Karlsruhe Institute of Technology

Probability & Computing

Concentration

Expectation Management

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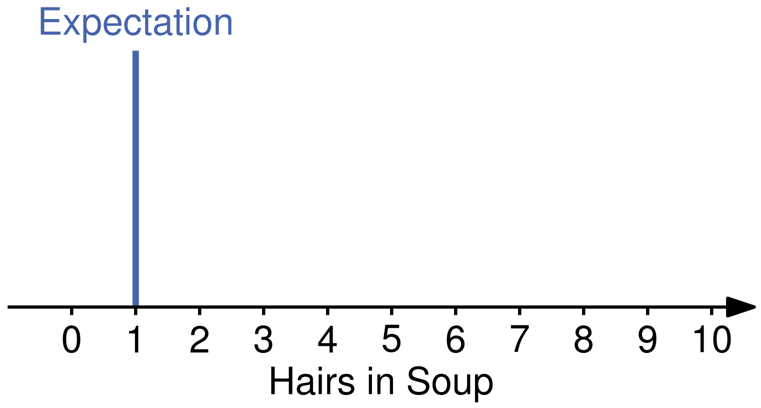

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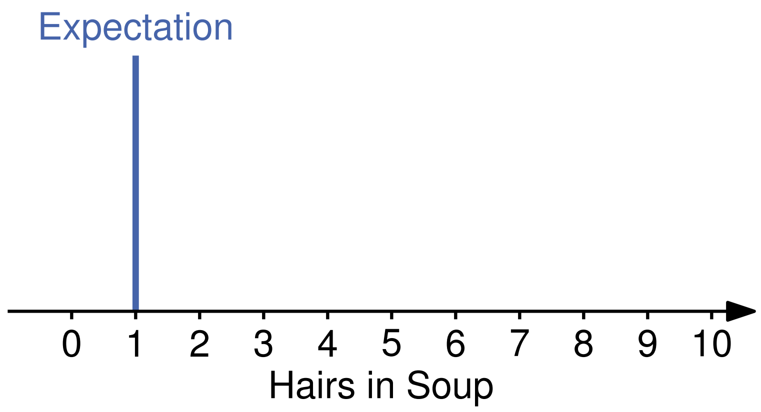

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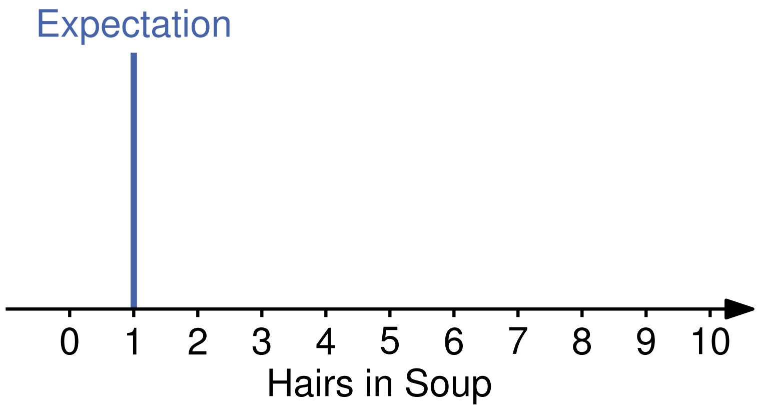

I "expect" the sniper to hit the target...

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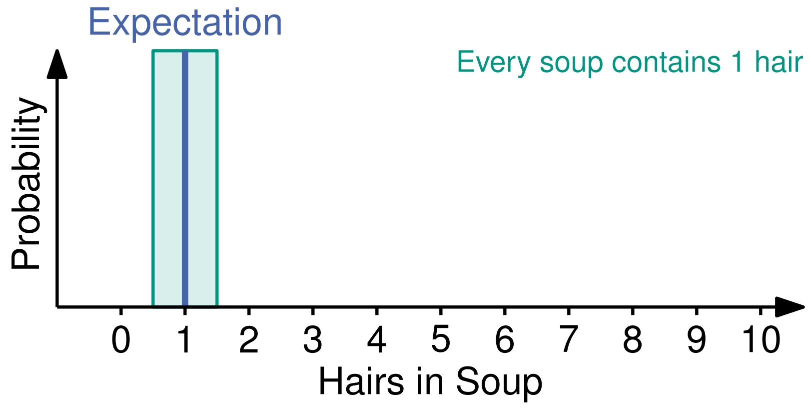

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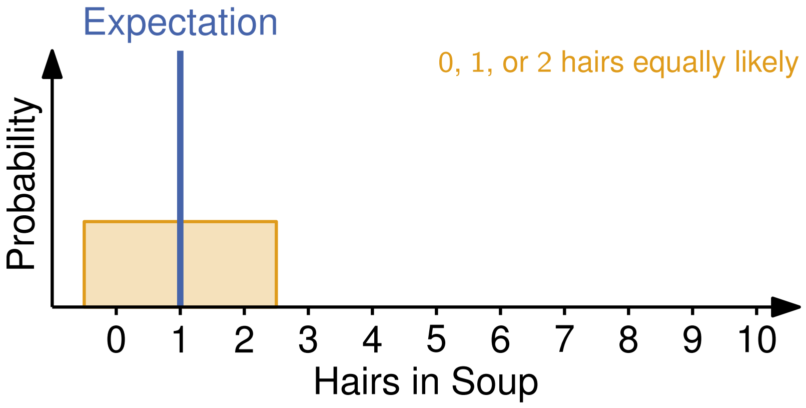

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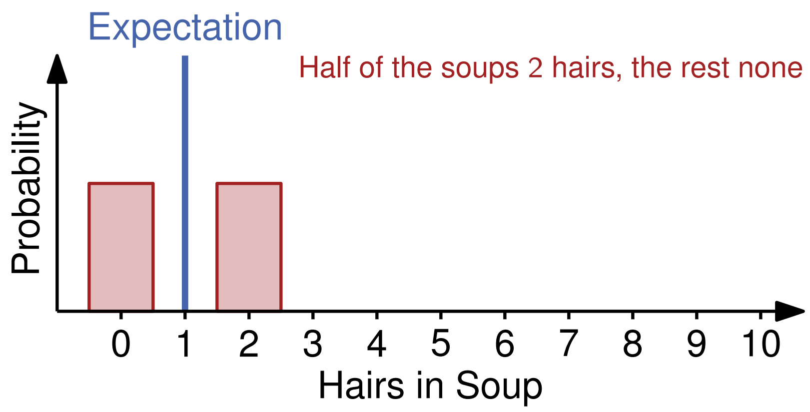

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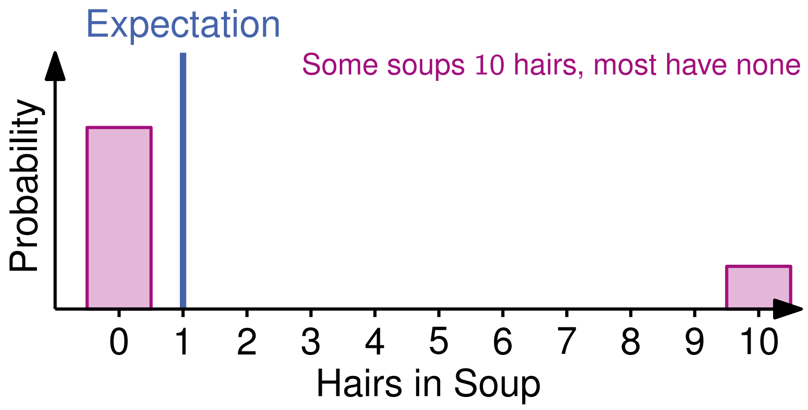

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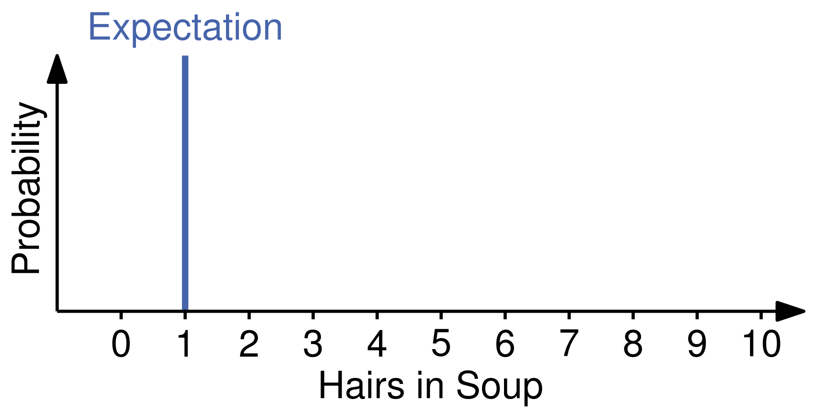

Knowing that the expected value is 1 hair:

How likely is it that I get at least 10?

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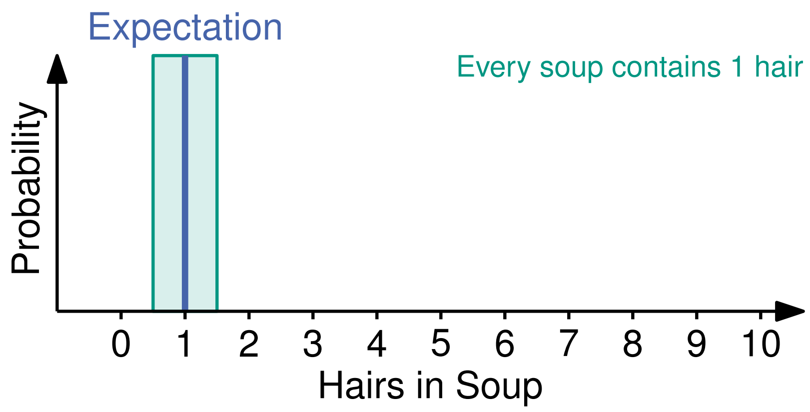

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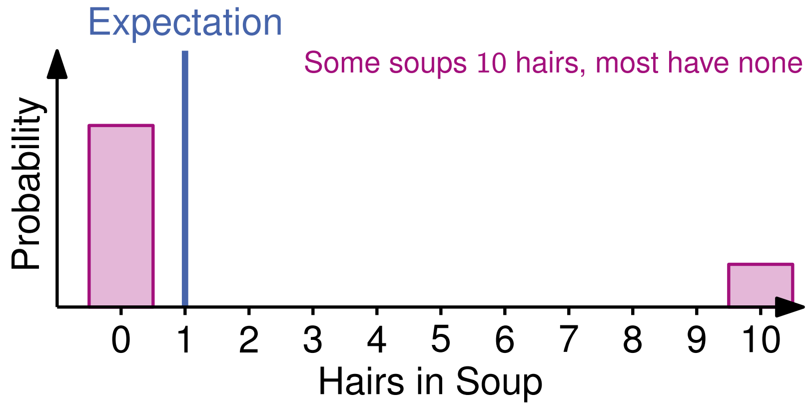

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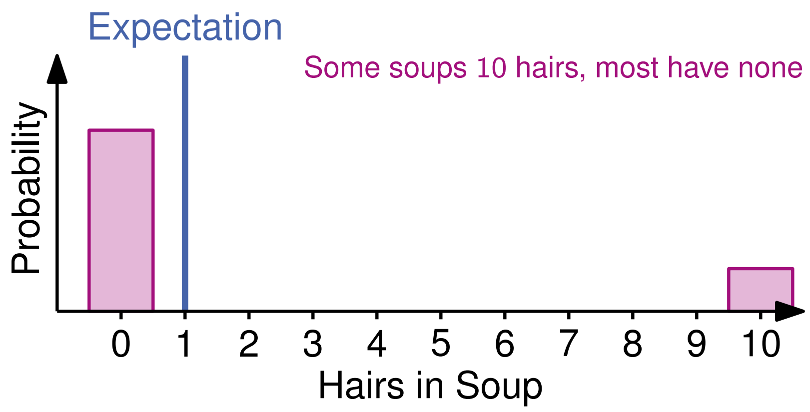

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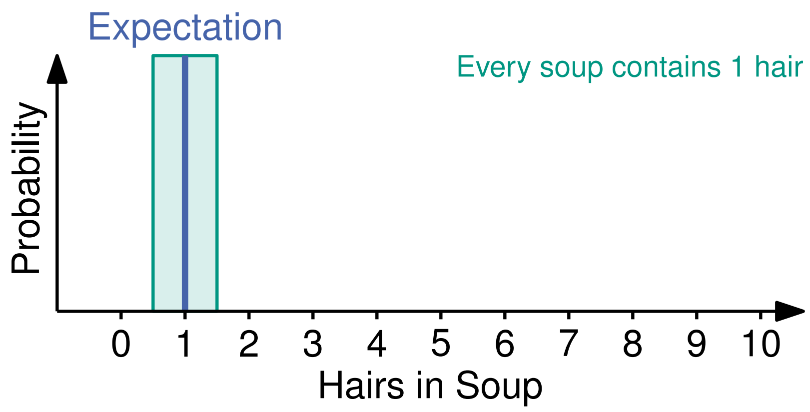

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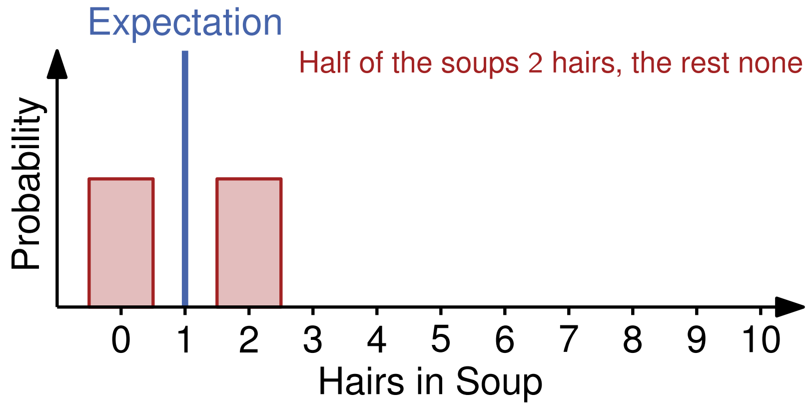

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- How useful is that information in practice?
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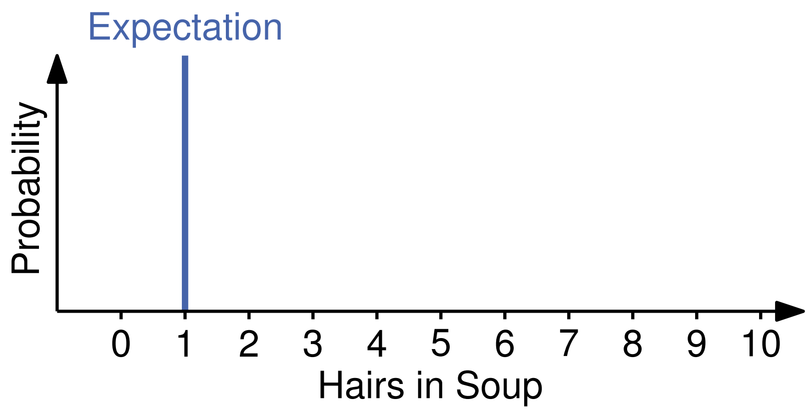

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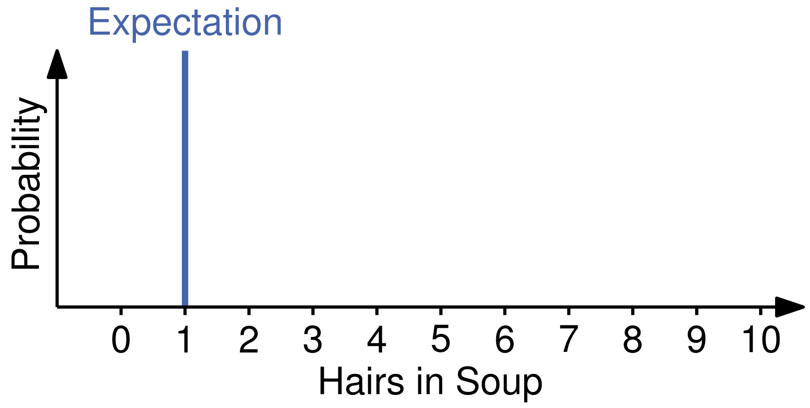

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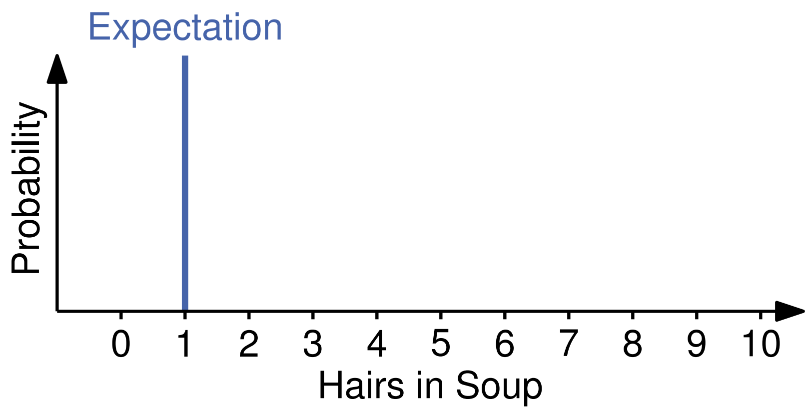

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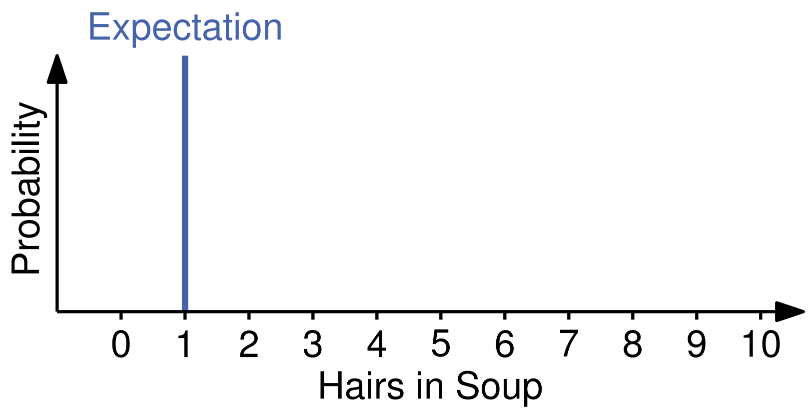

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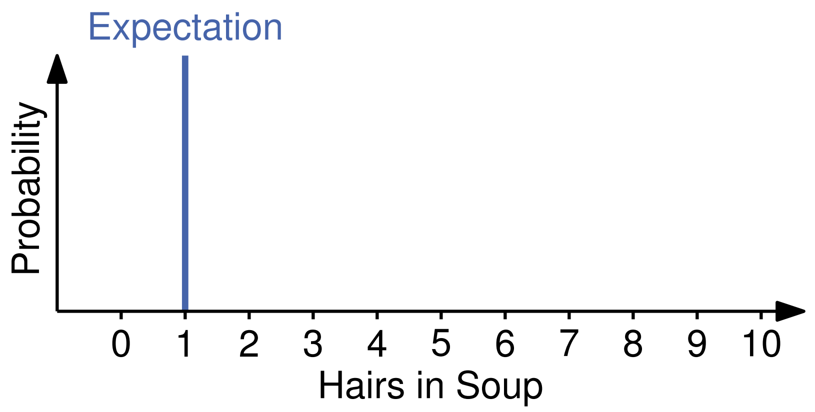

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> **Definition**: A **concentration inequality** bounds the probability of a random variable to deviate from a given value (typically its expectation) by a certain amount.

Markov's Inequality

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About Markov

- Andrei "The Furious" Andreyevich Markov (Russian mathematician)
- Unhappy with the state of living at the time (1921)
- Informed St. Petersburg Academy of Sciences that he could not attend meetings due to not having shoes
- After getting shoes from the Communist Party he replied:

  Finally, I received footwear. However, it is stupidly stitched together and does not accord with my measurements. Thus I cannot attend the meetings. I propose placing the footwear in a museum, as an example of the material culture of the current time.

"Shape, The hidden geometry of absolutely everything", Jordan Ellenberg

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Theorem (Markov's inequality): Let X be a non-negative random variable and let $a > 0$. Then, $\Pr[X \geq a] \leq \mathbb{E}[X]/a$.

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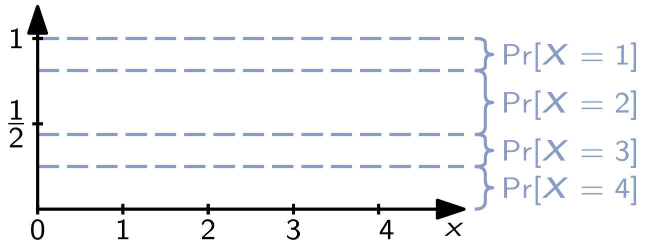

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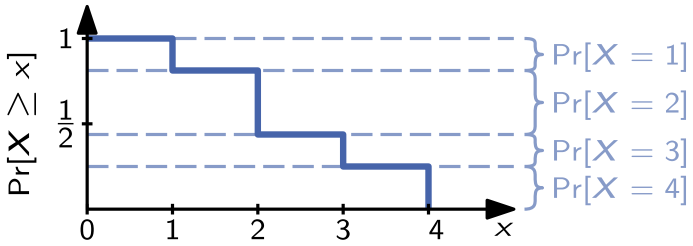

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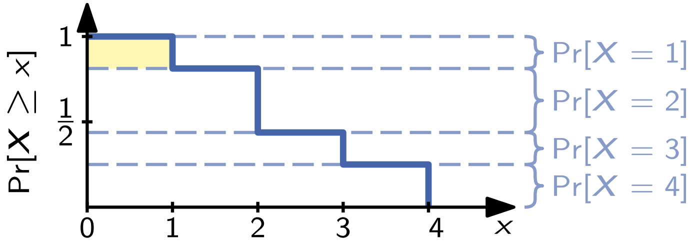

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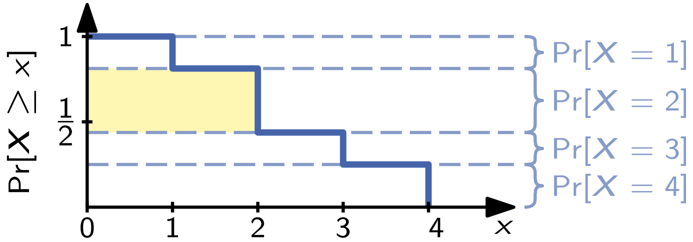

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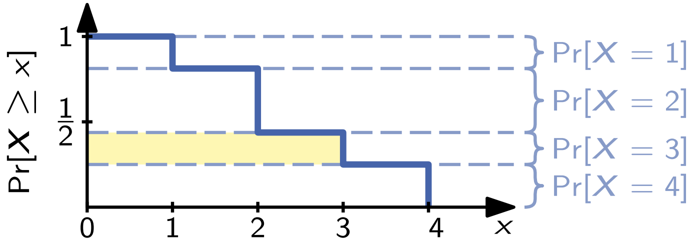

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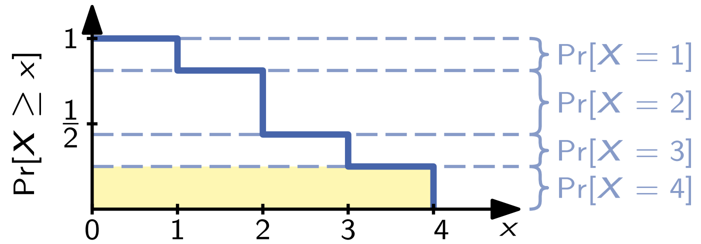

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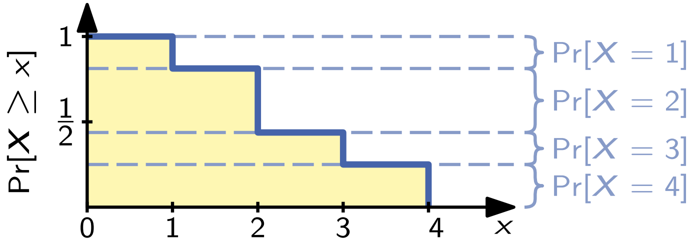

$\sum_x x \cdot \Pr[X = x]$

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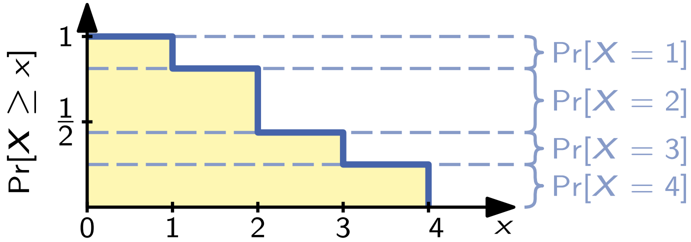

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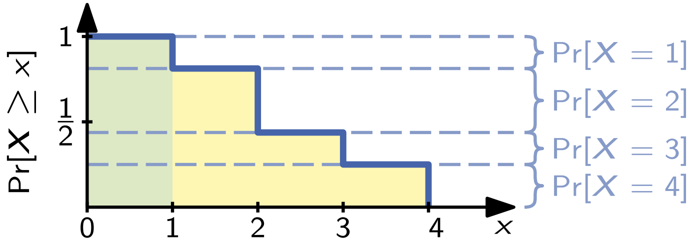

$\mathbb{E}[X] = \sum_x x \cdot \Pr[X = x]$ $\quad$ $1 \cdot \Pr[X \geq 1]$

fits into

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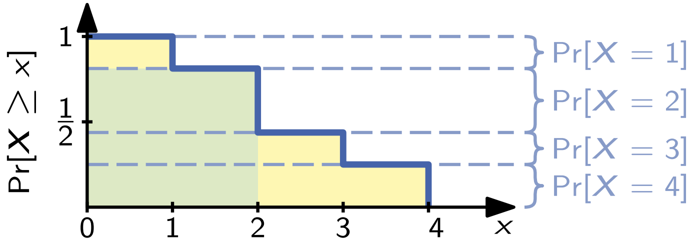

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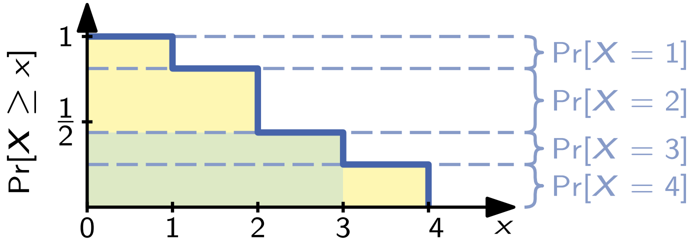

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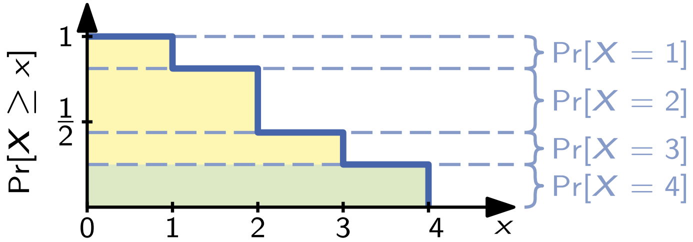

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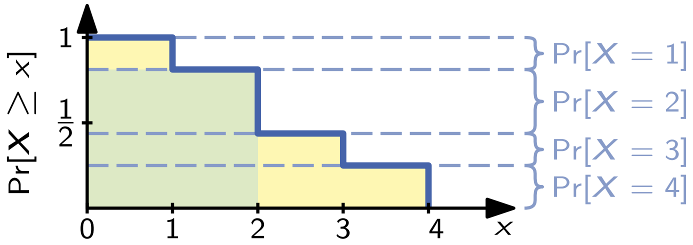

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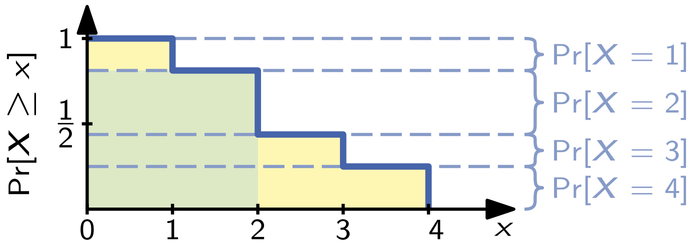

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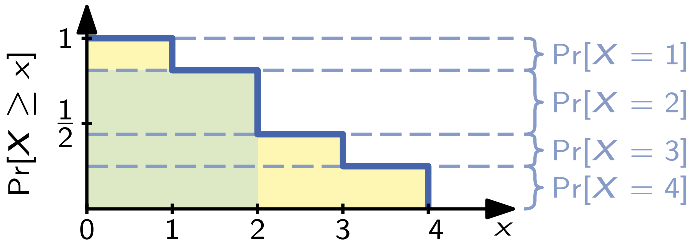

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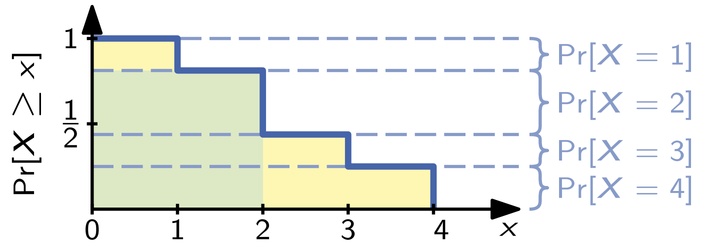

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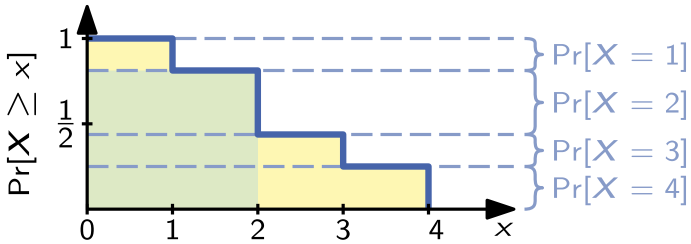

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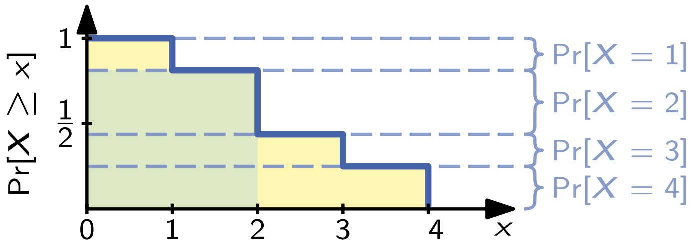

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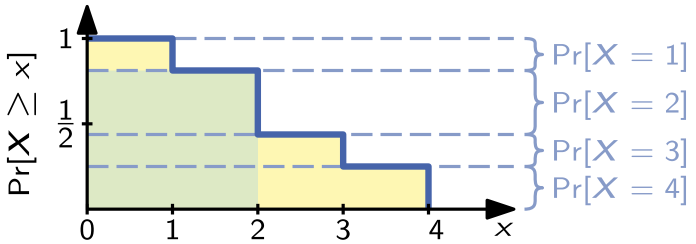

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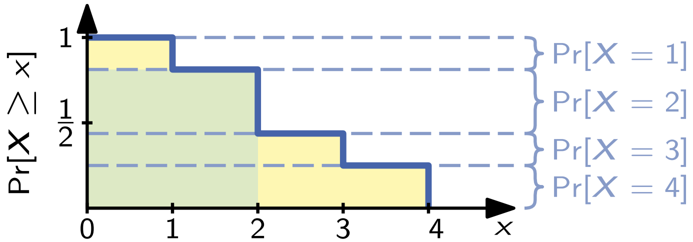

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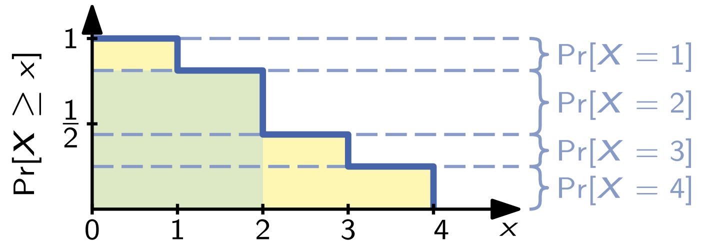

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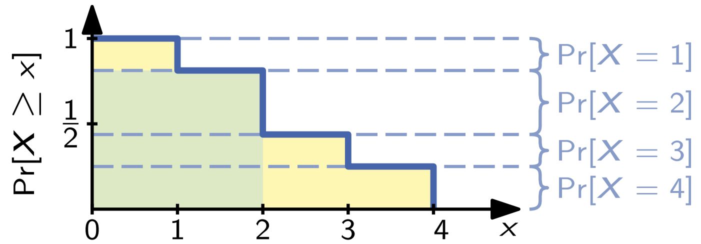

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Oh no...

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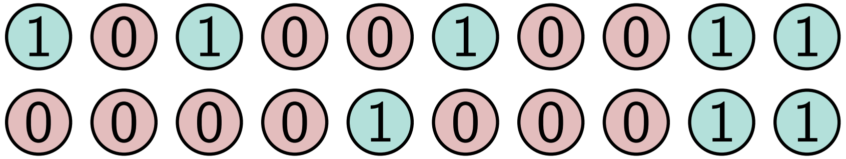

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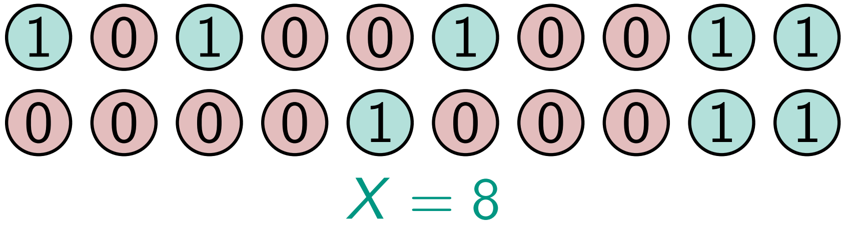

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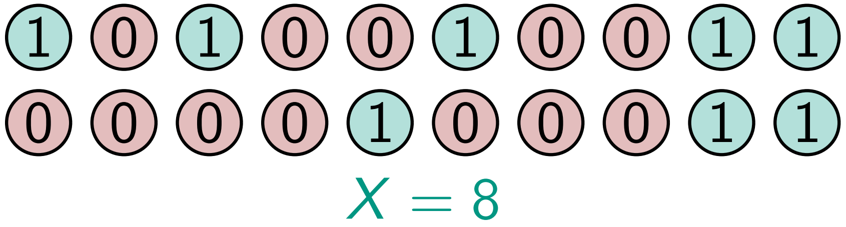

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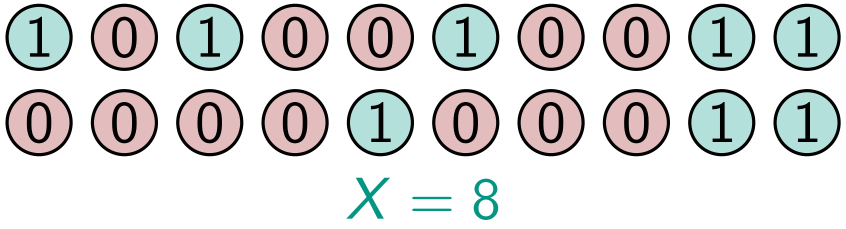

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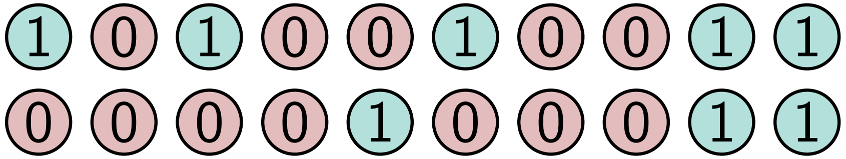

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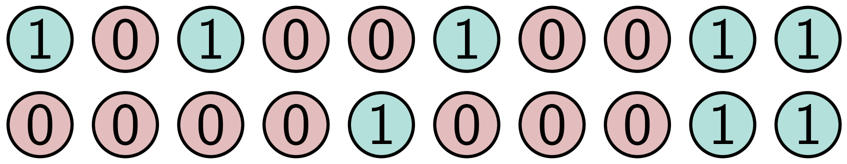

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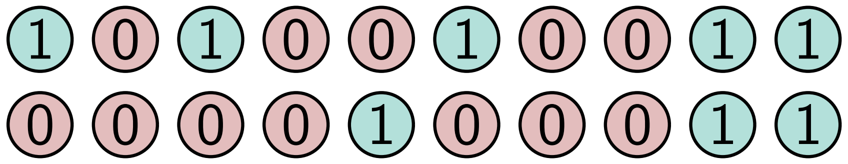

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Karlsruhe Institute of Technology

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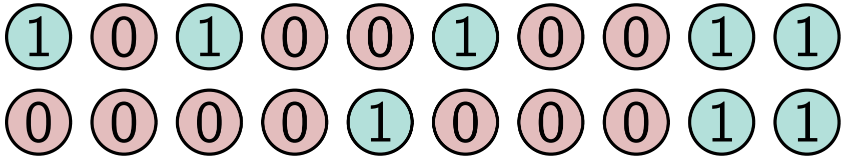

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$\Rightarrow$ There is no better bound (that relies only on the expected value)

We need more information about the shape of the distribution!

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- How much information do we need to characterize the shape of a distribution?

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Example

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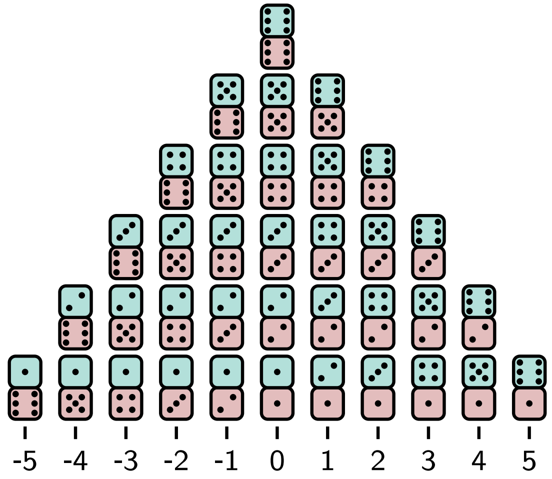

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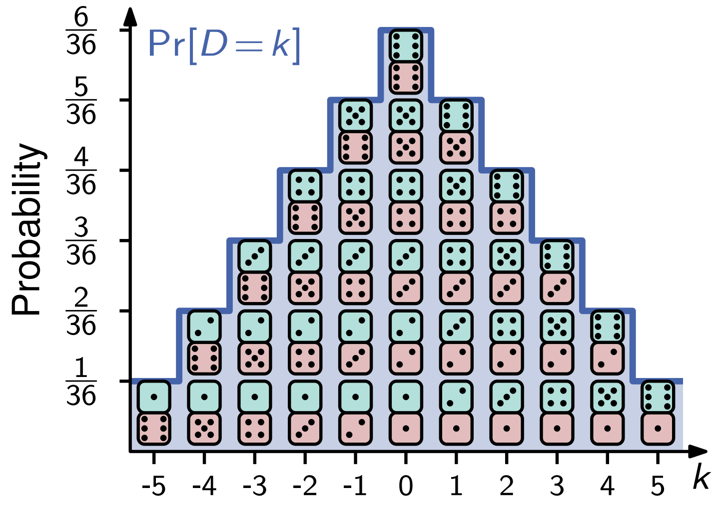

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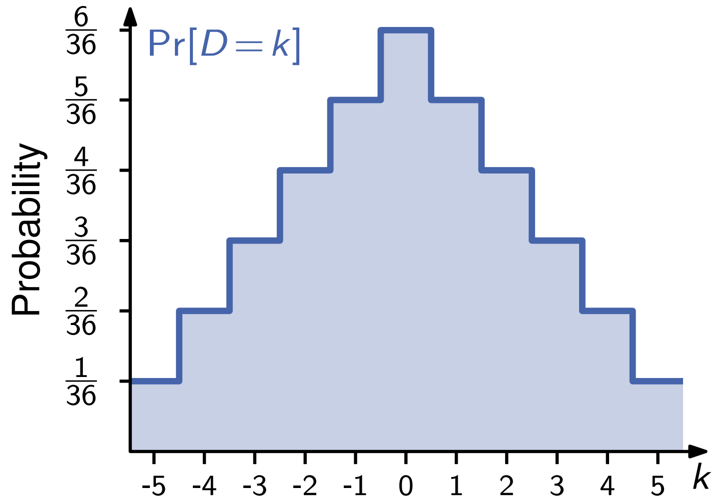

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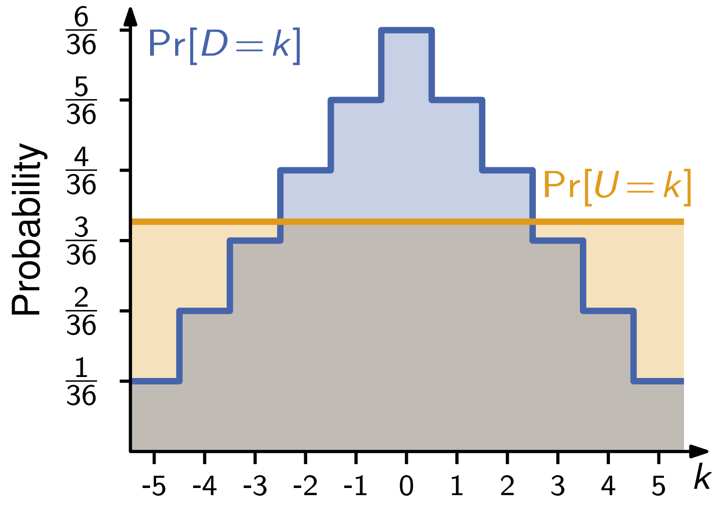

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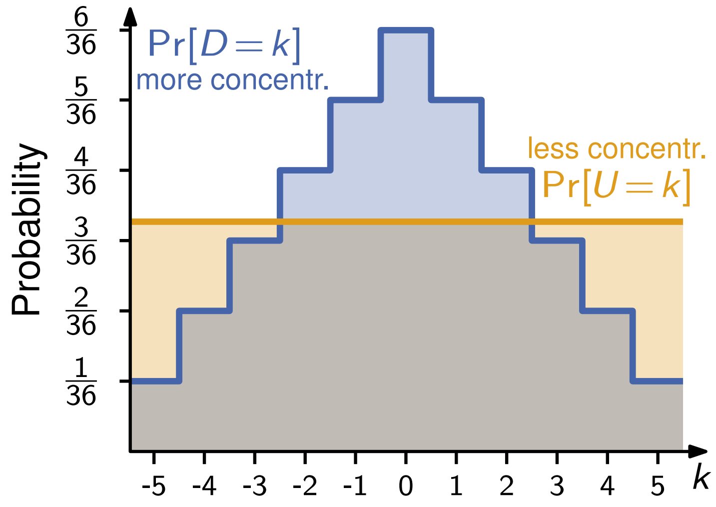

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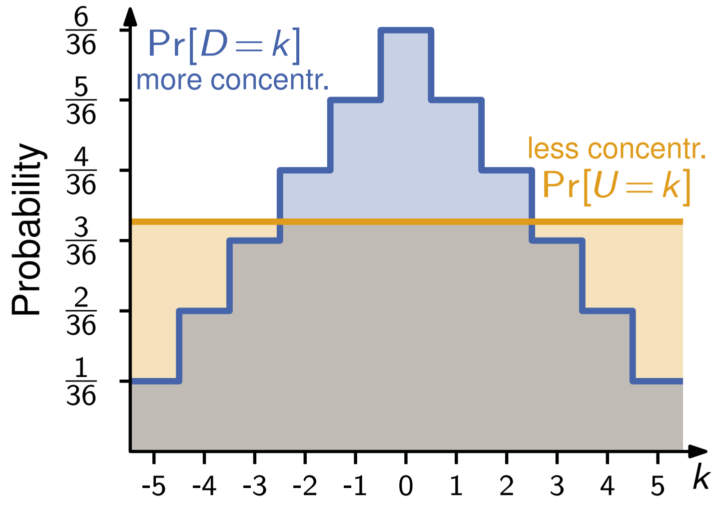

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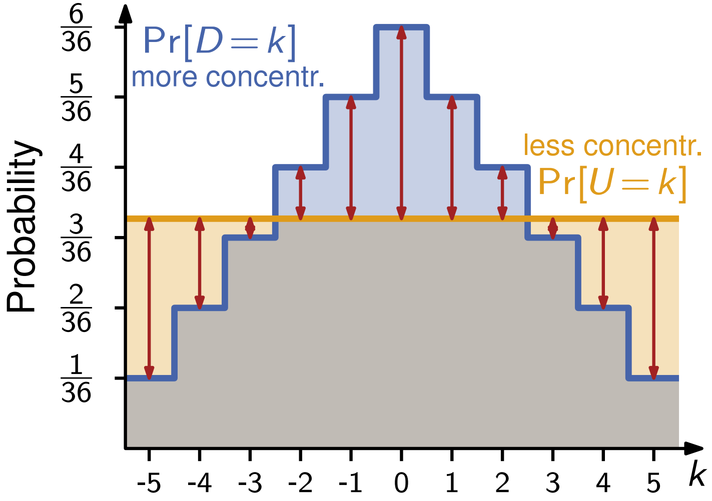

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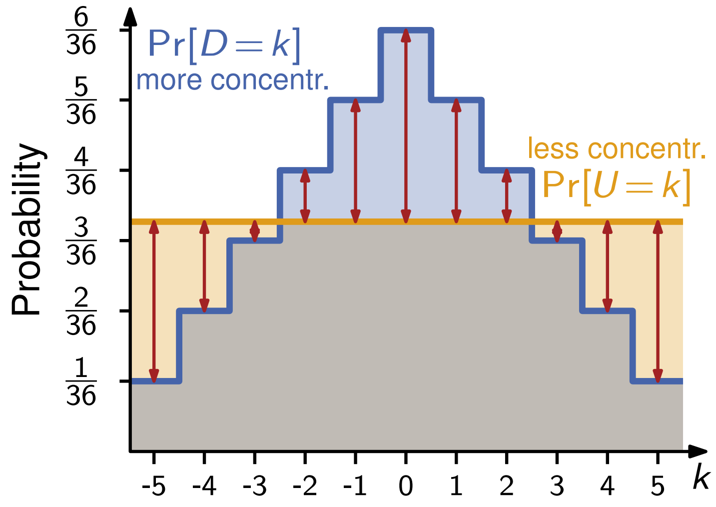

Characterizing the Shape of a Distribution

- How much information do we need to characterize the shape of a distribution?

Example

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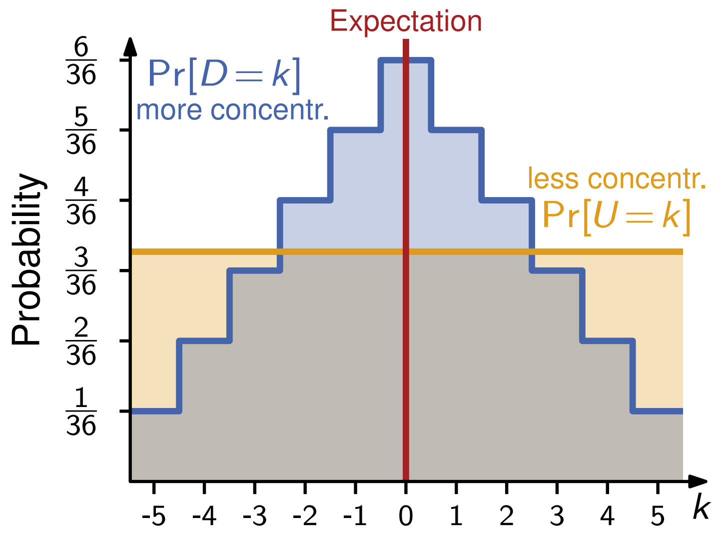

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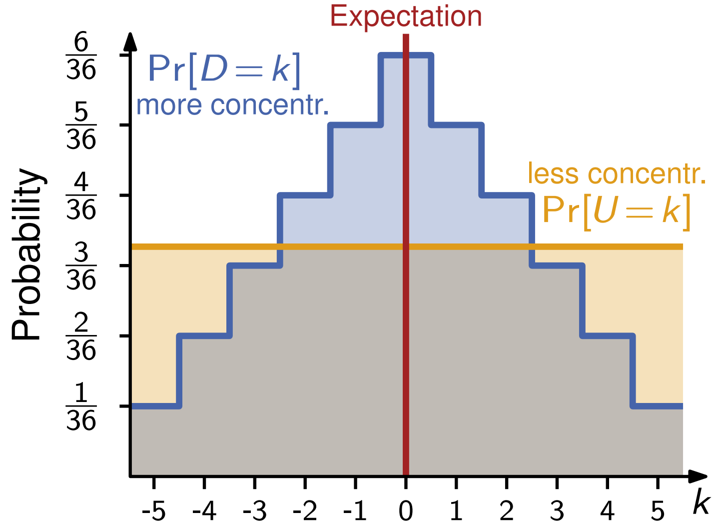

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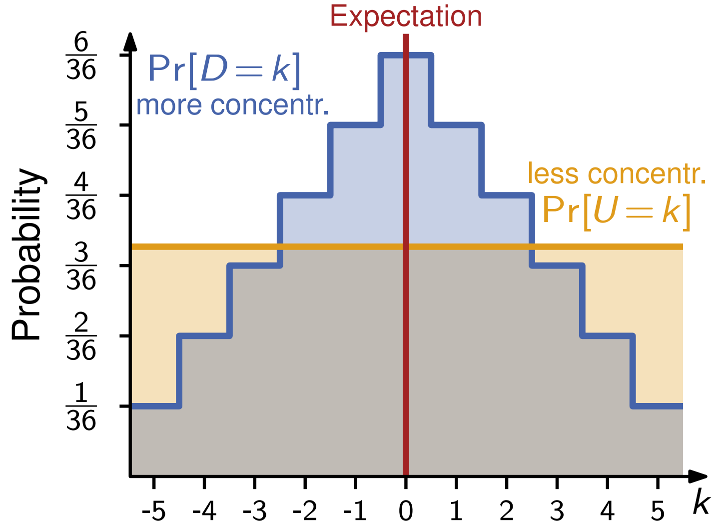

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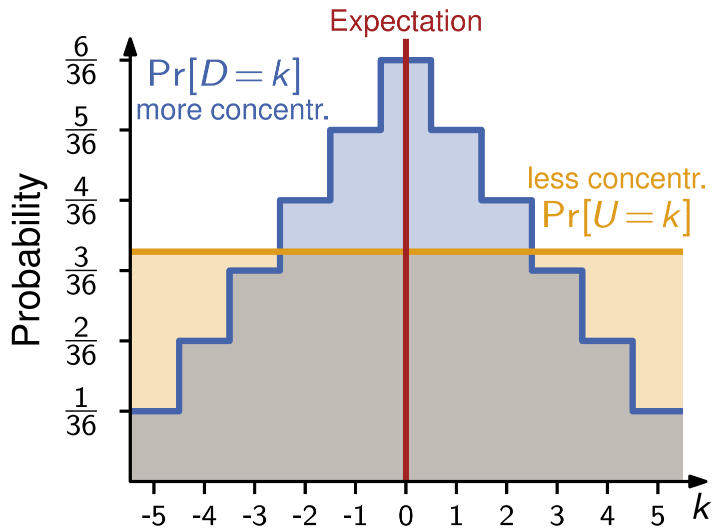

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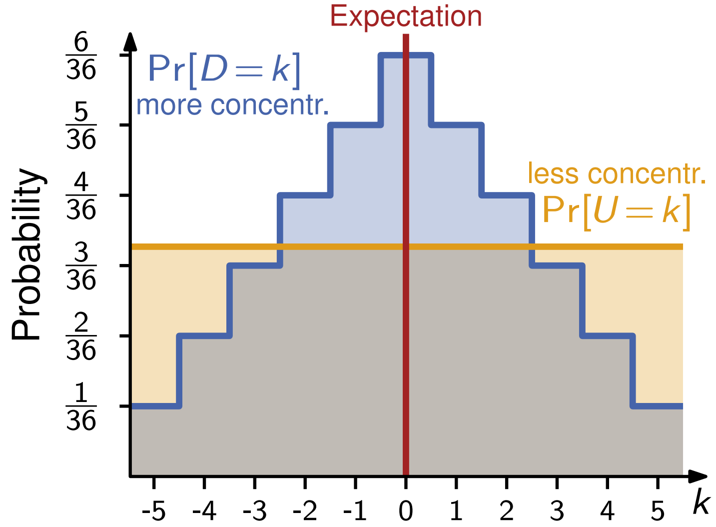

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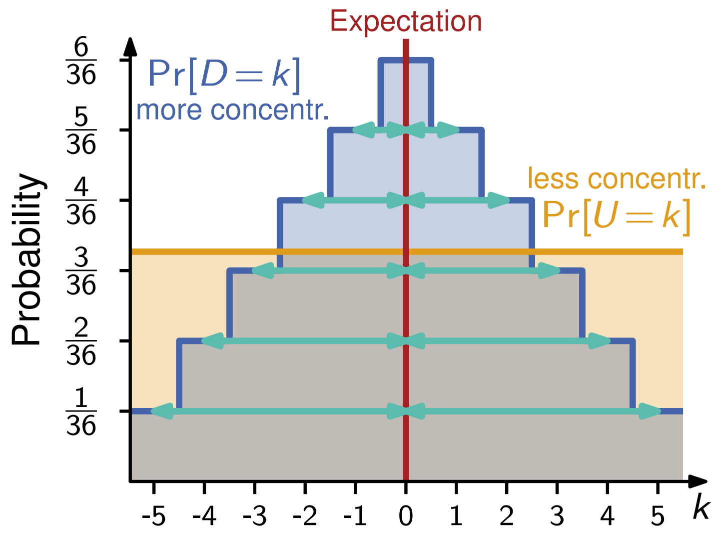

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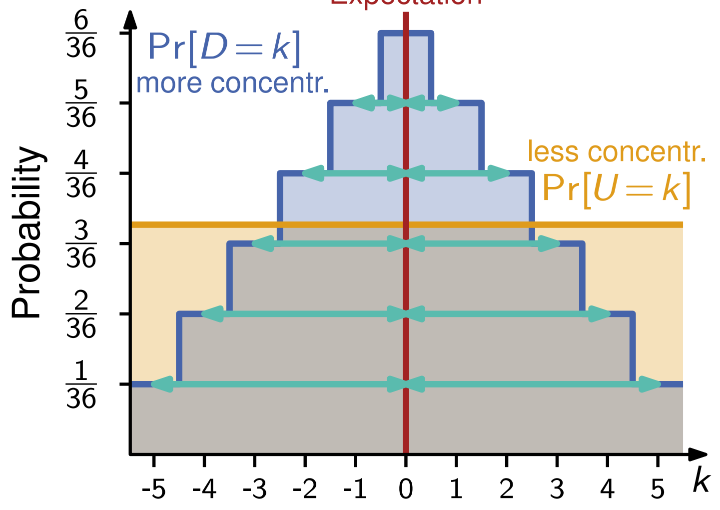

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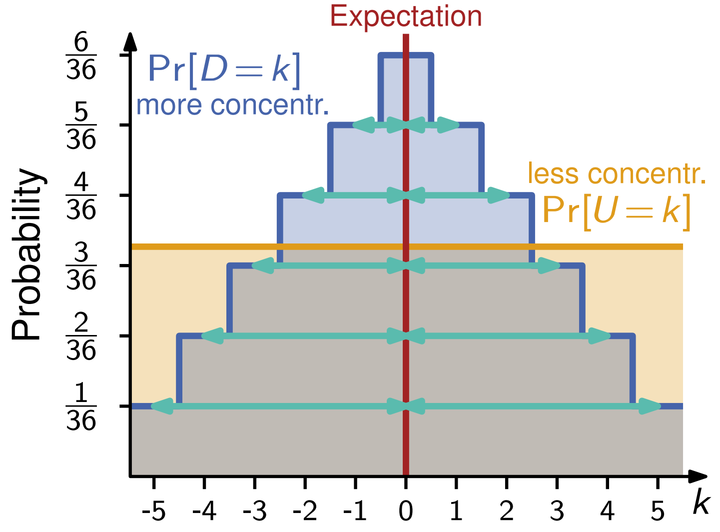

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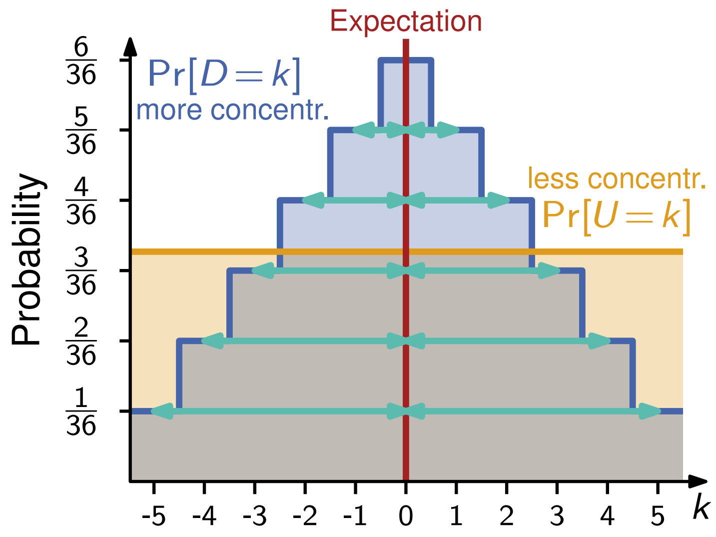

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- Problem: Nobody likes absolute value

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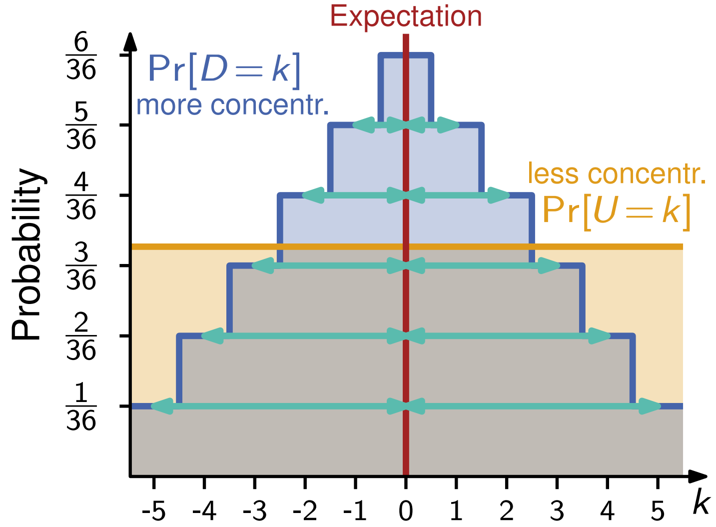

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  $\Rightarrow$ Fix: square instead

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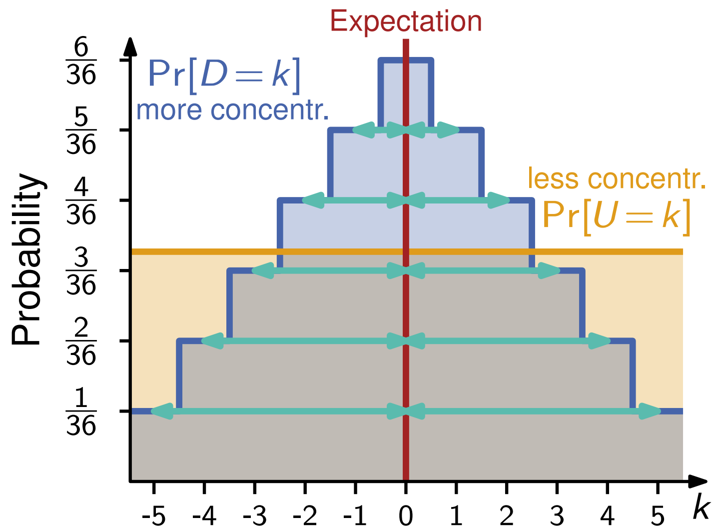

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$$\mathbb{E}[D^2] = \sum_k \Pr[D = k] \cdot k^2 \approx 5.833$$
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Squared distance to $\mathbb{E}$

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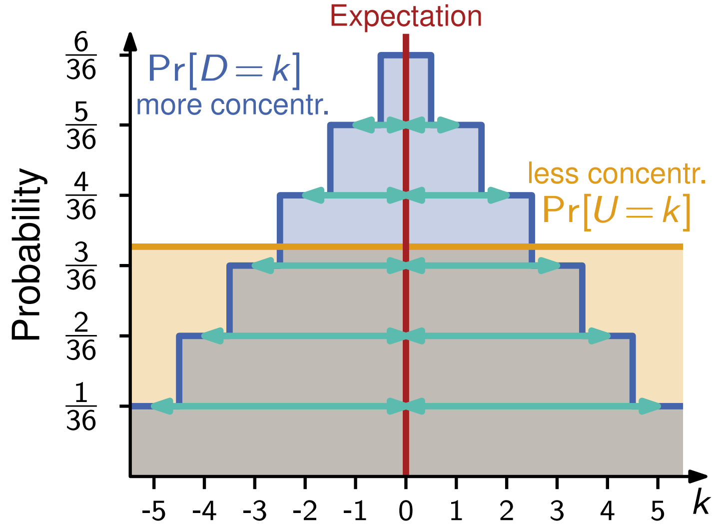

Expectation? $f(k) = k$

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Squared distance to $\mathbb{E}$

These are just expectations of functions of random variables!

Do you have a Moment?

Expectation and Functions

- Random variable X taking values in a set S
- A function f, e.g. $f(X) = X$, $f(X) = |X|$, $f(X) = X^2$, $f(X) = \sqrt{X}$, $f(X) = X^3$, $f(X) = e^X$
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$$\mathbb{E}[(X - \mathbb{E}[X])^2] = \text{Var}[X]$$

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- The smaller the variance, the more concentrated the random variable

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Karlsruhe Institute of Technology

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- The smaller the variance, the more concentrated the random variable

... and with Markov's help, we can turn that insight into a concentration inequality!

Chebychev's Inequality

Markov's teacher! (Markov's inequality actually appeared earlier in Chebychev's works)

Theorem (Chebychev's inequality): Let X be a random variable with finite variance and let $b > 0$. Then, $\Pr[|X - \mathbb{E}[X]| \geq b] \leq \mathrm{Var}[X]/b^2$.

Markov: $Y \geq 0$, $a > 0$: $\Pr[Y \geq a] \leq \mathbb{E}[Y]/a$

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Application: Unfair Coins

1 0 1 0 0 1 0 0 1 1 0 0 0 0 1 0 0 0 1 1

- $X \sim \mathrm{Bin}(20, \frac{1}{5})$, $\Pr[X \geq 16]$? | $\mathbb{E}[X] = 20 \cdot \frac{1}{5} = 4$

  $\Pr[X \geq 16] = \sum_{k=16}^{20} \binom{20}{k} \left(\frac{1}{5}\right)^k \cdot \left(1 - \frac{1}{5}\right)^{20-k} \approx 0.0000000138$
- Markov: $\Rightarrow \Pr[X \geq 16] \leq \mathbb{E}[X]/16 = 0.25$

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Theorem (Chebychev's inequality): Let X be a random variable with finite variance and let $b > 0$. Then, $\Pr[|X - \mathbb{E}[X]| \geq b] \leq \mathrm{Var}[X]/b^2$.

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Application: Unfair Coins

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Order of magnitude better than Markov!

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Application: ER – Degree Distribution

Karlsruhe Institute of Technology

Recap

- $G(n, p)$: Start with n nodes, connect any two with fixed probability p, independently
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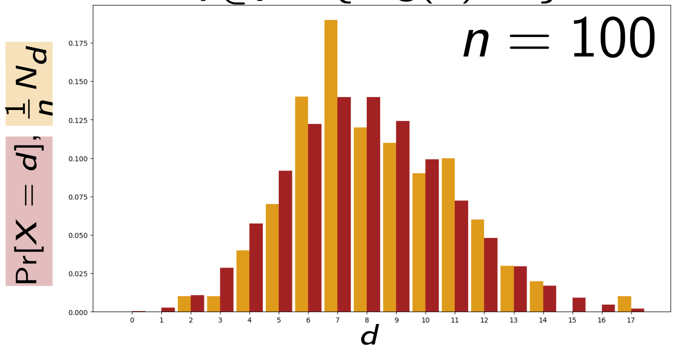

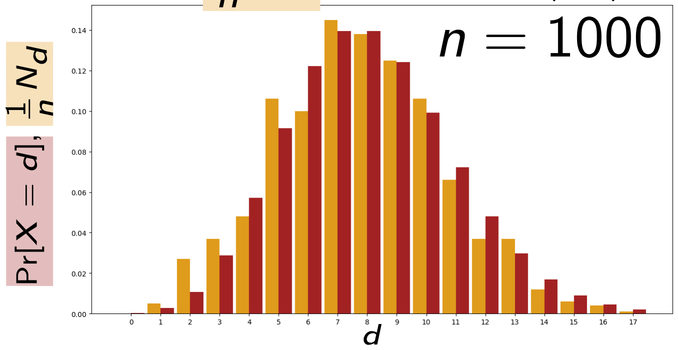

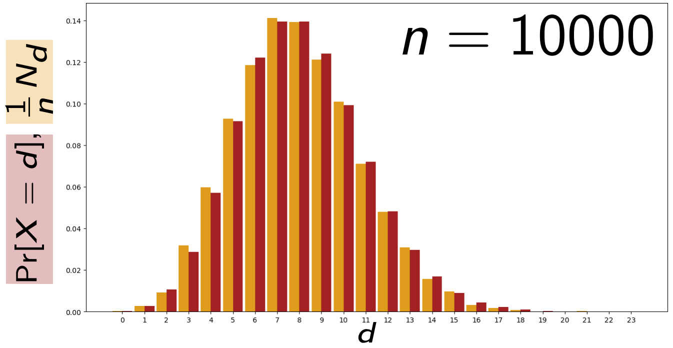

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Theorem: Consider a $G(n, p)$ with $p = c/n$ for constant $c > 0$. For $\lambda = -n \log(1 - p)$, let $X \sim \text{Pois}(\lambda)$. Then for all $d > 0$ and every $\varepsilon > 0$ we have

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Chebychev: X finite variance, $b > 0$
$\Pr[|X - \mathbb{E}[X]| \geq b] \leq \mathrm{Var}[X]/b^2$

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$$\Pr\left[\left|\mathbb{E}\left[\frac{1}{n}N_d\right] - \frac{1}{n}N_d\right| \geq \varepsilon\right]$$

$$\lim_{n\to\infty} \Pr\left[\left|\mathbb{E}\left[\frac{1}{n}N_d\right] - \frac{1}{n}N_d\right| \geq \varepsilon\right] = 0$$

$N_d \in \{0, \ldots, n\}$

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$= (n\Pr[\deg(v) = d])^2$ (see Step 1)

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&= \tfrac{1}{n^2}(n\Pr[\deg(v) = d] \\
&\quad + n(n-1)\Pr[\deg(v) = d \wedge \deg(u) = d] \\
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&= \tfrac{1}{n}\Pr[\deg(v)=d] \\
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&= \tfrac{1}{n^2}\left(\mathbb{E}[(N_d)^2] - \mathbb{E}[N_d]^2\right) \\
&= \tfrac{1}{n^2}\big(n\Pr[\deg(v)=d] \\
&\qquad + n(n-1)\Pr[\deg(v)=d \wedge \deg(u)=d] \\
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&= \tfrac{1}{n}\Pr[\deg(v)=d] \qquad\qquad \leq 1 \\
&\qquad + \tfrac{n-1}{n}\Pr[\deg(v)=d \wedge \deg(u)=d] \\
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\end{aligned}$$

$$\lim_{n\to\infty} \Pr\left[\left|\mathbb{E}\left[\tfrac{1}{n}N_d\right] - \tfrac{1}{n}N_d\right| \geq \varepsilon\right] = 0$$

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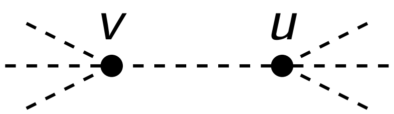
Couplings

- Consider $\deg(u)$ and $\deg(v)$



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Couplings

- Consider $\deg(u)$ and $\deg(v)$
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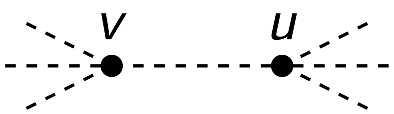

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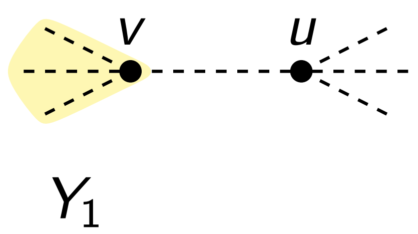

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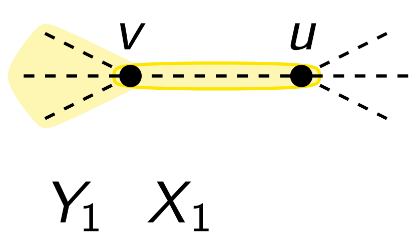

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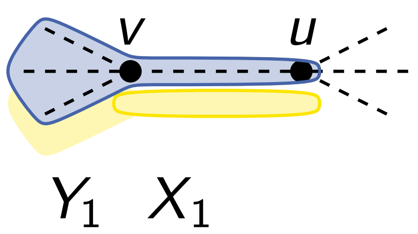

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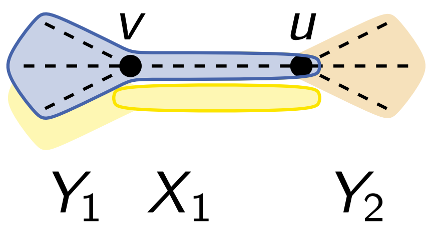

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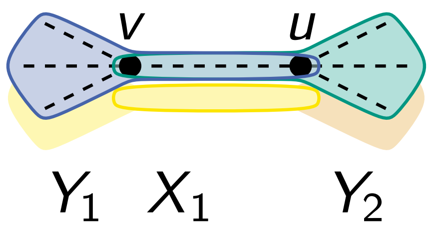

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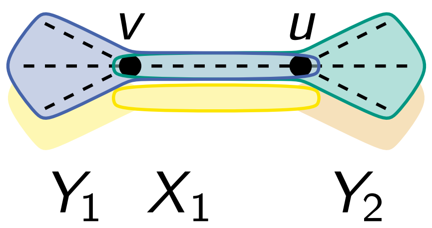

$\deg(v)$ ← dependent → $\deg(u)$
$\overset{d}{=}$ $\overset{d}{=}$
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Step 2: Concentration of $\frac{1}{n} N_d$

$$\Pr\left[\left|\mathbb{E}\left[\tfrac{1}{n}N_d\right] - \tfrac{1}{n}N_d\right| \geq \varepsilon\right] \leq \underbrace{\mathrm{Var}\left[\tfrac{1}{n}N_d\right]}/\varepsilon^2$$

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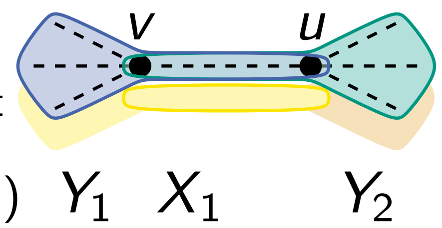

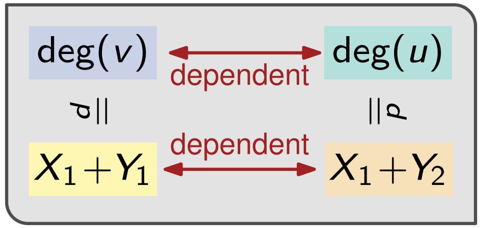

Step 2: Concentration of $\frac{1}{n}N_d$

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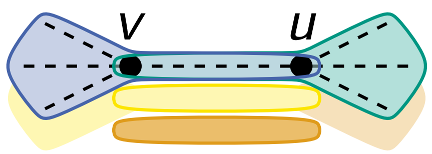

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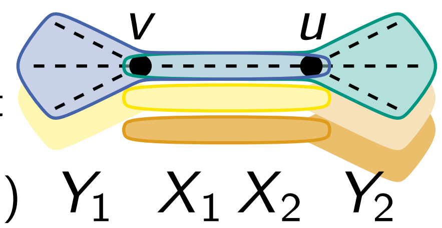

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X_1+Y_1	← dependent →	X_1+Y_2

$\deg(u)$
$\stackrel{d}{=}$
X_2+Y_2

Step 2: Concentration of $\frac{1}{n}N_d$

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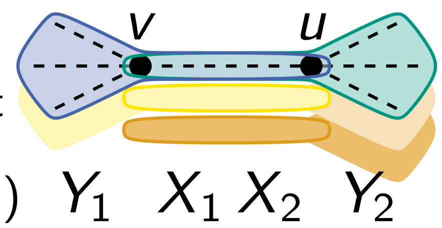

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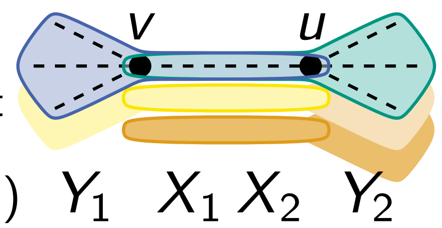

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$\overset{d}{\parallel}$		$\overset{d}{\parallel}$
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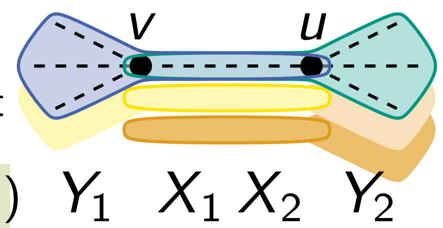

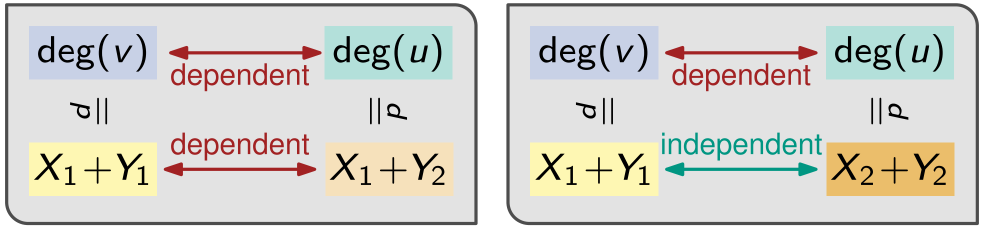

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Step 2: Concentration of $\frac{1}{n} N_d$

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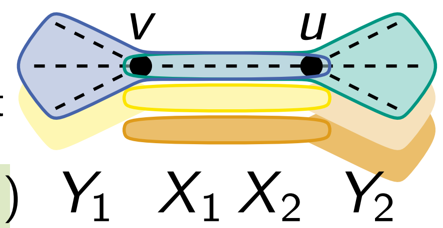

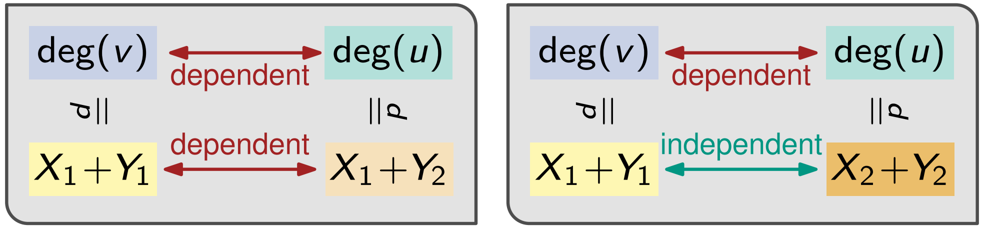

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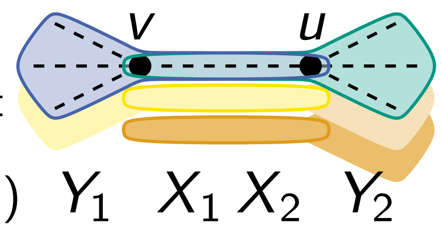

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X_1+Y_1	dependent	X_1+Y_2

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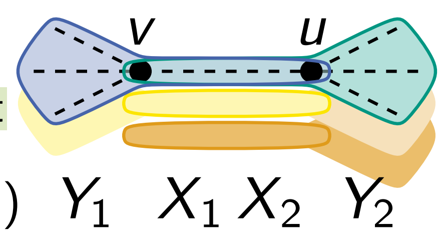

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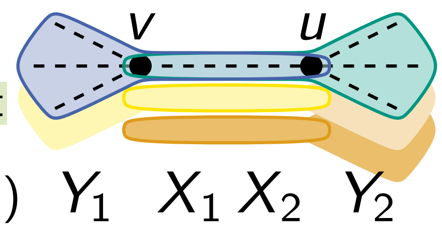

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&= \tfrac{1}{n} + \Pr[X_1 + Y_1 = d \wedge X_1 + Y_2 = d] \\
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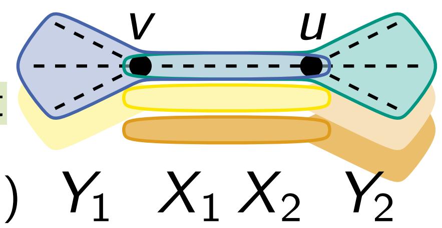

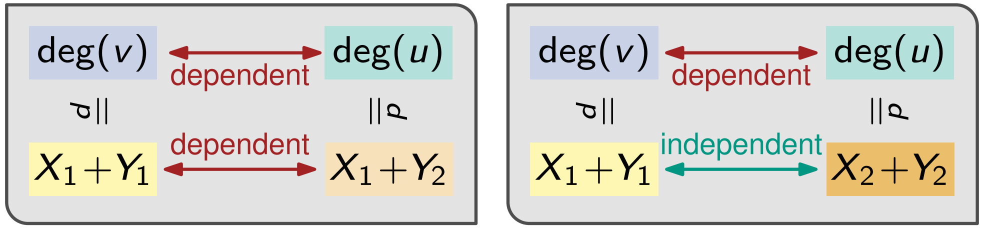

Karlsruhe Institute of Technology

Step 2: Concentration of $\frac{1}{n}N_d$

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\end{aligned}$$

For the whole event to occur, this needs to happen

Which excludes this from happening

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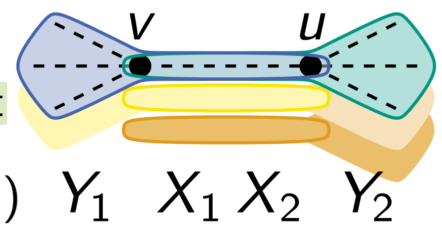

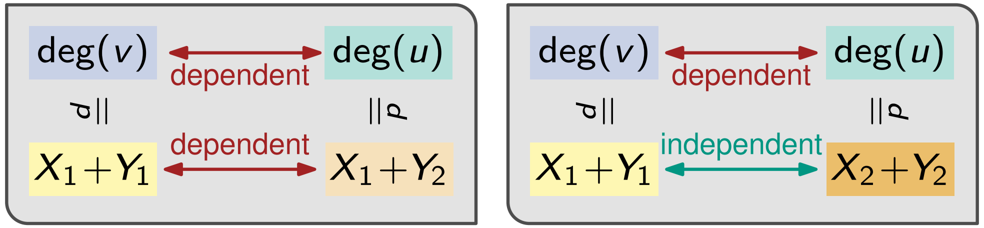

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&\quad + \Pr[X_1 + Y_1 = d \wedge X_1 + Y_2 = d \wedge X_2 + Y_2 \neq d \mid X_1 = 1]\Pr[X_1 = 1]
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$\Rightarrow X_2 = 1$

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$$\Pr\left[\left|\mathbb{E}\left[\tfrac{1}{n}N_d\right] - \tfrac{1}{n}N_d\right| \geq \varepsilon\right] \leq \underbrace{\mathrm{Var}\left[\tfrac{1}{n}N_d\right]}/\varepsilon^2$$

$$\begin{aligned}
\mathrm{Var}[\tfrac{1}{n}N_d] &= \mathbb{E}[(\tfrac{1}{n}N_d)^2] - \mathbb{E}[\tfrac{1}{n}N_d]^2 \\
&= \tfrac{1}{n^2}\left(\mathbb{E}[(N_d)^2] - \mathbb{E}[N_d]^2\right) \\
&\leq \tfrac{1}{n} + \Pr[\deg(v) = d \wedge \deg(u) = d] \\
&\quad - \Pr[\deg(v) = d]\Pr[\deg(u) = d] \quad \deg(v) \overset{d}{=} \deg(u) \\
&\leq \tfrac{1}{n} + \Pr[X_1 + Y_1 = d \wedge X_1 + Y_2 = d \wedge X_2 + Y_2 \neq d] \\
&= \tfrac{1}{n} + \Pr[X_1 + Y_1 = d \wedge X_1 + Y_2 = d \wedge X_2 + Y_2 \neq d \mid X_1 = 0]\,\overbrace{\Pr[X_1 = 0]}^{\leq 1} \\
&\quad + \underbrace{\Pr[X_1 + Y_1 = d \wedge X_1 + Y_2 = d \wedge X_2 + Y_2 \neq d \mid X_1 = 1]}_{\leq 1}\Pr[X_1 = 1] \\
&\leq \tfrac{1}{n} + \Pr[Y_1 = d \wedge Y_2 = d \wedge X_2 + Y_2 \neq d \mid X_1 = 0] \\
&\quad + \Pr[X_1 = 1] \\
&= \tfrac{1}{n} + \Pr[Y_1 = d \wedge Y_2 = d \wedge X_2 = 1 \mid X_1 = 0] + \Pr[X_1 = 1]
\end{aligned}$$

Law of total probability

$\Rightarrow X_2 = 1$

- $Y_1, Y_2 \sim \mathrm{Bin}(n-2, p)$
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independent

$$\lim_{n\to\infty} \Pr\left[\left|\mathbb{E}\left[\tfrac{1}{n}N_d\right] - \tfrac{1}{n}N_d\right| \geq \varepsilon\right] = 0$$

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&= \tfrac{1}{n} + \Pr[X_1 + Y_1 = d \wedge X_1 + Y_2 = d \wedge X_2 + Y_2 \neq d \mid X_1 = 0]\, \overbrace{\Pr[X_1 = 0]}^{\leq 1} \qquad \text{Law of total probability} \\
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\end{aligned}$$

Law of total probability

$\Rightarrow X_2 = 1$

independent

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independent

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$$\mathrm{Var}[\tfrac{1}{n}N_d] = \mathbb{E}[(\tfrac{1}{n}N_d)^2] - \mathbb{E}[\tfrac{1}{n}N_d]^2 \leq \tfrac{1}{n} + 2p$$

$$= \tfrac{1}{n^2}\left(\mathbb{E}[(N_d)^2] - \mathbb{E}[N_d]^2\right)$$

$$\leq \tfrac{1}{n} + \Pr[\deg(v) = d \wedge \deg(u) = d]$$

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Law of total probability

$$+ \underbrace{\Pr[X_1 + Y_1 = d \wedge X_1 + Y_2 = d \wedge X_2 + Y_2 \neq d \mid X_1 = 1]}_{\leq 1}\Pr[X_1 = 1]$$

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Law of total probability

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Law of total probability

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Fréchet: $\Pr[A]-\Pr[B]\leq\Pr[A\wedge\bar{B}]$

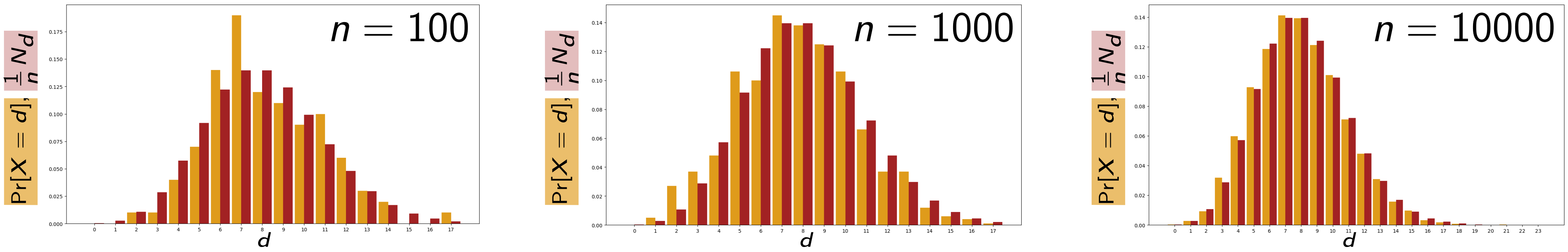
Application: ER – Degree Distribution

Theorem: Consider a $G(n, p)$ with $p = c/n$ for constant $c > 0$. For $\lambda = -n \log(1-p)$, let $X \sim \mathrm{Pois}(\lambda)$. Then for all $d > 0$ and every $\varepsilon > 0$ we have $\lambda = c + O(1/n) \to c$ for $n \to \infty$

$$\lim_{n\to\infty} \Pr\left[\left|\Pr[X = d] - \tfrac{1}{n} N_d\right| \geq \varepsilon\right] = 0.$$

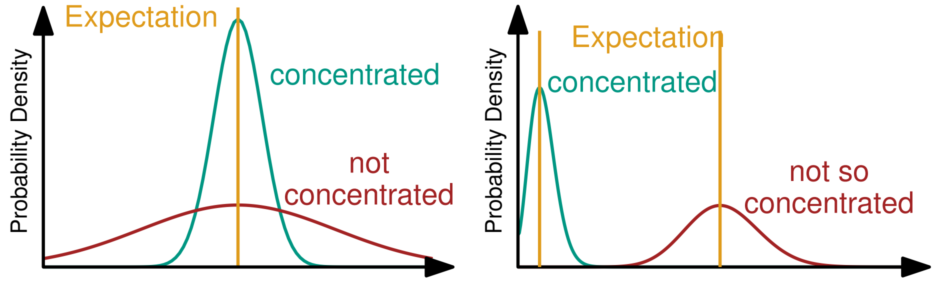
Proof

- Step 1: $\Pr[X = d]$ is close to the expectation of $\frac{1}{n} N_d$ $\lim_{n\to\infty} \left|\Pr[X = d] - \mathbb{E}\left[\tfrac{1}{n} N_d\right]\right| = 0$ ✓
- Step 2: $\frac{1}{n} N_d$ is concentrated (via Chebychev) $\lim_{n\to\infty} \Pr\left[\left|\mathbb{E}\left[\tfrac{1}{n} N_d\right] - \tfrac{1}{n} N_d\right| \geq \varepsilon\right] = 0$ ✓



Concentration Bounds So Far

Definition: A **concentration inequality** bounds the probability of a random variable to deviate from a given value (typically its expectation) by a certain amount.



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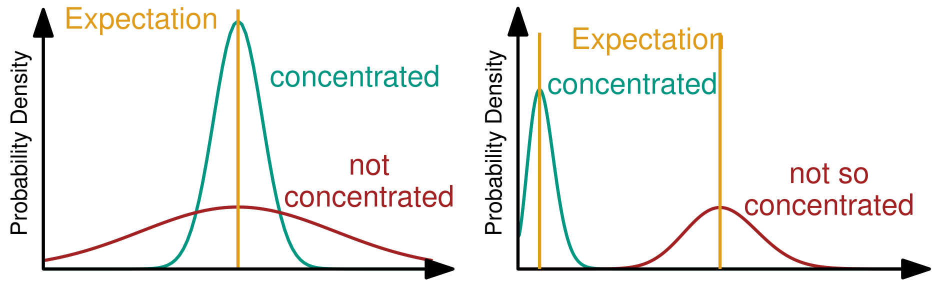

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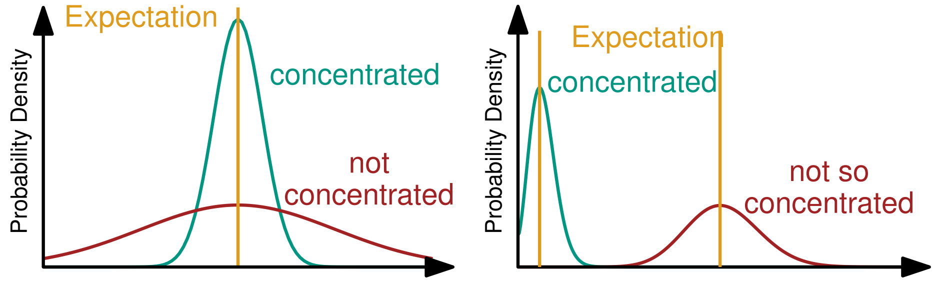

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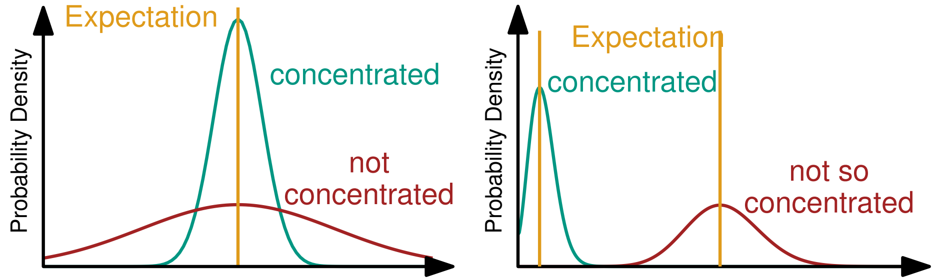

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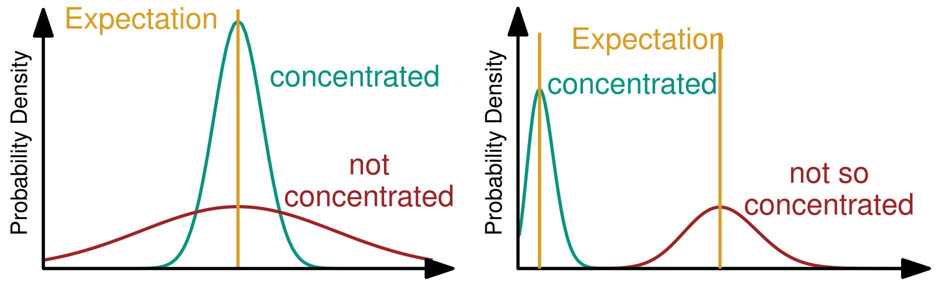

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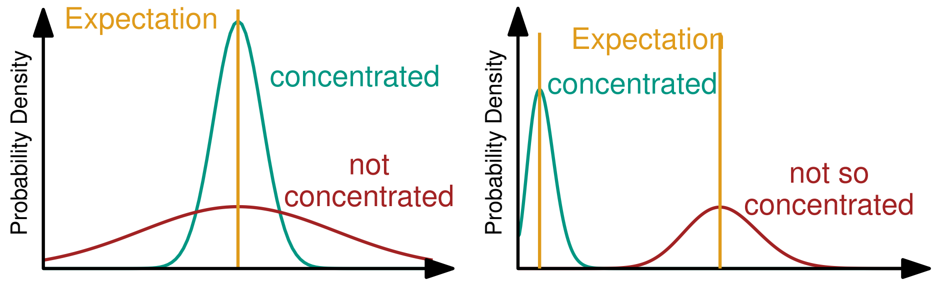

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Can we utilize higher-order moments for even stronger bounds?

Another Moment Please

- The n-th raw moment of a random variable X is $\mathbb{E}[X^n]$

Another Moment Please

Karlsruhe Institute of Technology

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Looks scary, but is again just $\mathbb{E}[f(X)]$ for $f(X) = e^{tX}$

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Get bounds for specific random variables by finding a good t!

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Theorem: Let $X \sim \text{Bin}(n, p)$. Then for any $\varepsilon > 0$

$$\Pr[X \geq (1+\varepsilon)\mathbb{E}[X]] \leq \left(\frac{e^{\varepsilon}}{(1+\varepsilon)^{(1+\varepsilon)}}\right)^{\mathbb{E}[X]}.$$

Proof Consider X as the sum of independent $X_i \sim \text{Ber}(p)$

$$M_{X_i}(t) = \mathbb{E}[e^{tX_i}] = \Pr[X_i = 0] \cdot e^{t \cdot 0} + \Pr[X_i = 1] \cdot e^{t \cdot 1}$$
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Chernoff: Random variable X and $a > 0$: $\Pr[X \geq a] \leq \min_{t>0} \mathbb{E}[e^{tX}]/e^{ta}$.

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- Sum of 20 unfair $\{0, 1\}$-coin tosses: $X \sim \text{Bin}(20, \frac{1}{5})$, $\mathbb{E}[X] = 4$

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Markov:	≤ 0.25
Chebychev:	$\lessapprox 0.022$
Actual:	≈ 0.0000000138

Chernoff – Simpler Versions

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Corollary: Let $X \sim \text{Bin}(n, p)$. Then for any $t \geq 6\mathbb{E}[X]$, $\Pr[X \geq t] \leq 2^{-t}$.

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Corollary: Let $X \sim \text{Bin}(n, p)$. Then for any $t \geq 6\mathbb{E}[X]$, $\Pr[X \geq t] \leq 2^{-t}$.

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Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

Chernoff – Simpler Versions

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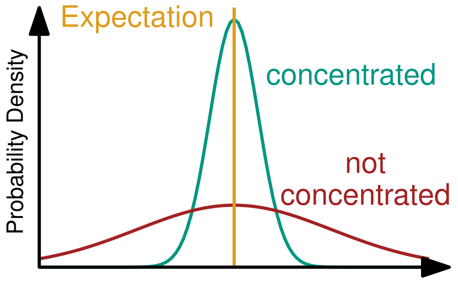
- In fact, these also work when the X_i are Bernoulli random variables with different success probabilities

Conclusion

Concentration

- Is a random variable likely to yield values close to the expectation?
- Concentration inequalities bound the probability for a random variable to deviate from its expectation



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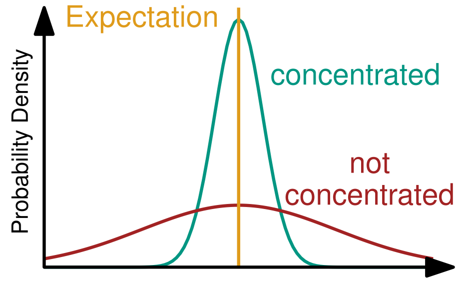

Moments

- Used to characterize the shape of a distribution
- First moment: expected value
- Second moment: variance
- Moment generating functions to determine higher-order moments

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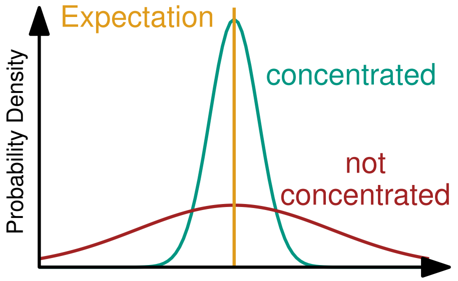

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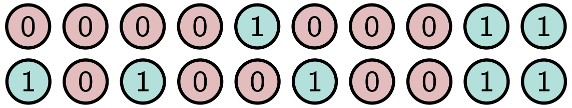

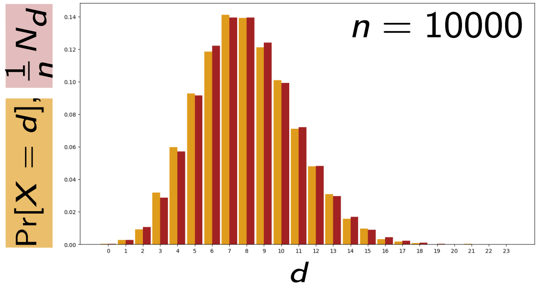

Concentration Inequalities
- Markov: Based on first moment
- Chebychev: Squaring within Markov (utilizing second moment)
- Chernoff: Exponentiating within Markov (utilizing moment generating functions)
- Examples: Sum of coin flips, empirical degree distribution of ER graphs