BLATT 6

1) PDF von Pareto: $f_X(x) = \begin{cases} \alpha\, x_{min}^{\alpha} \cdot x^{-(\alpha+1)} & x \geq x_{min} \\ 0 & \text{sonst} \end{cases}$

Sei $X \sim Par(\alpha, x_{min})$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_{x_{min}}^{\infty} x \alpha\, x_{min}^{\alpha}\, x^{-(\alpha+1)}\, dx$$

$$= \alpha\, x_{min}^{\alpha} \int_{x_{min}}^{\infty} x^{-\alpha}\, dx$$

Fall 1: $\alpha = 1$

$$\mathbb{E}(X) = \alpha\, x_{min}^{\alpha} \Big[\log(x)\Big]_{x_{min}}^{\infty}$$

$\Rightarrow$ nicht endlich

Fall 2: sonst

$$\mathbb{E}(X) = \alpha\, x_{min}^{\alpha} \left[\frac{1}{-\alpha+1}\, x^{-\alpha+1}\right]_{x_{min}}^{\infty}$$

Endlich, falls Exponent von $x \leq 0$,

dann geht es gegen 0

$\Rightarrow \quad -\alpha+1 \leq 0 \iff \alpha \geq 1$

$\Rightarrow \alpha > 1$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{x_{min}}^{\infty} x^2 \alpha\, x_{min}^{\alpha}\, x^{-(\alpha+1)}\,dx$$

$$= \alpha x_{min}^{\alpha} \int_{x_{min}}^{\infty} x^{(-\alpha)+1}\,dx$$

Fall 1: $\alpha = 2$

$$\mathbb{E}(X^2) = \alpha\, x_{min}^{\alpha} \left[\log x\right]_{x_{min}}^{\infty} \rightarrow \infty$$

Fall 2: Sonst

$$\mathbb{E}(X^2) = \alpha\, x_{min}^{\alpha} \left[\frac{1}{-\alpha+2}\, x^{-\alpha+2}\right]_{x_{min}}^{\infty}$$

unendlich falls Exponent von $x \geq 0$

$\Rightarrow \alpha < 2$

$\Rightarrow$ unendlich falls $\alpha \leq 2$

$\Rightarrow \alpha \in (1, 2]$

2a)

$$\mathbb{P}(X=n+k \mid X>n) = \frac{\mathbb{P}(X=n+k \wedge X>n)}{\mathbb{P}(X>n)}$$

$$\mathbb{P}(X=n+k \wedge X>n) = \mathbb{P}(X=n+k) = p(1-p)^{n+k-1}$$

$$\mathbb{P}(X>n) = 1-\mathbb{P}(X\le n) = 1-\sum_{i=1}^{n} p(1-p)^{i-1}$$

$$= 1-p\cdot\sum_{i=0}^{n-1}(1-p)^{i}$$ ↖ geom. Reihe

$$= 1-p\cdot\frac{1-(1-p)^n}{1-(1-p)} = 1-(1-(1-p)^n)$$

$$=(1-p)^n$$

$$\Rightarrow \mathbb{P}(X=n+k \mid X>n) = \frac{p(1-p)^{n+k-1}}{(1-p)^n} = p(1-p)^{k-1} = \mathbb{P}(X=k)$$

2b) $\mathbb{P}(Z > t) = 1 - \mathbb{P}(Z \leq t) = 1 - F_{Exp(\lambda)}(t)$

$= 1 - (1 - e^{-t\lambda}) = e^{-t\lambda}$

$\mathbb{P}(Y_n > t) = 1 - \mathbb{P}(Y_n \leq t) = 1 - \mathbb{P}(X_n \leq tn)$

$= 1 - \sum_{i=1}^{tn} p \cdot (1-p)^{i-1}$

$= 1 - p \sum_{i=0}^{tn-1} (1-p)^i = 1 - p \sum_{i=0}^{tn} (1-p)^i + p(1-p)^{tn-1}$

$\underset{\substack{\uparrow \\ \text{geom.} \\ \text{Reihe}}}{=} \left(1 - \frac{\lambda}{n}\right)^{tn} + p(1-p)^{tn-1}$

$\lim_{n \to \infty} \mathbb{P}(Y_n > t) = \lim_{n \to \infty} \left(\left(1 - \frac{\lambda}{n}\right)^{tn} + p \cdot (\underbrace{1-p}_{<1})^{tn-1} \right)$

für $p > 0$. (für $p = 0$ sind sie trivial gleich)

$= e^{-t\lambda} = \mathbb{P}(Z > t)$

$\lim_{n \to \infty} \left(1 + \frac{\alpha}{n}\right)^n = e^{\alpha}$