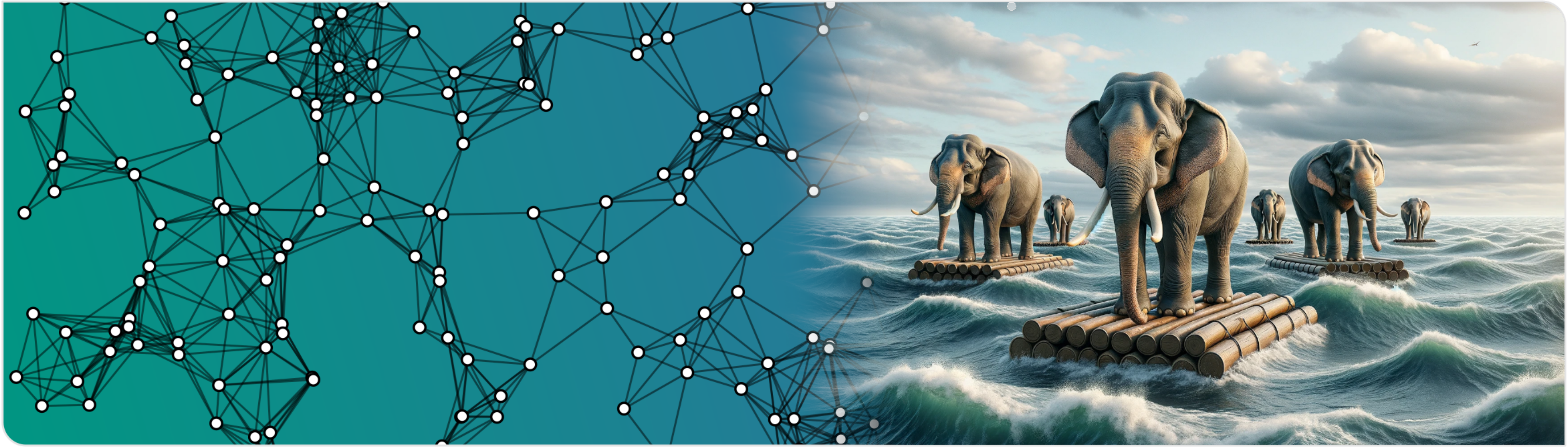
Probability & Computing

Bounded Differences & Geometric Inhomogeneous Random Graphs



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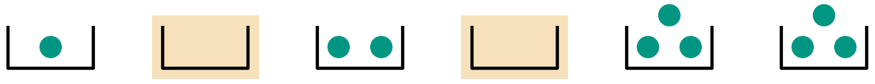

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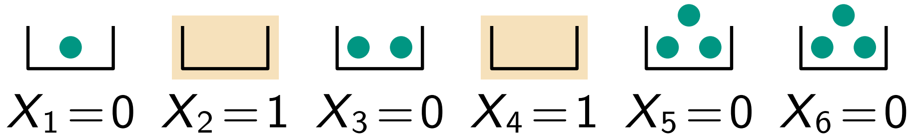

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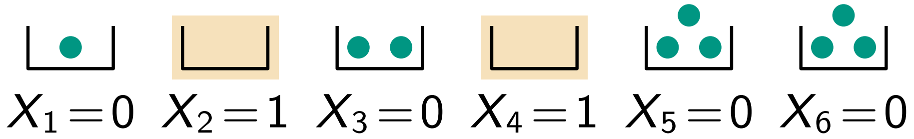

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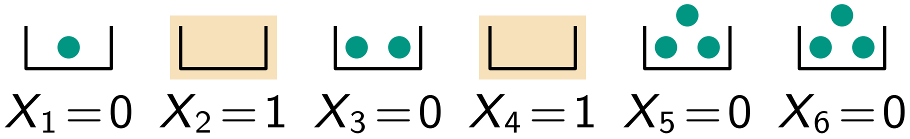

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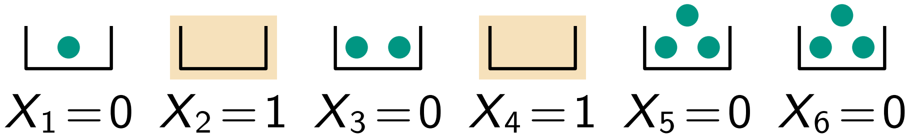

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  - Chernoff: ✗ (our Bernoulli random variables are *not* independent)

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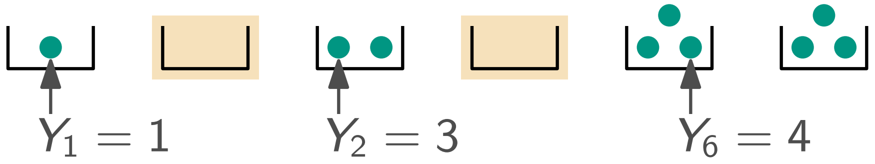

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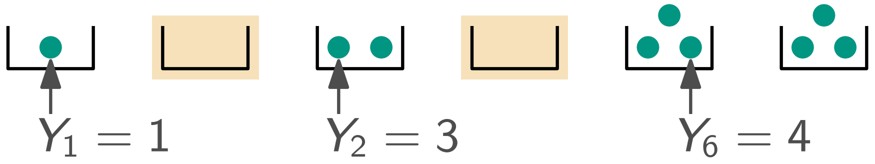

$\Rightarrow X = f(Y_1, \ldots, Y_k) = \sum_{i \in [n]} \mathbb{1}_{\{\nexists j : Y_j = i\}}$ (summands not independent, but the Y_j are)

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(simpler variants work, e.g., for sums of independent random variables)

Example

Today: similarly strong but beyond sums of independent Bernoulli random variables

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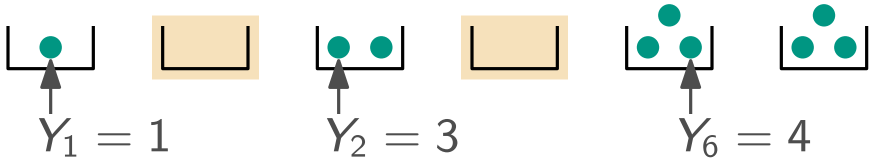

$$\Rightarrow X = f(Y_1, \dots, Y_k) = \sum_{i \in [n]} \mathbb{1}_{\{\nexists j : Y_j = i\}} \quad \text{(summands not independent, but the } Y_j \text{ are)}$$

$$= \sum_{i \in [n]} \max_{j \in [k]} \{2 - |\{Y_j, i\}|\} \quad \text{(“not” a sum Bernoulli random variables)}$$

Recall: Concentration

Concentration Inequalities

- Bound the probability for a random variable to deviate from its expectation
- Markov: generally applicable, but not very strong
- Chebychev: stronger, but requires knowledge about variance
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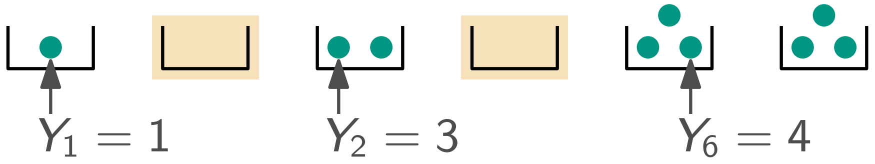

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Can we show concentration for some arbitrary function of independent random variables?

... under certain conditions!

Method of Bounded Differences

Aka ... Bounded differences inequality, McDiarmid’s inequality, Azuma-Hoeffding inequality

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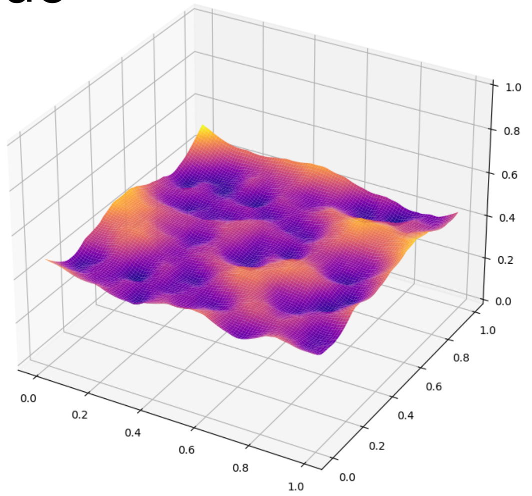

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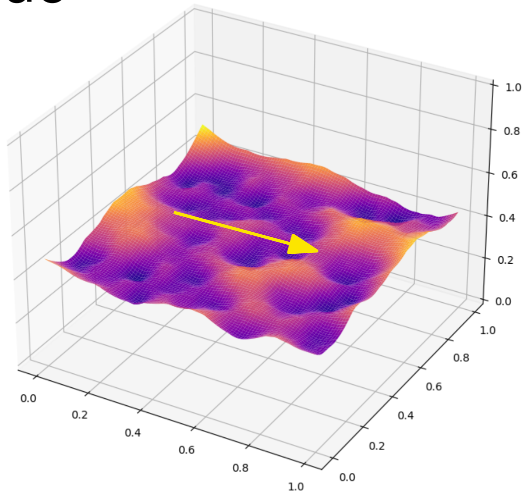

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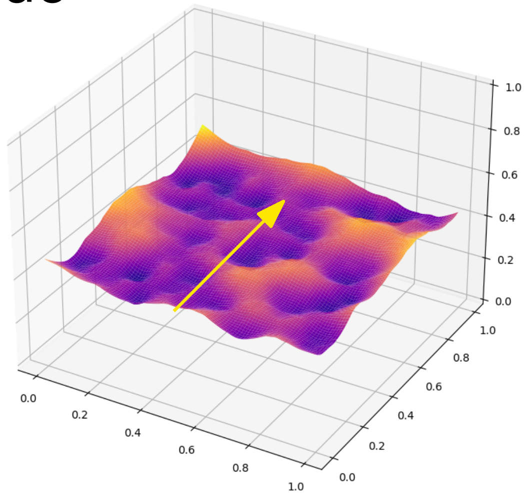

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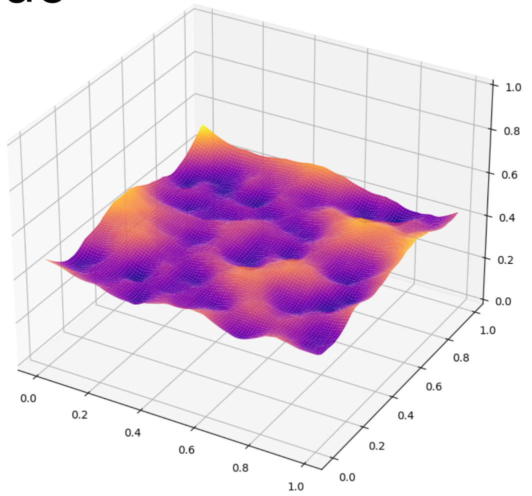
How do we measure this?



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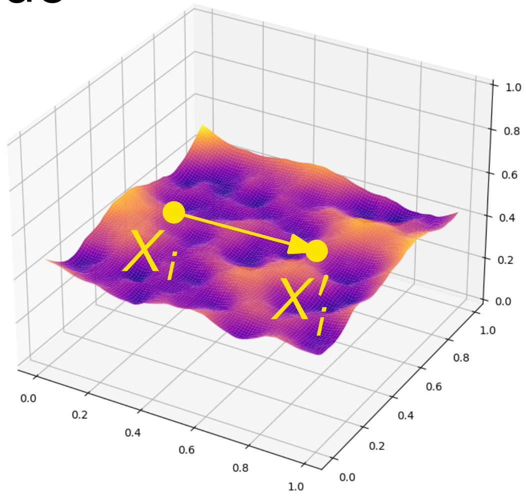

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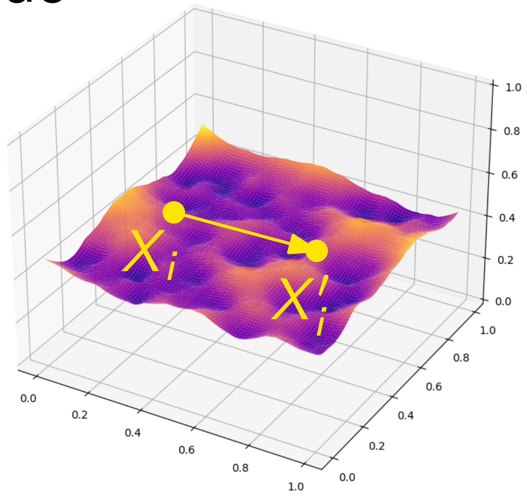

Theorem: Let $X_1, ..., X_n$ be independent random variables taking values in a set S. Let $f : S^n \to \mathbb{R}$ satisfy the bounded differences condition with parameters Δ_i. Then, for $\Delta = \sum_{i \in [n]} \Delta_i^2$:

$$\Pr[|f - \mathbb{E}[f]| \geq t] \leq 2e^{-2t^2/\Delta}.$$

(write f for $f(X_1, ..., X_n)$)

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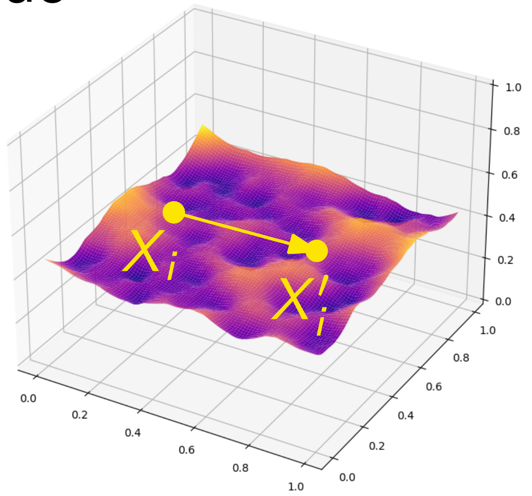

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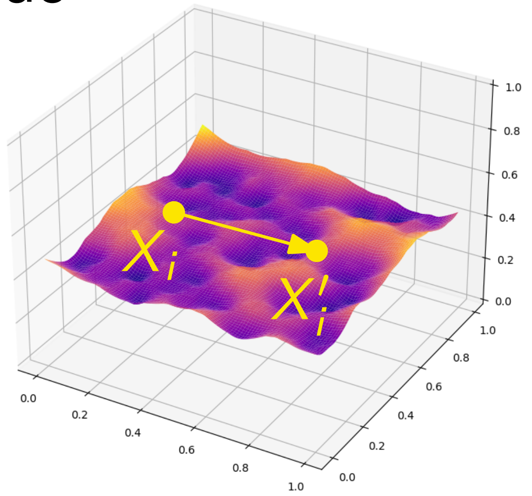

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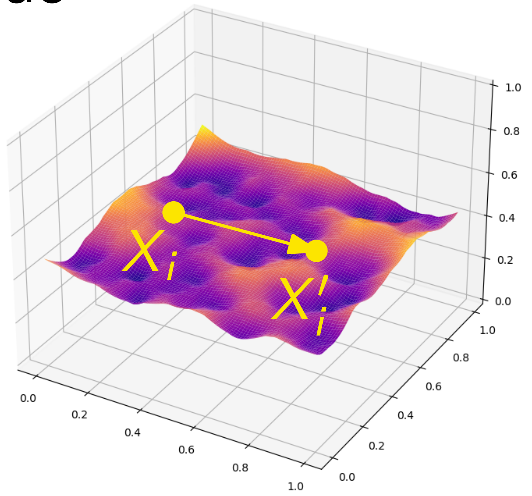

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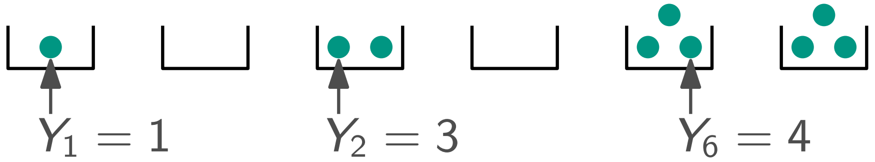
Cor. $\mathbb{E}[f] \leq g(n)$: $\Pr[f \geq cg(n)] \leq e^{-2((c-1)g(n))^2/\Delta}$.

Application: Balls into Bins



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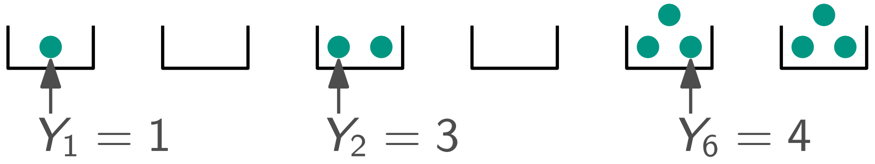

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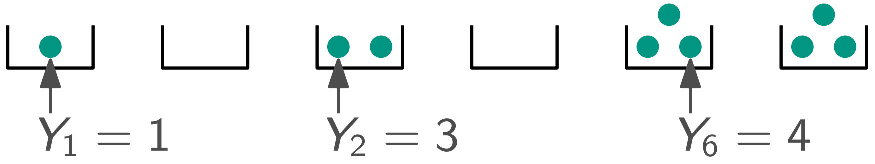

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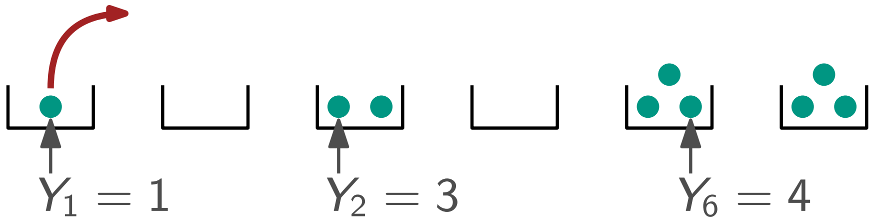

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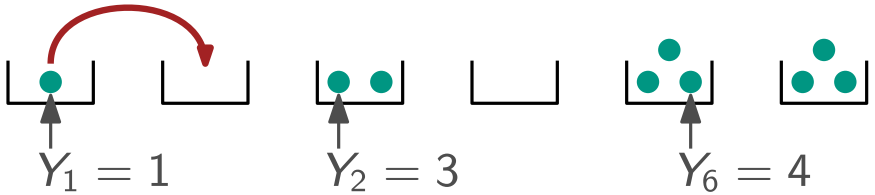
Karlsruhe Institute of Technology



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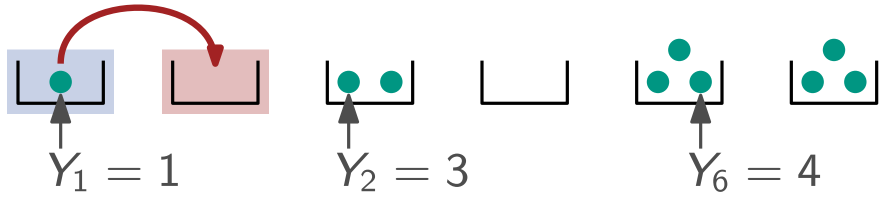

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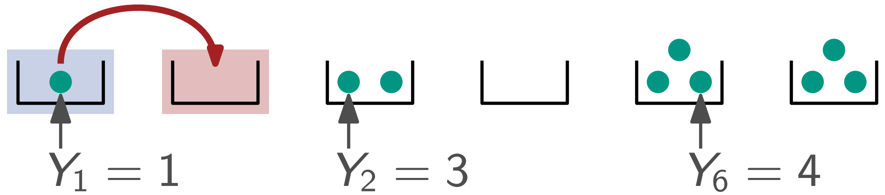

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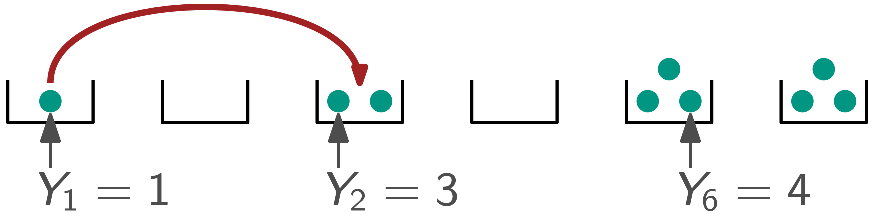

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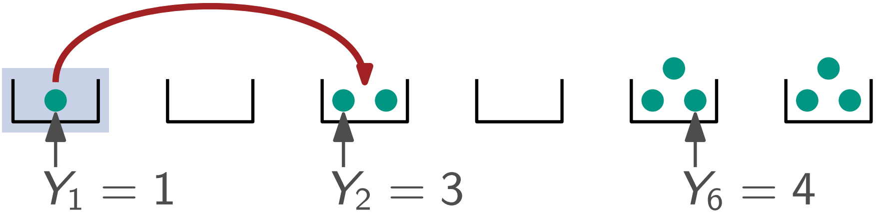

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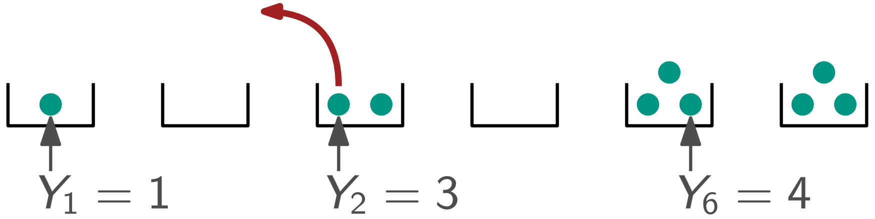

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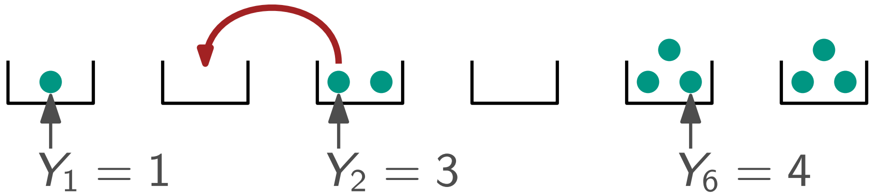

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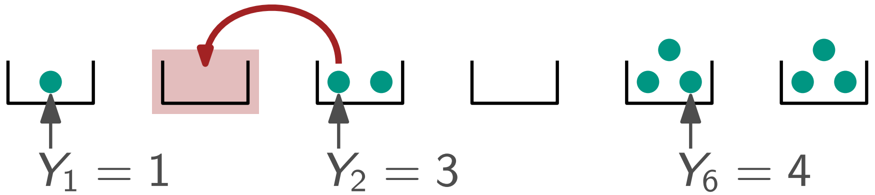

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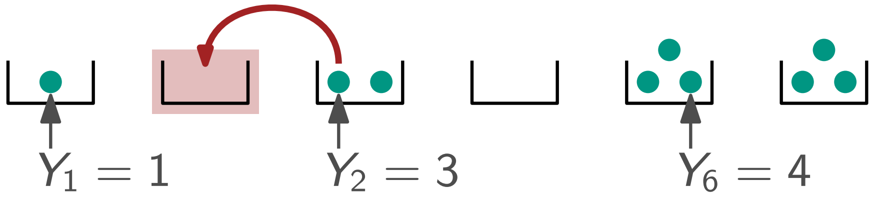

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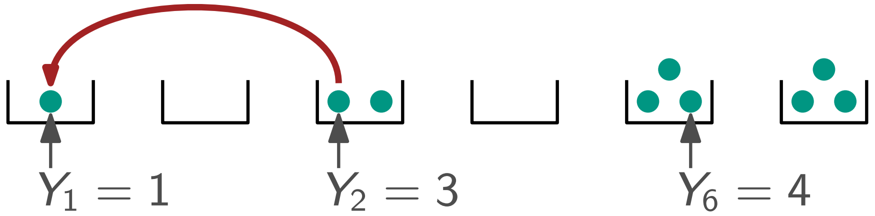

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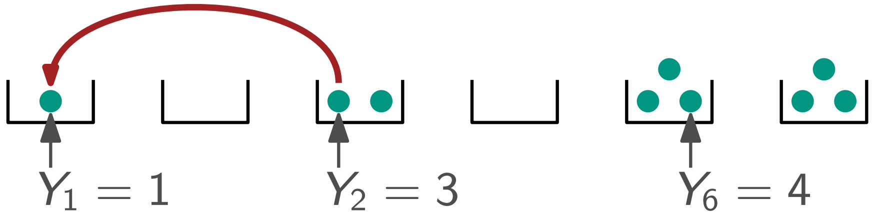
Application: Balls into Bins



- k balls distributed uniformly at random over n bins
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$|f(..., Y_i, ...) - f(..., Y_i', ...)| \leq \Delta_i$
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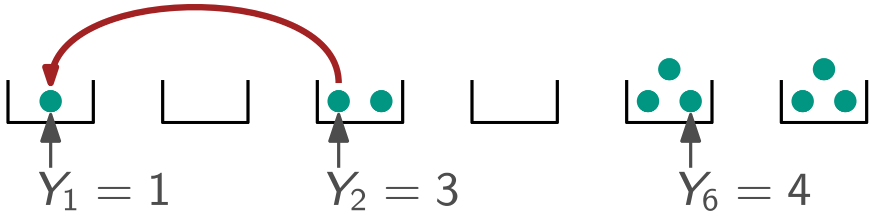

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Concentration via bounded differences

Application: Balls into Bins

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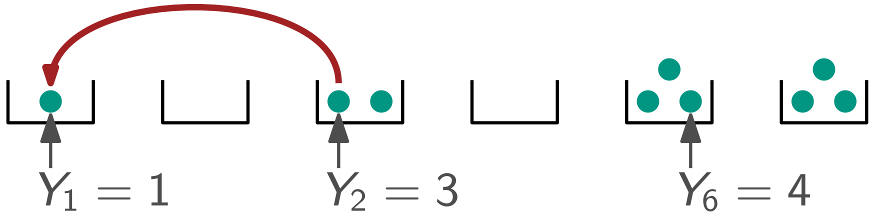

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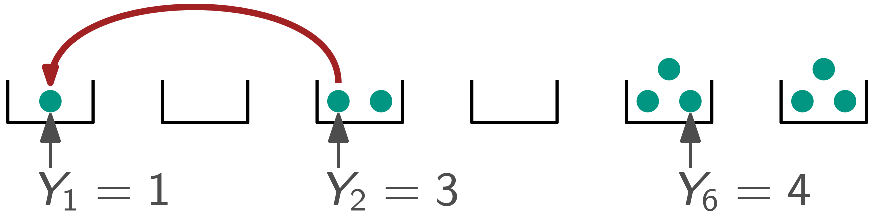

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$$\Delta = \sum_{i=1}^{k} \Delta_i^2 \leq \sum_{i=1}^{k} 1^2 = k \quad \Rightarrow \Pr[f \geq \mathbb{E}[f] + 5\sqrt{k}] \leq e^{-2(5\sqrt{k})^2/k}$$

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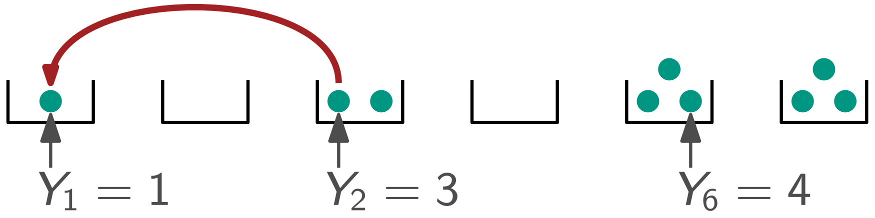

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Much better than Markov's $\rightarrow 1$

Application: The Factory

- Products are distributed uniformly at random over boxes on a conveyor belt

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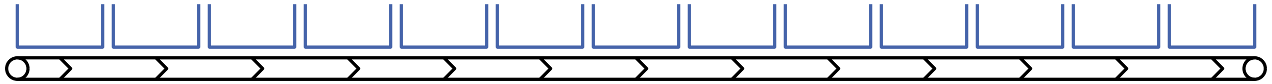

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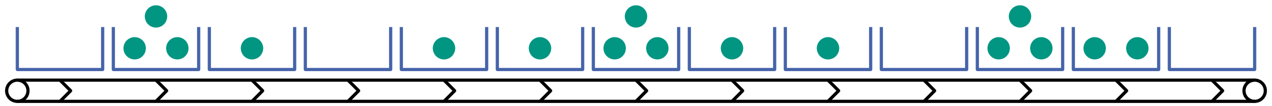

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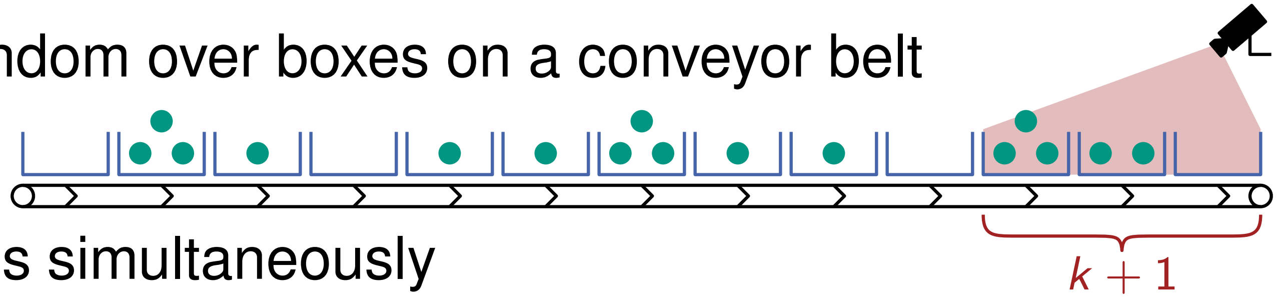

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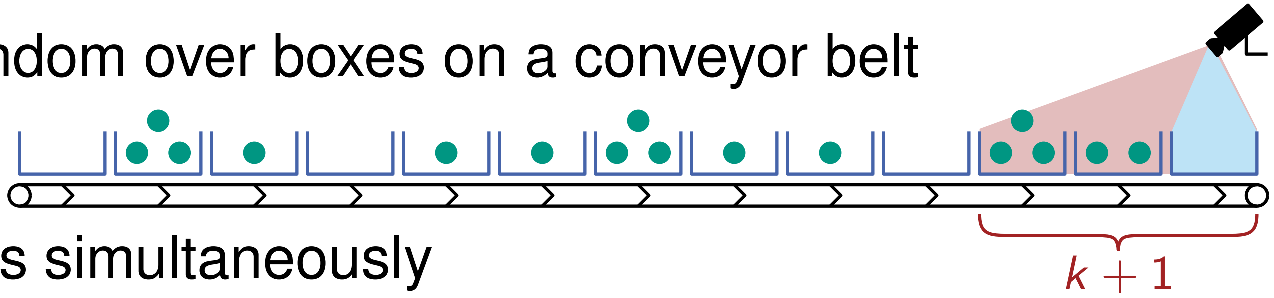

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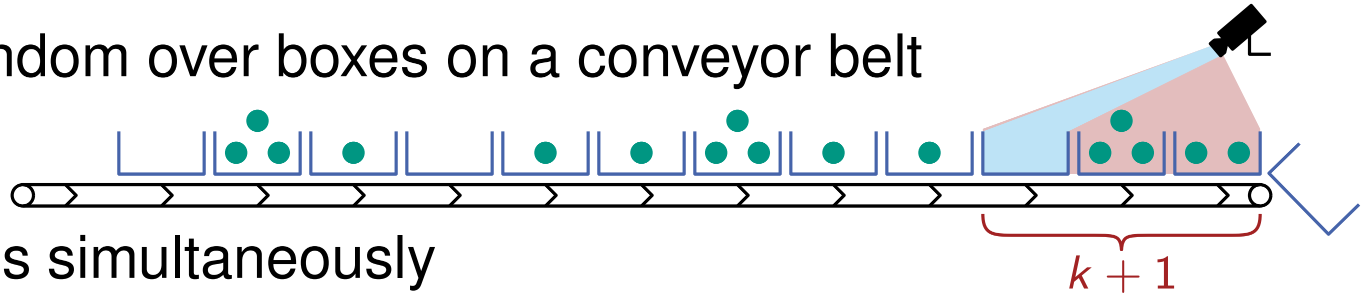

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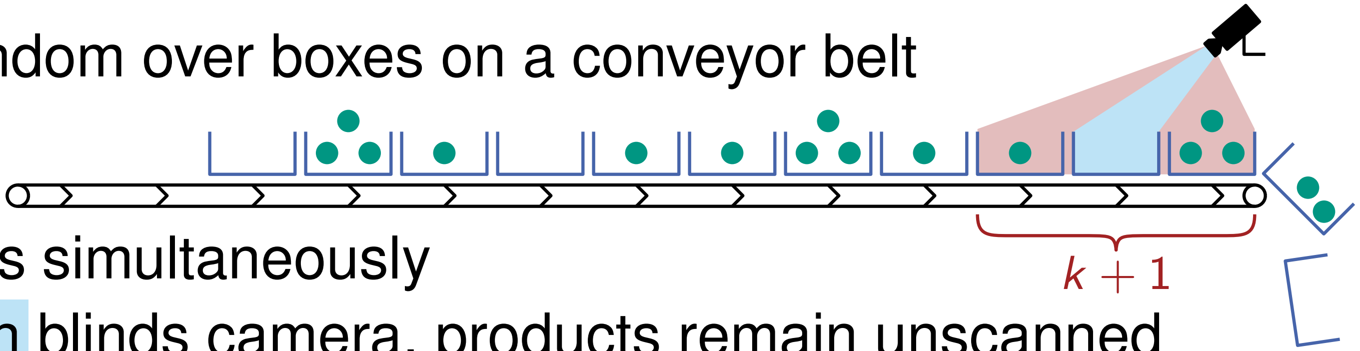

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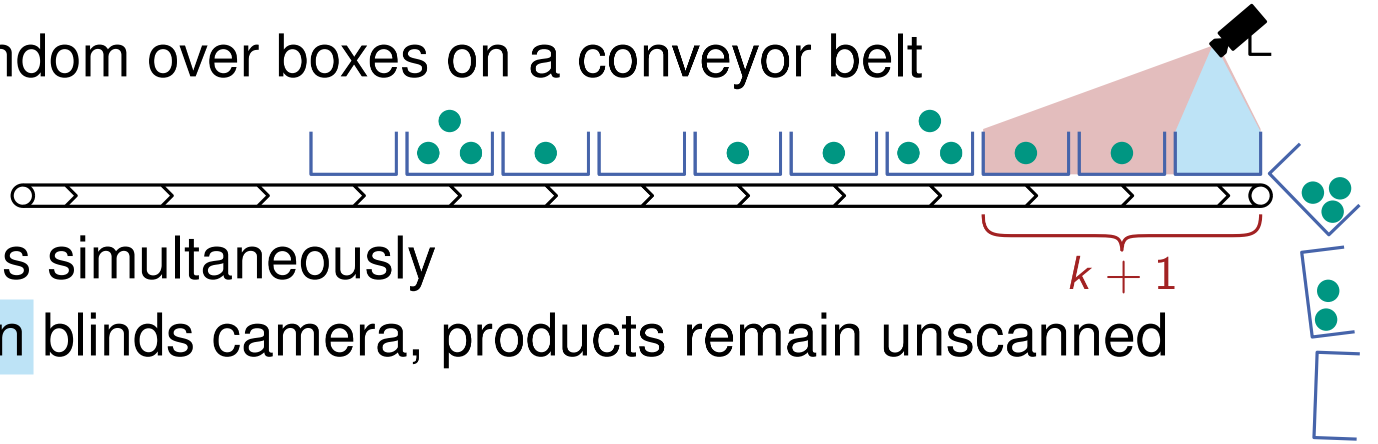

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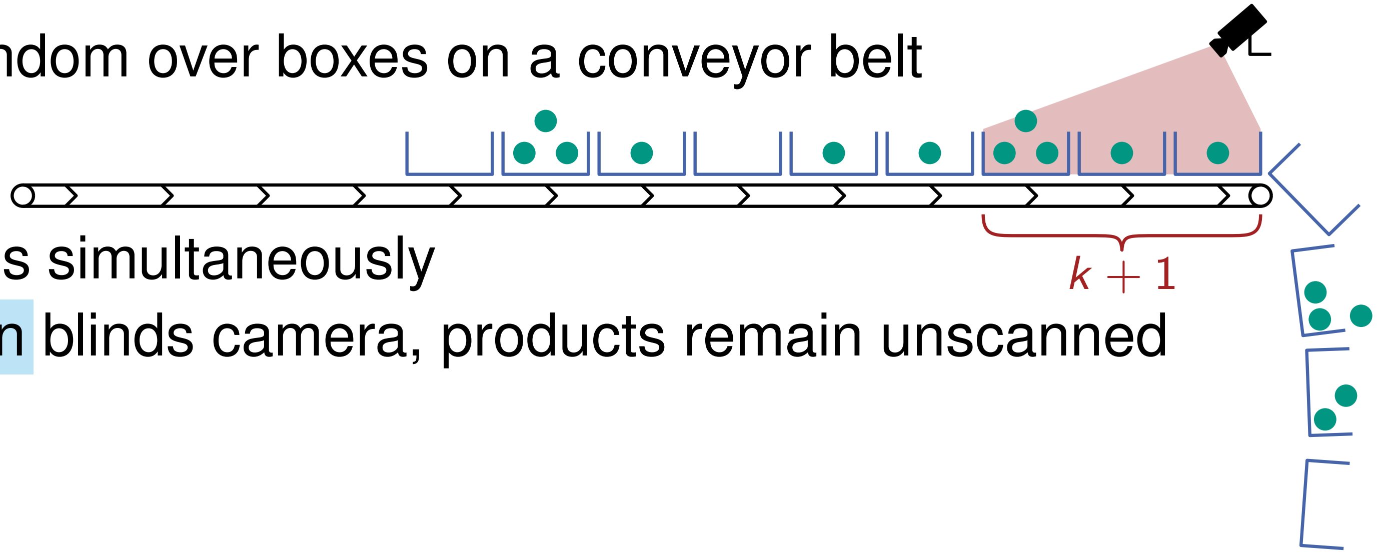

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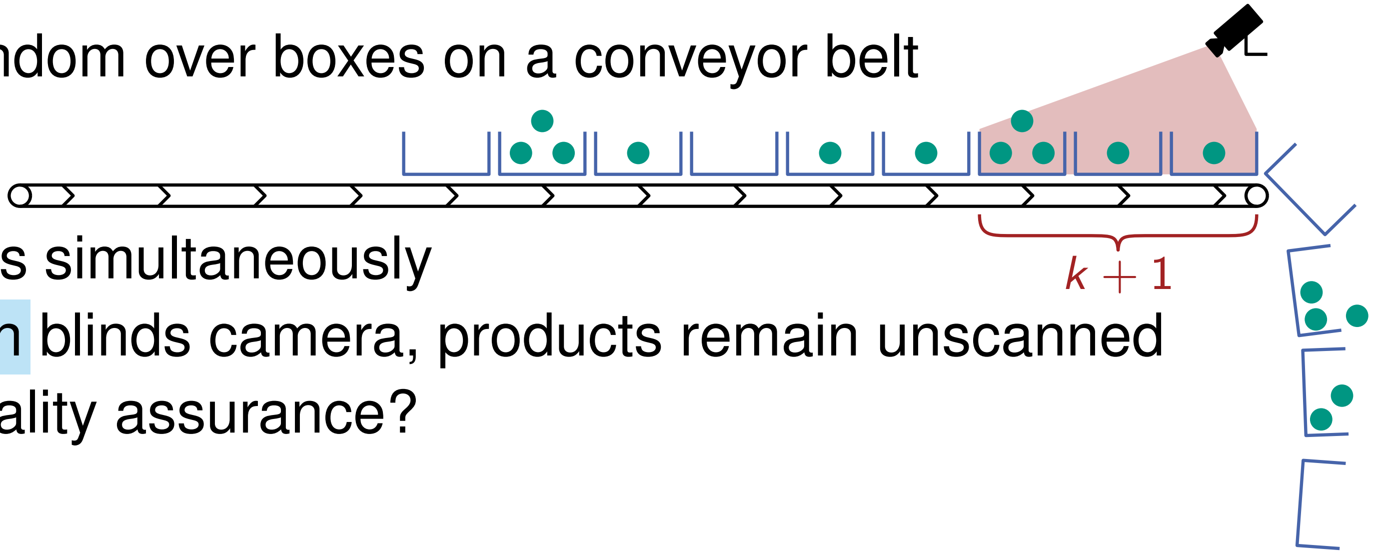

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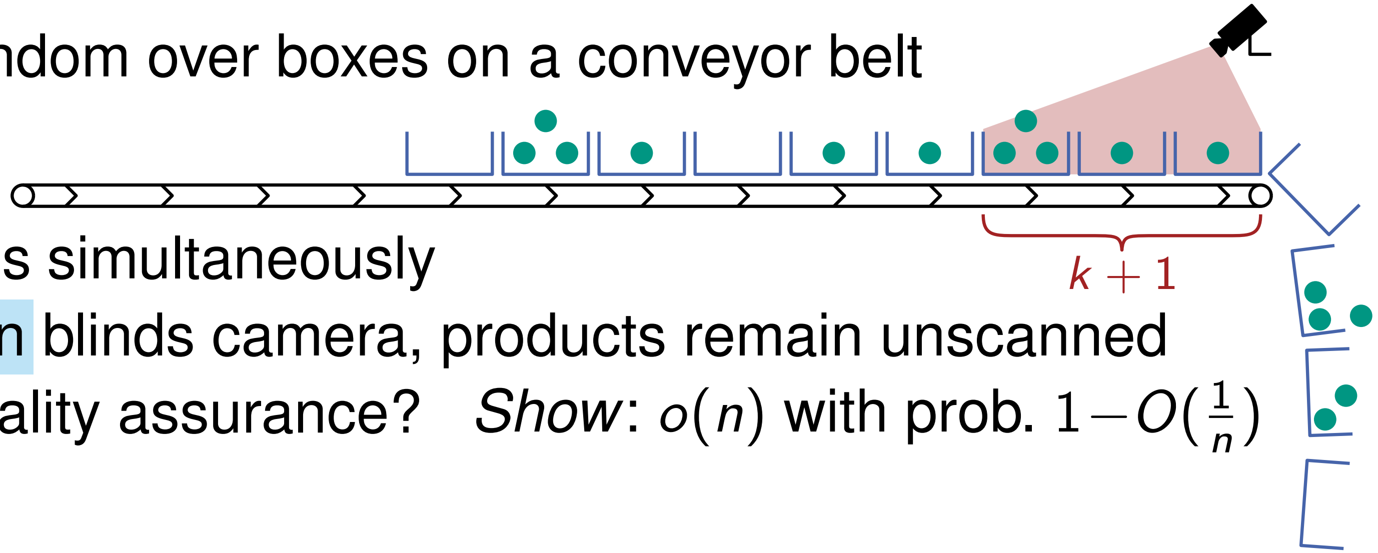

- A camera scans $k+1$ consecutive boxes simultaneously
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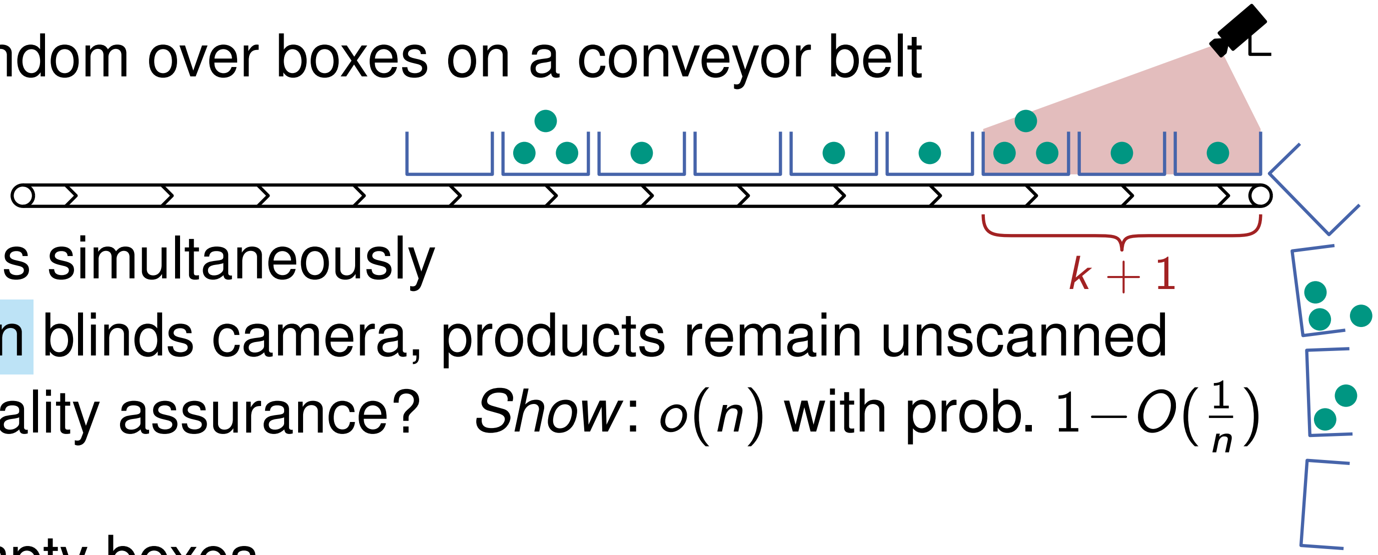

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Formalize

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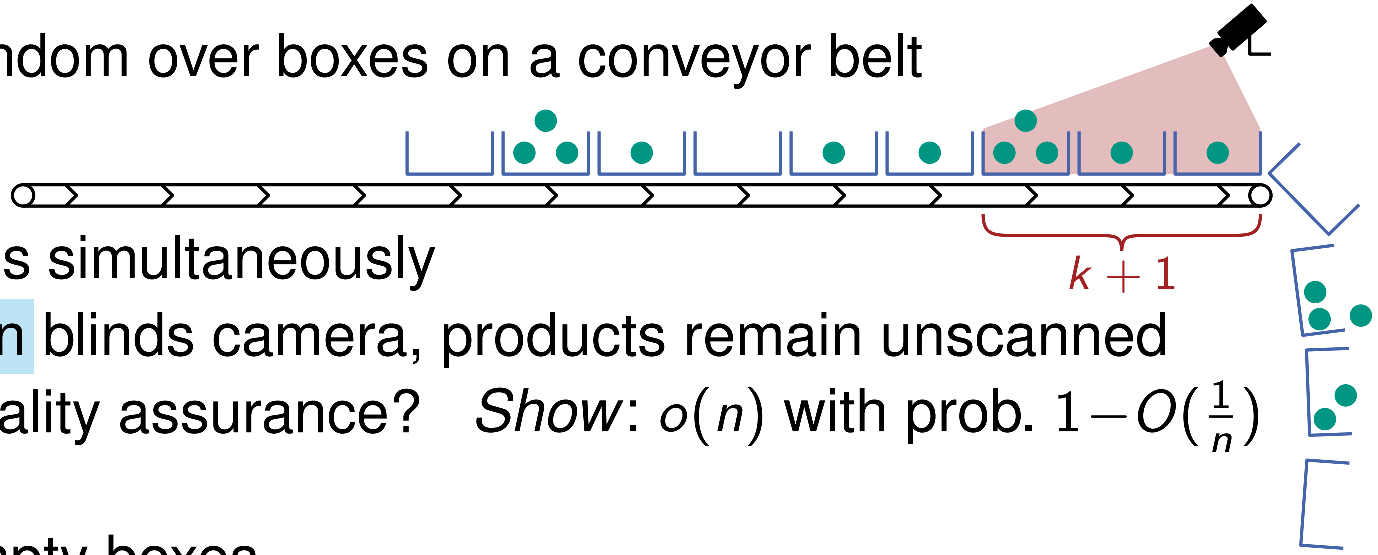

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Application: The Factory

Karlsruhe Institute of Technology

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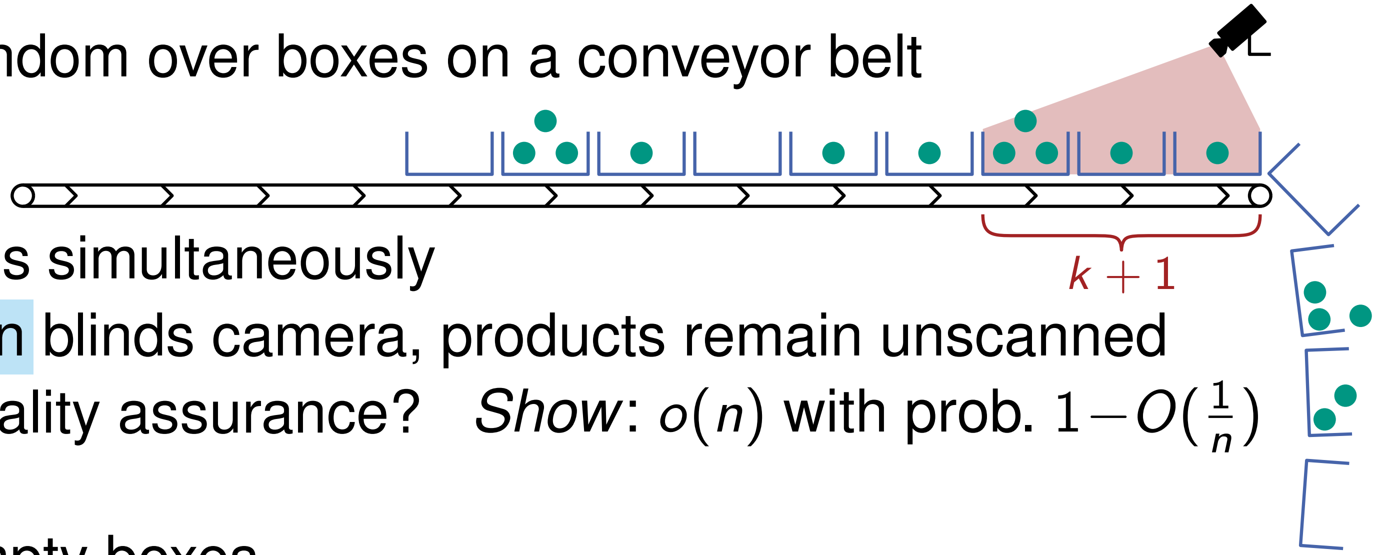

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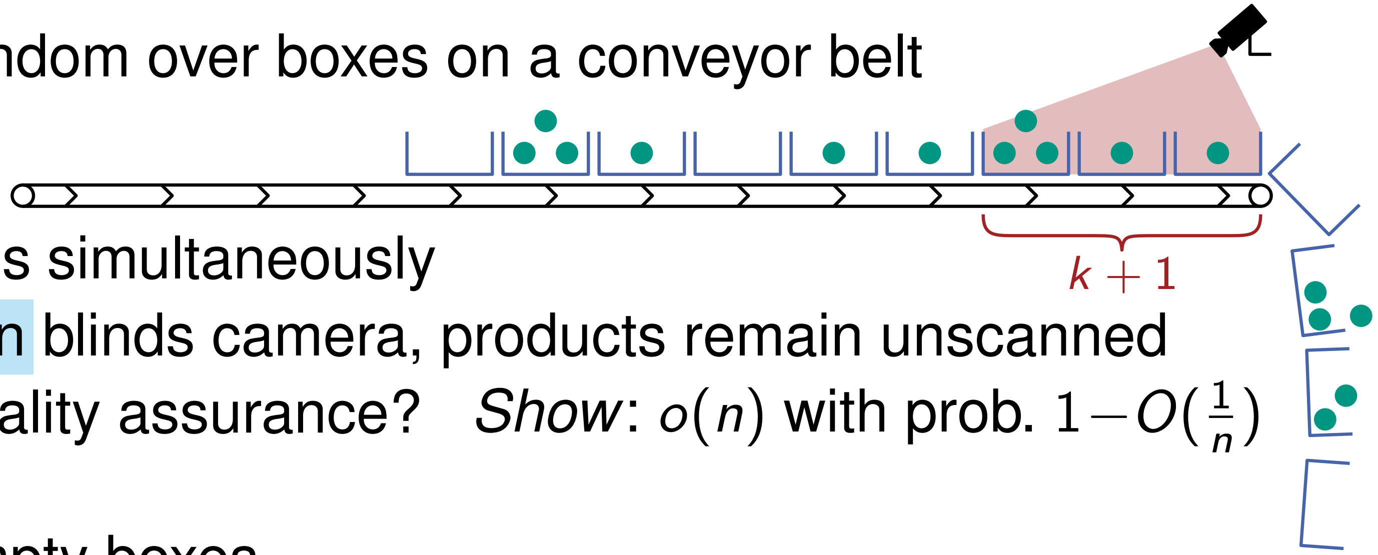

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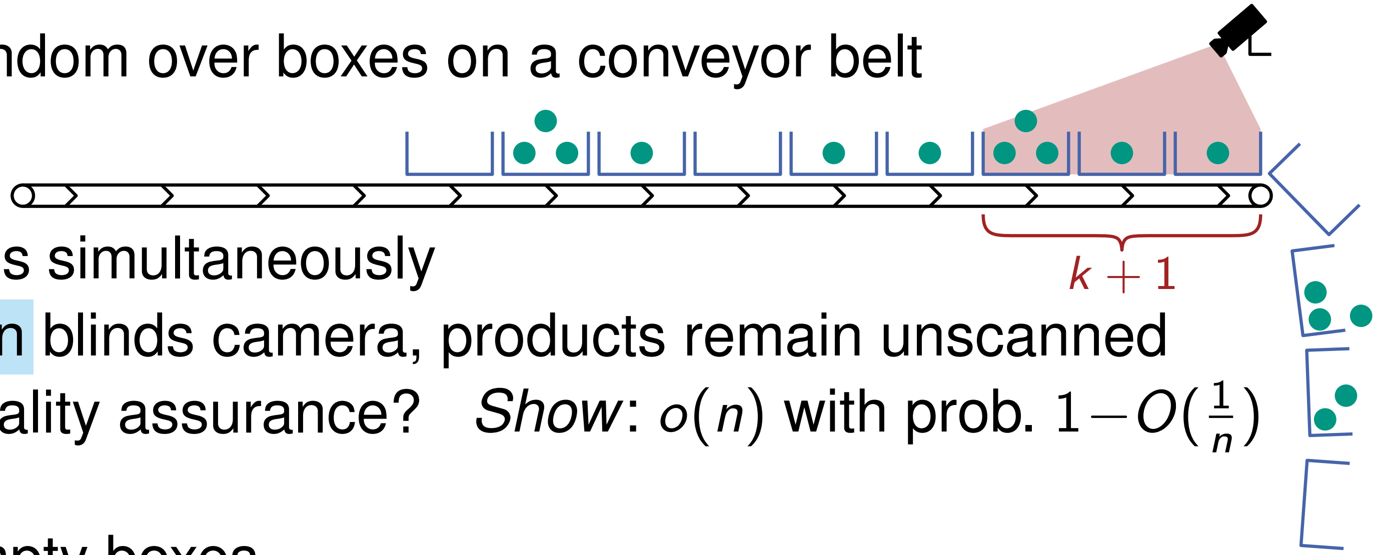

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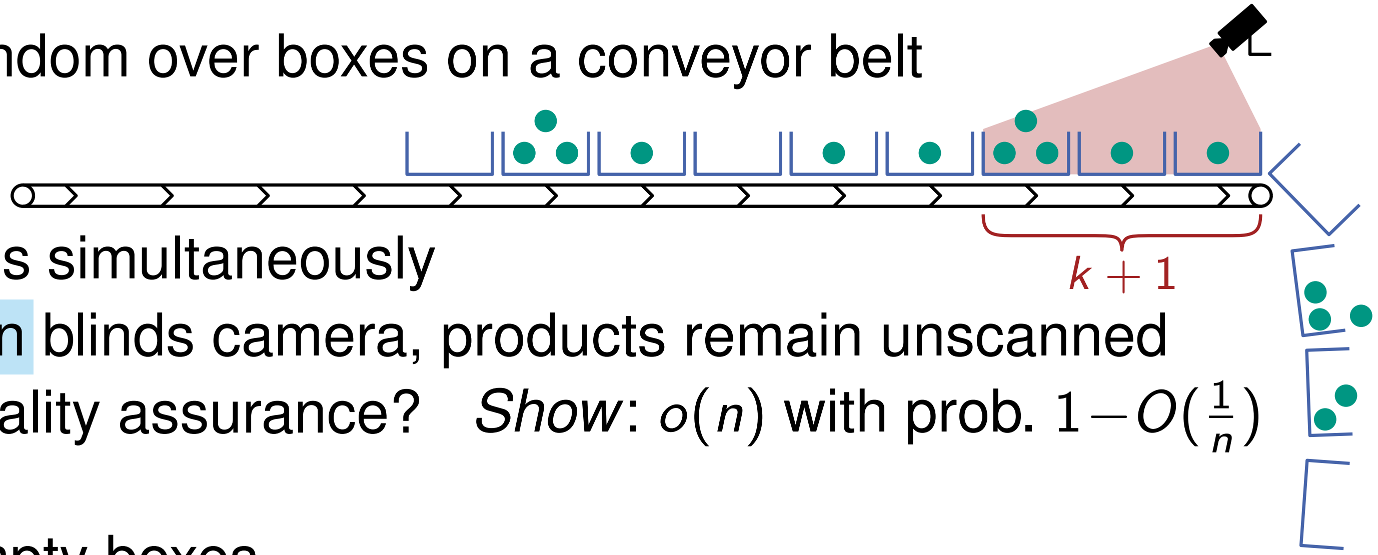

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- Solution: Relax dependencies and compute upper bound instead

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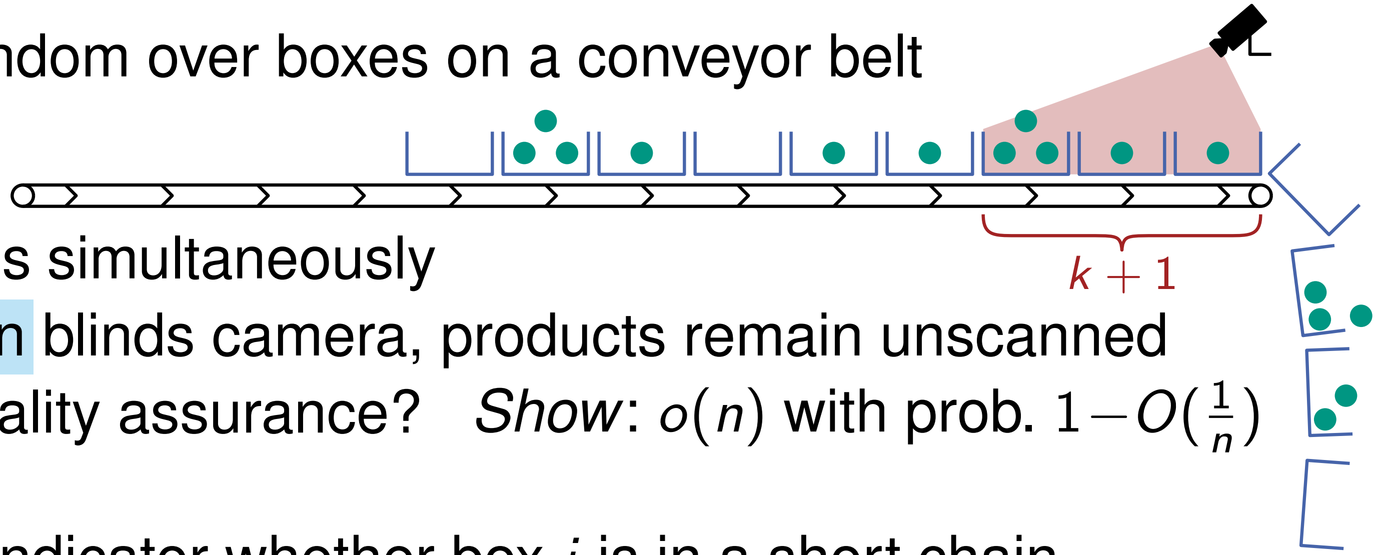

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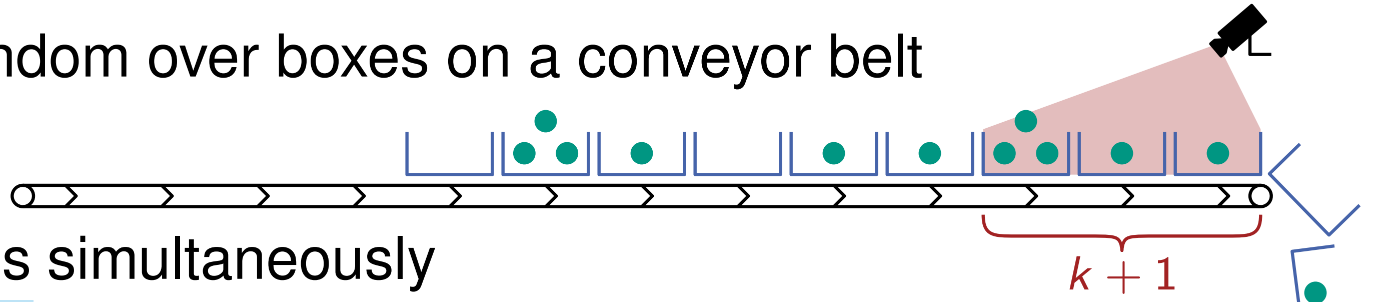

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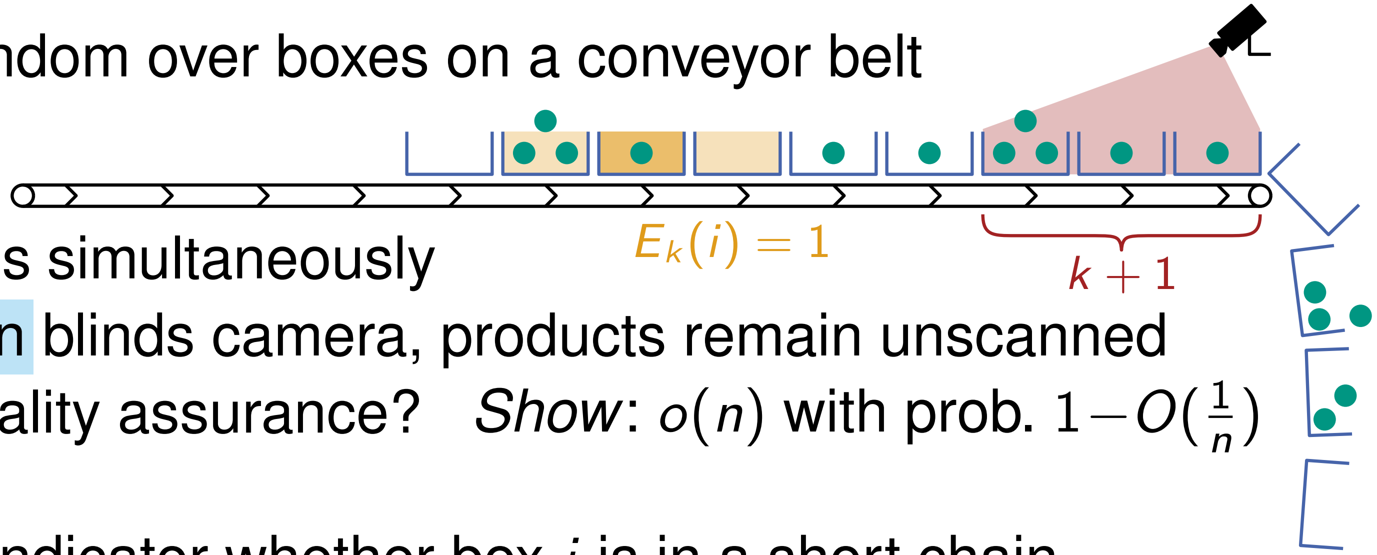

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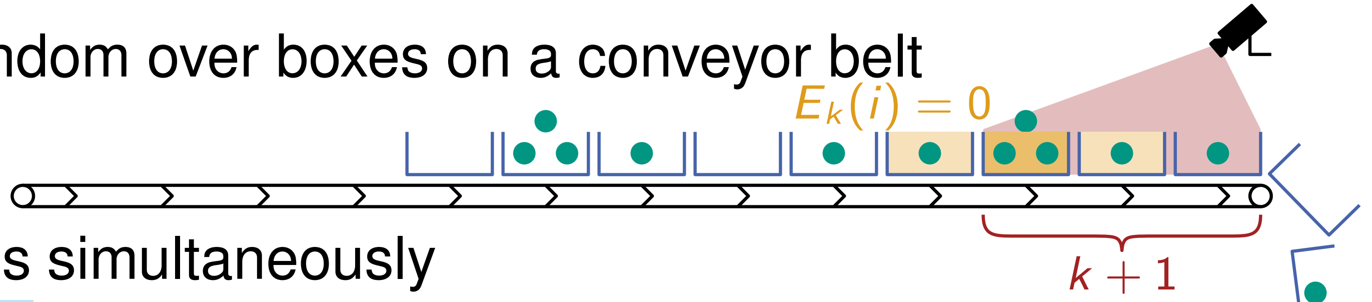

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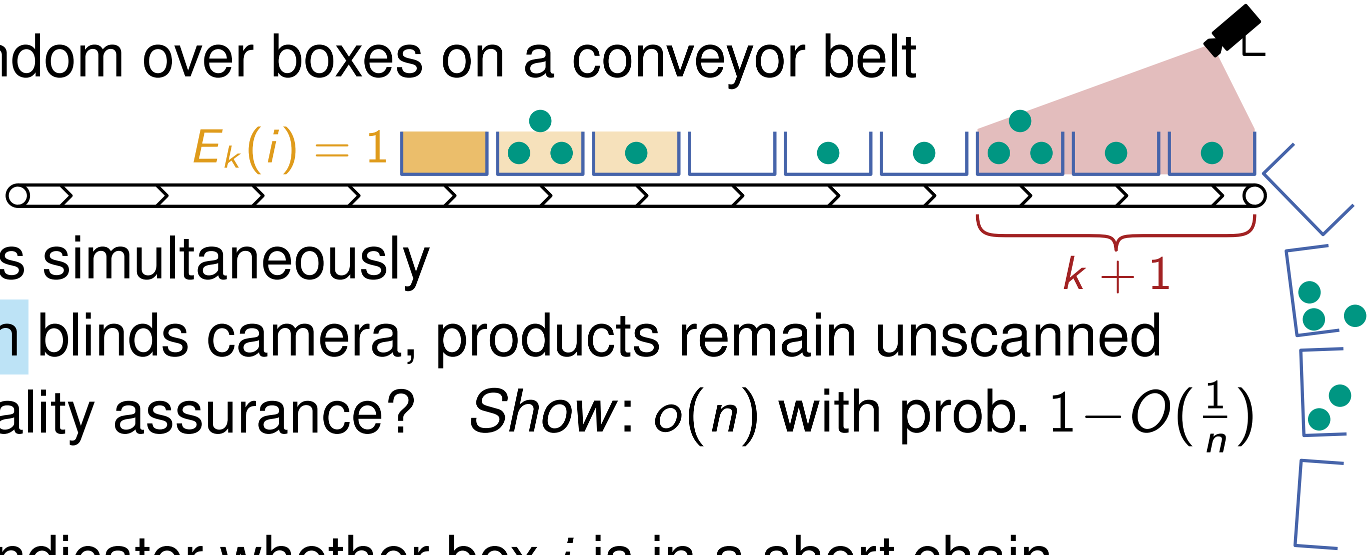

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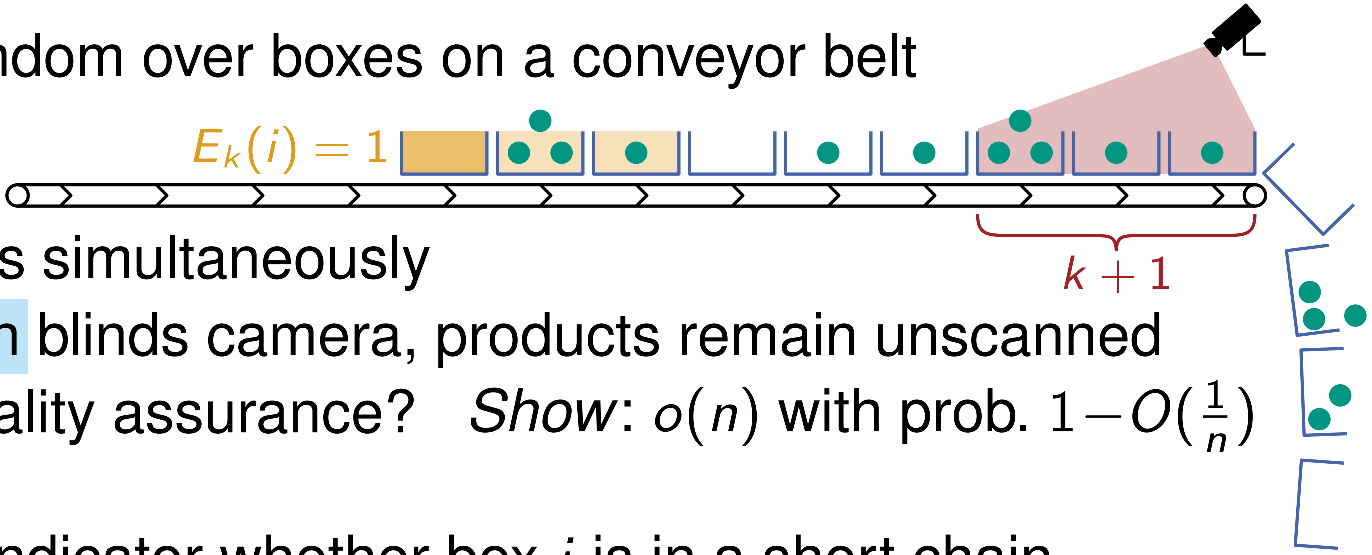

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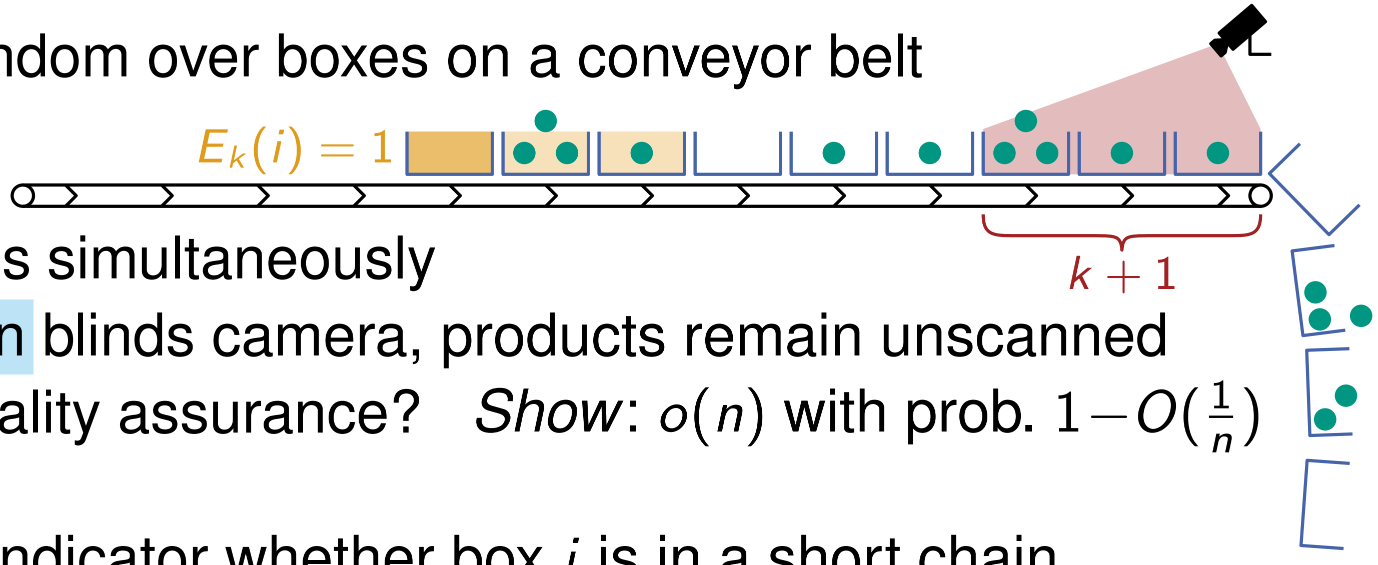
- $E_k(i)$ = number of empty boxes in box i and k closest (assuming k even)

  ↳ Box i in short chain $\Rightarrow E_k(i) > 0$

Application: The Factory

- Products are distributed uniformly at random over boxes on a conveyor belt

n products $m = n/k$ boxes $k = \log\log(n)$



- A camera scans $k+1$ consecutive boxes simultaneously
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Relax and bound

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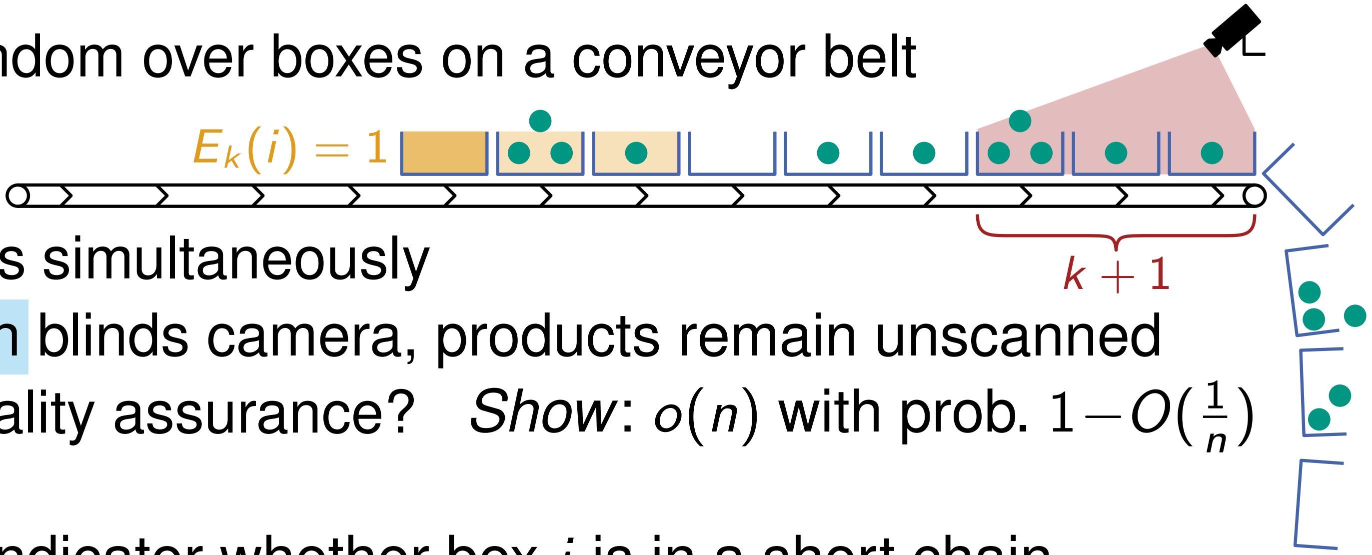

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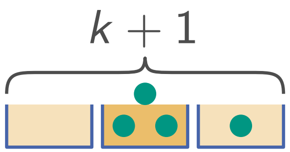
  - ↳ Box i in short chain $\Rightarrow E_k(i) > 0$
- Y_i' = indicator whether $E_k(i) > 0 \Rightarrow Y_i \leq Y_i'$
  - ↳ $X = \sum_{i=1}^{m} X_i \cdot Y_i \leq \sum_{i=1}^{m} X_i \cdot Y_i' =: X'$

Expectation of X'



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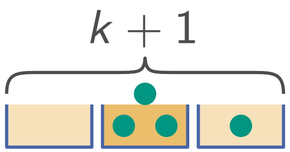

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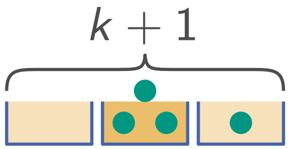

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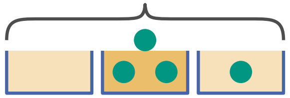
All $k+1$ empty
$\Rightarrow$ Box i empty $\Rightarrow X_i = 0$



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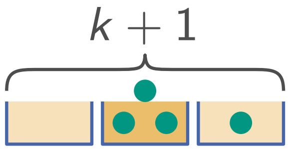

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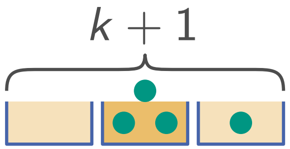

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No empty box $\Rightarrow Y_i' = 0$

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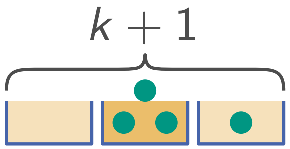

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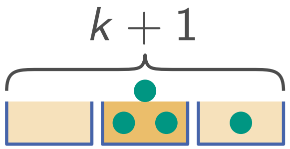

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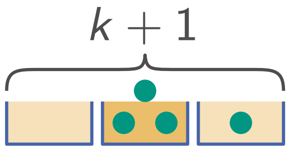

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knowing that exactly ℓ boxes are empty

- Box i empty? $\Rightarrow X_i = 0$

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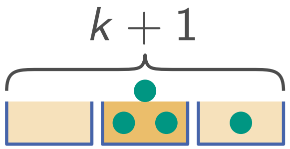

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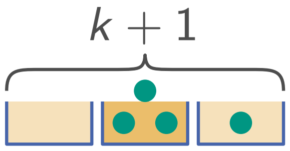

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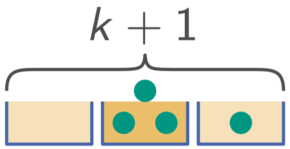

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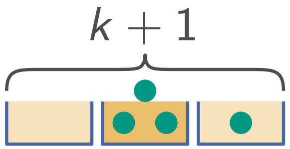
Expected number of products in box i, knowing that exactly ℓ boxes are empty

- Box i empty? $\Rightarrow X_i = 0$
- Else: n products distributed u.a.r. over $m' = m - \underbrace{\ell}_{\leq k}$ boxes

$$m' \geq \frac{n}{\log\log(n)} - \log\log(n)$$

Expectation of X' (for n large enough)



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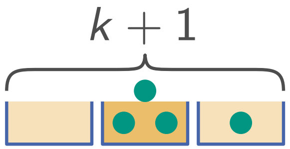
(for n large enough) $$\geq \frac{1}{2} \frac{n}{\log\log(n)}$$

Expectation of X' (for n large enough)



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$\hookrightarrow \mathbb{E}[X_i \mid E_k(i) = \ell] = \frac{n}{m'} \leq 2\log\log(n)$

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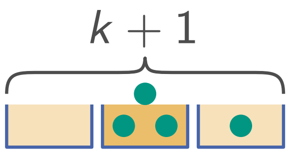

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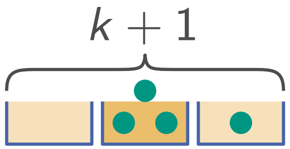

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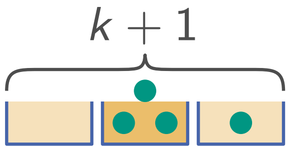

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&\leq \sum_{\ell=1}^{k} 2\log\log(n) \cdot \Pr[E_k(i) = \ell] \\
&= 2\log\log(n) \underbrace{\sum_{\ell=1}^{k} \Pr[E_k(i) = \ell]}_{\leq \Pr[\text{“Exists an empty box among } k+1\text{”}]}
\end{aligned}$$

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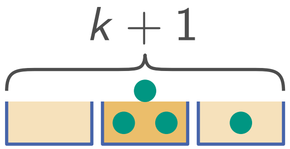

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$$\mathbb{E}[X'] = \sum_{i=1}^{m} \mathbb{E}[X_i \cdot Y_i']$$

(law of total expectation)
$$= \sum_{\ell=0}^{k+1} \mathbb{E}[X_i \cdot Y_i' \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell]$$

$$= \sum_{\ell=0}^{k} \mathbb{E}[X_i \cdot Y_i' \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell]$$

$$= \sum_{\ell=1}^{k} \mathbb{E}[X_i \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell]$$

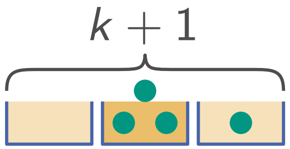
$$\leq \sum_{\ell=1}^{k} 2\log\log(n) \cdot \Pr[E_k(i) = \ell]$$

$$= 2\log\log(n) \underbrace{\sum_{\ell=1}^{k} \Pr[E_k(i) = \ell]}$$

$$\leq \Pr[\text{“Exists an empty box among } k+1\text{”}]$$

(union bound)
$$\leq (k+1) \cdot \Pr[\text{“A given box is empty”}]$$

Expectation of X' (for n large enough)



- n products
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- X_i, products in box i
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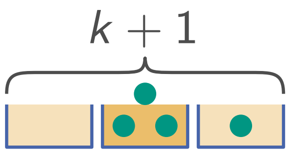
$$
\begin{aligned}
\mathbb{E}[X'] &= \textstyle\sum_{i=1}^{m} \mathbb{E}[X_i \cdot Y_i'] \\
\text{(law of total expectation)} \quad &= \textstyle\sum_{\ell=0}^{k+1} \mathbb{E}[X_i \cdot Y_i' \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell] \\
&= \textstyle\sum_{\ell=0}^{k} \mathbb{E}[X_i \cdot Y_i' \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell] \\
&= \textstyle\sum_{\ell=1}^{k} \mathbb{E}[X_i \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell] \\
&\leq \textstyle\sum_{\ell=1}^{k} 2\log\log(n) \cdot \Pr[E_k(i) = \ell] \\
&= 2\log\log(n) \underbrace{\textstyle\sum_{\ell=1}^{k} \Pr[E_k(i) = \ell]}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Pr[\text{“Exists an empty box among } k+1\text{”}] \\
\text{(union bound)} \quad &\leq (k+1) \cdot \Pr[\text{“A given box is empty”}] \\
&\leq 2k\left(1 - \frac{1}{m}\right)^{n}
\end{aligned}
$$

Expectation of X' (for n large enough)



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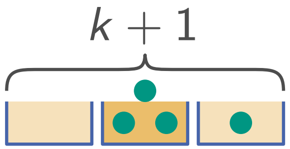
&\leq \sum_{\ell=1}^{k} 2\log\log(n) \cdot \Pr[E_k(i) = \ell] \\
&= 2\log\log(n) \sum_{\ell=1}^{k} \Pr[E_k(i) = \ell]
\end{aligned}$$

$$\begin{aligned}
\sum_{\ell=1}^{k} \Pr[E_k(i) = \ell] &\leq \Pr[\text{“Exists an empty box among } k+1\text{”}] \\
\text{(union bound)} \quad &\leq (k+1) \cdot \Pr[\text{“A given box is empty”}] \\
&\leq 2k \Big(1 - \underbrace{\tfrac{1}{m}}_{\text{a product hits a given box}}\Big)^{n}
\end{aligned}$$

Expectation of X' (for n large enough)



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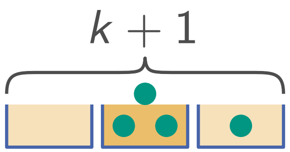
$$= 2\log\log(n) \underbrace{\textstyle\sum_{\ell=1}^{k} \Pr[E_k(i) = \ell]}$$

$$\le \Pr[\text{“Exists an empty box among } k+1\text{”}]$$

(union bound)
$$\le (k+1) \cdot \Pr[\text{“A given box is empty”}]$$

$$\le 2k \underbrace{\left(1 - \frac{1}{m}\right)}_{\text{a product does not hit a given box}}{}^{n}$$

Expectation of X' (for n large enough)



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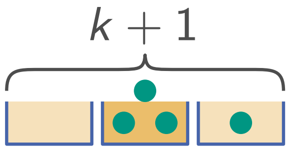
&= \sum_{\ell=1}^{k} \mathbb{E}[X_i \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell] \\
&\leq \sum_{\ell=1}^{k} 2 \log\log(n) \cdot \Pr[E_k(i) = \ell] \\
&= 2 \log\log(n) \underbrace{\sum_{\ell=1}^{k} \Pr[E_k(i) = \ell]}
\end{aligned}$$

$$\leq \Pr[\text{“Exists an empty box among } k+1\text{”}]$$

(union bound) $\leq (k+1) \cdot \Pr[\text{“A given box is empty”}]$

$$\leq 2k \underbrace{\left(1 - \frac{1}{m}\right)^n}_{\text{none of the } n \text{ products hit a given box}}$$

Expectation of X' (for n large enough)



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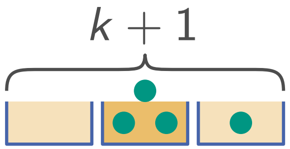
(union bound)
$$\leq (k+1) \cdot \Pr[\text{“A given box is empty”}]$$
$$\leq 2k\left(1 - \frac{1}{m}\right)^n$$
$$= 2k\left(1 - \frac{k}{n}\right)^n$$

Expectation of X' (for n large enough)



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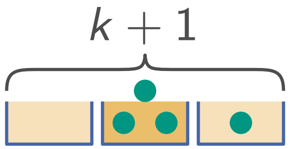
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Pr[\text{“Exists an empty box among } k+1\text{”}] \\
\text{(union bound)} \quad &\leq (k+1) \cdot \Pr[\text{“A given box is empty”}] \\
&\leq 2k\left(1 - \frac{1}{m}\right)^n \\
&= 2k\left(1 - \frac{k}{n}\right)^n \overset{(1+x \leq e^x)}{\leq} 2k \cdot e^{-k}
\end{aligned}
$$

Expectation of X' (for n large enough)



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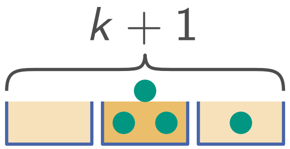
$$\leq 2k\left(1 - \frac{1}{m}\right)^n$$
$$= 2k\left(1 - \frac{k}{n}\right)^n \overset{(1+x \leq e^x)}{\leq} 2k \cdot e^{-k} = 2\frac{\log\log(n)}{\log(n)}$$

Expectation of X' (for n large enough)



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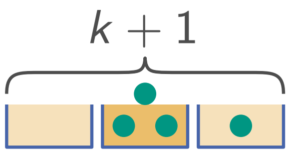
Expectation of X' (for n large enough)



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$$\leq \sum_{\ell=1}^{k} 2\log\log(n) \cdot \Pr[E_k(i) = \ell]$$
$$= 2\log\log(n) \sum_{\ell=1}^{k} \Pr[E_k(i) = \ell]$$
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Expectation of X' (for n large enough)

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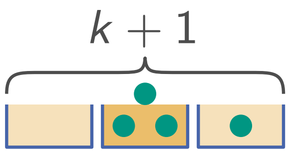

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&= \sum_{\ell=1}^{k} \mathbb{E}[X_i \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell] \\
&\leq \sum_{\ell=1}^{k} 2 \log\log(n) \cdot \Pr[E_k(i) = \ell] \\
&= 2\log\log(n) \sum_{\ell=1}^{k} \Pr[E_k(i) = \ell] \\
&\leq 2\log\log(n) \cdot 2\frac{\log\log(n)}{\log(n)} \\
&= 4\frac{\log\log(n)^2}{\log(n)}
\end{aligned}
$$

$$\mathbb{E}[X'] = \sum_{i=1}^{m} 4\frac{\log\log(n)^2}{\log(n)}$$

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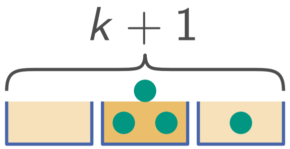

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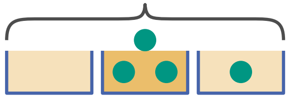
&\leq \sum_{\ell=1}^{k} 2\log\log(n) \cdot \Pr[E_k(i) = \ell] \\
&= 2\log\log(n) \sum_{\ell=1}^{k} \Pr[E_k(i) = \ell] \\
&\leq 2\log\log(n) \cdot 2\frac{\log\log(n)}{\log(n)} \\
&= 4\frac{\log\log(n)^2}{\log(n)}
\end{aligned}$$

$$\mathbb{E}[X'] = \sum_{i=1}^{m} 4\frac{\log\log(n)^2}{\log(n)} = m \cdot 4\frac{\log\log(n)^2}{\log(n)}$$

Expectation of X' (for n large enough)

$k+1$



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&\leq \sum_{\ell=1}^{k} 2 \log\log(n) \cdot \Pr[E_k(i) = \ell] \\
&= 2 \log\log(n) \sum_{\ell=1}^{k} \Pr[E_k(i) = \ell] \\
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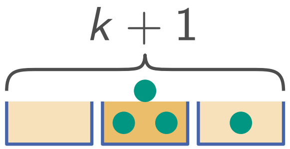
\end{aligned}$$

$$\mathbb{E}[X'] = \sum_{i=1}^{m} 4 \frac{\log\log(n)^2}{\log(n)} = m \cdot 4 \frac{\log\log(n)^2}{\log(n)} = \frac{n}{\log\log(n)} \cdot 4 \frac{\log\log(n)^2}{\log(n)}$$

Expectation of X' (for n large enough)



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$$\leq 2 \log\log(n) \cdot 2 \frac{\log\log(n)}{\log(n)}$$

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$$\mathbb{E}[X'] = \sum_{i=1}^{m} 4 \frac{\log\log(n)^2}{\log(n)} = m \cdot 4 \frac{\log\log(n)^2}{\log(n)} = \frac{n}{\cancel{\log\log(n)}} \cdot 4 \frac{\log\log(n)^{\cancel{2}}}{\log(n)}$$

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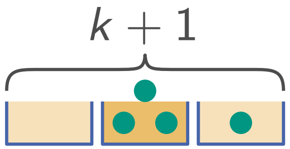
Expectation of X' (for n large enough)



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(law of total expectation)

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&\leq 2\log\log(n) \cdot 2\frac{\log\log(n)}{\log(n)} \\
&= 4\frac{\log\log(n)^2}{\log(n)}
\end{aligned}$$

$$\mathbb{E}[X'] = \sum_{i=1}^{m} 4\frac{\log\log(n)^2}{\log(n)} = m \cdot 4\frac{\log\log(n)^2}{\log(n)} = \frac{n}{\cancel{\log\log(n)}} \cdot 4\frac{\log\log(n)^{\cancel{2}}}{\log(n)} = n \cdot 4\frac{\log\log(n)}{\log(n)}$$

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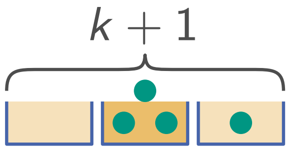

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Concentration of $\underline{X}$ (for n large enough)

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Karlsruhe Institute of Technology

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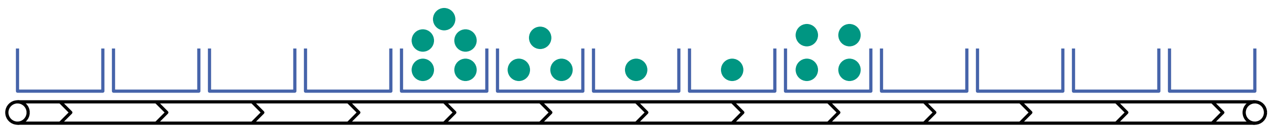

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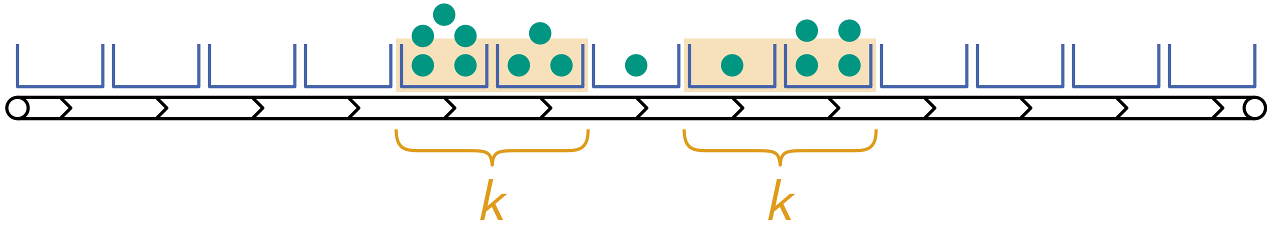

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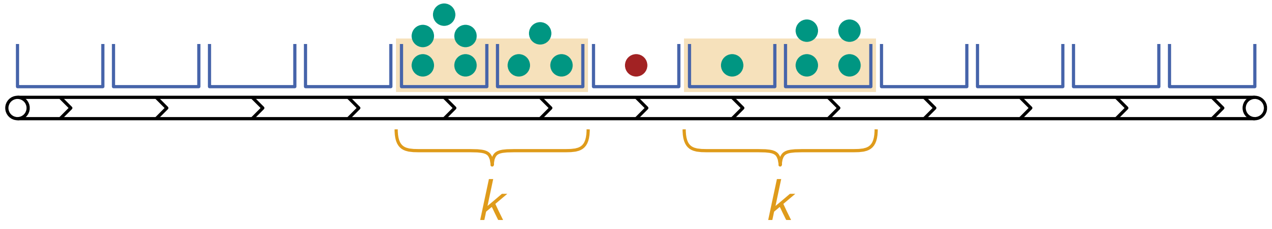

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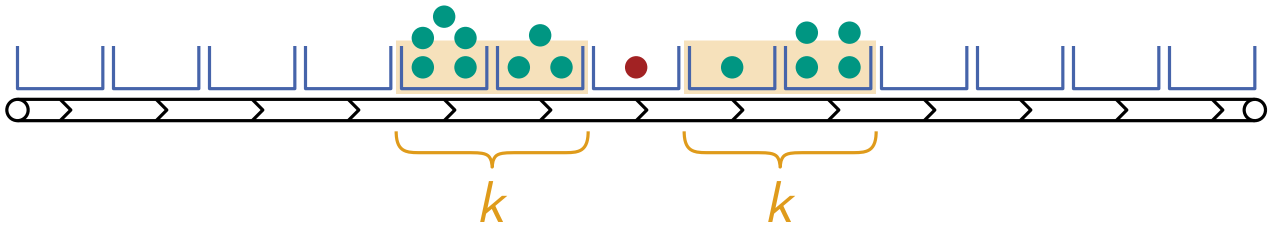

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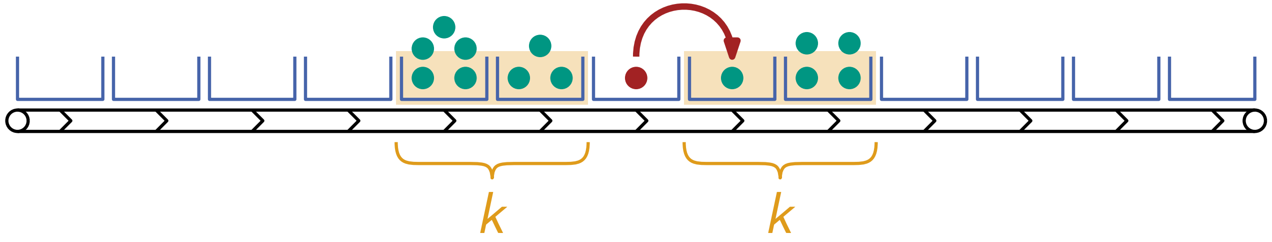

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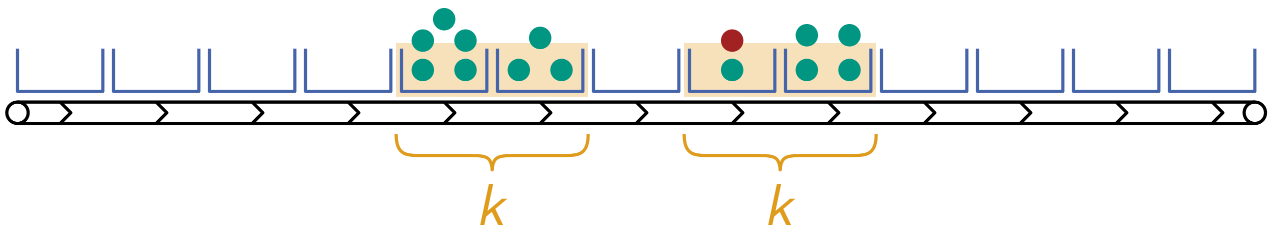

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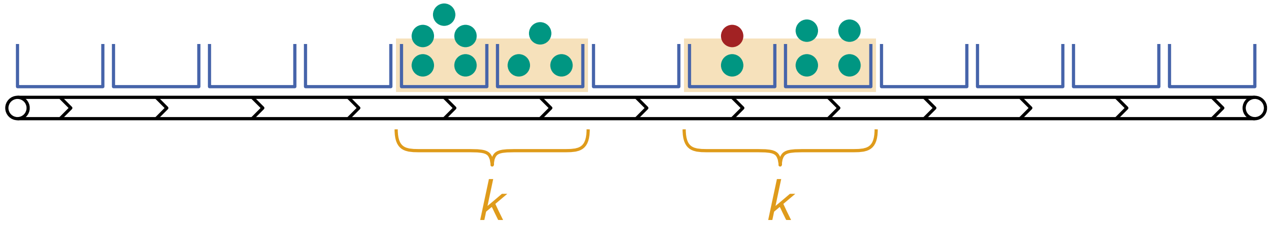

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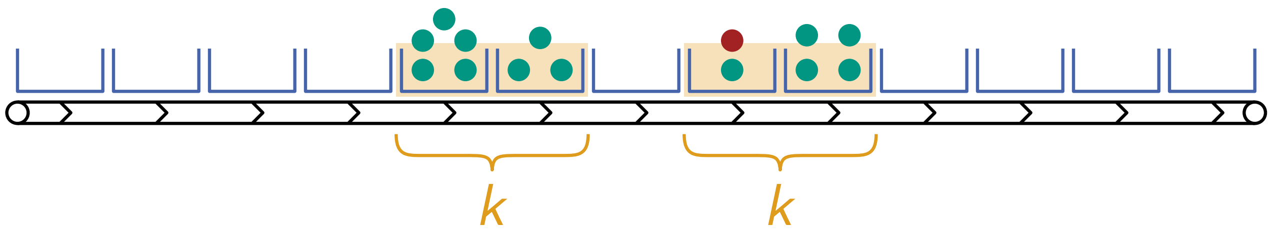

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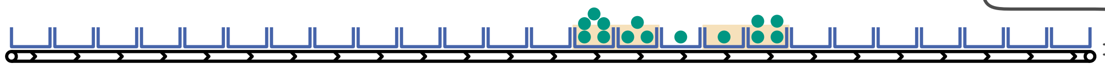

But this case (all products in few boxes) is super unlikely...

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Corollary of "On the Method of Typical Bounded Differences", Warnke, Comb. Probab. Comput. 2015

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Karlsruhe Institute of Technology

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The more we need to mitigate, the higher the price!

Not too bad if A is very likely to occur!

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- View X as a function $f(Z_1, ..., Z_n)$ of independent rand. var. where Z_j for $j \in [n]$ denotes the box of the j-th product
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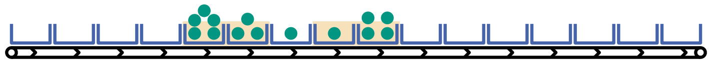

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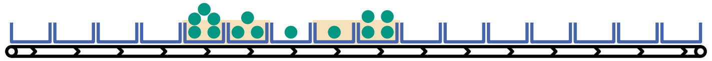

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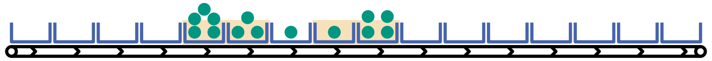

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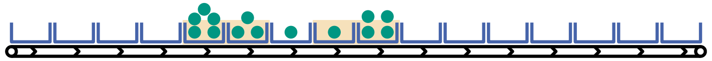

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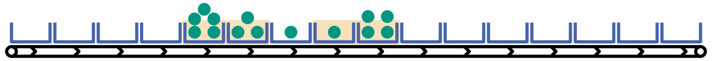

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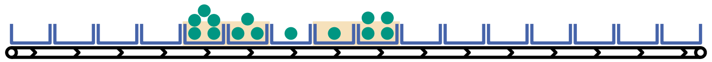

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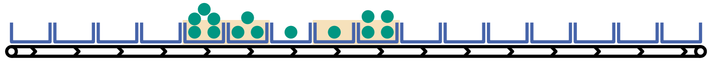

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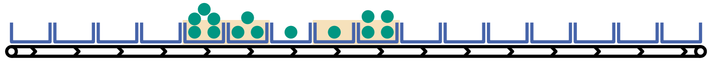

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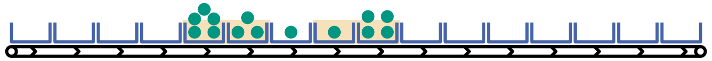

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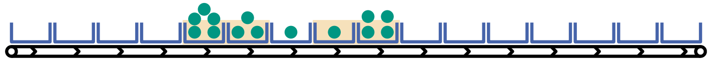

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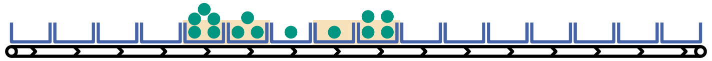

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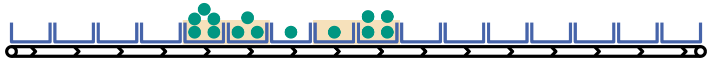

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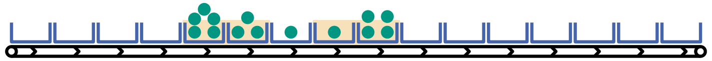

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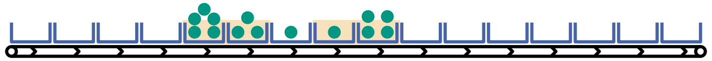
  - Union bound over $\leq n$ sequences: $\Pr[\neg A] \leq n^{-\delta\varepsilon^2/3+1} \leq n^{-\lambda}$ (for arbitrarily large λ)



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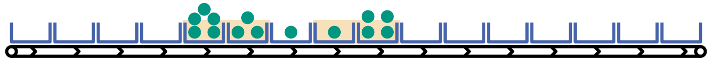

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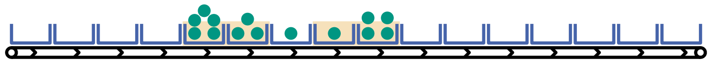

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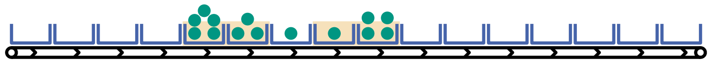

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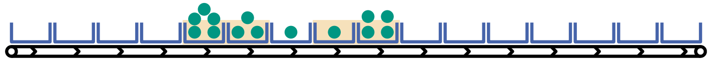

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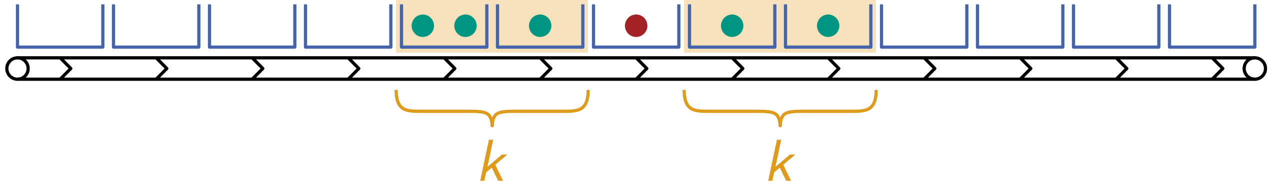

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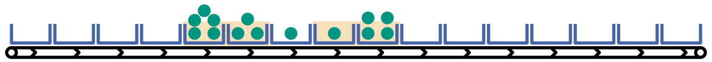

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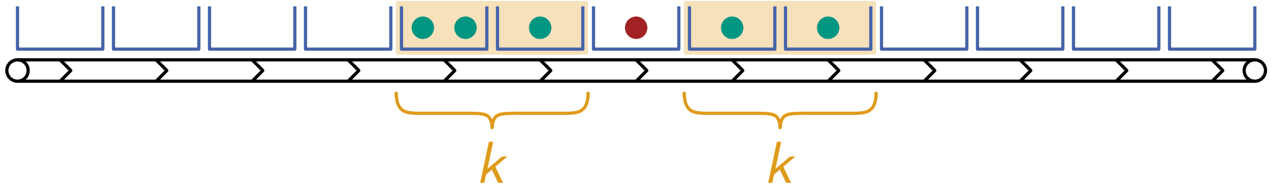

  - Moving one product empties at most one box $\Rightarrow$ at most two new short chains

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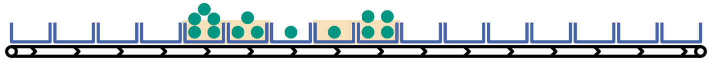

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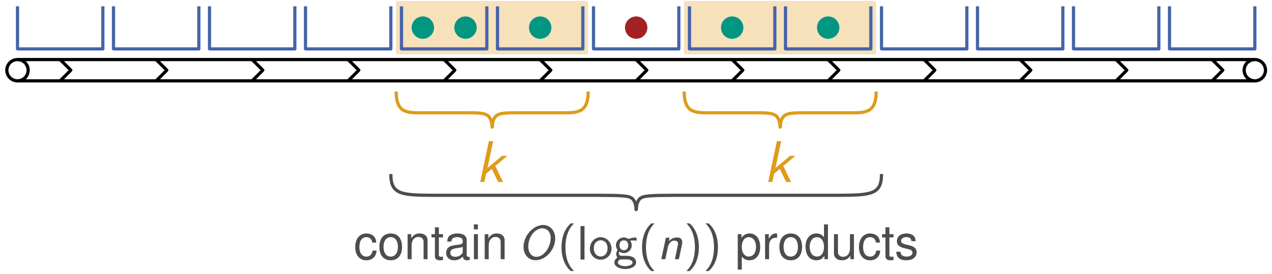

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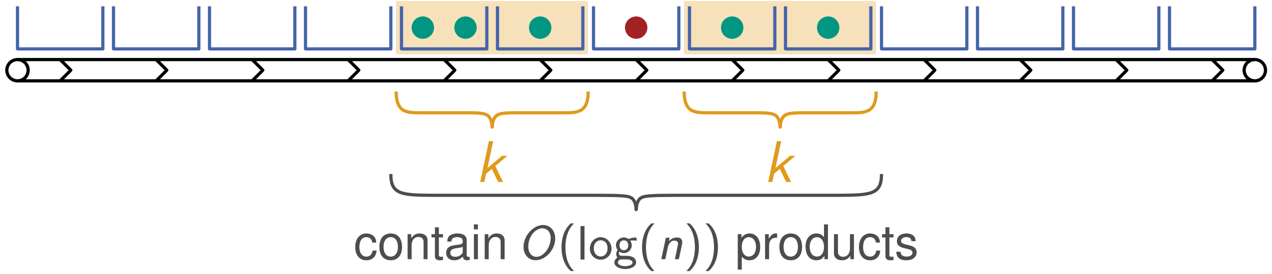

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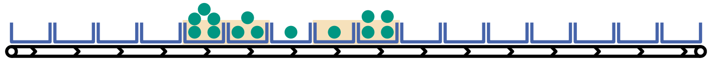

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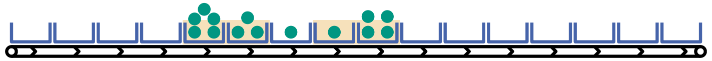

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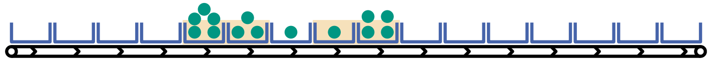

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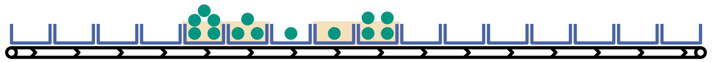

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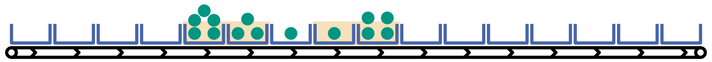

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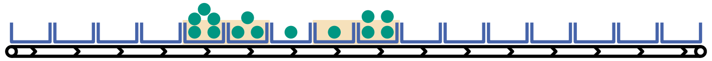

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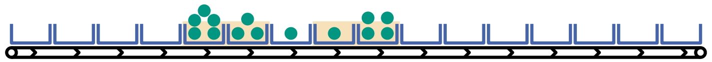

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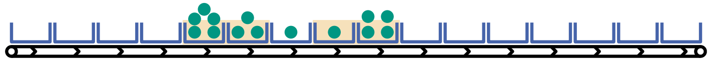

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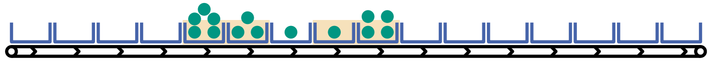

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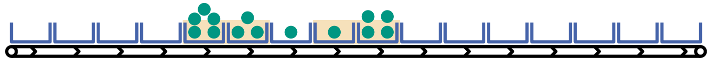

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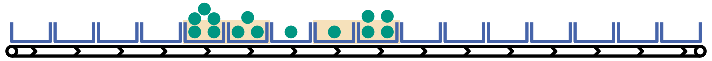

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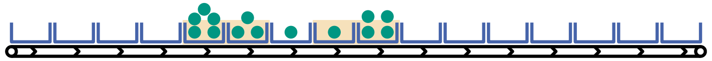

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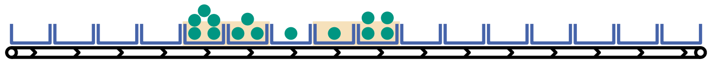

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&= \textstyle\sum_{j=1}^{n} (O(\log(n)) + 1)^2 \\
&= O(n \log(n)^2)
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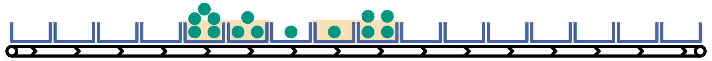
Much better than n^3 from before!



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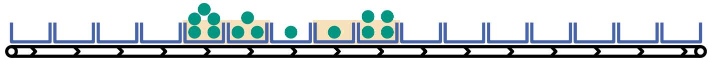

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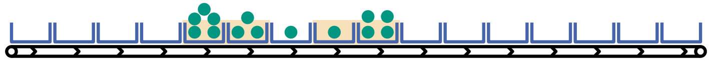

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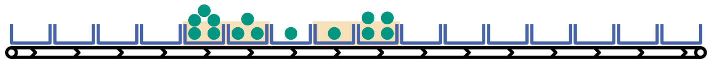

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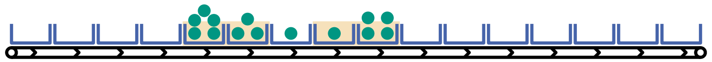

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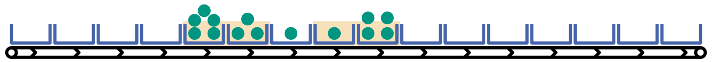

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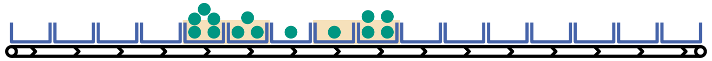

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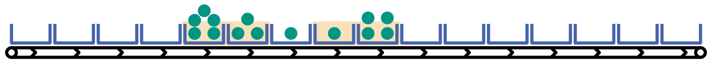

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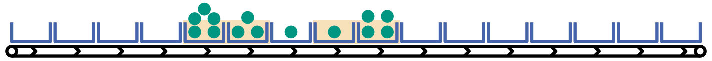

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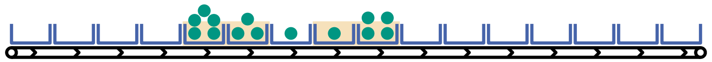

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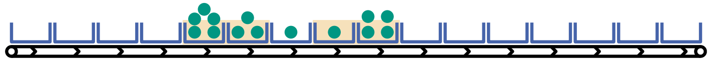

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Motivation

- Average-case analysis: analyze models that represent the real world
- Models seen so far
  - Erdős-Rényi random graphs: simple but no locality

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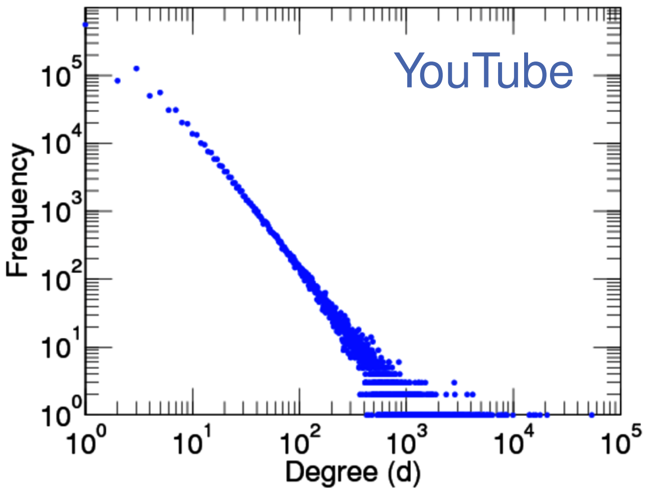

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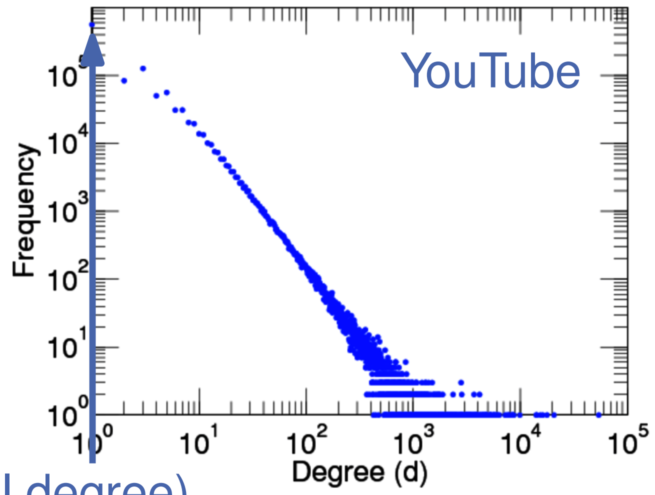

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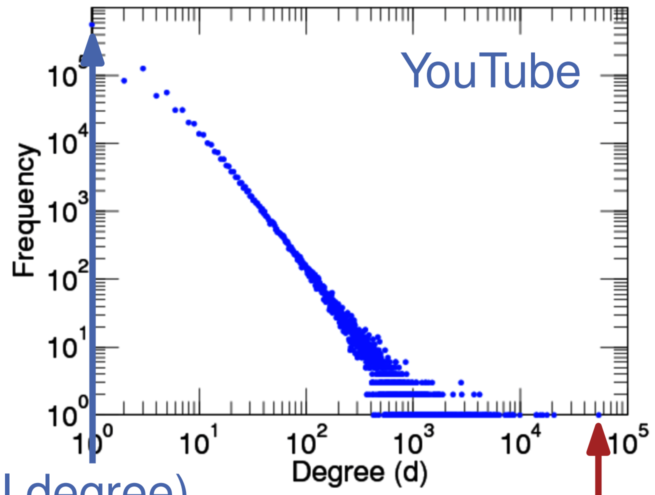

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- Realistic representation: power-law distribution

"Scale-free networks well done", Voitalov, van der Hoorn, van der Hofstad, Krioukov, Phys. Rev. Research. 2019

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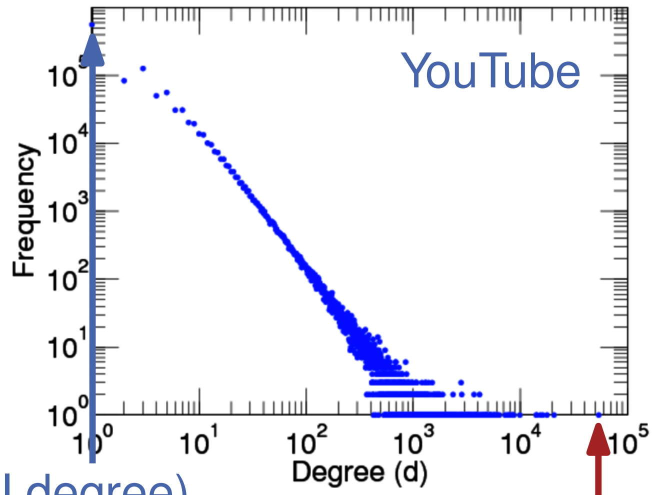

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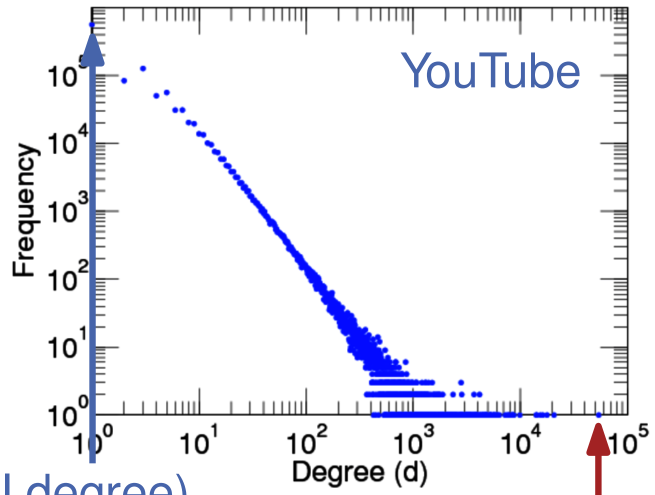
  - Erdős-Rényi random graphs: simple but no locality
  - Random geometric graphs: locality but no heterogeneity (all vertices roughly same degree)

Not realistic: celebrities are very-high-degree vertices in social networks

konect.cc/plot/degree.a.youtube-links.full.png



(most vertices small degree)

(few vertices very high degree)

- Realistic representation: power-law distribution

"Scale-free networks well done", Voitalov, van der Hoorn, van der Hofstad, Krioukov, Phys. Rev. Research. 2019

- Pareto distribution: $X \sim \mathrm{Par}(\alpha, x_{\min})$

$$f_X(x) = \begin{cases} \alpha x_{\min}^{\alpha} \cdot x^{-(\alpha+1)}, & \text{if } x \geq x_{\min} \\ 0, & \text{otherwise} \end{cases}$$

Geometric Inhomogeneous Random Graphs

Karlsruhe Institute of Technology

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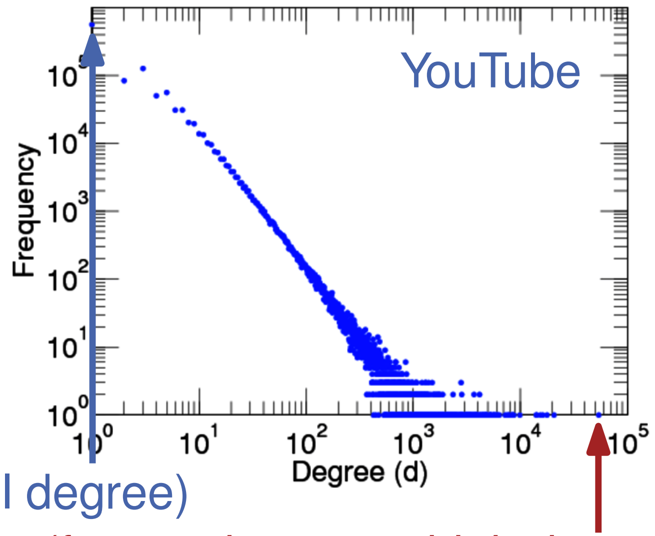

(most vertices small degree)

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Idea

- Add Pareto distribution to RGGs

Geometric Inhomogeneous Random Graphs

Definition

- Consider n vertices

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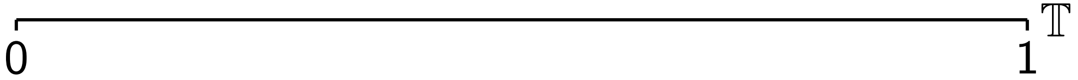

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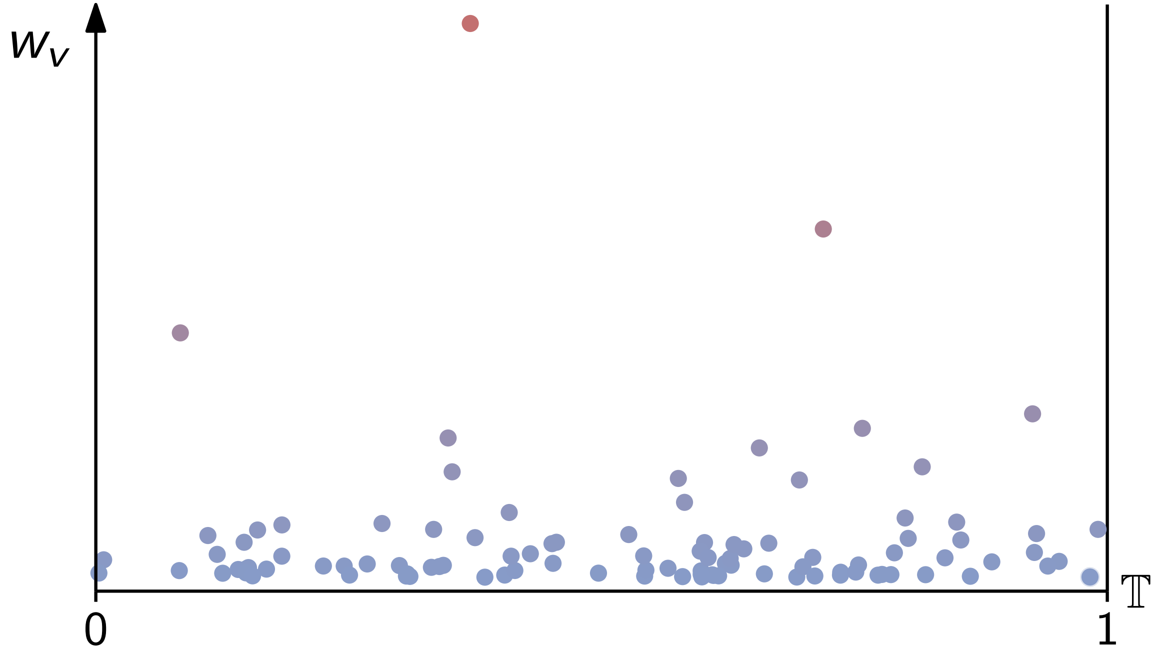

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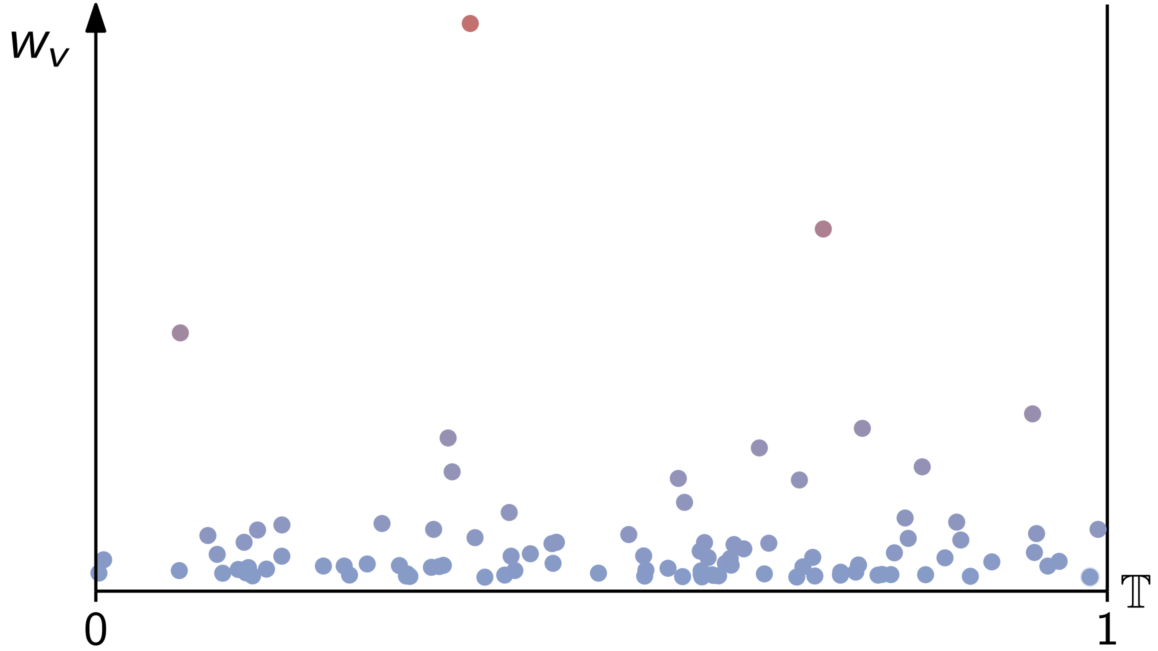

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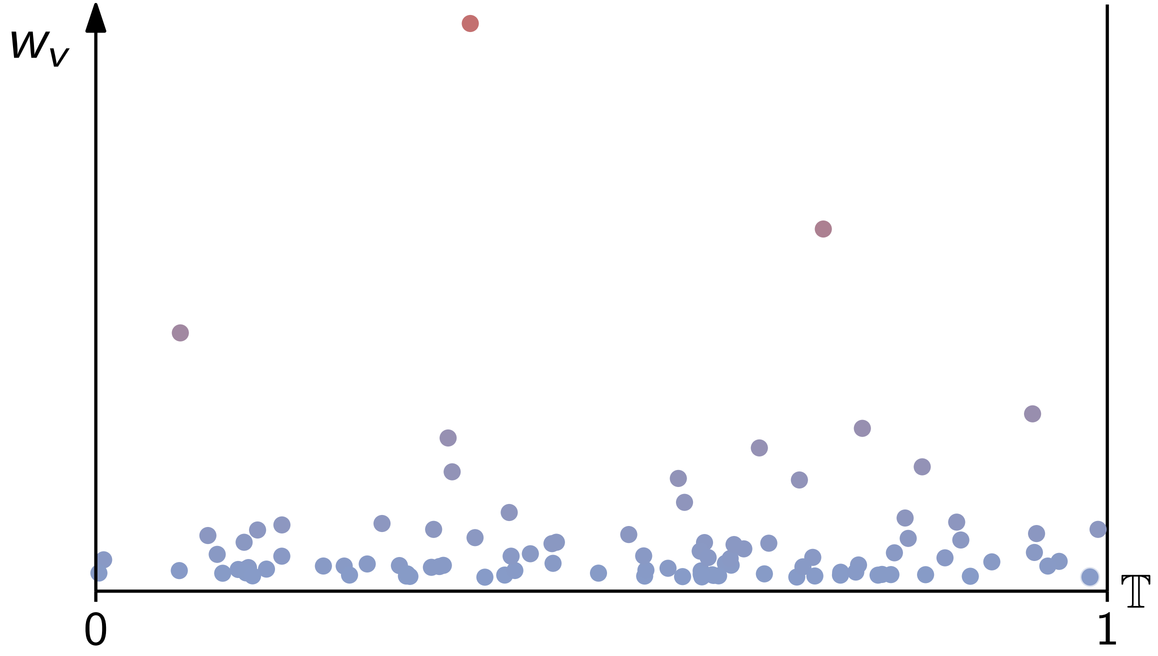

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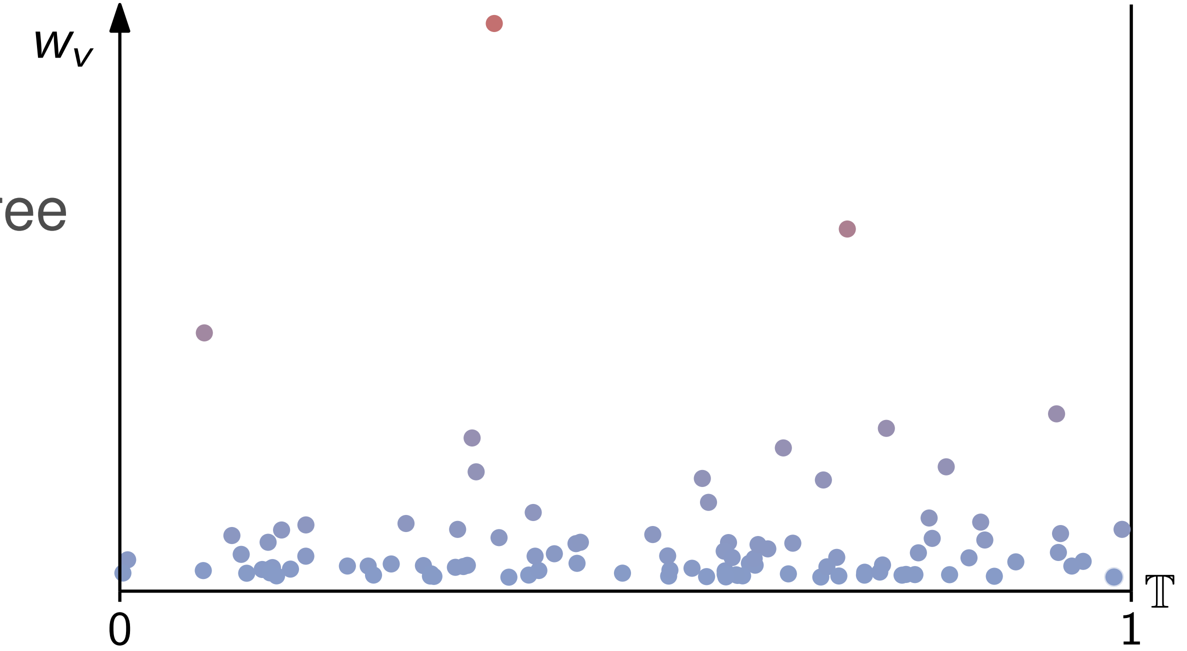

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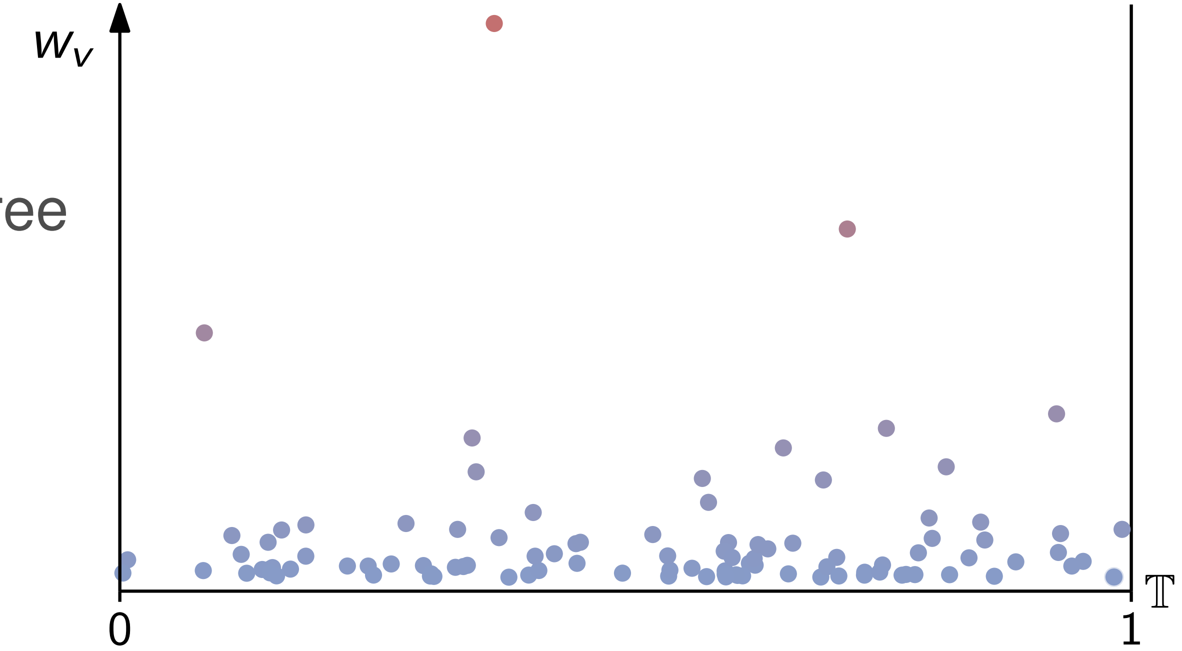

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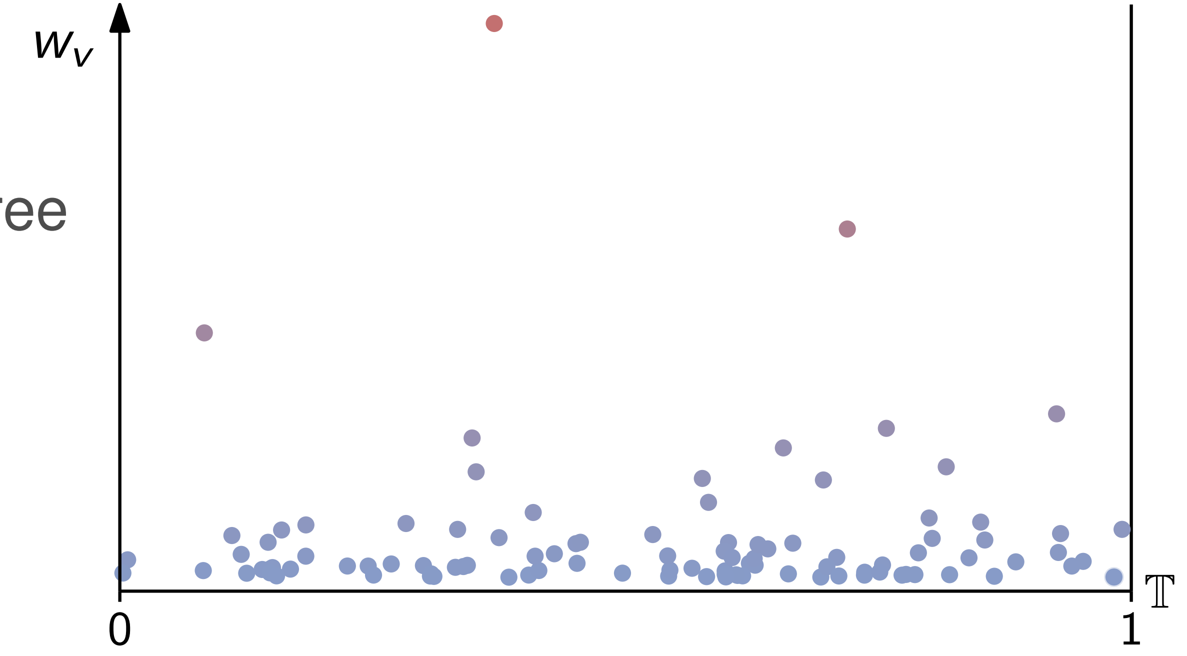

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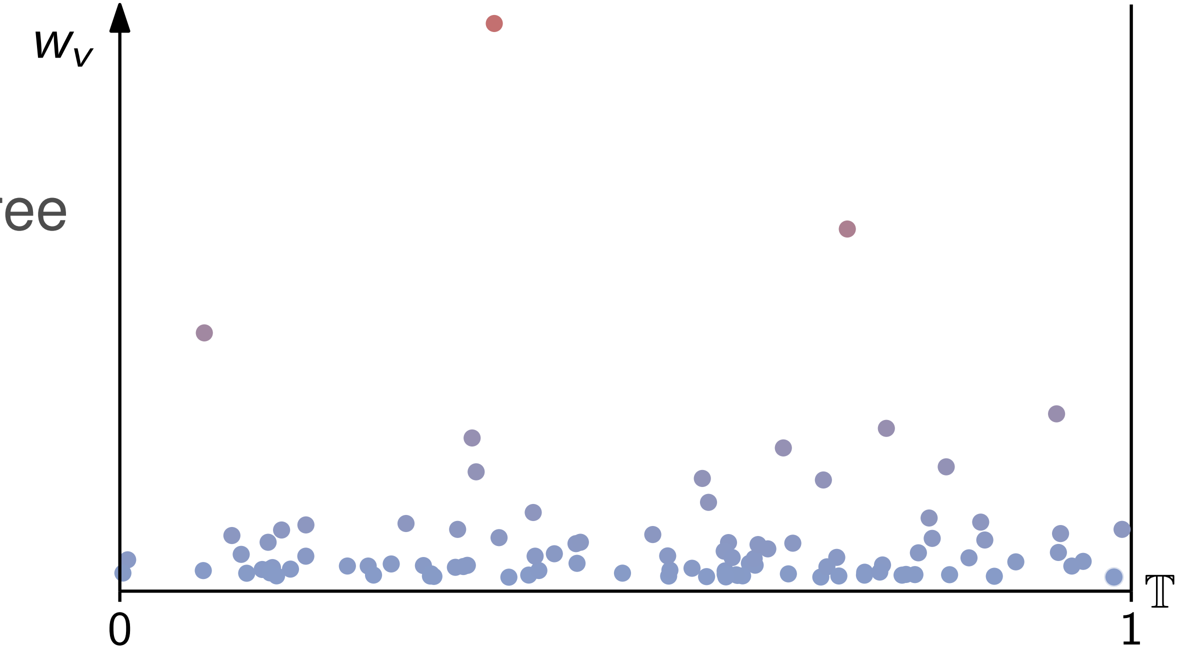

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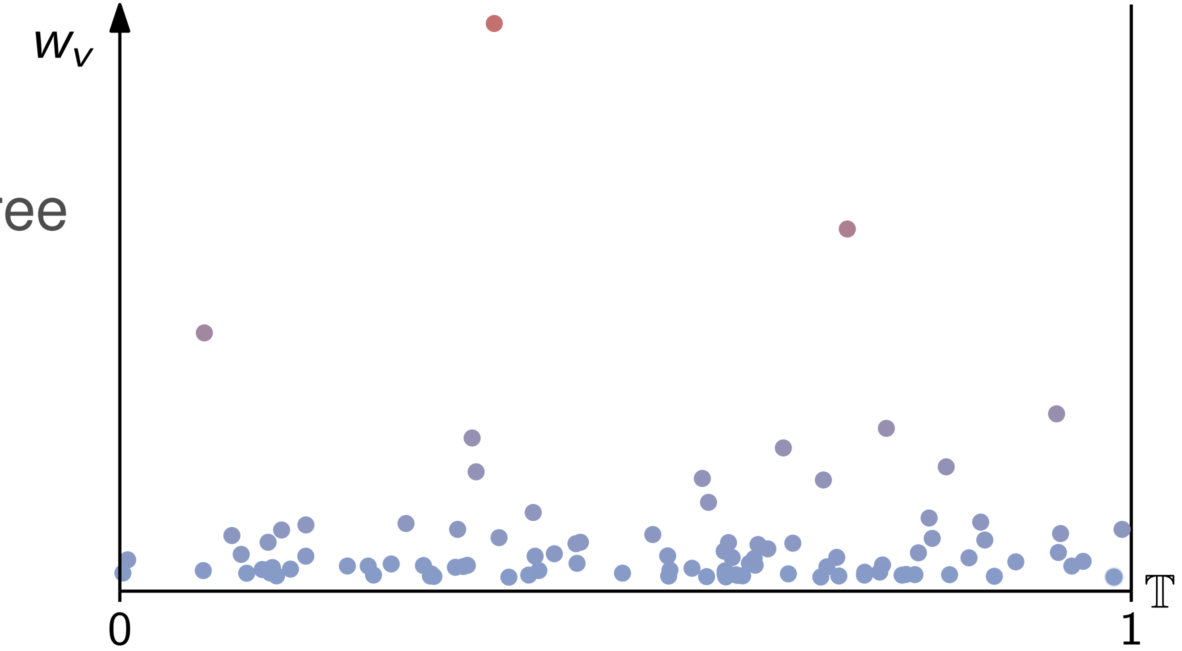

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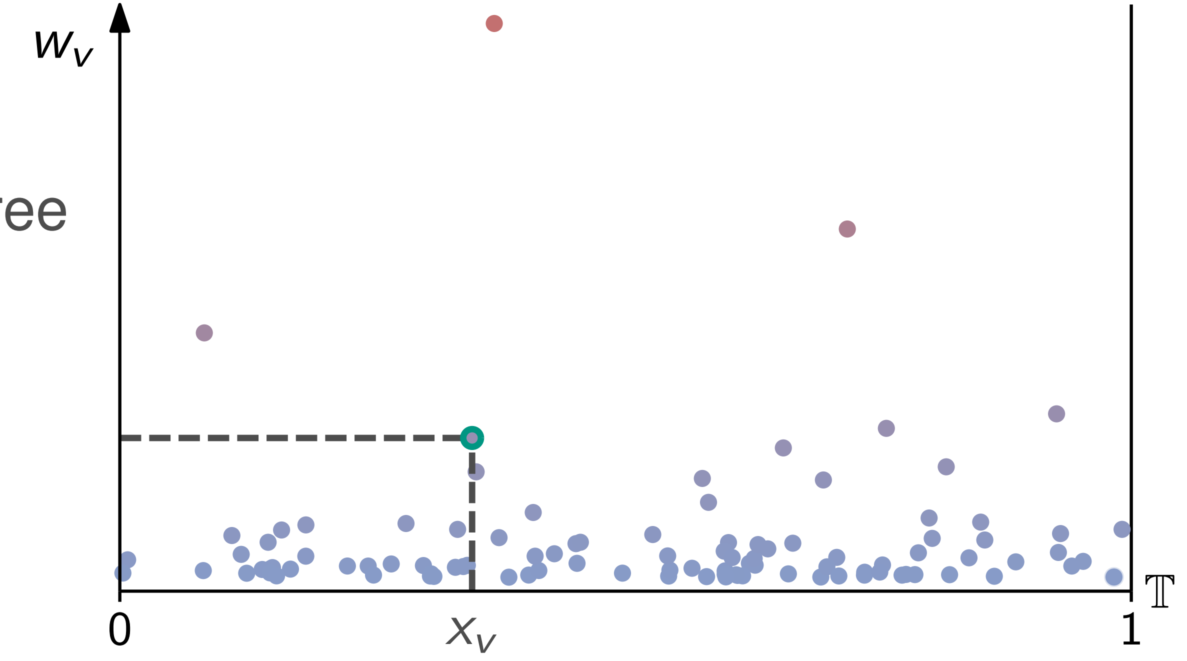

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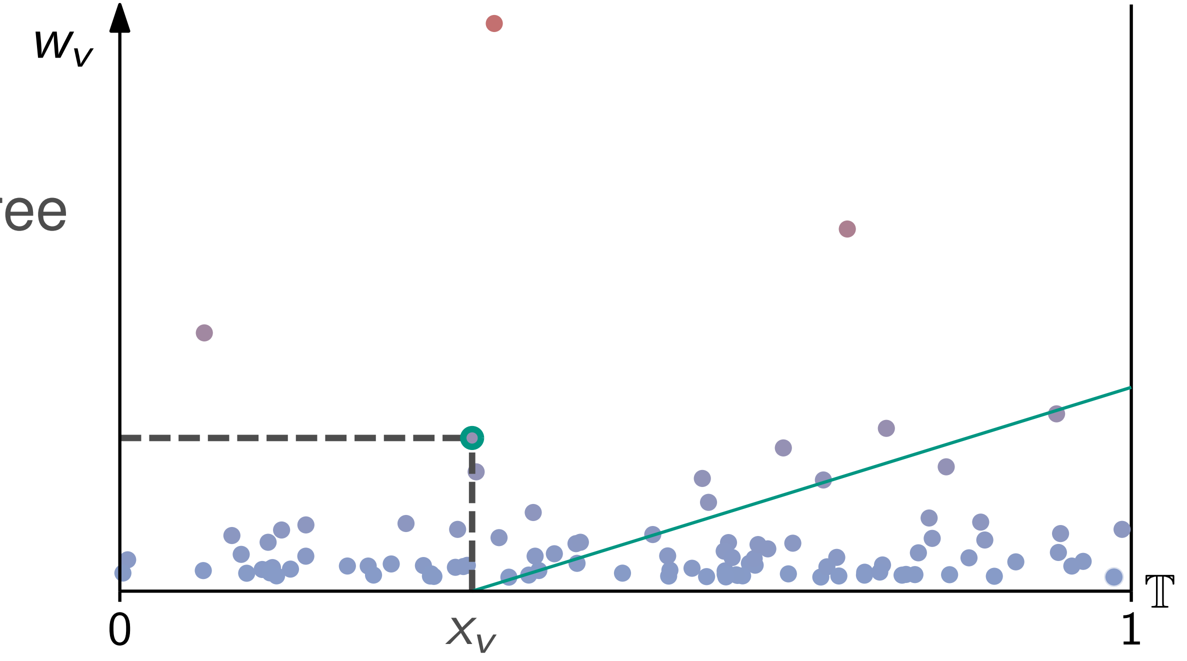

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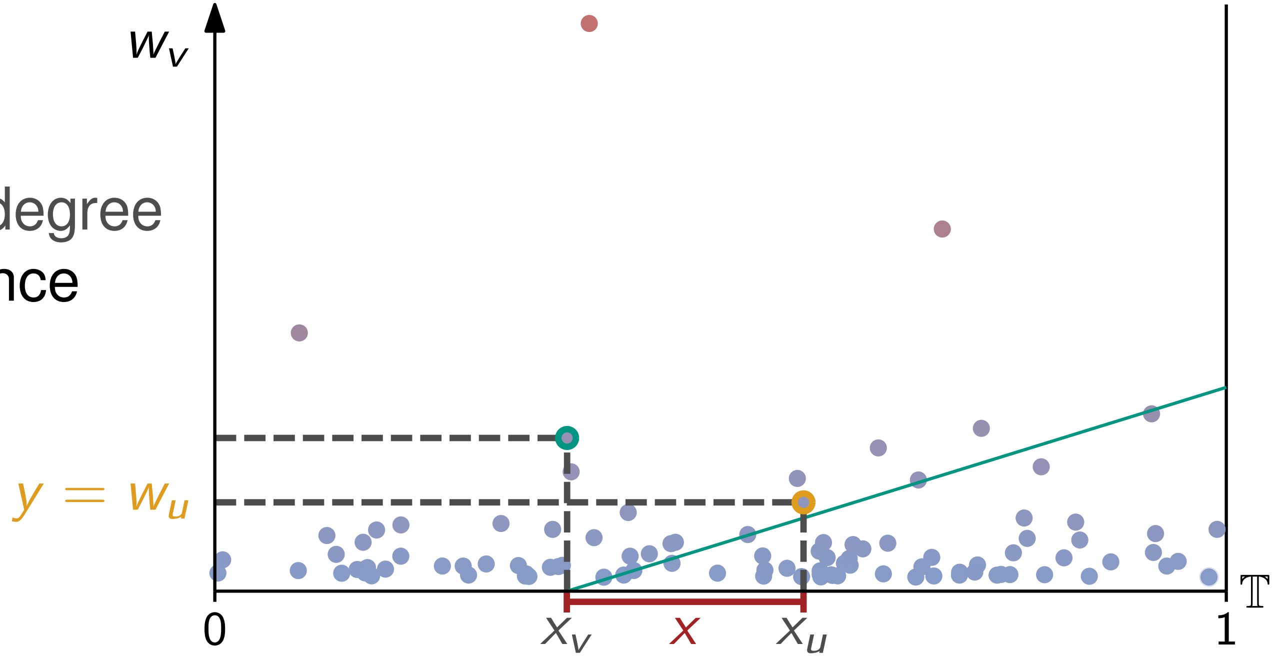

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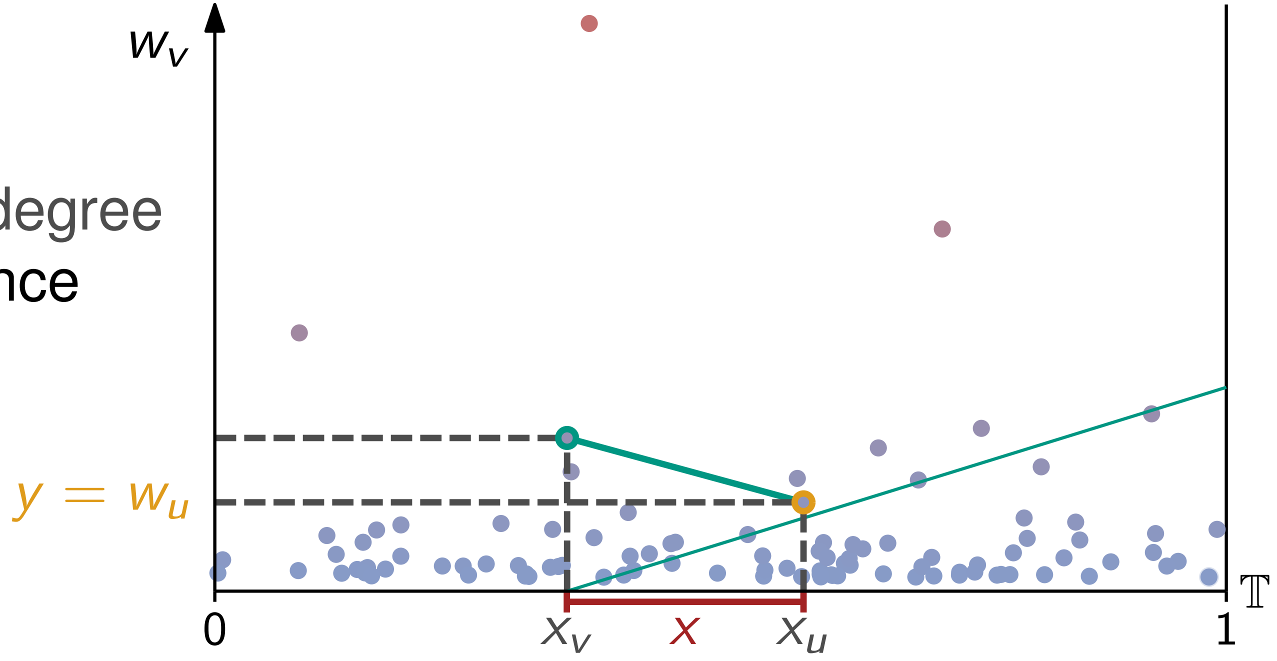

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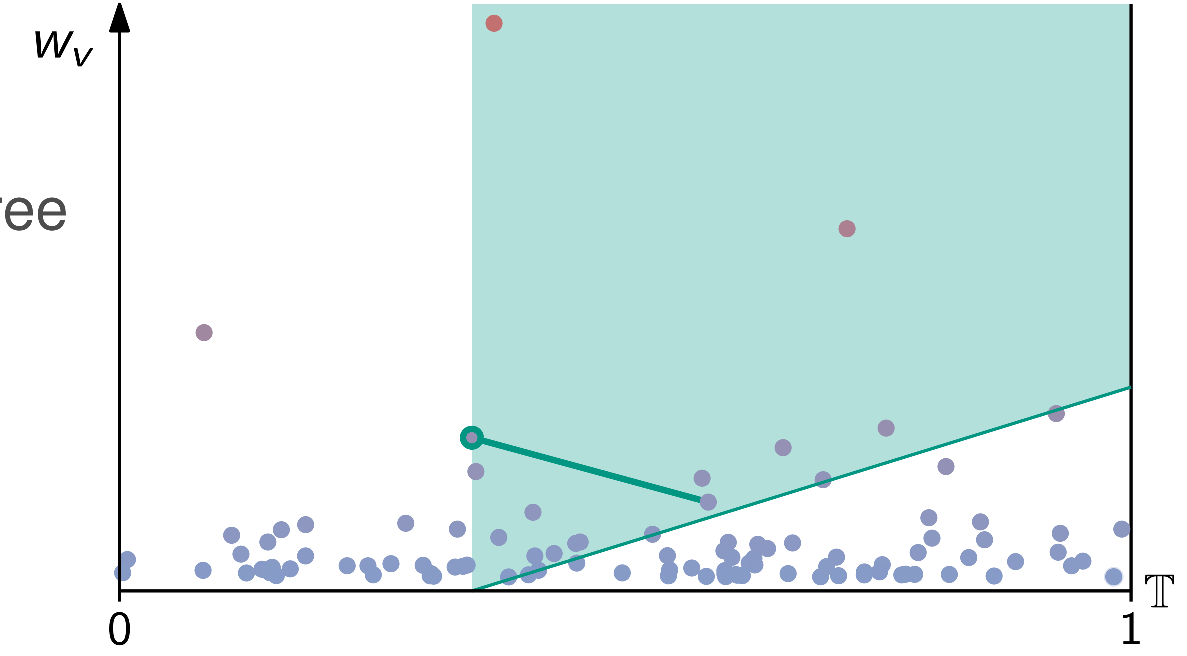

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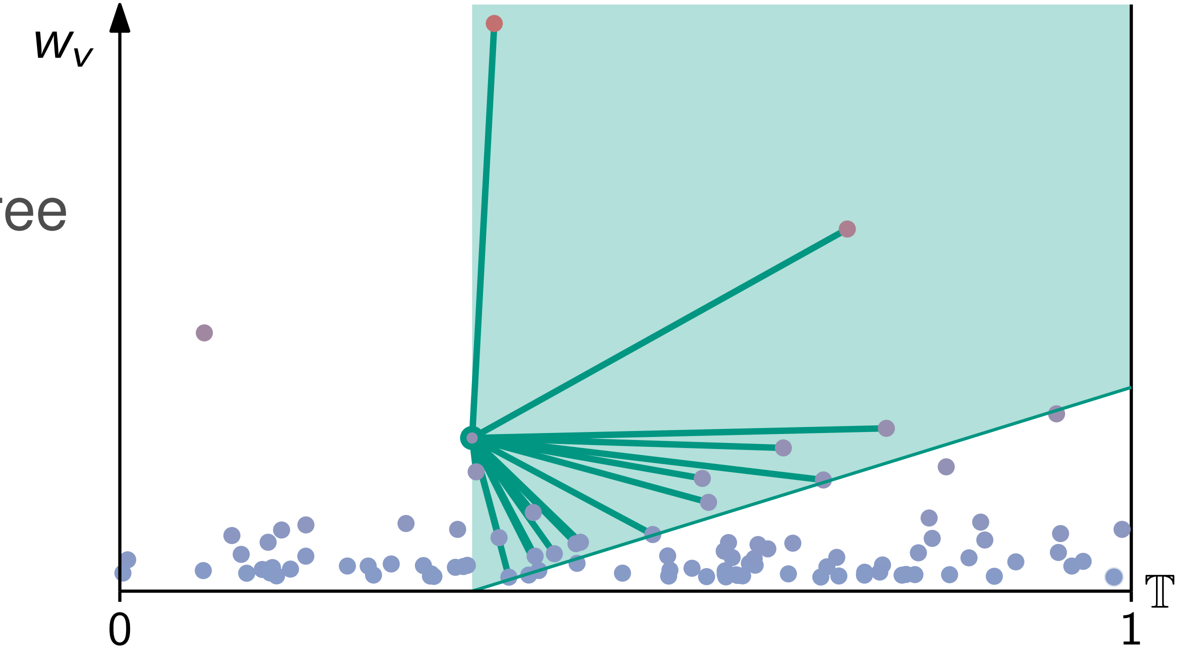

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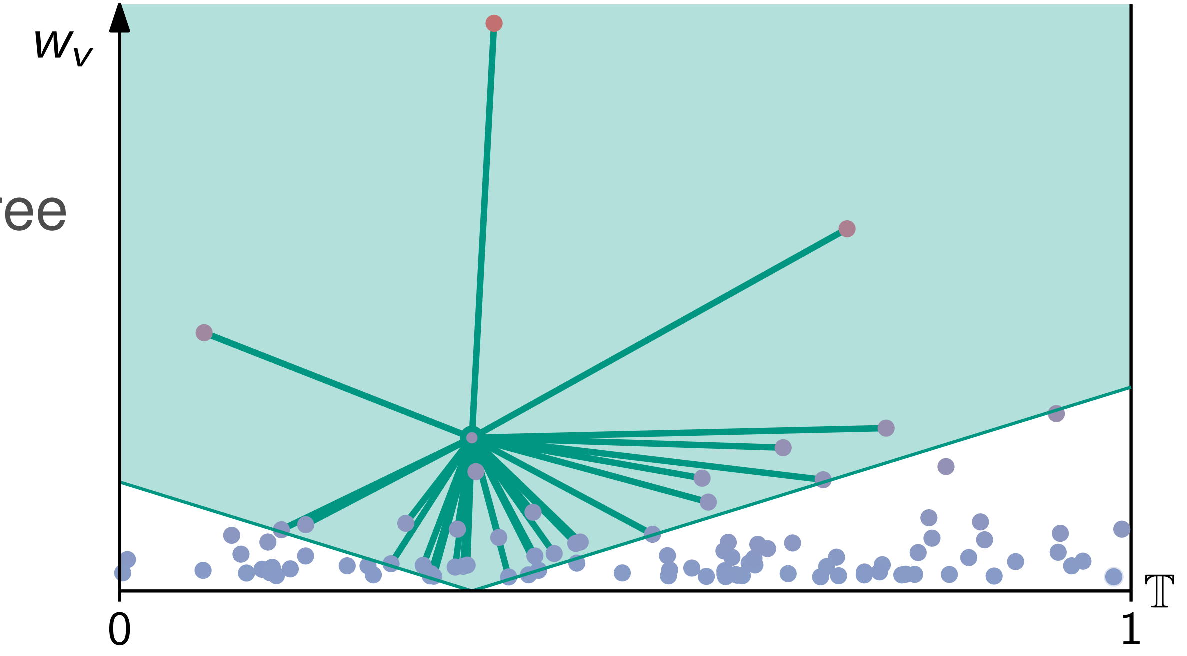

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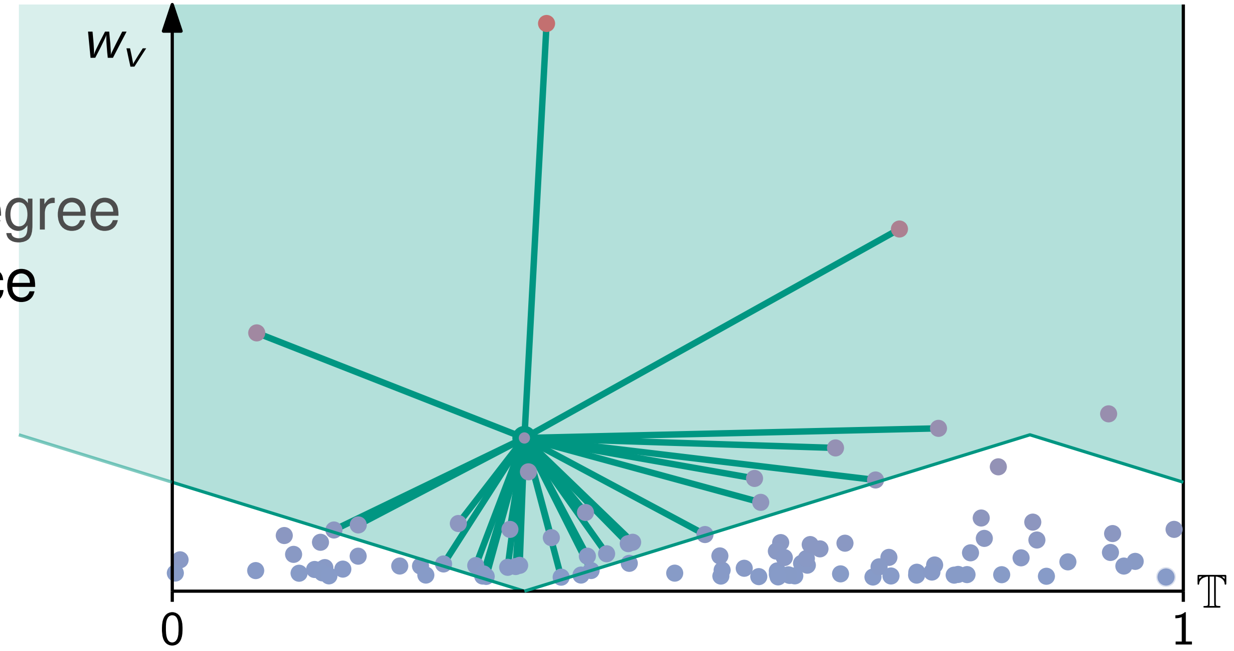

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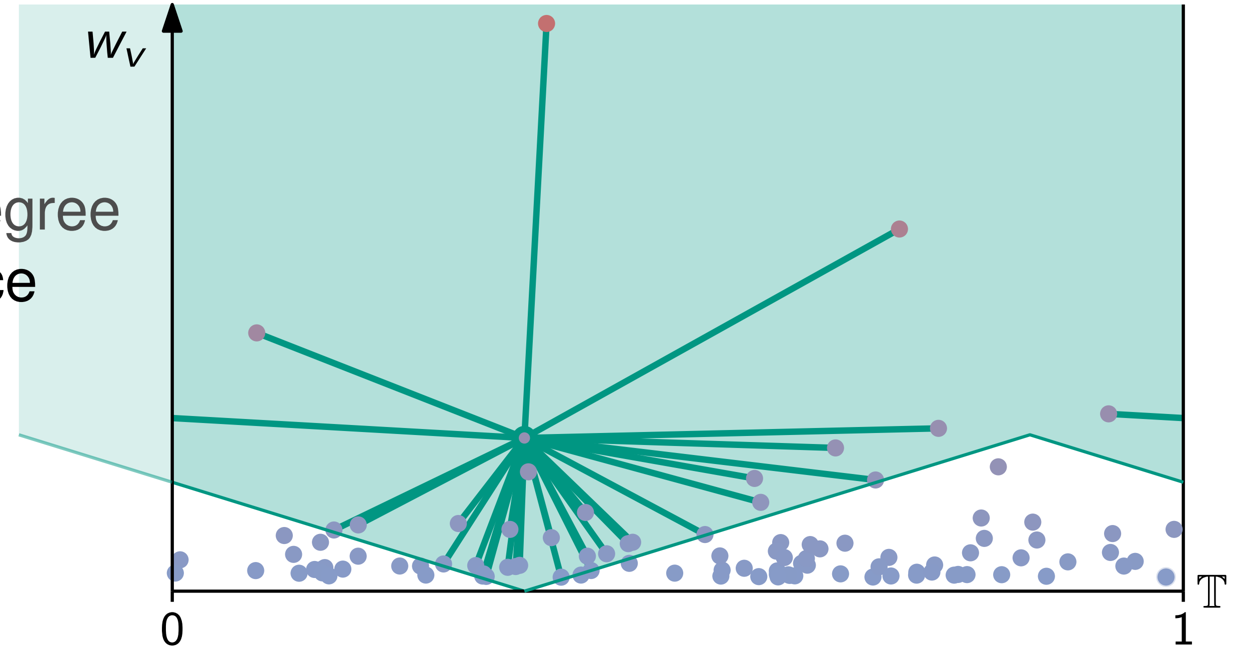

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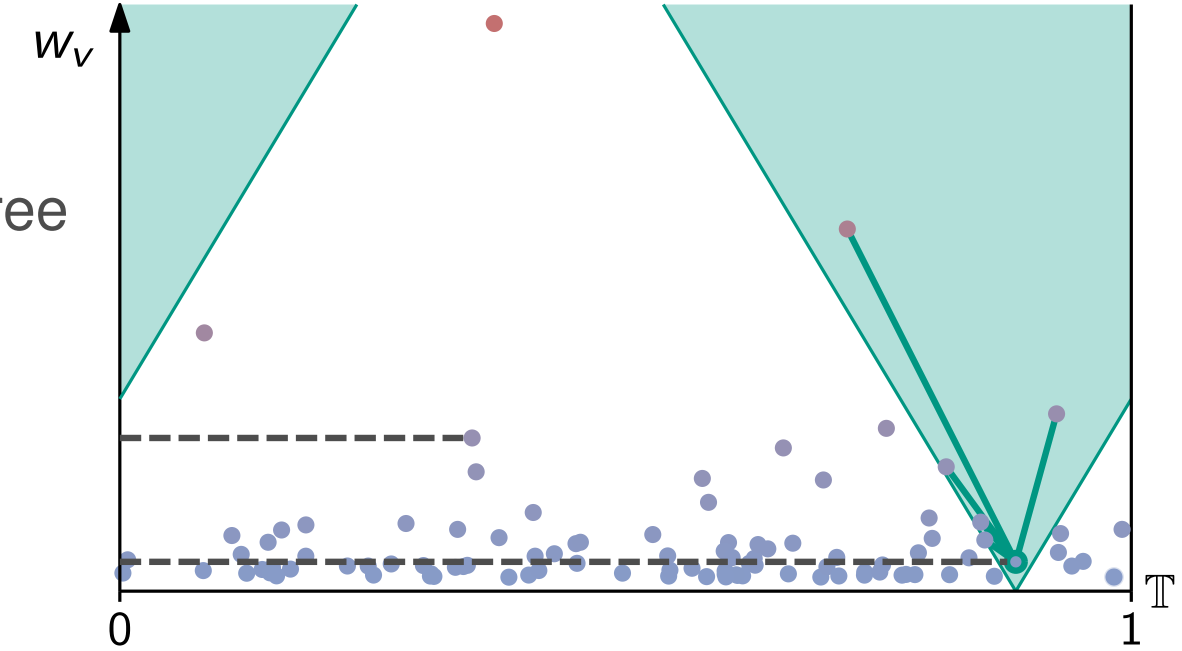
- The lower w_v, the steeper the wedge

  ↳ The lower the degree



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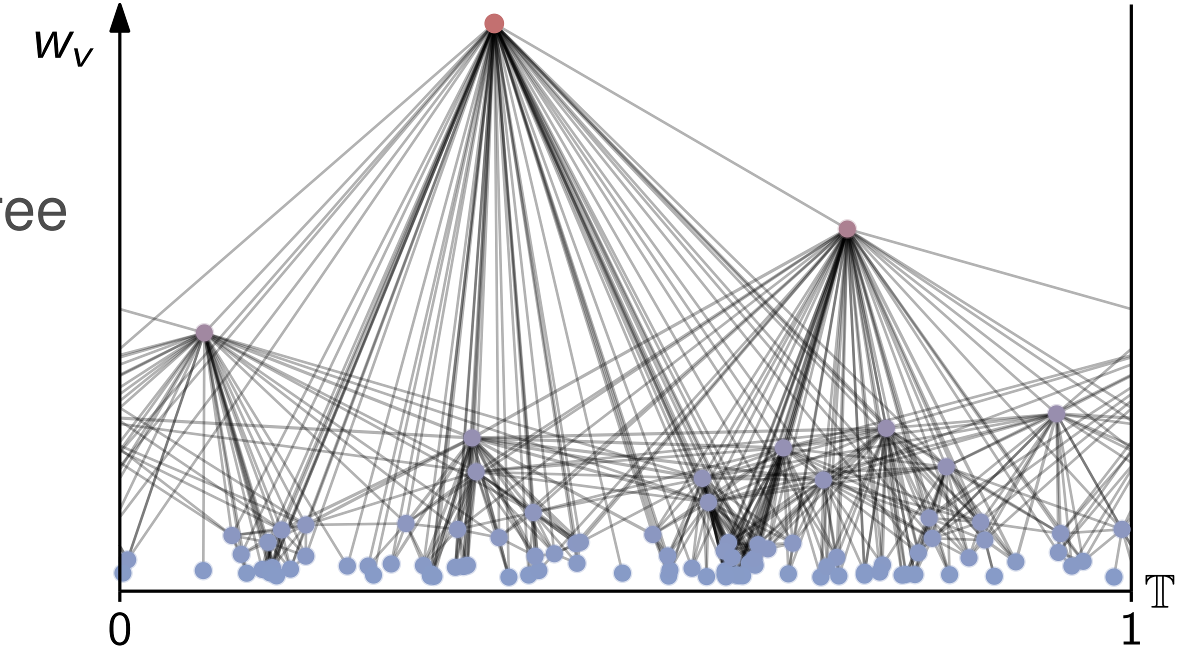

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Expected Degree ($d = 1$)

- Consider vertex v with weight w_v

GIRG
- n independent vertices
- $x_v \sim \mathcal{U}([0,1])$
- $w_v \sim \mathrm{Par}(\tau - 1, 1)$ for $\tau \in (2,3)$
  $f_{w_v}(w) = (\tau - 1)w^{-\tau}$
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This is a random variable

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Expected Degree ($d = 1$)

- Consider vertex v with weight w_v
- We want to compute $\underbrace{\mathbb{E}[\deg(v) \mid w_v]}_{\text{This is a random variable}}$

GIRG

- n independent vertices
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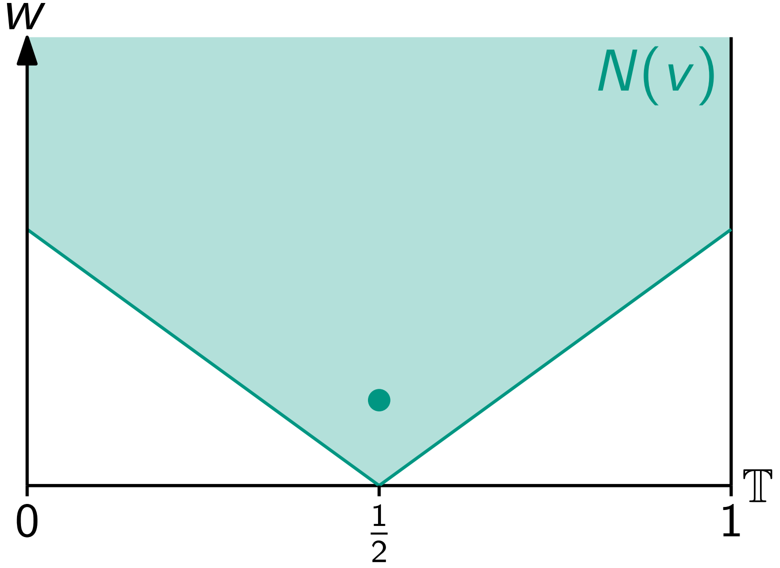

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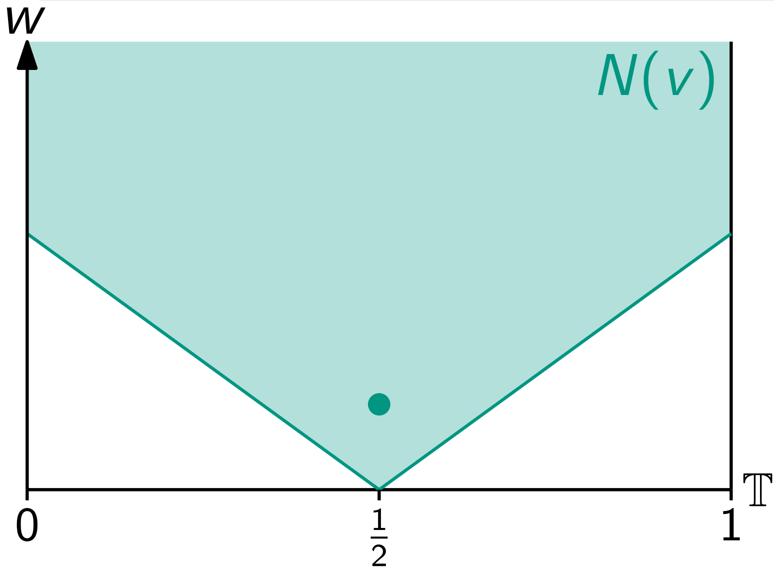

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This is *not* the area of the shape,
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$\Rightarrow$ Use law of total probability to account for that

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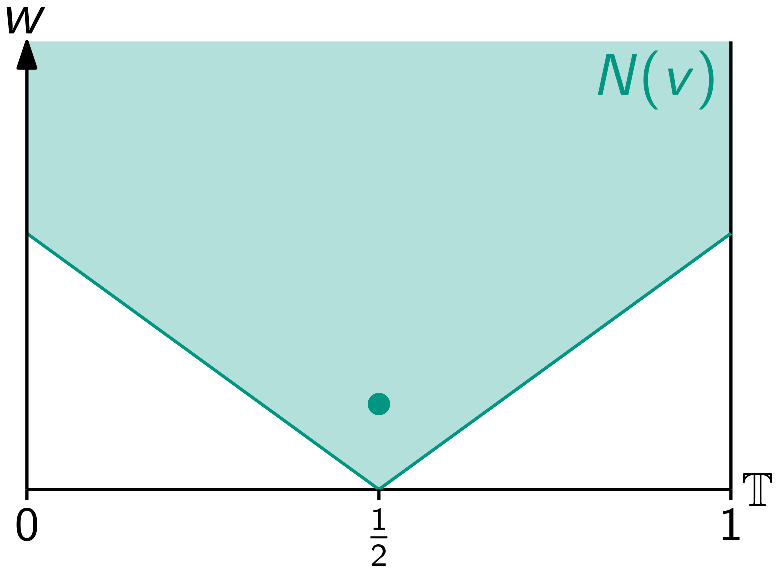

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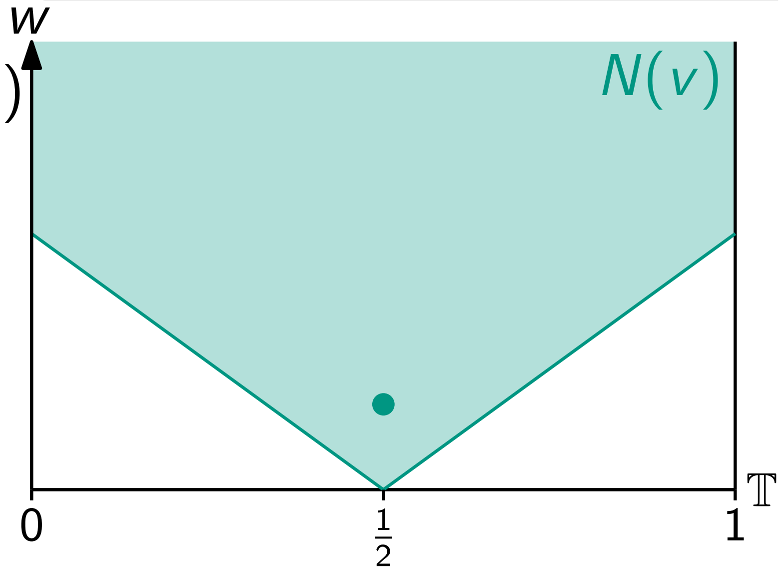

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Density of w_u conditional on w_v but weights are drawn independently

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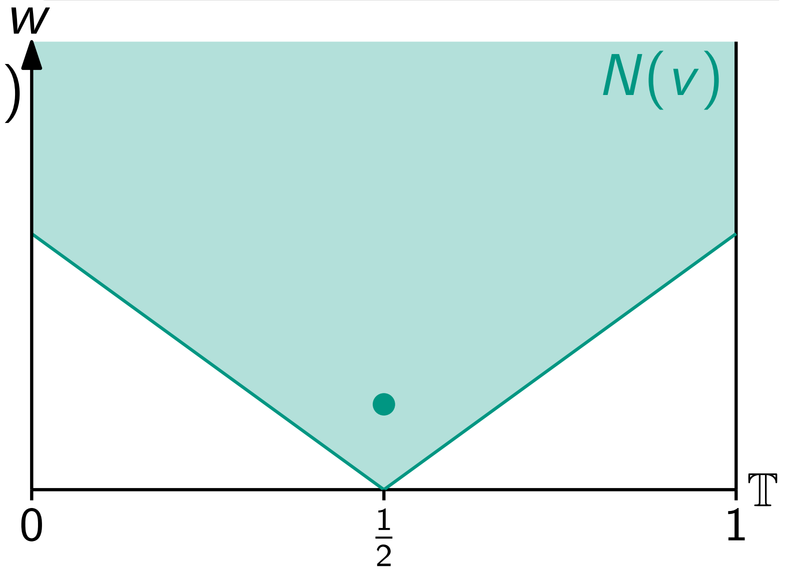

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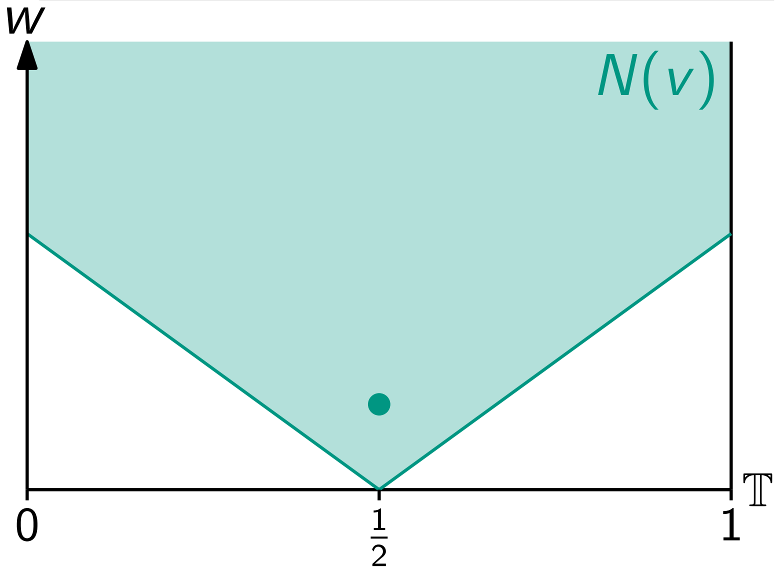

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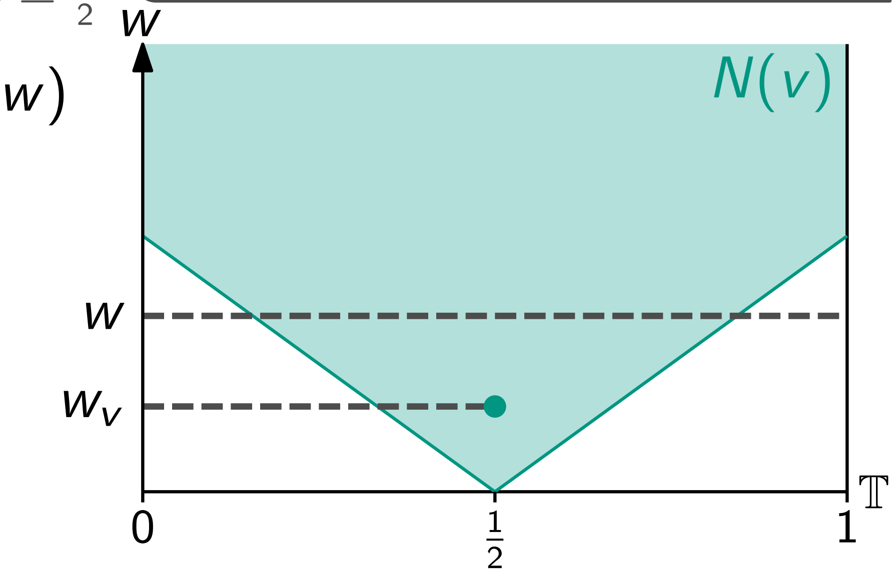

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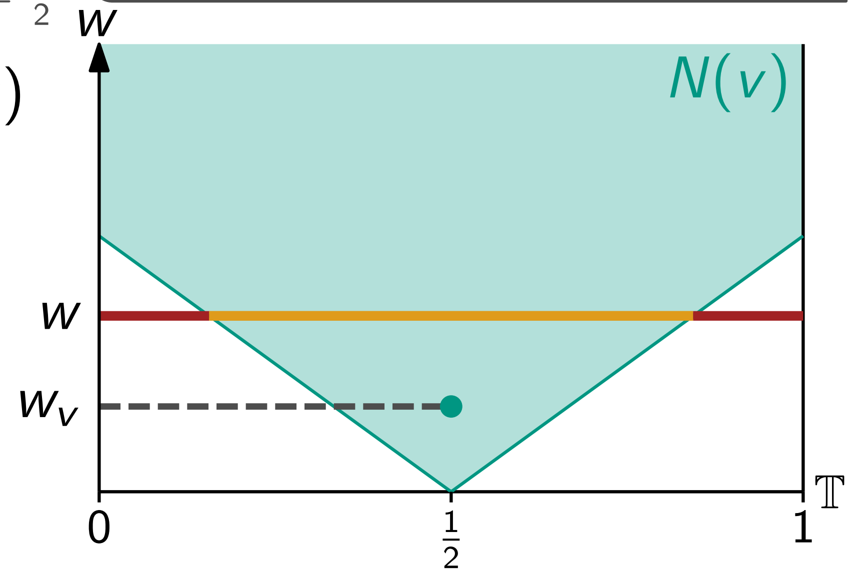

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

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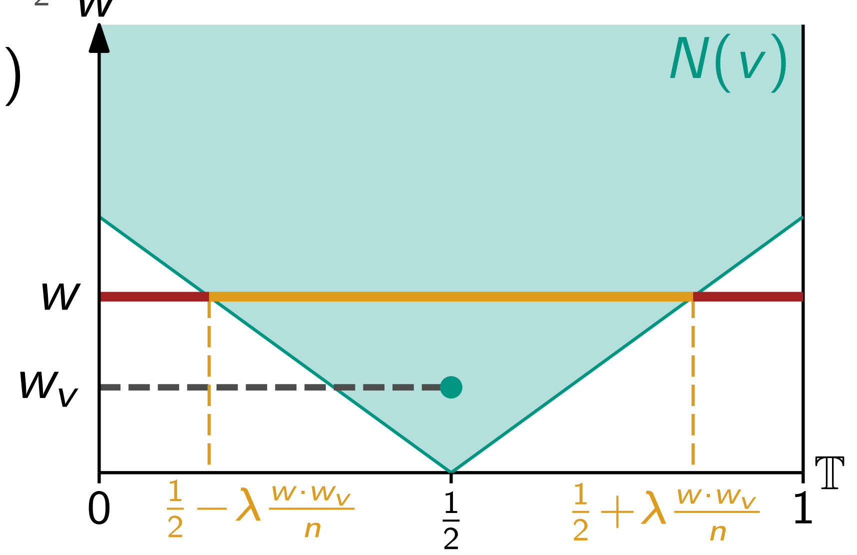

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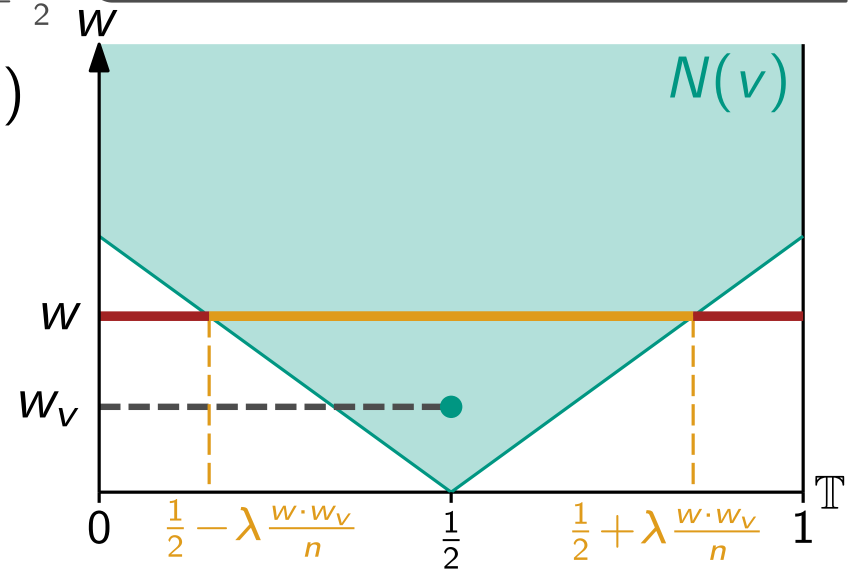

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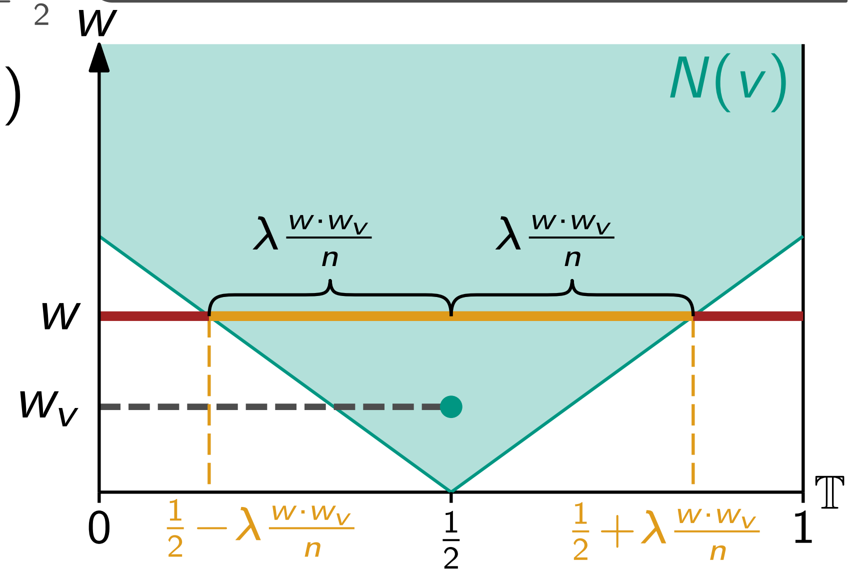

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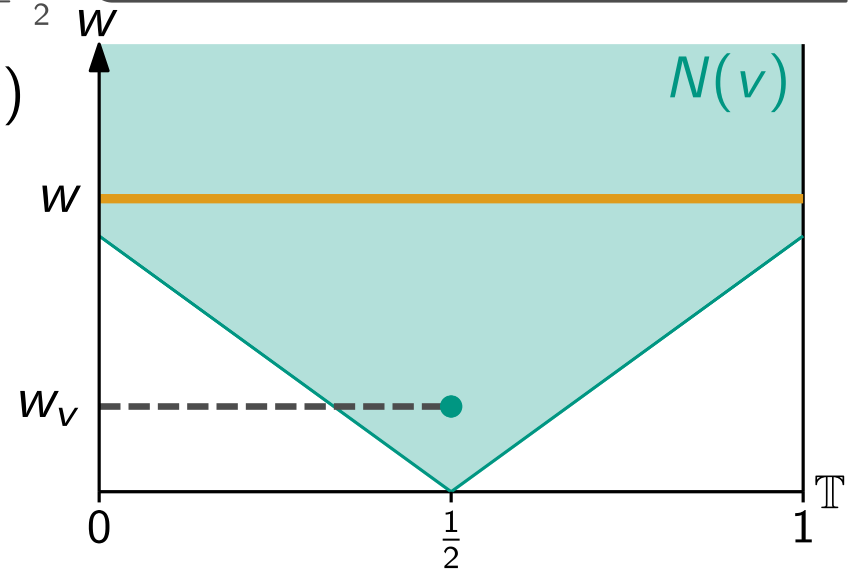

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w.l.o.g $x_v = \frac{1}{2}$

Case 1: $w \leq \frac{n}{2\lambda w_v} \Rightarrow \lambda \frac{w \cdot w_v}{n} \leq \frac{1}{2} \longrightarrow = 2\lambda \frac{w \cdot w_v}{n} = \Theta(\frac{w \cdot w_v}{n})$

Case 2: $w > \frac{n}{2\lambda w_v} \Rightarrow \lambda \frac{w \cdot w_v}{n} > \frac{1}{2} \longrightarrow = 1$

GIRG

- n independent vertices
- $x_v \sim \mathcal{U}([0, 1])$
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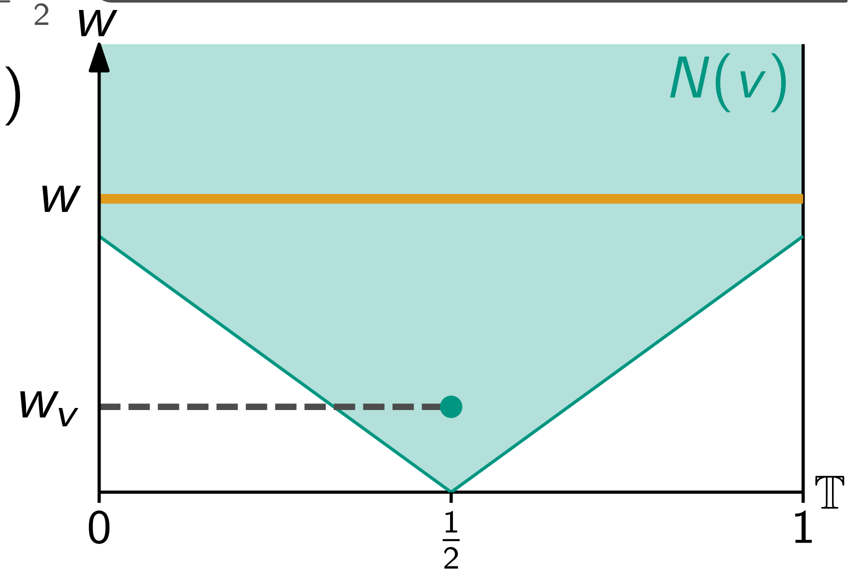

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If $w_v \geq \frac{n}{2\lambda}$, then $\boxed{\frac{n}{2\lambda w_v}} \leq 1$

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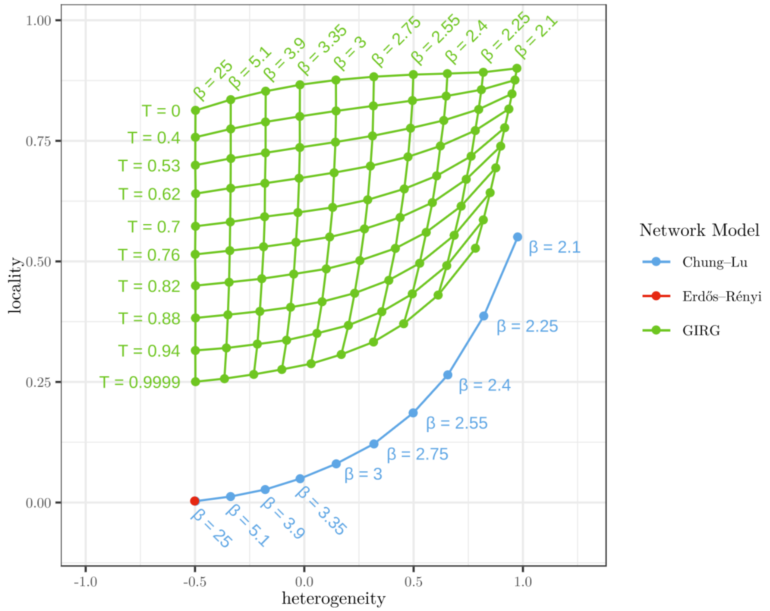

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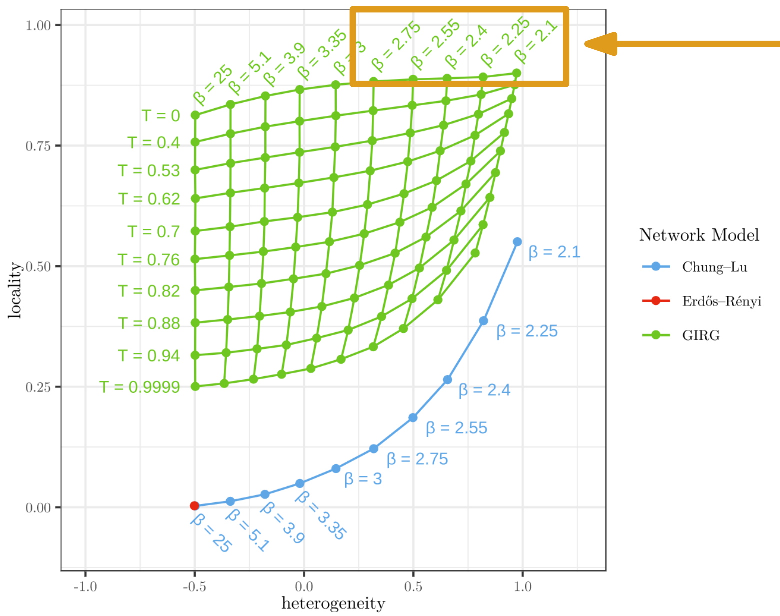

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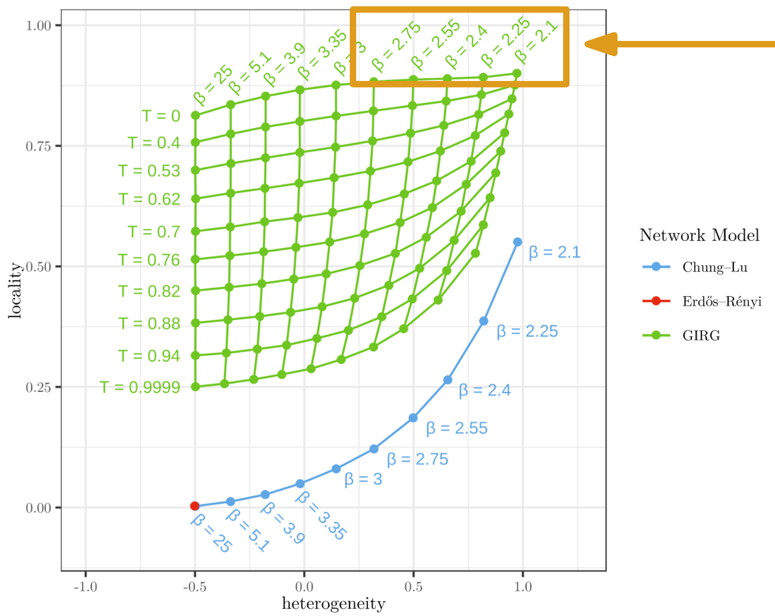

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GIRGs without geometry / ER with weights

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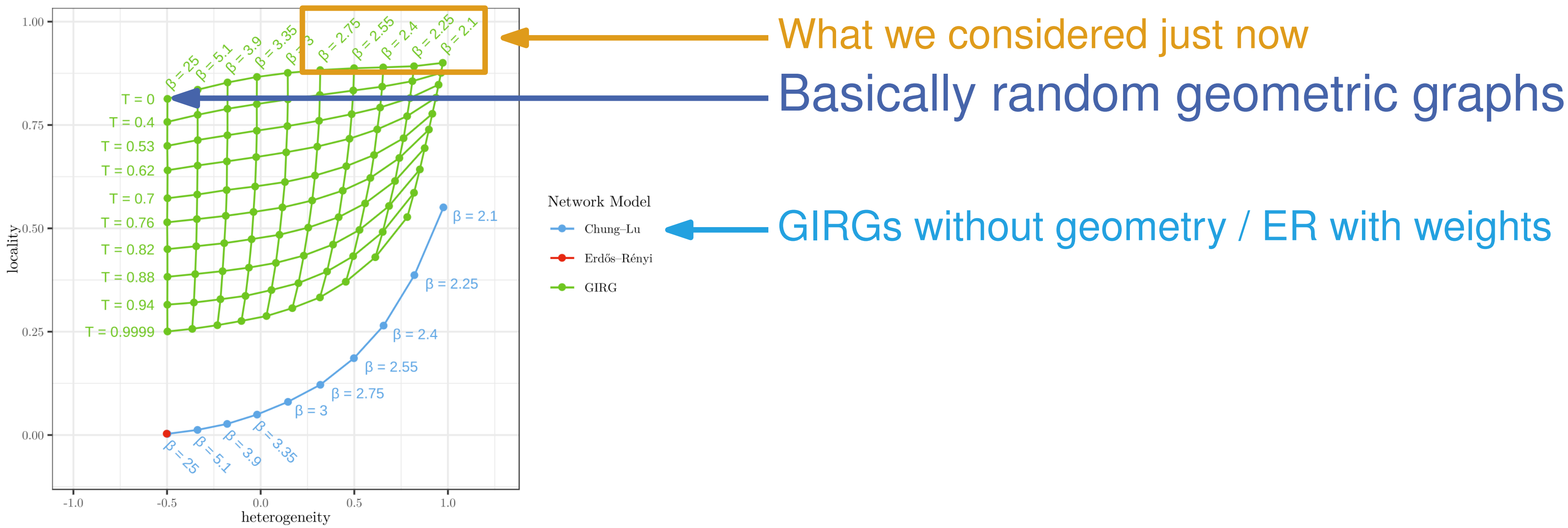

Institute of Theoretical Informatics, Algorithm Engineering & Scalable Algorithms

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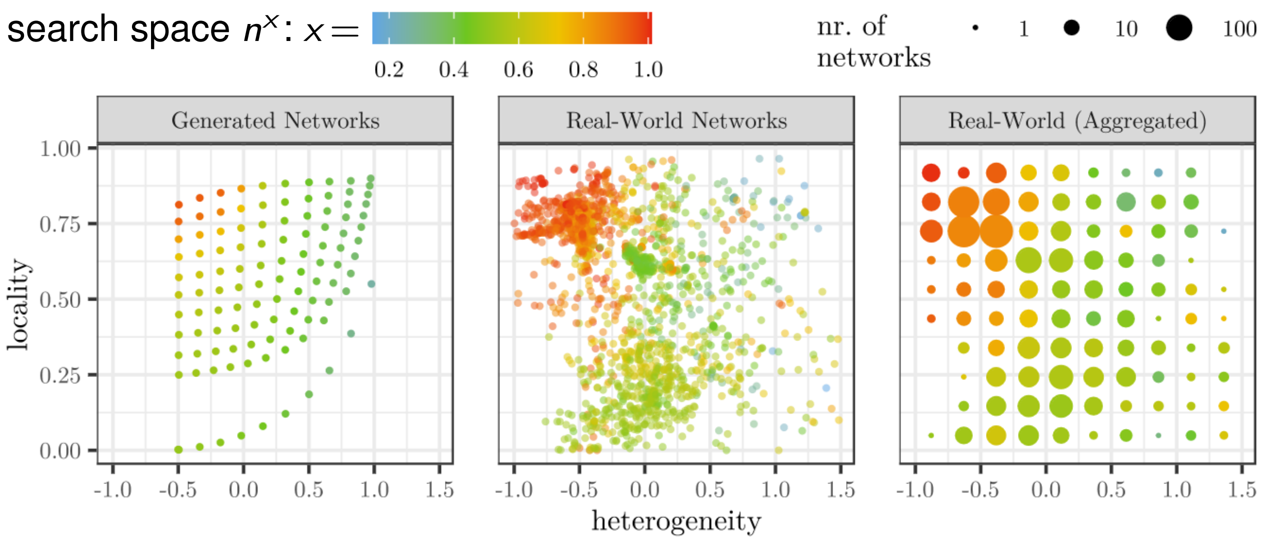

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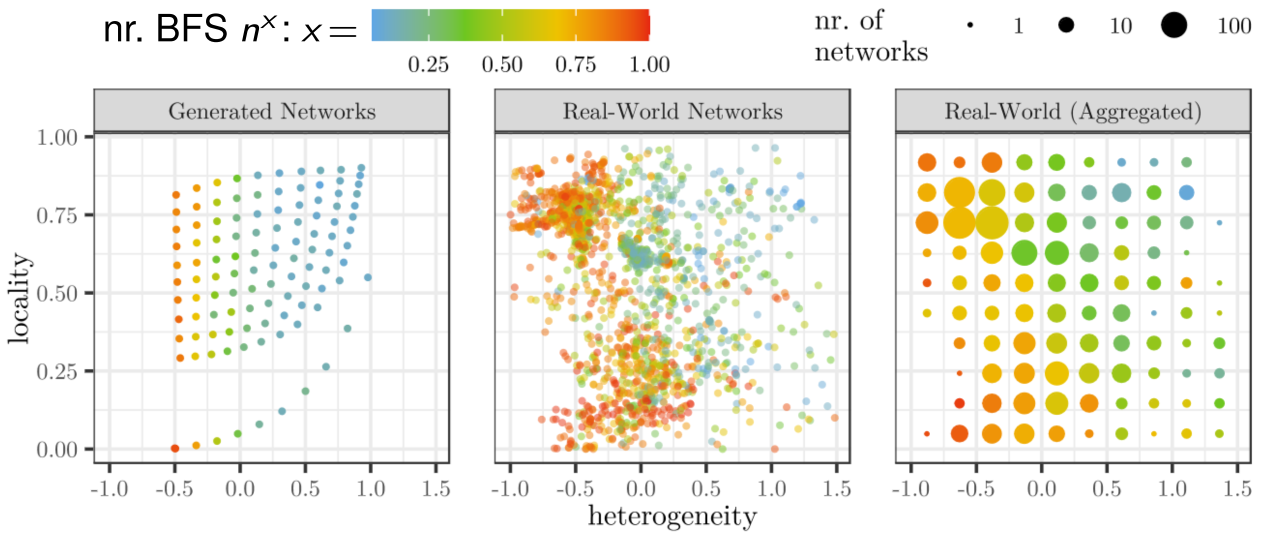

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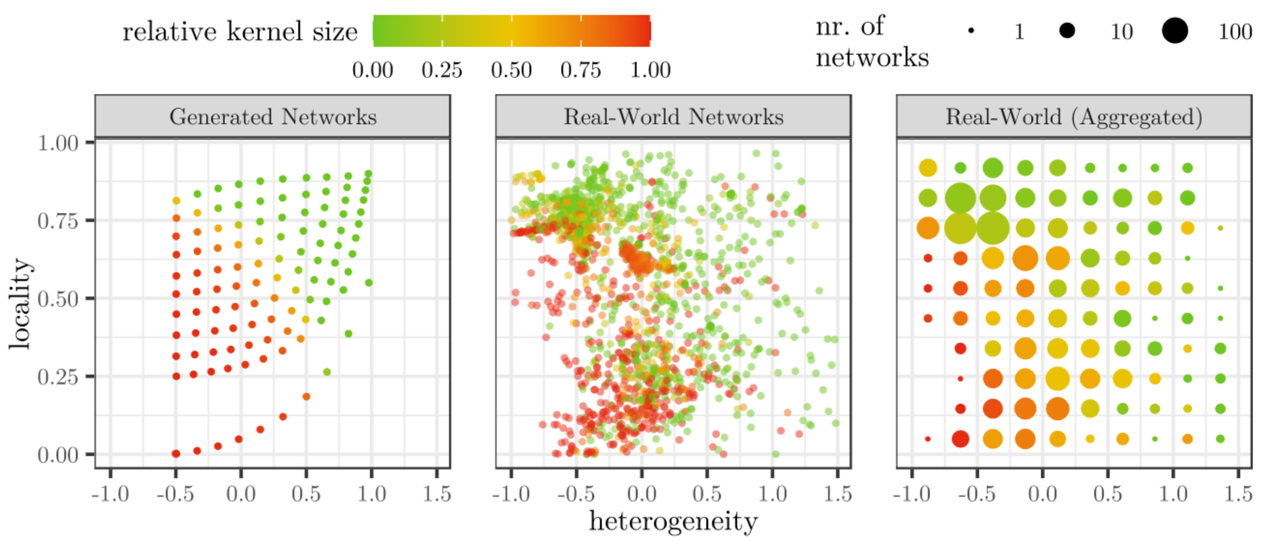

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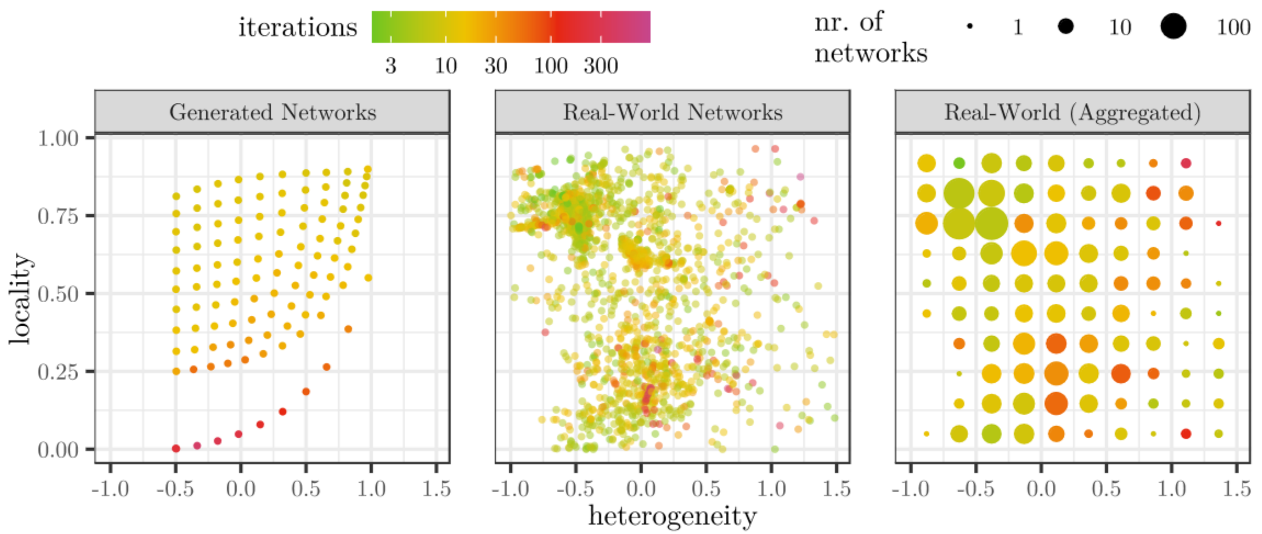

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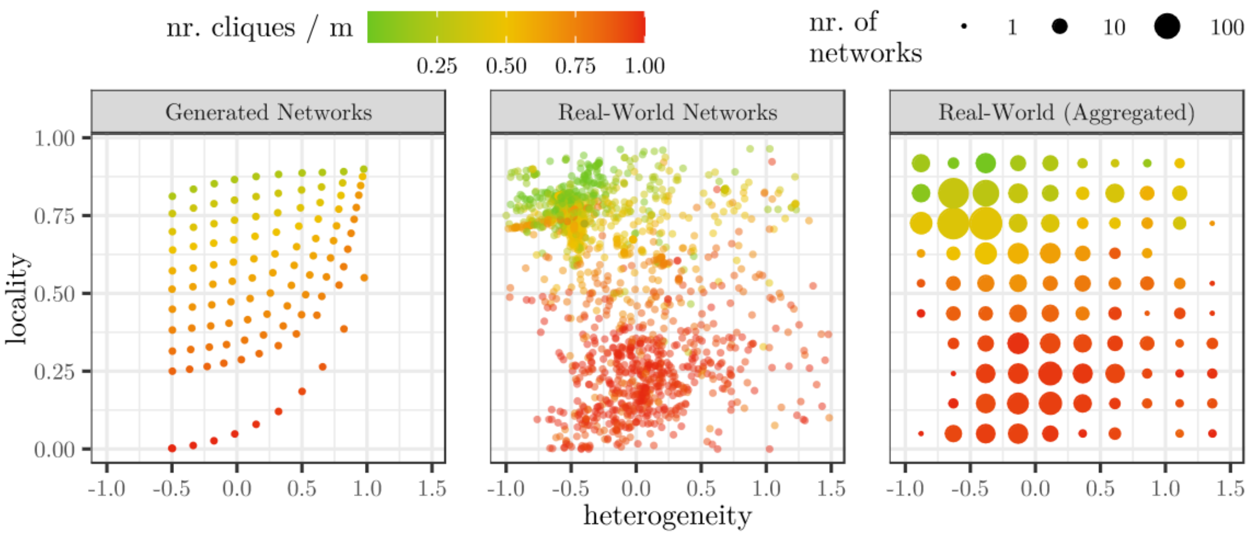

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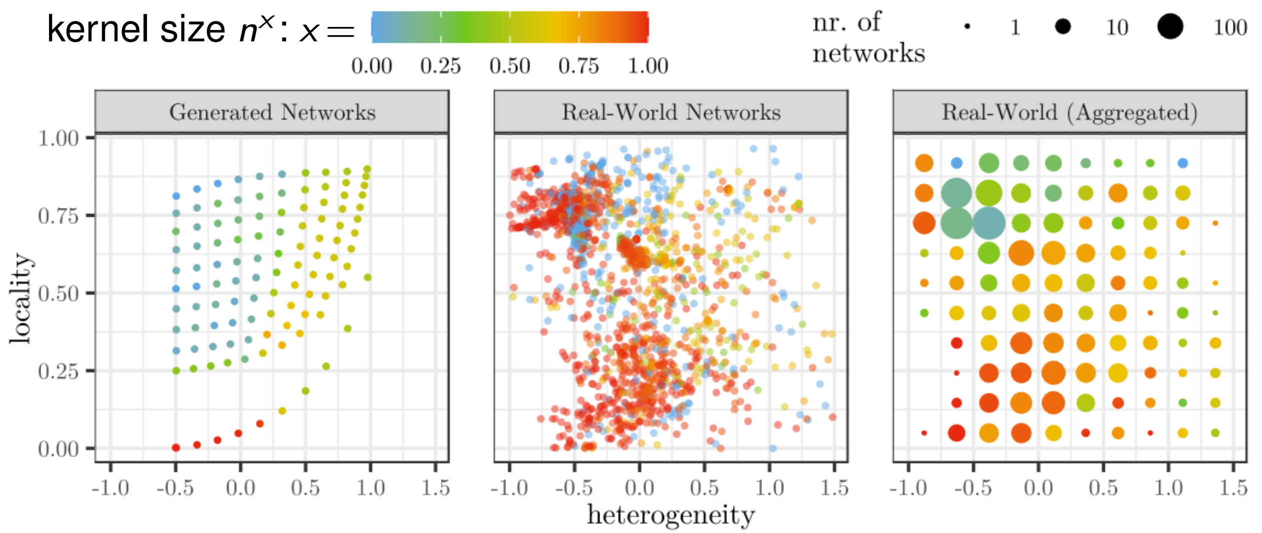

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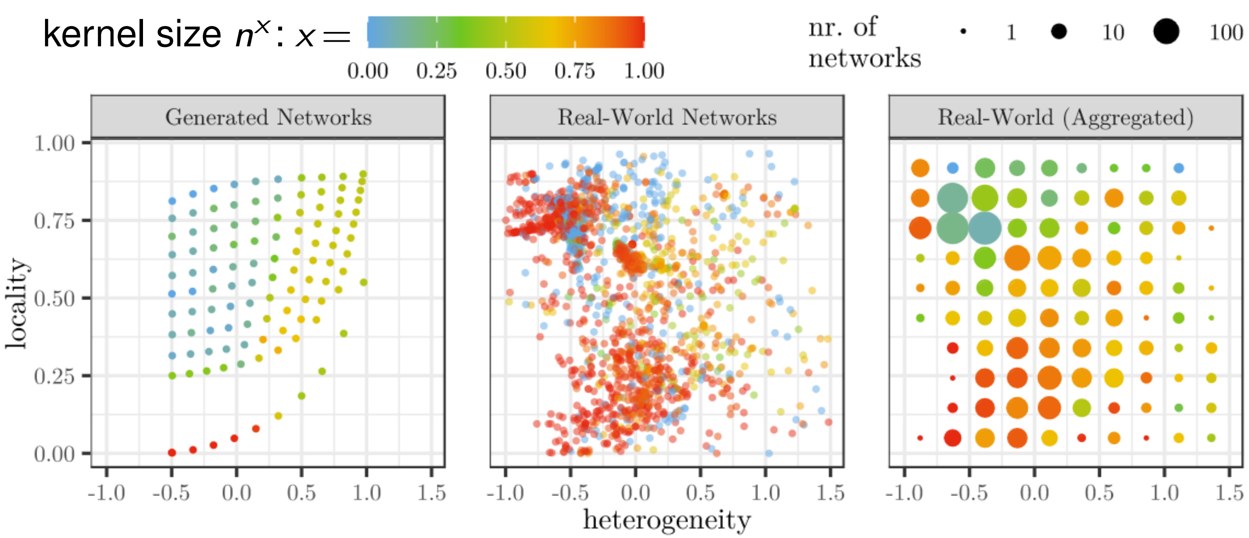
Use GIRGs for average-case analysis!



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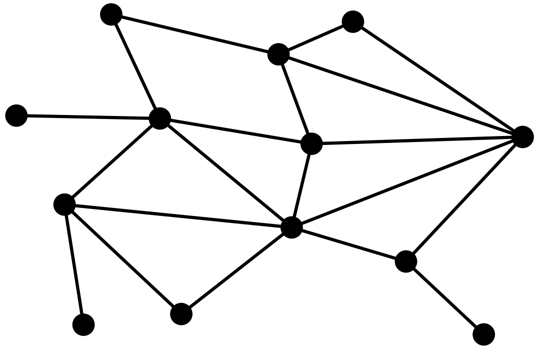

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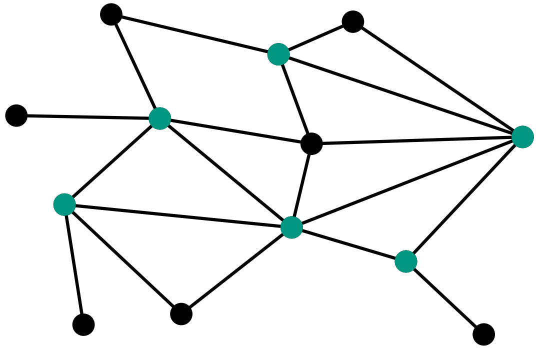

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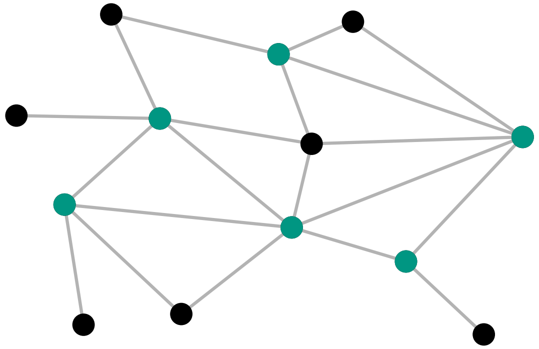

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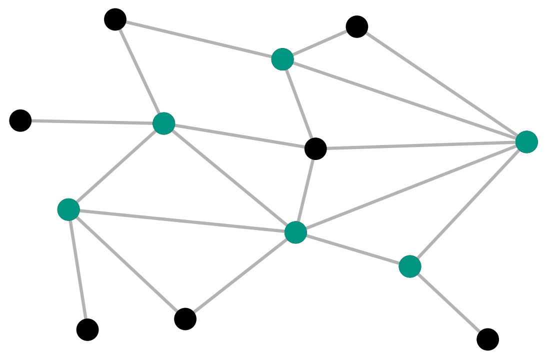

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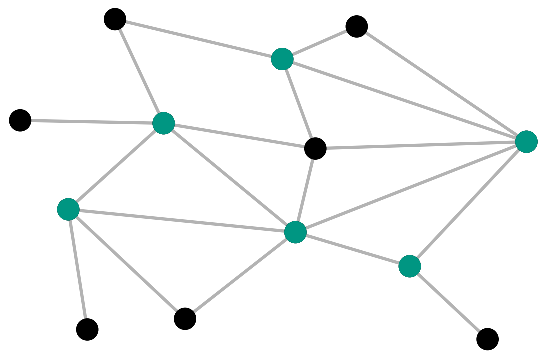
- Find a *small* vertex cover S' *fast*



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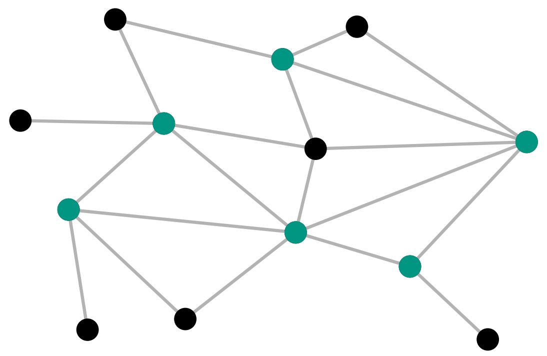

Vertex Cover

- Given undirected graph $G = (V, E)$
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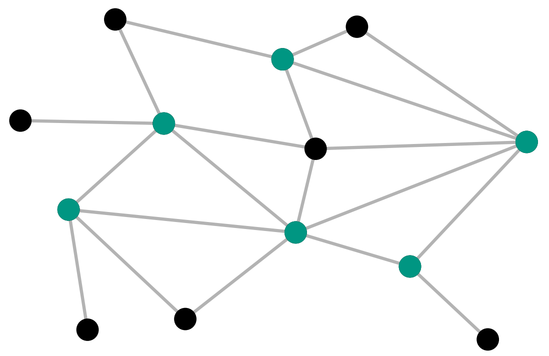

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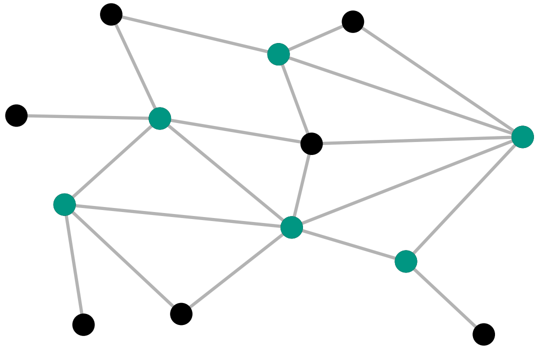

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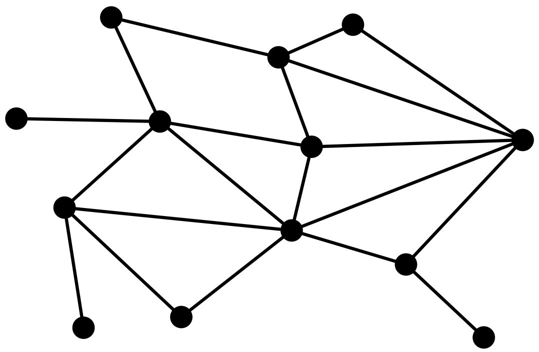

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- Simple approximation algorithm repeatedly takes/deletes vertex of largest degree

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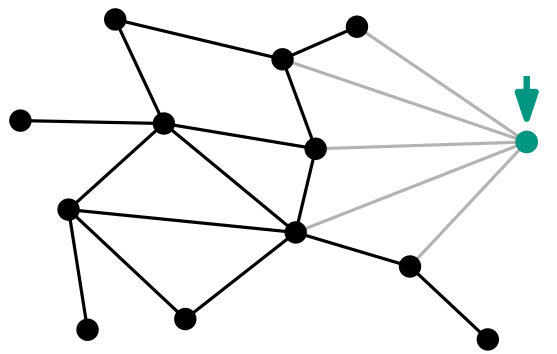

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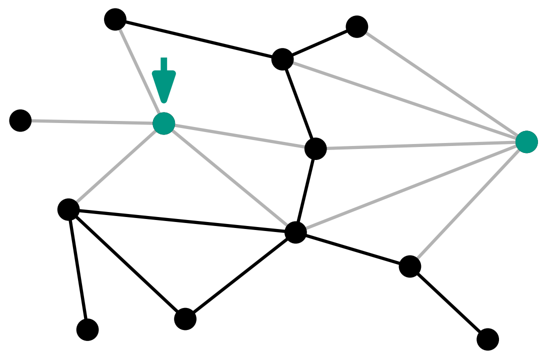

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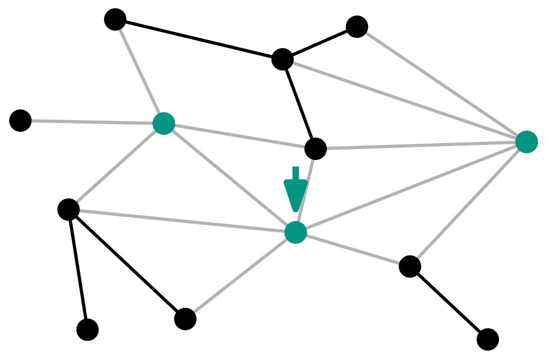

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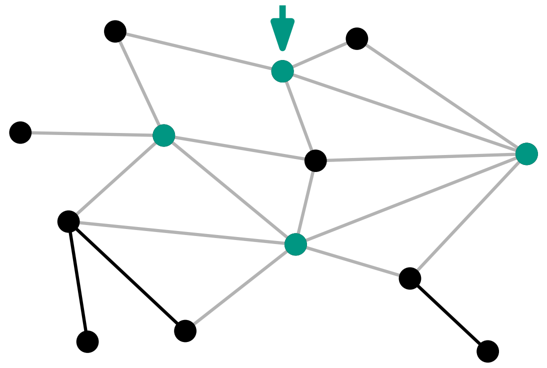

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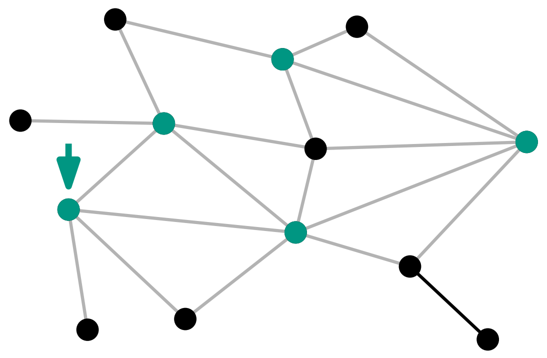

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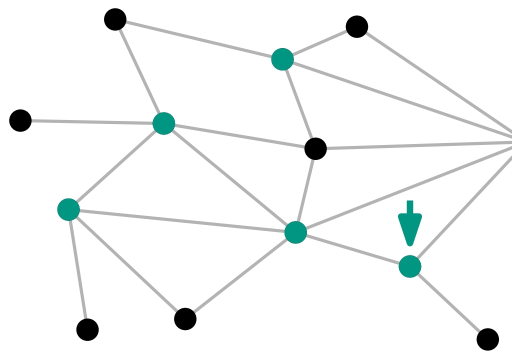

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Practice

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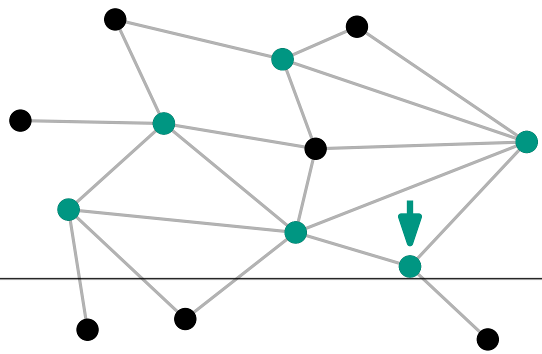
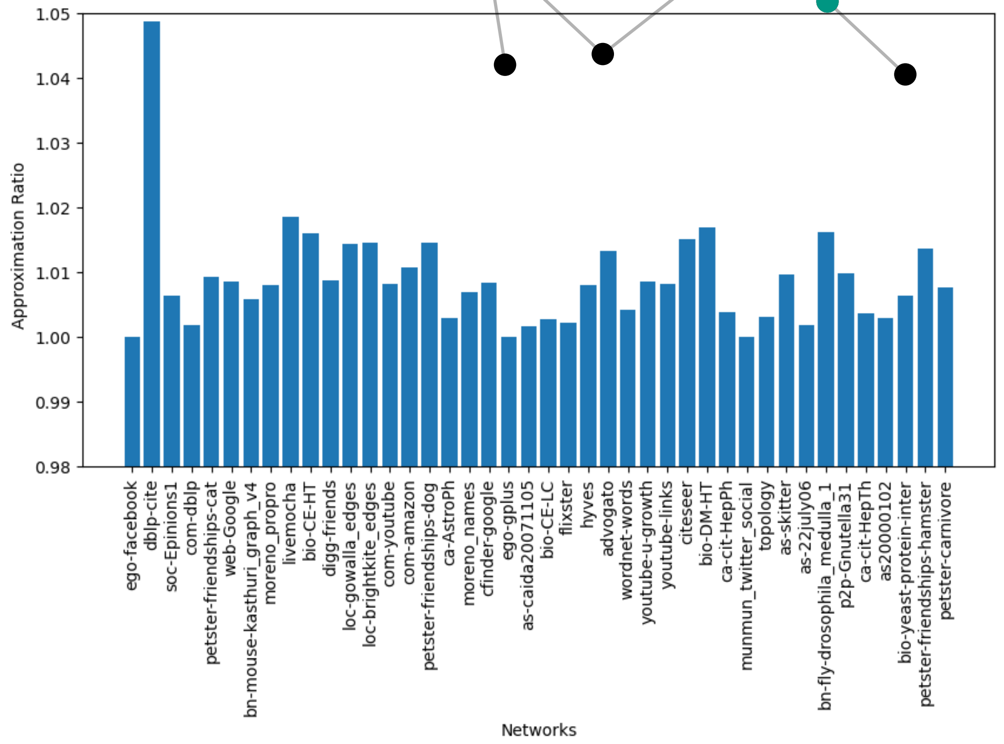
- Close to optimal ratios on real graphs

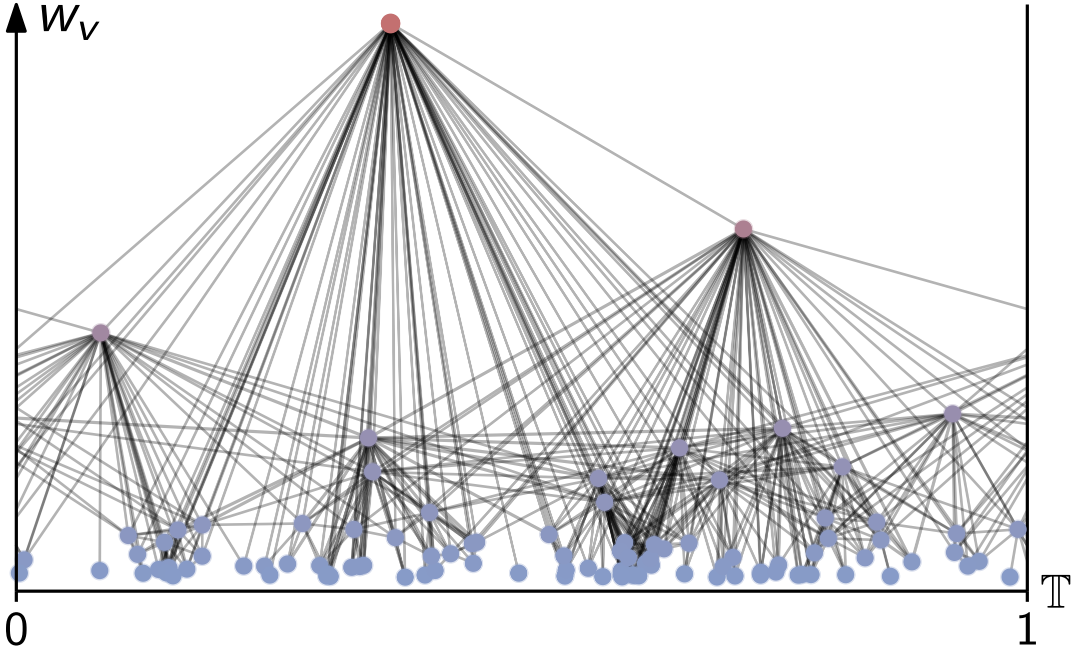
"Vertex Cover on Complex Networks", Da Silva, Gimenez-Lugo, Da Silva, IJMPC 2013





Analsysis on GIRGs

(based on) "Efficiently Approximating Vertex Cover on Scale-Free Networks with Underlying Hyperbolic Geometry", Bläsius, Friedrich, K., Algorithmica 2023



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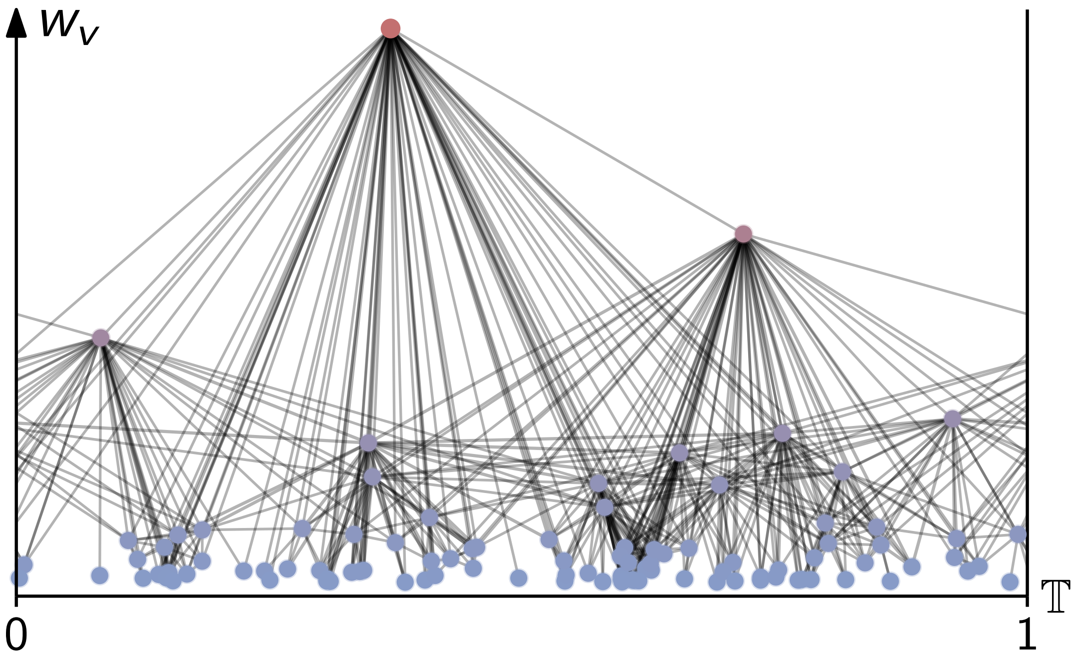

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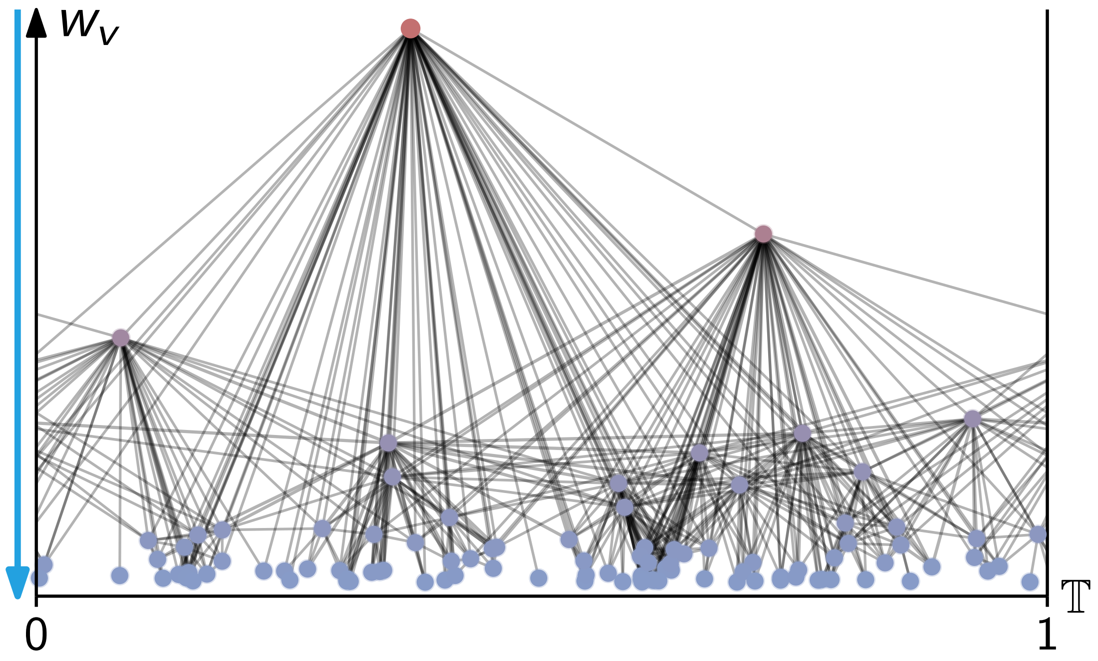

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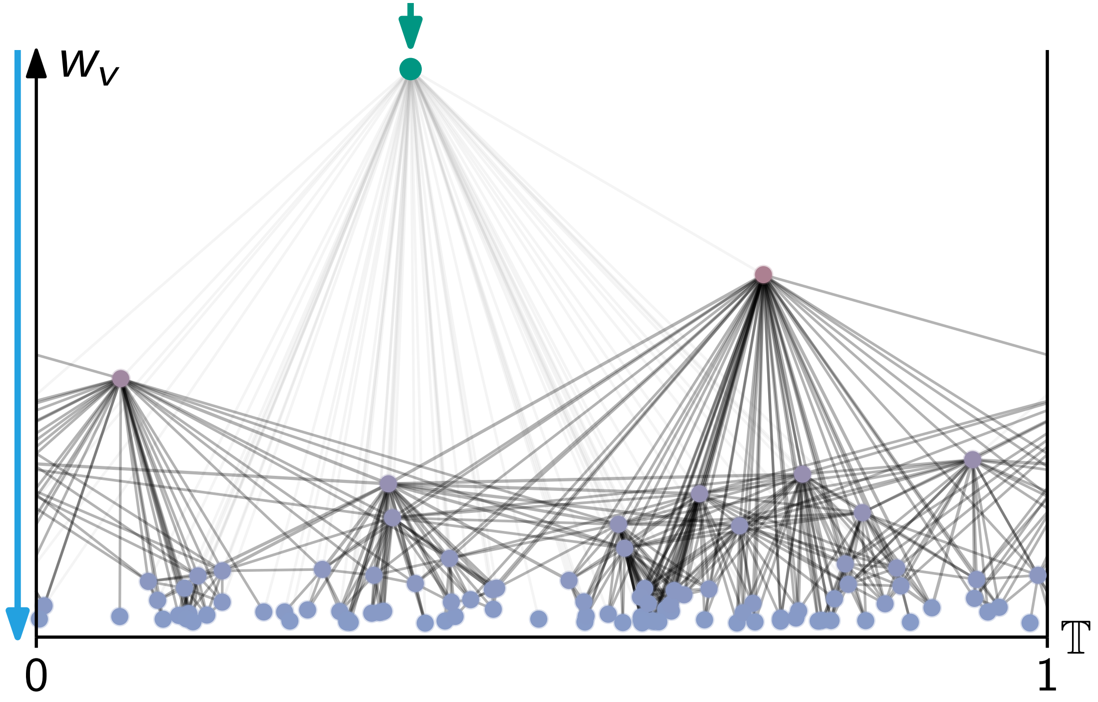

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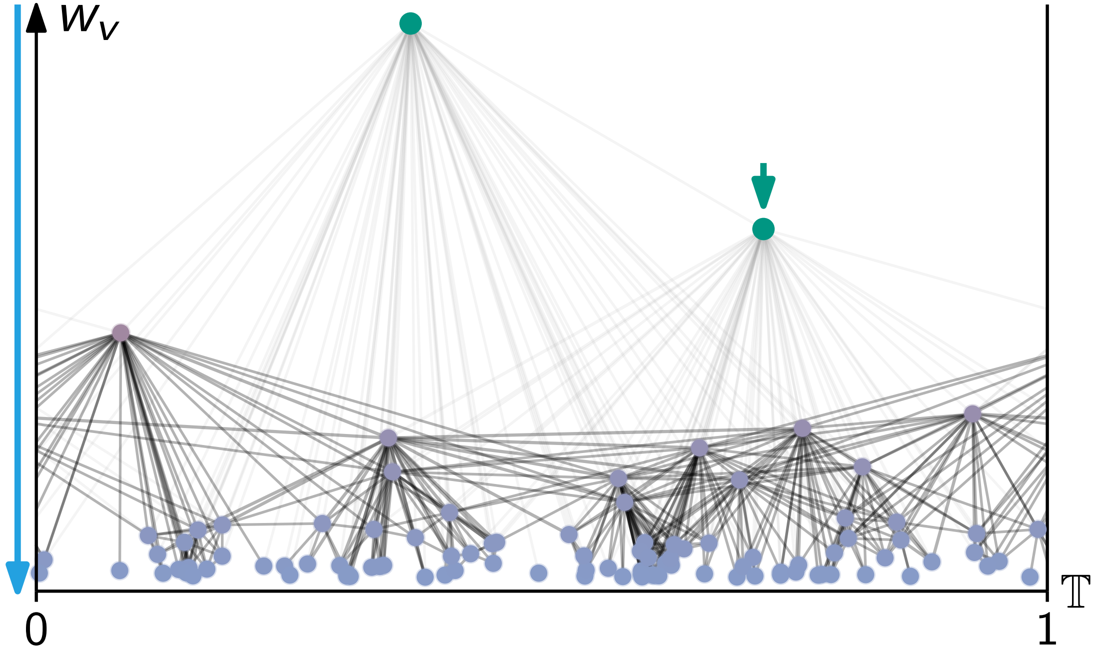

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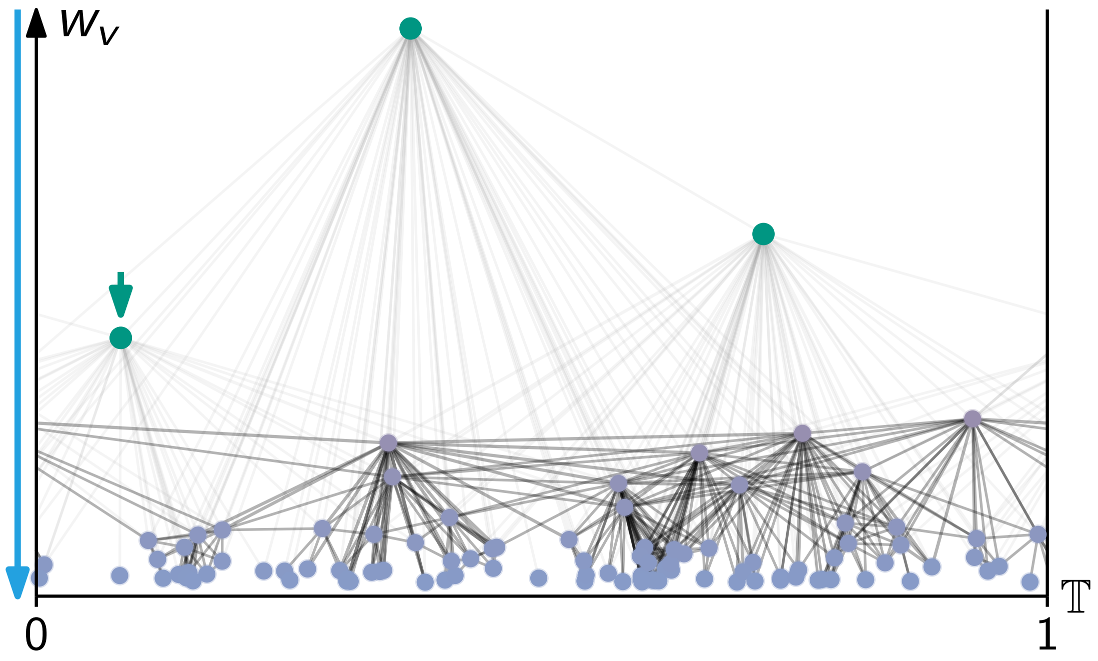

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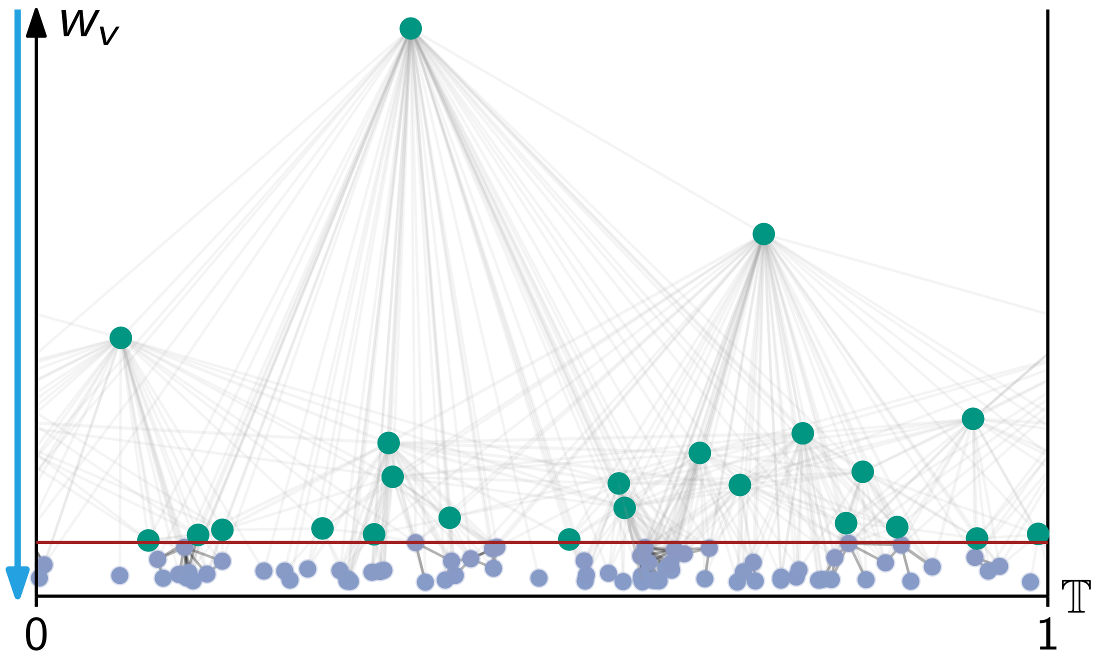
Learn from the Model

- Once high-degree vertices are taken/removed, remaining vertices have roughly equal weight/degree



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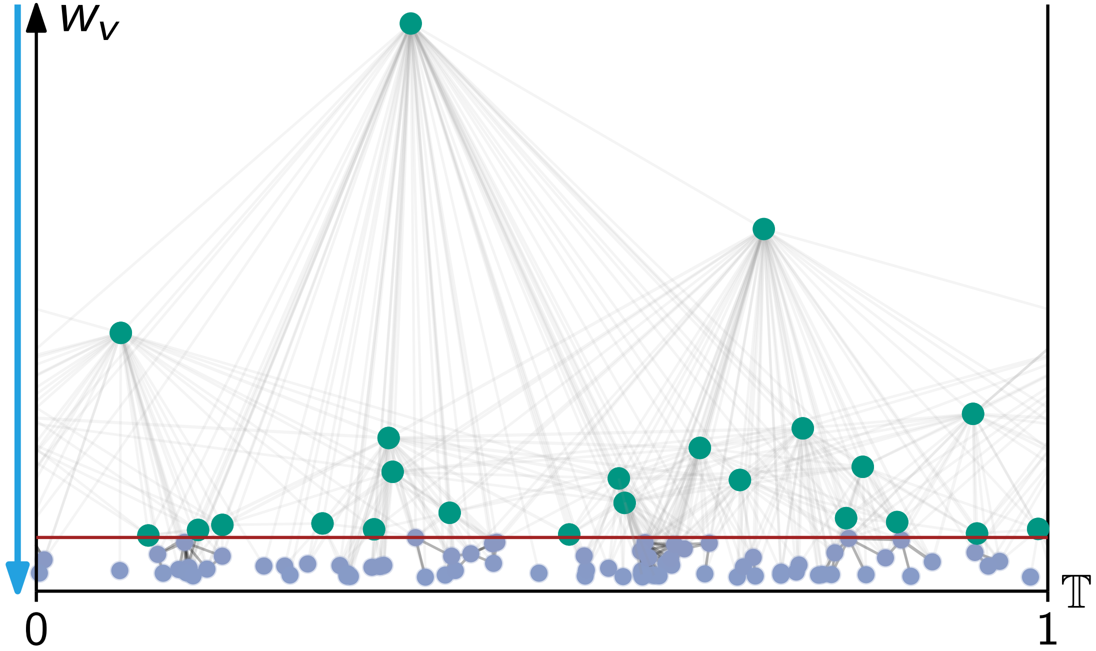

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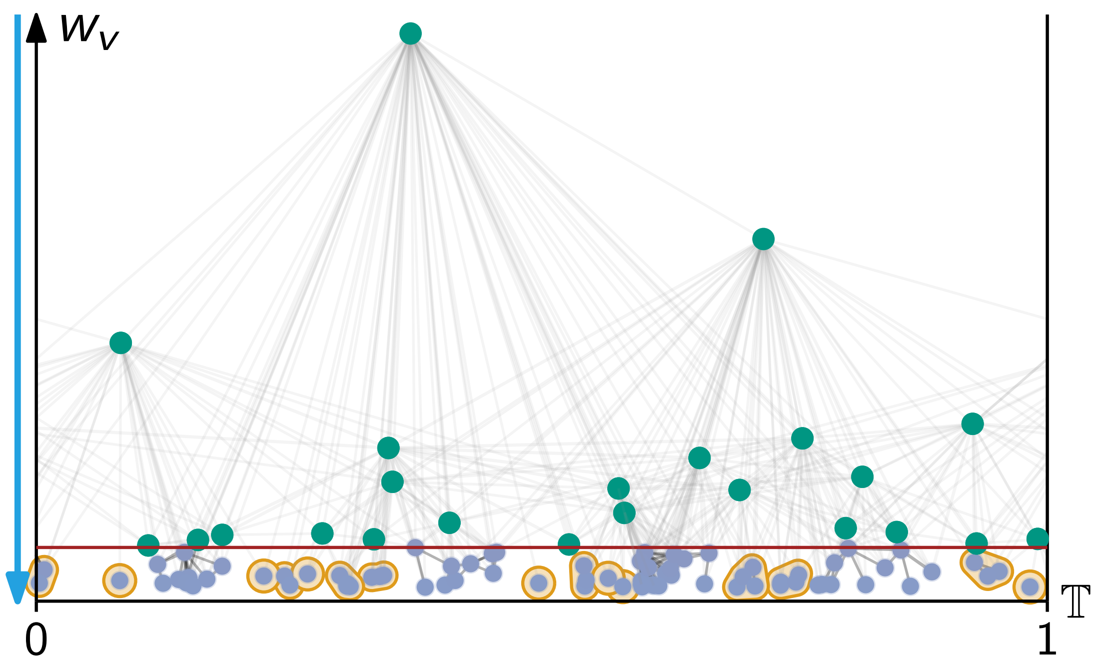

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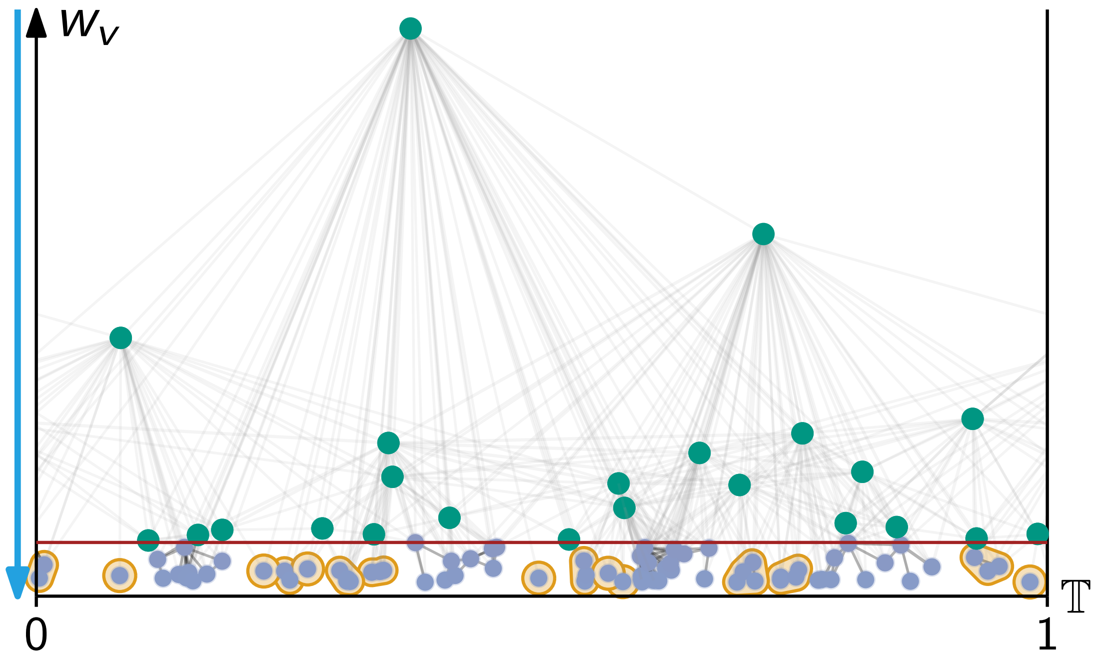
  - Search and solve small components after each greedily taken vertex



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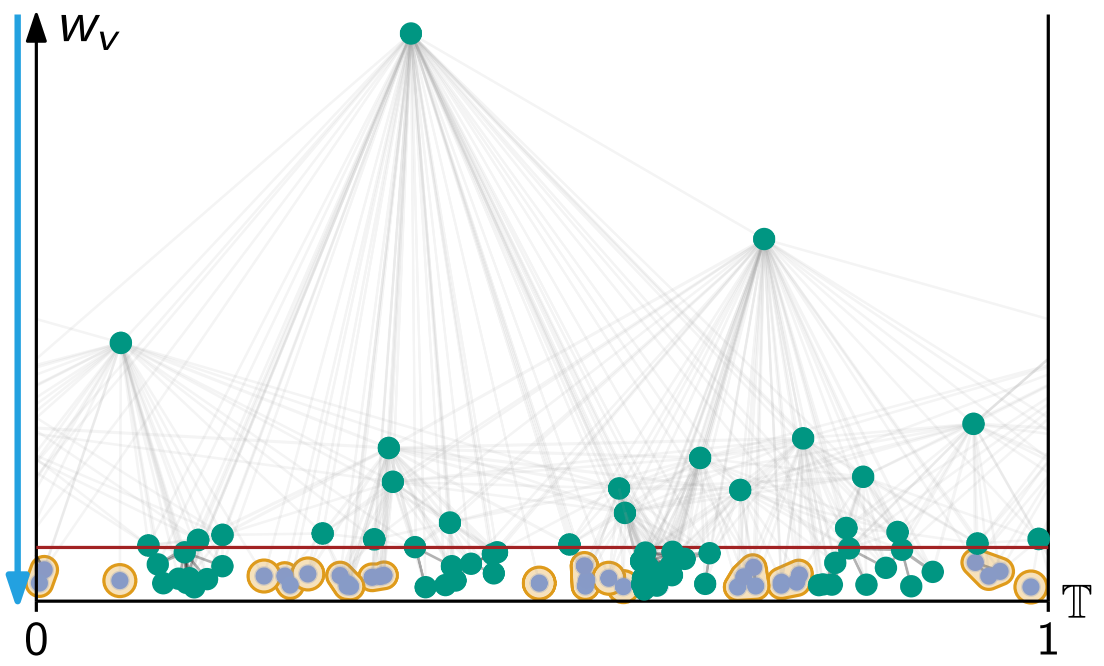

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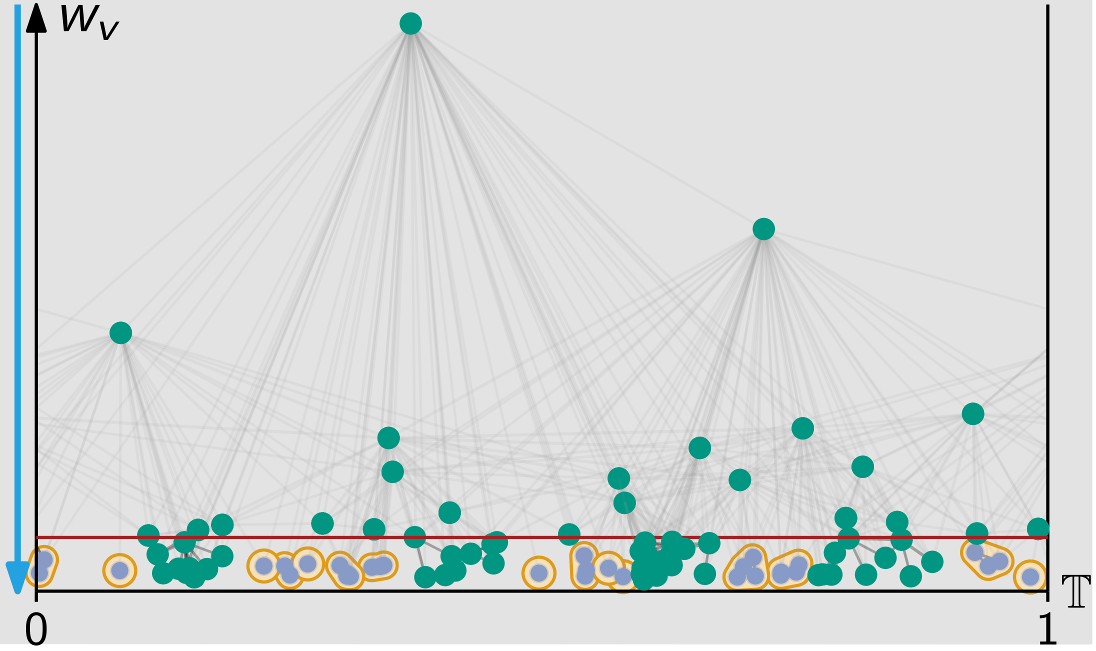
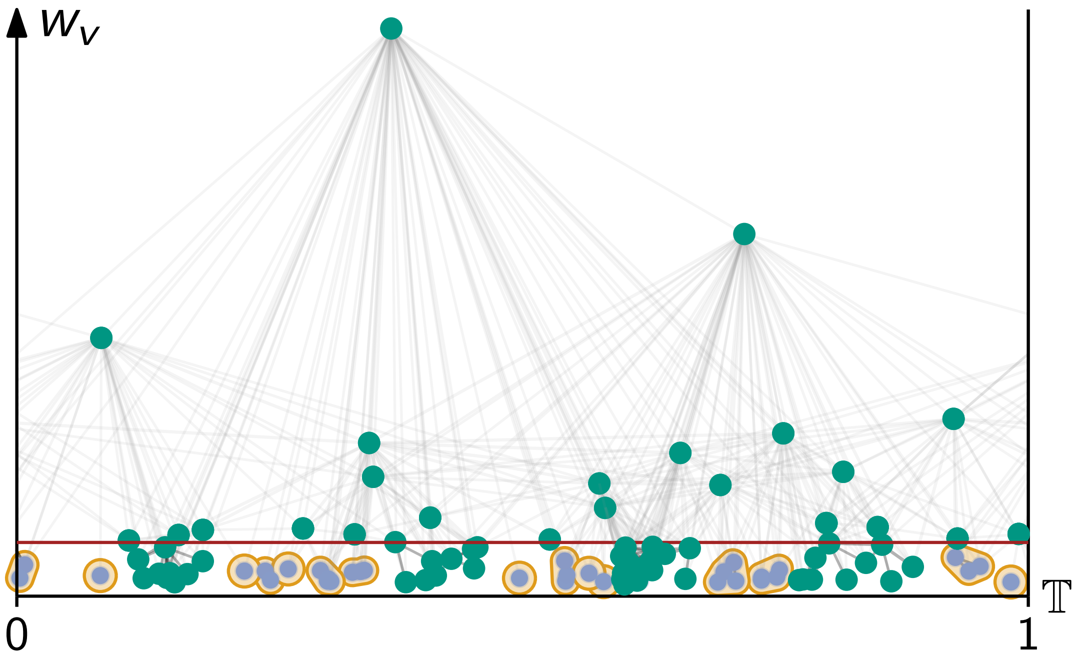
This variant yields an upper bound on the quality of the other



Analysis on GIRGs – Approximation Ratio

Theorem: Let G be GIRG with n vertices and m edges. Then, an approximate vertex cover S' of G can be computed in time $O(m \log(n))$ such that the approximation ratio is $(1 + o(1))$ $\underbrace{\text{asymptotically almost surely.}}_{\text{with probability } 1 - o(1)}$



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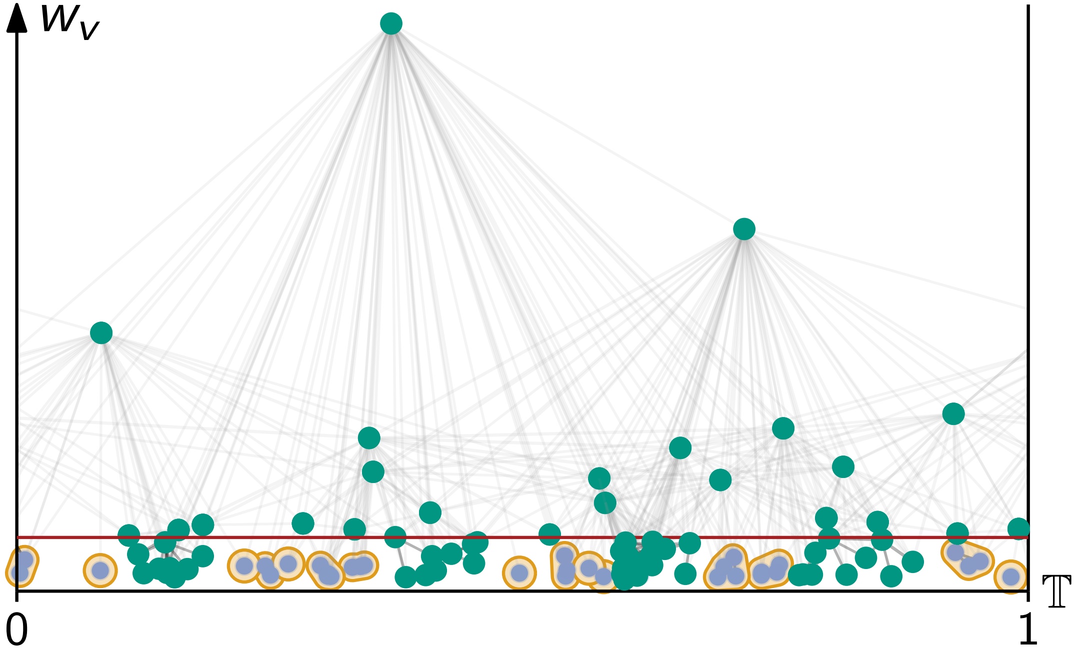

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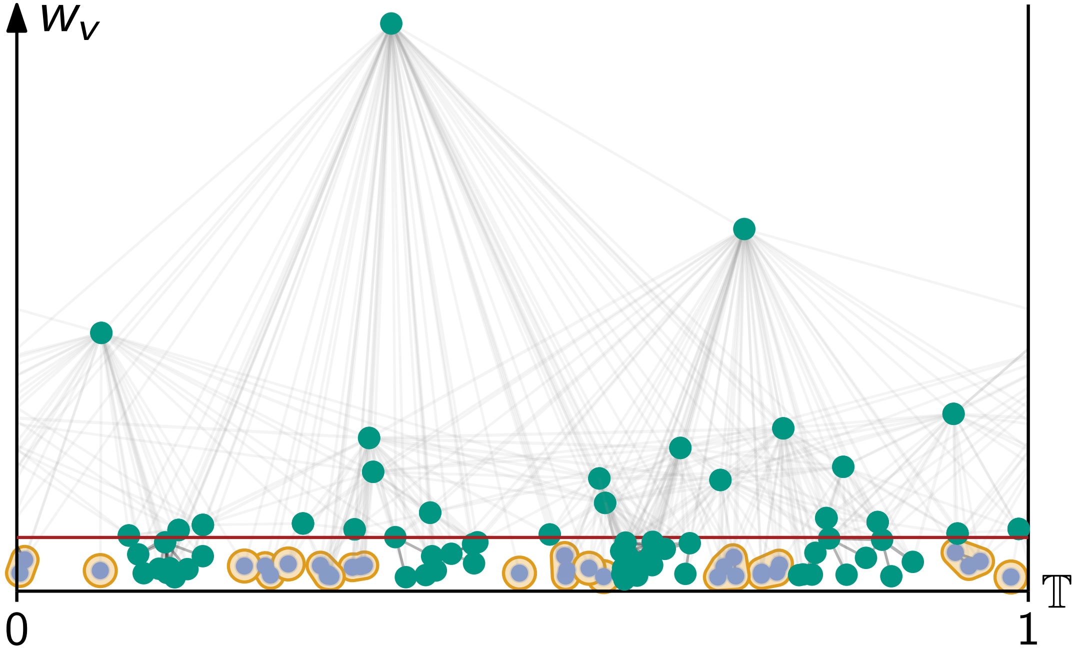

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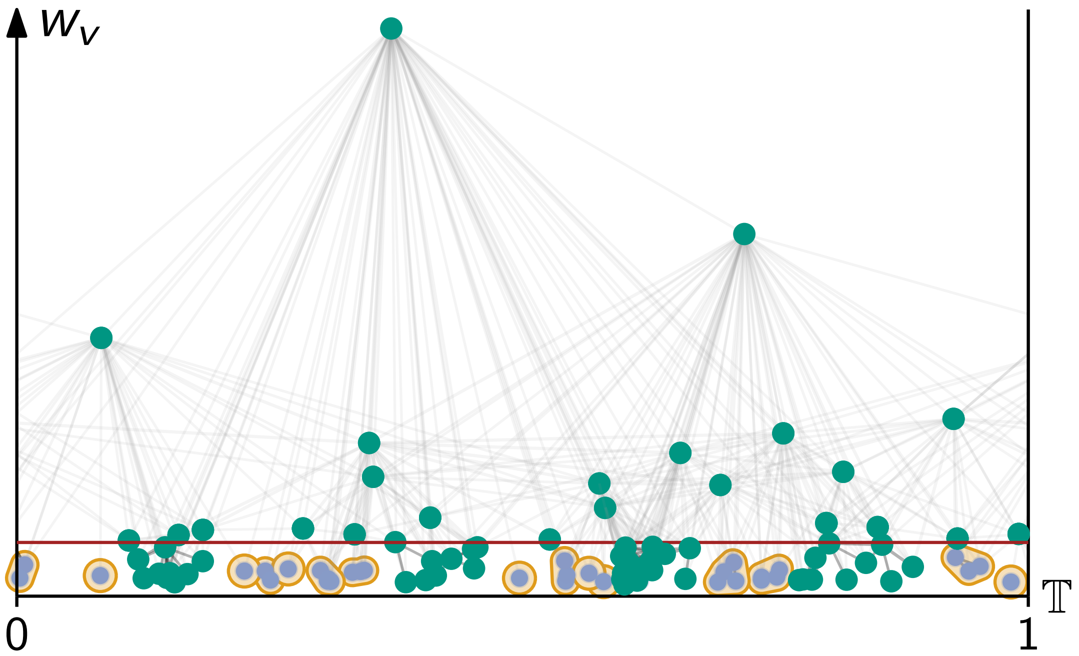

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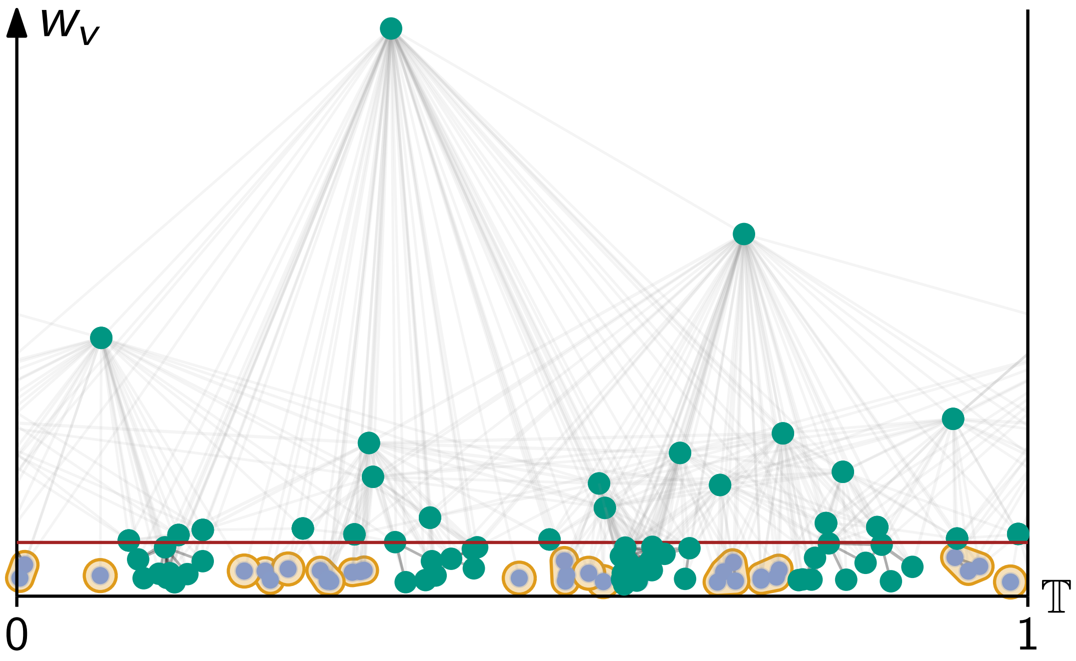

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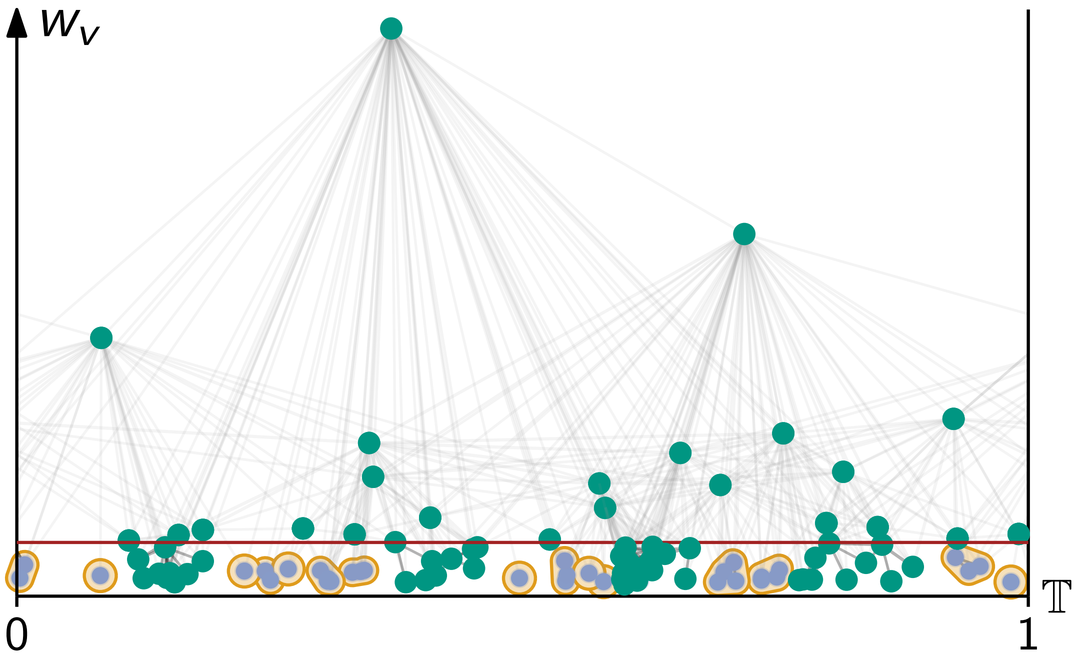
- $|S| = \Omega(n)$ with prob $1 - o(1)$

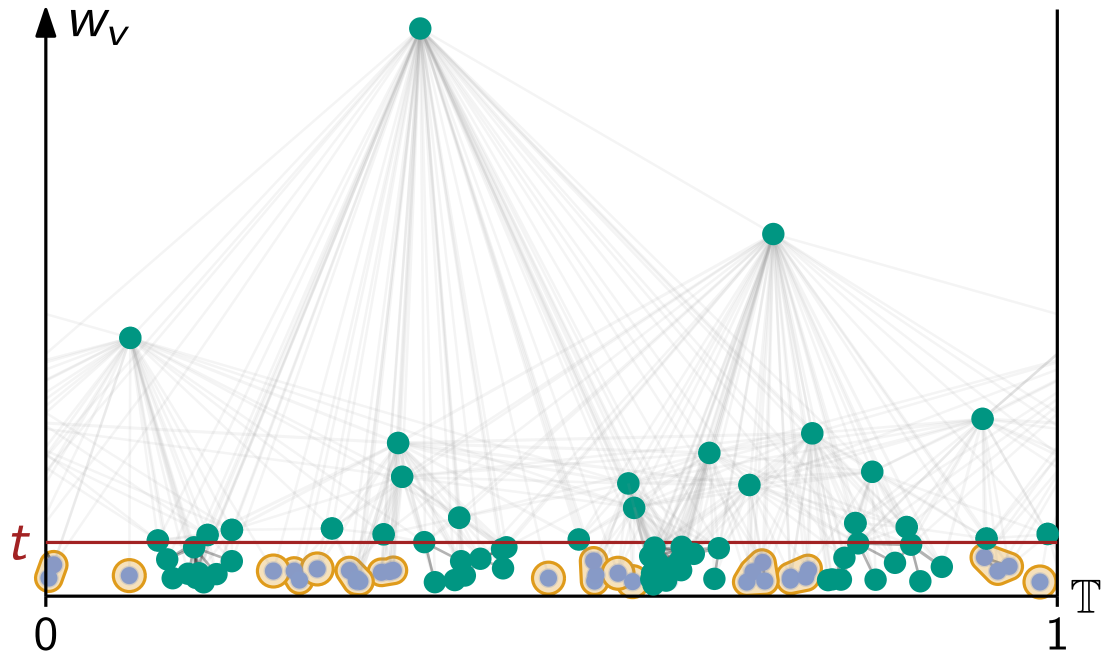
“Greed is Good for Deterministic Scale-Free Networks”, Chauhan et al. FSTTCS 2016



Remains to show: $|S'_g| = o(n)$

Analysis on GIRGs – Greedy Vertices $\geq t$

Lemma: Let G be a GIRG with n vertices, let $t = \omega(1)$, and let $N_{w \geq t}$ be the number of vertices with weight at least t. Then, $N_{w \geq t} = o(n)$ with probability $1 - O(1/n)$.



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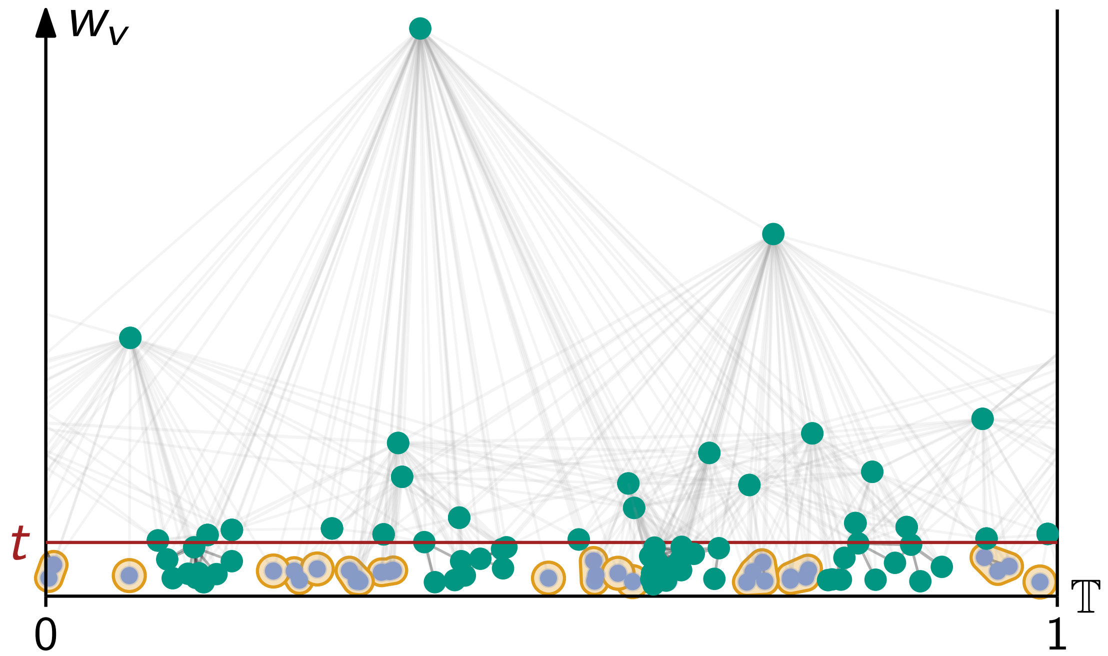
GIRG

- n independent vertices
- $w_v \sim \mathrm{Par}(\tau - 1, 1)$ for $\tau \in (2, 3)$



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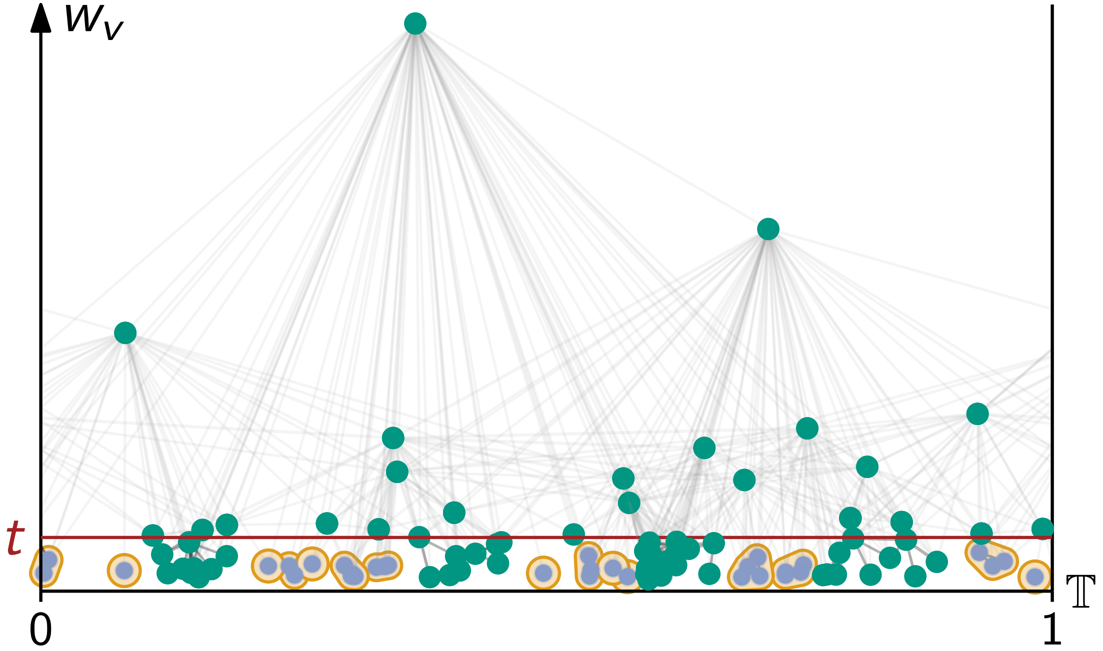

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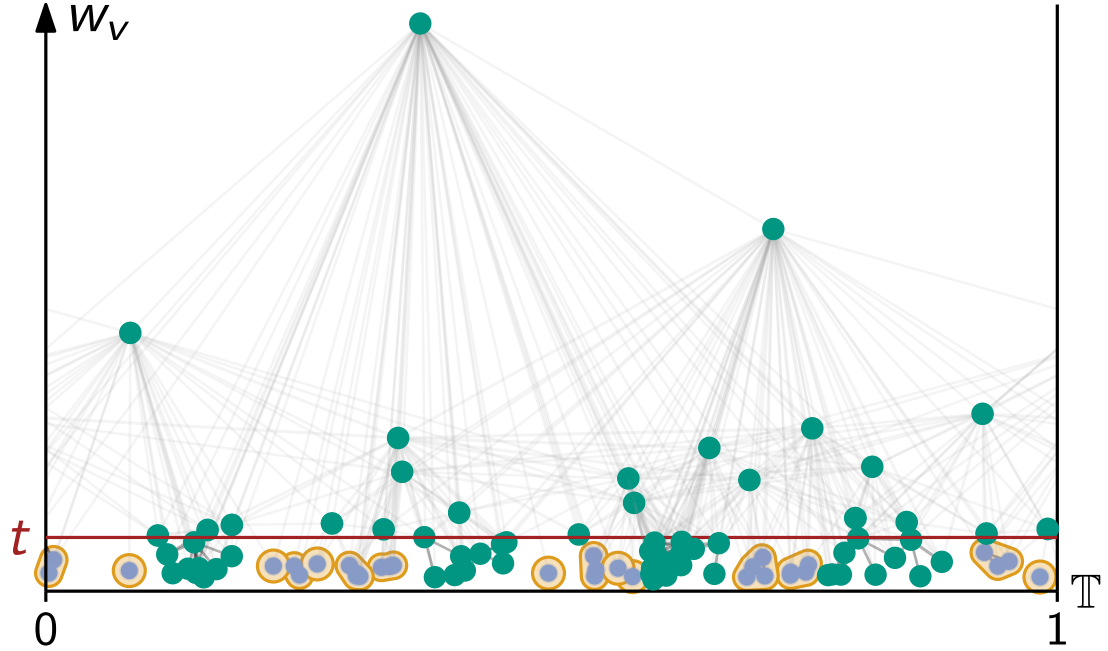
GIRG

- n independent vertices
- $w_v \sim \mathrm{Par}(\tau - 1, 1)$ for $\tau \in (2, 3)$



Analysis on GIRGs – Greedy Vertices $\geq t$

Lemma: Let G be a GIRG with n vertices, let $t = \omega(1)$, and let $N_{w \geq t}$ be the number of vertices with weight at least t. Then, $N_{w \geq t} = o(n)$ with probability $1 - O(1/n)$.

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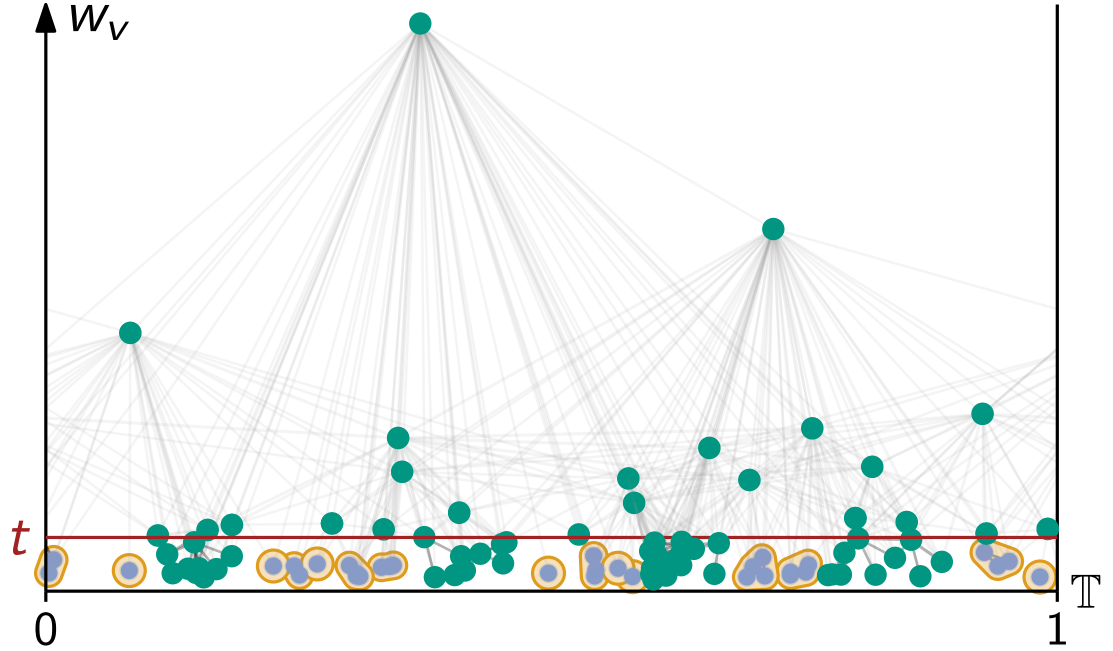

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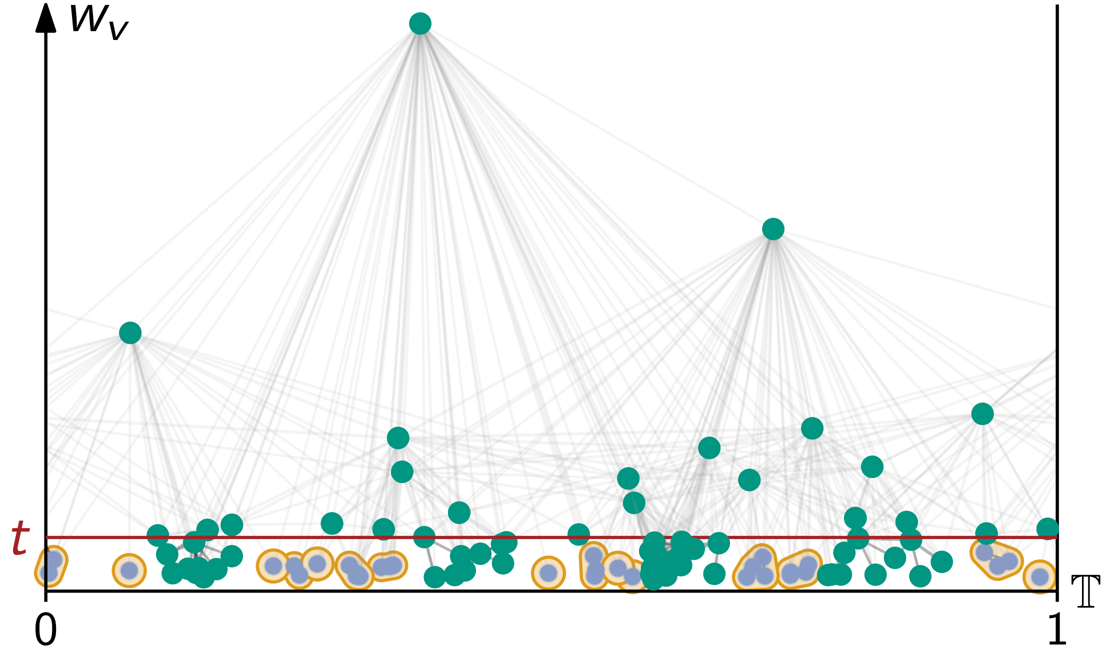

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- Since there is a $g(n) \in o(n) \cap \Omega(\log(n))$ with $g(n) \geq \mathbb{E}[N_{w \geq t}]$, Chernoff gives concentration ✓

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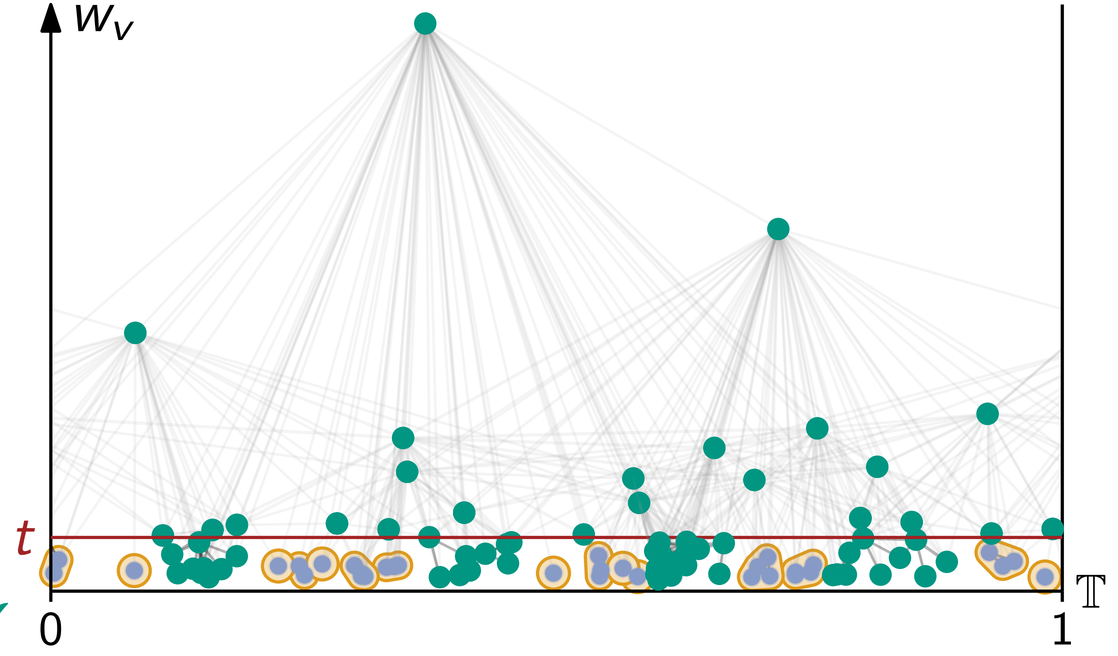

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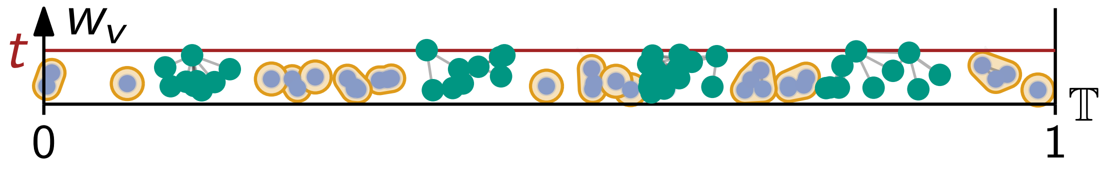

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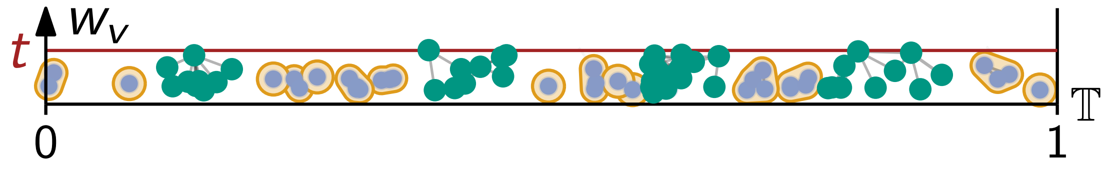

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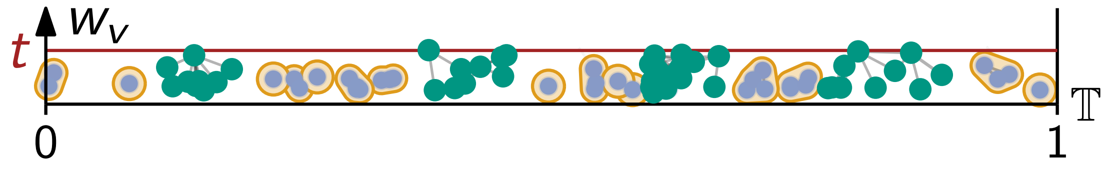

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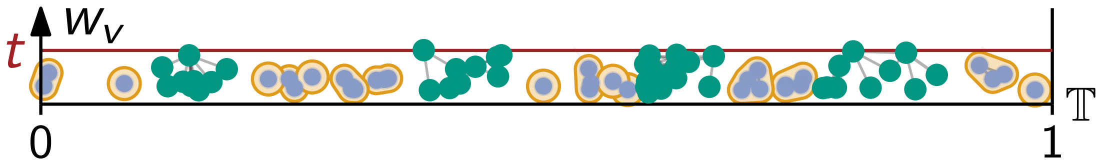

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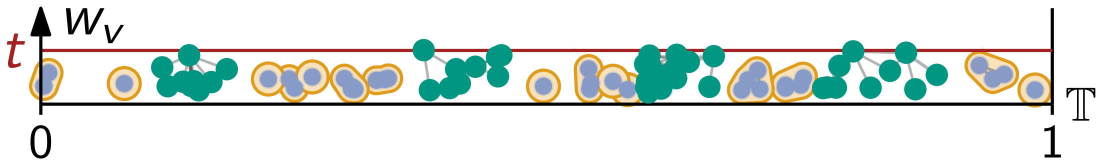

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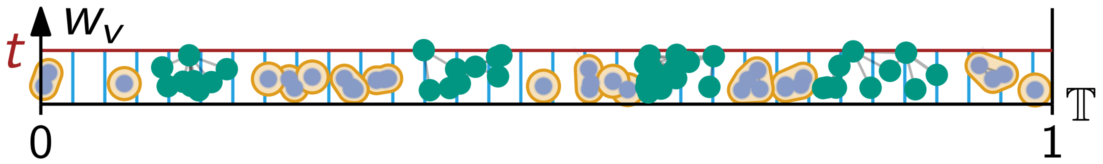
Idea

- Discretize ground space into cells such that edges cannot span empty cells



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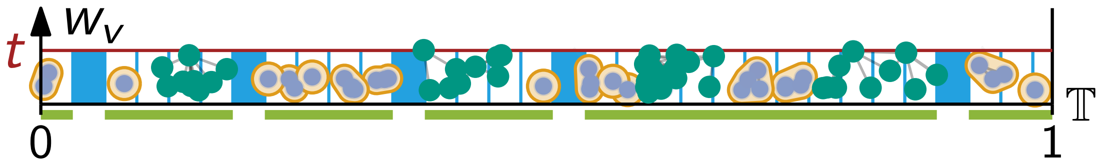
- Use empty cells as delimiters between components
- Regard chains of non-empty cells as one component



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- Use empty cells as delimiters between components
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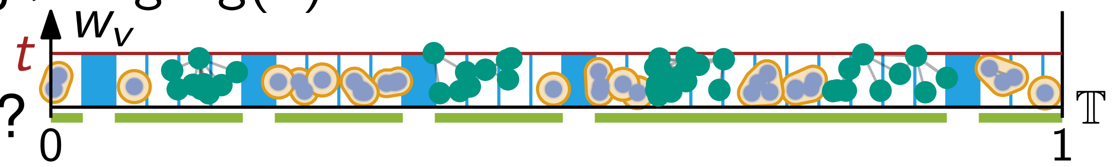
- Count all vertices that are in chains containing $> \log\log(n)$ vertices

  (also potentially counting small components)
- When does a chain contain too many vertices?



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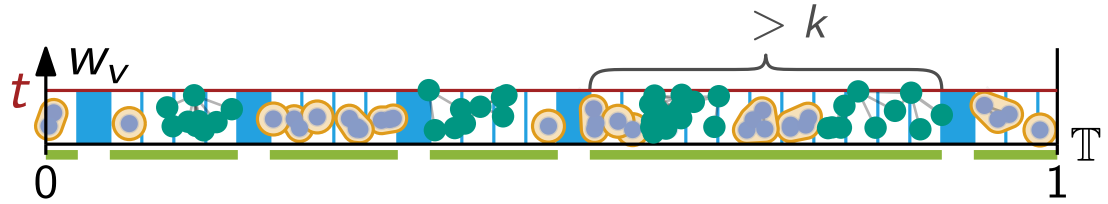
Case 1 Too many cells in long chains, say $> k$ cells



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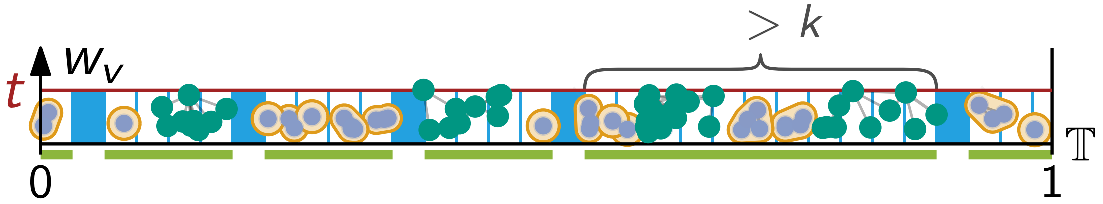

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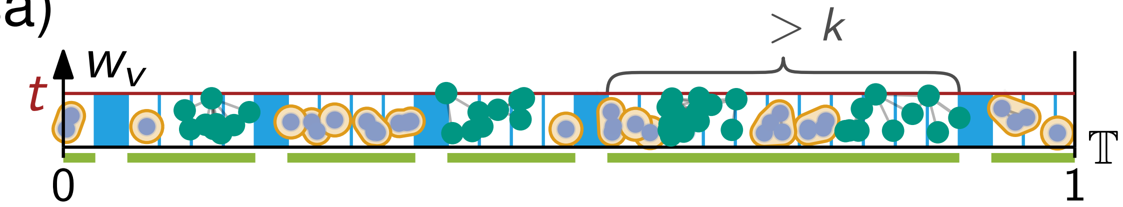
- Proof via method of bounded differences!
  Total number of cells in long chains does not change much ($\leq 2k+1$) when one cell moves from empty to non-empty (or vice versa)



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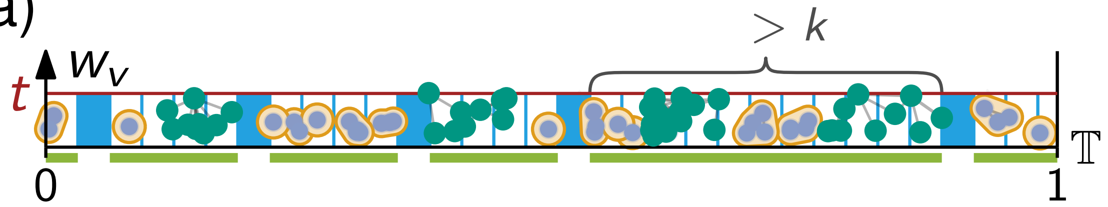
- Use Poissonization to get rid of dependencies



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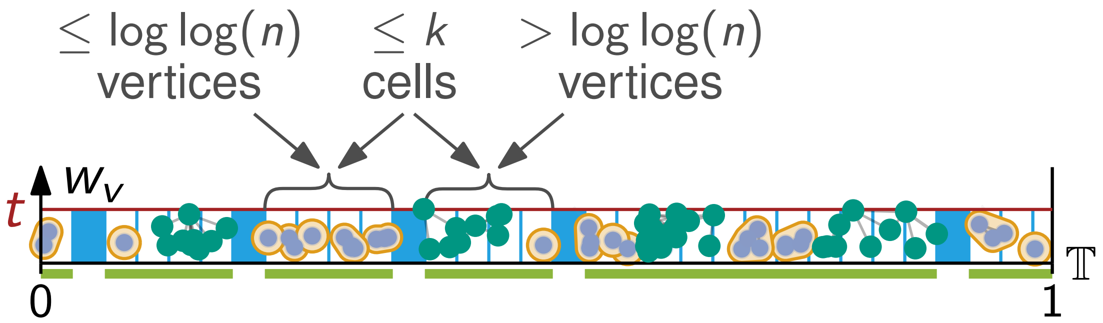

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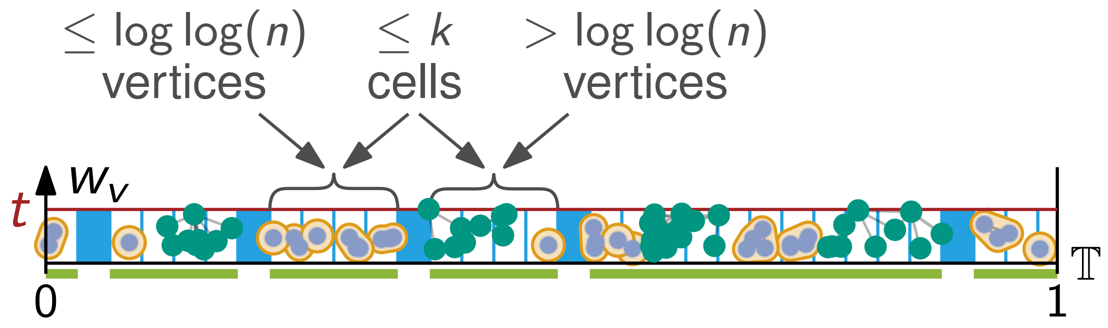

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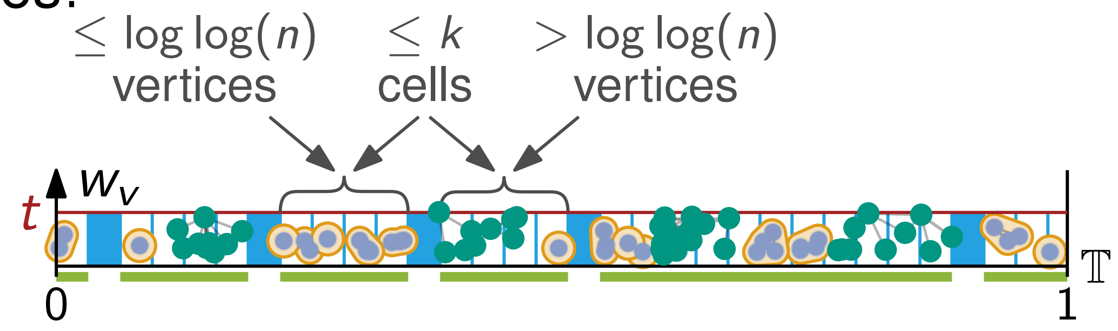

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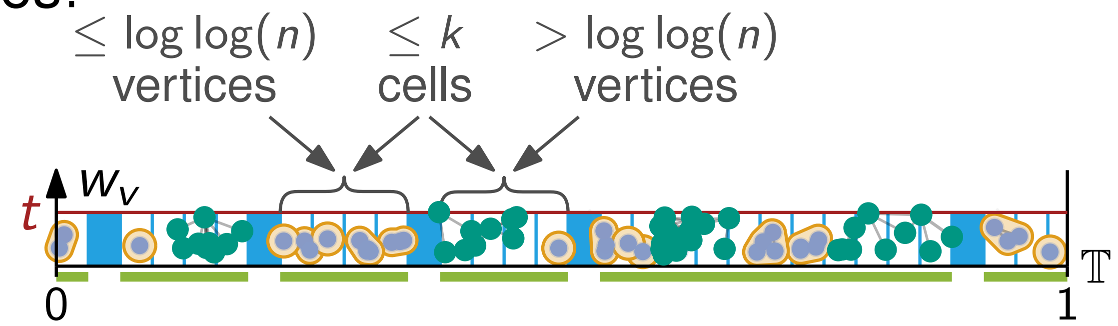

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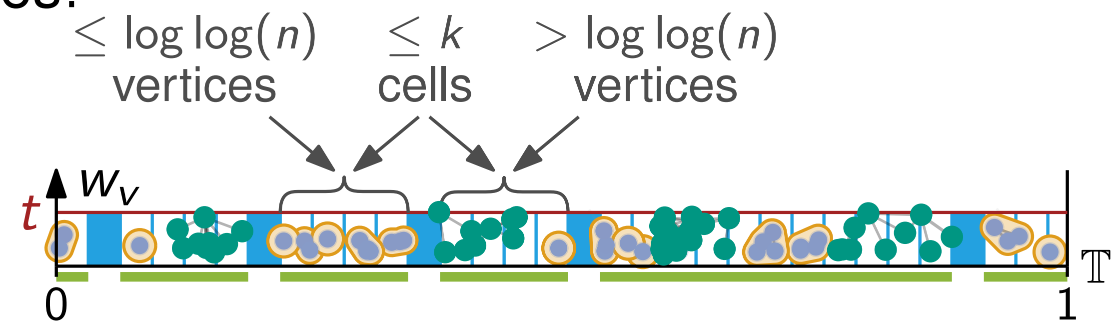

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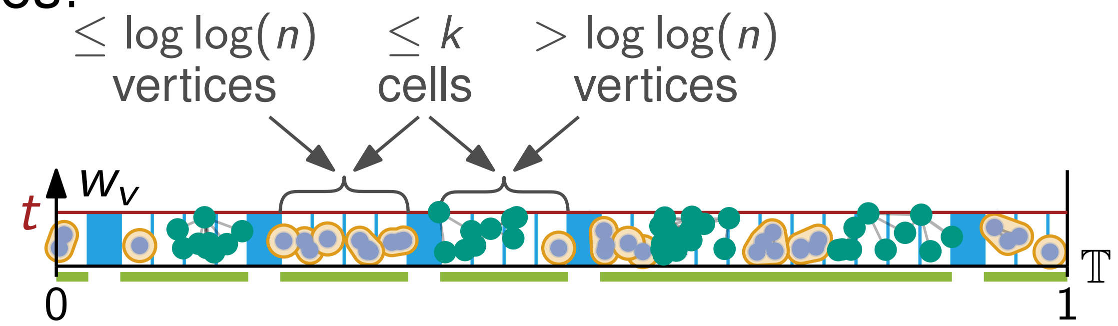

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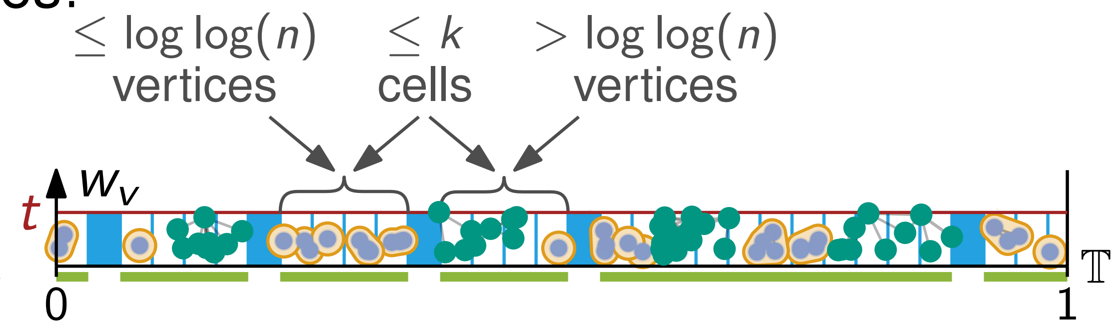
$\rightsquigarrow$ w.h.p., $o(n)$ vertices in large components ✓



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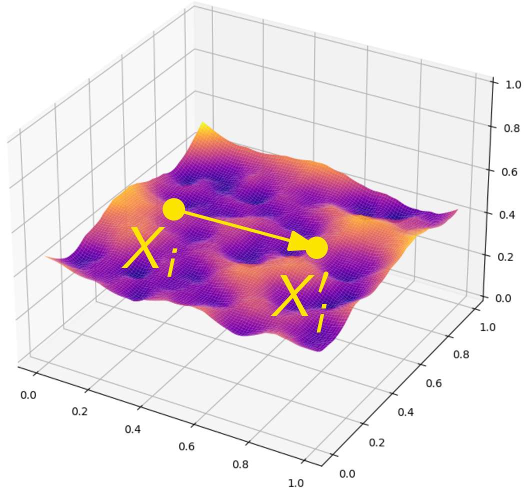
  - What is the worst that can happen when changing one input?



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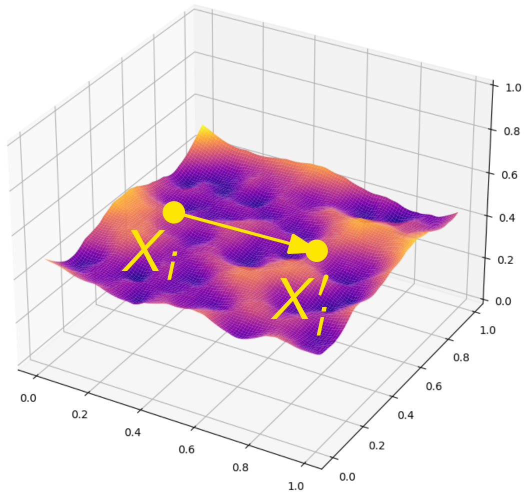

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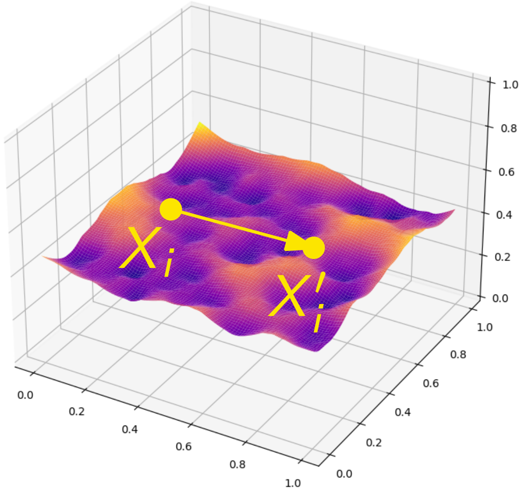

Method of Typical Bounded Differences

- Define typical event, distinguish worst changes depending on whether event occurred
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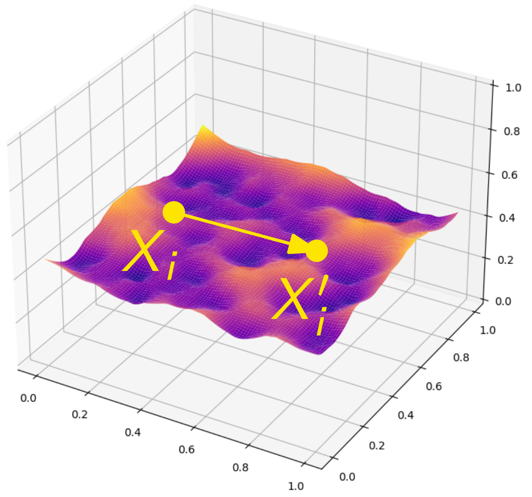

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Geometric Inhomogeneous Random Graphs

- Pretty realistic graph model (heterogeneity, locality)
- Not too hard to analyze
- Used for average-case analysis (e.g. vertex cover approximation)

(not discussed in lecture)