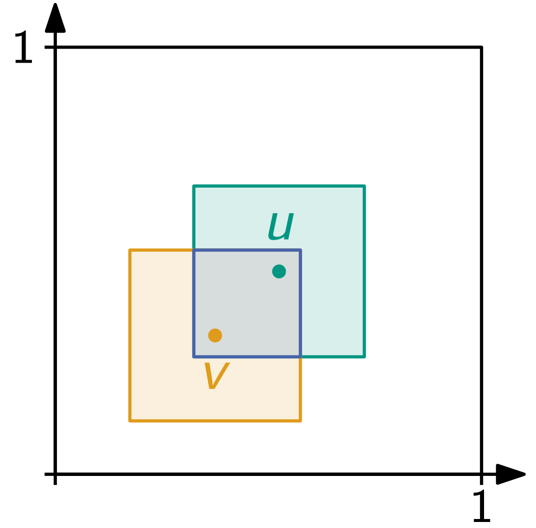

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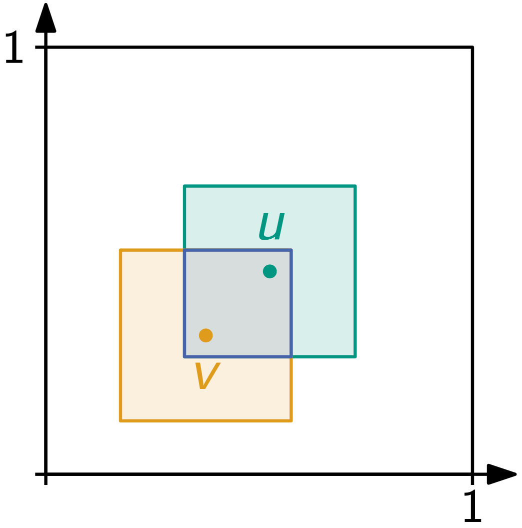

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Simple Random Geometric Graphs – Locality

Locality

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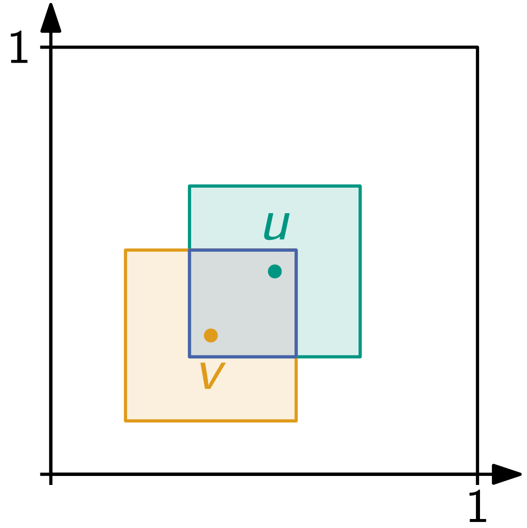

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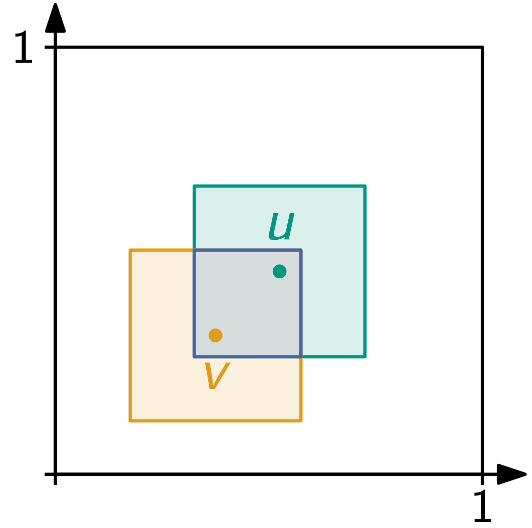

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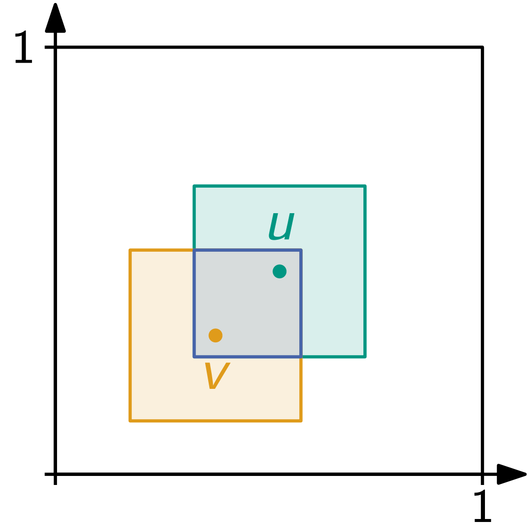

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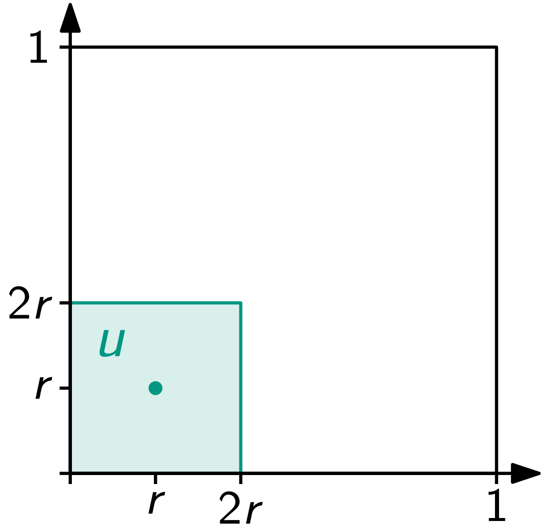

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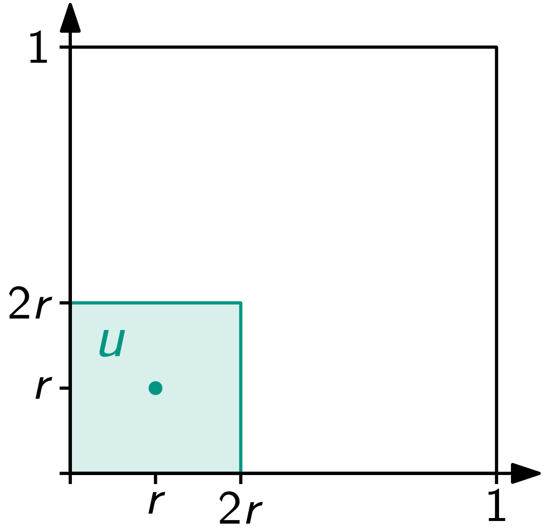

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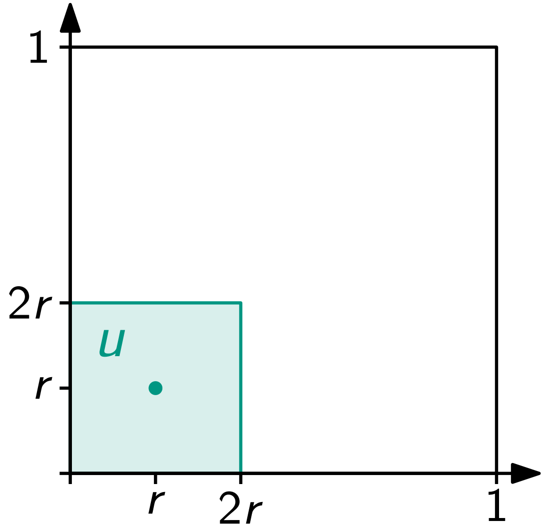

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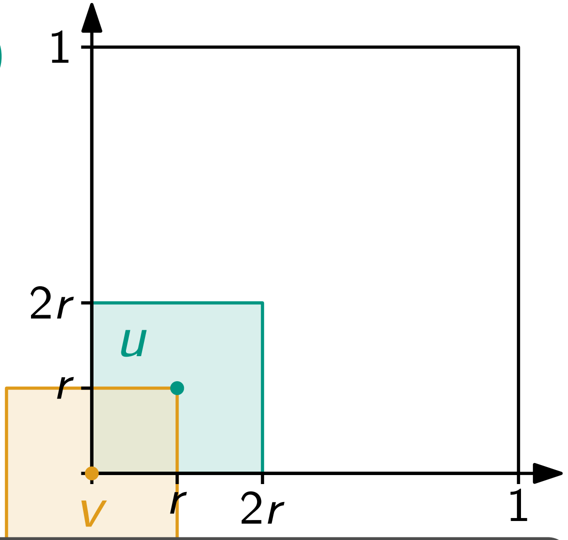

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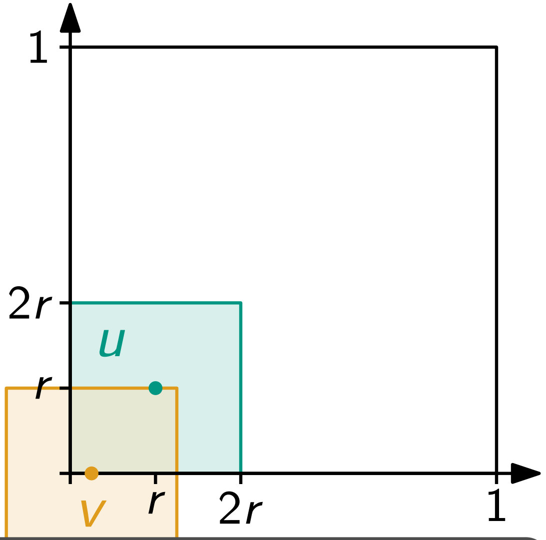

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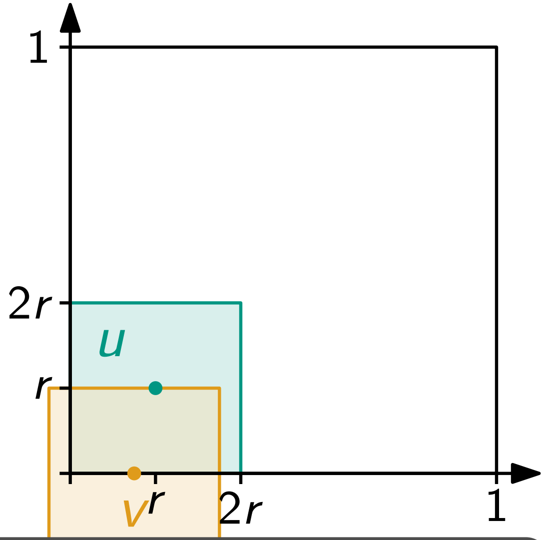

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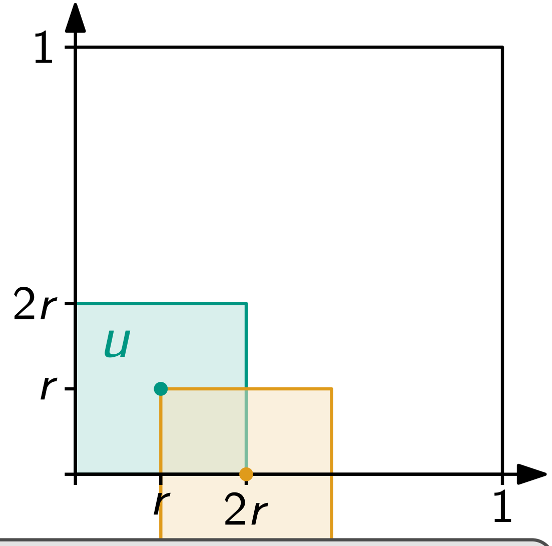

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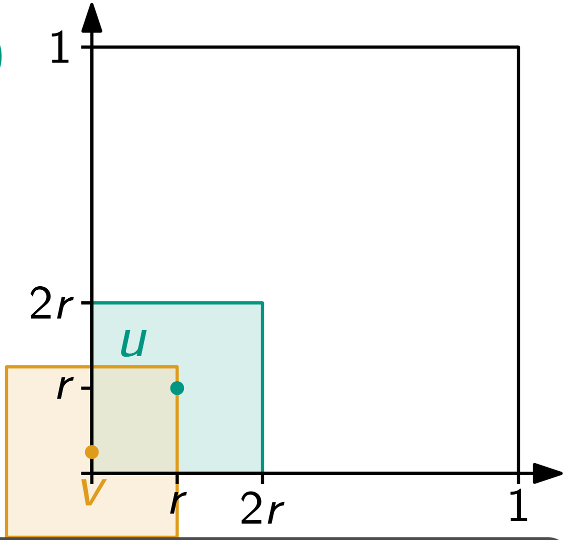

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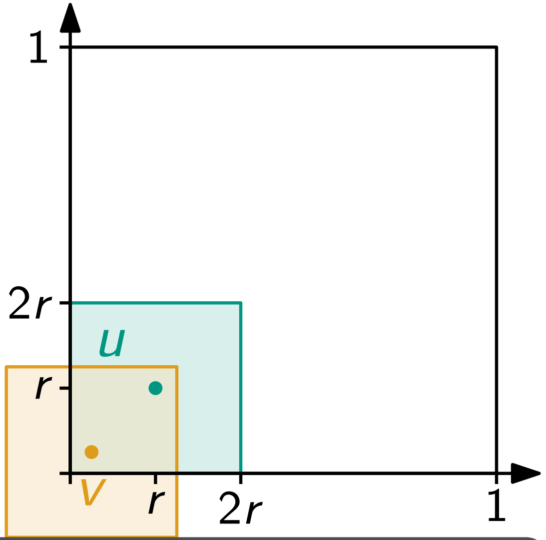

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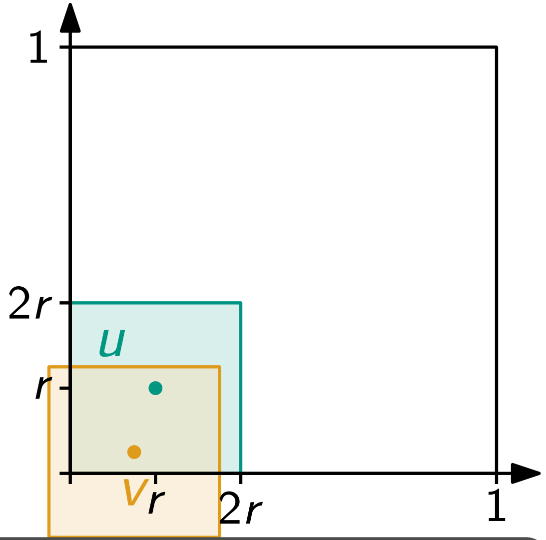

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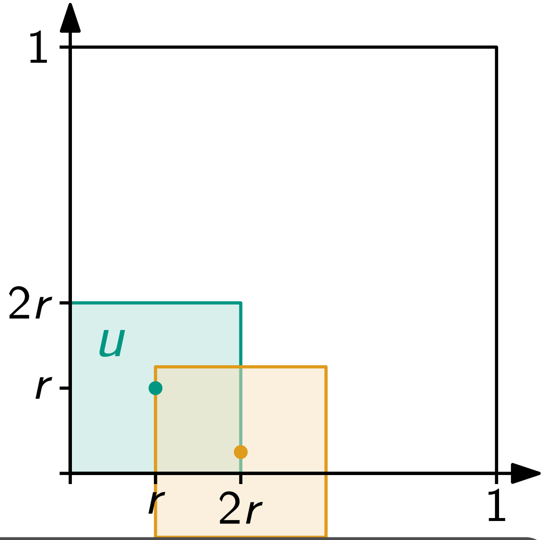

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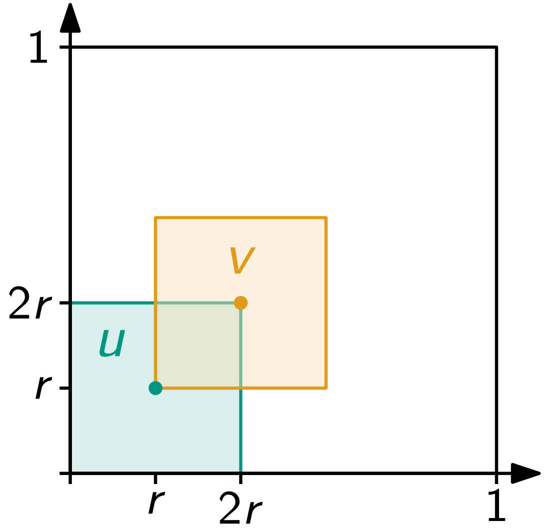

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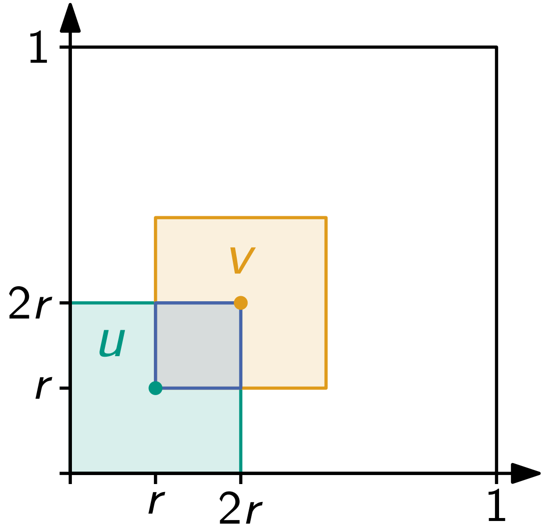

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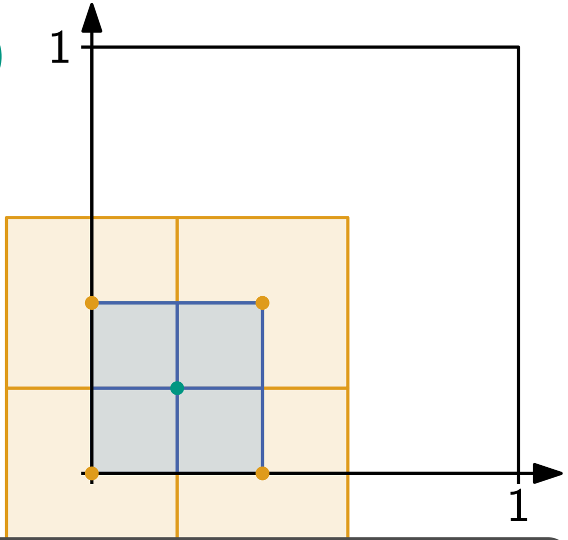

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Karlsruhe Institute of Technology

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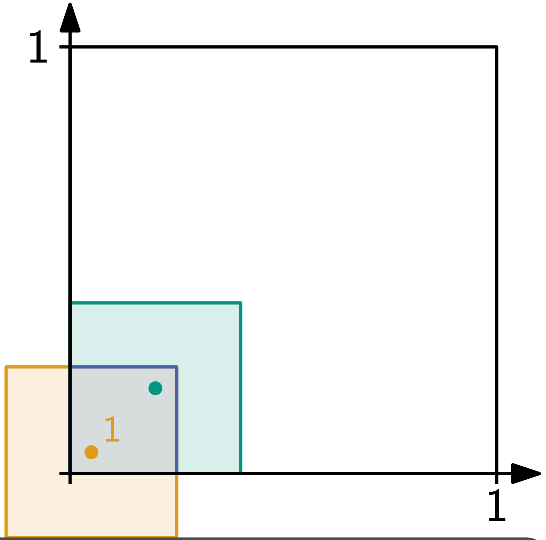

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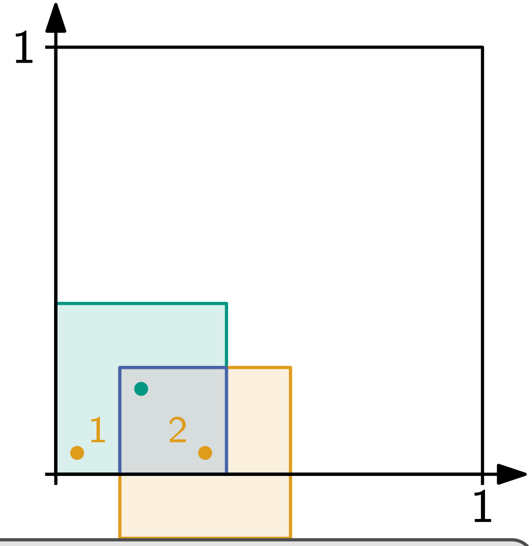

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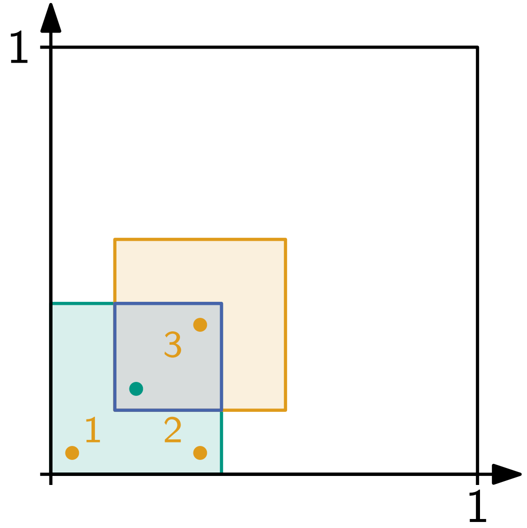

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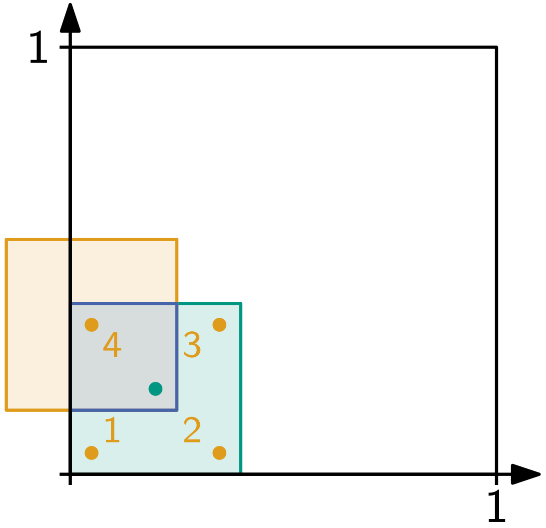

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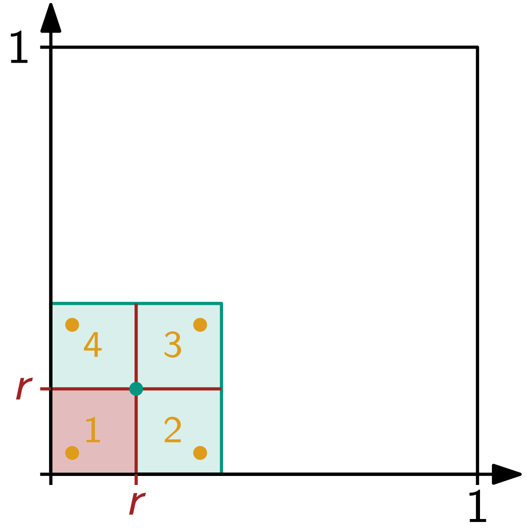

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$\Rightarrow$ Integrate only one quarter and multiply by 4

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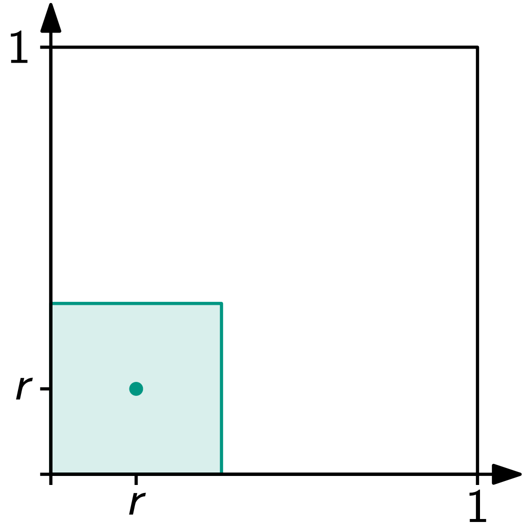

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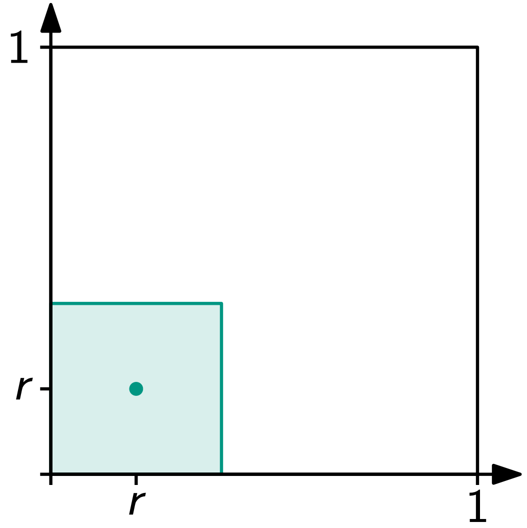
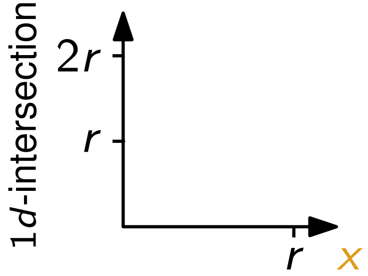


Consider size of intersection in *one* dimension depending on position of v



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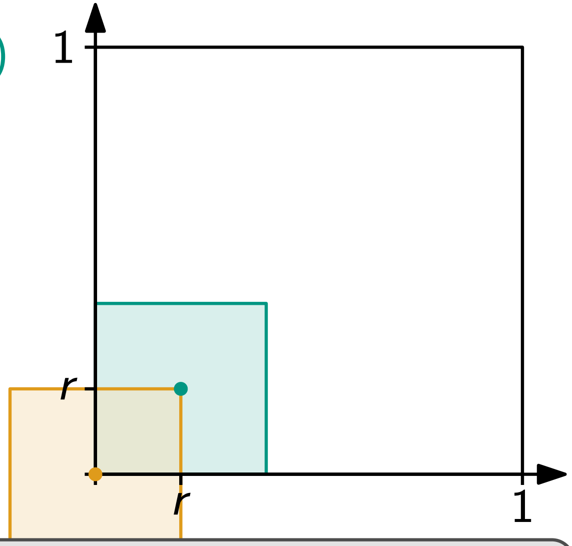

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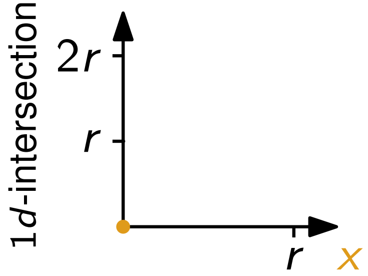

Law of Total Probability
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Simple Random Geometric Graphs – Locality

Locality

Realistic assumption: $r = \Theta(n^{-1/2})$ such that $\mathbb{E}[\deg(v)] = \Theta(1)$

Convention: $v = P_v$

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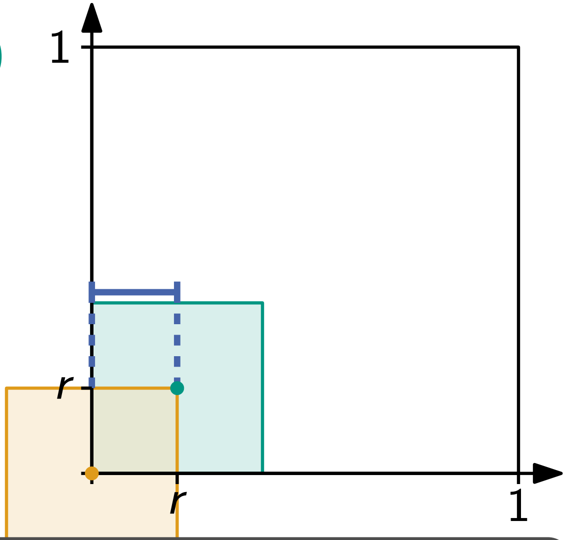

Numerator $\Pr[w \in N(v) \wedge v \in N(u) \wedge w \in N(u)]$

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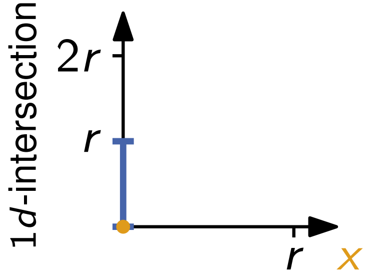

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Simple Random Geometric Graphs – Locality

Locality

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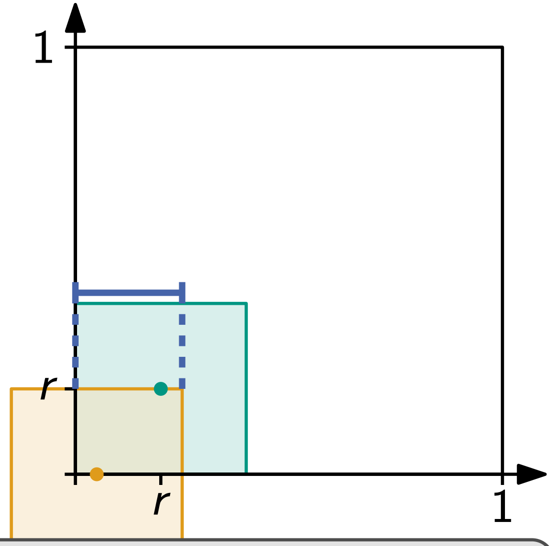
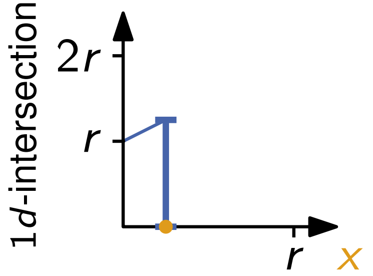


Consider size of intersection in *one* dimension depending on position of v



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Simple Random Geometric Graphs – Locality

Locality

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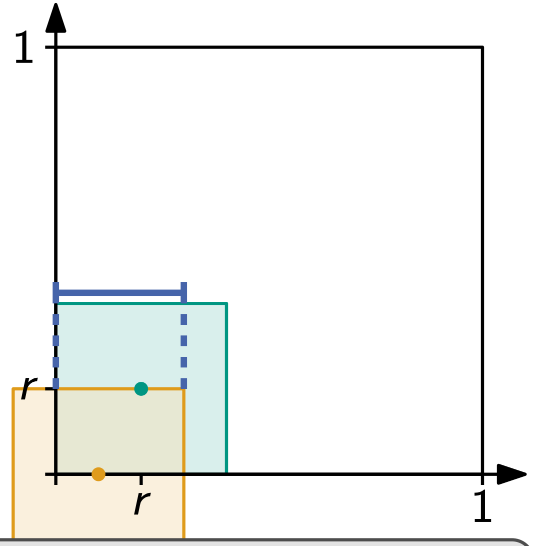

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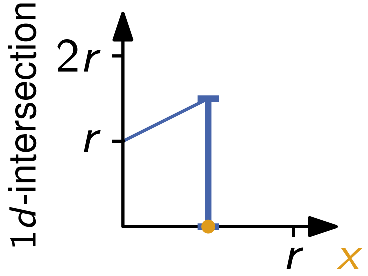

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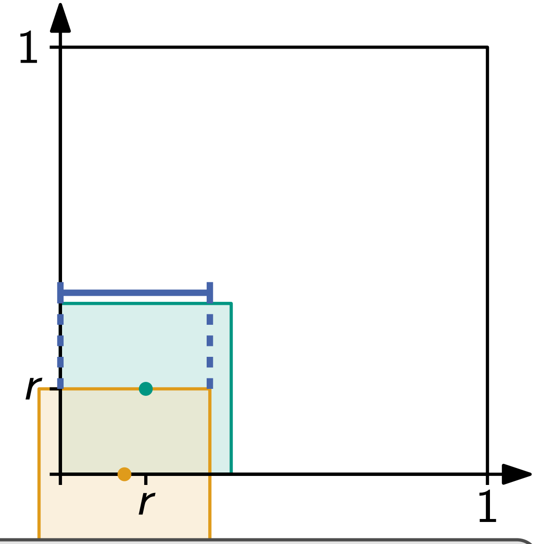

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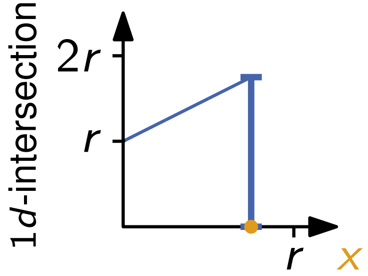

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Simple Random Geometric Graphs – Locality

Locality

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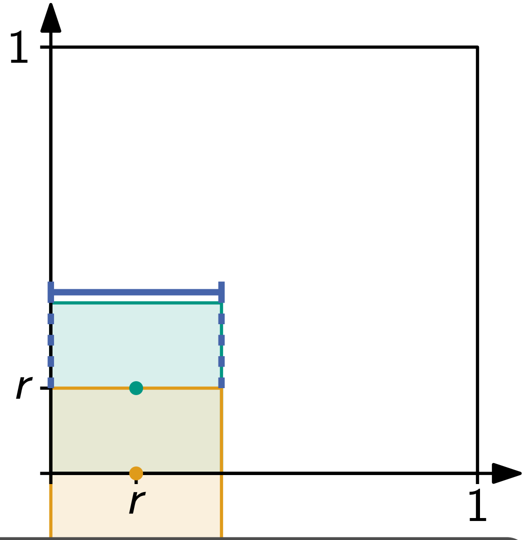
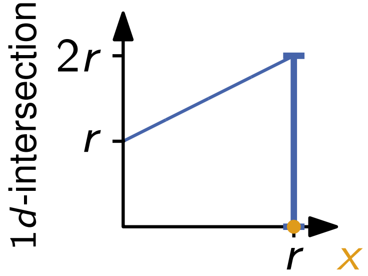


Consider size of intersection in *one* dimension depending on position of v



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Simple Random Geometric Graphs – Locality

Locality

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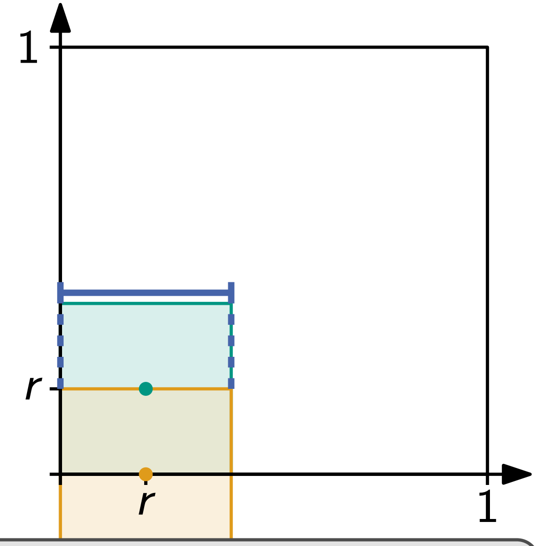
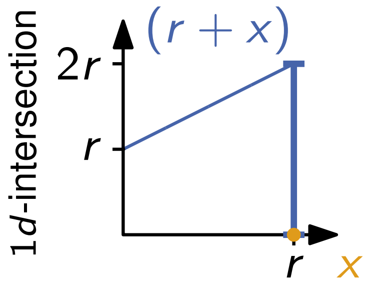


Consider size of intersection in *one* dimension depending on position of v



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Simple Random Geometric Graphs – Locality

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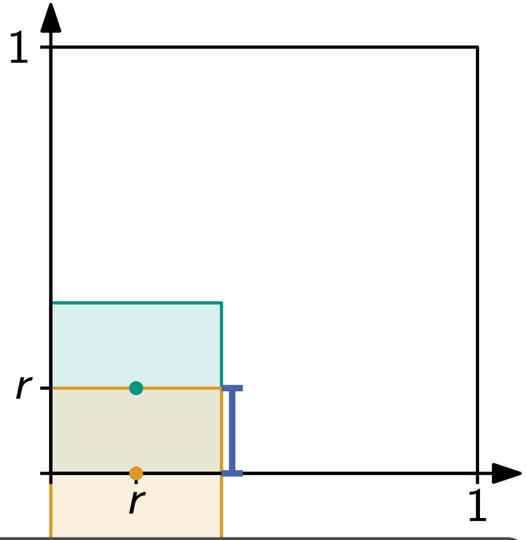
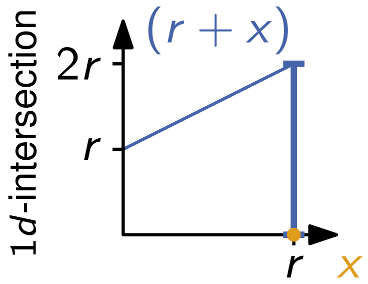
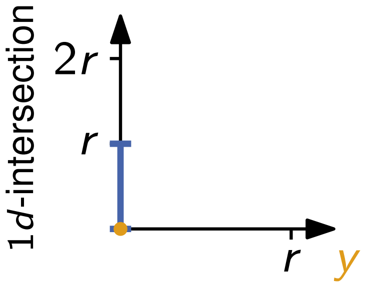


Consider size of intersection in *one* dimension depending on position of v





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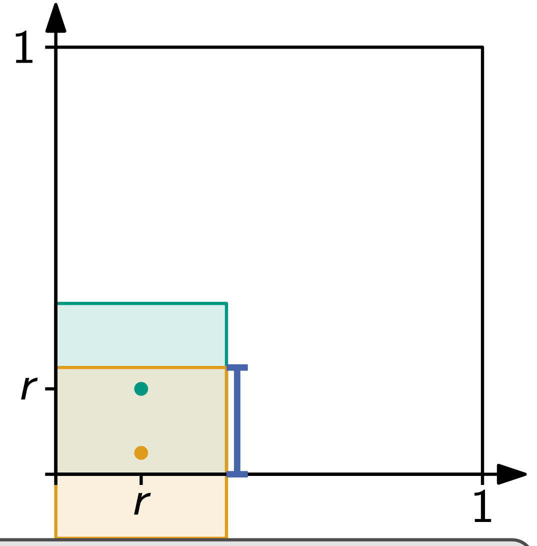
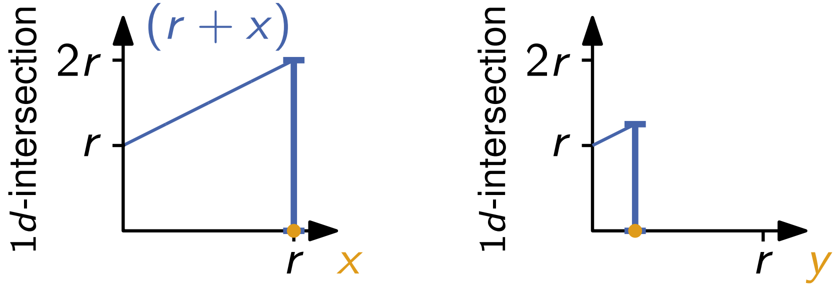


Consider size of intersection in *one* dimension depending on position of v



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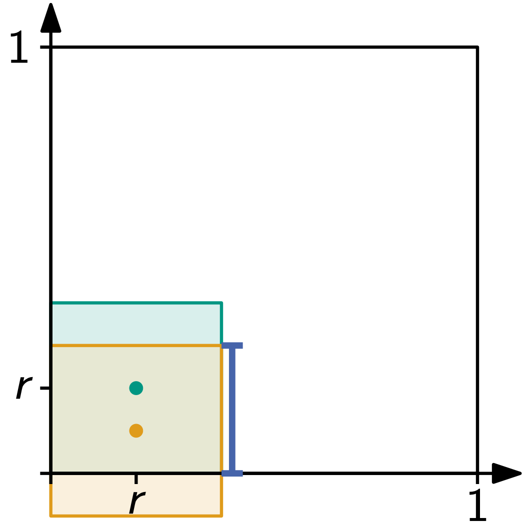
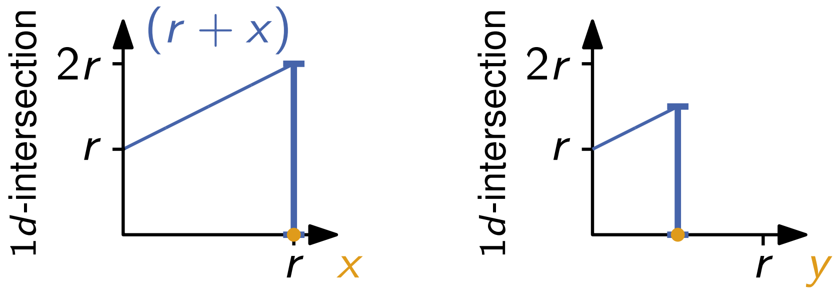


Consider size of intersection in *one* dimension depending on position of v



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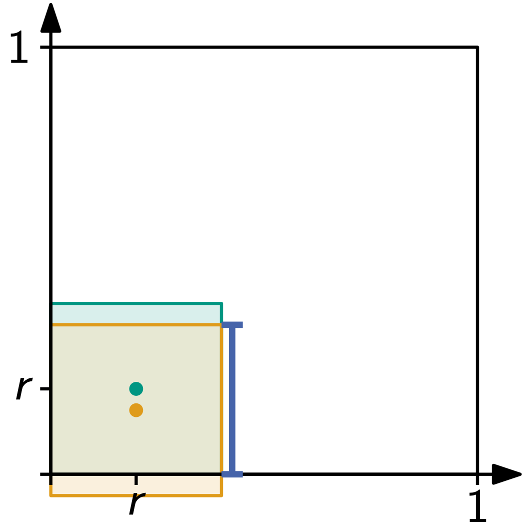

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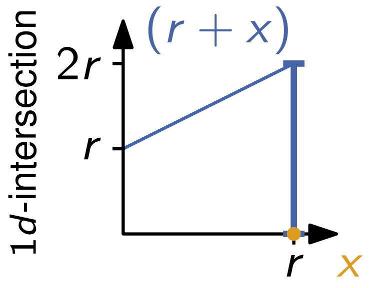

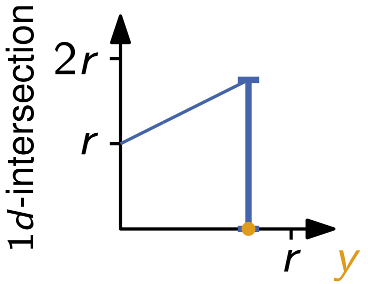

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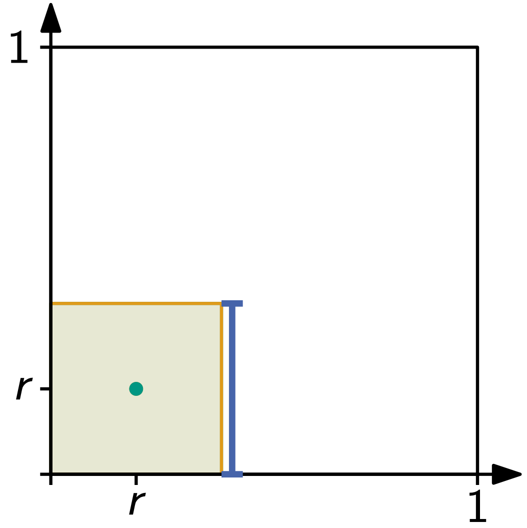

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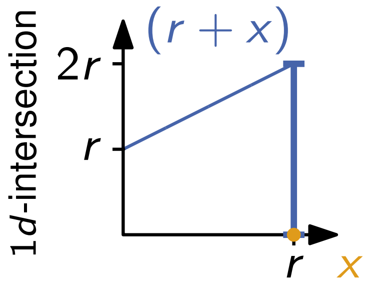

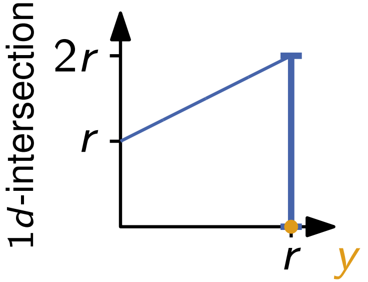

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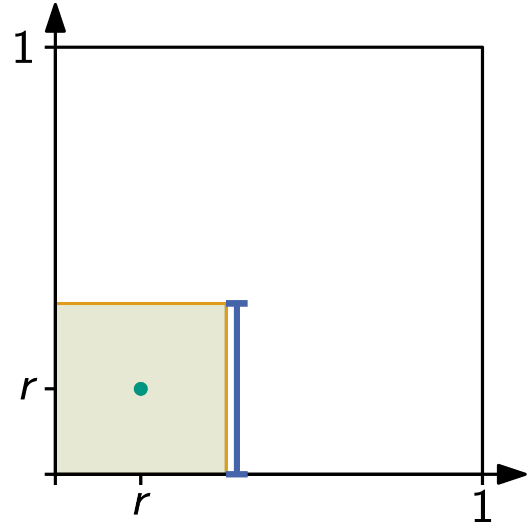

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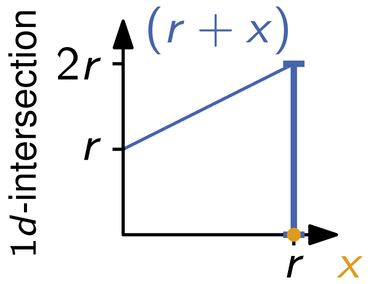

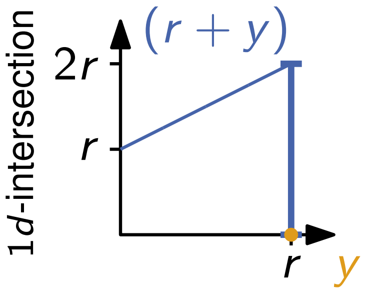

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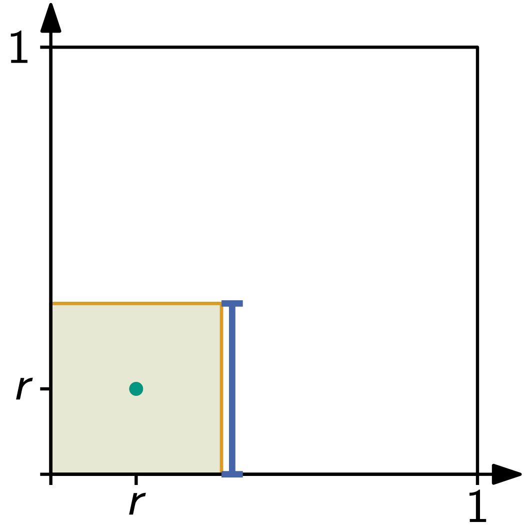

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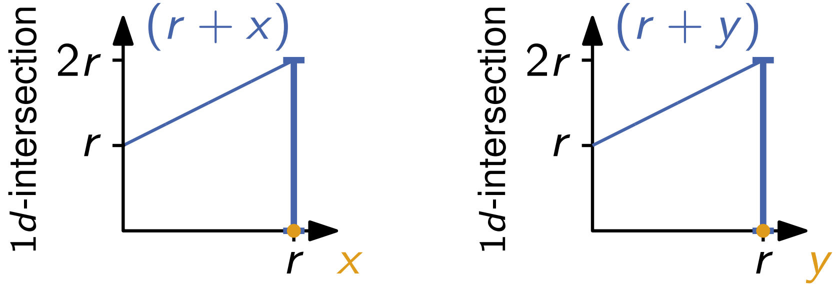

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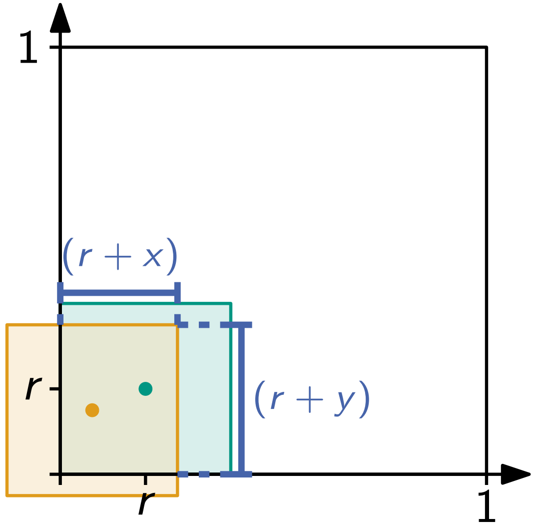

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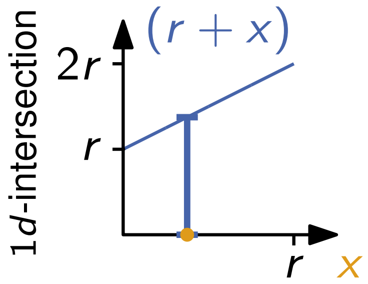

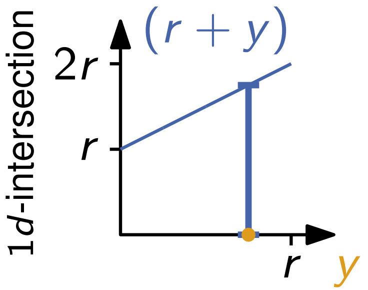

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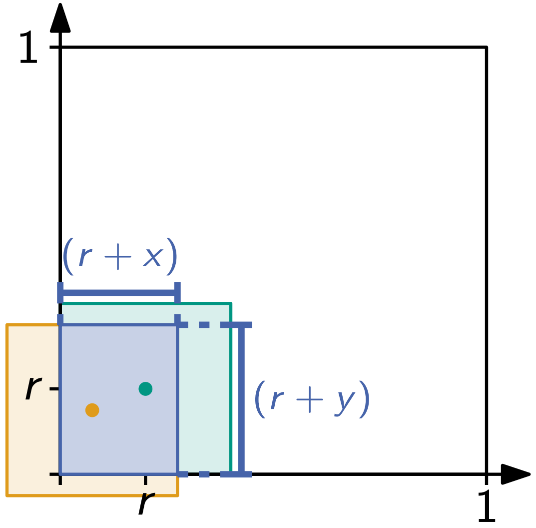

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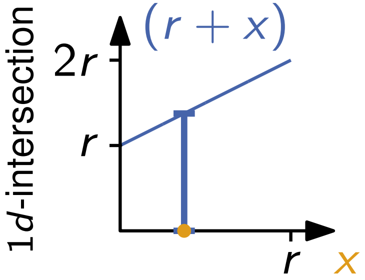

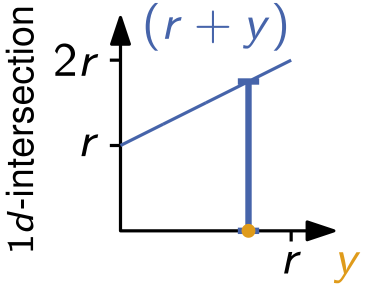

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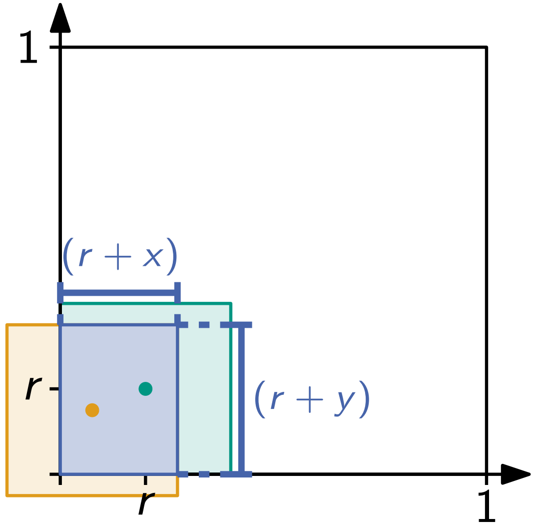

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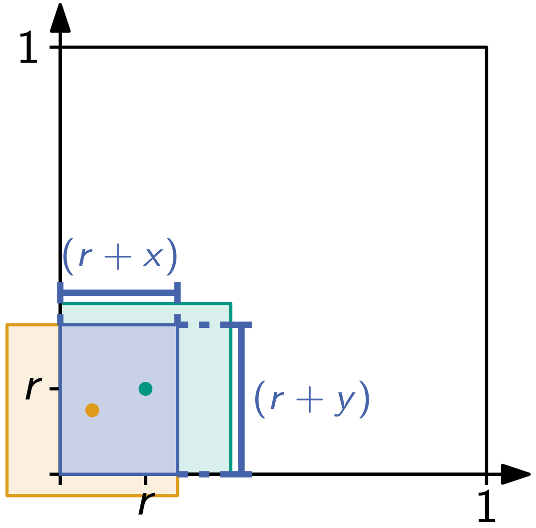

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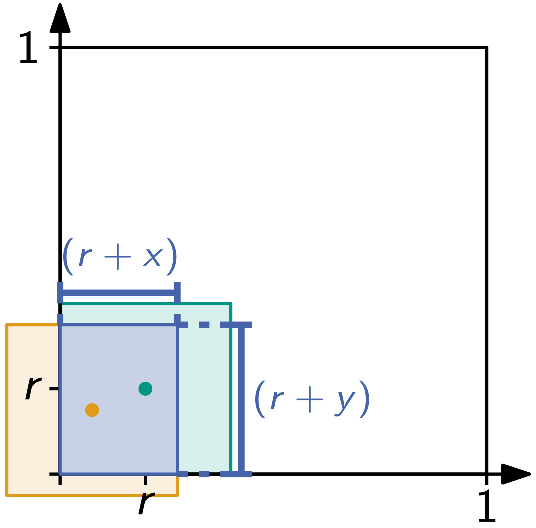

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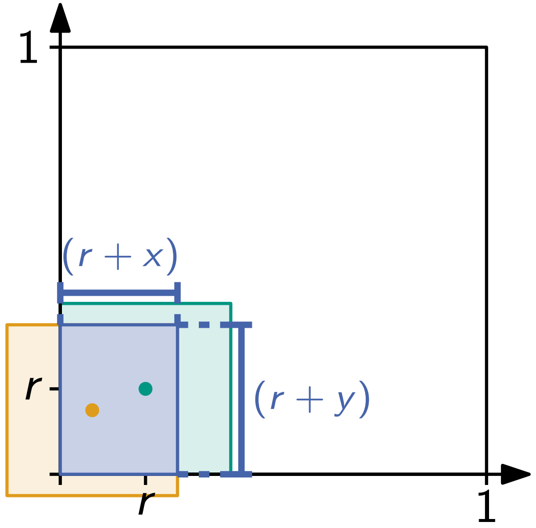

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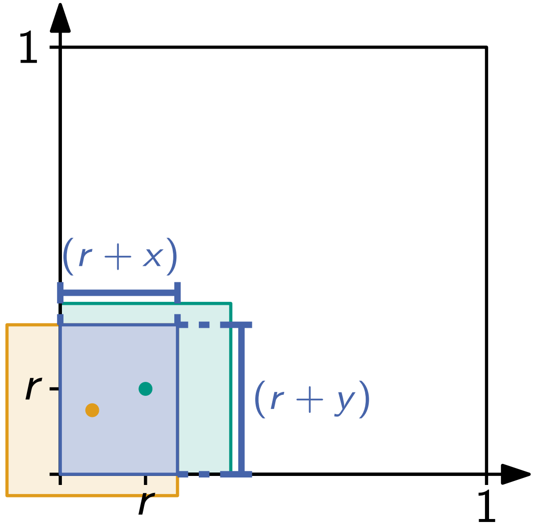

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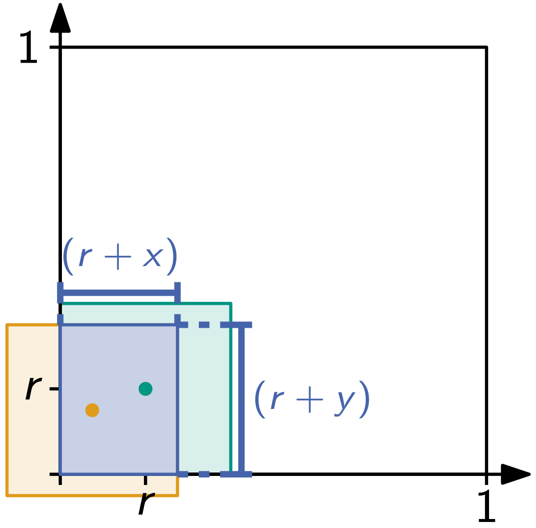
same integral



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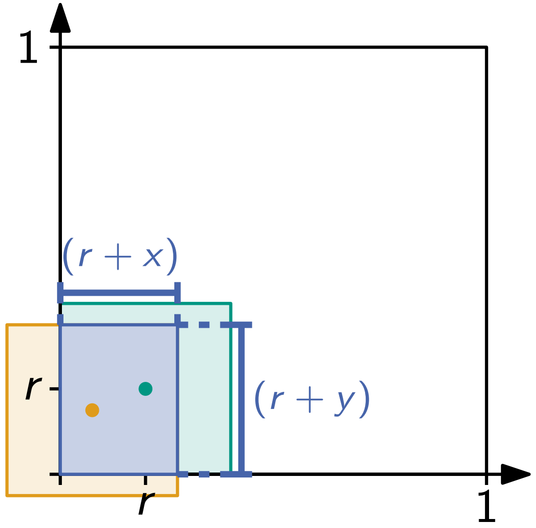

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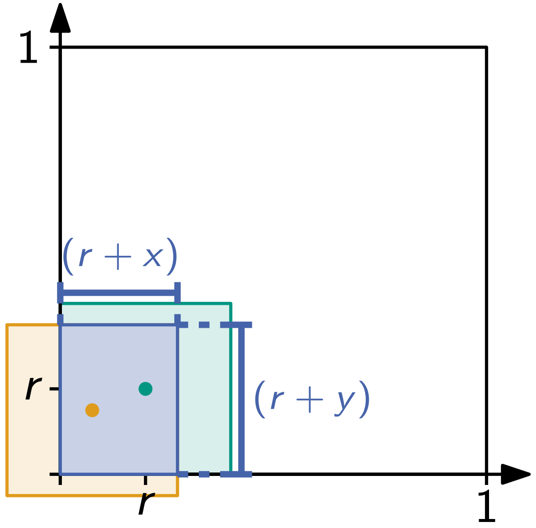

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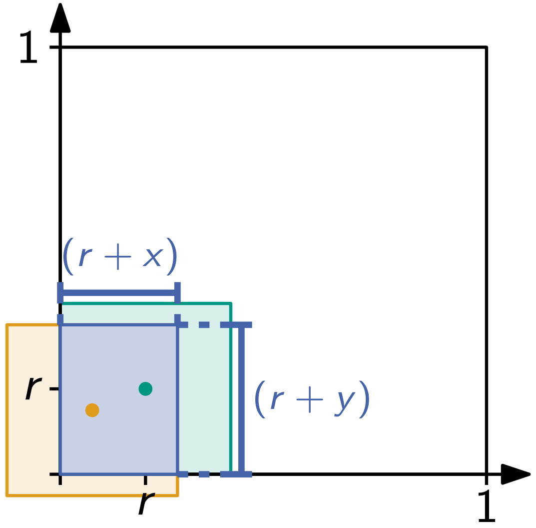

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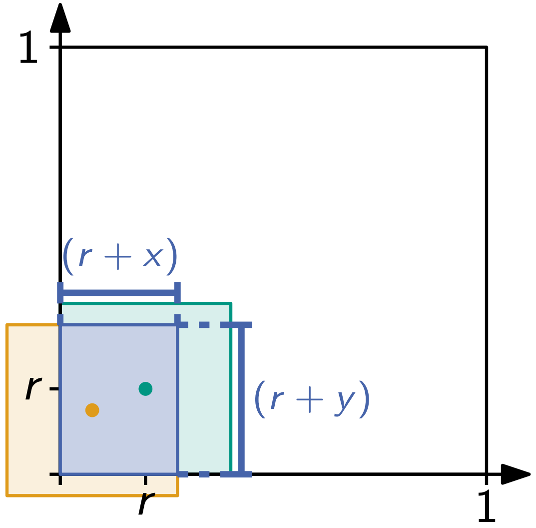

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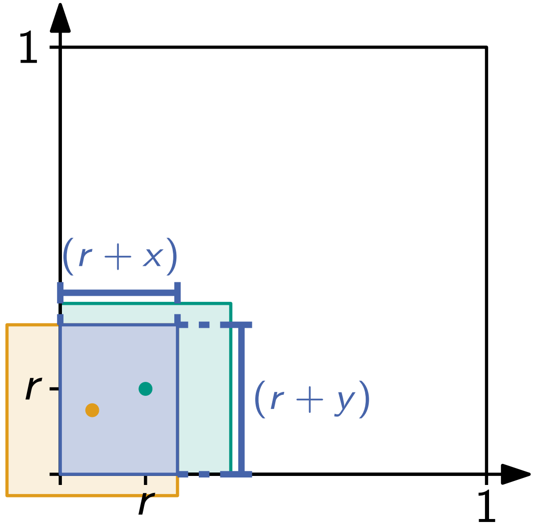

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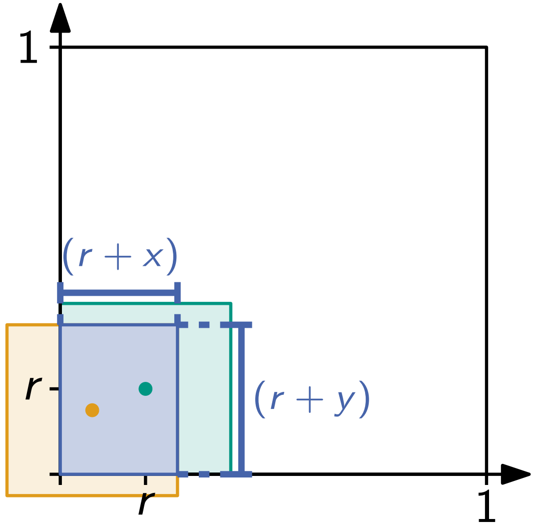

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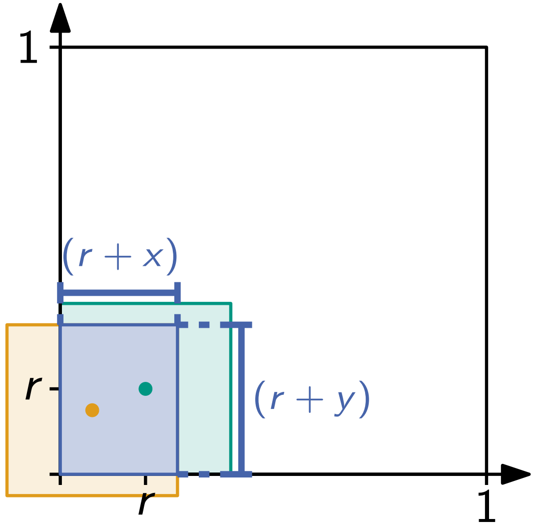

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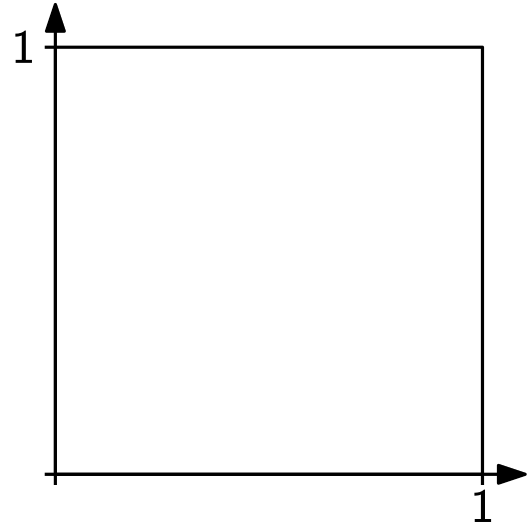

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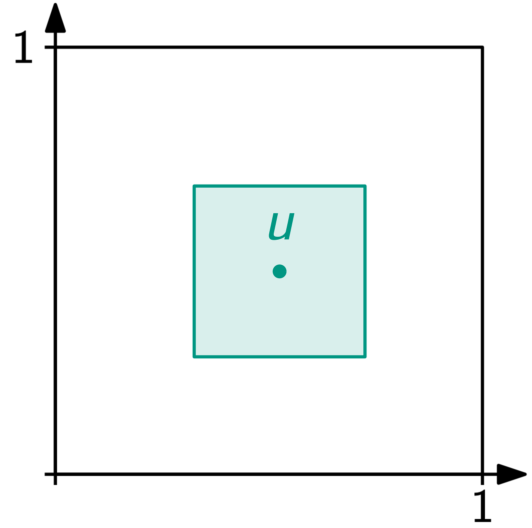

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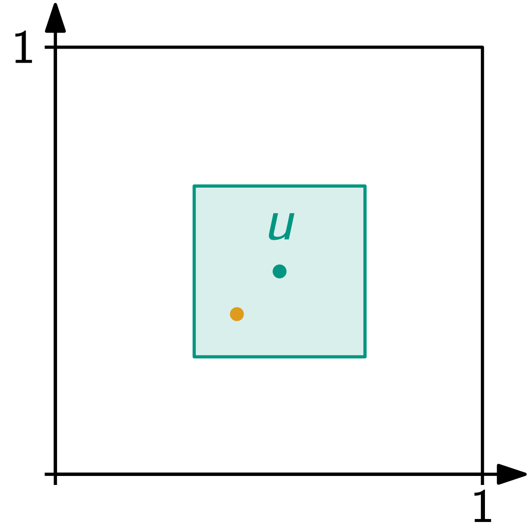

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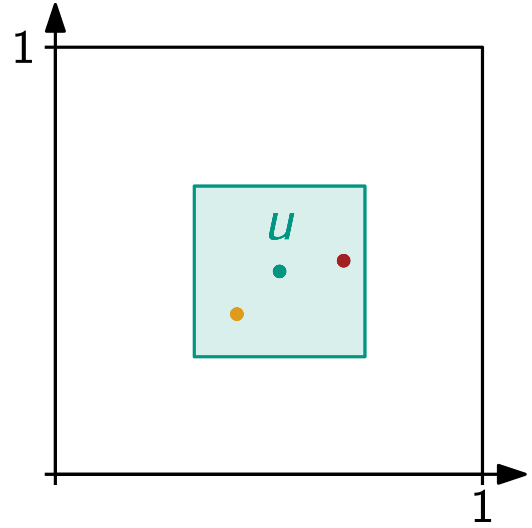

Numerator $\Pr[w \in N(v) \wedge v \in N(u) \wedge w \in N(u)] = 9r^4$

Denominator

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Simple Random Geometric Graphs – Locality

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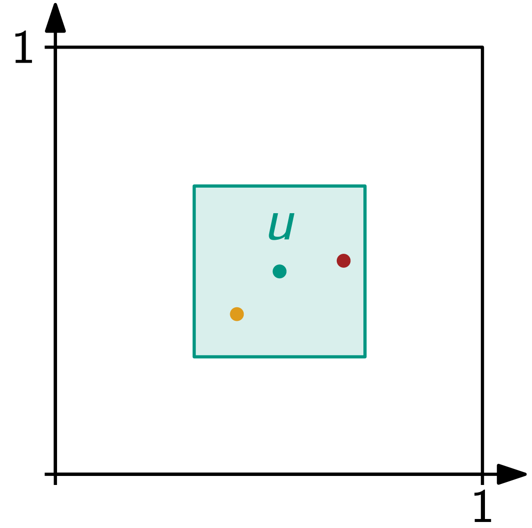

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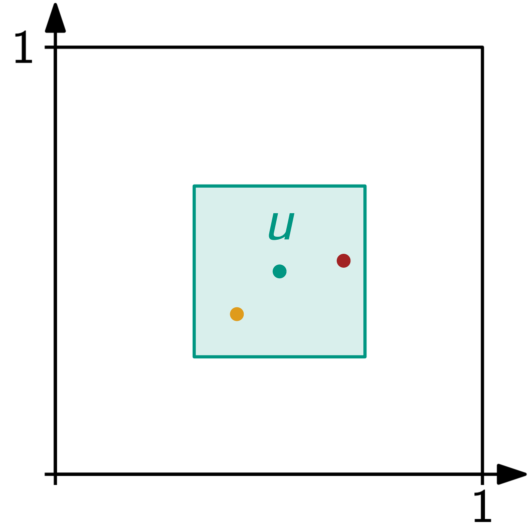

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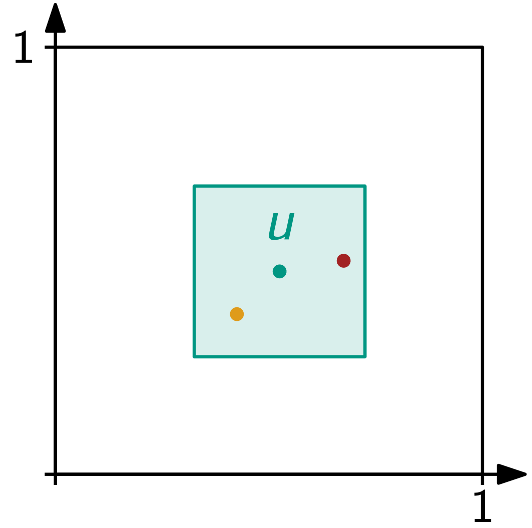

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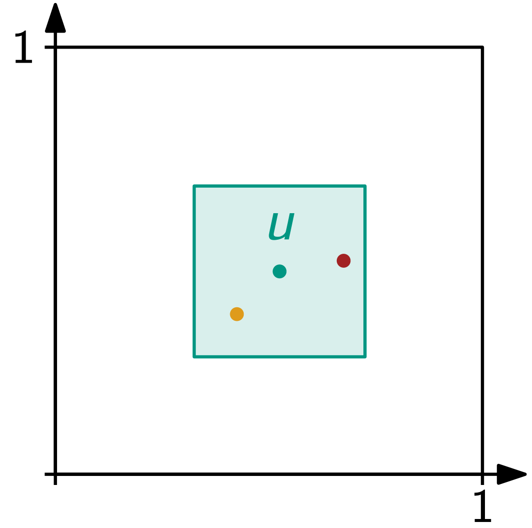

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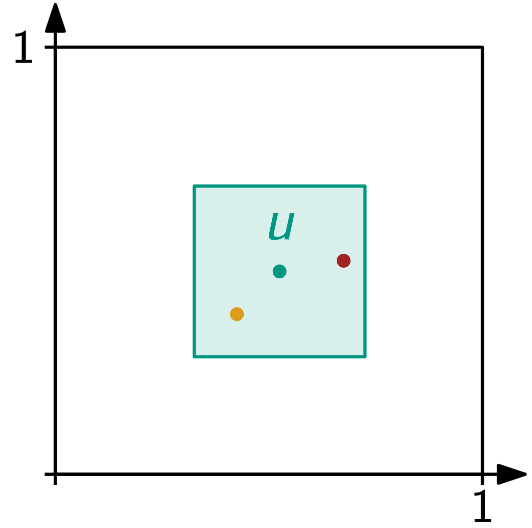

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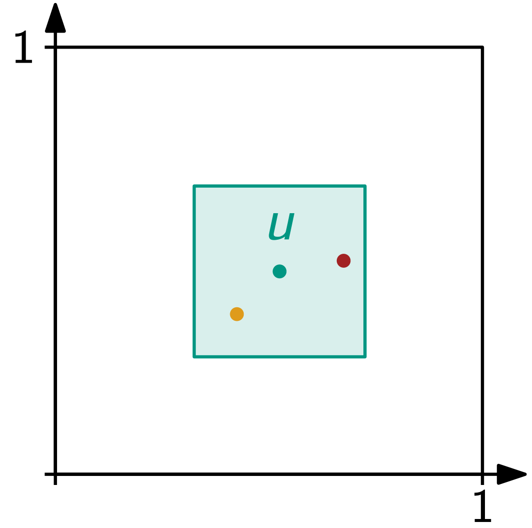

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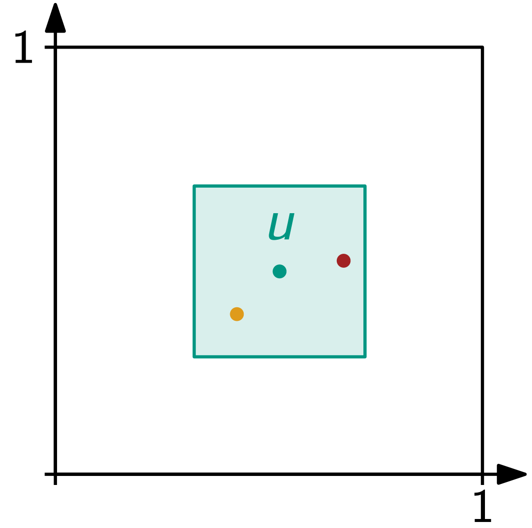

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Karlsruhe Institute of Technology

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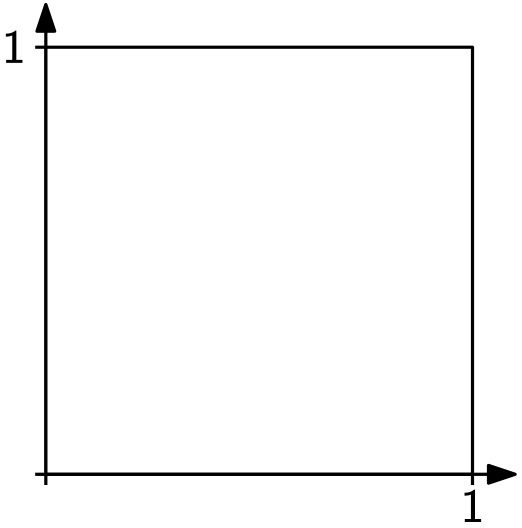

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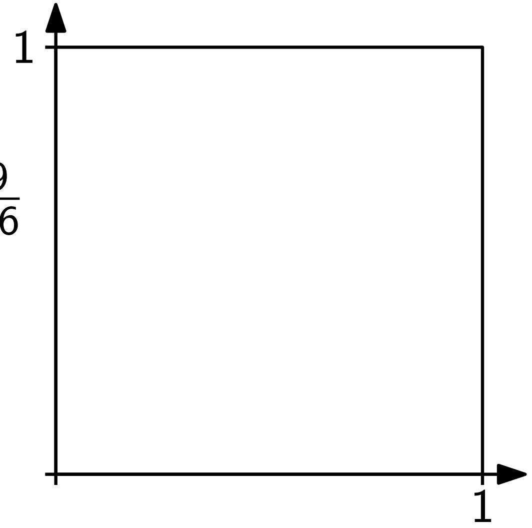

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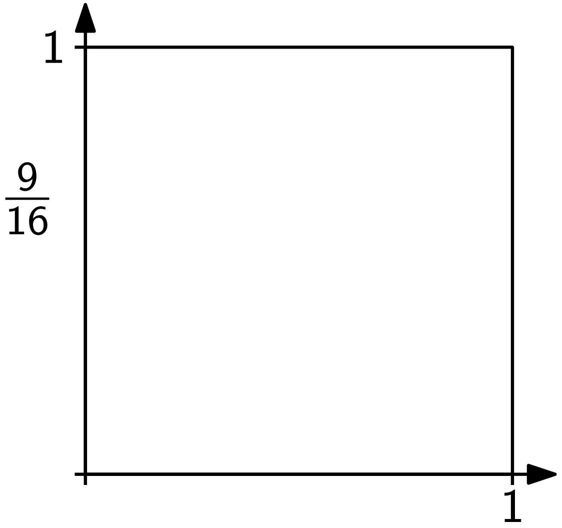

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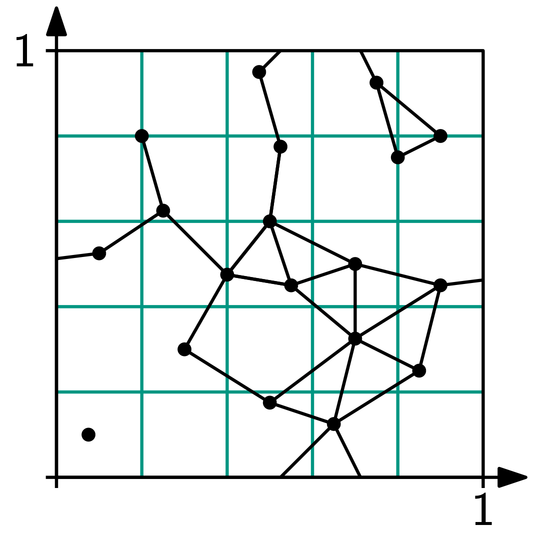
Application: Simple RGGs – Fair Distribution

- Discretize the space into equally sized grid cells, such that the expected number of vertices in each cell is $\log(n)$



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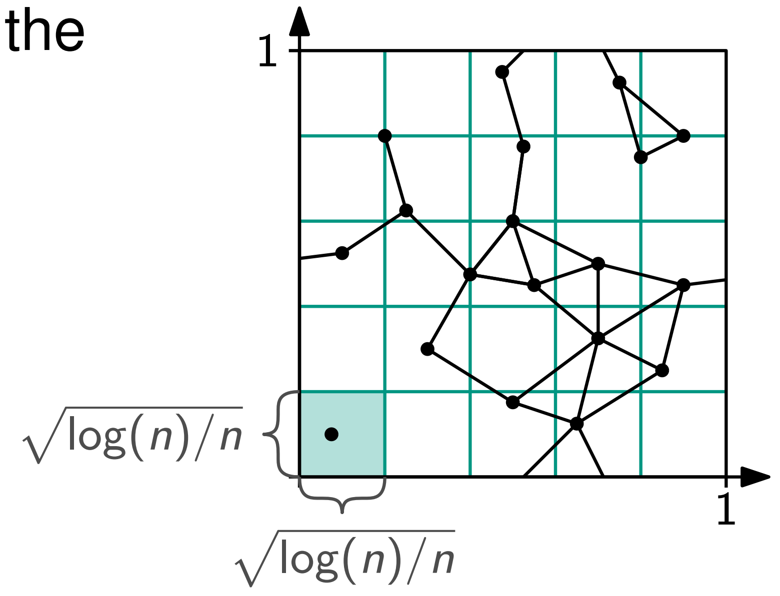

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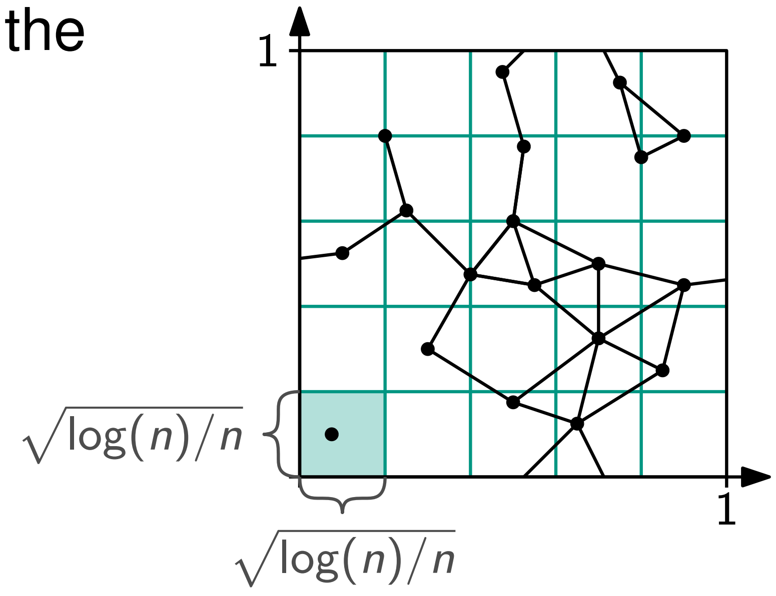

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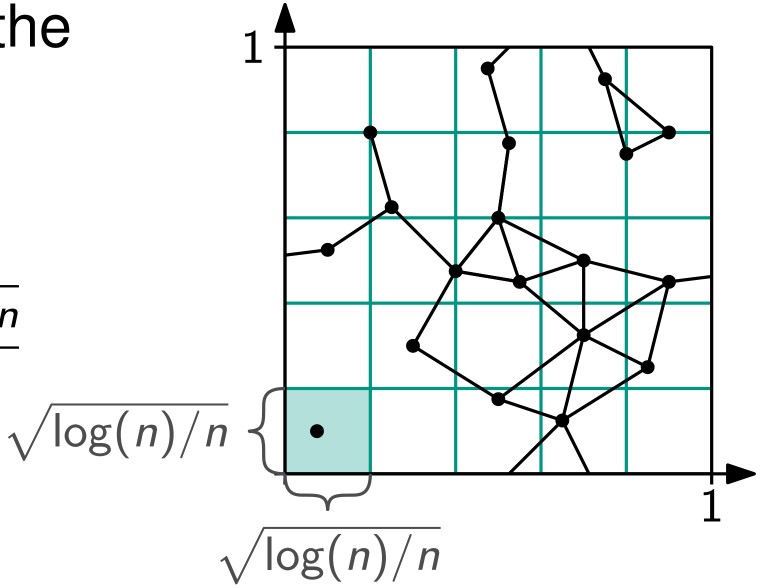

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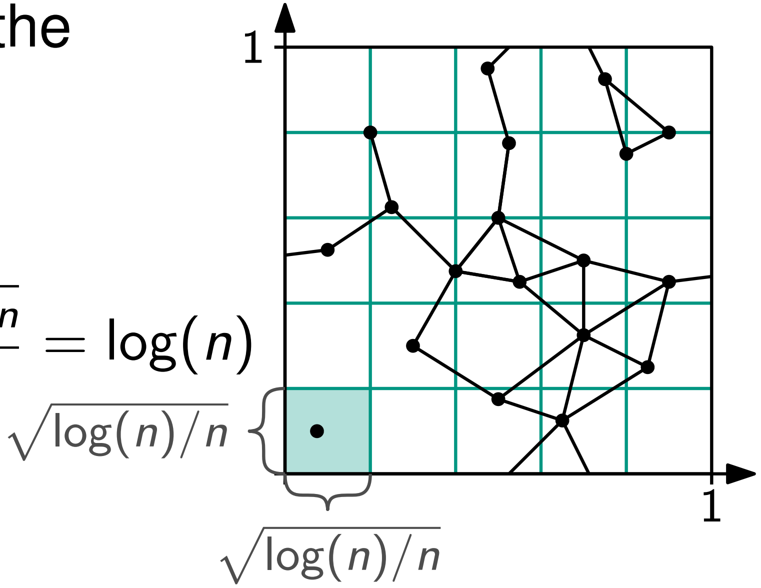

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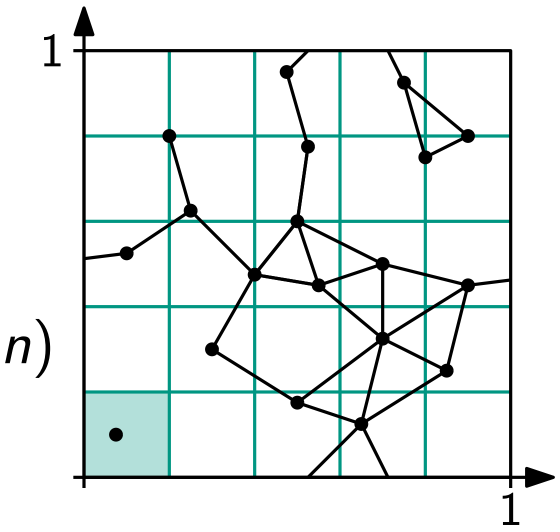

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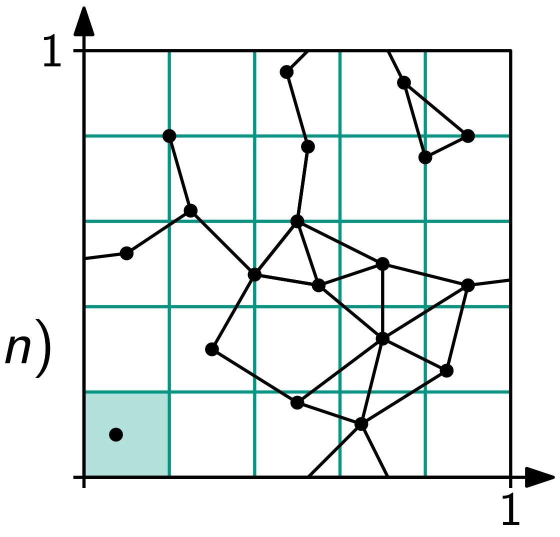

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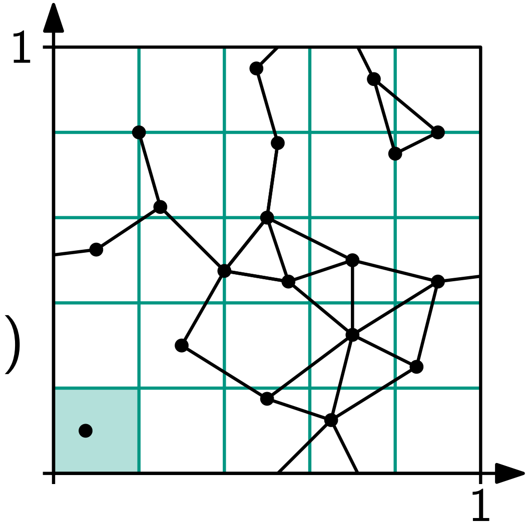

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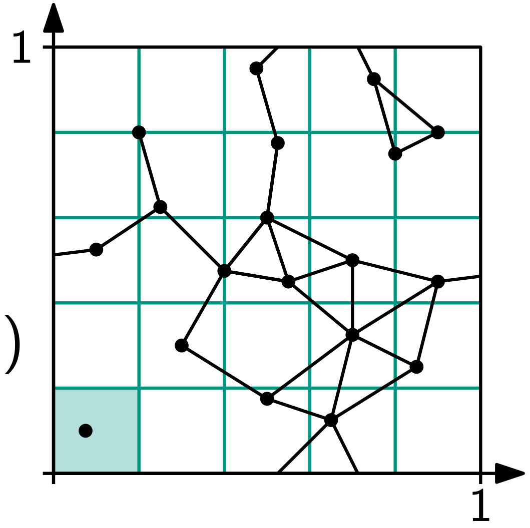

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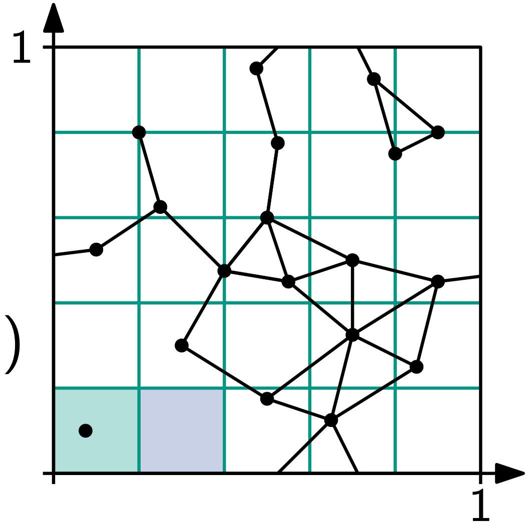

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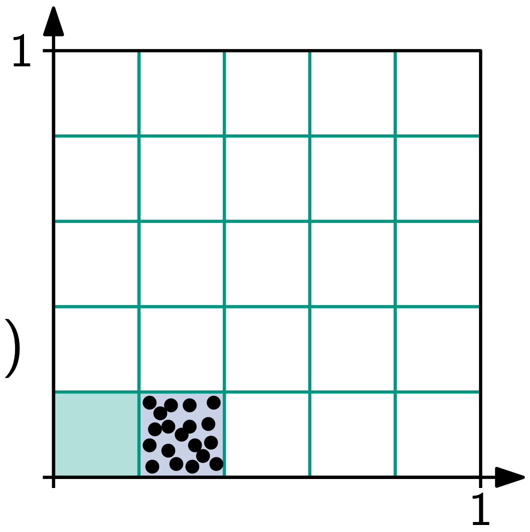

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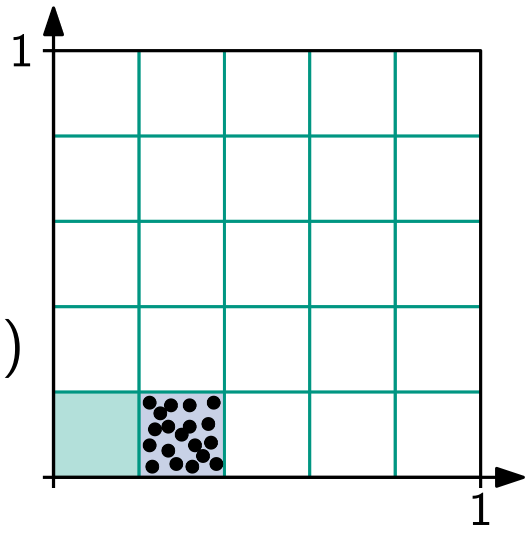

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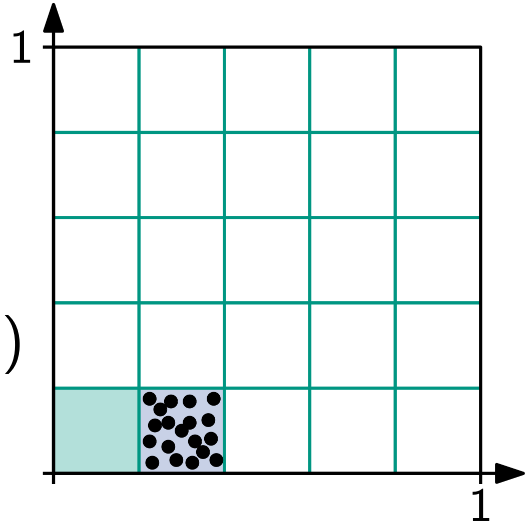

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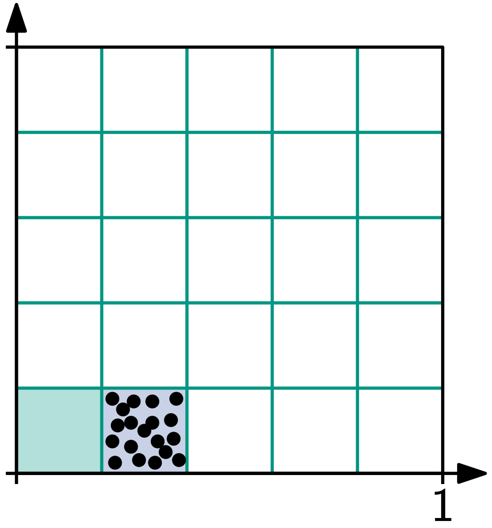

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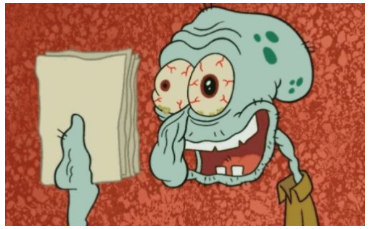

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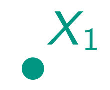

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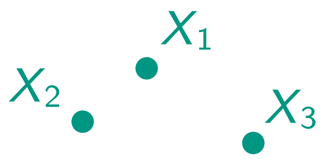

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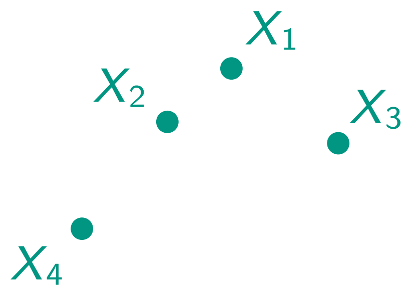

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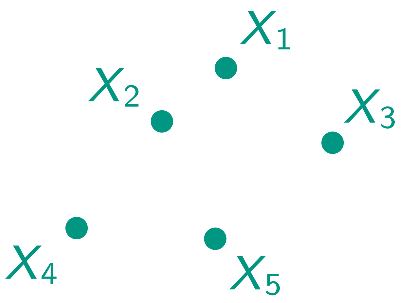

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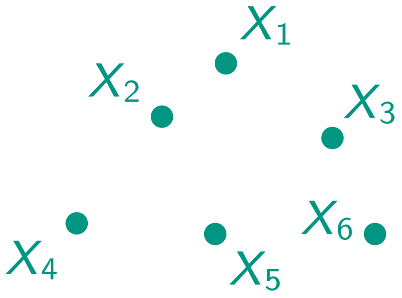

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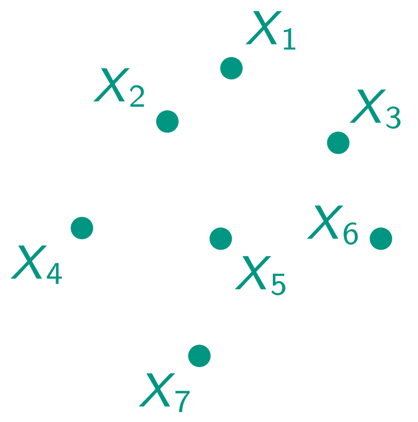

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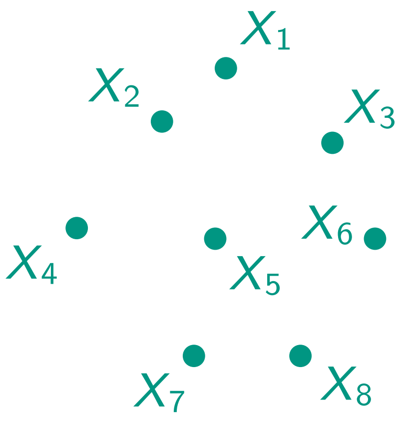

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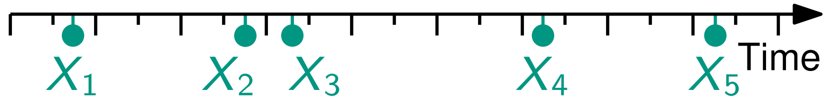

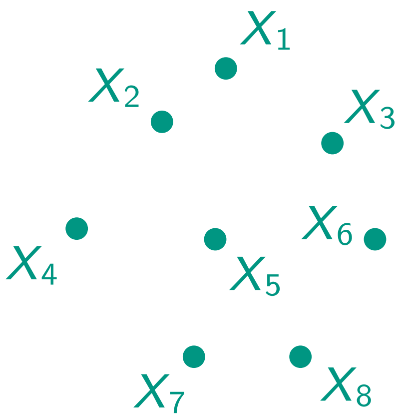

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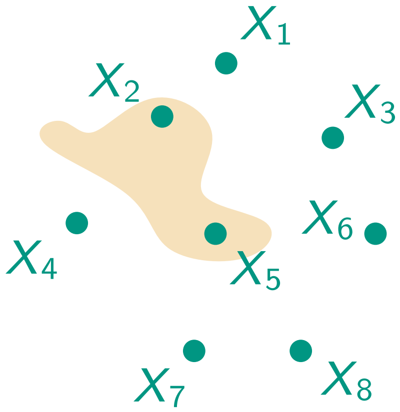

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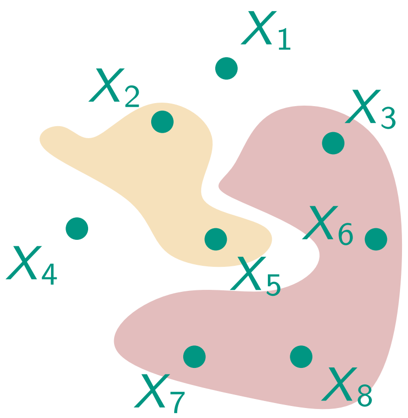

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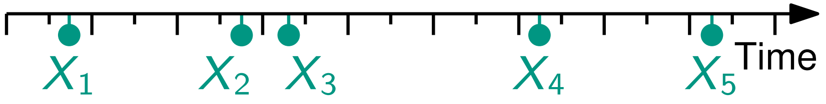

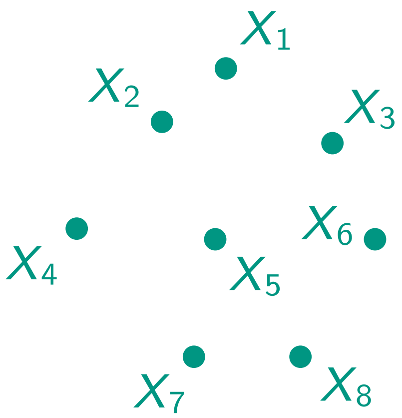

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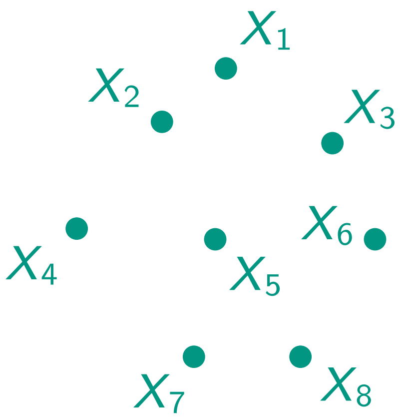

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Karlsruhe Institute of Technology

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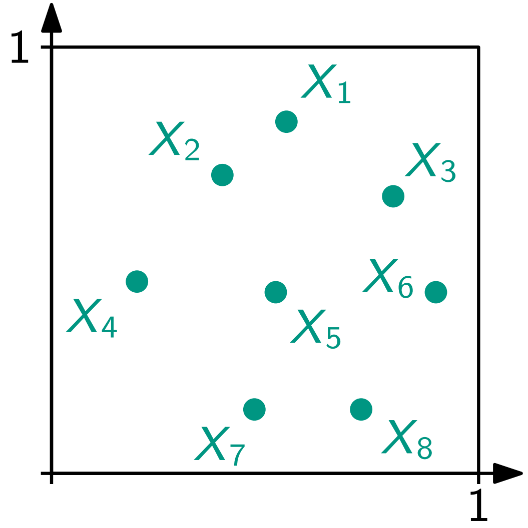

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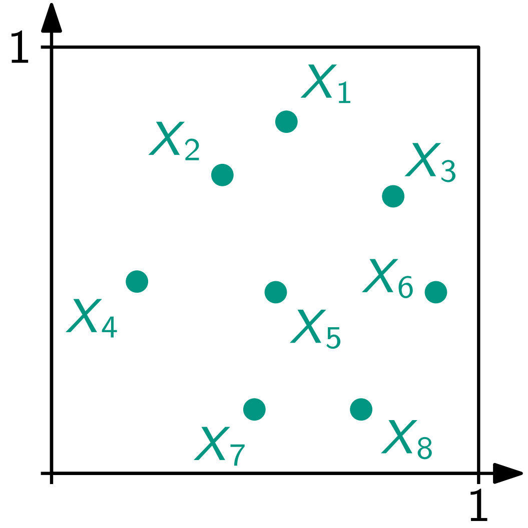

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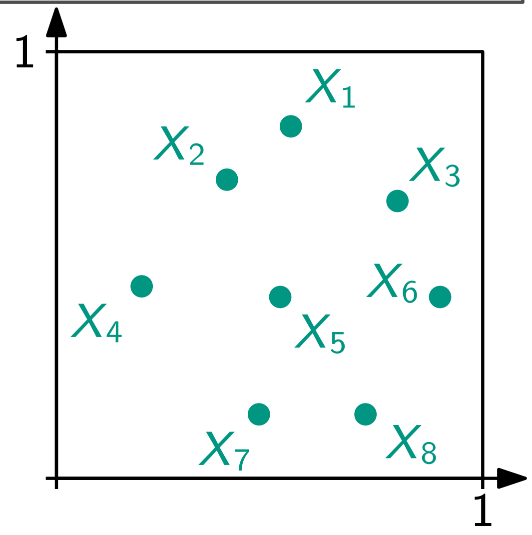

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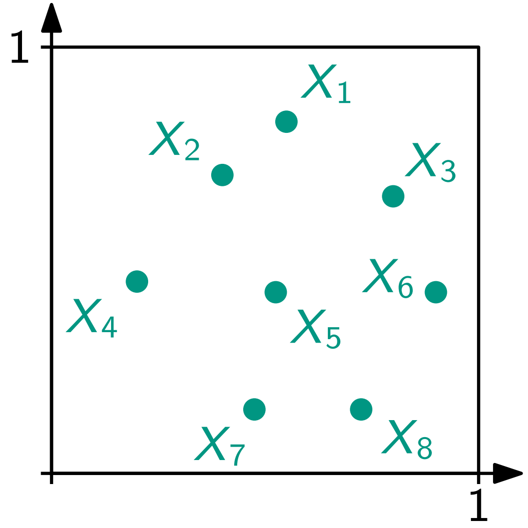

Poissonization

Idea

- Avoid dependencies by replacing uniform point sampling with a Poisson point process

Definition: A **Poisson *Point* process** with *intensity* λ is a collection of random variables $X_1, X_2, \ldots \in \mathbb{R}^2$ such that, if $|A|$ is the area of A and $N(A) = |\{i \mid X_i \in A\}|$, then

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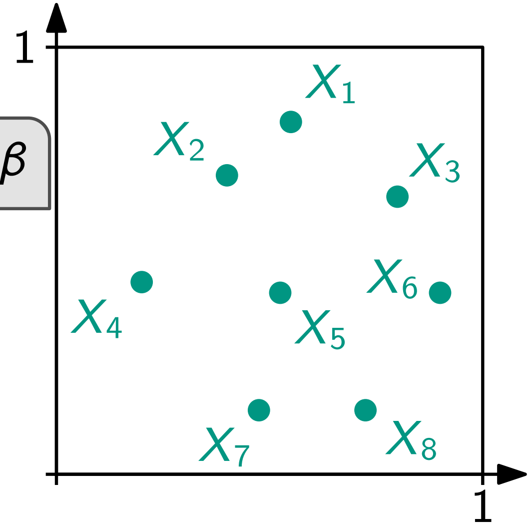

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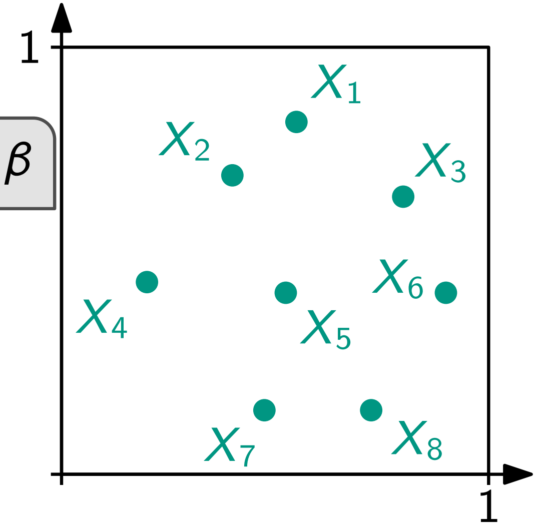

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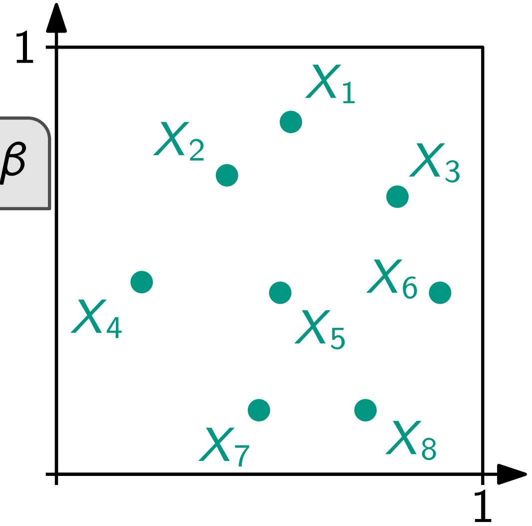

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Karlsruhe Institute of Technology

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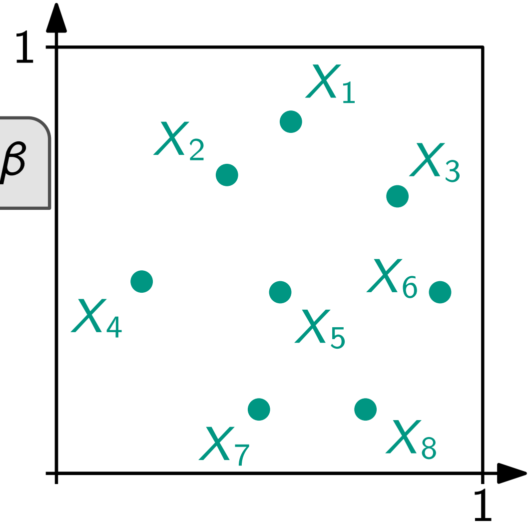

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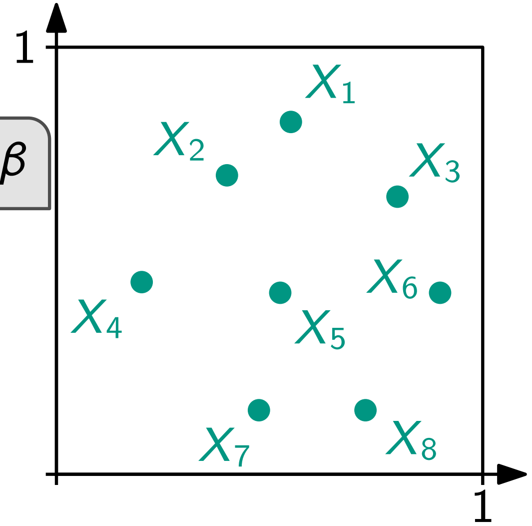

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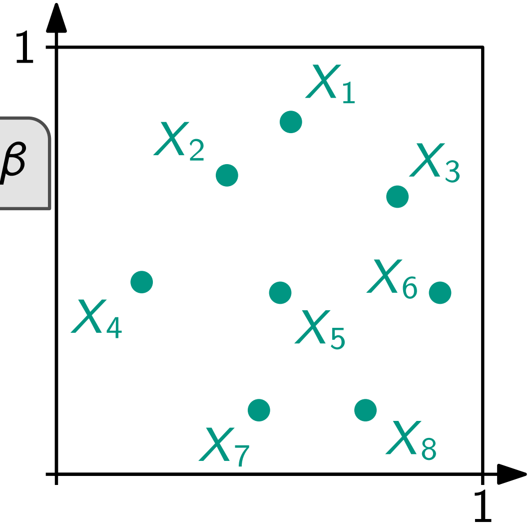

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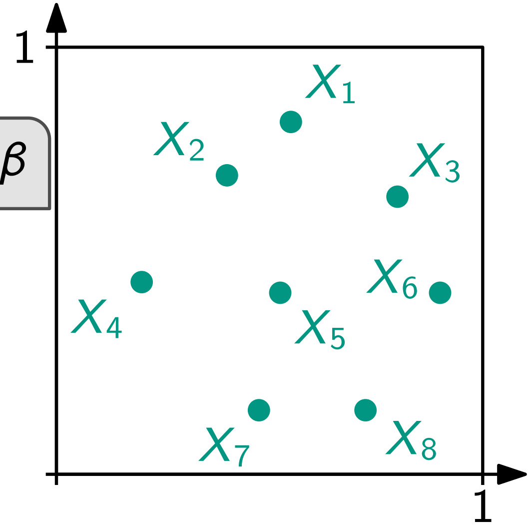
- The resulting **Poissonized RGG** has n vertices in expectation



Application: Poissonized RGGs – Fair Distribution

- Vertices of RGG distributed using Poisson point process with intensity $\lambda = n$
- Discretize the space into equally sized grid cells, such that the expected number of vertices in each cell is $\log(n)$
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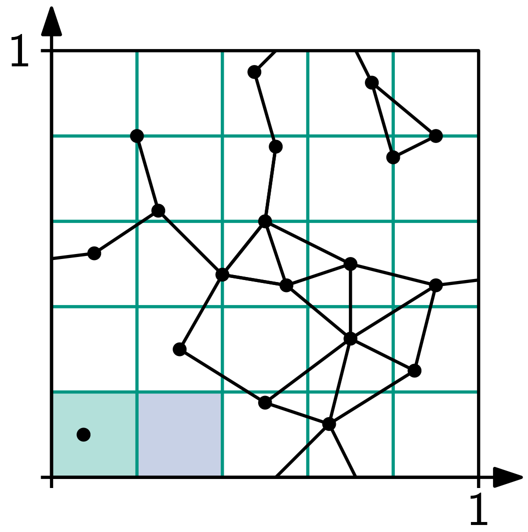

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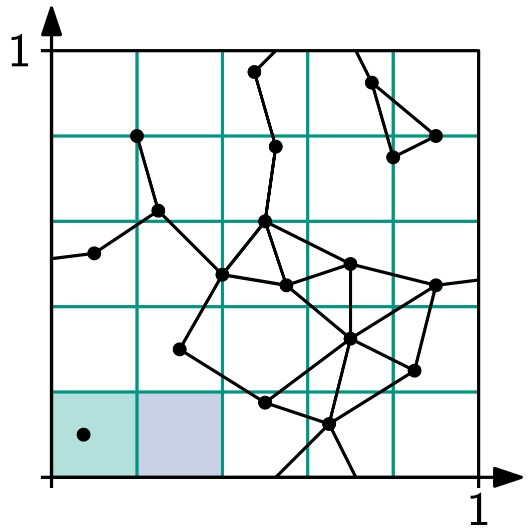

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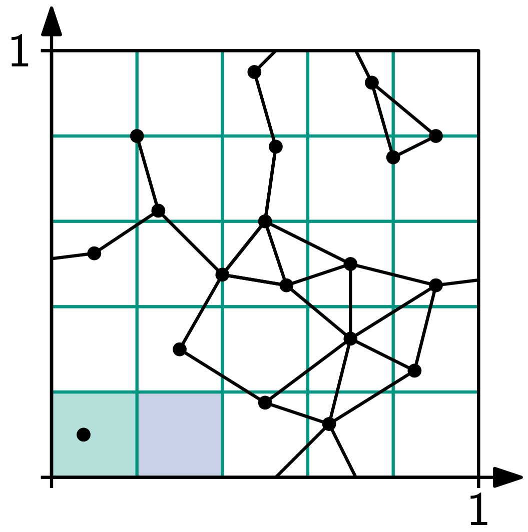

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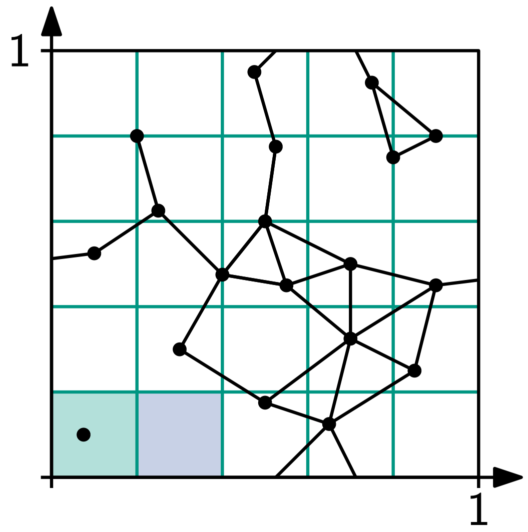

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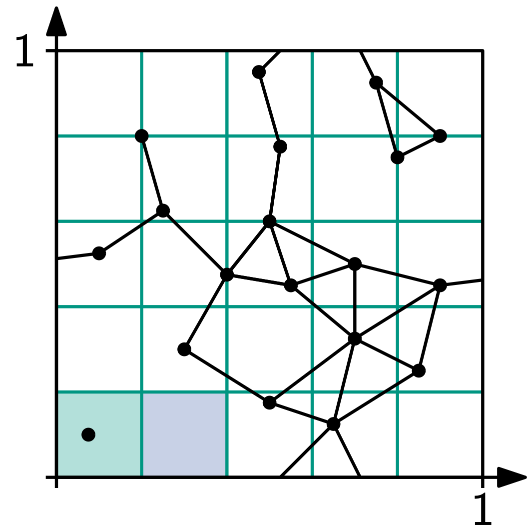

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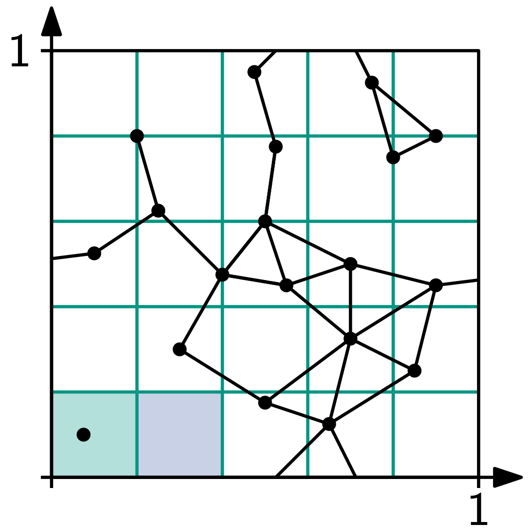

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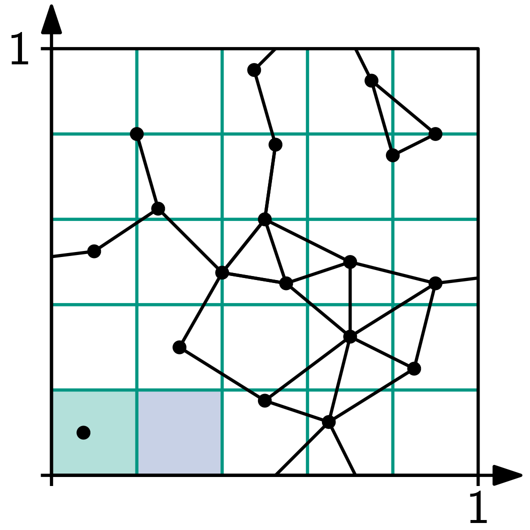

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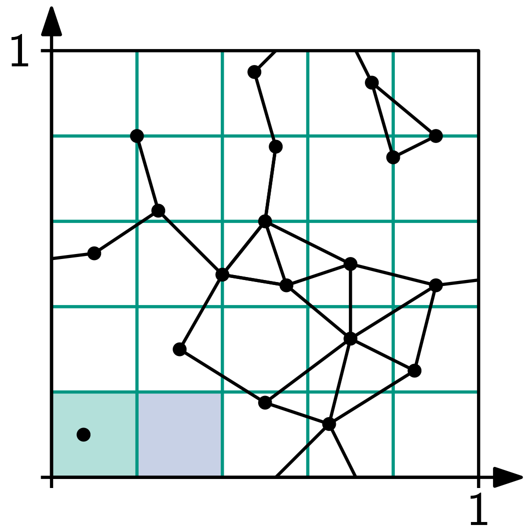

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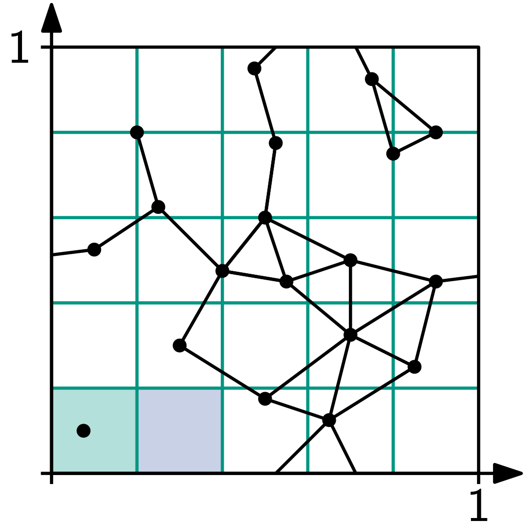

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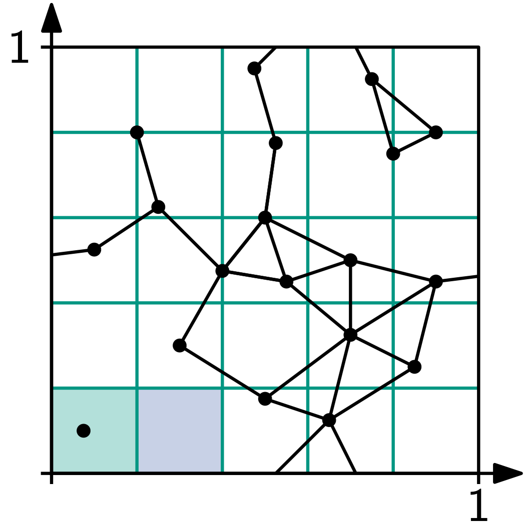

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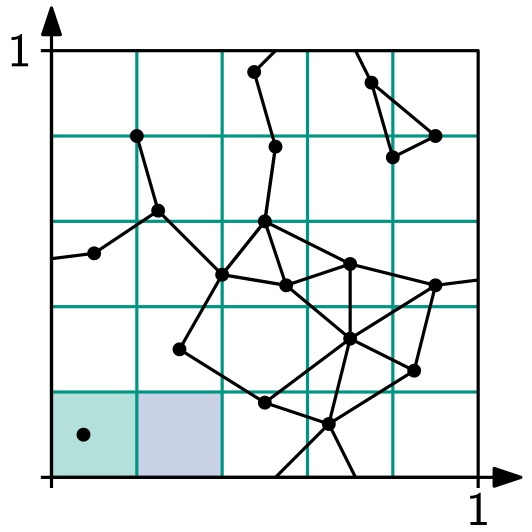

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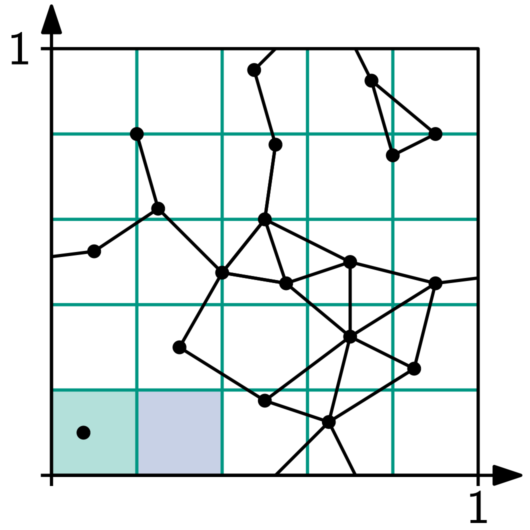

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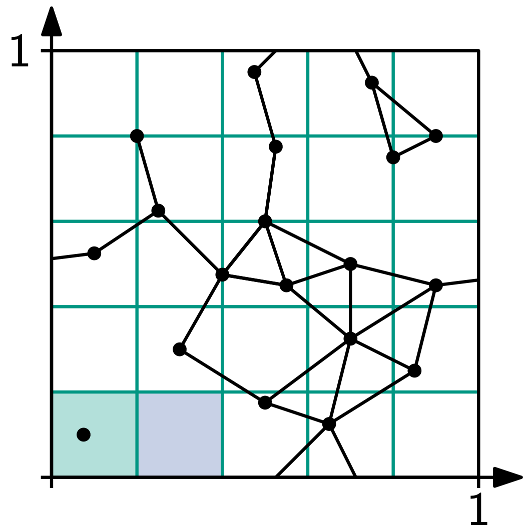

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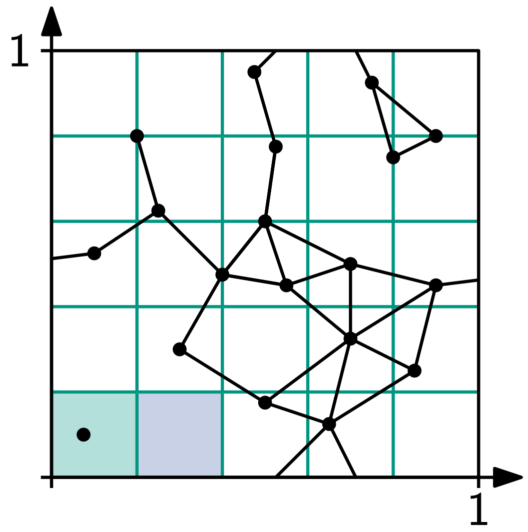

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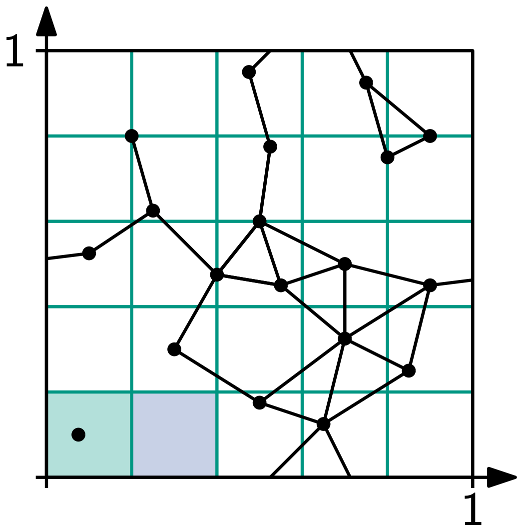

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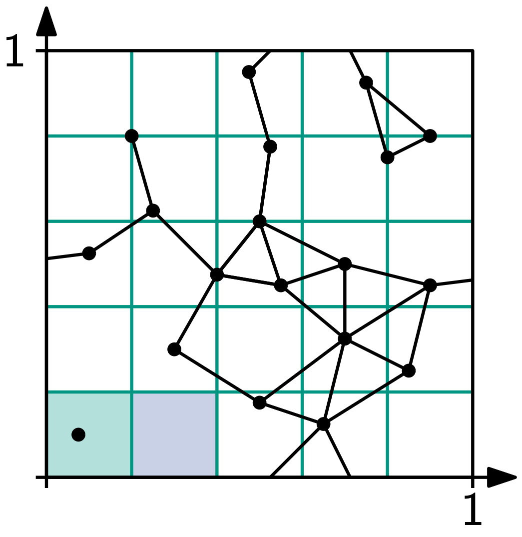

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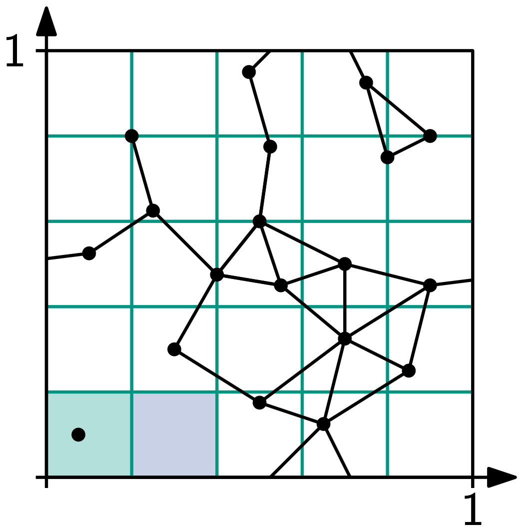

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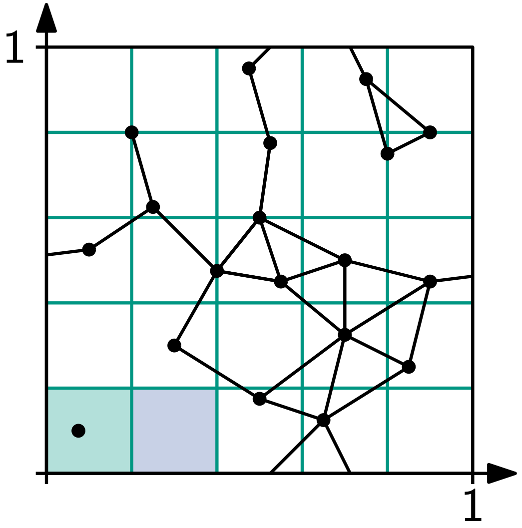

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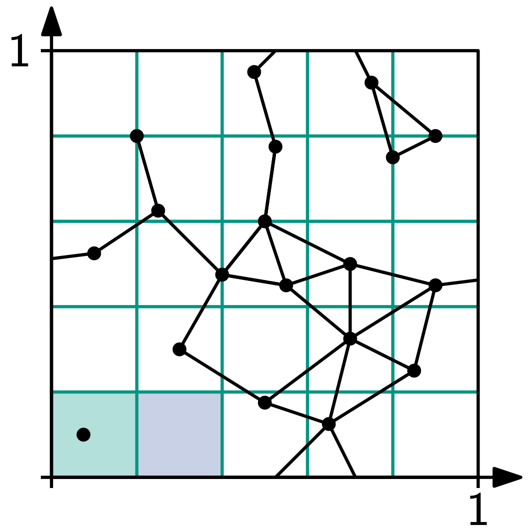

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but we cheated...

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Situation

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$$= \frac{\Pr_{\text{RGG}(\text{Pois}(n))}[\forall i : X_i = \log(n) \wedge N = n]}{\Pr_{\text{RGG}(\text{Pois}(n))}[N = n]}$$

$N \sim \text{Pois}(\lambda|A|)$

$$\Pr[N = k] = \frac{(\lambda|A|)^k e^{-\lambda|A|}}{k!}$$

Stirling

$$n! \geq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}}$$

De-Poissonization

Karlsruhe Institute of Technology

Situation

- We started with a simple RGG (n, $\mathbb{T}^2$, L_∞-norm, $P_i \sim \mathcal{U}([0,1]^2)$, $\Pr[\{u,v\} \in E] = \mathbb{1}_{\{d(u,v) \le r\}}$)
- Switched to Poissonized RGG (n is replaced by Pois(n)) and obtained $\Pr[\forall i : X_i = \log(n)] \le e^{-n/\log(n)}$
- How can we translate this result to the original model?

Recall

- Conditioned on the number of points in area A, the points are distributed uniformly in A
- So we get from the poissonized RGG to the original, by conditioning on the fact that the number of points N in $[0,1]^2$ obtained in the Poisson point process is exactly $N = n$

$$\Pr[N = n] = \frac{(\lambda|A|)^n e^{-\lambda|A|}}{n!} = \frac{n^n e^{-n}}{n!} \le \cancel{\left(\frac{n}{e}\right)^n} \cdot \frac{1}{\sqrt{2\pi n}\cancel{\left(\frac{n}{e}\right)^n} \underbrace{e^{\frac{1}{12n+1}}}_{\Theta(1)}} = \Theta(n^{-1/2})$$

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Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

RGG – The Bigger Picture

Seen so far

- Simple RGG
- n, $\mathbb{T}^2$, L_∞-norm, $P_i \sim \mathcal{U}([0,1]^2)$, $\Pr[\{u,v\} \in E] = \mathbb{1}_{\{d(u,v) \le r\}}$

Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

RGG – The Bigger Picture

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- Expected degree of a vertex is $(n-1)4r^2$

RGG – The Bigger Picture

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More commonly used model

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Random Geometric Graph
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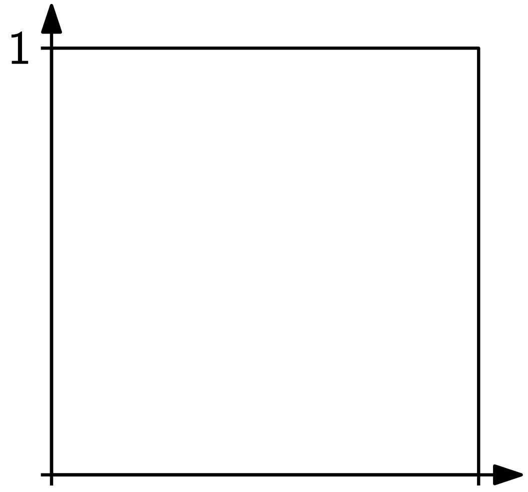

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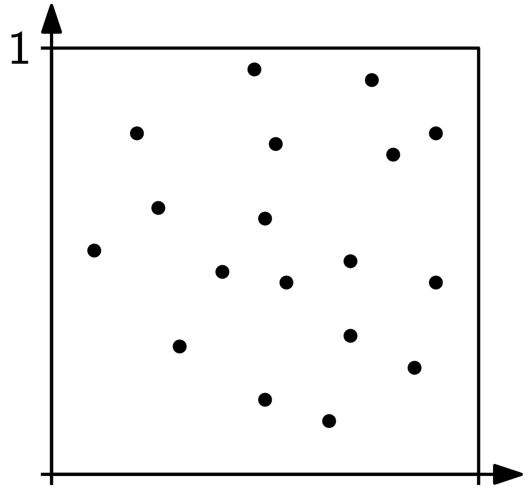

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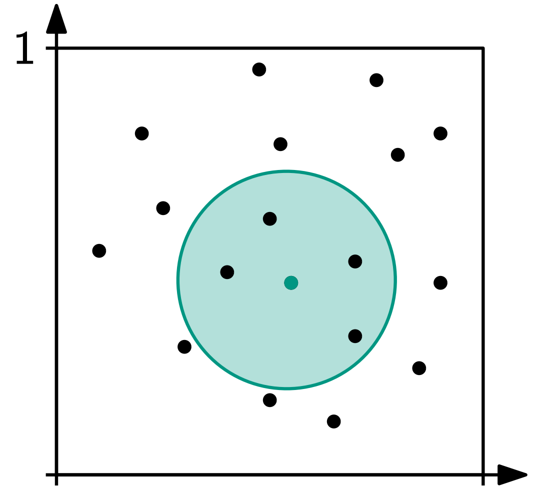
Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

$N(v)$ is a disk



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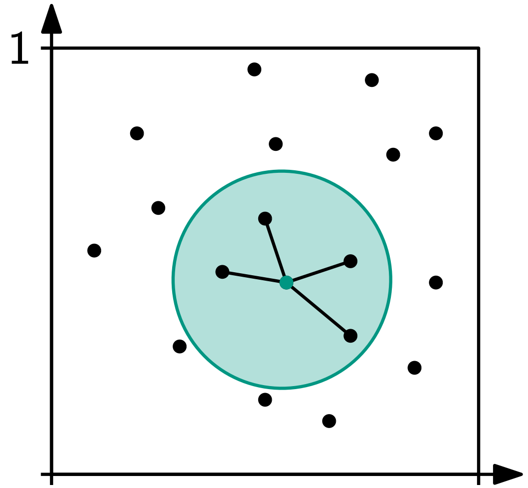
Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

$\boxed{N(v)}$ is a disk



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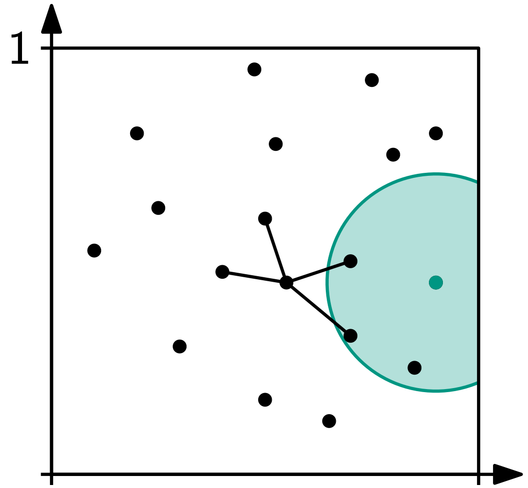
Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

$\boxed{N(v)}$ is a disk
No wrap-around!



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Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

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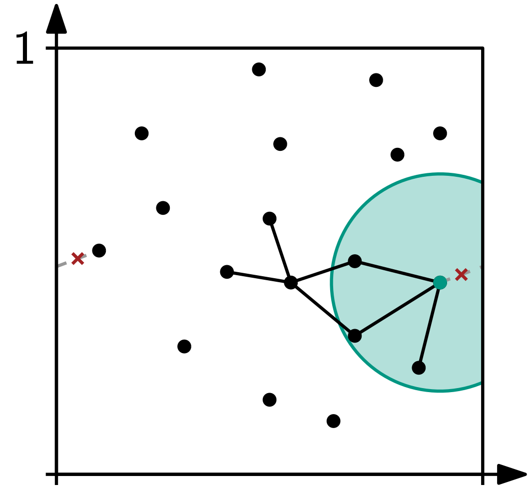

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Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

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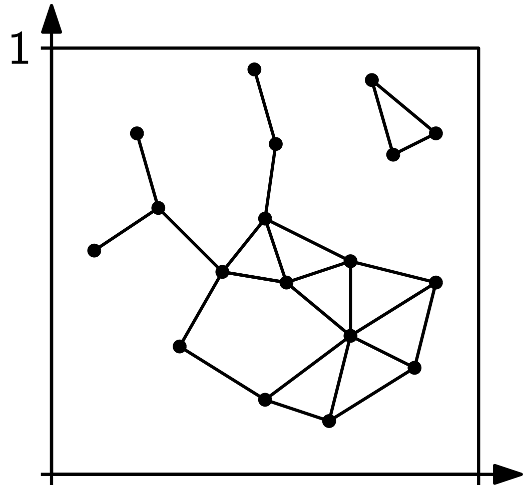

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- Complications
  - Vertices near the boundary / corners behave differently

Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

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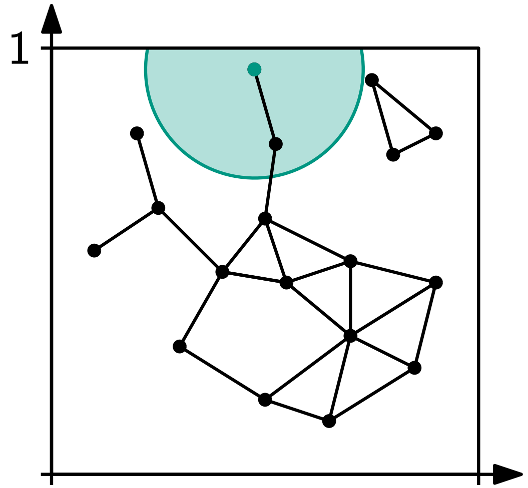

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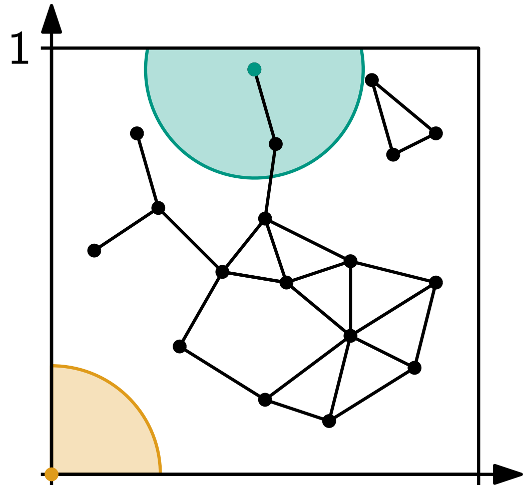

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  - Vertices near the boundary / corners behave differently
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Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

$\boxed{N(v)}$ is a disk
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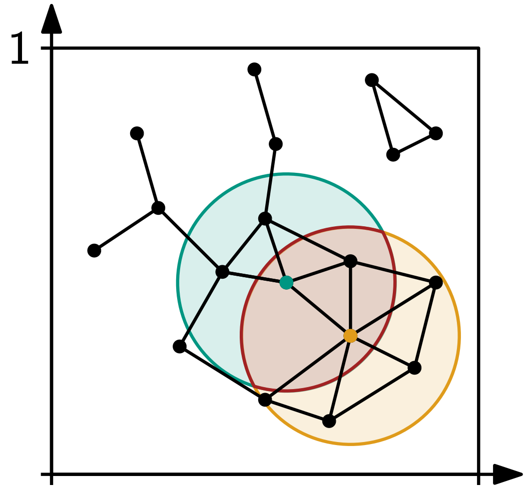

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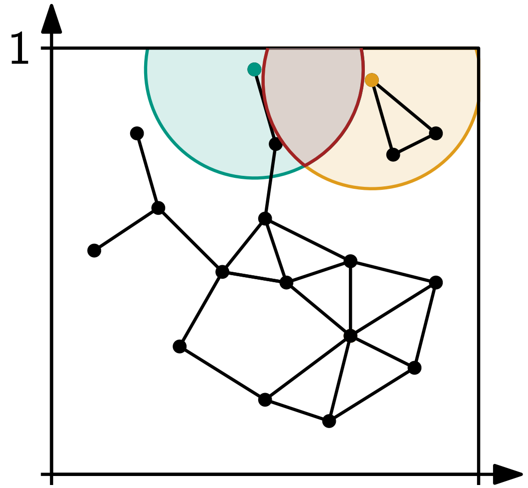

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- Still $\mathbb{E}[\deg(v)] = \Theta(nr^2)$
- Still probability to connect given common neighbor non-vanishing

Random Geometric Graph
Nodes distributed in metric space
Connection probability depends on distance

$\boxed{N(v)}$ is a disk
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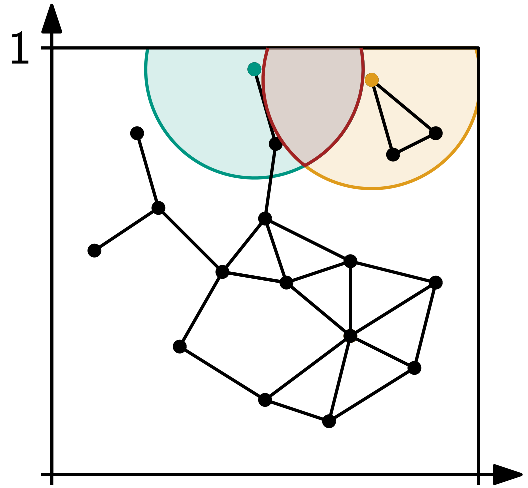

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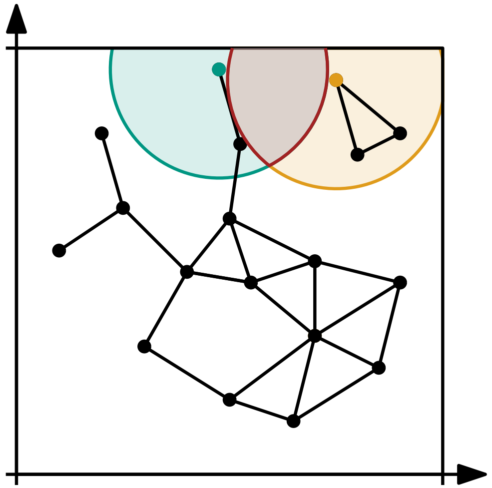
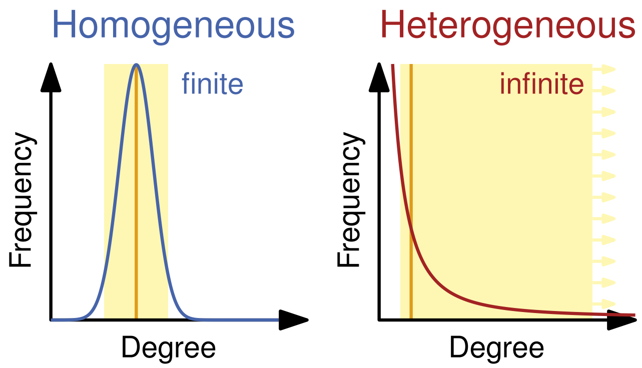


Problem: Homogeneous degree distribution does not match many real-world graphs

A Heterogeneous Distribution

Motivation

- Distributions seen so far have finite variance
- Graphs with corresponding degree distributions are *homogeneous*



A Heterogeneous Distribution

Motivation

- Distributions seen so far have finite variance
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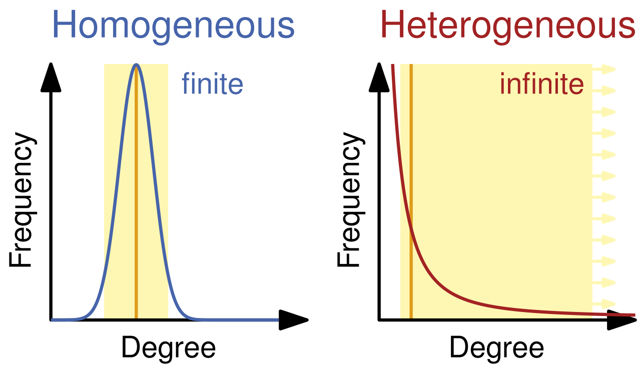
  $\Rightarrow$ For constant expected degree, it is very unlikely to find a high-degree vertex



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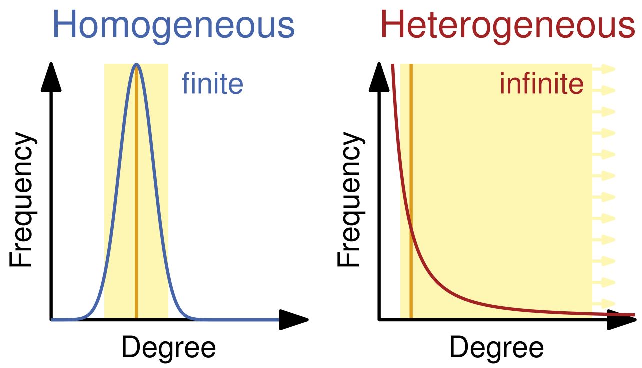
- In real-world graphs high-degree vertices are not too rare (think of celebrities in a social network)



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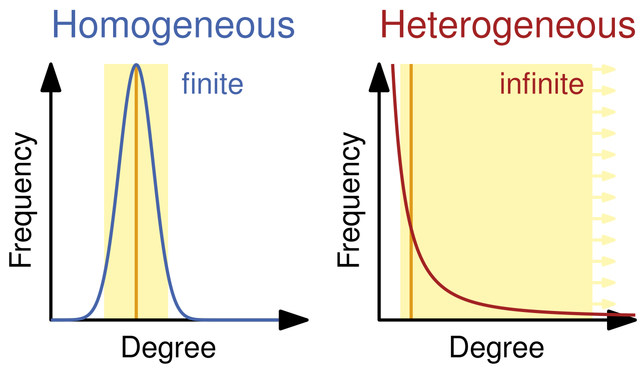

Pareto Distribution

- $X \sim \mathrm{Par}(\alpha, x_{\min})$

A Heterogeneous Distribution

Karlsruhe Institute of Technology

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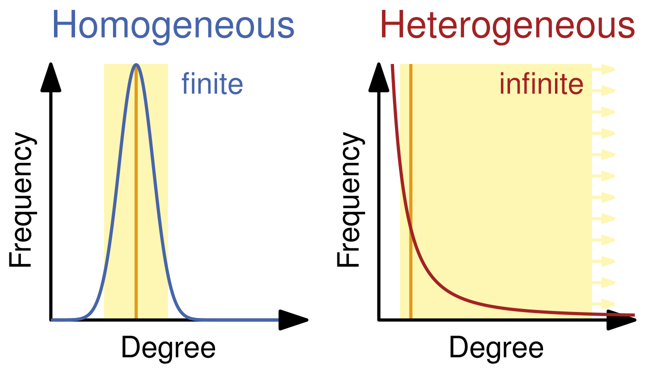


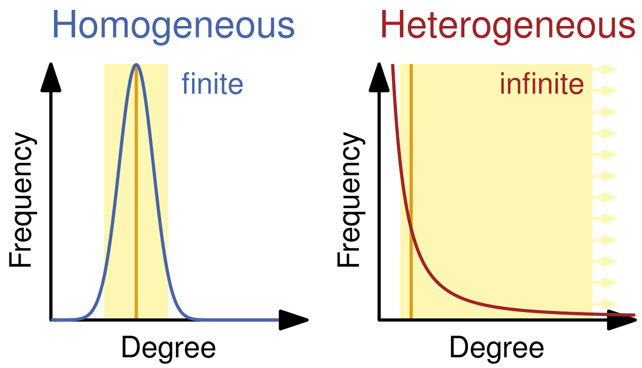
Pareto Distribution

- $X \sim \mathrm{Par}(\alpha, x_{\mathrm{min}})$

α: shape parameter

A Heterogeneous Distribution

Karlsruhe Institute of Technology



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  $x_{\min}$: minimum attainable value

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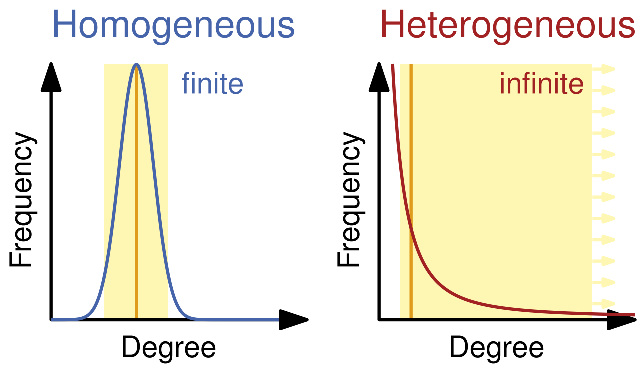
A Heterogeneous Distribution



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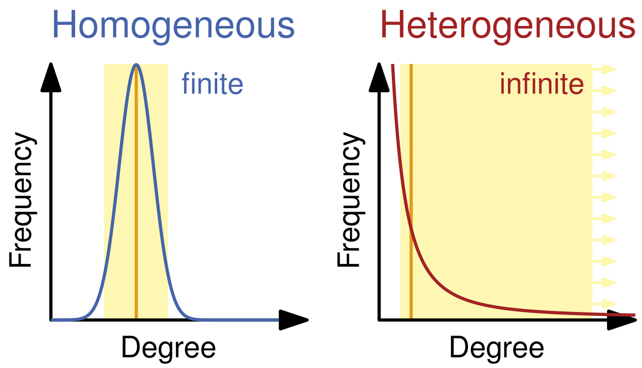
Pareto Distribution

- $X \sim \text{Par}(\alpha, x_{\text{min}})$
  - x_{min}: minimum attainable value
  - α: shape parameter
- Probability density function: $f_X(x) = \begin{cases} \alpha x_{\text{min}}^{\alpha} \cdot x^{-(\alpha+1)}, & \text{if } x \geq x_{\text{min}} \\ 0, & \text{otherwise} \end{cases}$

A Heterogeneous Distribution



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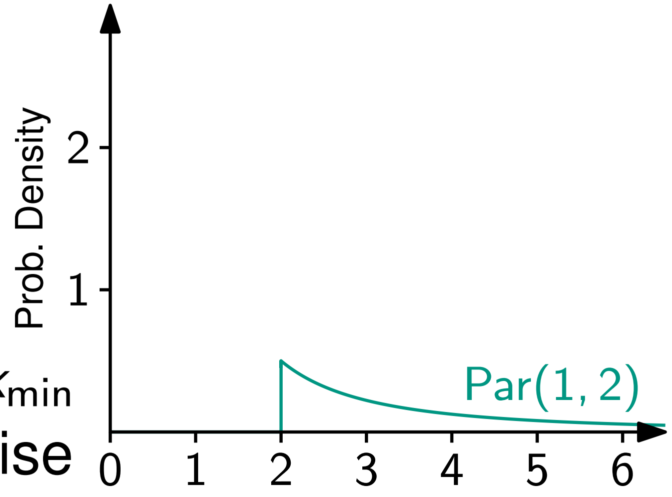
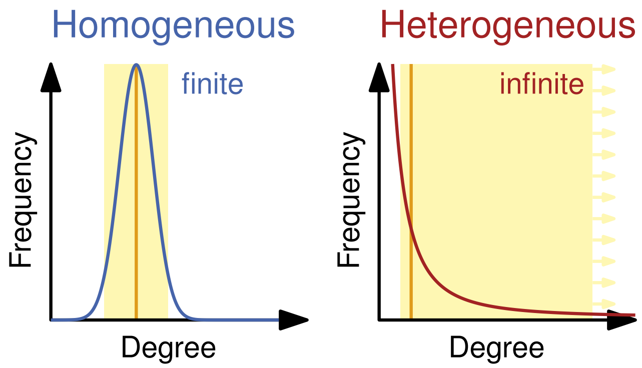


A Heterogeneous Distribution



Motivation

- Distributions seen so far have finite variance
- Graphs with corresponding degree distributions are *homogeneous*

  $\Rightarrow$ For constant expected degree, it is very unlikely to find a high-degree vertex
- In real-world graphs high-degree vertices are not too rare (think of celebrities in a social network)

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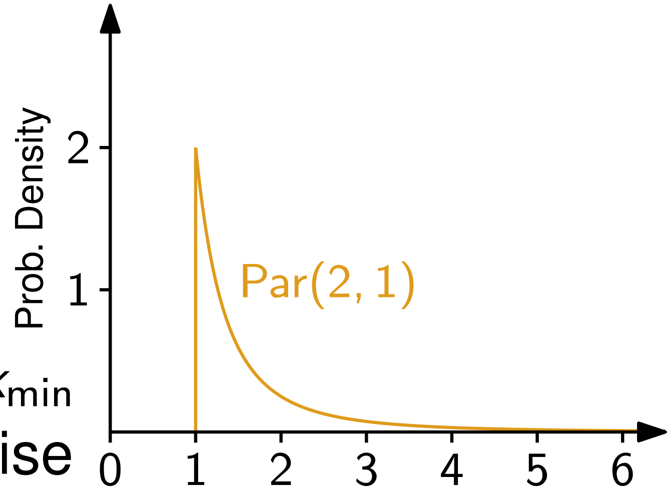

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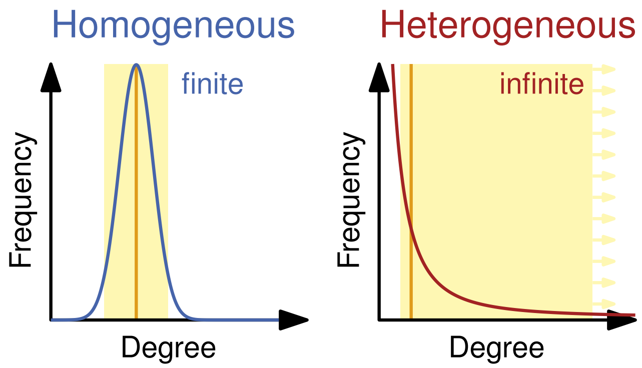

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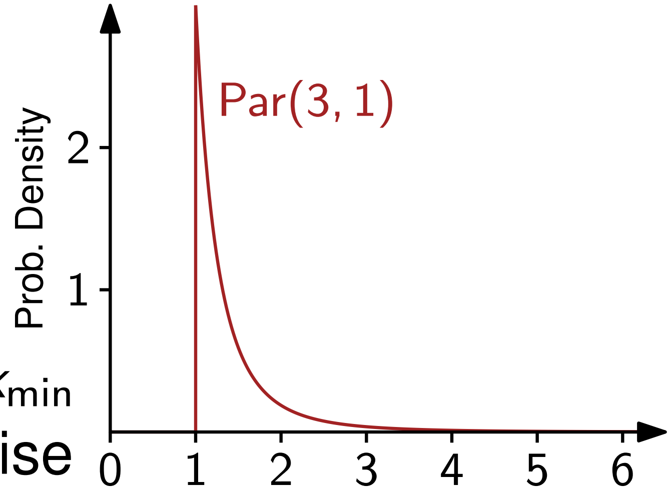

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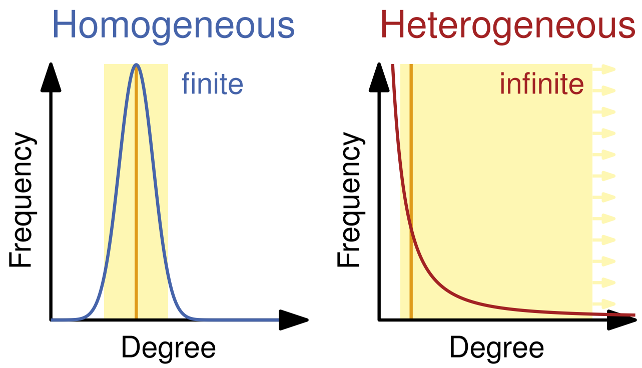

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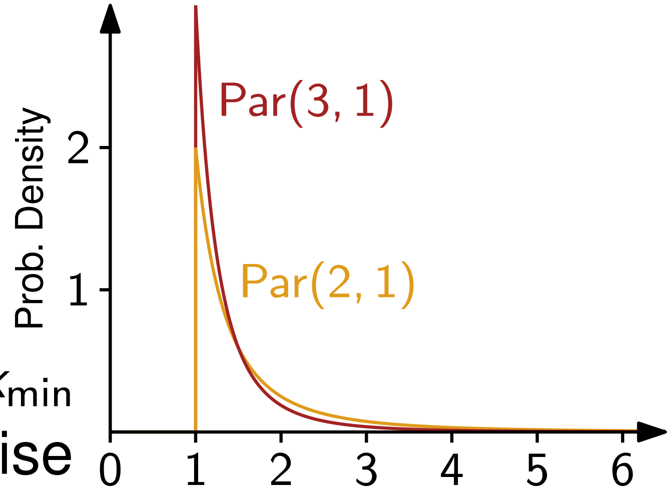

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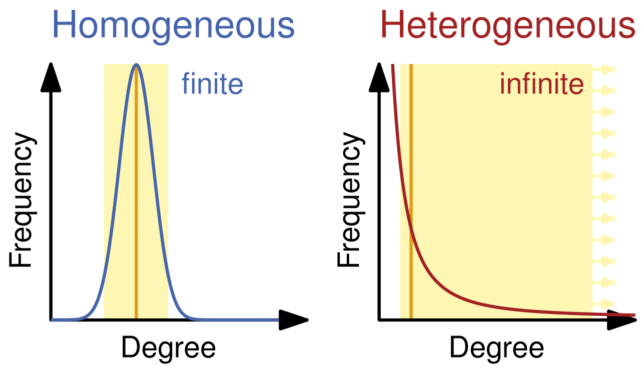

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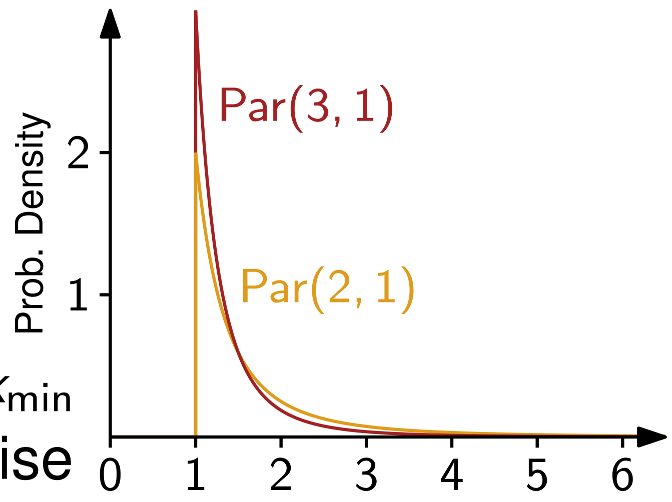

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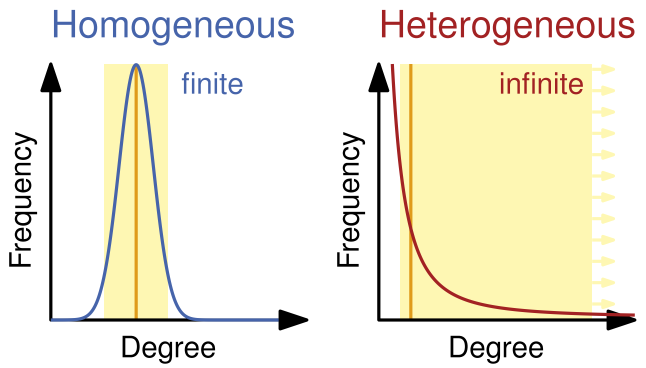

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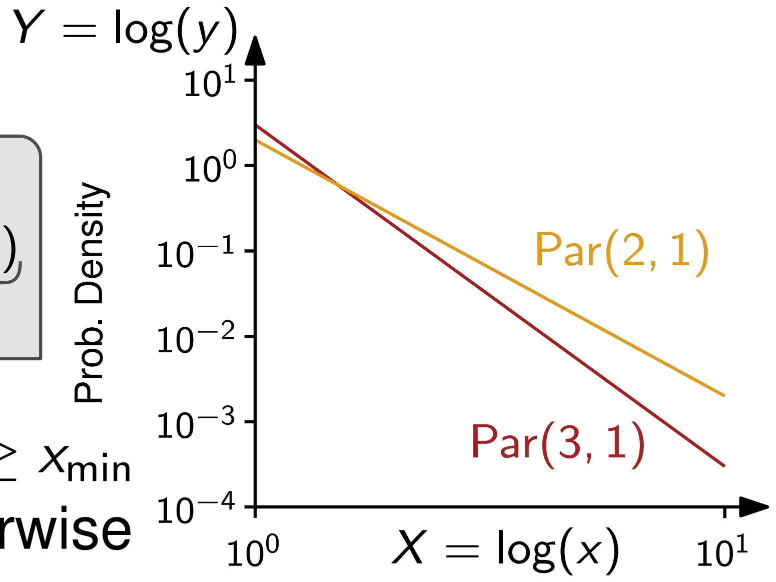

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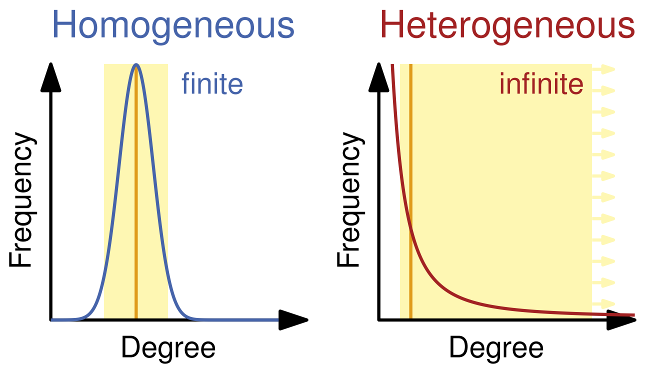
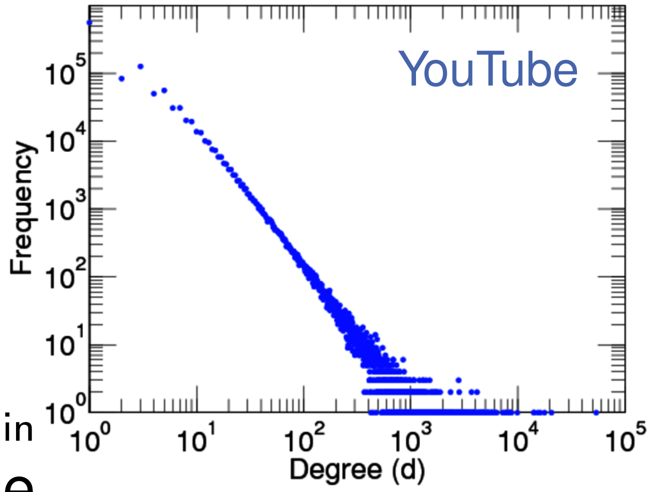


konect.cc/plot/degree.a.youtube-links.full.png



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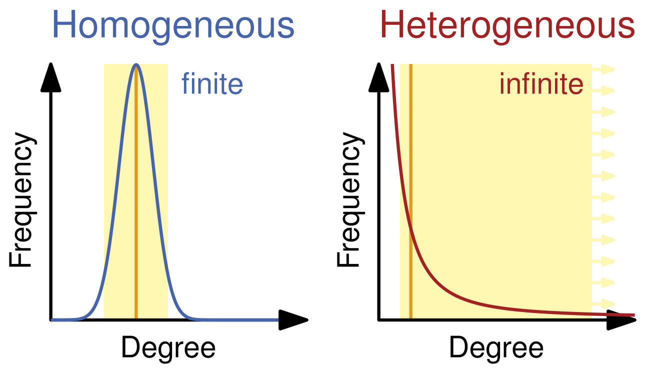

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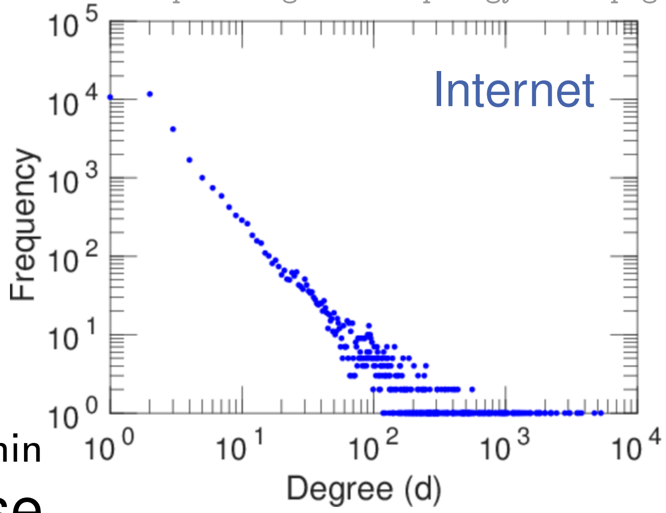

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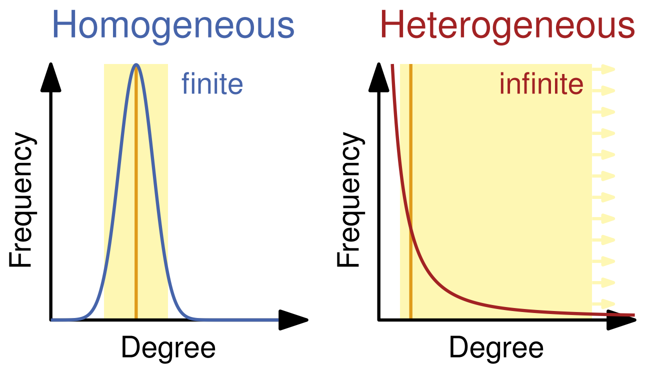

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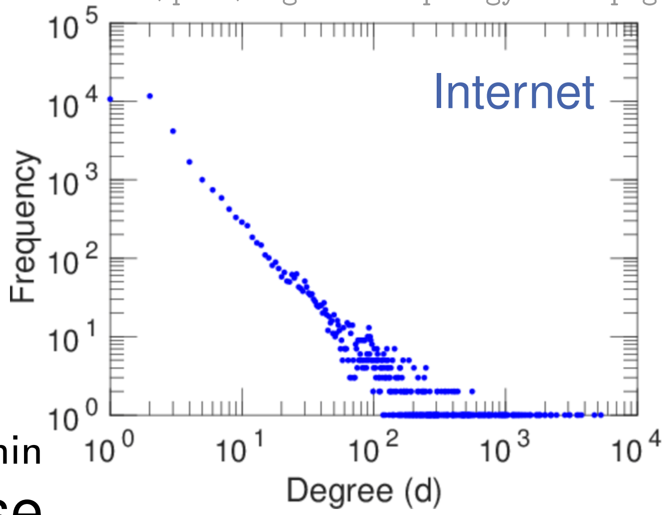

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Exercise: Determine for which values of α we have $\mathbb{E}[X] < \infty$ but $\mathrm{Var}[X] = \infty$

Conclusion

Continuous Distributions

- For our purposes they are handled like discrete versions (replacing sums with integrals)
- Seen today: Uniform distribution, exponential distribution, Pareto distribution, joint distributions

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- Number of points is a random variable
- Conditioned on certain number, points are distributed uniformly

We can simulate a PPP by drawing number according to Poisson distribution and distributing as many points uniformly

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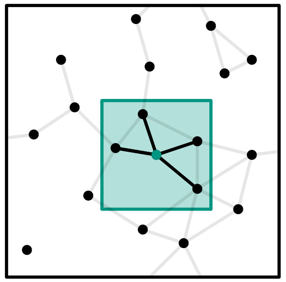

Random Geometric Graphs

- Vertices distributed at random in metric space
- Edges form with probability depending on distances

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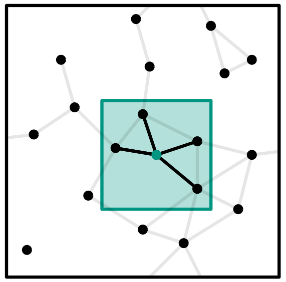

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Poisson (Point) Process (not discussed in lecture)

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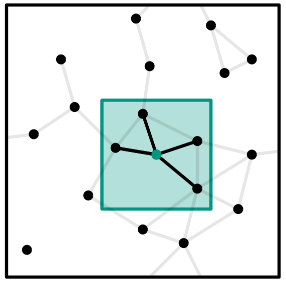

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Outlook: More realistic extension of RGGs featuring a heterogeneous degree distribution