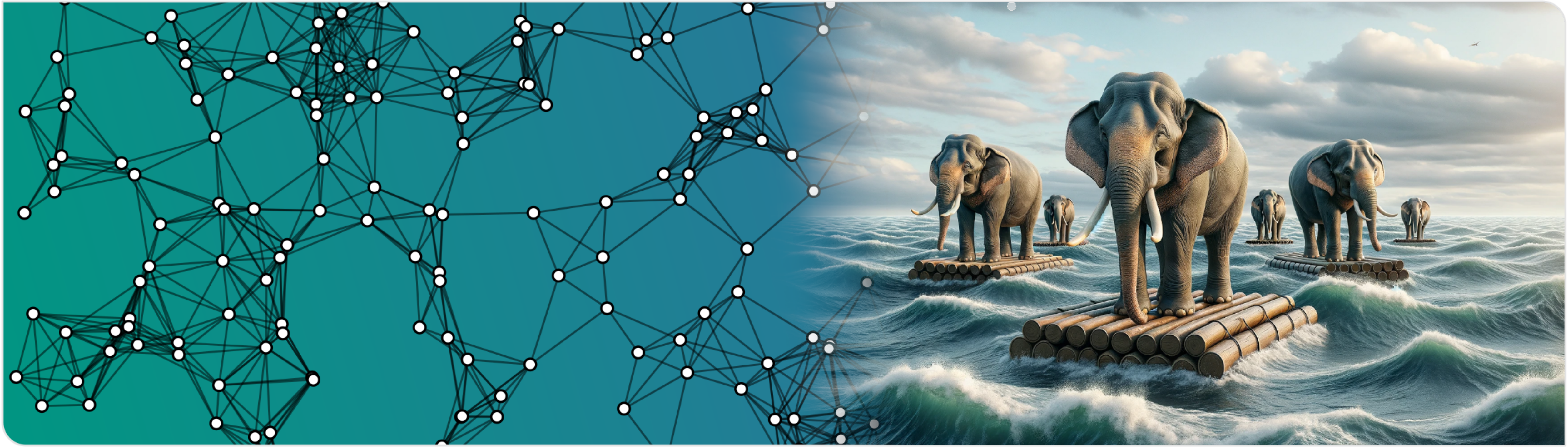
Karlsruhe Institute of Technology

# Probability & Computing

## Bounded Differences & Geometric Inhomogeneous Random Graphs



# Recall: Concentration

## Concentration Inequalities

- Bound the probability for a random variable to deviate from its expectation
- Markov: generally applicable, but not very strong
- Chebychev: stronger, but requires knowledge about variance
- Chernoff: even stronger, but requires knowledge about moment generating functions

**Markov**: $X$ non-negative, $a > 0$:
$\Pr[X \geq a] \leq \mathbb{E}[X]/a$.

(simpler variants work, e.g., for sums of independent random variables)

*Today: similarly strong but beyond sums of independent Bernoulli random variables*

## Example

- $k$ balls distributed uniformly at random over $n$ bins
- Random variable $X$ counts empty bins

$X_1 = 0$ $X_2 = 1$ $X_3 = 0$ $X_4 = 1$ $X_5 = 0$ $X_6 = 0$

- Let $X_i = \mathbb{1}_{\{\text{Bin } i \text{ is empty}\}}$ for $i \in [n] \Rightarrow X = \sum_{i=1}^{n} X_i$

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n \cdot \Pr[X_i = 1]$$
$$= n \cdot \left(1 - \frac{1}{n}\right)^k$$
$$\underset{n \to \infty}{\approx} n \cdot e^{-k/n}$$

- Concentration: $\Pr[X \geq \mathbb{E}[X] + 5\sqrt{k}]$
  - Markov: $\Pr[X \geq \mathbb{E}[X] + 5\sqrt{k}] \leq \frac{\mathbb{E}[X]}{\mathbb{E}[X]+5\sqrt{k}} = 1 - \frac{5\sqrt{k}}{\mathbb{E}[X]+5\sqrt{k}} \xrightarrow{n \to \infty} 1$ ✗
  - Chebychev: tedious... ✗
  - Chernoff: ✗ (our Bernoulli random variables are *not* independent)

# Recall: Concentration

## Concentration Inequalities

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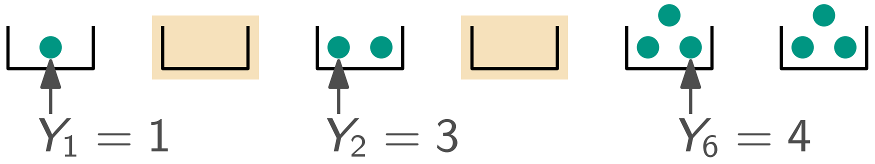
(simpler variants work, e.g., for sums of independent random variables)

## Example

*Today: similarly strong but beyond sums of independent Bernoulli random variables*

- $k$ balls distributed uniformly at random over $n$ bins
- Random variable $X$ counts empty bins
- Let independent $Y_j \sim \mathcal{U}([n])$ for $j \in [k]$ denote the bin of the $j$-th ball



$\Rightarrow X = f(Y_1, ..., Y_k) = \sum_{i \in [n]} \mathbb{1}_{\{\nexists j : Y_j = i\}}$ (summands not independent, but the $Y_j$ are)

$= \sum_{i \in [n]} \max_{j \in [k]} \{2 - |\{Y_j, i\}|\}$ (“not” a sum Bernoulli random variables)

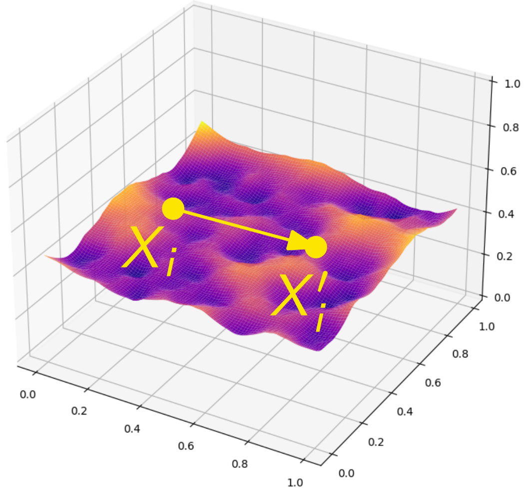
*Can we show concentration for some arbitrary function of independent random variables?*

*... under certain conditions!*

# Method of Bounded Differences

**Aka ...** Bounded differences inequality, McDiarmid's inequality, Azuma-Hoeffding inequality

**Idea** If changing one of the random inputs of $f(X_1, \dots, X_k)$ does not change $f(\cdot)$ much then a lot has to go wrong for $f(\cdot)$ to deviate from its expected value



**Definition**: A function $f : S^n \to \mathbb{R}$ satisfies the **bounded differences condition** ("Lipschitz condition") with parameters $\Delta_i$, if $|f(X_1, \dots, X_i, \dots, X_n) - f(X_1, \dots, X_i', \dots, X_n)| \leq \Delta_i$ for all $i \in [n]$ and $X_i, X_i' \in S$.

**Theorem**: Let $X_1, \dots, X_n$ be independent random variables taking values in a set $S$. Let $f : S^n \to \mathbb{R}$ satisfy the bounded differences condition with parameters $\Delta_i$. Then, for $\Delta = \sum_{i \in [n]} \Delta_i^2$:

$$\Pr[|f - \mathbb{E}[f]| \geq t] \leq 2e^{-2t^2/\Delta}.$$

(write $f$ for $f(X_1, \dots, X_n)$)

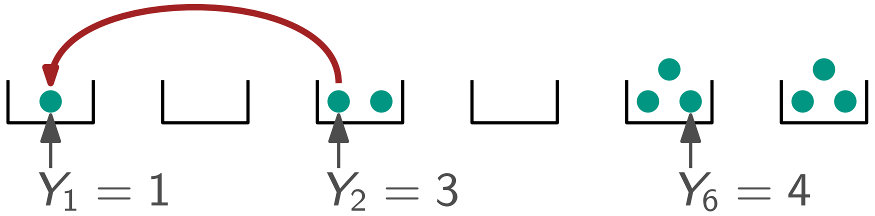
**Lemma**: $\Pr[f \geq \mathbb{E}[f] + t] \leq e^{-2t^2/\Delta}$.

also for $\Pr[f \leq \mathbb{E}[f] - t]$

**Cor.** $\mathbb{E}[f] \leq g(n)$: $\Pr[f \geq cg(n)] \leq e^{-2((c-1)g(n))^2/\Delta}$.

# Application: Balls into Bins

- $k$ balls distributed uniformly at random over $n$ bins
- Random variable $X$ counts empty bins
- Let independent $Y_j \sim \mathcal{U}([n])$ for $j \in [k]$ denote the bin of the $j$-th ball, and $X = f(Y_1, \ldots Y_k)$



## Bounded differences condition

$|f(\ldots, Y_i, \ldots) - f(\ldots, Y_i', \ldots)| \le \Delta_i$
for all $i$ and $Y_i, Y_i'$

- Intuition: How much can the number of empty bins change if we move a ball from one bin to another?
  - A ball is moved from an almost empty bin to...
    - ... an empty bin $\Rightarrow +1 - 1 \Rightarrow \Delta_i = 0$
    - ... a non-empty bin $\Rightarrow +1 \Rightarrow \Delta_i = 1$
  - A ball is moved from a not almost empty bin to...
    - ... an empty bin $\Rightarrow -1 \Rightarrow \Delta_i = 1$
    - ... a non-empty bin $\Rightarrow \Delta_i = 0$

$\Delta_i \le 1$

Function $f(Y_1, \ldots, Y_k)$:
- $Y_1, \ldots, Y_k$ independent
- bounded differences $\Delta_i$
- $\Delta = \sum_{i=1}^{k} \Delta_i^2$

Then $\Pr[f \ge \mathbb{E}[f] + t] \le e^{-2t^2/\Delta}$

## Concentration via bounded differences

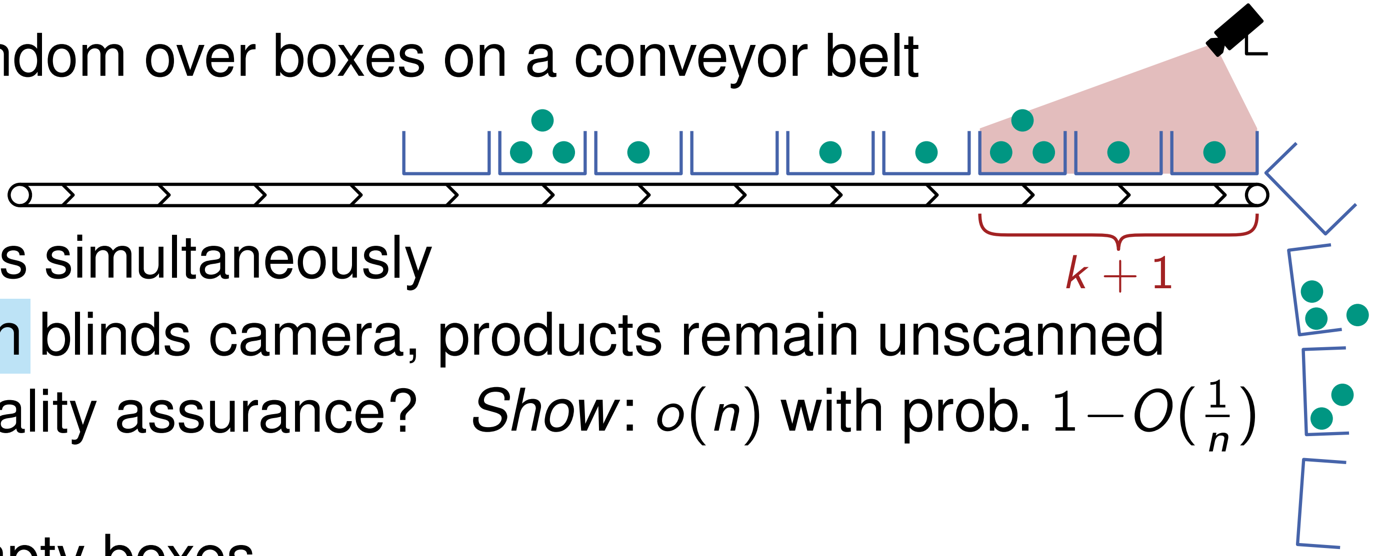
$$\Delta = \sum_{i=1}^{k} \Delta_i^2 \le \sum_{i=1}^{k} 1^2 = k \quad \Rightarrow \Pr[f \ge \mathbb{E}[f] + 5\sqrt{k}] \le e^{-2(5\sqrt{k})^2/k} = e^{-50}$$

Much better than Markov's → 1

# Application: The Factory

- Products are distributed uniformly at random over boxes on a conveyor belt

$n$ products $m = n/k$ boxes $k = \log\log(n)$



- A camera scans $k+1$ consecutive boxes simultaneously
- Problem: Empty box in view $\Rightarrow$ reflection blinds camera, products remain unscanned
- Question: How many products avoid quality assurance? *Show*: $o(n)$ with prob. $1 - O(\frac{1}{n})$

**Formalize**

- *chain*: consecutive sequence of non-empty boxes
- *short chain*: incl. max. chain of length $\leq k$ $\Rightarrow$ *exactly* products in short chains unscanned
- $X_i$ = number of products in box $i$, $Y_i$ = indicator whether box $i$ is in a short chain
- Then $X = \sum_{i=1}^{m} X_i \cdot Y_i$ is the number of unscanned products
- Problem: Dependencies (between $X_i$'s, between $X_i$ and $Y_i$)
- Solution: Relax dependencies and compute upper bound instead

# Application: The Factory

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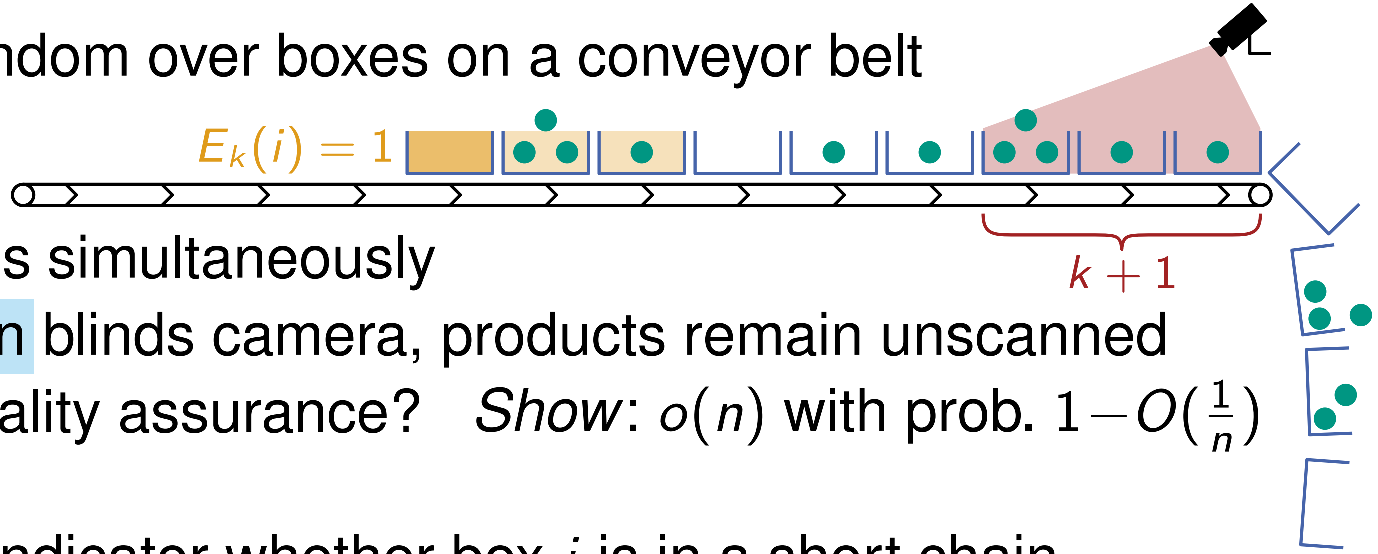

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**Relax and bound**

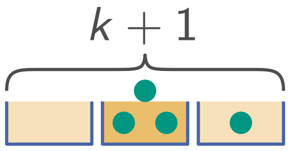
- $X_i$ = number of products in box $i$, $Y_i$ = indicator whether box $i$ is in a short chain
- Then $X = \sum_{i=1}^{m} X_i \cdot Y_i$ is the number of unscanned products
- $E_k(i)$ = number of empty boxes in box $i$ and $k$ closest (assuming $k$ even)

  ↳ Box $i$ in short chain $\Rightarrow E_k(i) > 0$

- $Y_i'$ = indicator whether $E_k(i) > 0 \Rightarrow Y_i \leq Y_i'$

  ↳ $X = \sum_{i=1}^{m} X_i \cdot Y_i \leq \sum_{i=1}^{m} X_i \cdot Y_i' =: X'$

# Expectation of $X'$ (for $n$ large enough)



$$\mathbb{E}[X'] = \textstyle\sum_{i=1}^{m} \underbrace{\mathbb{E}[X_i \cdot Y_i']}$$

(law of total expectation) $= \sum_{\ell=0}^{k+1} \mathbb{E}[X_i \cdot Y_i' \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell]$

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$$= \textstyle\sum_{\ell=1}^{k} \underbrace{\mathbb{E}[X_i \mid E_k(i) = \ell]} \cdot \Pr[E_k(i) = \ell]$$

Expected number of products in box $i$, knowing that exactly $\ell$ boxes are empty

- Box $i$ empty? $\Rightarrow X_i = 0$
- Else: $n$ products distributed u.a.r. over $m' = m - \underbrace{\ell}_{\leq k}$ boxes

$\hookrightarrow \mathbb{E}[X_i \mid E_k(i) = \ell] = \frac{n}{m'} \leq 2 \log\log(n)$

$$m' \geq \frac{n}{\log\log(n)} - \log\log(n)$$

(for $n$ large enough) $\geq \frac{1}{2} \frac{n}{\log\log(n)}$

- $n$ products
- $m = n/k$ boxes, $k = \log\log(n)$
- $X' = \sum X_i \cdot Y_i'$
- $X_i$, products in box $i$
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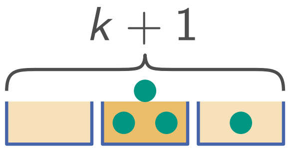

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$$= \sum_{\ell=0}^{k} \mathbb{E}[X_i \cdot Y_i' \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell]$$

$$= \sum_{\ell=1}^{k} \mathbb{E}[X_i \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell]$$

$$\leq \sum_{\ell=1}^{k} 2\log\log(n) \cdot \Pr[E_k(i) = \ell]$$

$$= 2\log\log(n) \underbrace{\sum_{\ell=1}^{k} \Pr[E_k(i) = \ell]}$$

$$\leq \Pr[\text{“Exists an empty box among } k+1\text{”}]$$

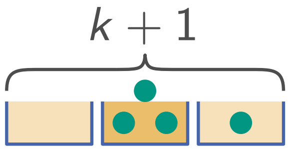
(union bound) $\leq (k+1) \cdot \Pr[\text{“A given box is empty”}]$

$$\leq 2k\left(1 - \frac{1}{m}\right)^n$$

$(1 + x \leq e^x)$

$$= 2k\left(1 - \frac{k}{n}\right)^n \leq 2k \cdot e^{-k} = 2\frac{\log\log(n)}{\log(n)}$$

# Expectation of $X'$ (for $n$ large enough)



- $n$ products
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- $Y_i'$, indicator $E_k(i) > 0$

$$\mathbb{E}[X'] = \sum_{i=1}^{m} \underbrace{\mathbb{E}[X_i \cdot Y_i']}$$

(law of total expectation)

$$\begin{aligned}
&= \sum_{\ell=0}^{k+1} \mathbb{E}[X_i \cdot Y_i' \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell] \\
&= \sum_{\ell=0}^{k} \mathbb{E}[X_i \cdot Y_i' \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell] \\
&= \sum_{\ell=1}^{k} \mathbb{E}[X_i \mid E_k(i) = \ell] \cdot \Pr[E_k(i) = \ell] \\
&\leq \sum_{\ell=1}^{k} 2\log\log(n) \cdot \Pr[E_k(i) = \ell] \\
&= 2\log\log(n) \sum_{\ell=1}^{k} \Pr[E_k(i) = \ell] \\
&\leq 2\log\log(n) \cdot 2\frac{\log\log(n)}{\log(n)} \\
&= 4\frac{\log\log(n)^2}{\log(n)}
\end{aligned}$$

$$\mathbb{E}[X'] = \sum_{i=1}^{m} 4\frac{\log\log(n)^2}{\log(n)} = m \cdot 4\frac{\log\log(n)^2}{\log(n)} = \frac{n}{\cancel{\log\log(n)}} \cdot 4\frac{\log\log(n)^{\cancel{2}}}{\log(n)} = n \cdot 4\frac{\log\log(n)}{\log(n)} = o(n) \checkmark$$

# Concentration of $X$ (for $n$ large enough)

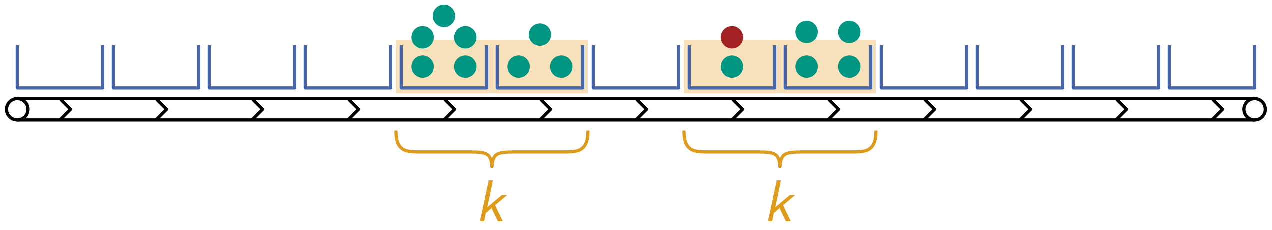
- $n$ products
- $m = n/k$ boxes, $k = \log\log(n)$
- $X = \sum X_i \cdot Y_i$
- $X_i$, products in box $i$
- $Y_i$, indicator $i$ in short chain

$$\mathbb{E}[X] \leq \mathbb{E}[X'] \leq 4n\frac{\log\log(n)}{\log(n)}$$

$|f(\ldots, Z_j, \ldots) - f(\ldots, Z_j', \ldots)| \leq \Delta_j$ for all $j$ and $Z_j, Z_j'$

## Bounded Differences

- View $X$ as a function $f(Z_1, \ldots, Z_n)$ of independent rand. var. where $Z_j$ for $j \in [n]$ denotes the box of the $j$-th product
- Bounded differences condition:
  - Worst change in number of products in short chains when moving a single product from one box to another
  - Consider chain of $2k + 1$ boxes containing *all* $n$ products and one box contains only one of them



$\Rightarrow X = 0$, since no short chain and, thus, no products in short chains

- Move product to next box

$\Rightarrow X = n$, since all products in short chains now

$\Big\}\ \Delta_j \leq n$

# Concentration of $X$ (for $n$ large enough)

## Bounded Differences

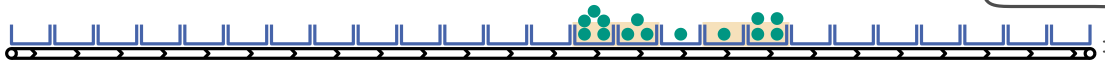
- View $X$ as a function $f(Z_1, \ldots, Z_n)$ of independent rand. var. where $Z_j$ for $j \in [n]$ denotes the box of the $j$-th product
- Bounded differences condition: $\Delta_j \leq n$
- Bounded differences inequality:

$$\Delta = \sum_{j=1}^{n} \Delta_j^2 \leq \sum_{j=1}^{n} n^2 = n^3 \qquad g(n) = 4n\frac{\log\log(n)}{\log(n)}$$

$$\Pr\left[X \geq c4n\frac{\log\log(n)}{\log(n)}\right] \leq \exp\left(-\frac{2(c-1)^2\left(4n\frac{\log\log(n)}{\log(n)}\right)^2}{n^3}\right)$$

This bound is useless, since worst-case changes are too big

$$= \exp\left(-\Theta\left(\underbrace{\frac{\log\log(n)^2}{n\log(n)^2}}_{= o(1)}\right)\right) \xrightarrow{n \to \infty} 1$$



*But this case (all products in few boxes) is super unlikely...*

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$$\mathbb{E}[X] \leq \mathbb{E}[X'] \leq 4n\frac{\log\log(n)}{\log(n)}$$

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Function $f(Z_1, \ldots, Z_n)$:

- $Z_1, \ldots, Z_n$ independent
- bounded differences $\Delta_j$
- $\Delta = \sum_{j=1}^{n} \Delta_j^2$
- $g(n) \geq \mathbb{E}[f]$

$\Pr[f \geq cg(n)] \leq e^{-2((c-1)g(n))^2/\Delta}$.

# Method of Typical Bounded Differences

**Definition**: A function $f\colon S^n \to \mathbb{R}$ satisfies the **typical bounded differences condition** with respect to

- an event $A \subseteq S^n$ and
- parameters $\Delta_i^A \leq \Delta_i$ for $i \in [n]$,

if $$|f(X_1, ..., X_i, ..., X_n) - f(X_1, ..., X_i', ..., X_n)| \leq \begin{cases} \Delta_i^A, & \text{if } (X_1, ..., X_i, ..., X_n) \in A, \\ \Delta_i, & \text{otherwise} \end{cases}$$

for all $i \in [n]$ and $X_i, X_i' \in S$.

- $\Delta_i^A$ is worst-case change, assuming $A$ held before the change

**Theorem**: Let $X_1, ..., X_n$ be independent random variables taking values in a set $S$, let $A \subseteq S^n$ be an event, and let $f\colon S^n \to \mathbb{R}$ satisfy the typical bounded differences condition w.r.t. $A$ and parameters $\Delta_i^A \leq \Delta_i$. Then, for $g(n) \geq \mathbb{E}[f]$, for all $\varepsilon_i \in (0, 1]$ and $\Delta = \sum_{i\in[n]}(\Delta_i^A + \varepsilon_i(\Delta_i - \Delta_i^A))^2$: $\Pr[f \geq cg(n)] \leq e^{-((c-1)g(n))^2/(2\Delta)} + \Pr[\neg A]\sum_{i\in[n]} \frac{1}{\varepsilon_i}$.

Corollary of “On the Method of Typical Bounded Differences”, Warnke, Comb. Probab. Comput. 2015

# Method of Typical Bounded Differences

**Theorem**: Let $X_1, \dots, X_n$ be independent random variables taking values in a set $S$, let $A \subseteq S^n$ be an event, and let $f\colon S^n \to \mathbb{R}$ satisfy the typical bounded differences condition w.r.t. $A$ and parameters $\Delta_i^A \leq \Delta_i$. Then, for $g(n) \geq \mathbb{E}[f]$, for all $\varepsilon_i \in (0, 1]$ and $\Delta = \sum_{i \in [n]} (\Delta_i^A + \varepsilon_i(\Delta_i - \Delta_i^A))^2$: $\Pr[f \geq cg(n)] \leq e^{-((c-1)g(n))^2/(2\Delta)} + \Pr[\neg A] \sum_{i \in [n]} \frac{1}{\varepsilon_i}$.

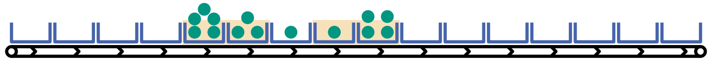
- Function of independent random variables as before
- $A$ is the good, typical event that should be very likely to occur
- $\Delta$ is sum of squared worst-case changes as before
  - We still consider general worst-case changes as before
  - But we can use the $\varepsilon_i$ to mitigate the worst-case effects
  - And focus on the worst-case changes, assuming $A$ held before the change
- But we have to pay for the mitigation!
  - With the probability that the good event $A$ does not occur
  - Multiplied with the inverse mitigators

The more we need to mitigate,
the higher the price!

Not too bad if $A$ is very
likely to occur!

# Application: The Factory (2nd Try)

- View $X$ as a function $f(Z_1, ..., Z_n)$ of independent rand. var. where $Z_j$ for $j \in [n]$ denotes the box of the $j$-th product
- Bounded differences condition: $\Delta_j \leq n$
  - When all $n$ products fall into $2k+1 = O(\log\log(n))$ boxes
  - But expected number of products in a single box $i$: $\mathbb{E}[B_i] = \frac{n}{m} = \frac{n}{\frac{n}{\log\log(n)}} = \log\log(n)$
  - And, thus, expected number in sequence of $2k+1$ boxes $\mathbb{E}[S] = \sum_{i=1}^{2k+1} \mathbb{E}[B_i] = O(\log\log(n)^2) \leq \delta\log(n) =: g(n)$ (for any $\delta > 0$ and suffciently large $n$)
  - So *typically* a sequence should contain way fewer than $n$ products
- Typical event $A = \{$“Every sequence of $2k+1$ boxes contains $O(\log(n))$ products”$\}$
  - See $S$ as sum of independent Bernoulli rand. var. (whether $j$-th product is in sequence)
  - Chernoff: For $g(n) \geq \mathbb{E}[S]$: $\Pr[S \geq (1+\varepsilon)g(n)] \leq e^{-\varepsilon^2/3\cdot g(n)} = e^{-\varepsilon^2/3\cdot\delta\log(n)} = n^{-\delta\varepsilon^2/3}$
  - Union bound over $\leq n$ sequences: $\Pr[\neg A] \leq n^{-\delta\varepsilon^2/3+1} \leq n^{-\lambda}$ (for arbitrarily large $\lambda$)



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$\mathbb{E}[X] \leq \mathbb{E}[X'] \leq 4n\frac{\log\log(n)}{\log(n)}$

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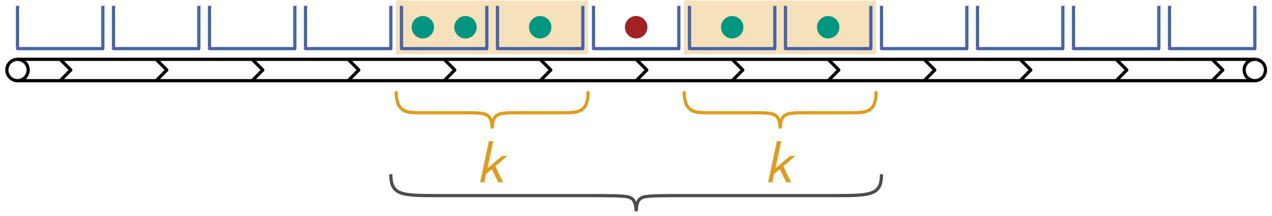
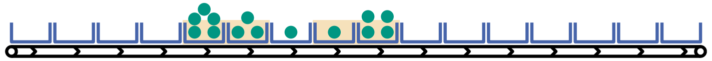
- Bounded differences condition: $\Delta_j \leq n$
- Typical event $A = \{$"Every sequence of $2k+1$ boxes contains $O(\log(n))$ products"$\}$, $\Pr[\neg A] \leq n^{-\lambda}$ (for arbitrary $\lambda$)
- Typical bounded differences condition:
  - Worst change in $f$ when moving a product from one box to another, assuming $A$ held before the move



contain $O(\log(n))$ products

  - Moving one product empties at most one box $\Rightarrow$ at most two new short chains
  - Assuming $A$, these short chains combined contain $O(\log(n))$ products $\Rightarrow \Delta_j^A = O(\log(n))$



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- $Y_i$, indicator $i$ in short chain

$$\mathbb{E}[X] \leq \mathbb{E}[X'] \leq 4n\frac{\log\log(n)}{\log(n)}$$

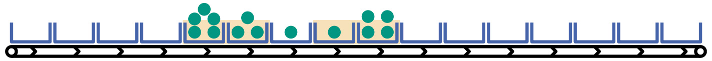
# Application: The Factory (2nd Try)

- View $X$ as a function $f(Z_1, \dots, Z_n)$ of independent rand. var. where $Z_j$ for $j \in [n]$ denotes the box of the $j$-th product
- Bounded differences condition: $\Delta_j \leq n$
- Typical event $A = \{$"Every sequence of $2k+1$ boxes contains $O(\log(n))$ products"$\}$, $\Pr[\neg A] \leq n^{-\lambda}$ (for arbitrary $\lambda$)
- Typical bounded differences condition: $\Delta_j^A = O(\log(n))$
- Typical bounded differences inequality:

$$
\begin{aligned}
\Delta &= \textstyle\sum_{j=1}^{n} (\Delta_j^A + \varepsilon_j(\Delta_j \cancel{- \Delta_j^A}))^2 \qquad\qquad \varepsilon_j = \frac{1}{n} \\
&\leq \textstyle\sum_{j=1}^{n} (\Delta_j^A + \varepsilon_j \Delta_j)^2 \\
&\leq \textstyle\sum_{j=1}^{n} (O(\log(n)) + \varepsilon_j n)^2 \\
&= \textstyle\sum_{j=1}^{n} (O(\log(n)) + 1)^2 \\
&= O(n \log(n)^2)
\end{aligned}
$$

Mitigators, arbitrary $\in (0, 1]$!

Much better than $n^3$ from before!



- $n$ products
- $m = n/k$ boxes, $k = \log\log(n)$
- $X = \sum X_i \cdot Y_i$
- $X_i$, products in box $i$
- $Y_i$, indicator $i$ in short chain

$\mathbb{E}[X] \leq \mathbb{E}[X'] \leq 4n \frac{\log\log(n)}{\log(n)}$

Function $f(Z_1, \dots, Z_n)$:

- $Z_1, \dots, Z_n$ independent
- typical event $A$
- bounded differences $\Delta_j^A \leq \Delta_j$
- $\Delta = \sum_{j=1}^{n} (\Delta_j^A + \varepsilon_j(\Delta_j - \Delta_j^A))^2$
- $g(n) \geq \mathbb{E}[f]$

$$\Pr[f \geq cg(n)] \leq e^{-((c-1)g(n))^2/(2\Delta)} + \Pr[\neg A] \sum_{j=1}^{n} \frac{1}{\varepsilon_j}$$

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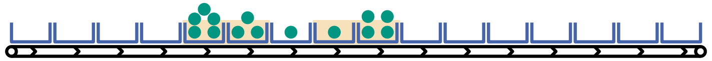
$$\Delta = O(n\log(n)^2) \qquad g(n) = 4n\frac{\log\log(n)}{\log(n)} \qquad \varepsilon_j = \frac{1}{n}$$

$$\Pr\left[X \geq c4n\frac{\log\log(n)}{\log(n)}\right] \leq \underbrace{\exp\left(-\Omega\left(n\frac{\log\log(n)^2}{\log(n)^4}\right)\right)}_{=O(1/n)} + \underbrace{\Pr[\neg A]\sum_{j=1}^{n}\frac{1}{\varepsilon_j}}_{\leq n^{-\lambda}\cdot n^2 = O(1/n)\ \checkmark \text{ for } \lambda = 3}$$



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# Geometric Inhomogeneous Random Graphs

## Motivation

- Average-case analysis: analyze models that represent the real world
- Models seen so far
  - Erdős-Rényi random graphs: simple but no locality
  - Random geometric graphs: locality but no heterogeneity (all vertices roughly same degree)

Not realistic: celebrities are very-high-degree vertices in social networks

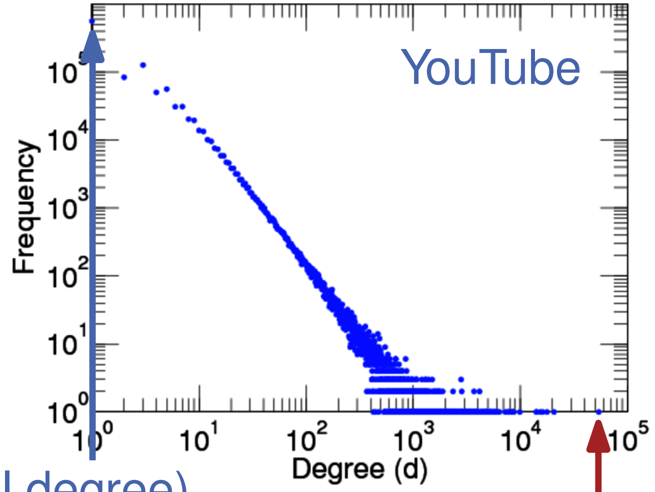
- Realistic representation: power-law distribution

"Scale-free networks well done", Voitalov, van der Hoorn, van der Hofstad, Krioukov, Phys. Rev. Research. 2019

- Pareto distribution: $X \sim \mathrm{Par}(\alpha, x_{\min})$

$$f_X(x) = \begin{cases} \alpha x_{\min}^{\alpha} \cdot x^{-(\alpha+1)}, & \text{if } x \geq x_{\min} \\ 0, & \text{otherwise} \end{cases}$$

konect.cc/plot/degree.a.youtube-links.full.png



(most vertices small degree)

(few vertices very high degree)

## Idea

- Add Pareto distribution to RGGs

# Geometric Inhomogeneous Random Graphs

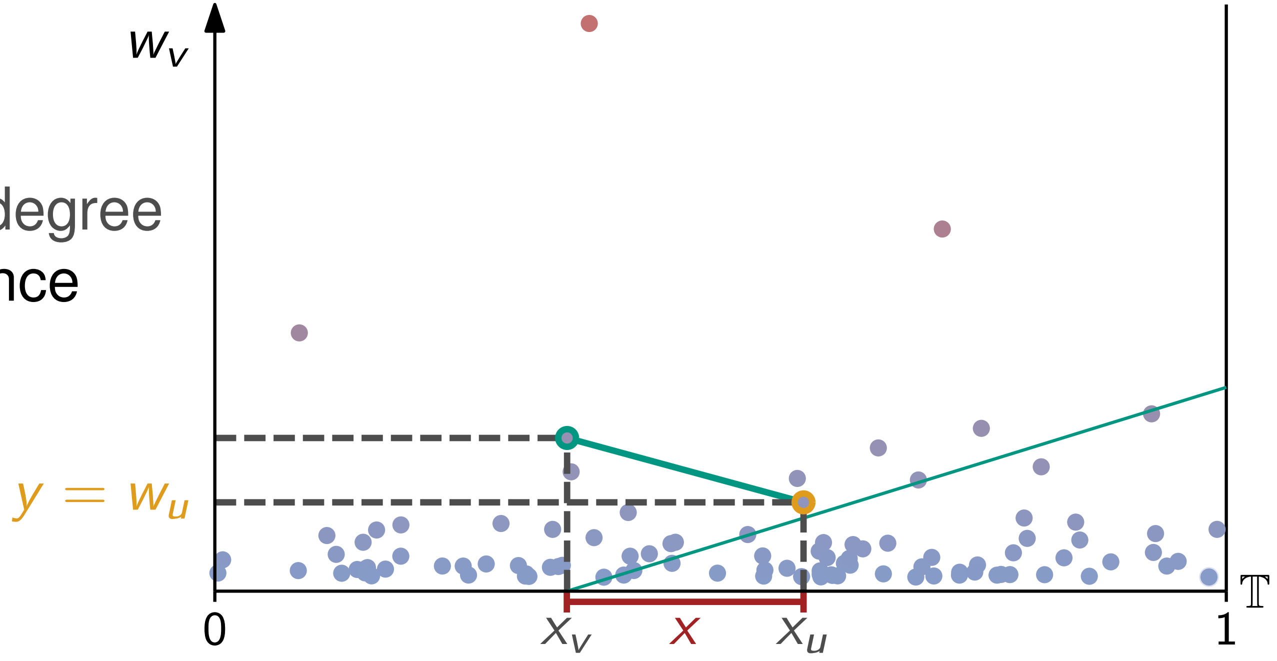
## Definition

- Consider $n$ vertices
- For each vertex $v$ independently:
  - Draw a *position* $x_v$ uniformly on $\mathbb{T}^d$
  - Draw a *weight* $w_v$ from $\text{Par}(\tau - 1, 1)$ for $\overset{\text{“Power-Law Exponent”}}{\tau} \in (2, 3) \Rightarrow f_{w_v}(w) = (\tau - 1)w^{-\tau}$
- Connect $u$ and $v$ with an edge, iff

$$\underbrace{\text{dist}(x_u, x_v)}_{L_\infty\text{-norm}} \leq \Big(\underset{\text{const. controls the avg. degree}}{\lambda} \frac{w_u \cdot w_v}{n}\Big)^{1/d}$$

- For $d = 1$, linear relation between distance and weight $\quad y = w_u, x = \text{dist}(x_u, x_v)$

$$x \leq \lambda \frac{w_v \cdot y}{n} \Leftrightarrow y \geq \frac{n}{\lambda w_v} x$$



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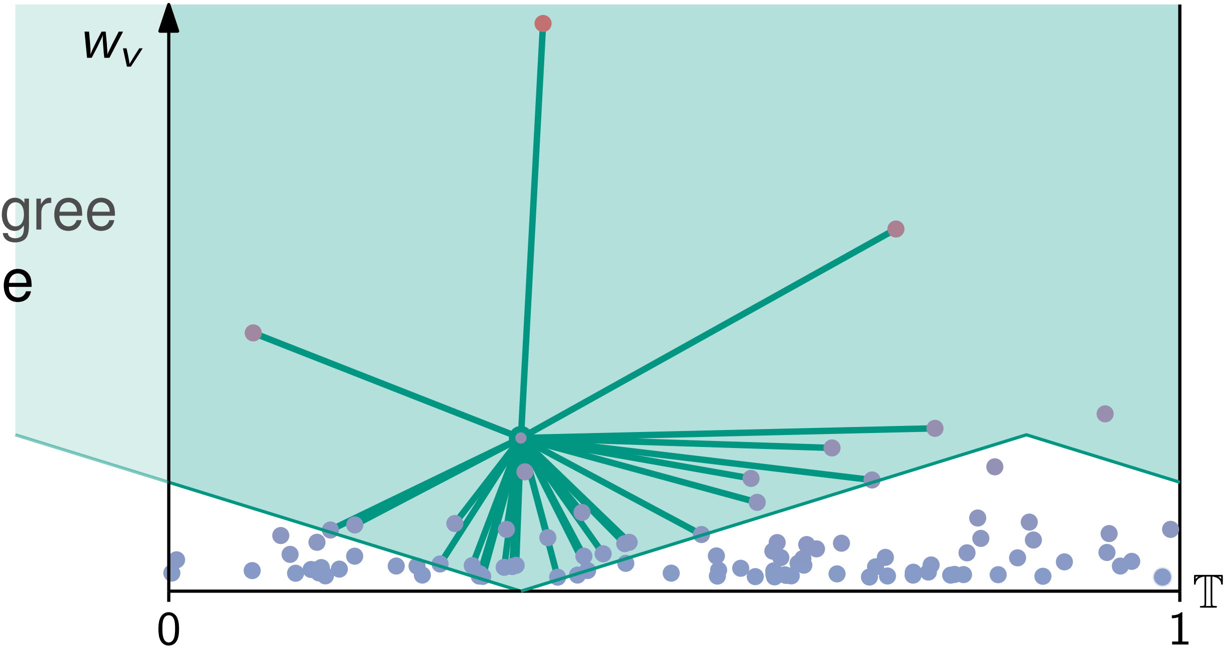

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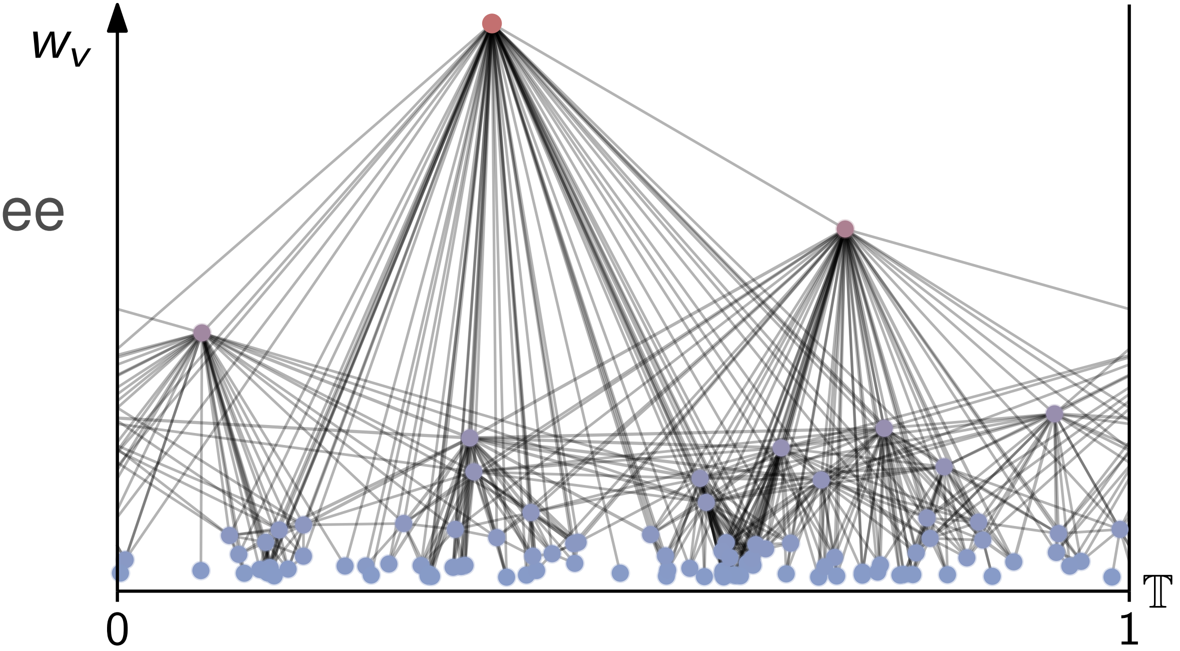
$$x \leq \lambda \frac{w_v \cdot y}{n} \Leftrightarrow y \geq \frac{n}{\lambda w_v} x$$

- The lower $w_v$, the steeper the wedge
  ↳ The lower the degree



# Expected Degree ($d = 1$)

- Consider vertex $v$ with weight $w_v$
- We want to compute $\mathbb{E}[\deg(v) \mid w_v]$
- Consider $X_u$ for $u \in V \setminus \{v\}$ indicating whether $\{u, v\} \in E$

$$\deg(v) = \sum_{u \in V \setminus \{v\}} X_u$$

$$\mathbb{E}[\deg(v) \mid w_v] = \sum_{u \in V \setminus \{v\}} \mathbb{E}[X_u \mid w_v]$$

$$= \Theta(n \underbrace{\Pr[\underbrace{\{u, v\} \in E}_{u \in \boxed{N(v)}} \mid w_v]}_{})$$

This is *not* the area of the shape,
since weights are *not* distributed uniformly!

$\Rightarrow$ Use law of total probability to account for that

**GIRG**
- $n$ independent vertices
- $x_v \sim \mathcal{U}([0, 1])$
- $w_v \sim \mathrm{Par}(\tau - 1, 1)$ for $\tau \in (2, 3)$
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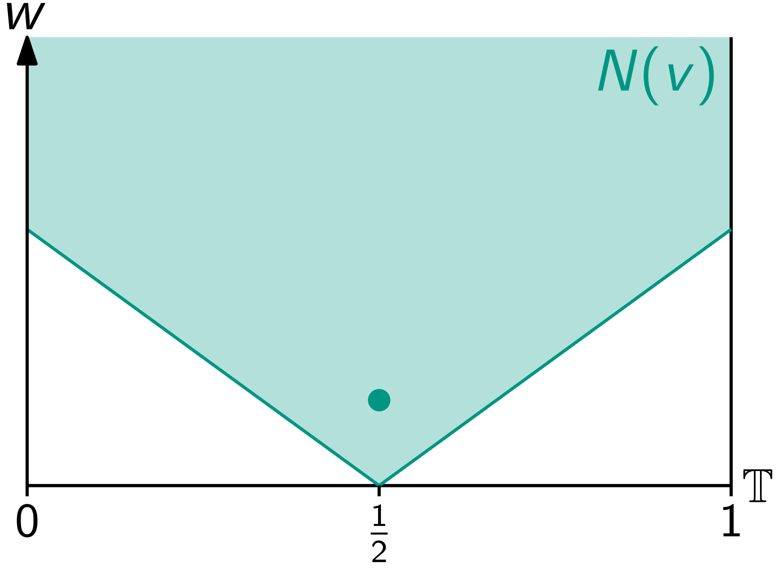
w.l.o.g $x_v = \frac{1}{2}$



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$$= \Pr[x_u \in [\tfrac{1}{2} - \lambda \tfrac{w \cdot w_v}{n}, \tfrac{1}{2} + \lambda \tfrac{w \cdot w_v}{n}]]$$

Case 1: $w \leq \frac{n}{2\lambda w_v} \Rightarrow \lambda \frac{w \cdot w_v}{n} \leq \frac{1}{2} \longrightarrow = 2\lambda \frac{w \cdot w_v}{n} = \Theta(\frac{w \cdot w_v}{n})$

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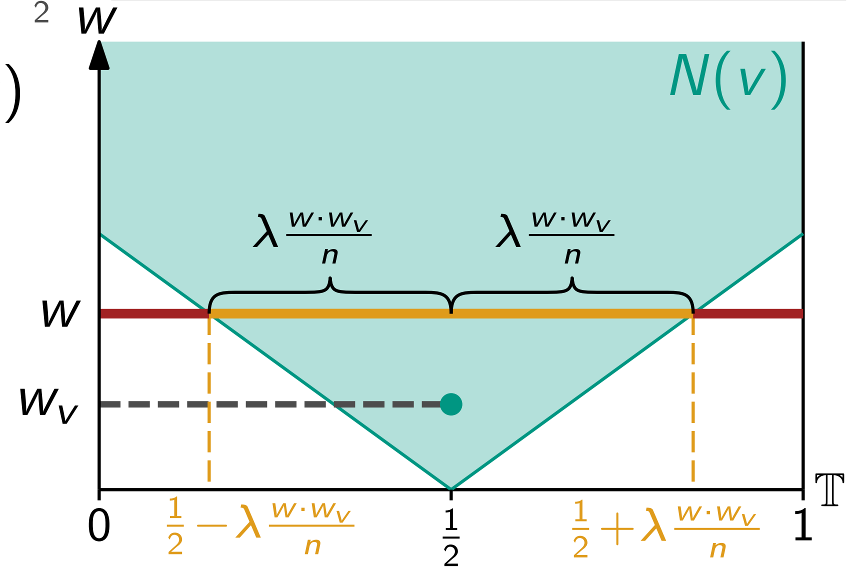

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$$\deg(v) = \sum_{u \in V\setminus\{v\}} X_u$$

$$\begin{aligned}\mathbb{E}[\deg(v) \mid w_v] &= \sum_{u \in V\setminus\{v\}} \mathbb{E}[X_u \mid w_v] \\ &= \Theta(n \Pr[\{u, v\} \in E \mid w_v]) \\ &= \Theta(n \int_1^\infty \underbrace{\Pr[u \in N(v) \mid w_u = w, w_v]}_{= \Pr[x_u \in [\frac{1}{2} - \lambda\frac{w\cdot w_v}{n}, \frac{1}{2} + \lambda\frac{w\cdot w_v}{n}]]} f_{w_u}(w) \mathrm{d}w)\end{aligned}$$

w.l.o.g $x_v = \frac{1}{2}$

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Case 2: $w > \frac{n}{2\lambda w_v} \Rightarrow \lambda\frac{w\cdot w_v}{n} > \frac{1}{2} \longrightarrow = 1$

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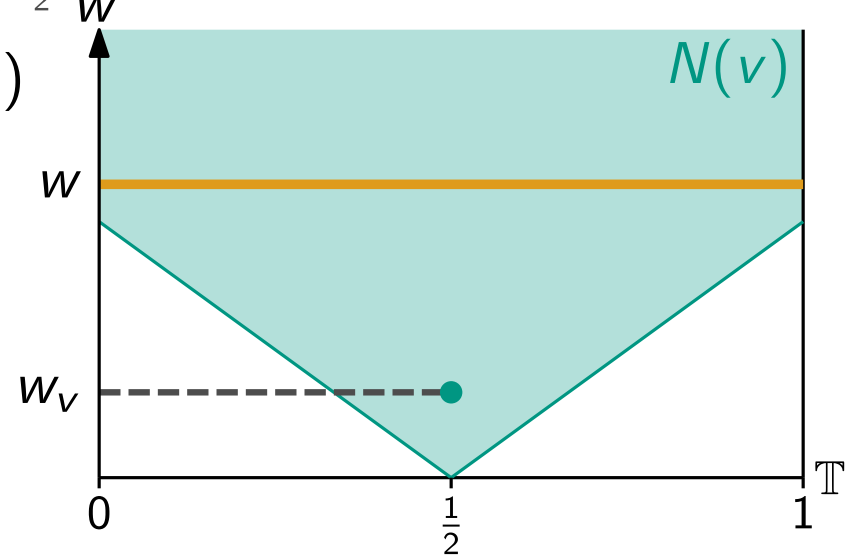

# Expected Degree ($d = 1$)

- Consider vertex $v$ with weight $w_v$
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$$= \Theta(n \int_1^\infty \Pr[u \in N(v) \mid w_u = w, w_v] f_{w_u}(w) \mathrm{d}w)$$

$$= \Theta\left(n\left(\cancel{\int_1^{\frac{n}{2\lambda w_v}} \frac{w \cdot w_v}{n} f_{w_u}(w)\mathrm{d}w} + \underbrace{\int_{\frac{n}{2\lambda w_v}}^\infty 1 \cdot f_{w_u}(w)\mathrm{d}w}\right)\right)$$

$$= \Pr[w_u \geq \tfrac{n}{2\lambda w_v}]$$

If $w_v \geq \frac{n}{2\lambda}$, then $\frac{n}{2\lambda w_v} \leq 1$

$$= \Pr[w_u \geq 1] = 1$$

$$= \Theta(n)$$

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If $w_v < \frac{n}{2\lambda}$

$$= \Theta\left(n\left(\int_1^{\frac{n}{2\lambda w_v}} \frac{w \cdot w_v}{n} f_{w_u}(w) \mathrm{d}w + \underbrace{\Pr[w_u \geq \frac{n}{2\lambda w_v}]}\right)\right)$$

(via CDF of Par) $= \left(\frac{n}{2\lambda w_v}\right)^{-(\tau-1)}$

$$= \left(\underbrace{\frac{2\lambda w_v}{n}}_{< 1}\right)^{\tau-1}$$

$$= O\left(\frac{w_v}{n}\right)$$

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$$\begin{aligned} &= \Theta\left(n\left(\int_1^{\frac{n}{2\lambda w_v}} \frac{w \cdot w_v}{n} f_{w_u}(w) \mathrm{d}w + \Pr\left[w_u \geq \frac{n}{2\lambda w_v}\right]\right)\right) \\ &= \Theta\left(n \int_1^{\frac{n}{2\lambda w_v}} \frac{w \cdot w_v}{n} f_{w_u}(w) \mathrm{d}w\right) + O(w_v) \\ &= \Theta\left(n \frac{w_v}{n} \int_1^{\frac{n}{2\lambda w_v}} w \cdot (\tau - 1) w^{-\tau} \mathrm{d}w\right) + O(w_v) \\ &= \Theta\left(w_v \int_1^{\frac{n}{2\lambda w_v}} w^{-(\tau-1)} \mathrm{d}w\right) + O(w_v) \end{aligned}$$

$$\begin{aligned} &= \Theta\left(w_v \left[\frac{1}{-(\tau-2)} w^{-(\tau-2)}\right]_1^{\frac{n}{2\lambda w_v}}\right) + O(w_v) \\ &= \Theta\left(w_v \left[w^{-(\tau-2)}\right]_{\frac{n}{2\lambda w_v}}^{1}\right) + O(w_v) \\ &= \Theta\left(w_v \left(1 - \underbrace{\left(\frac{n}{2\lambda w_v}\right)^{-(\tau-2)}}_{< 1 \text{ and } O(1)}\right)\right) + O(w_v) \\ &= \Theta(w_v) \end{aligned}$$

# Are GIRGs Realistic?

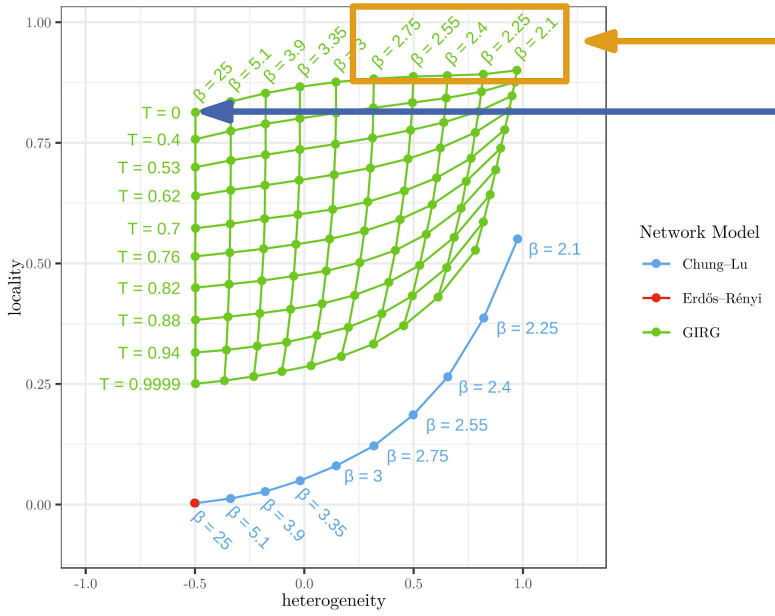
## Structural Properties

- Heterogeneity: $\deg(v) \approx w_v$, $w_v \sim \mathrm{Par}(\tau - 1, 1) \rightsquigarrow$ power-law degree distribution ✓
  (also works with other weight distributions)
- Locality (not seen here) ✓

## Algorithmic Properties

“On the External Validity of Average-Case Analyses of Graph Algorithms”, Bläsius, Fischbeck, ACM Trans. Algorithms 2023

- Setup: GIRGs with varying degrees of heterogeneity and locality (each dot is a graph)



What we considered just now

Basically random geometric graphs

GIRGs without geometry / ER with weights

# Are GIRGs Realistic?

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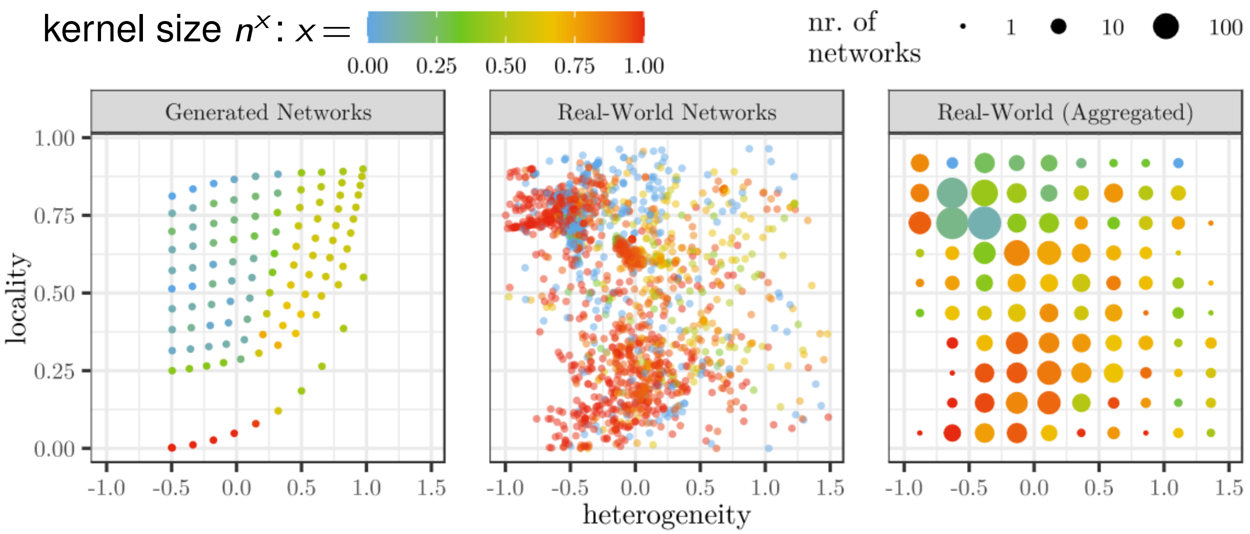
## Algorithmic Properties

“On the External Validity of Average-Case Analyses of Graph Algorithms”, Bläsius, Fischbeck, ACM Trans. Algorithms 2023

- Setup: GIRGs with varying degrees of heterogeneity and locality (each dot is a graph)
- Measure algorithmic properties on GIRGs and real graphs
  - Bidirectional breadth-first-search
  - Diameter computation via BFS
  - Vertex cover kernel size
  - Louvain clustering algorithm
  - Number of maximal cliques

    rather structural property
  - Chromatic number kernel size



*Use GIRGs for average-case analysis!*

# Vertex Cover Approximation

## Vertex Cover

- Given undirected graph $G = (V, E)$
- Find a smallest $S \subseteq V$ such that $\overbrace{G[V \setminus S]}^{\text{(induced subgraph)}}$ is edgeless
- NP-complete

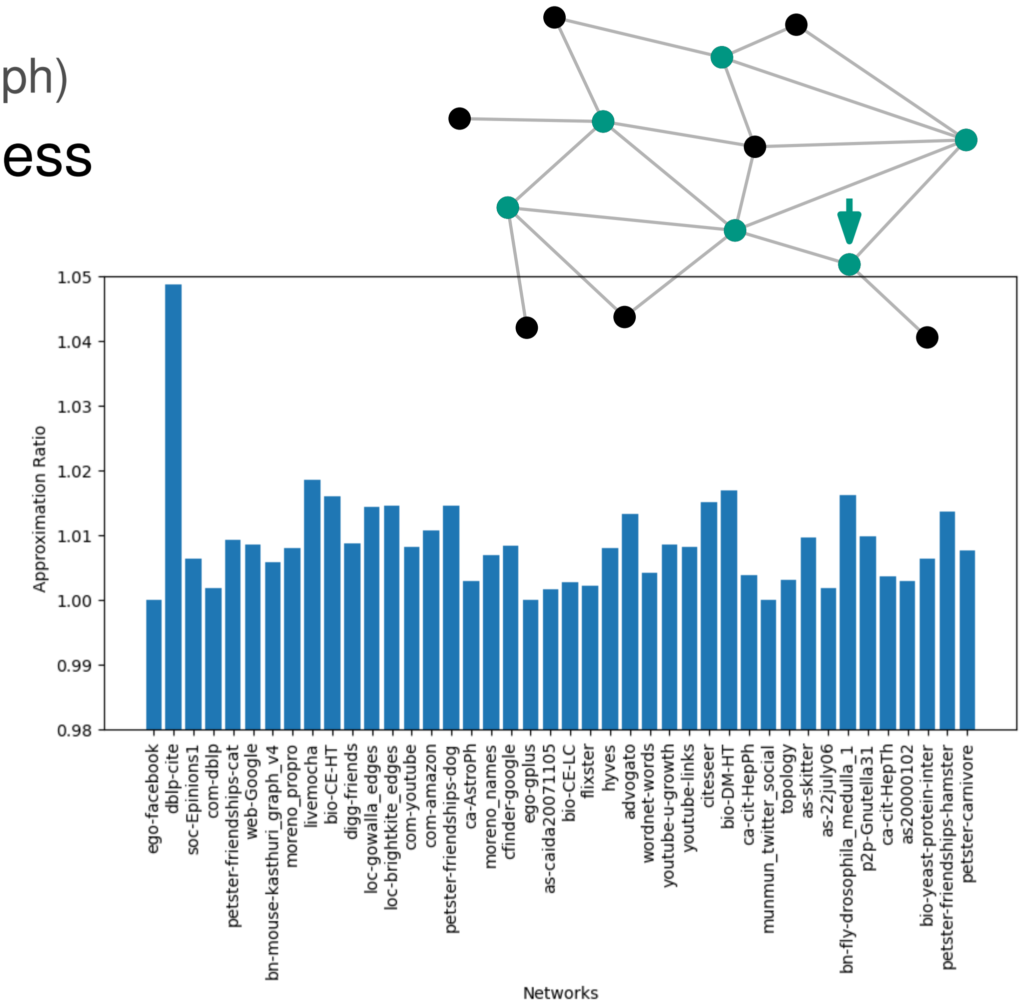
## Vertex Cover Approximation

- Find a *small* vertex cover $S'$ *fast*
- Approximation ratio: $r = |S'|/|S|$
- NP-hard to approximate with $r < \sqrt{2}$
- Believed to be NP-hard for $r < 2 - \varepsilon$ for const. $\varepsilon$

## Practice

- Simple approximation algorithm repeatedly takes/deletes vertex of largest degree
- Close to optimal ratios on real graphs

“Vertex Cover on Complex Networks”, Da Silva, Gimenez-Lugo, Da Silva, IJMPC 2013



# Analsysis on GIRGs

(based on) "Efficiently Approximating Vertex Cover on Scale-Free Networks with Underlying Hyperbolic Geometry", Bläsius, Friedrich, K., Algorithmica 2023

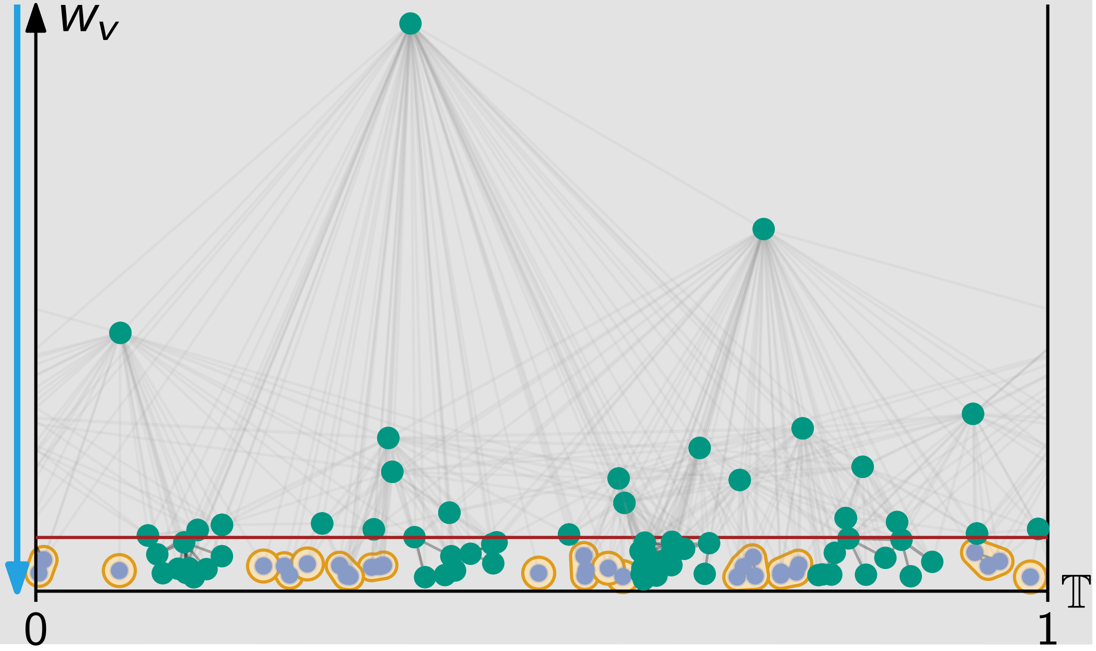
## Keep it simple

- Consider vertices in order of decreasing degree in original graph
- Consider vertices in order of decreasing *weight*

## Learn from the Model

- Once high-degree vertices are taken/removed, remaining vertices have roughly equal weight/degree
- Greedy algorithm picks vertices at random
- Improve quality by solving $\underbrace{\text{small}}_{\log\log(n)}$ separated components *exactly*
- Two variants
  - Search and solve small components after each greedily taken vertex
  - Take greedy until red line, solve small components exactly, take rest greedy too

This variant yields an upper bound on the quality of the other



# Analysis on GIRGs – Approximation Ratio

**Theorem**: Let $G$ be GIRG with $n$ vertices and $m$ edges. Then, an approximate vertex cover $S'$ of $G$ can be computed in time $O(m\log(n))$ such that the approximation ratio is $(1+o(1))$ asymptotically almost surely.

**Proof** *Approximation Ratio*

- Differentiate greedily taken vertices $S'_g$ from ones in exactly solved components $S'_e$
- For each small component, the optimal solution $S$ cannot contain fewer vertices than $S'_e$ does

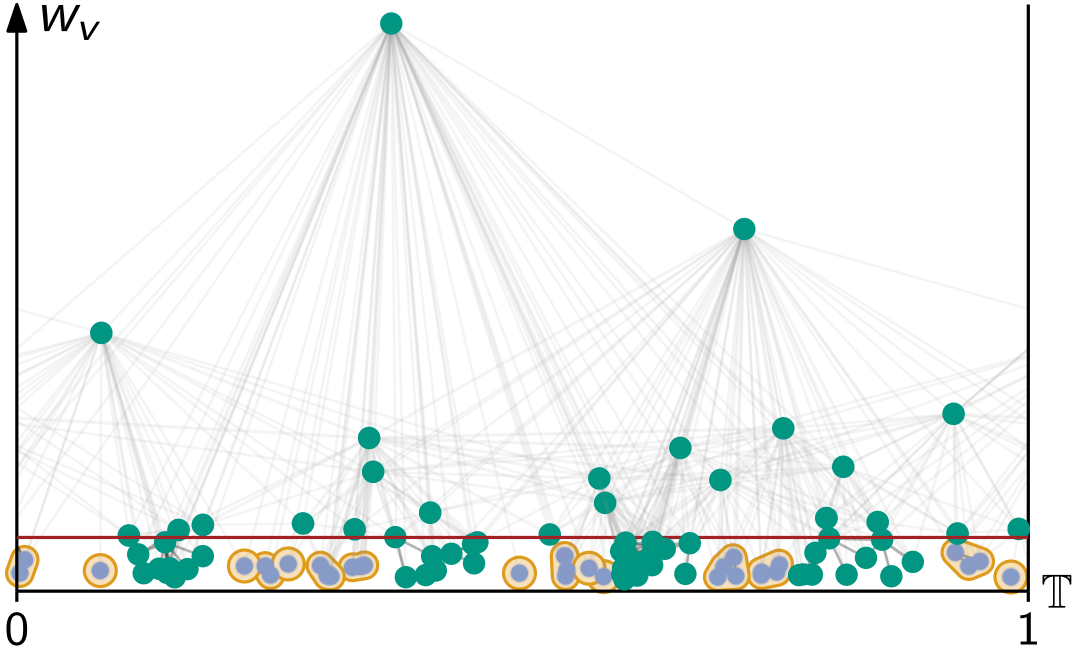
$$\Rightarrow |S'_e| \leq |S|$$

$$\Rightarrow r = \frac{|S'|}{|S|} = \frac{|S'_e|+|S'_g|}{|S|} \leq \frac{|S|+|S'_g|}{|S|} = 1 + \frac{|S'_g|}{|S|}$$

- $|S| = \Omega(n)$ with prob $1-o(1)$

"Greed is Good for Deterministic Scale-Free Networks", Chauhan et al. FSTTCS 2016

*Remains to show*: $|S'_g| = o(n)$



# Analysis on GIRGs – Greedy Vertices $\geq t$

**Lemma**: Let $G$ be a GIRG with $n$ vertices, let $t = \omega(1)$, and let $N_{w \geq t}$ be the number of vertices with weight at least $t$. Then, $N_{w \geq t} = o(n)$ with probability $1 - O(1/n)$.

**Proof**

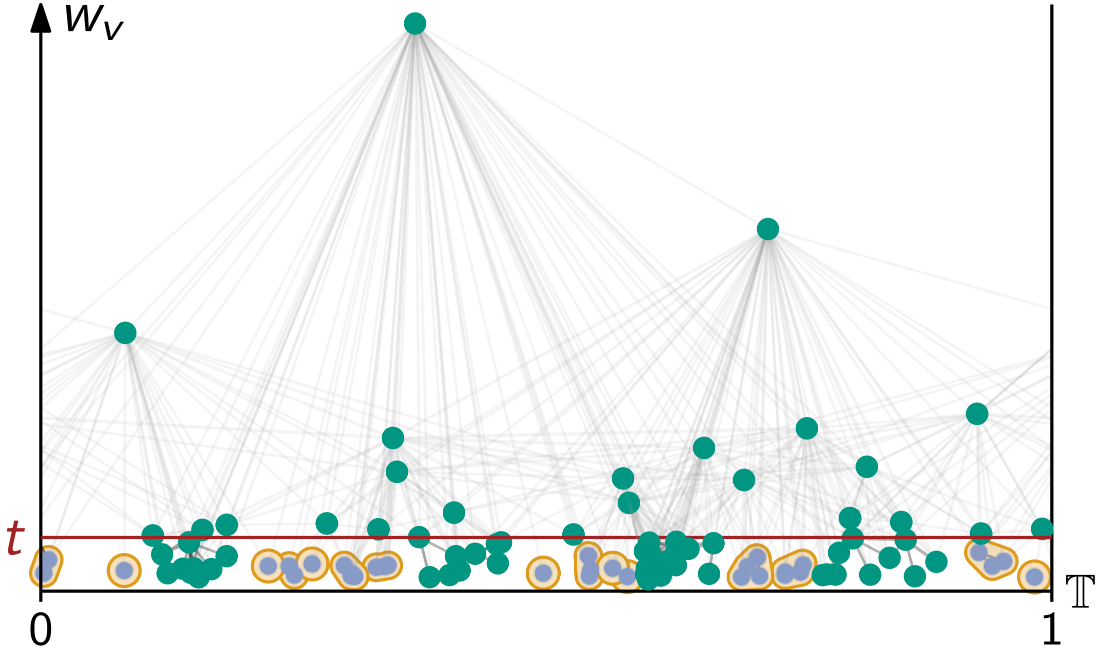
- Consider random variable $X_v = \mathbb{1}_{\{w_v \geq t\}}$
- $N_{w \geq t}$ is the sum of independent Bernoulli random variables $N_{w \geq t} = \sum_{v \in V} X_v$
- Expectation

$$\mathbb{E}[N_{w \geq t}] = \sum_{v \in V} \mathbb{E}[X_v] = n \Pr[w_v \geq t]$$
$$\text{(via CDF of Par)} = nt^{-(\tau - 1)}$$
$$(t = \omega(1), \tau \in (2,3)) = o(n)$$

- Since there is a $g(n) \in o(n) \cap \Omega(\log(n))$ with $g(n) \geq \mathbb{E}[N_{w \geq t}]$, Chernoff gives concentration ✓

**GIRG**

- $n$ independent vertices
- $w_v \sim \text{Par}(\tau - 1, 1)$ for $\tau \in (2,3)$



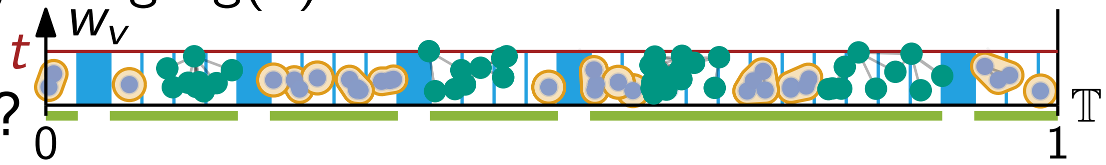
# Analysis on GIRGs – Greedy Vertices $< t$

- After (the $o(n)$) vertices with weight $\geq t$ are removed, the graph decomposes into several components
  - Components of size $\leq \log\log(n)$ are solved exactly
  - Larger components are assumed to be taken greedily (need to show: these are $o(n)$)
- Hard to determine how likely it is for a vertex to be in a large component
- Make use of geometry! Overestimate components by counting how many vertices are geometrically very close

**Idea**

- Discretize ground space into cells such that edges cannot span empty cells
- Use empty cells as delimiters between components
- Regard chains of non-empty cells as one component
- Count all vertices that are in chains containing $> \log\log(n)$ vertices

  (also potentially counting small components)
- When does a chain contain too many vertices?



# Analysis on GIRGs – Greedy Vertices $< t$

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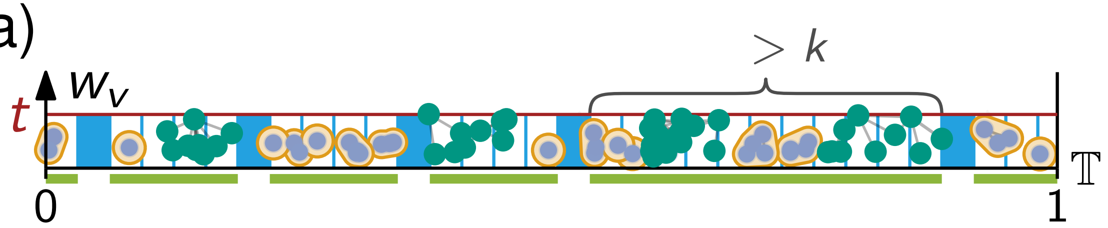
  - Components of size $\leq \log\log(n)$ are solved exactly
  - Larger components are assumed to be taken greedily (need to show: these are $o(n)$)
- Hard to determine how likely it is for a vertex to be in a large component
- Make use of geometry! Overestimate components by counting how many vertices are geometrically very close

**Case 1** Too many cells in long chains, say $> k$ cells

- Unlikely, if cells are small
- Proof via method of bounded differences!
  Total number of cells in long chains does not change much ($\leq 2k + 1$) when one cell moves from empty to non-empty (or vice versa)
- Use Poissonization to get rid of dependencies



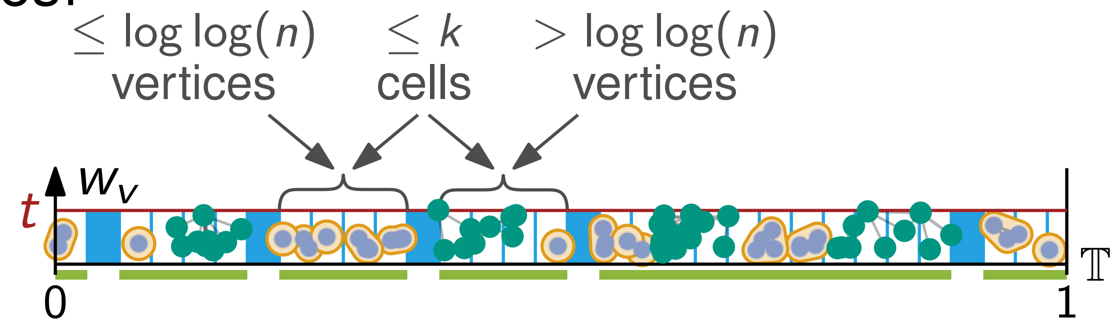
# Analysis on GIRGs – Greedy Vertices $< t$

- After (the $o(n)$) vertices with weight $\geq t$ are removed, the graph decomposes into several components
  - Components of size $\leq \log\log(n)$ are solved exactly
  - Larger components are assumed to be taken greedily (need to show: these are $o(n)$)
- Hard to determine how likely it is for a vertex to be in a large component
- Make use of geometry! Overestimate components by counting how many vertices are geometrically very close

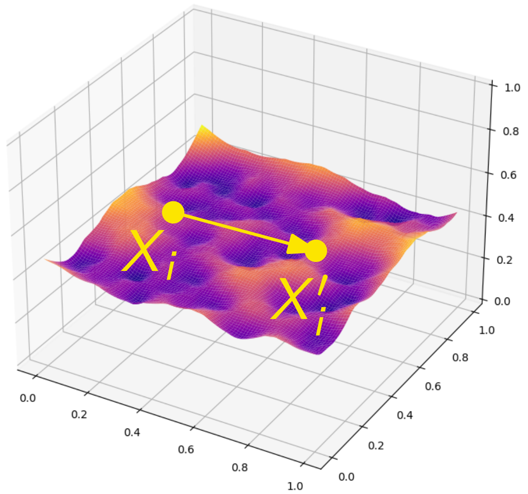
**Case 2** Short chains ($\leq k$ cells) contain too many vertices

- Unlikely, if cells are small
- Proof via method of *typical* bounded differences!
  - Imagine cells as boxes on conveyor belt
  - Imagine vertices as products
  - Typically not many vertices in few cells

$\rightsquigarrow$ w.h.p., $o(n)$ vertices in large components ✓



# Conclusion



**Method of Bounded Differences**

- Concentration for function of independent random variables
- Bounded differences (“Lipschitz”) condition
  - What is the worst that can happen when changing one input?
- Chernoff-like bound, weakened by sum of squared worst changes
- Useless if worst changes are too large

**Method of Typical Bounded Differences**

- Define typical event, distinguish worst changes depending on whether event occurred
- Use mitigators to weaken impact of general worst changes
- Pay with probability that typical event does not occur, multiplied with inverse mitigators

**Geometric Inhomogeneous Random Graphs**

- Pretty realistic graph model (heterogeneity, locality)
- Not too hard to analyze
- Used for average-case analysis (e.g. vertex cover approximation)

(not discussed in lecture)