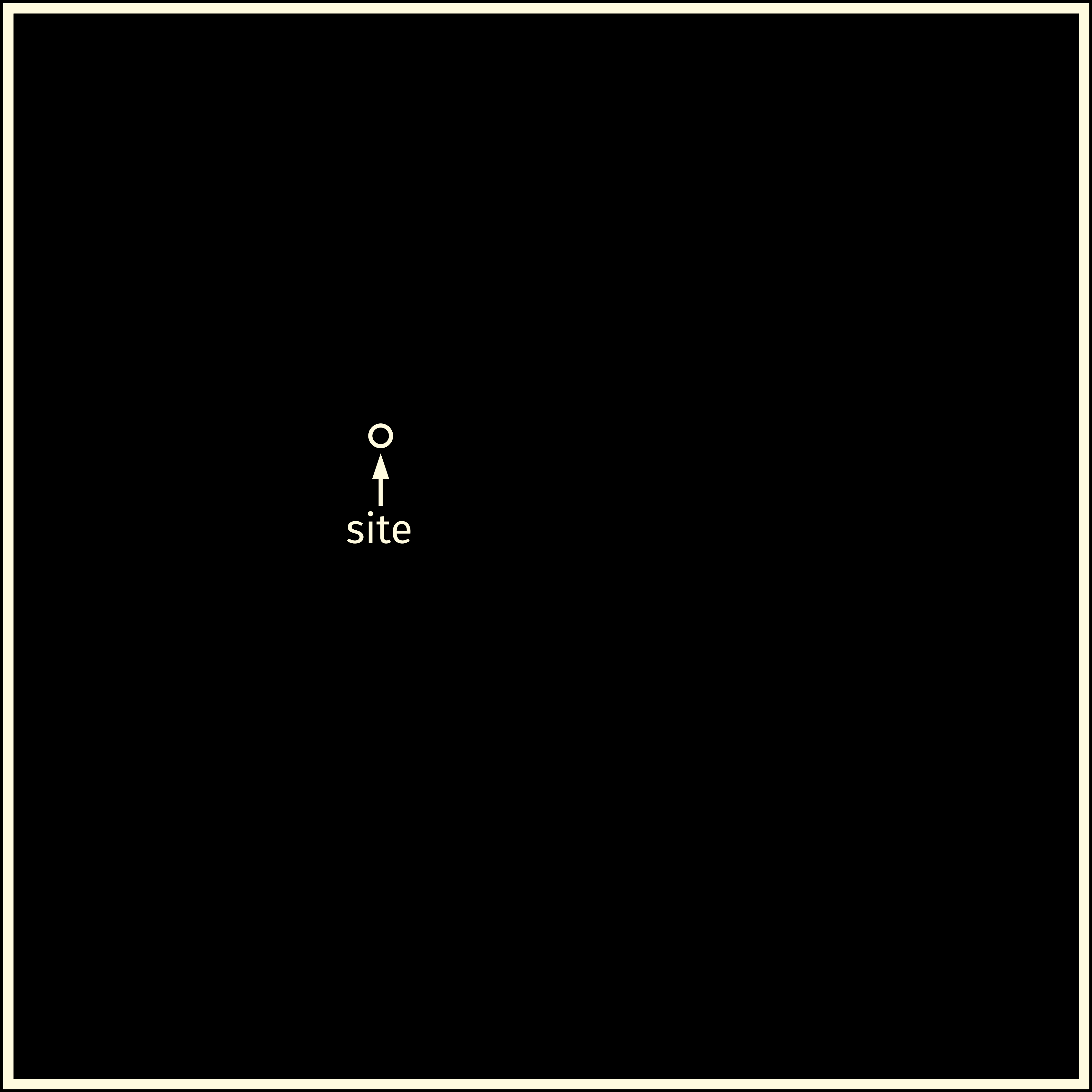
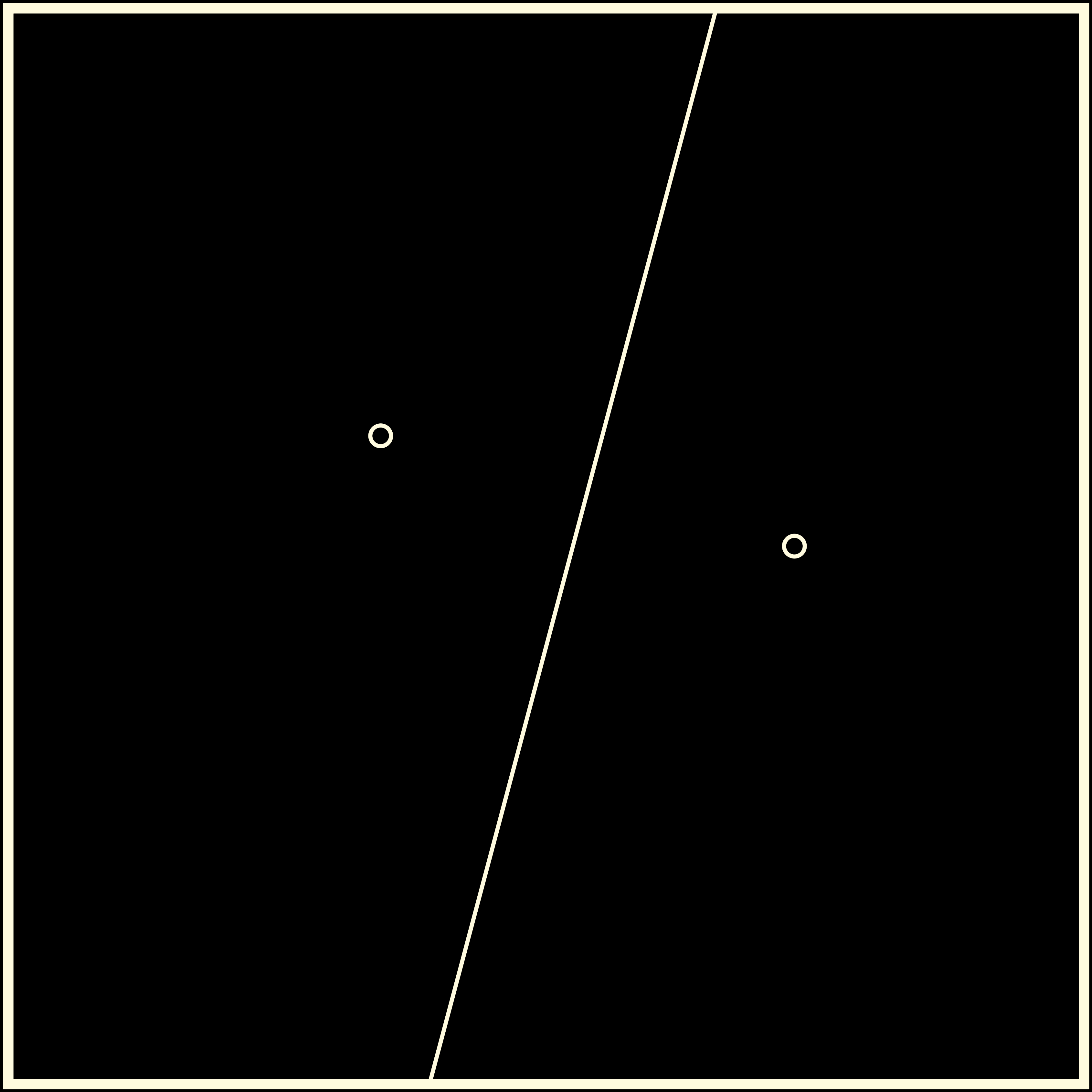
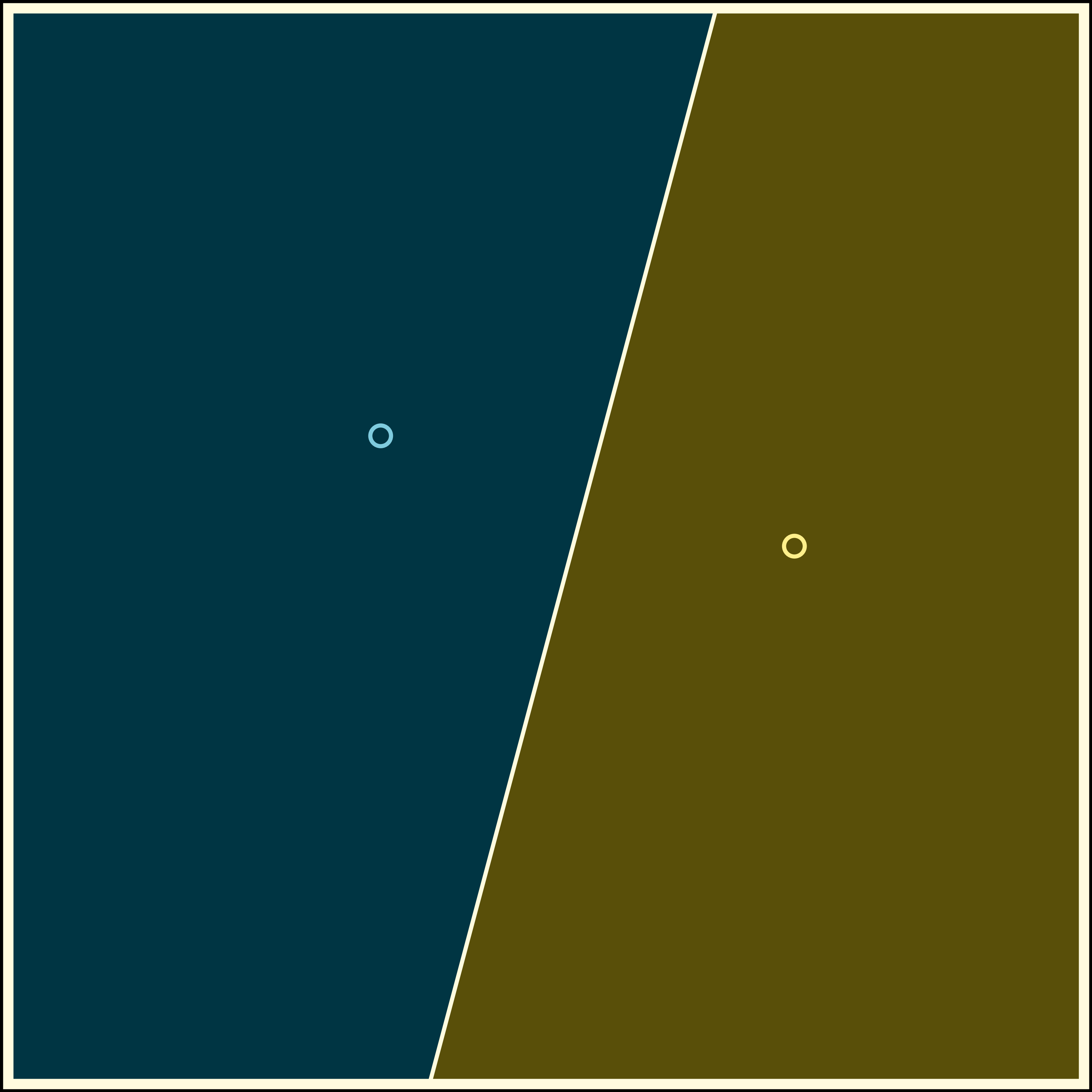
The Impact of Heterogeneity and Geometry on the Proof Complexity of Random Satisfiability

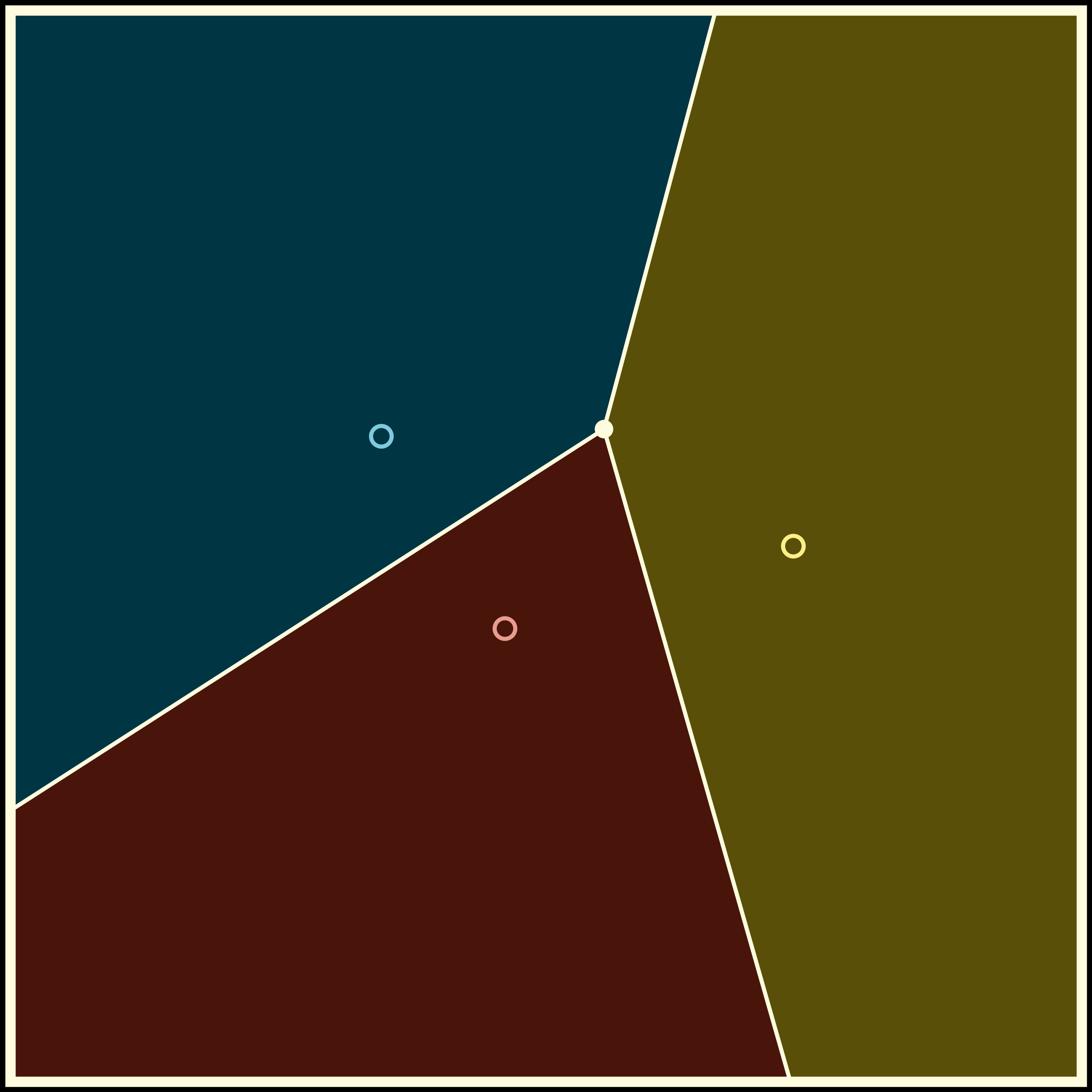
Thomas Bläsius, Tobias Friedrich, Andreas Göbel,
Jordi Levy, Ralf Rothenberger

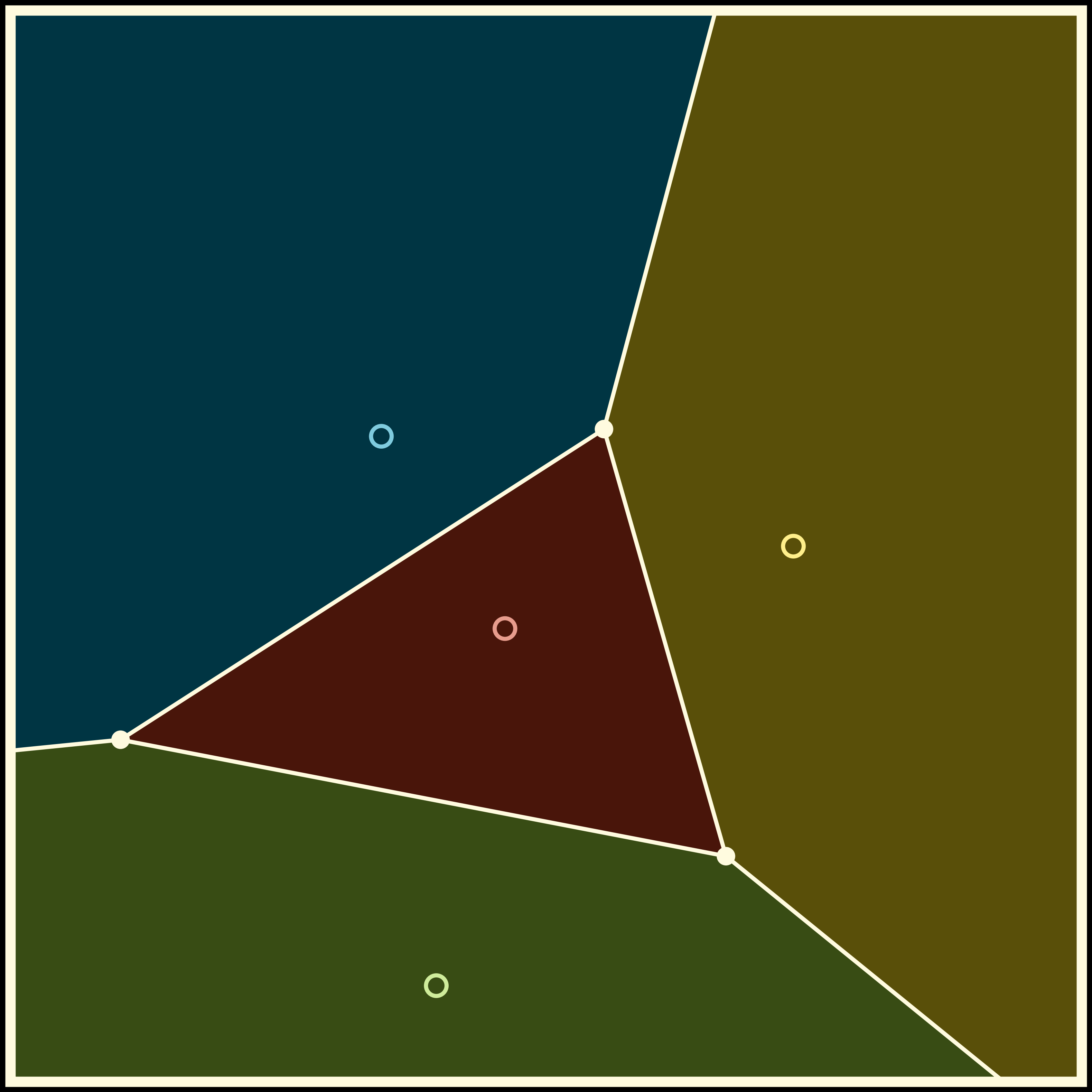
…, and the Complexity of Voronoi Diagrams

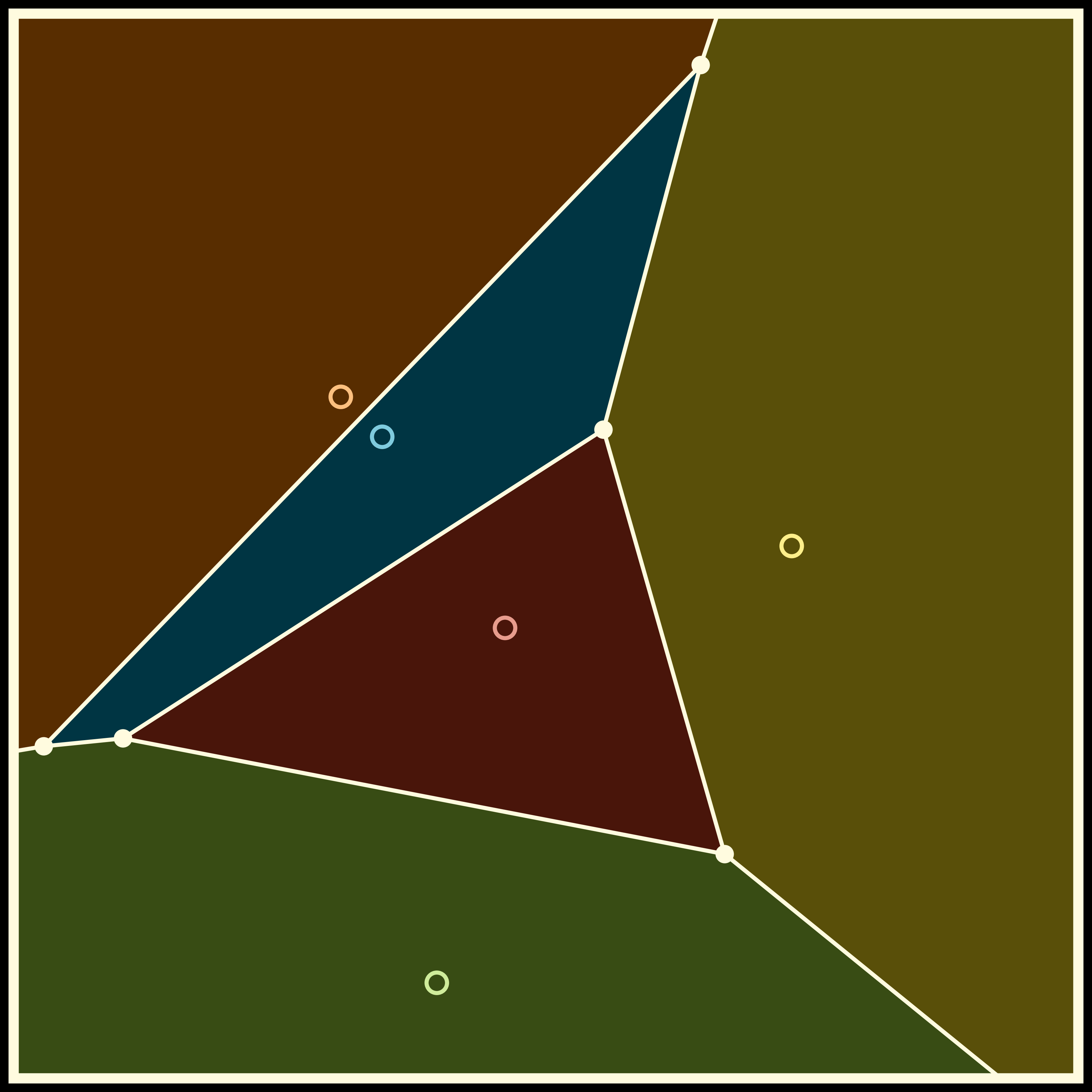

site

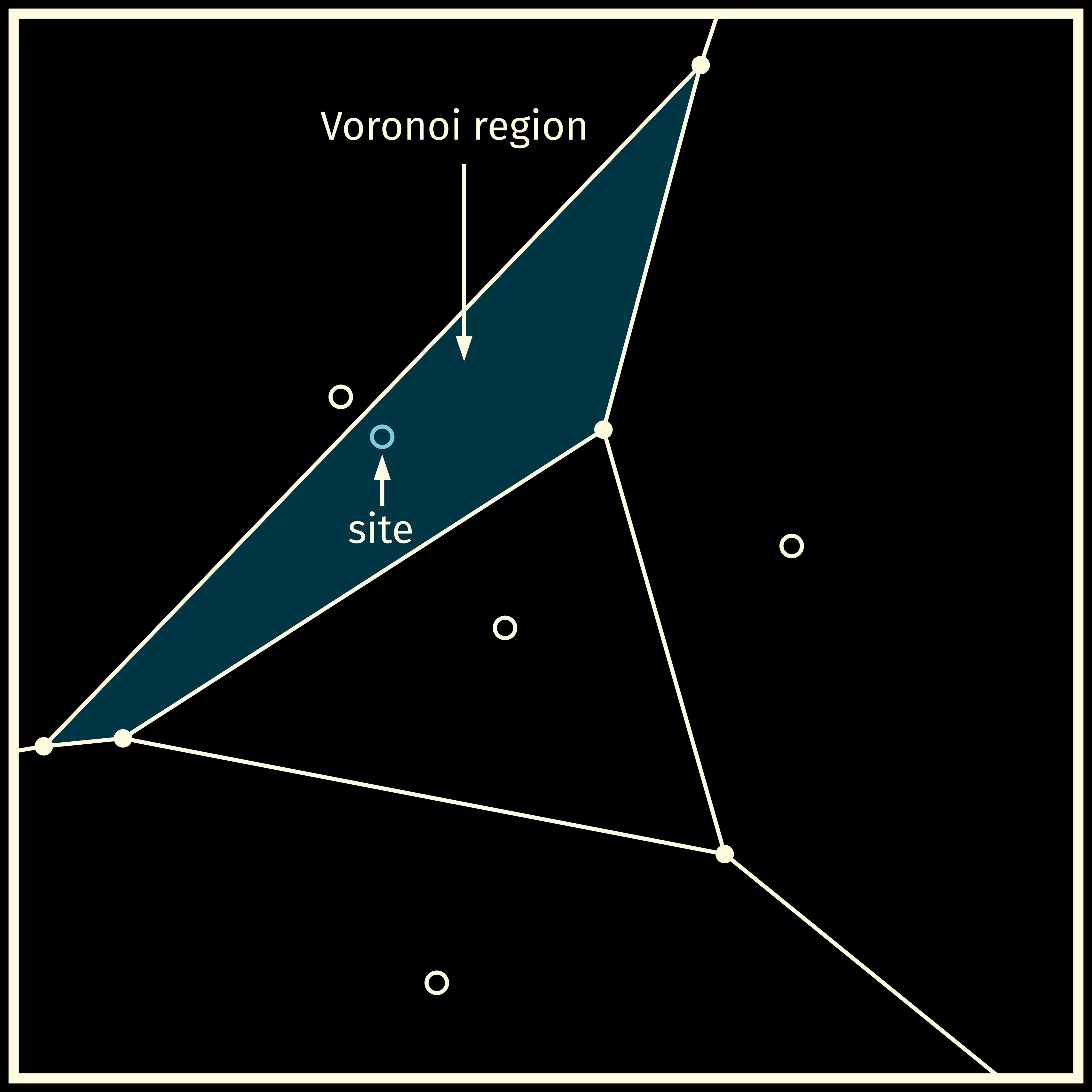


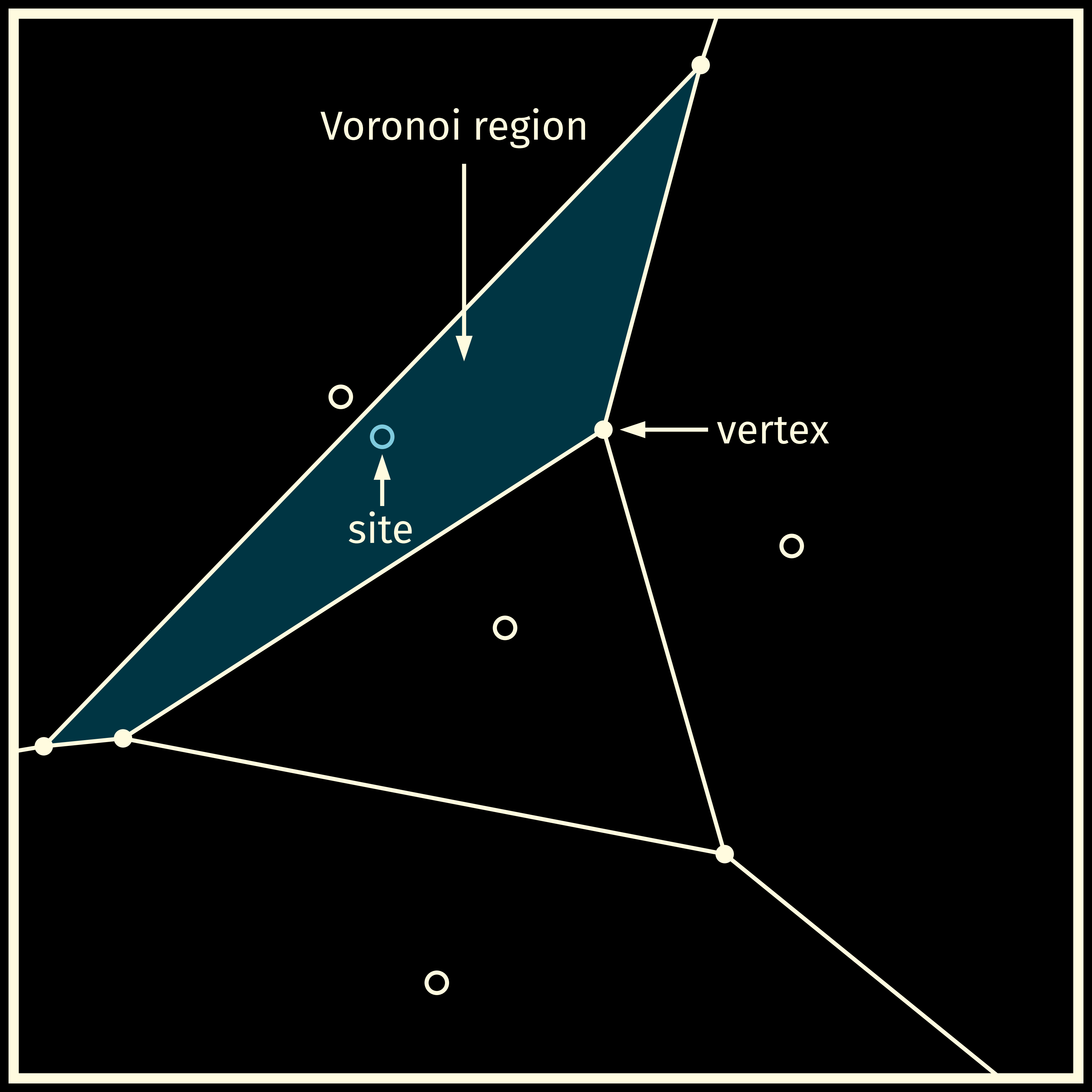


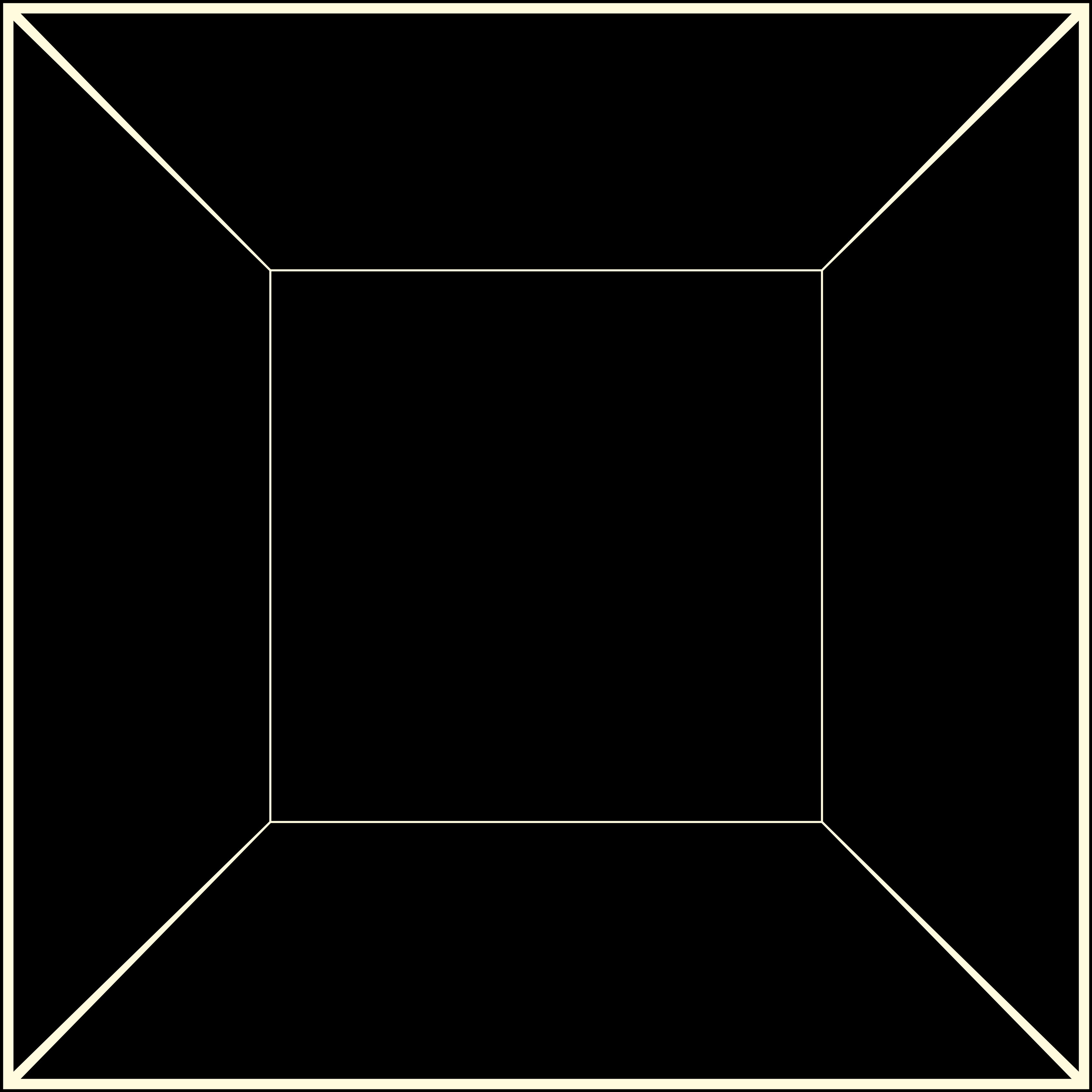


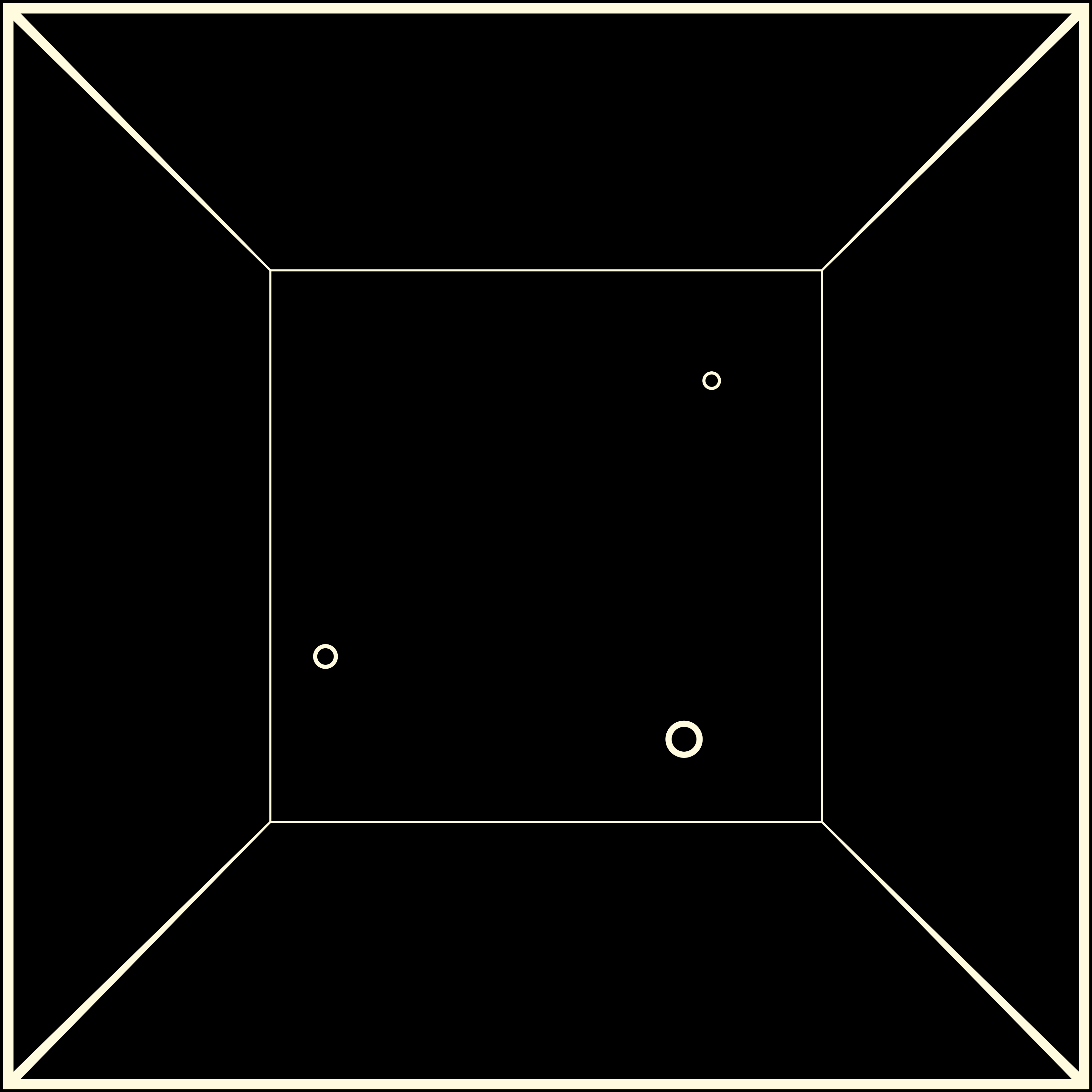


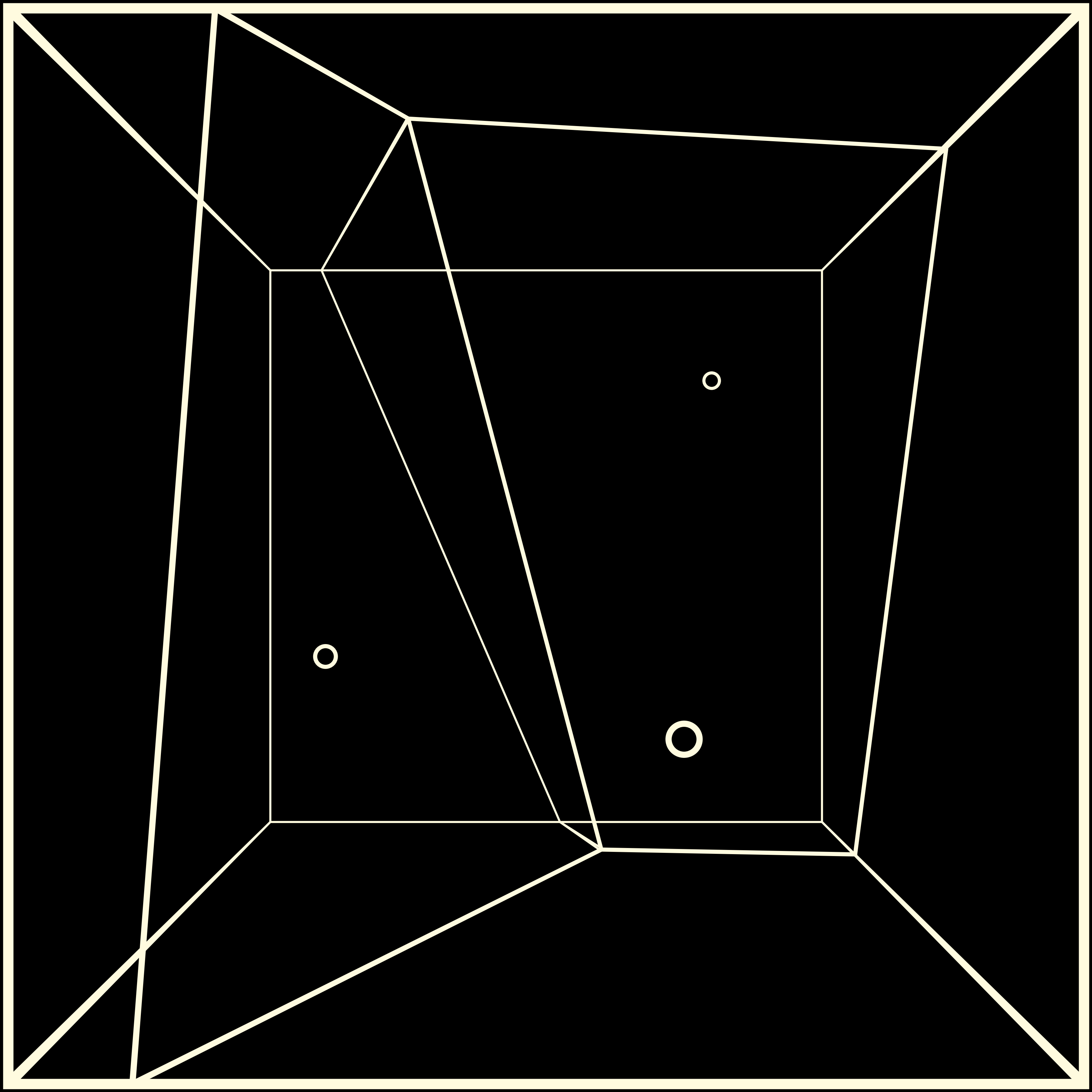


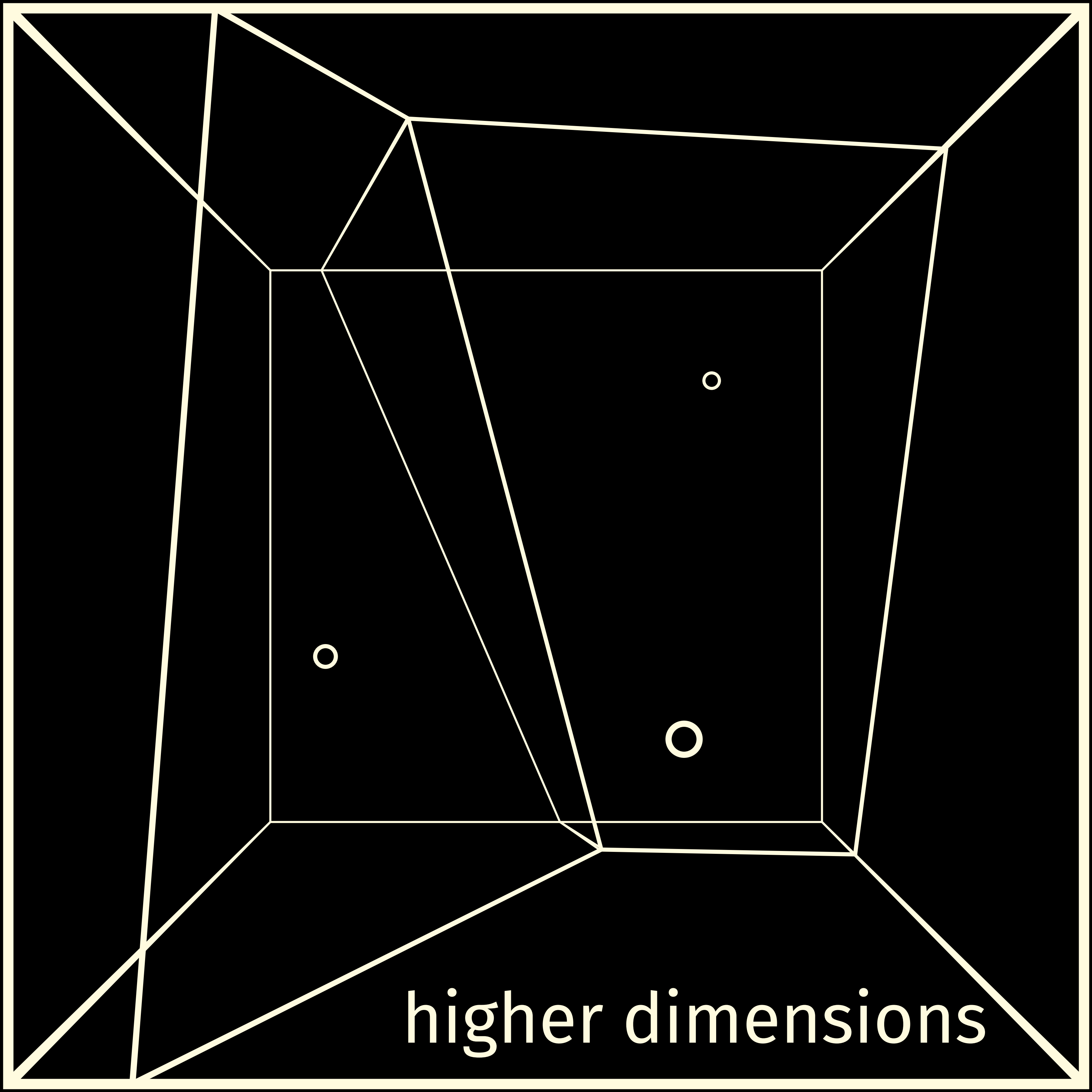

Voronoi region
site

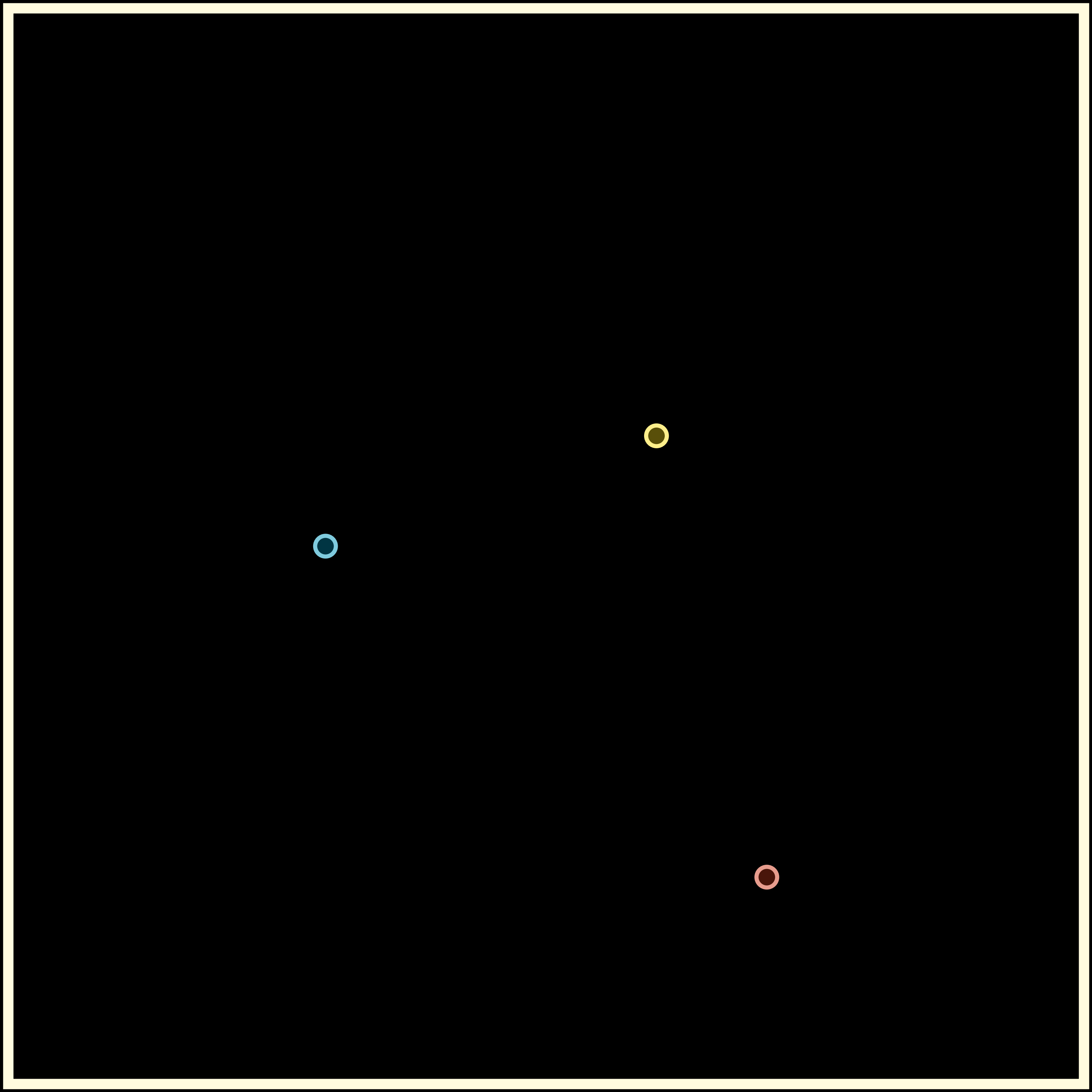

Voronoi region
vertex
site

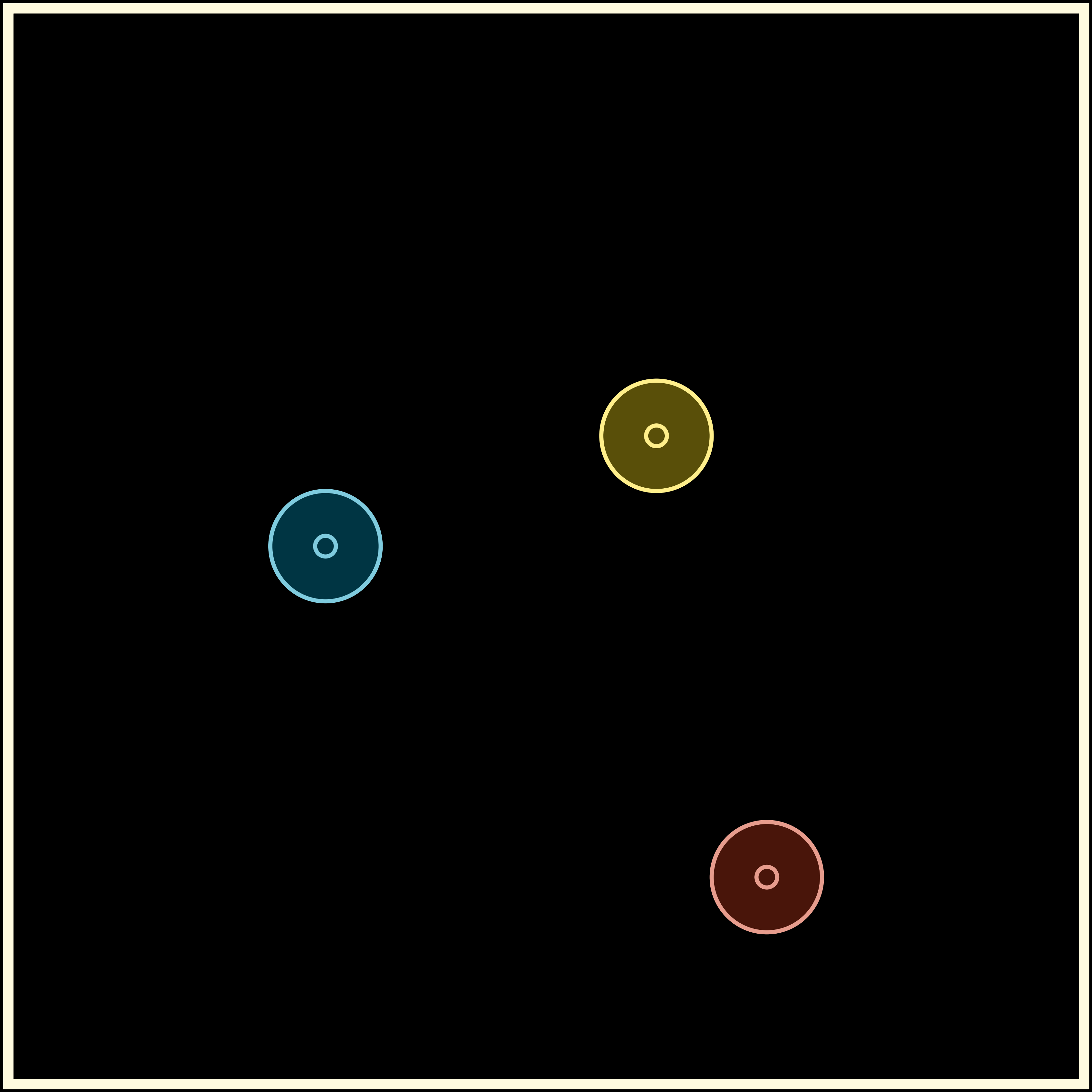


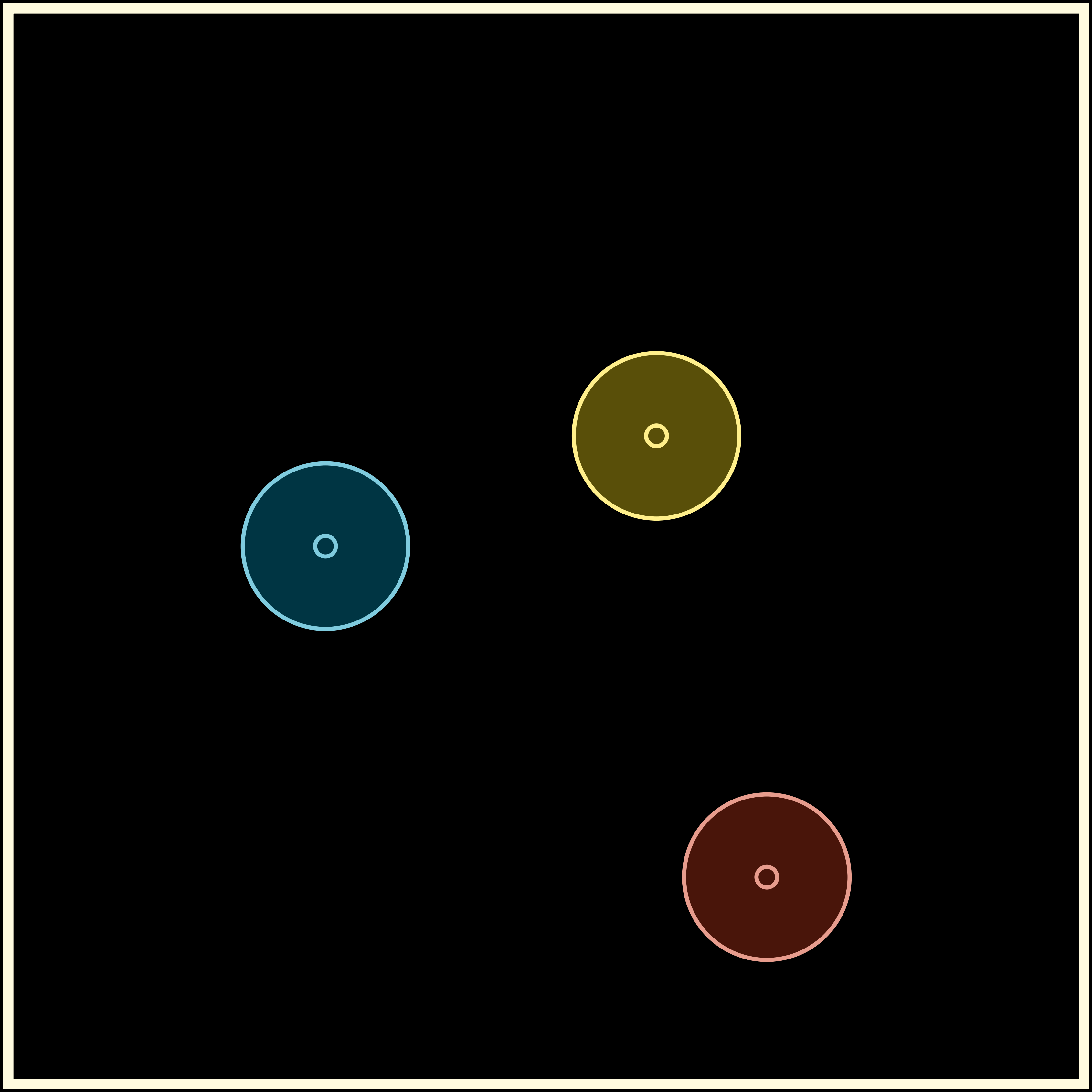


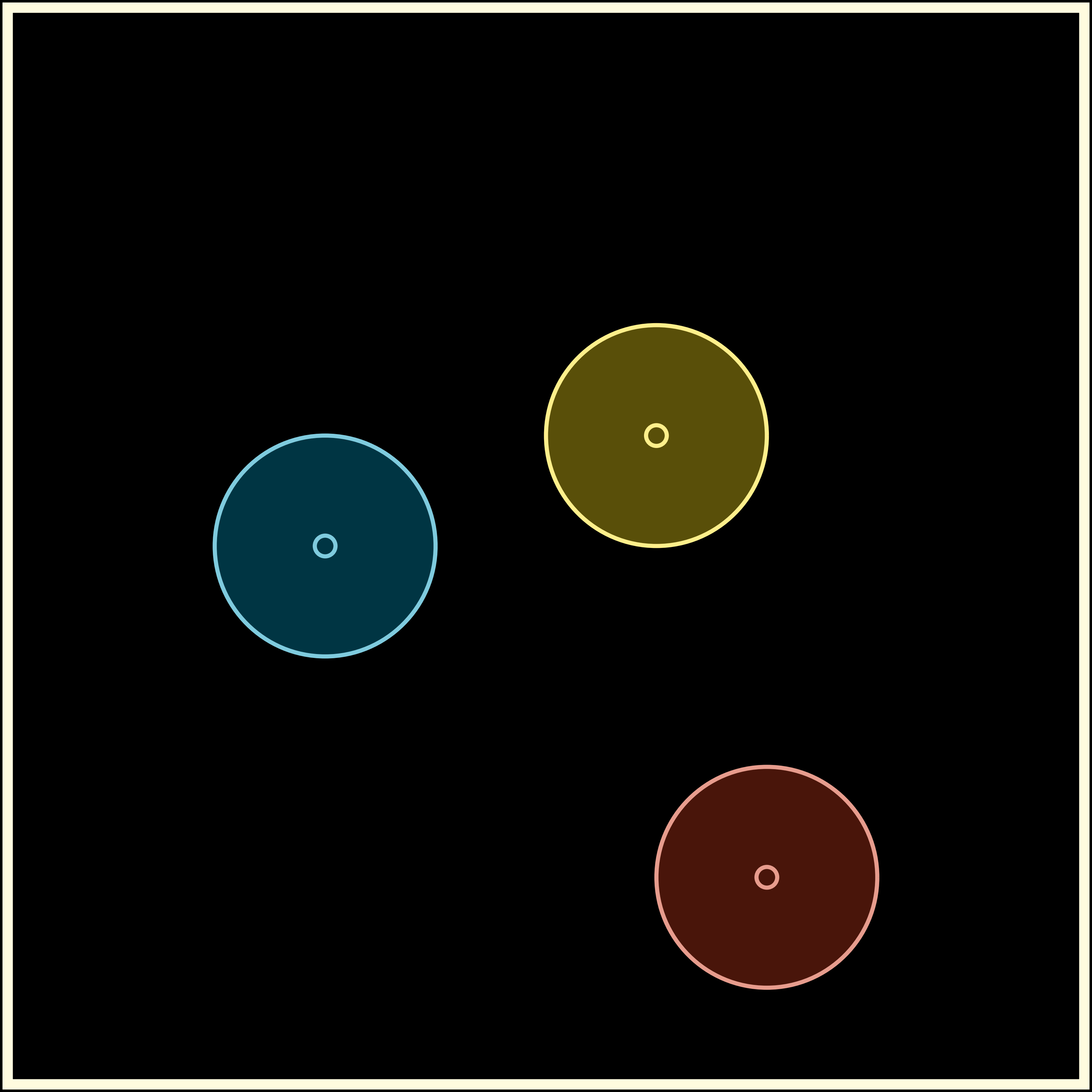


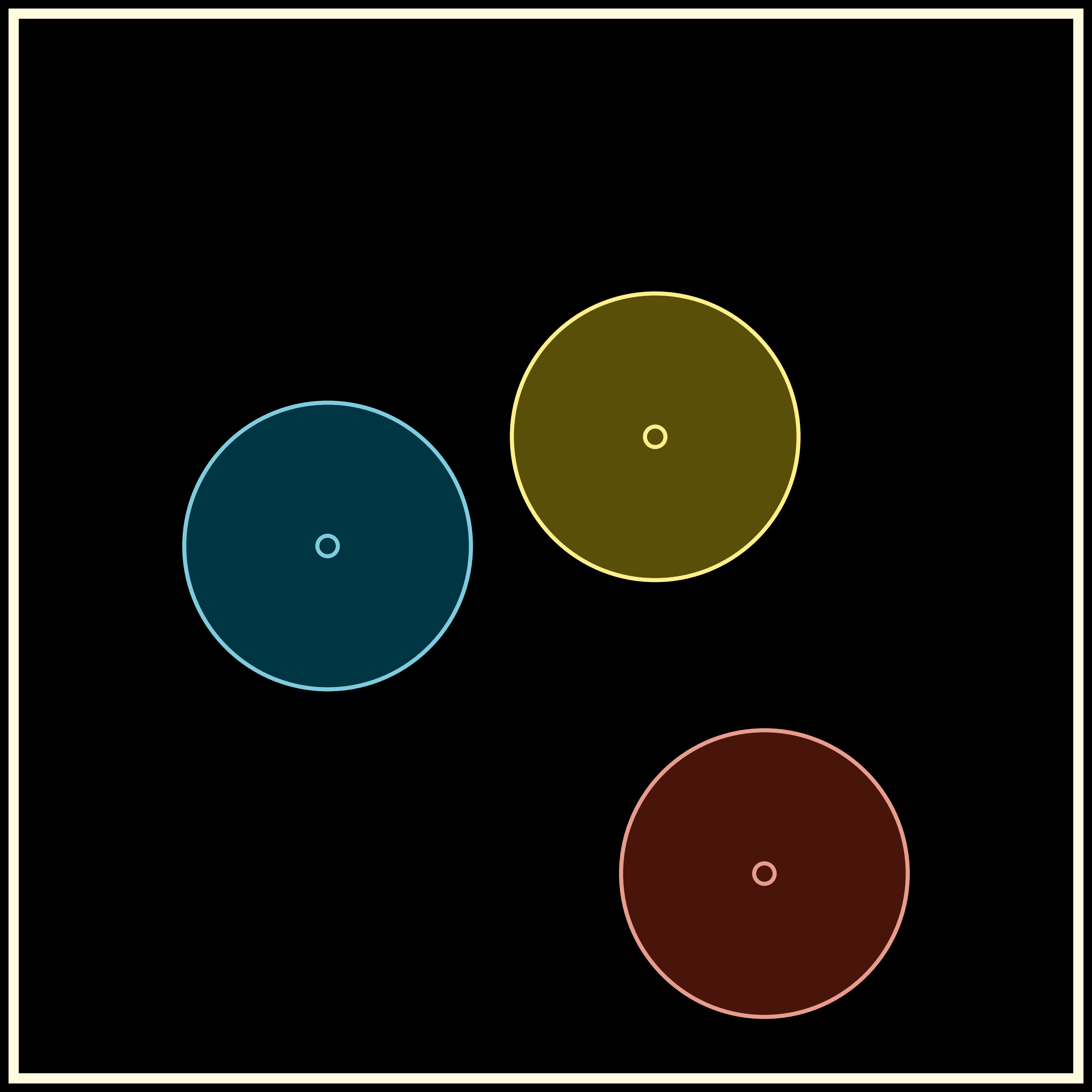

higher dimensions

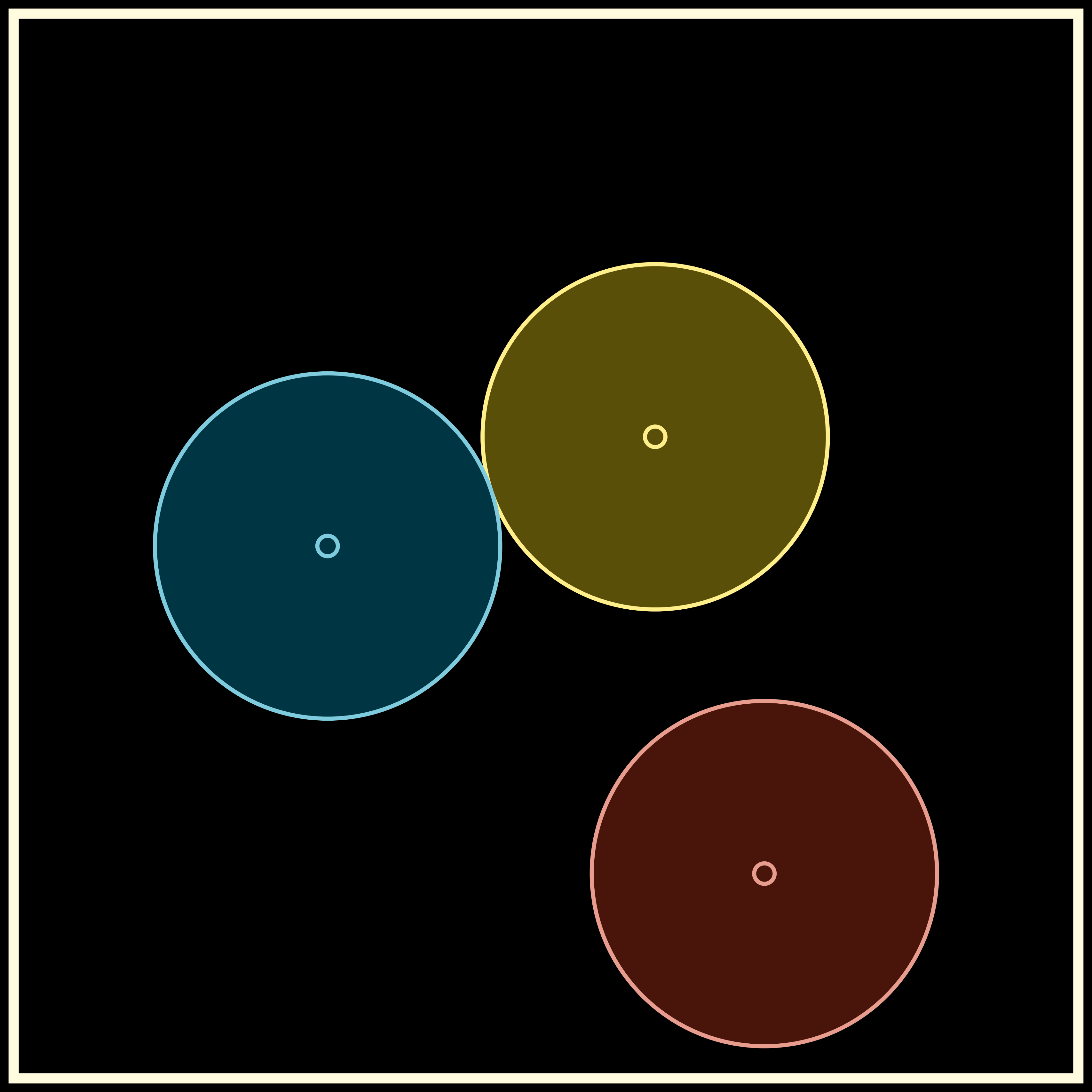


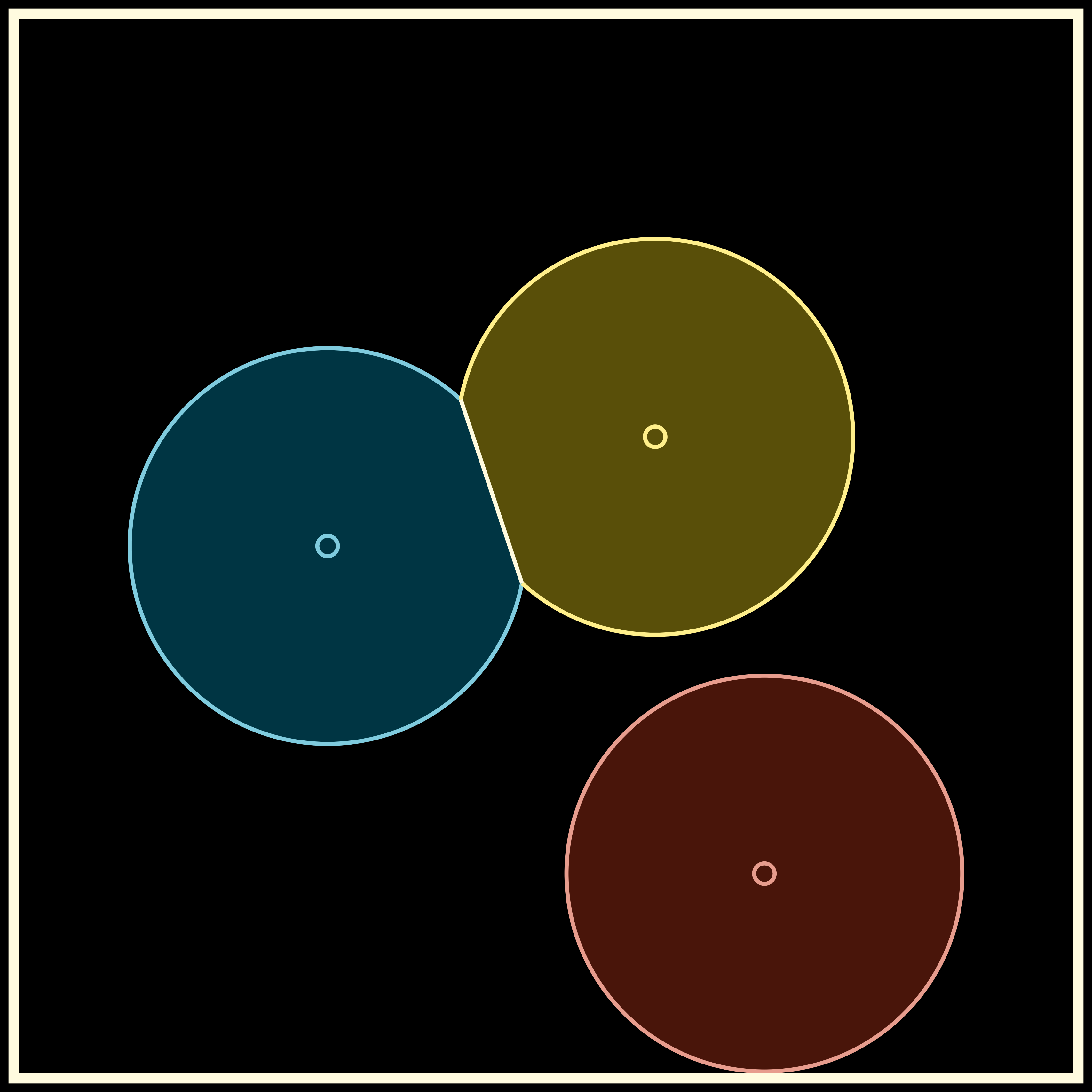


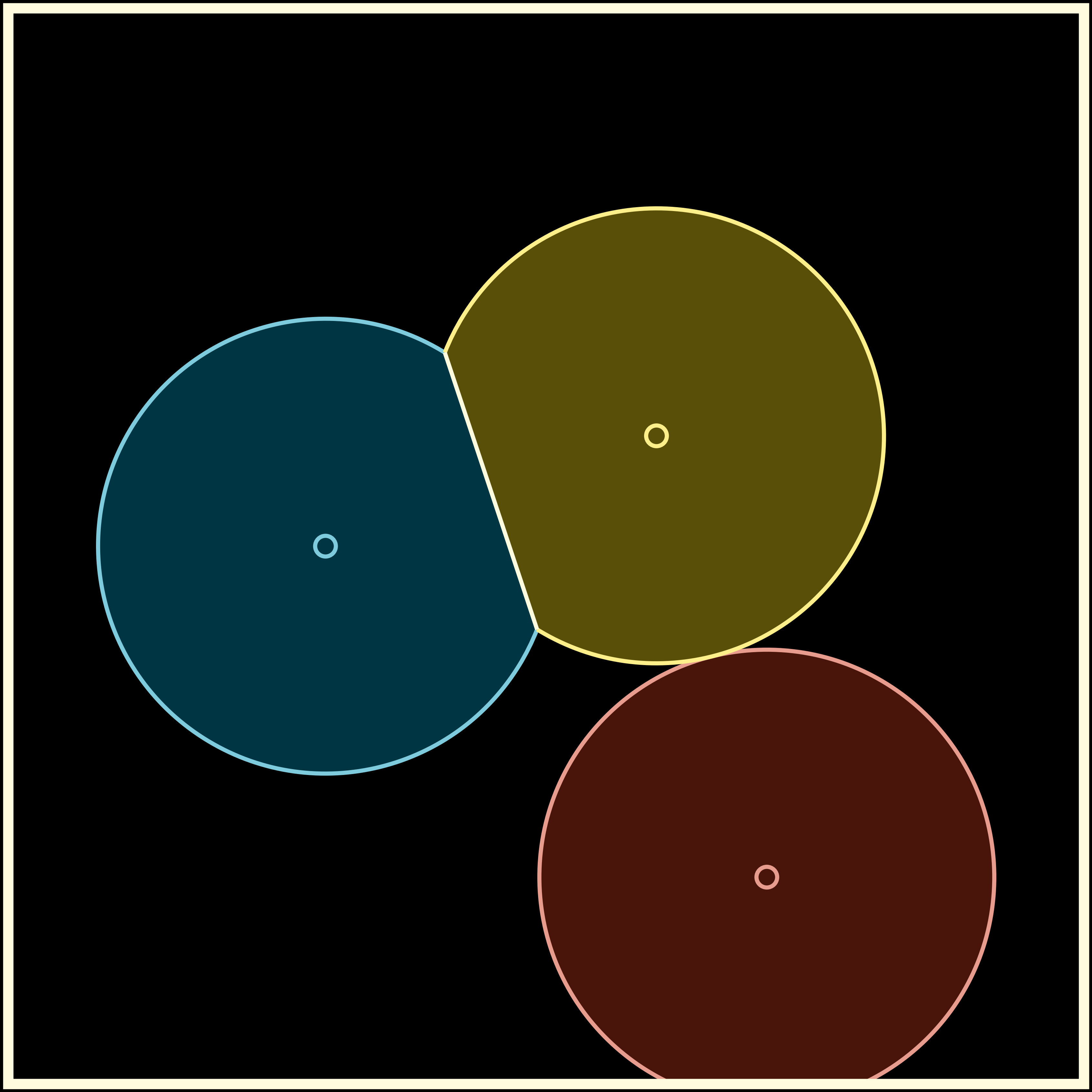


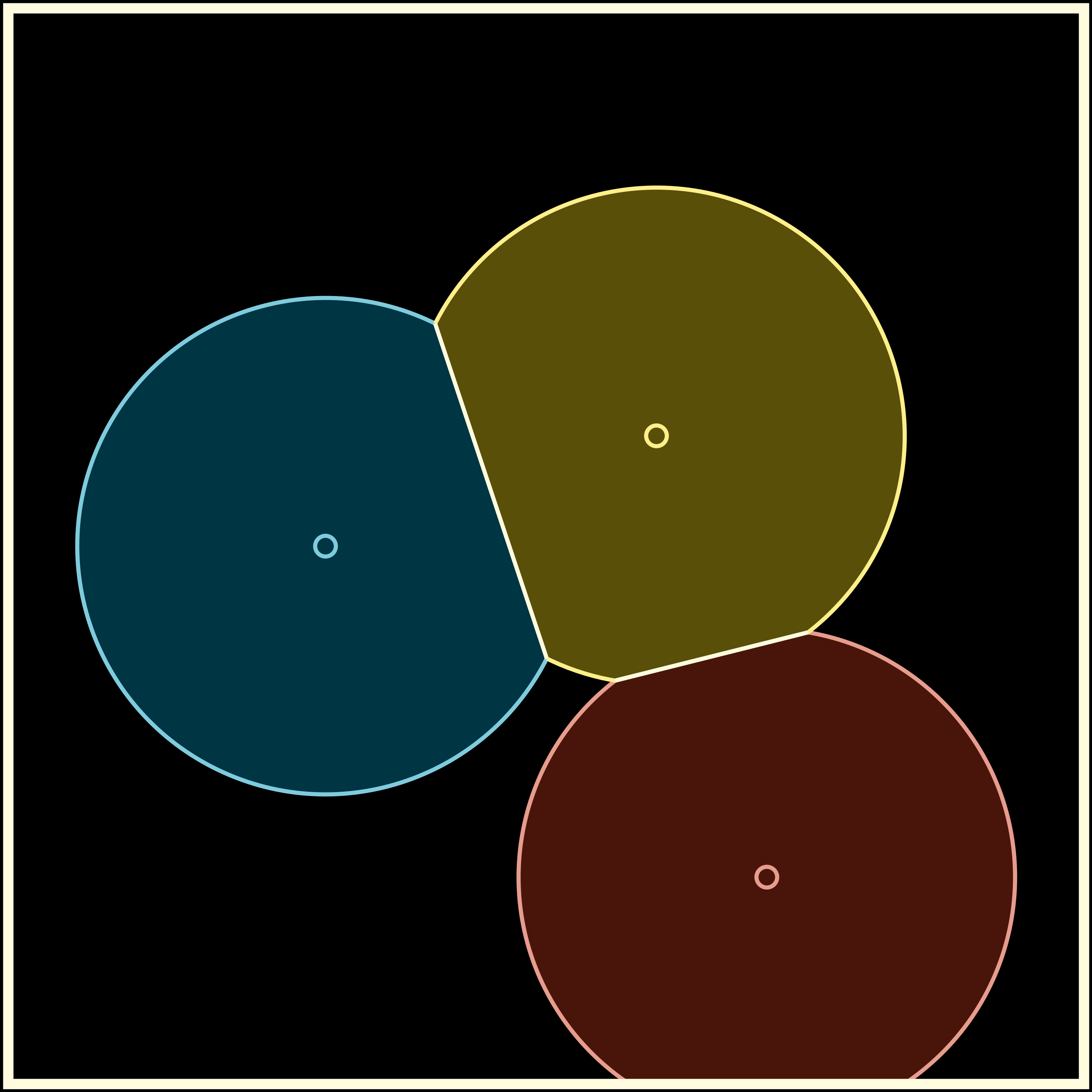


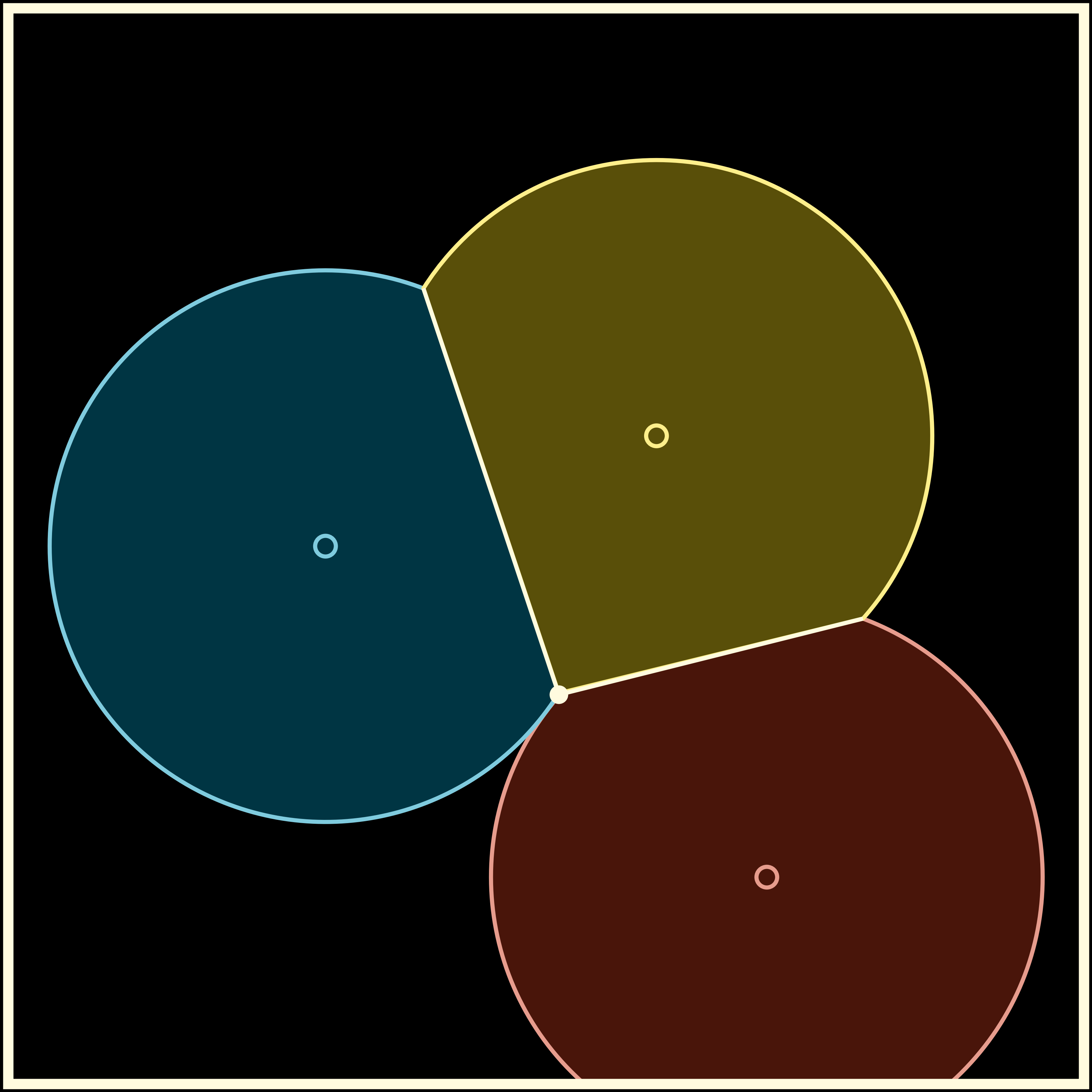


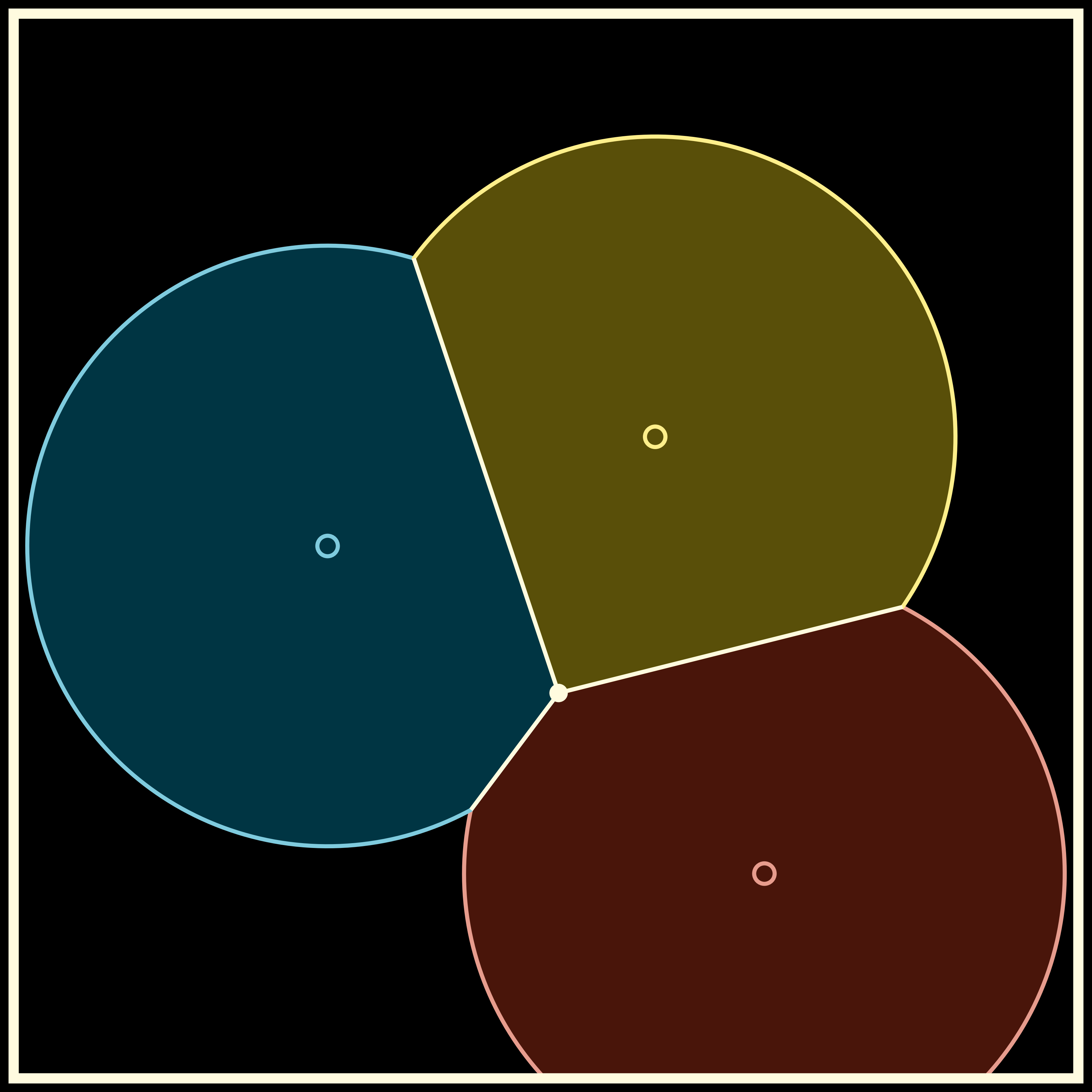


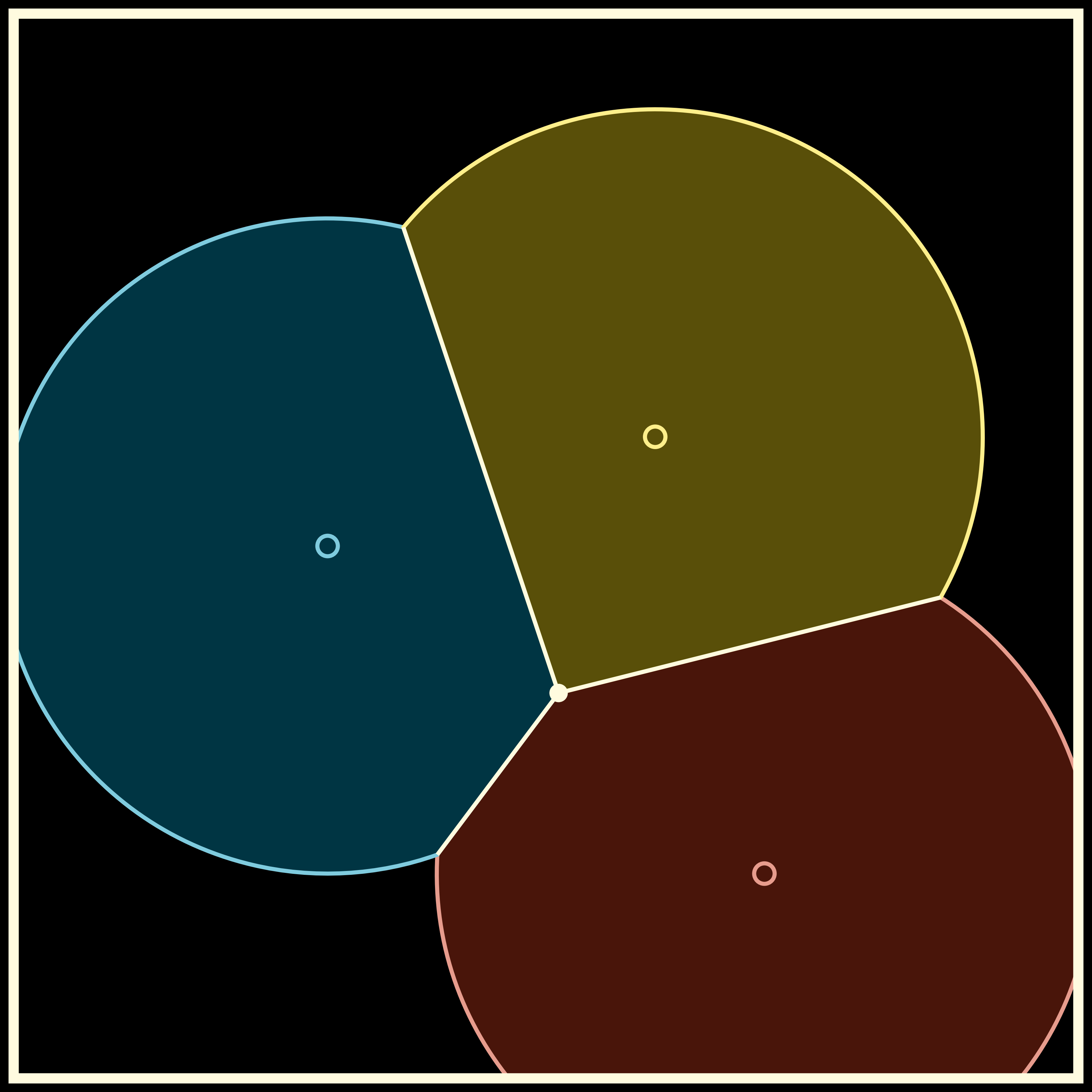


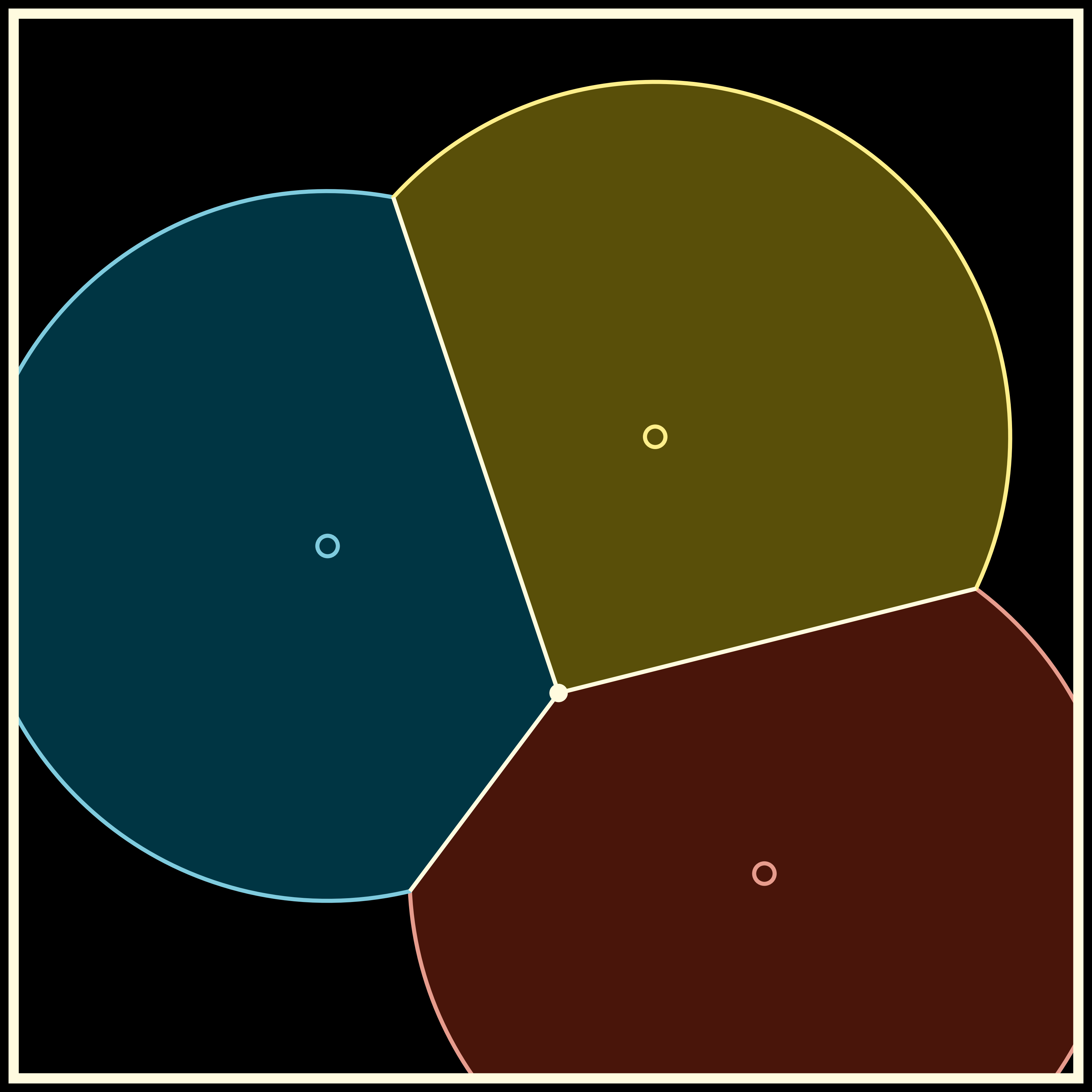


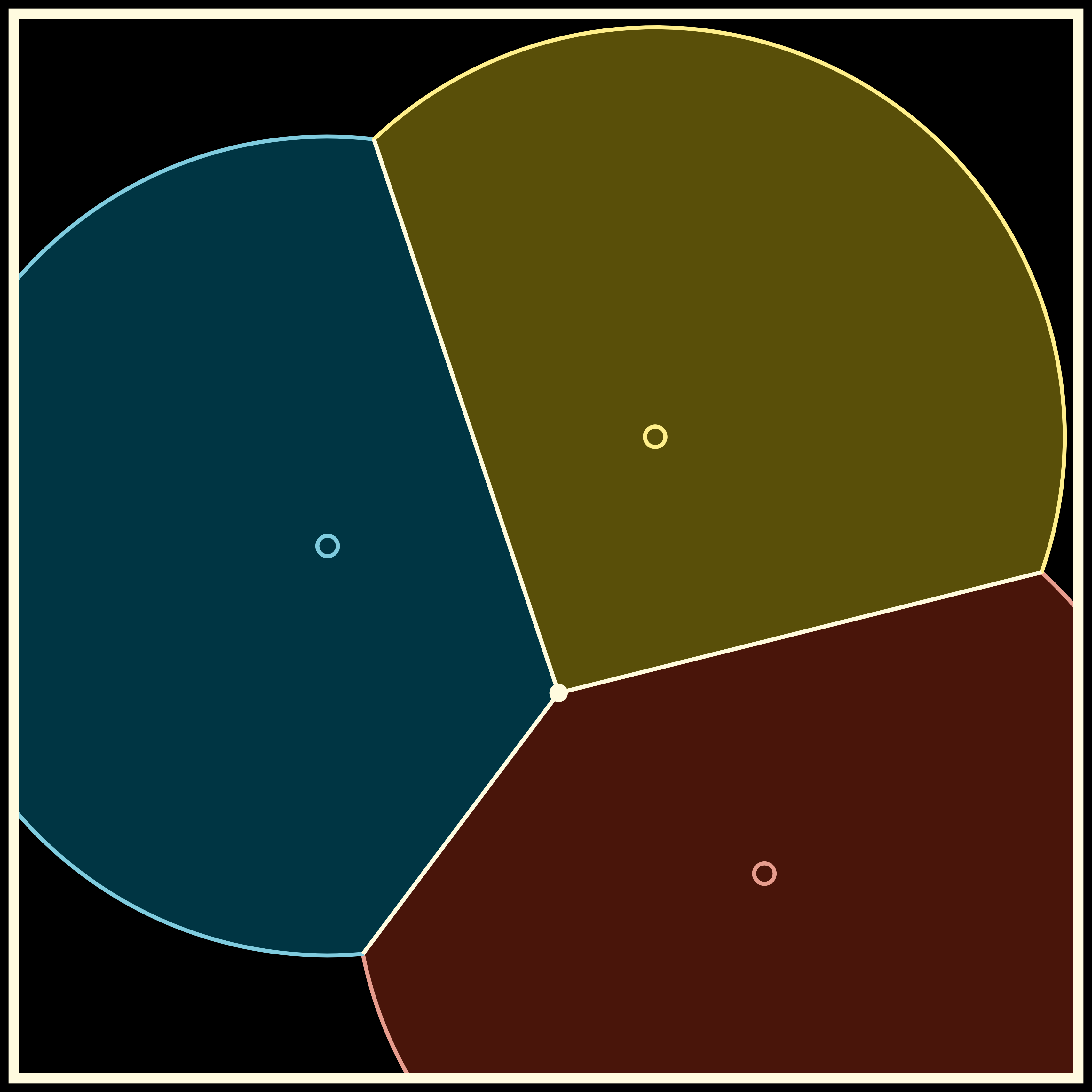


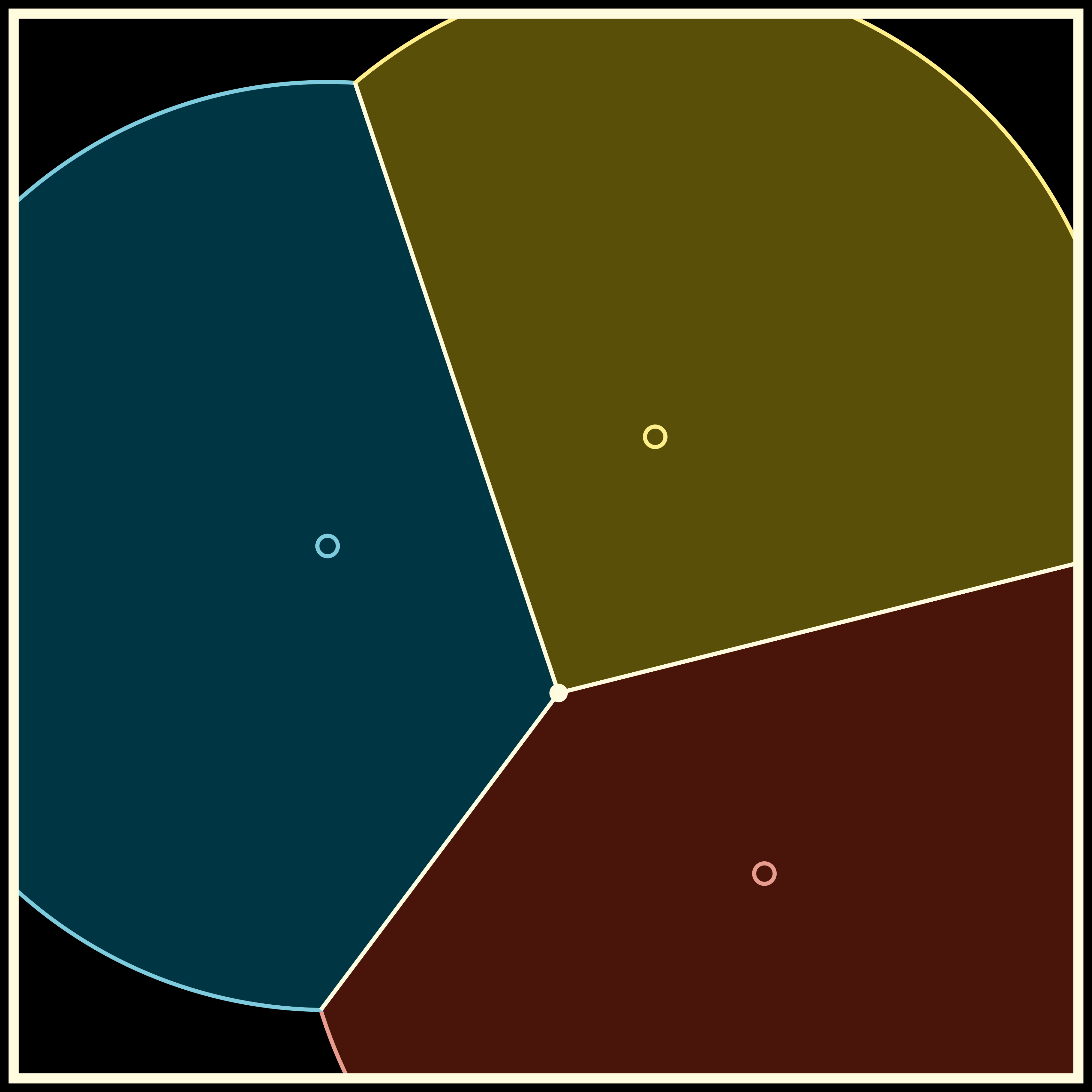


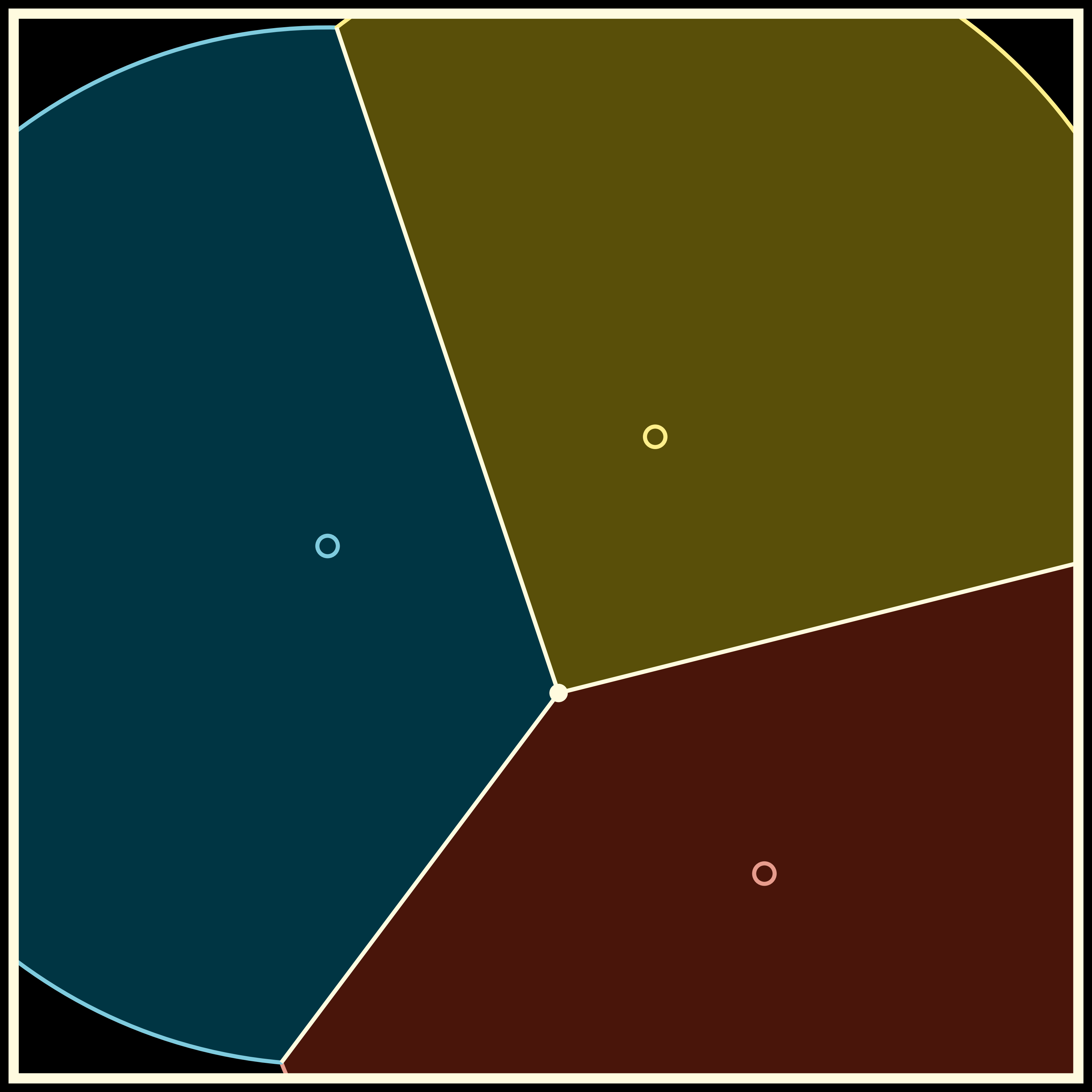


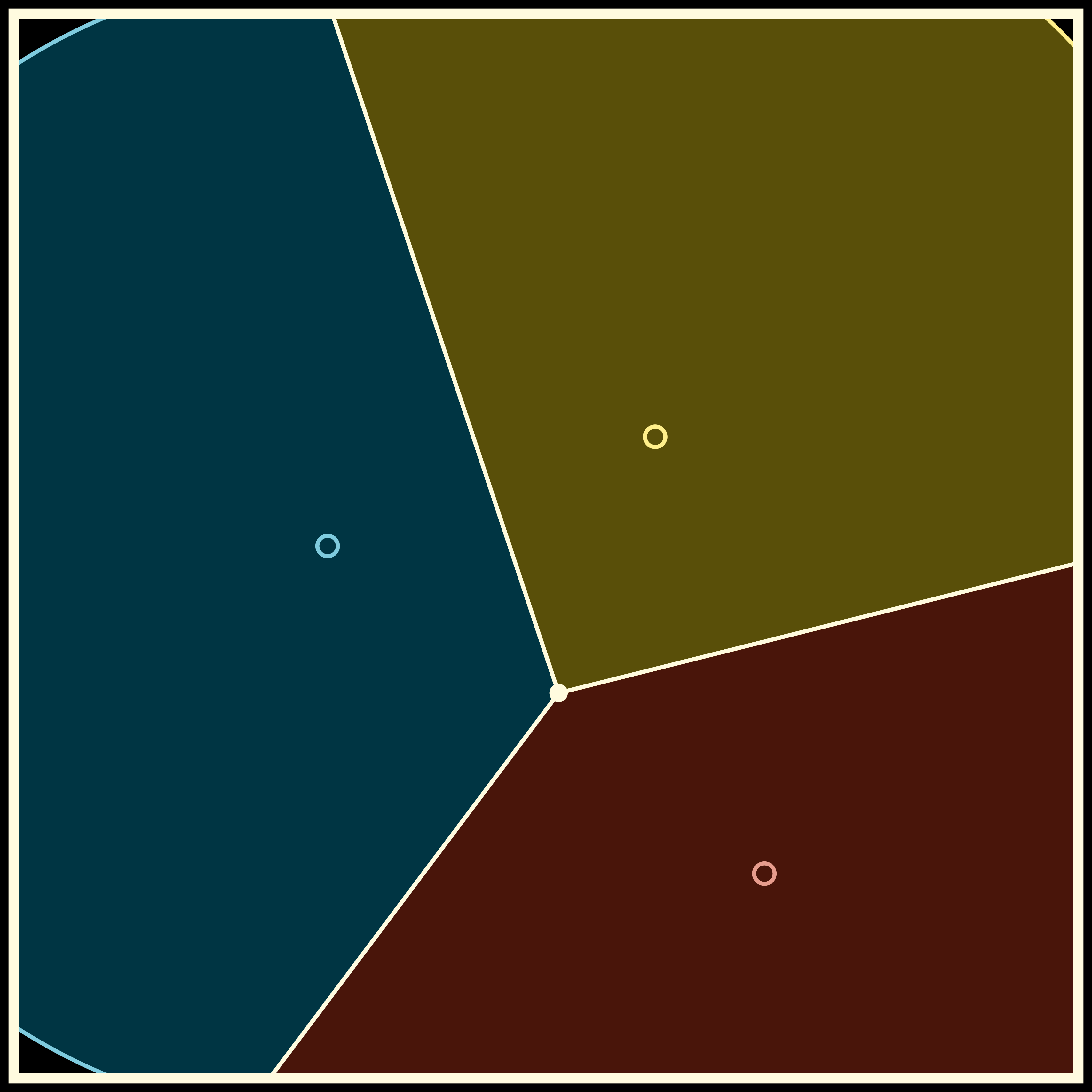


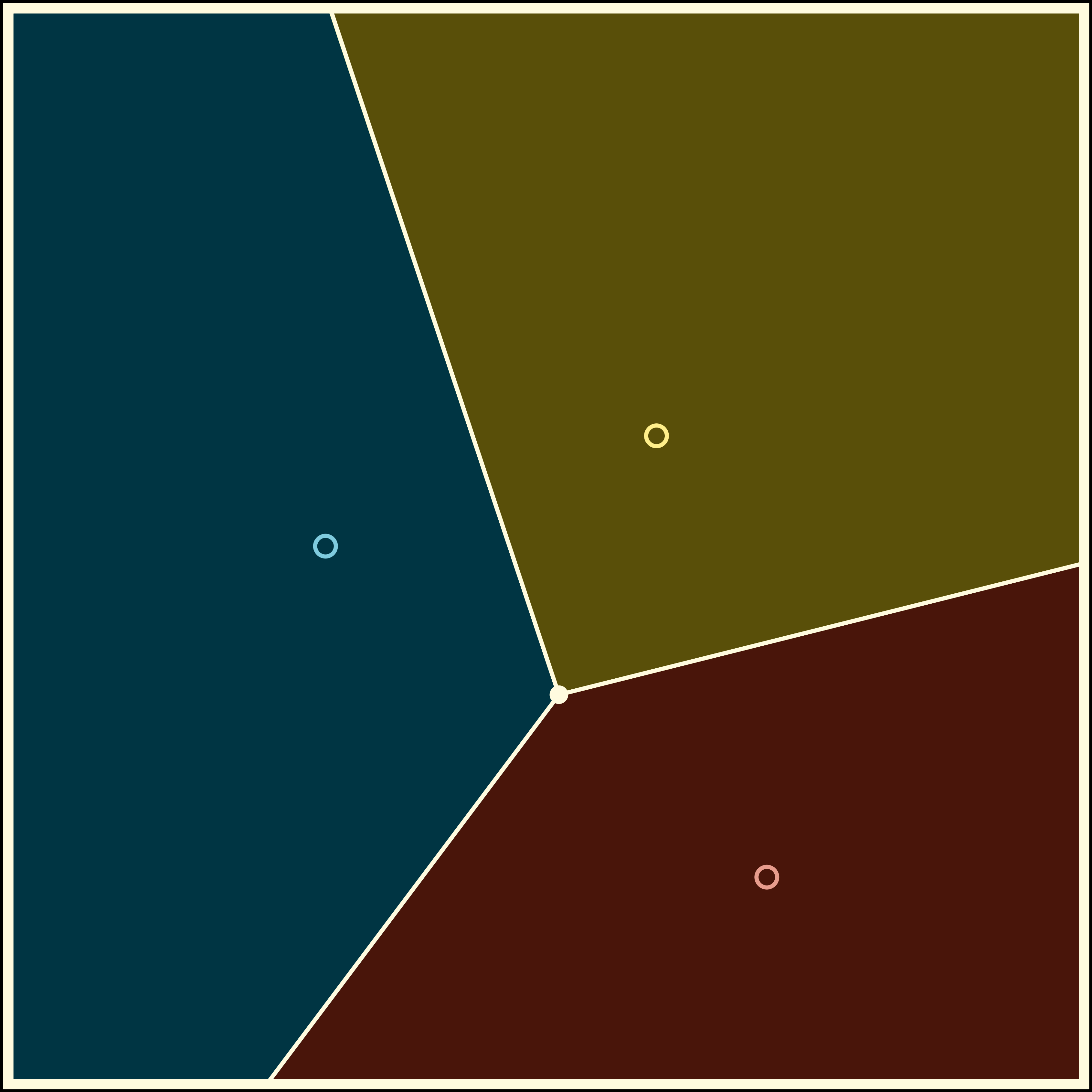


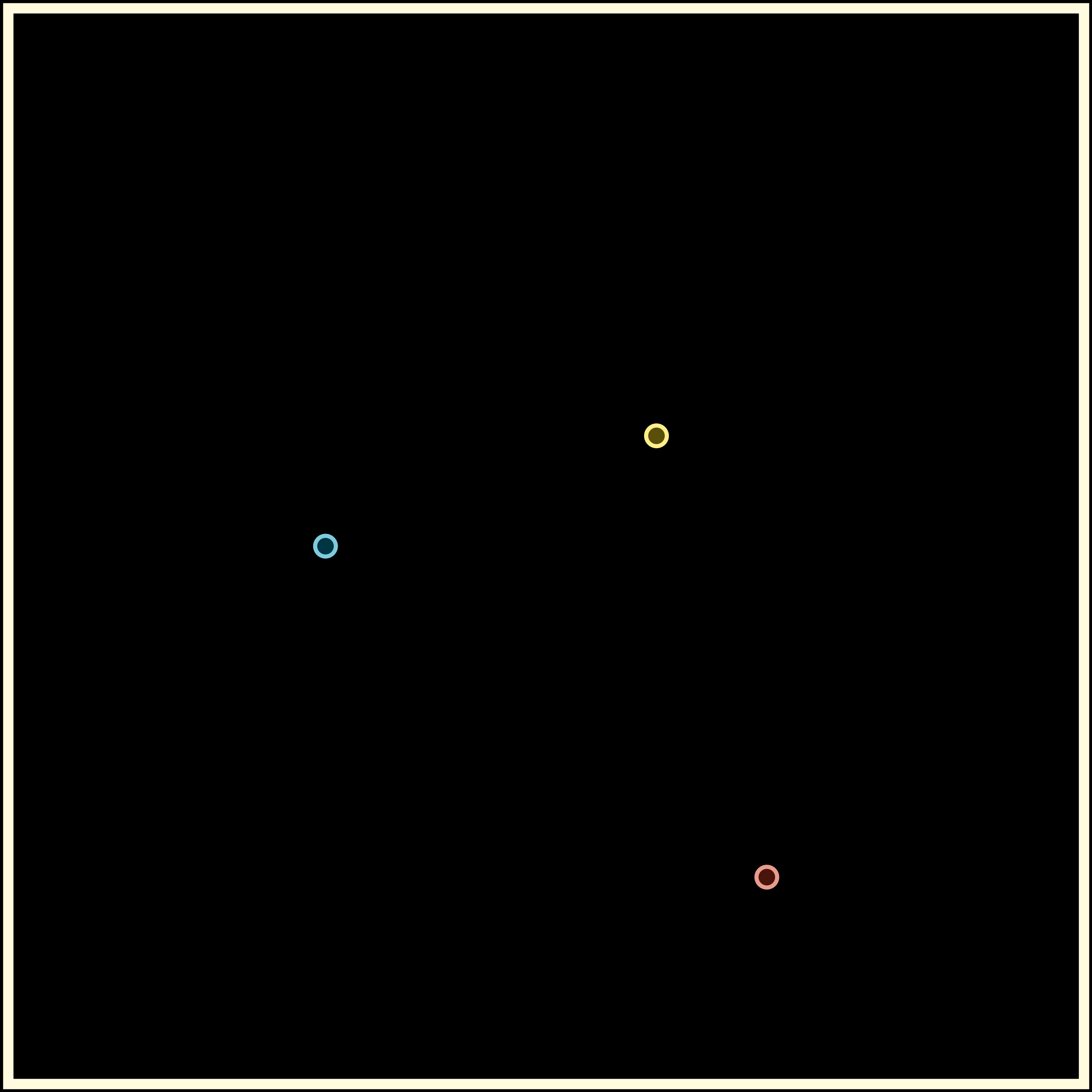


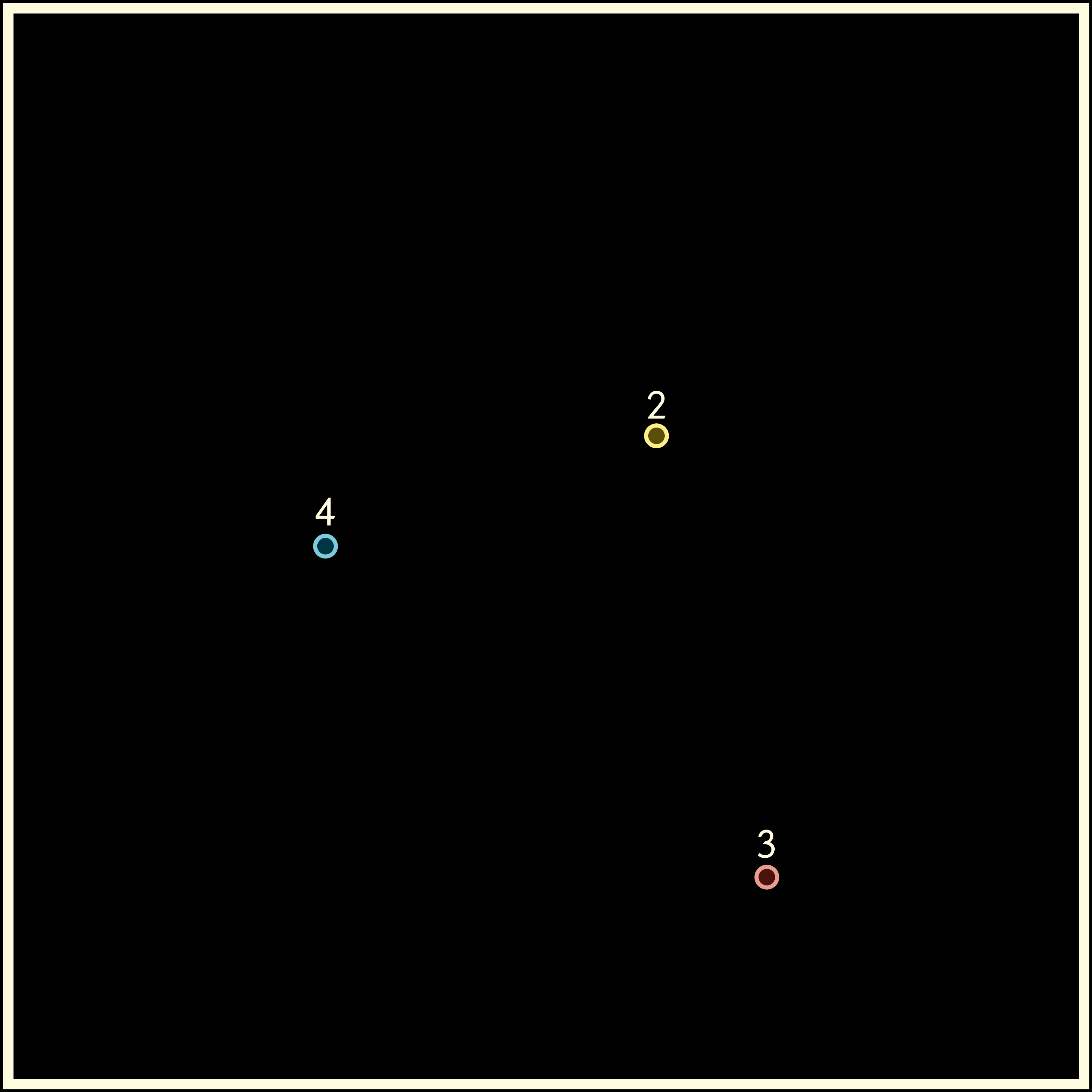


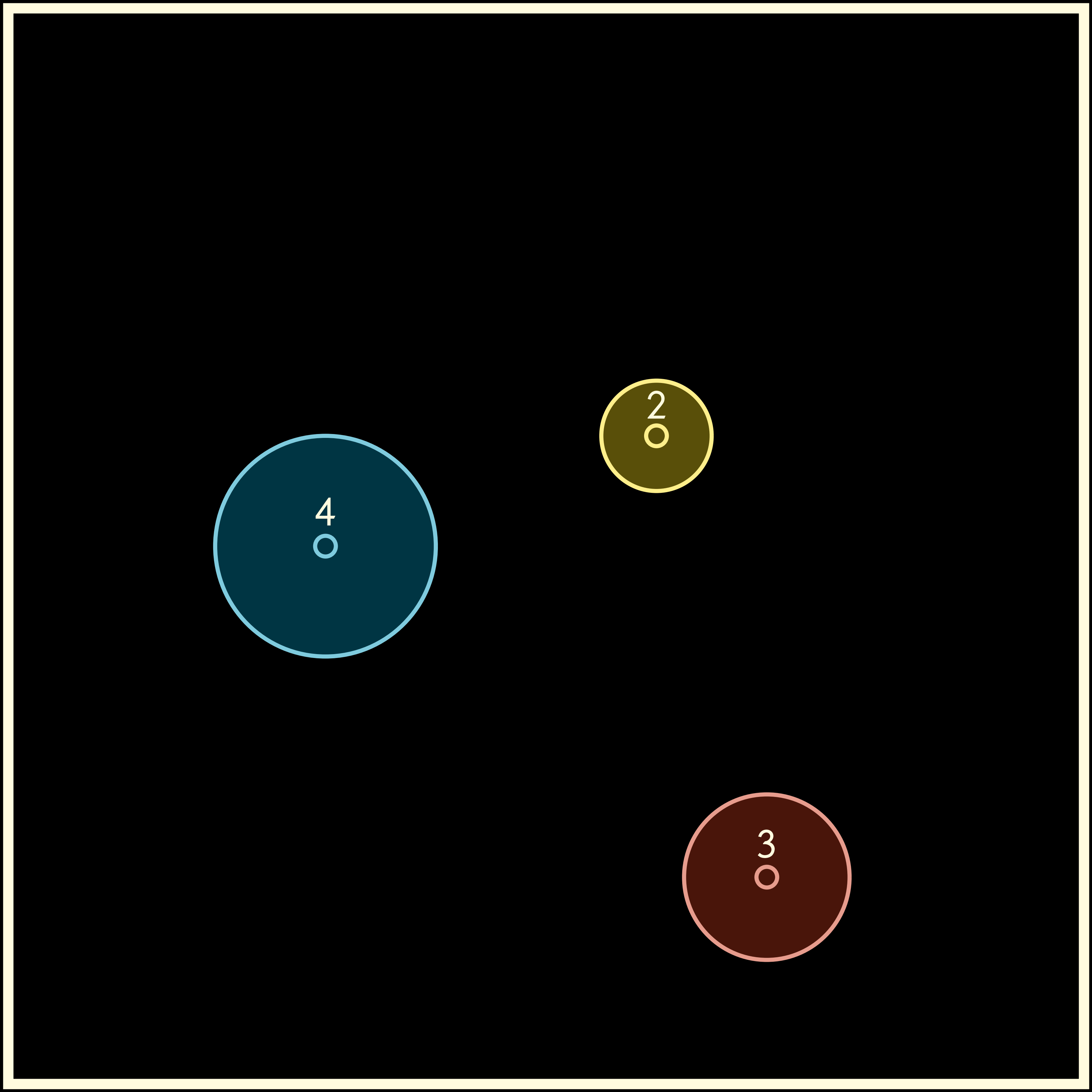


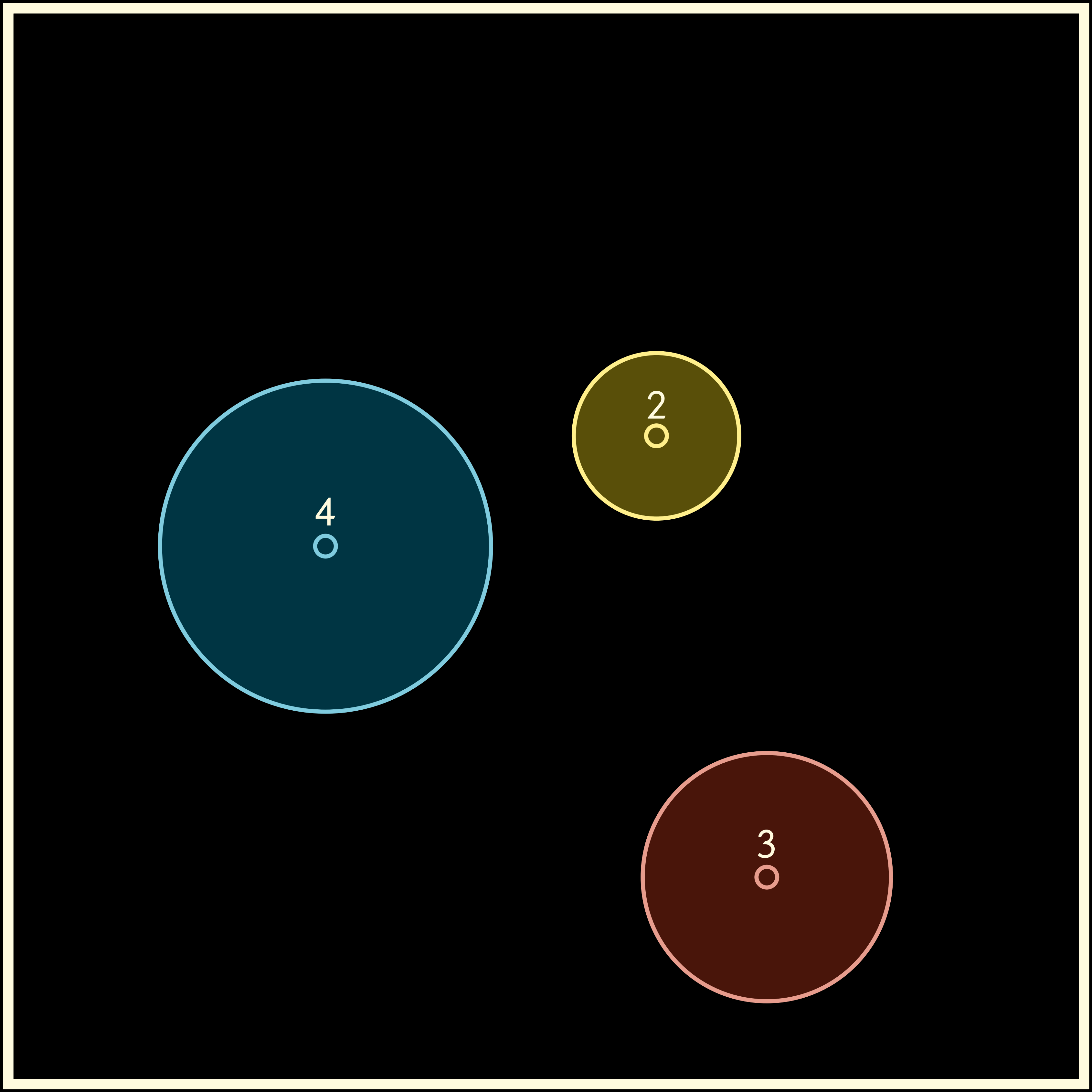


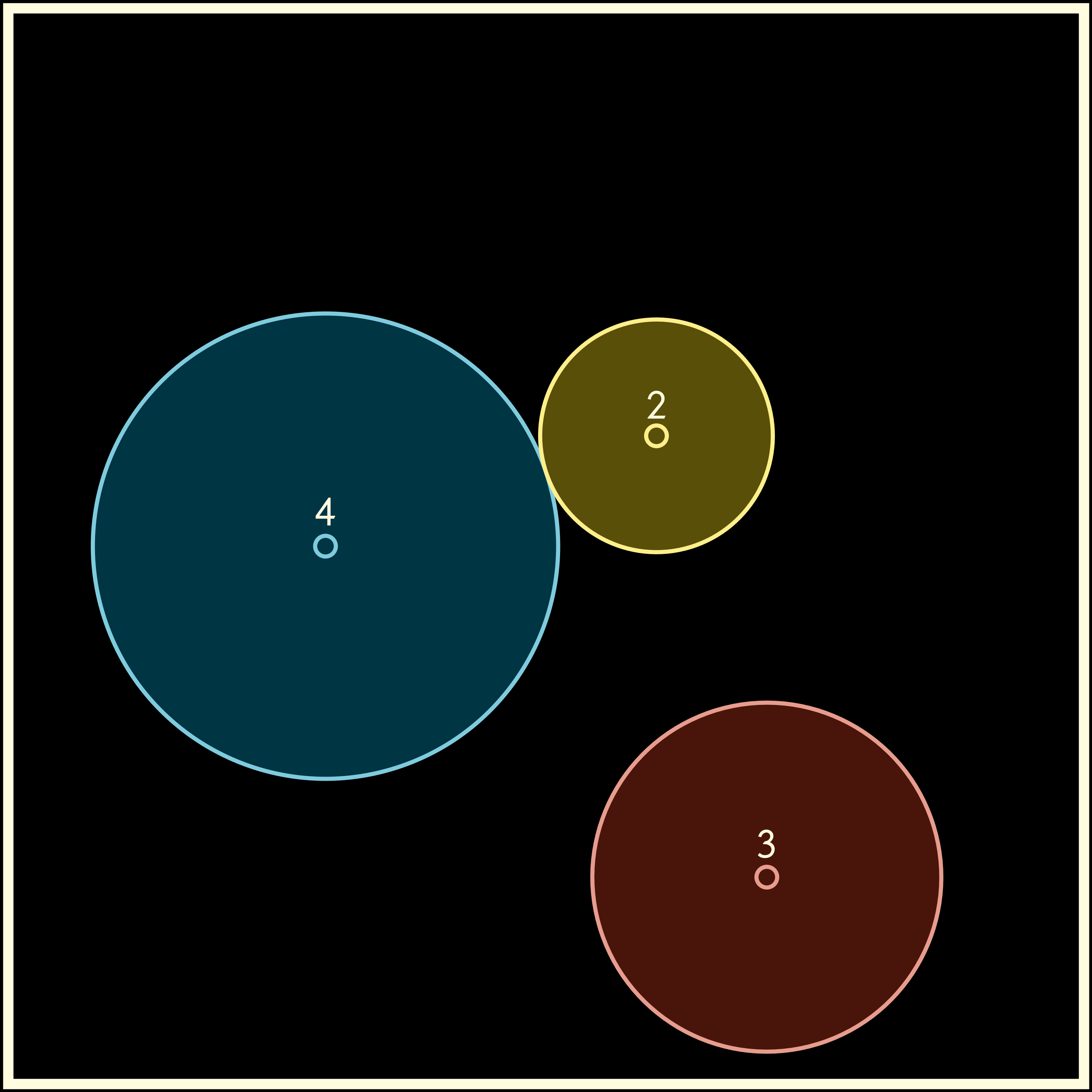


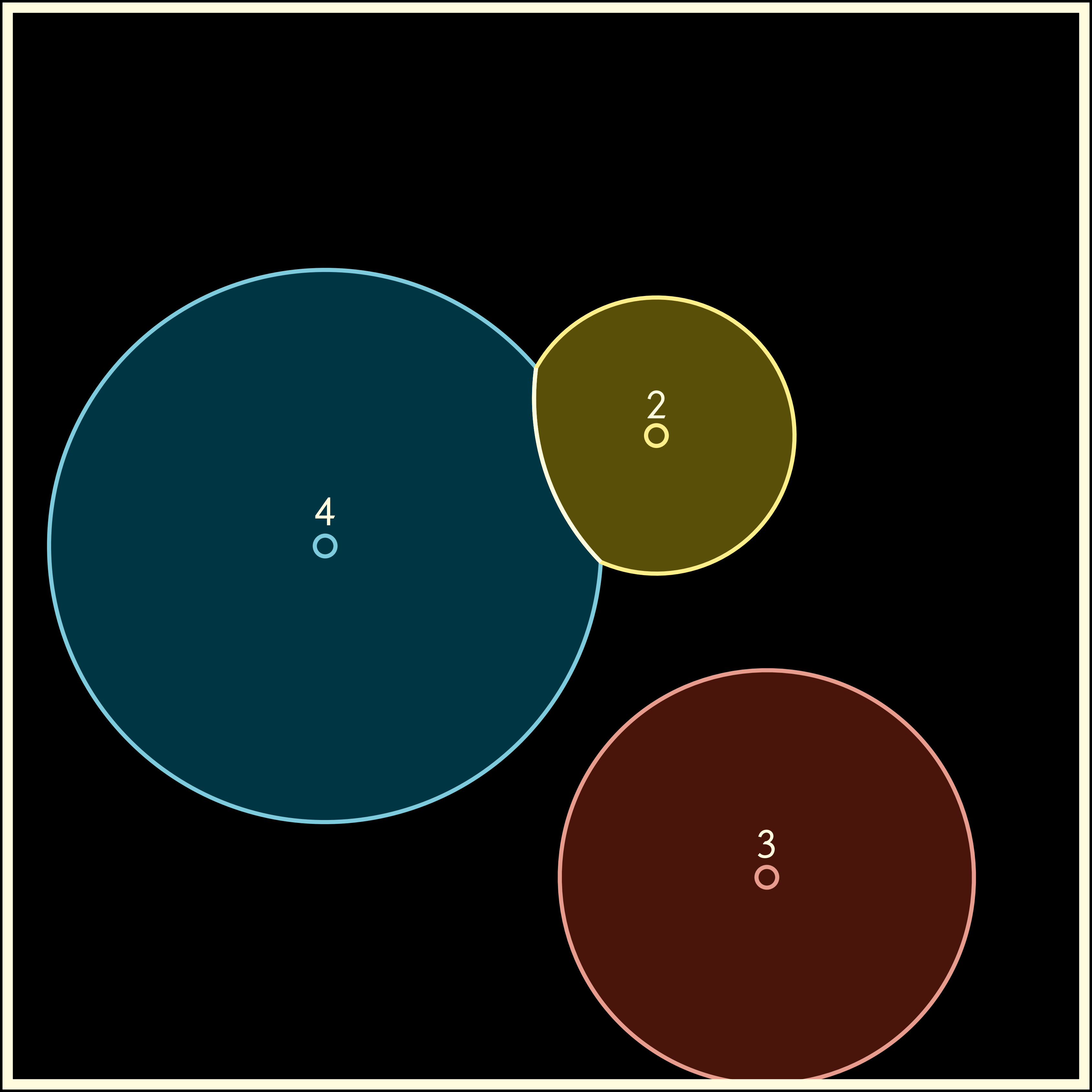



2
4
3

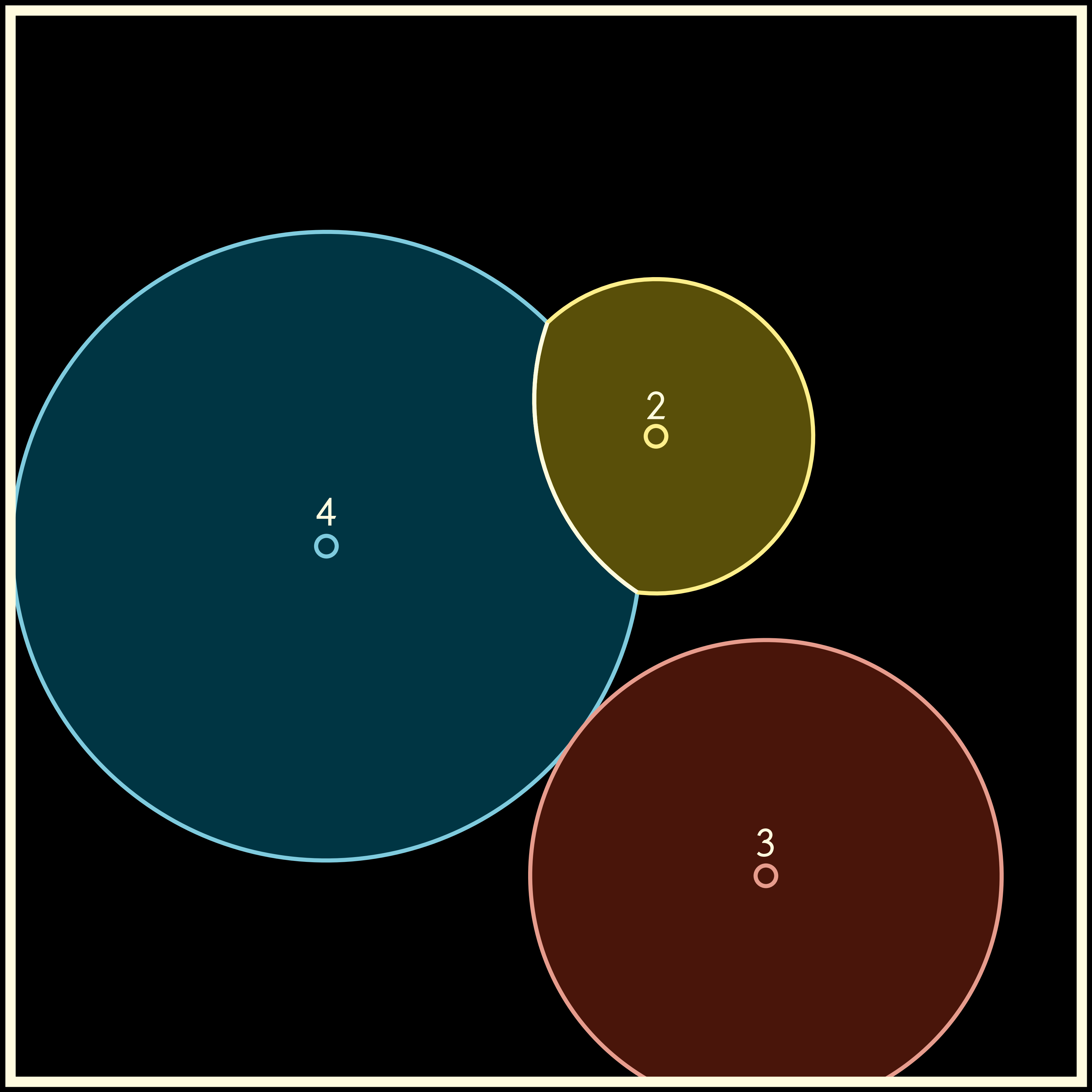

2
4
3

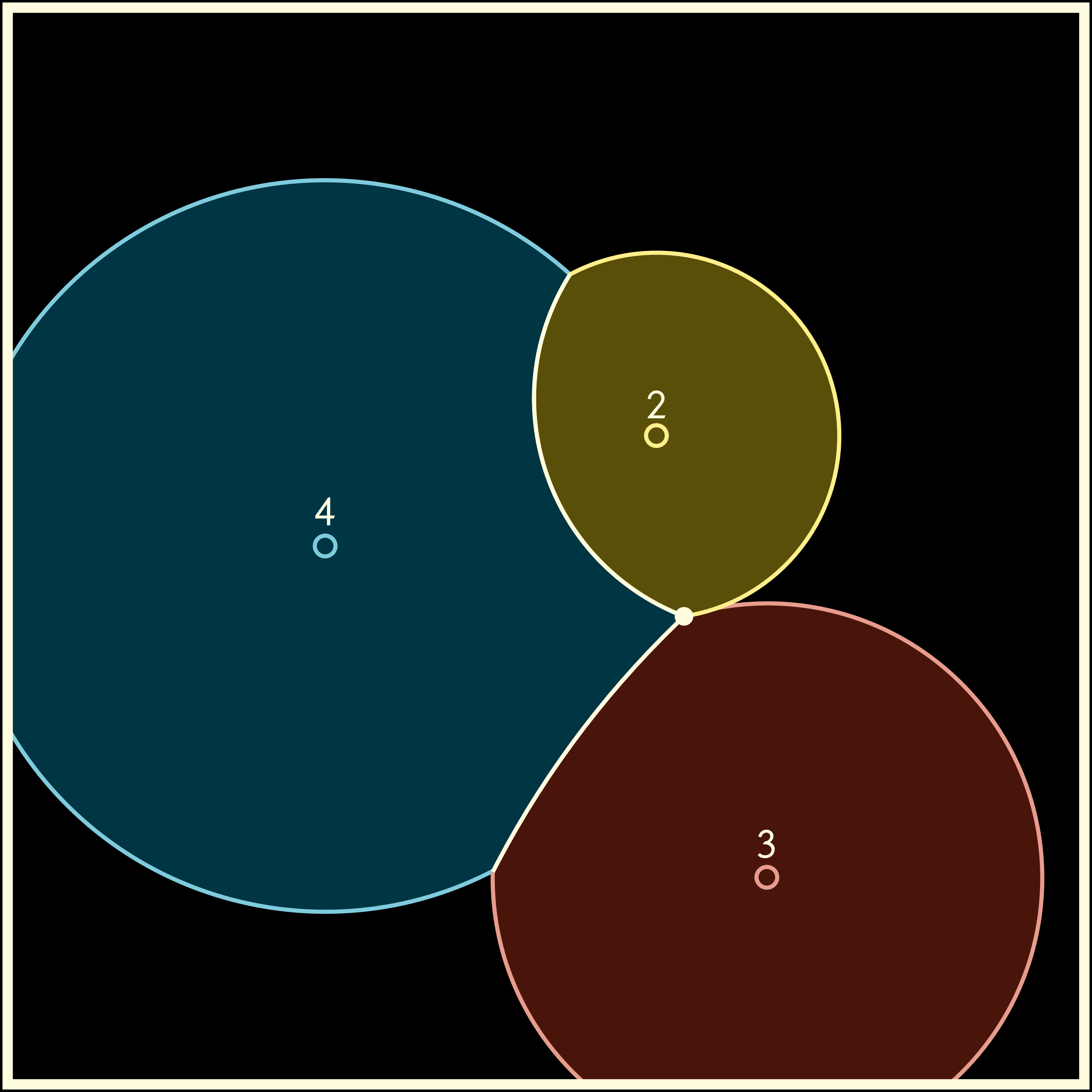

2
4
3

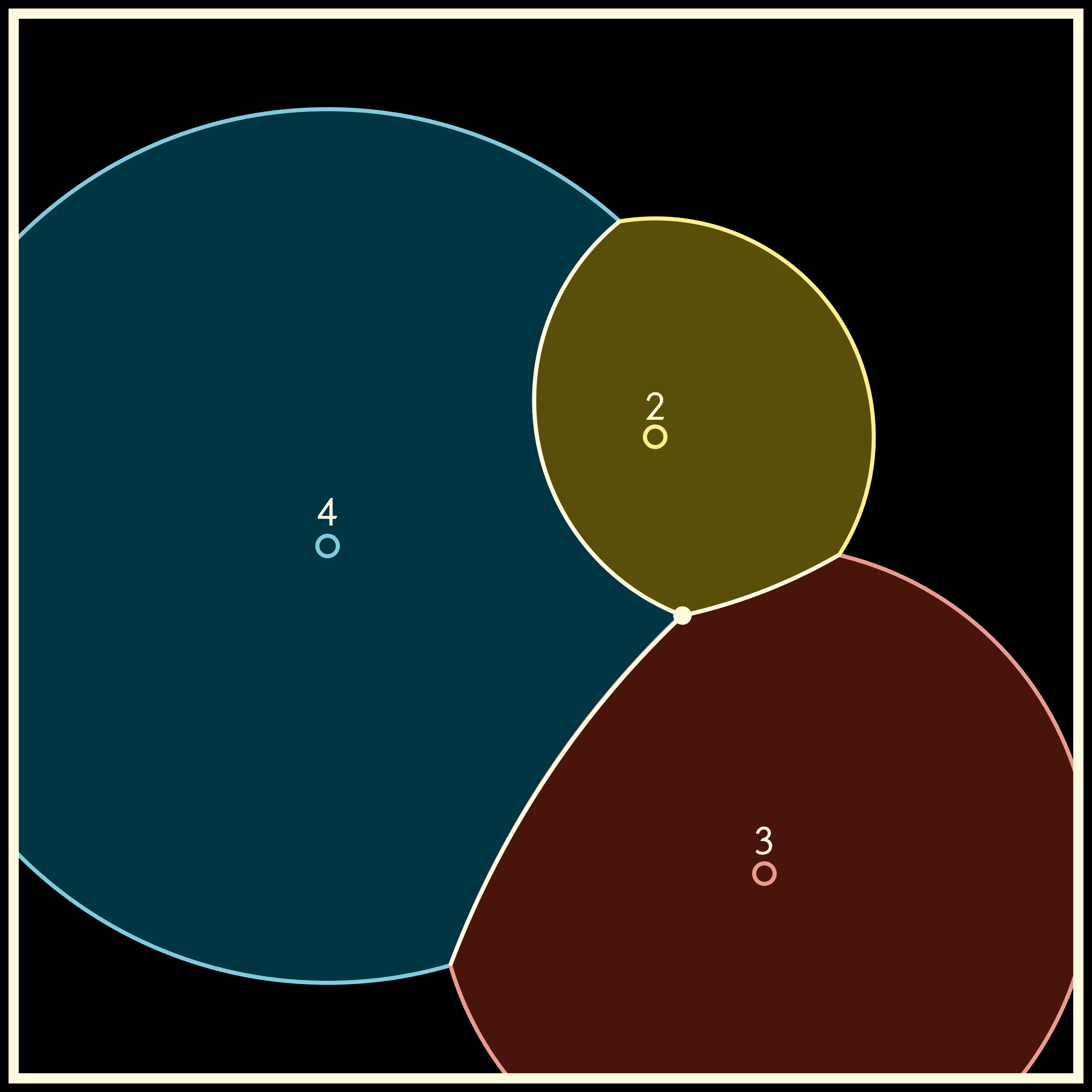

2
4
3



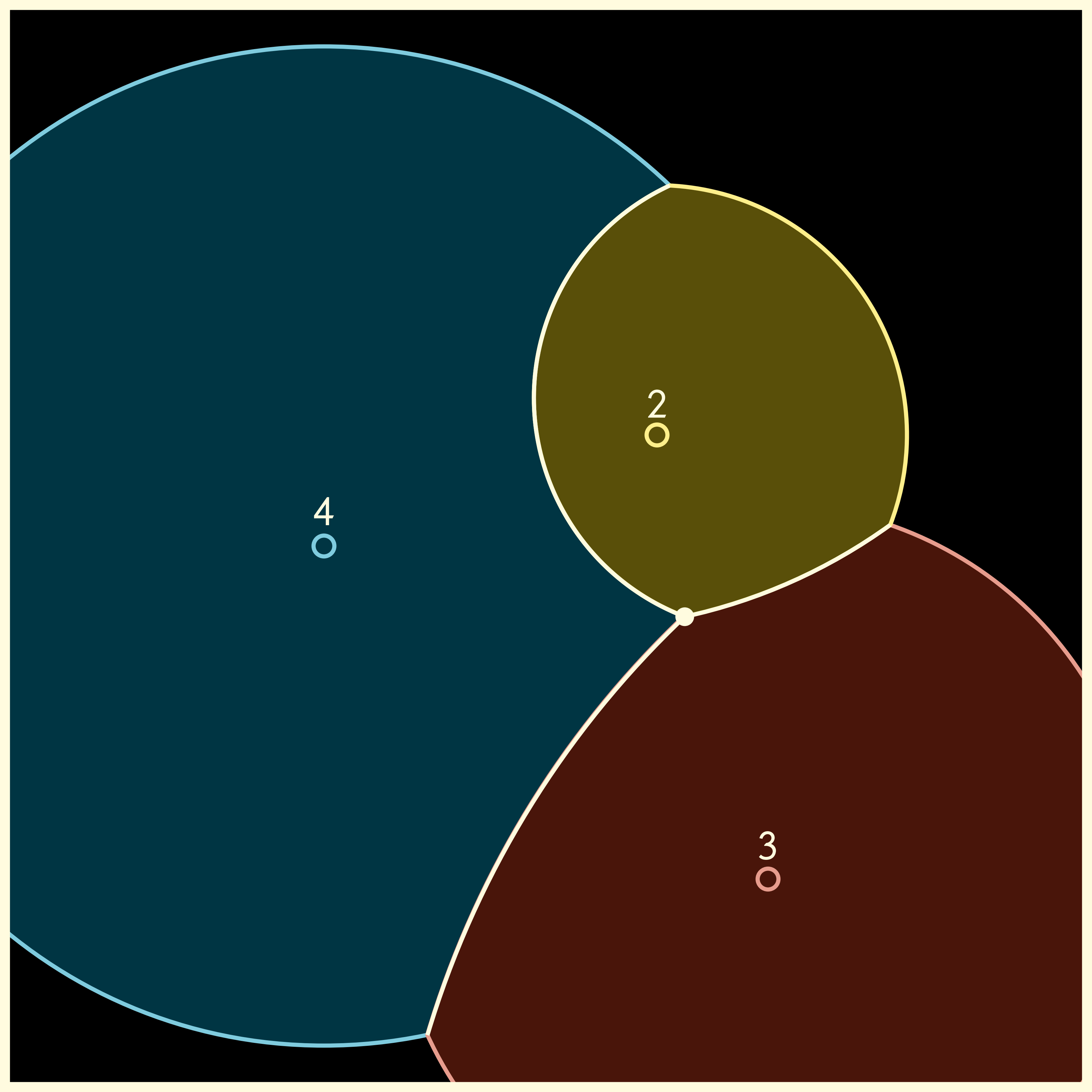
2
4
3


2
4
3

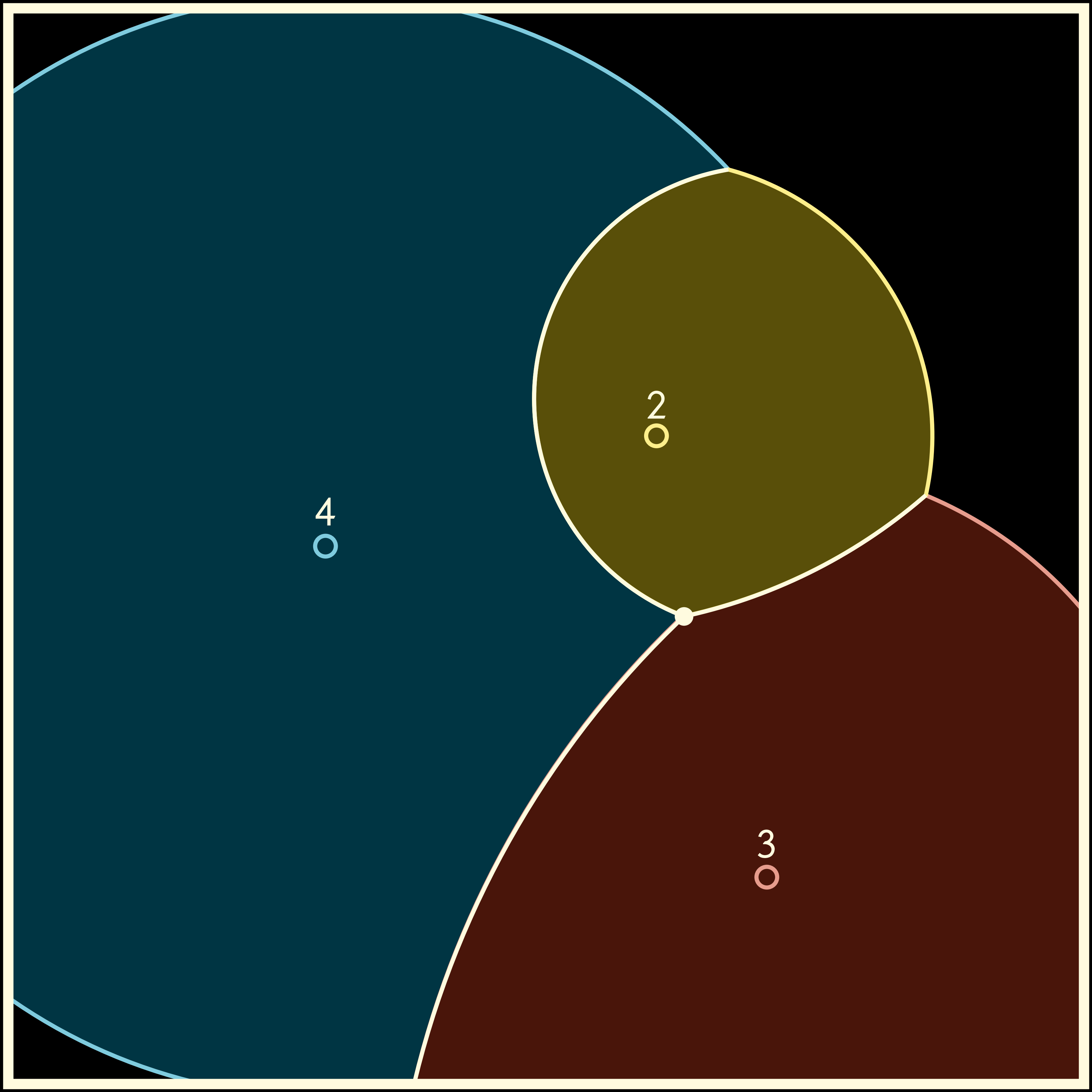

2
4
3



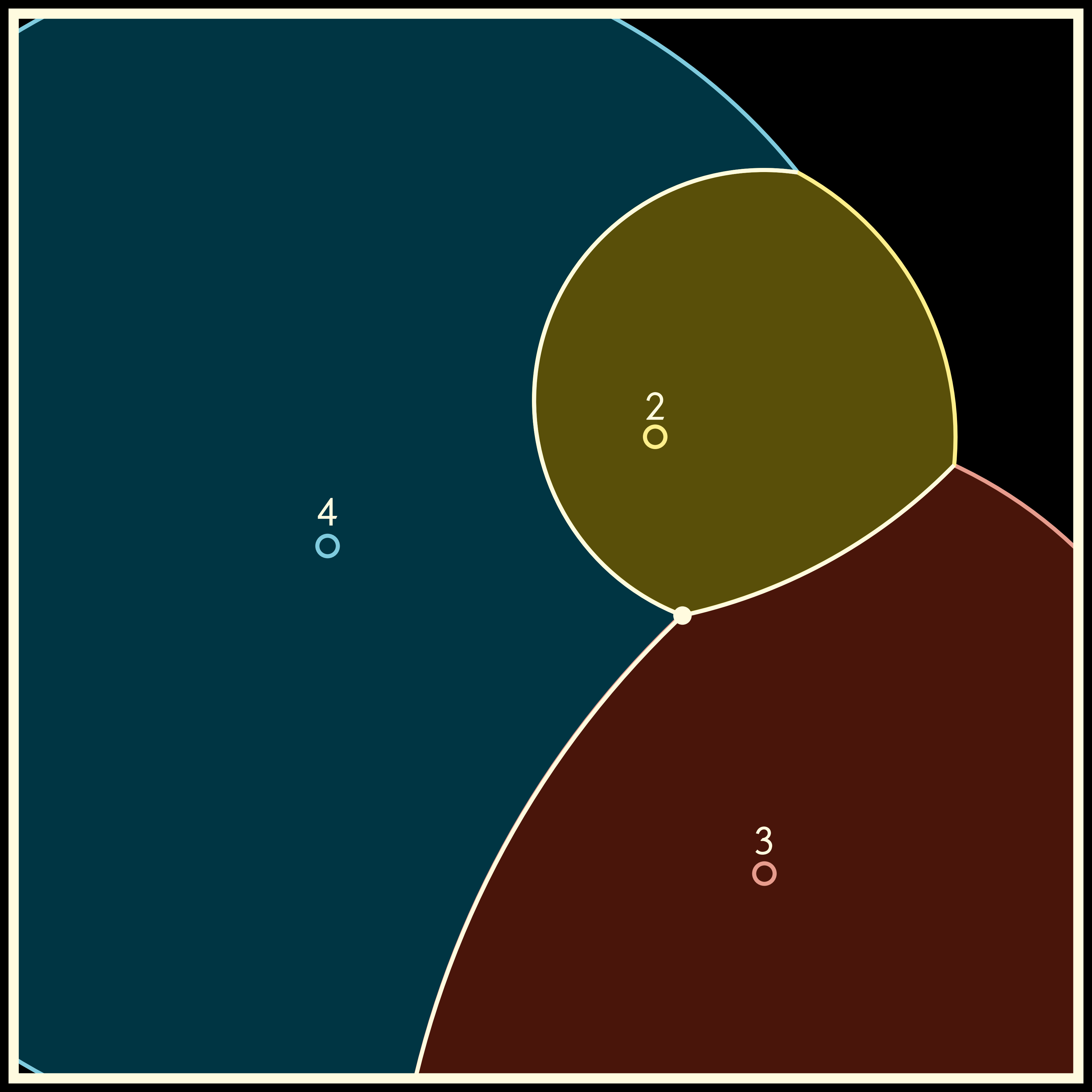
4
2
3

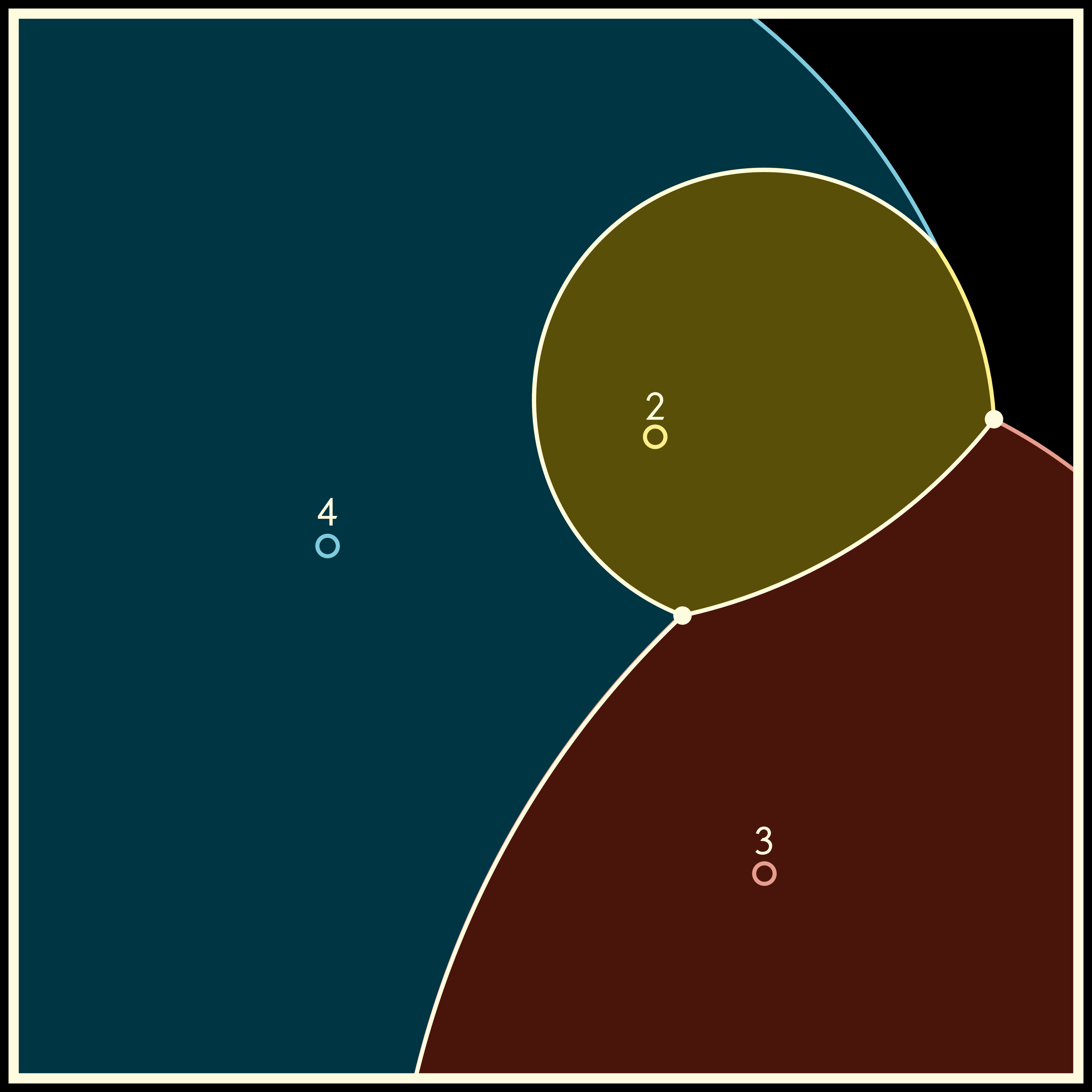


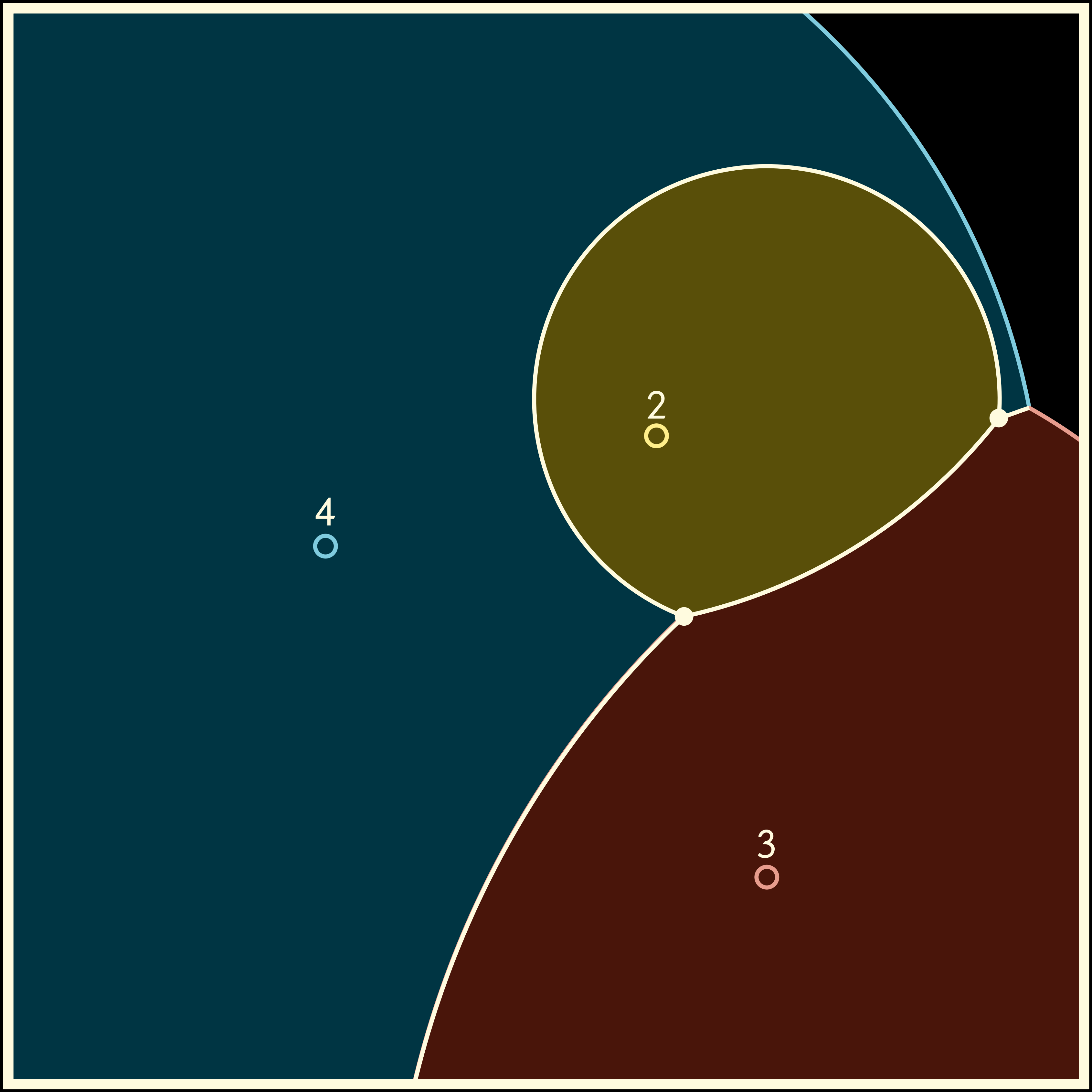
2
4
3

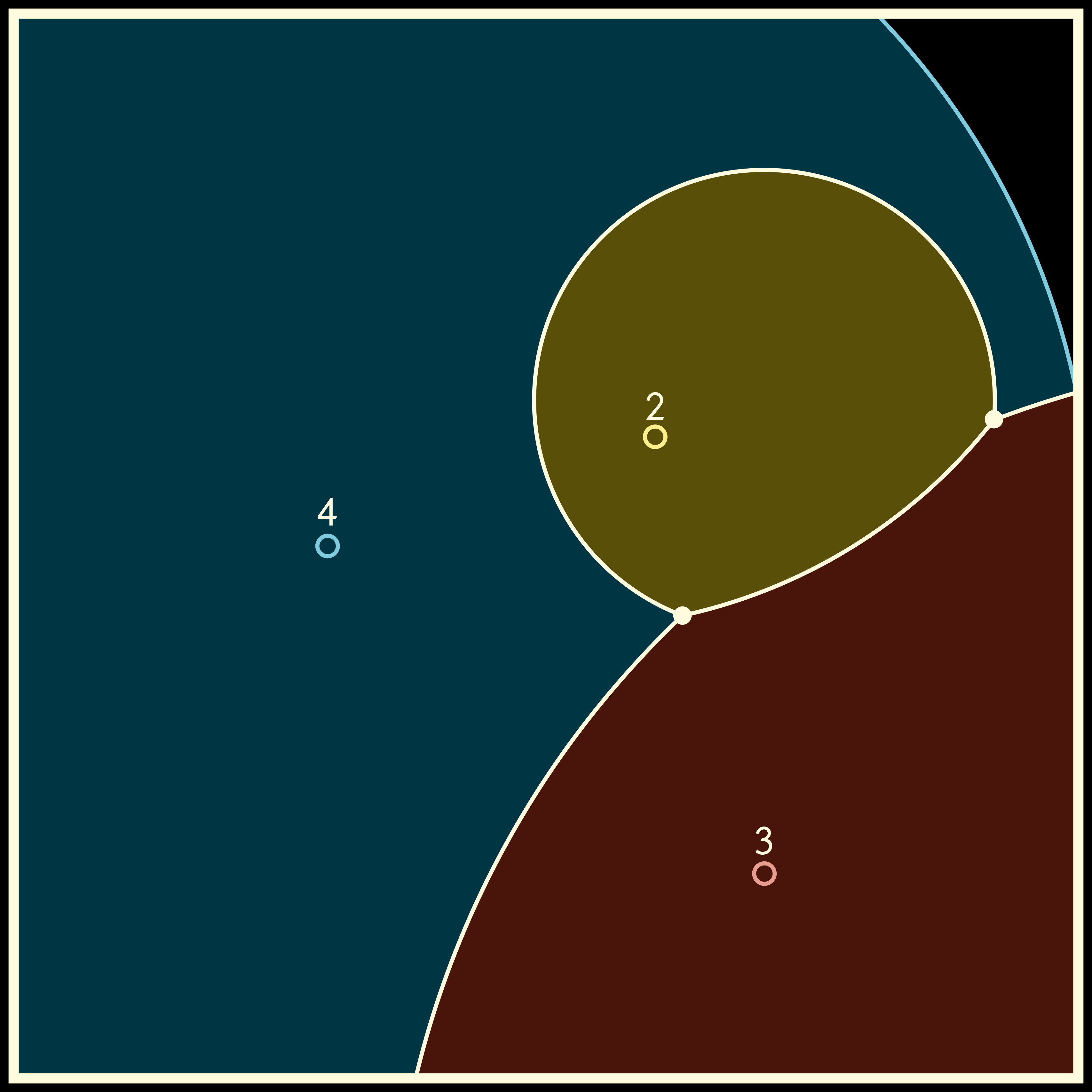


2
4
3

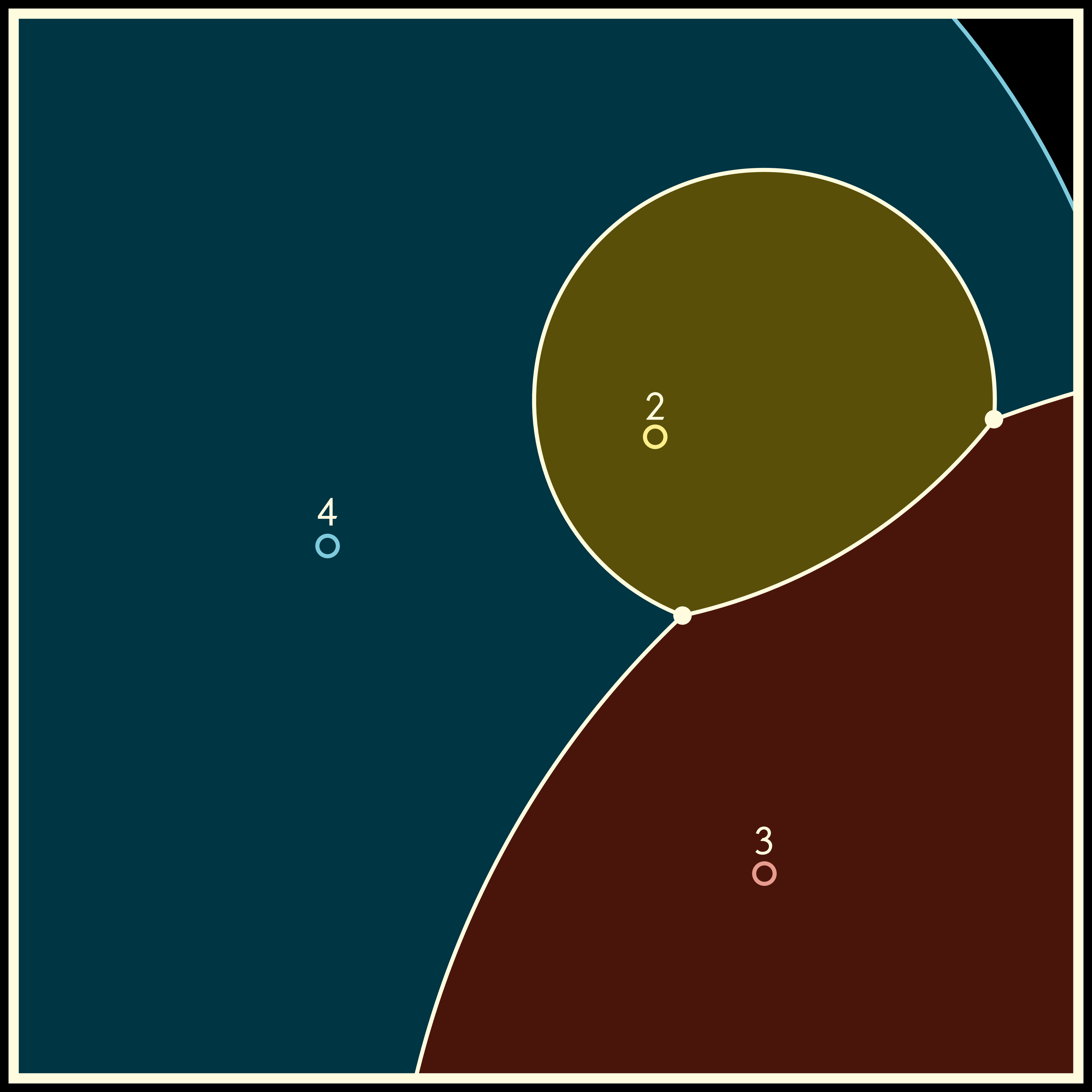

2
4
3

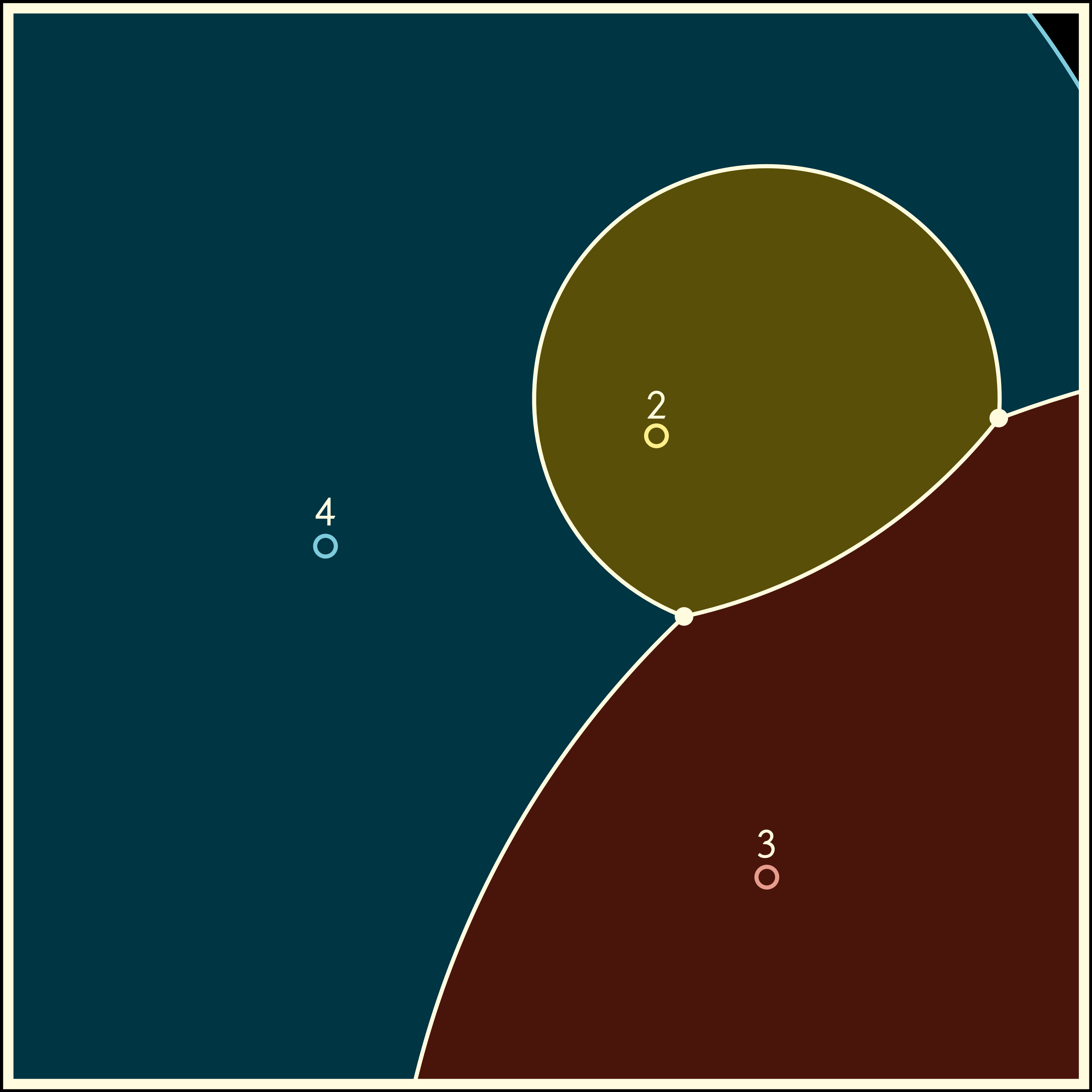

2
4
3

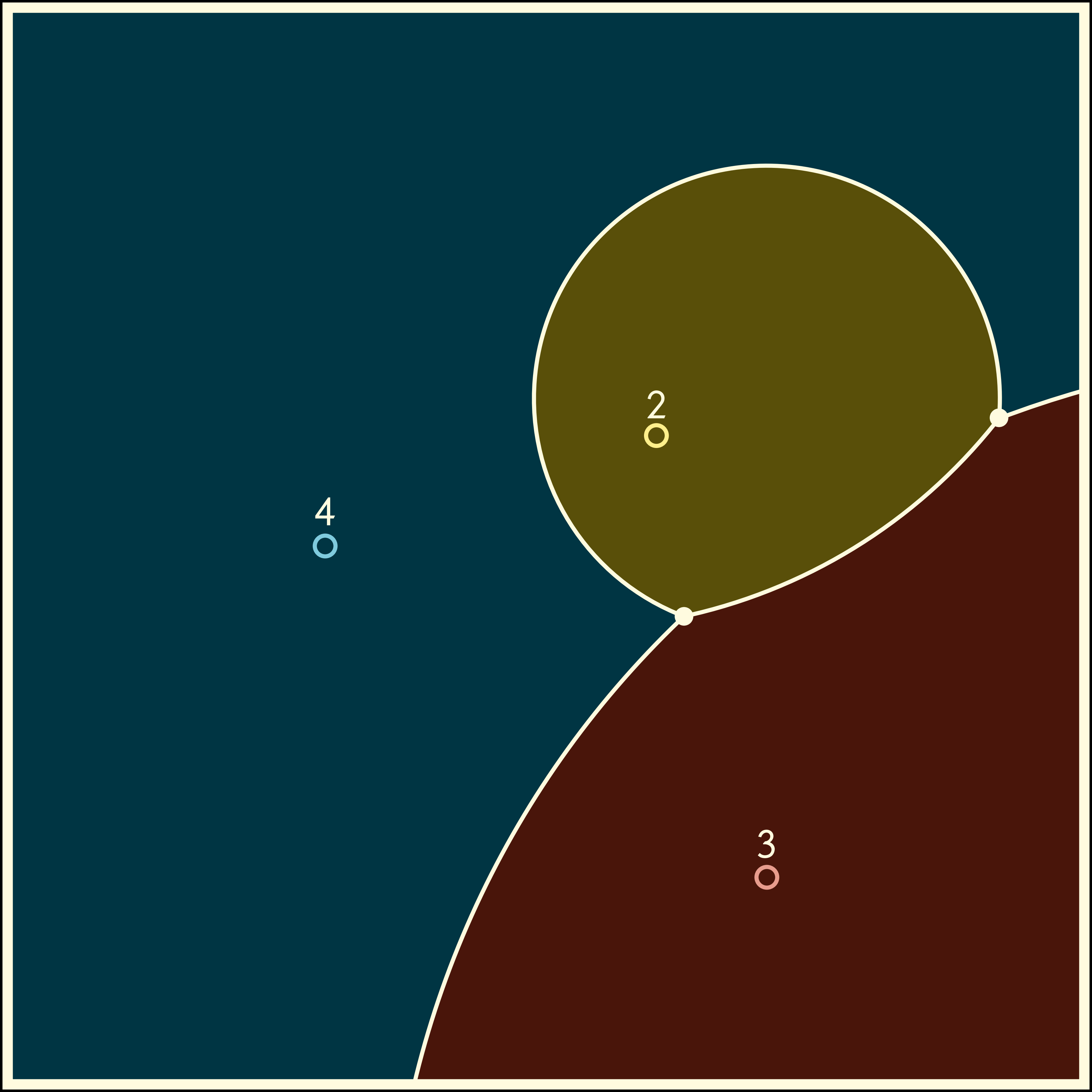

2
4
3



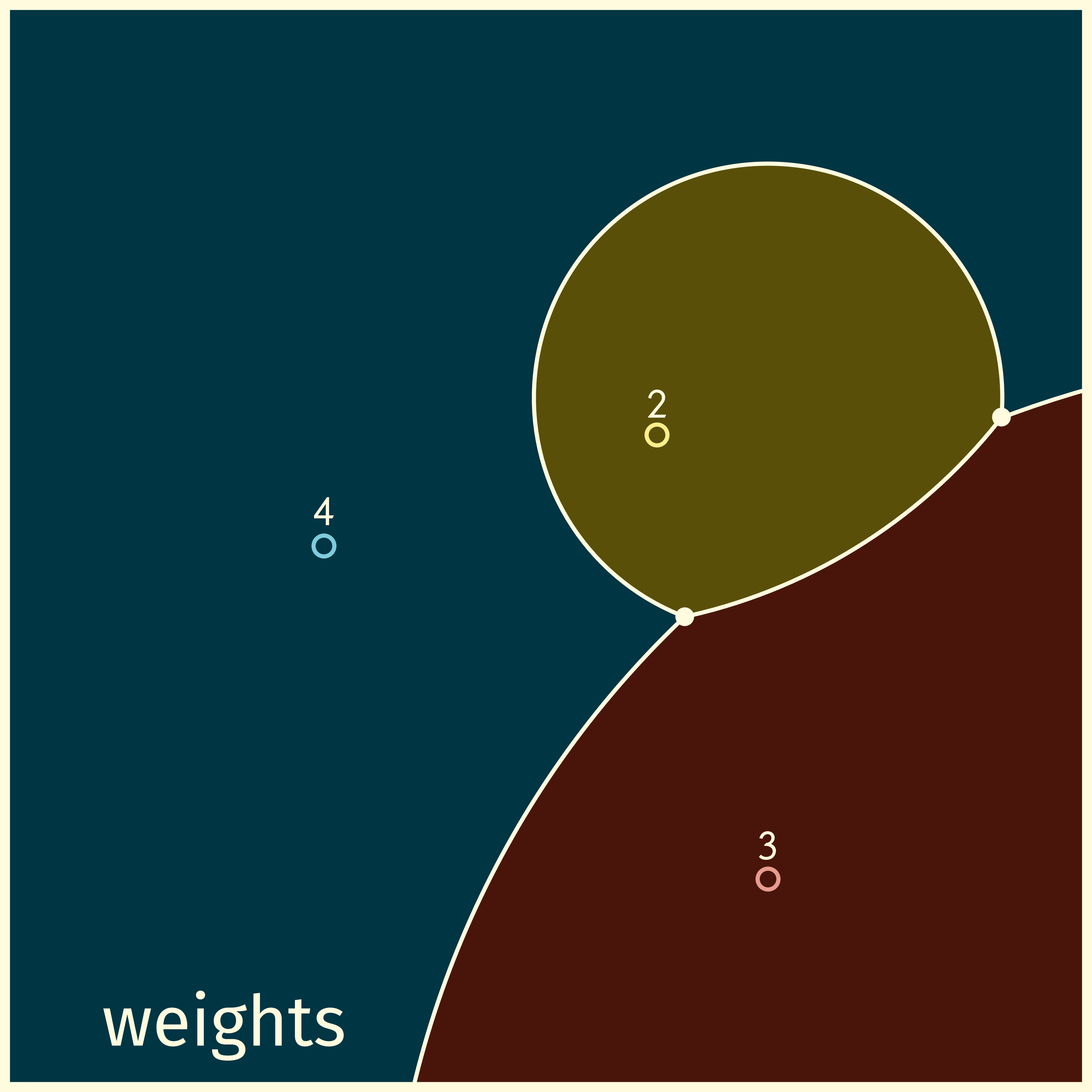
2
4
3

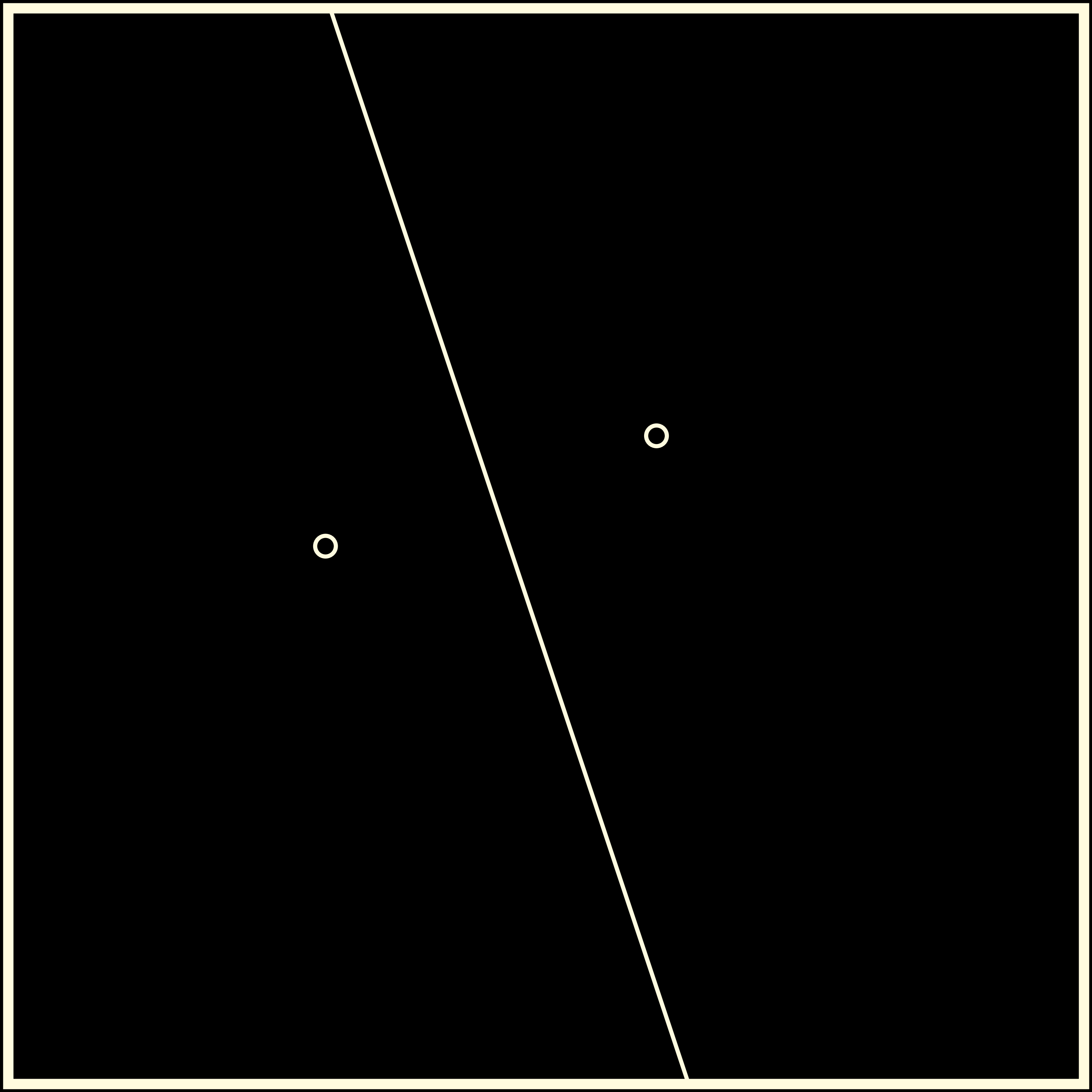

2
4
3

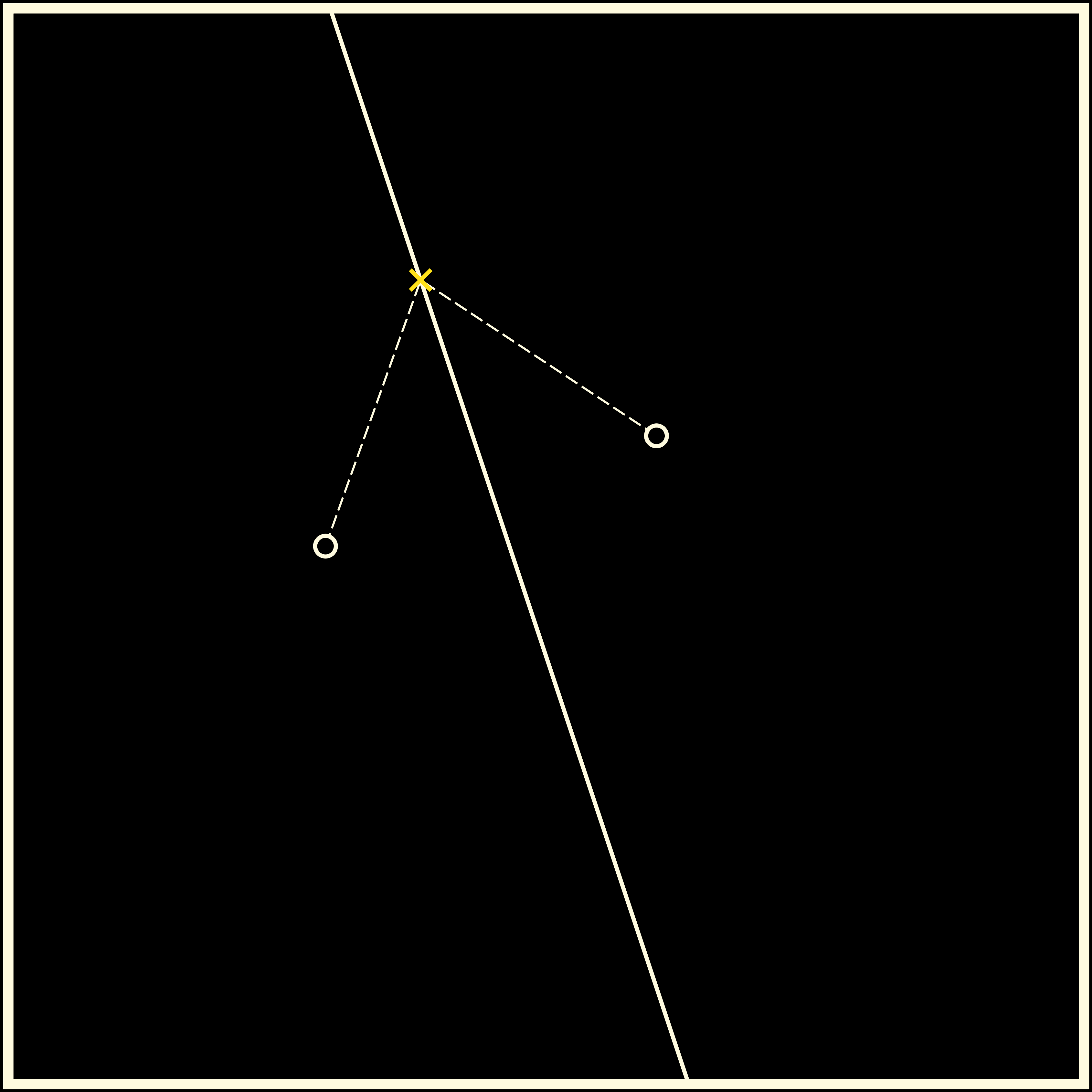

2
4
3

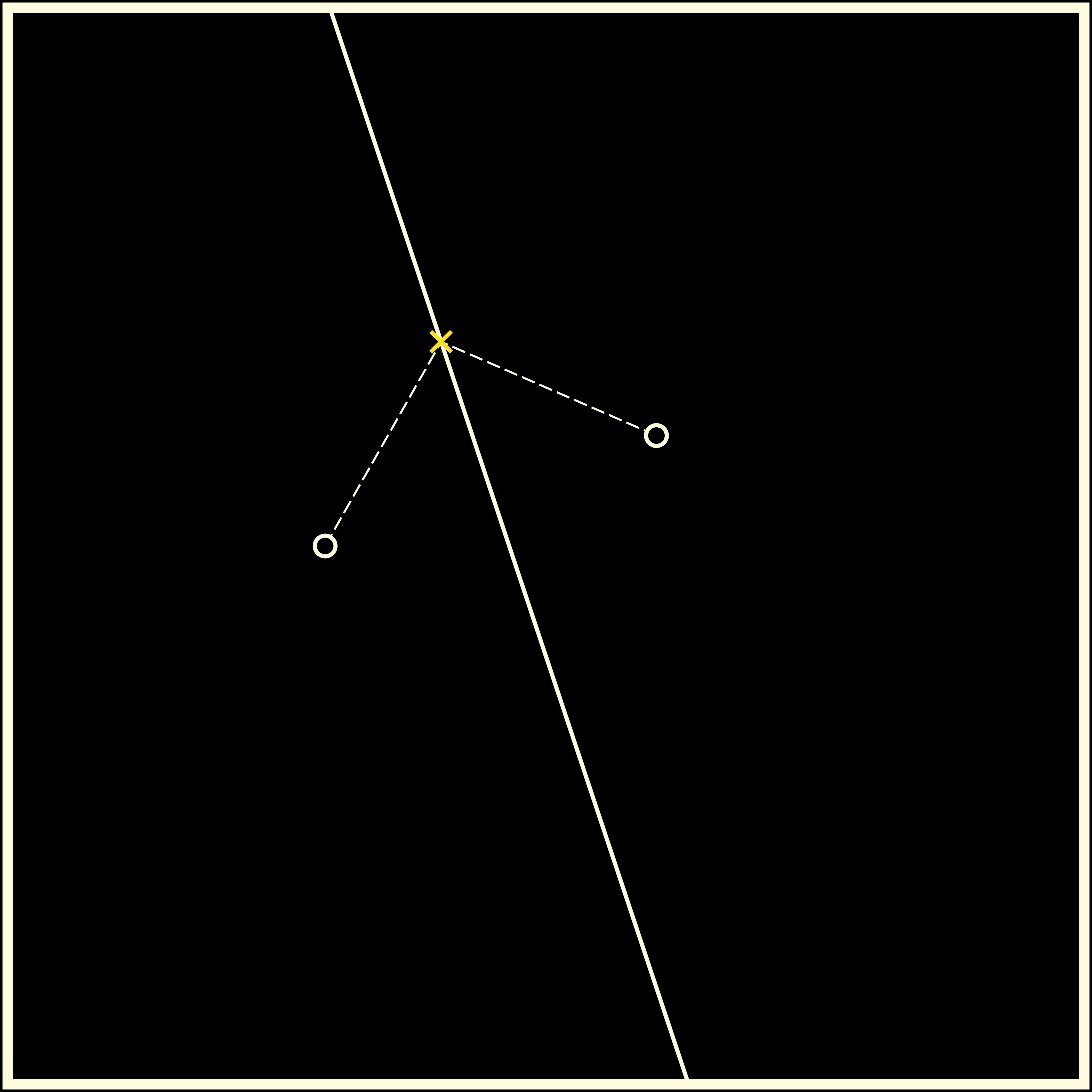


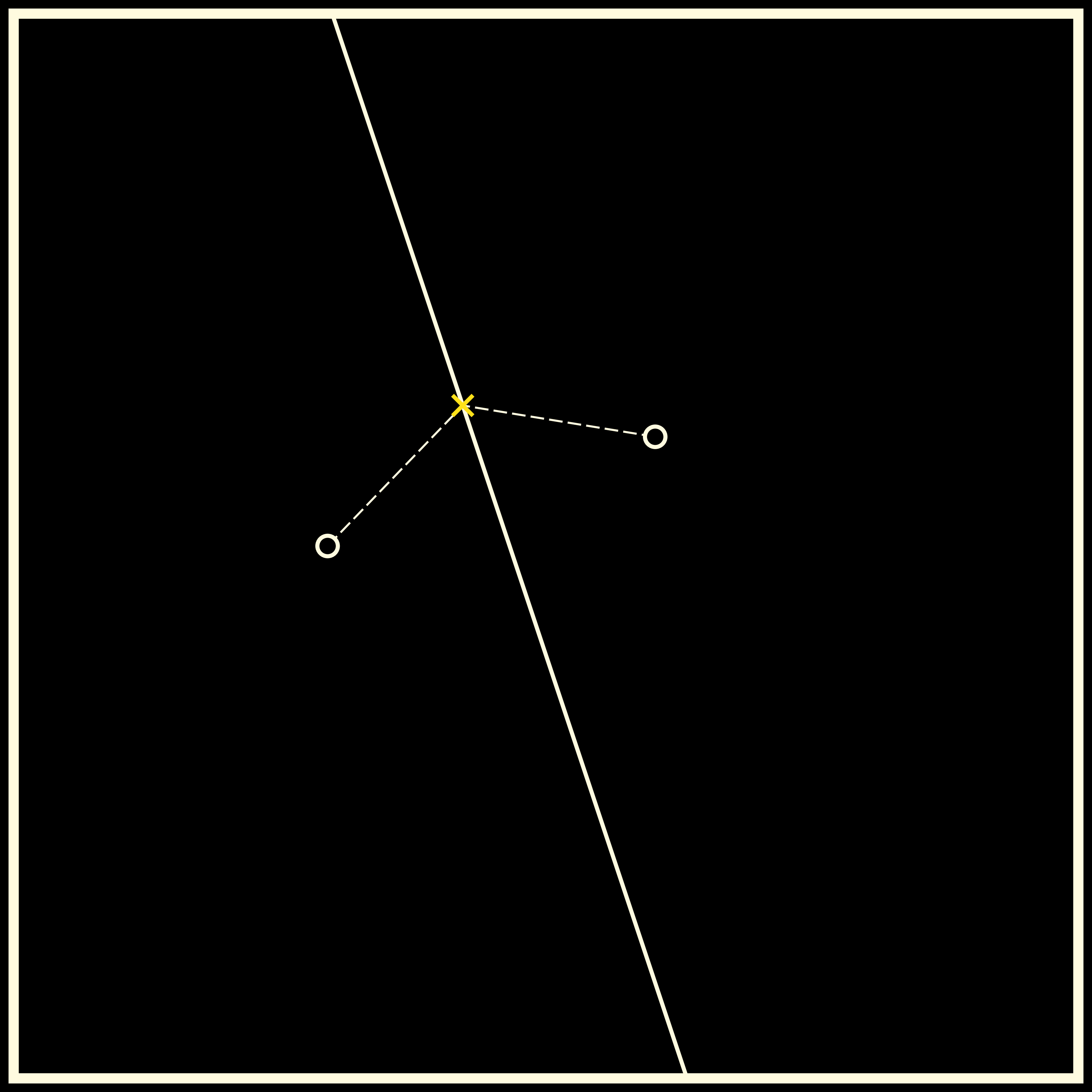
2
4
3

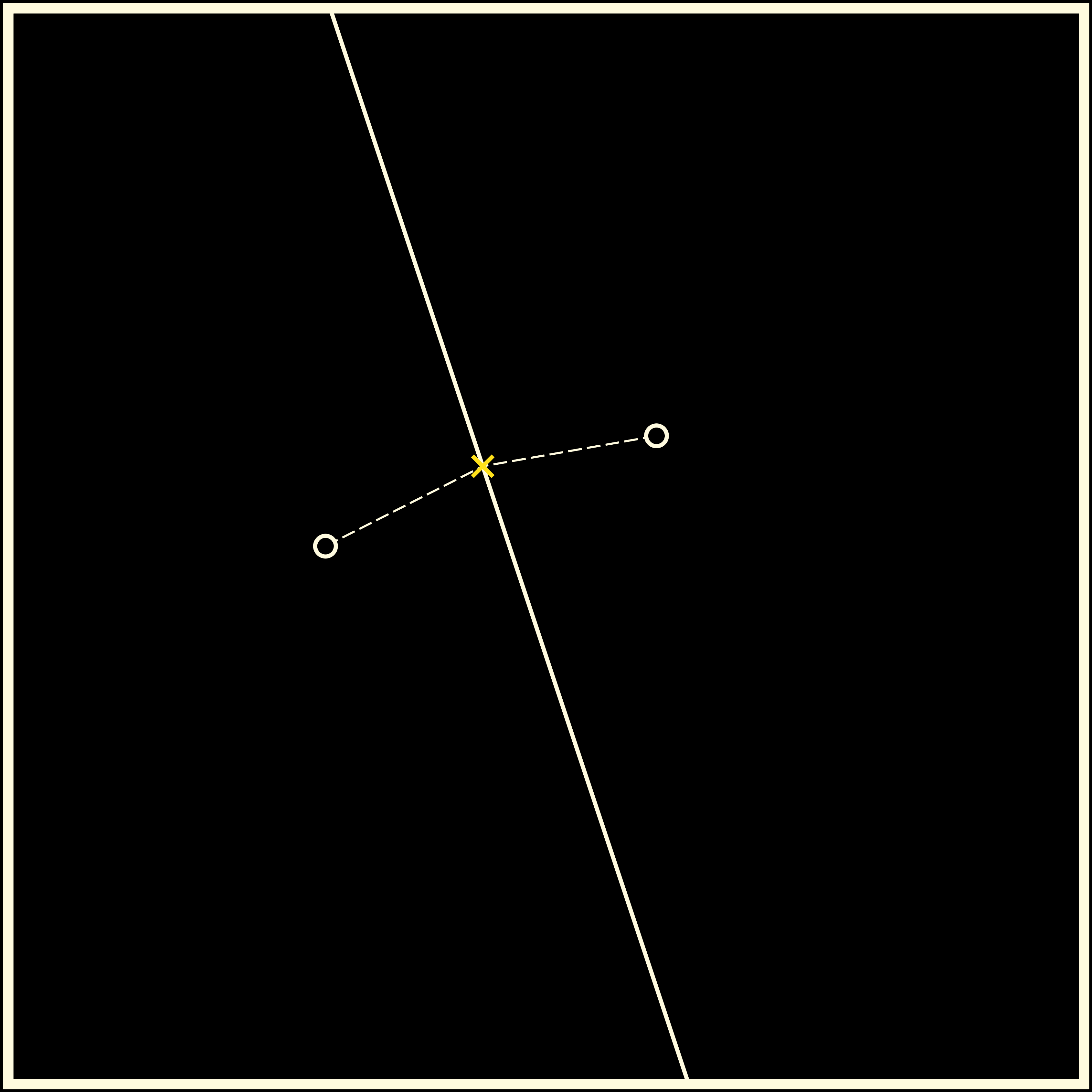

2
4
3
weights

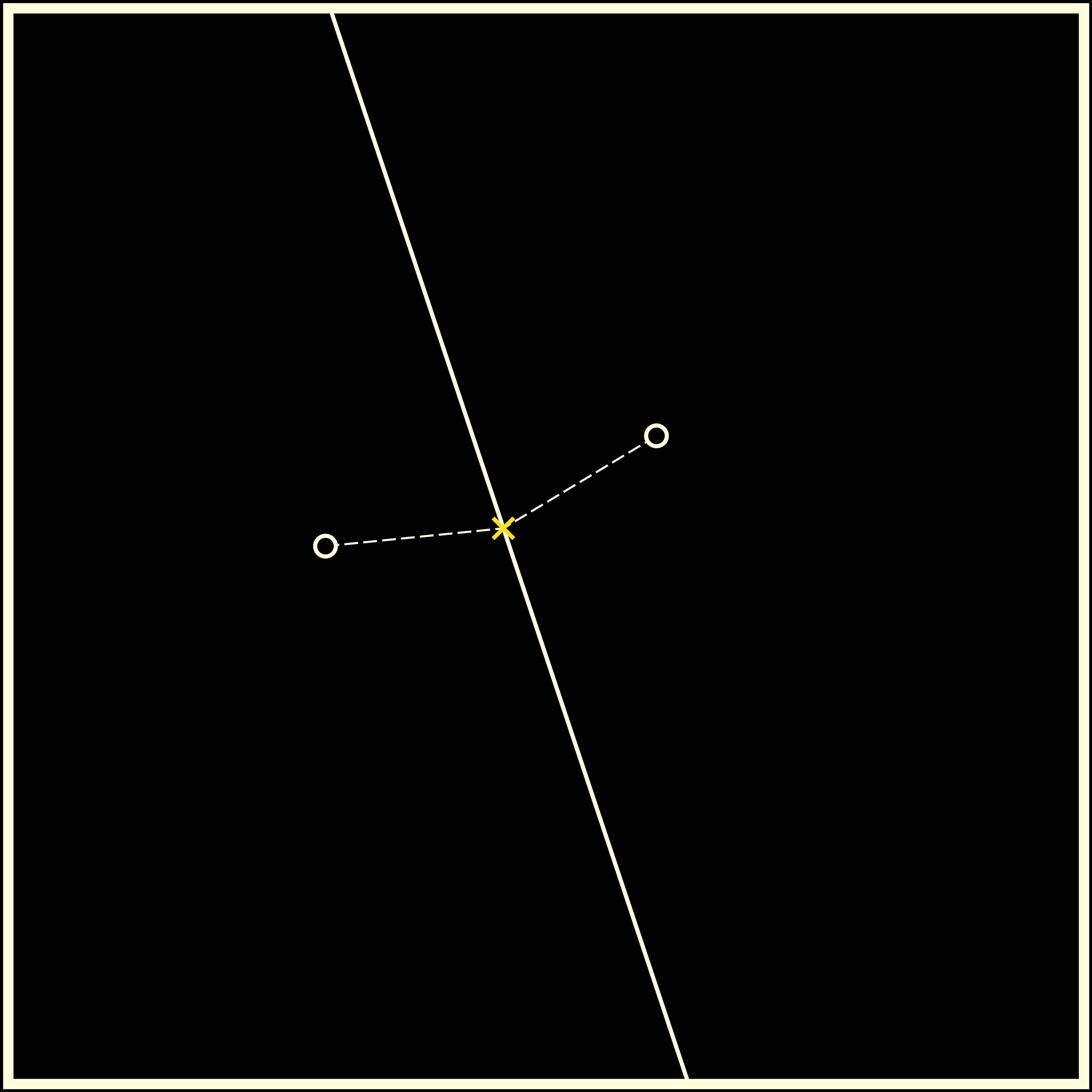


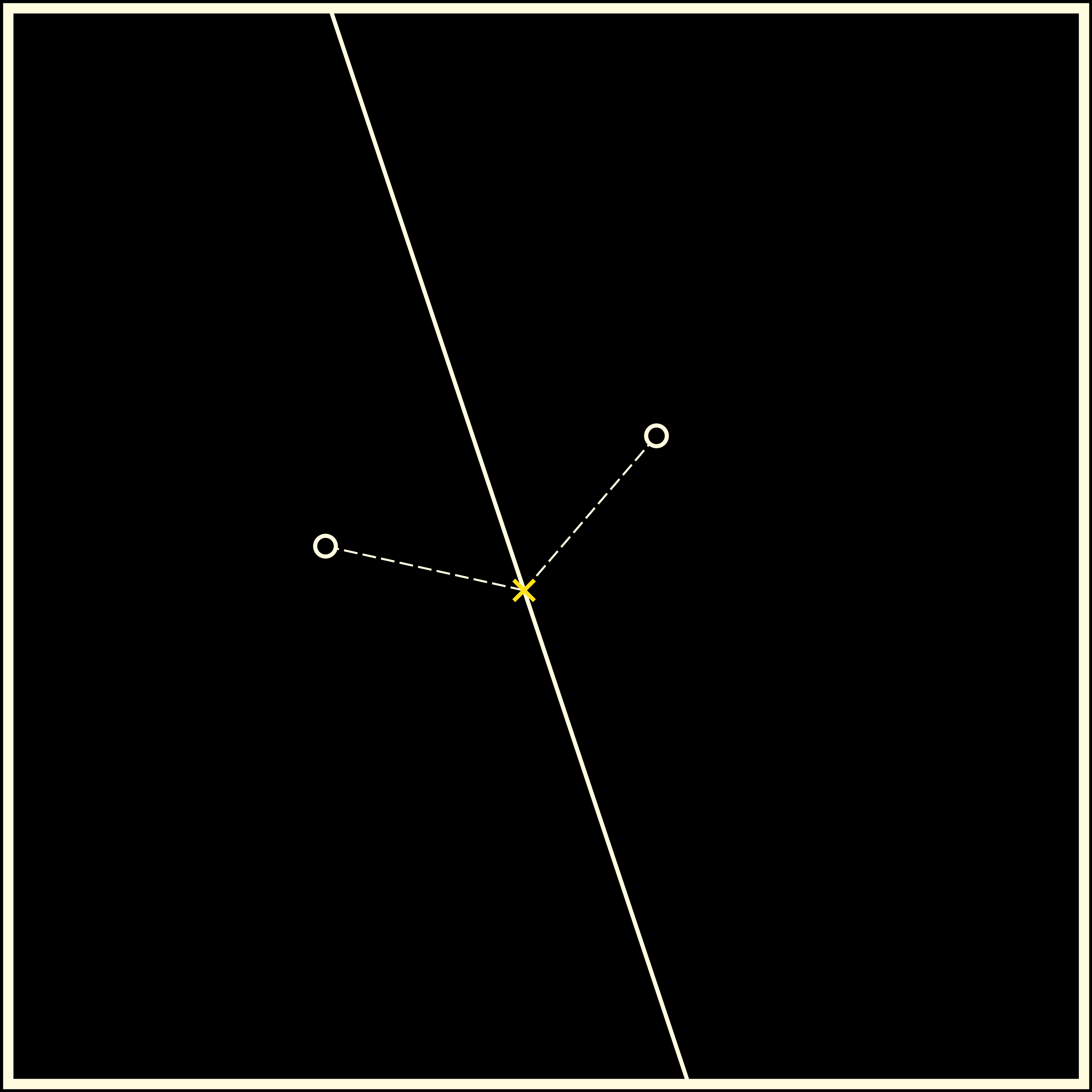


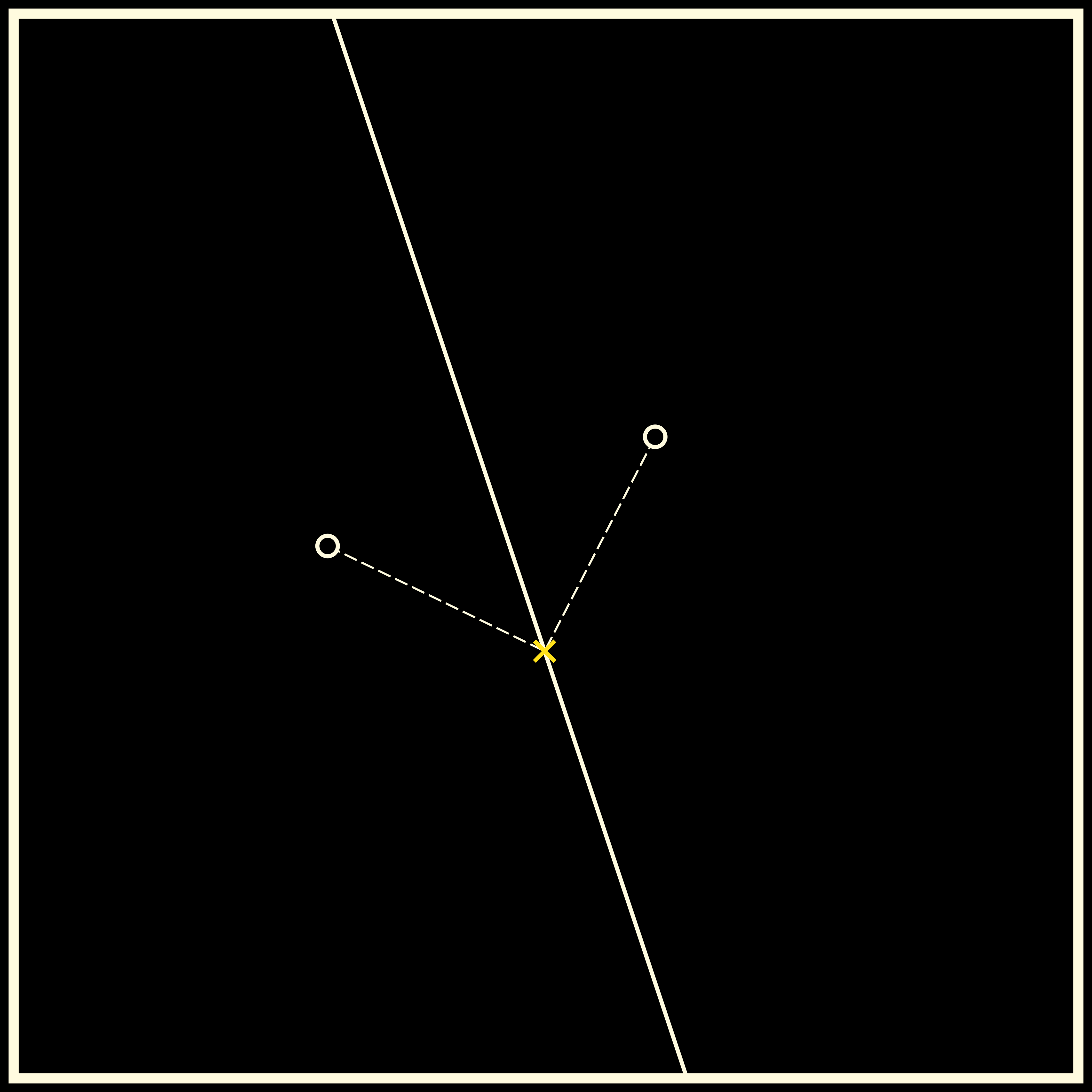


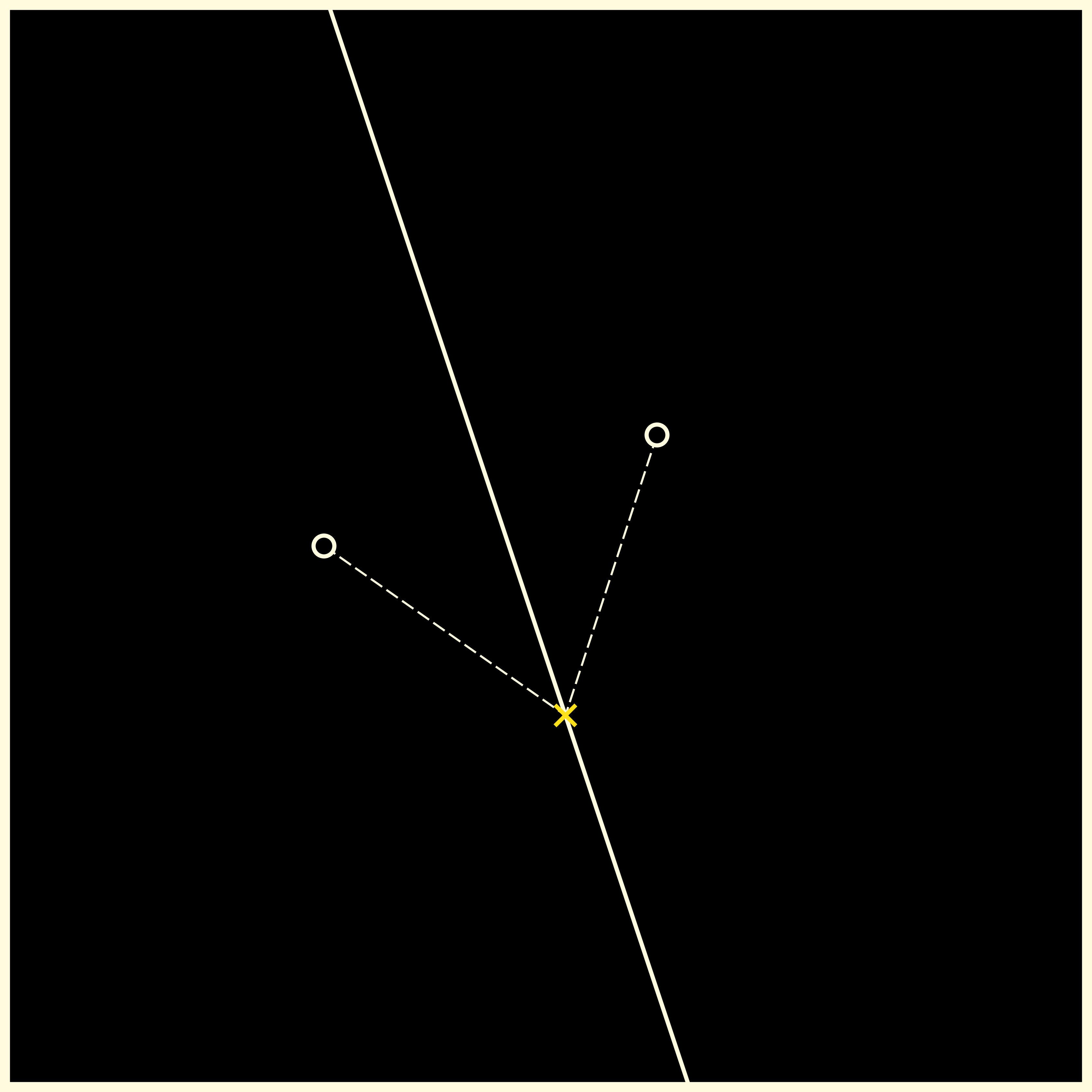


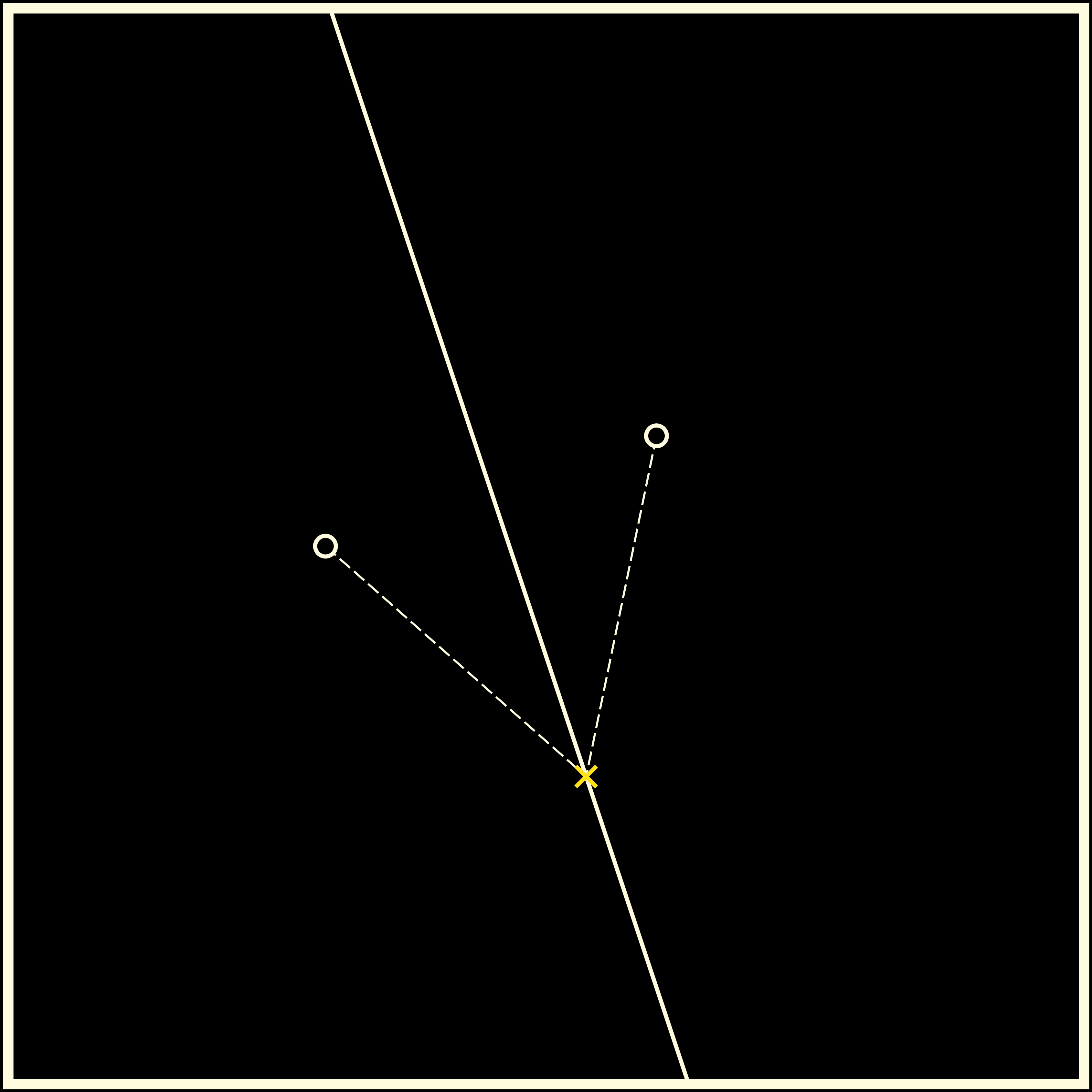


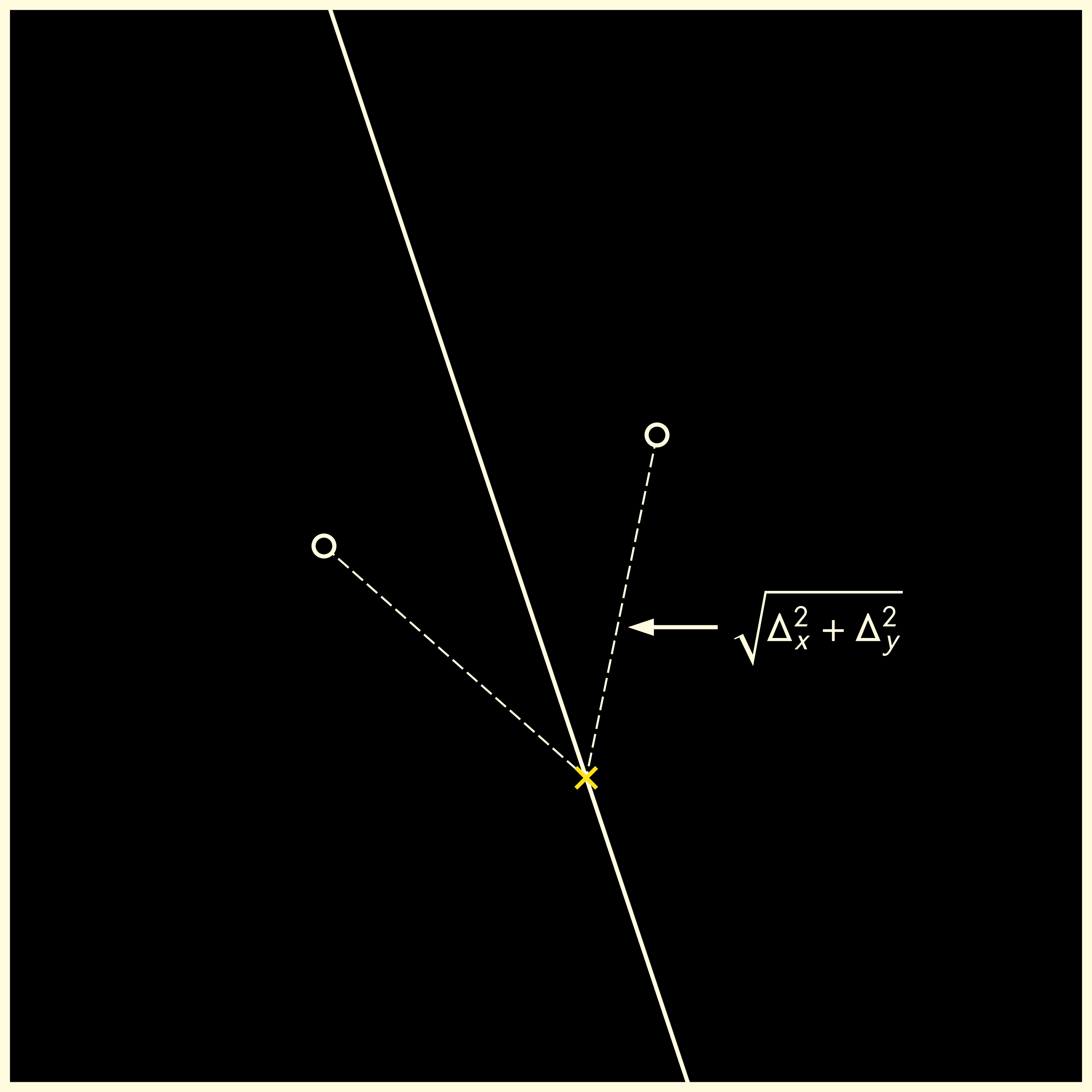


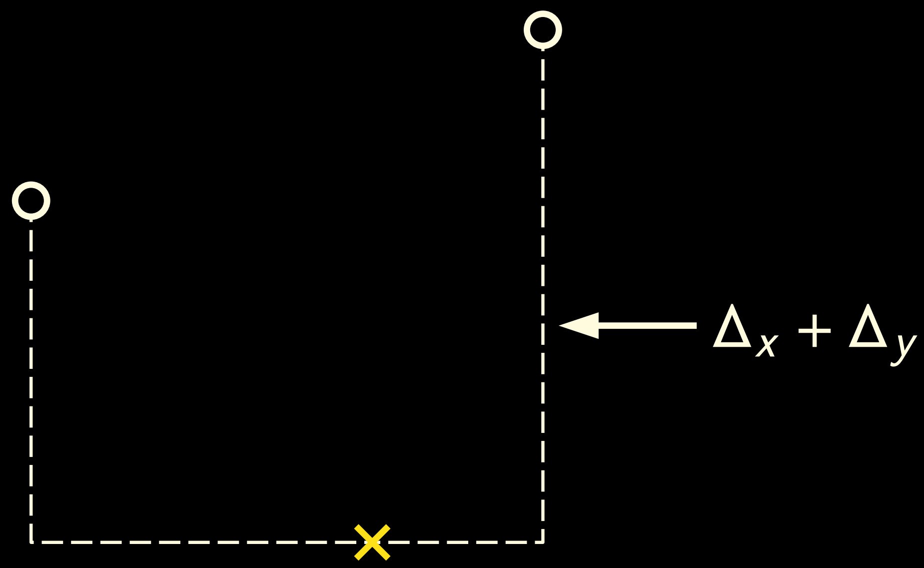


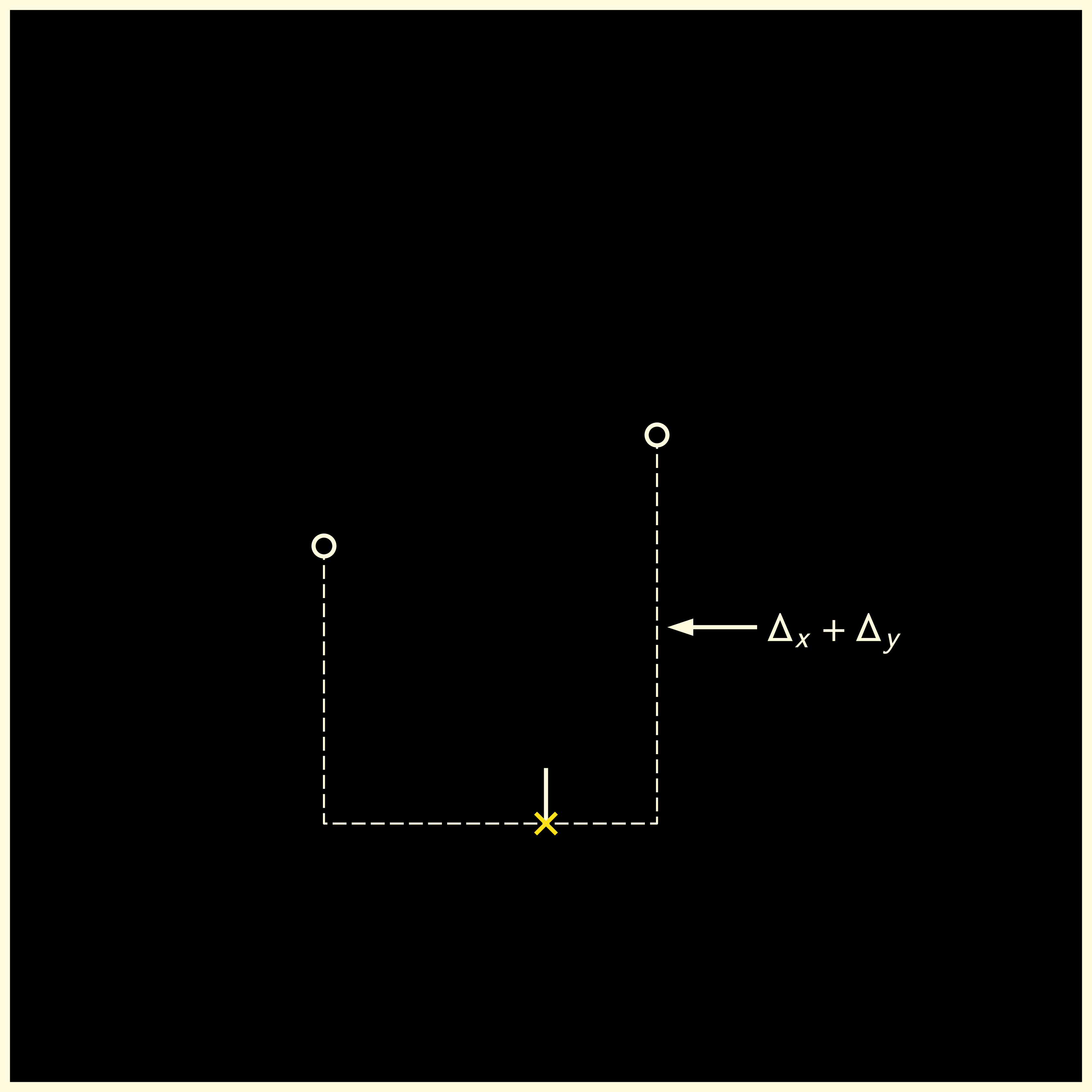




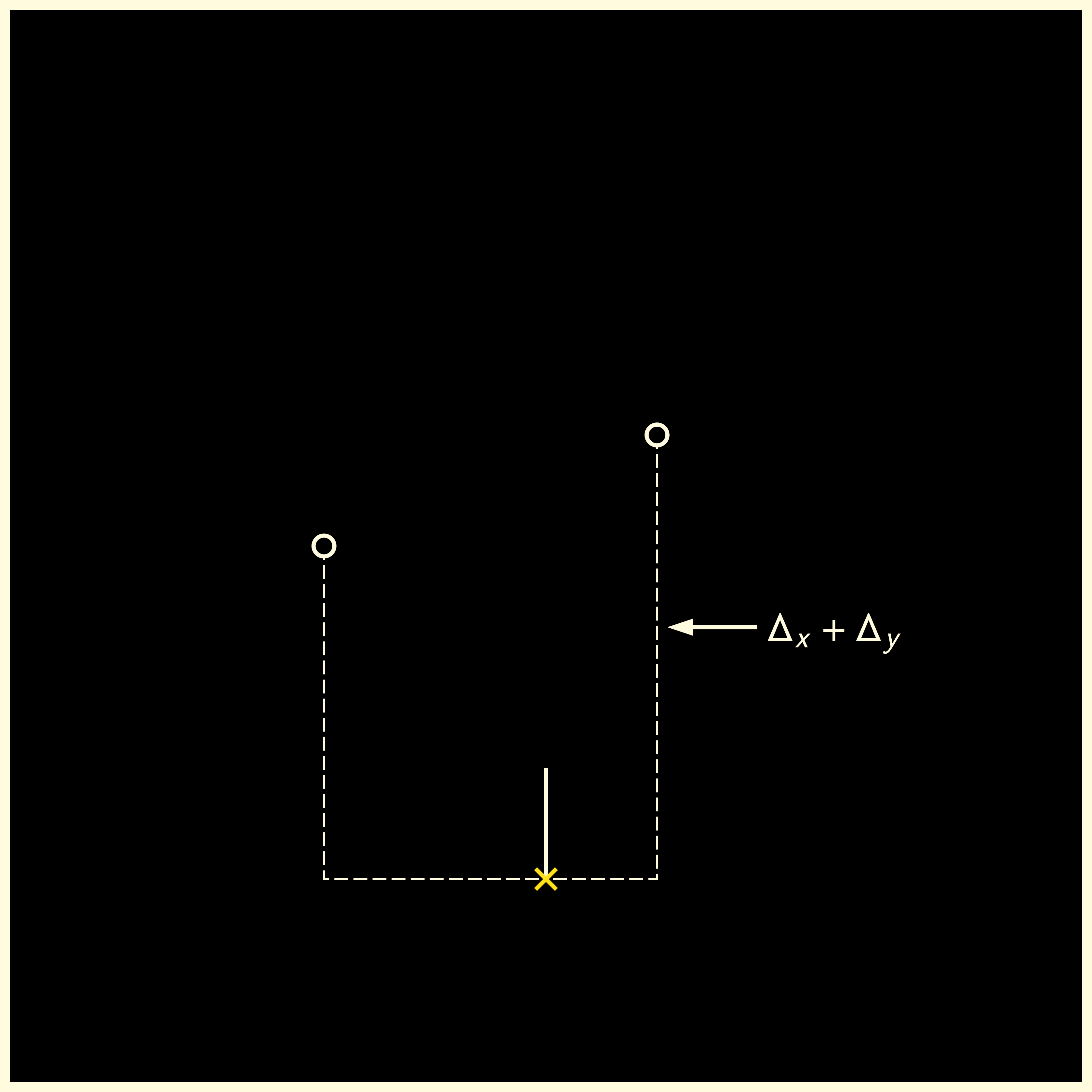


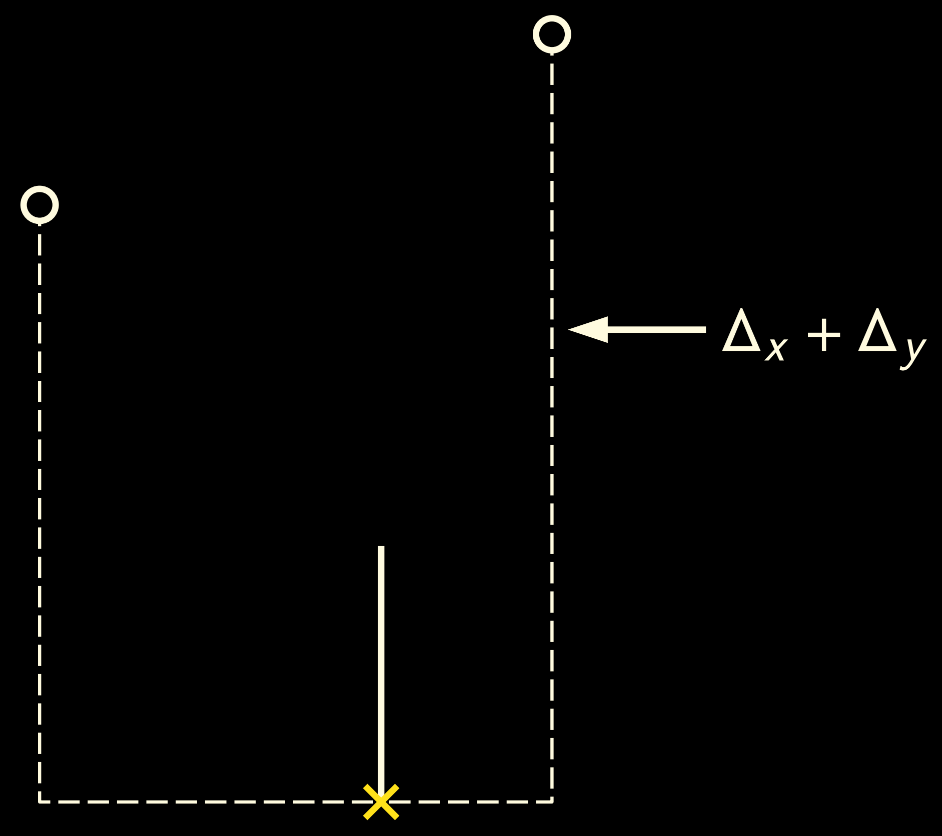

$\sqrt{\Delta_x^2 + \Delta_y^2}$

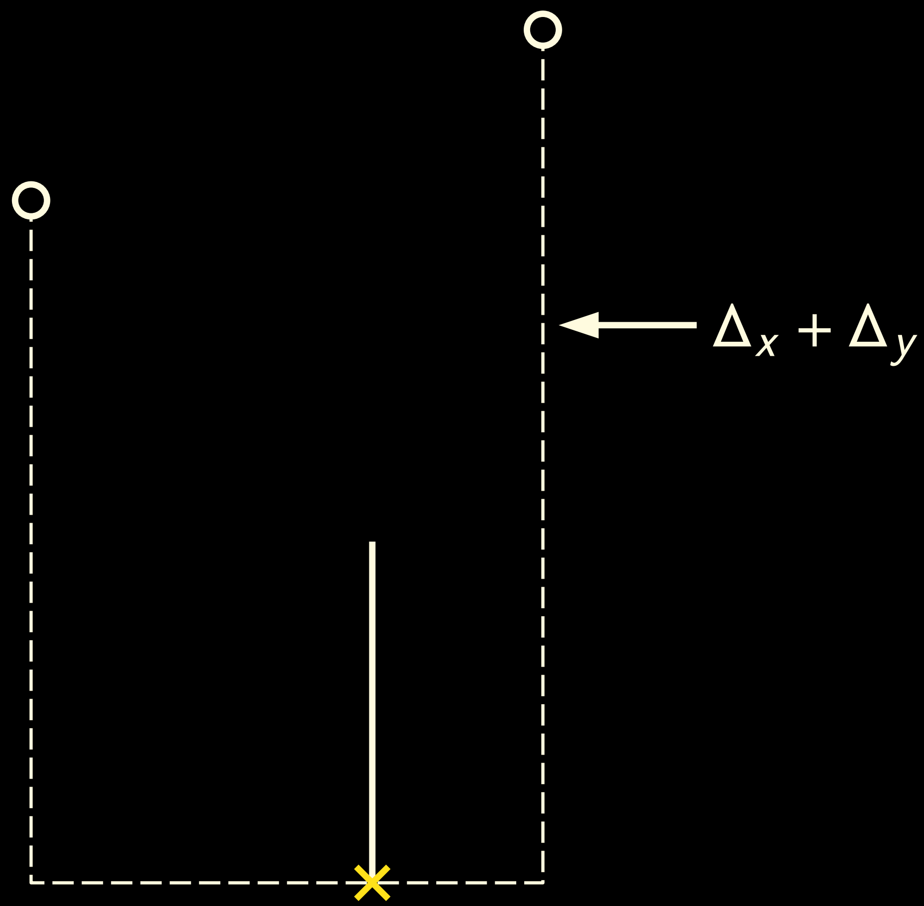

$\Delta_x + \Delta_y$



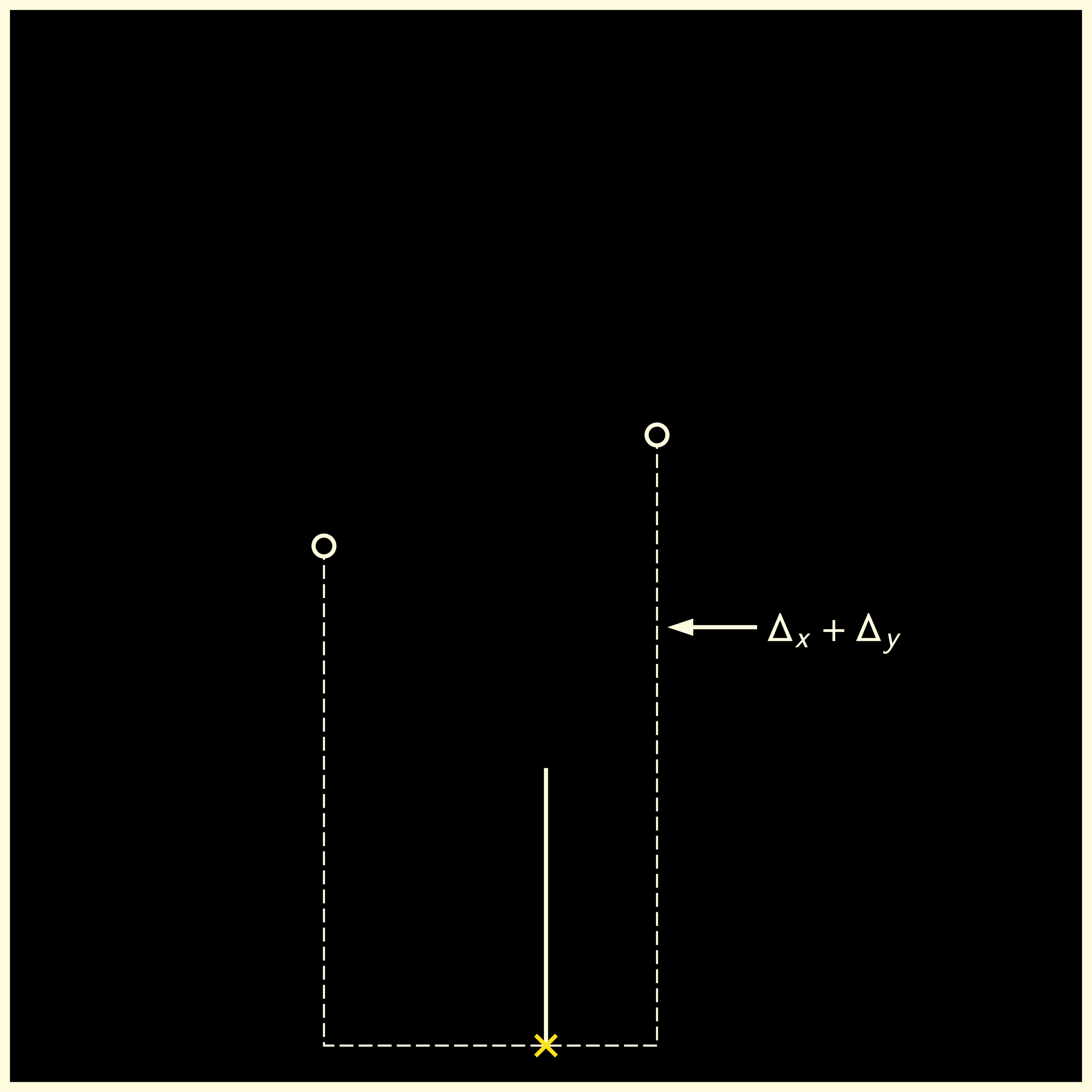
$\Delta_x + \Delta_y$

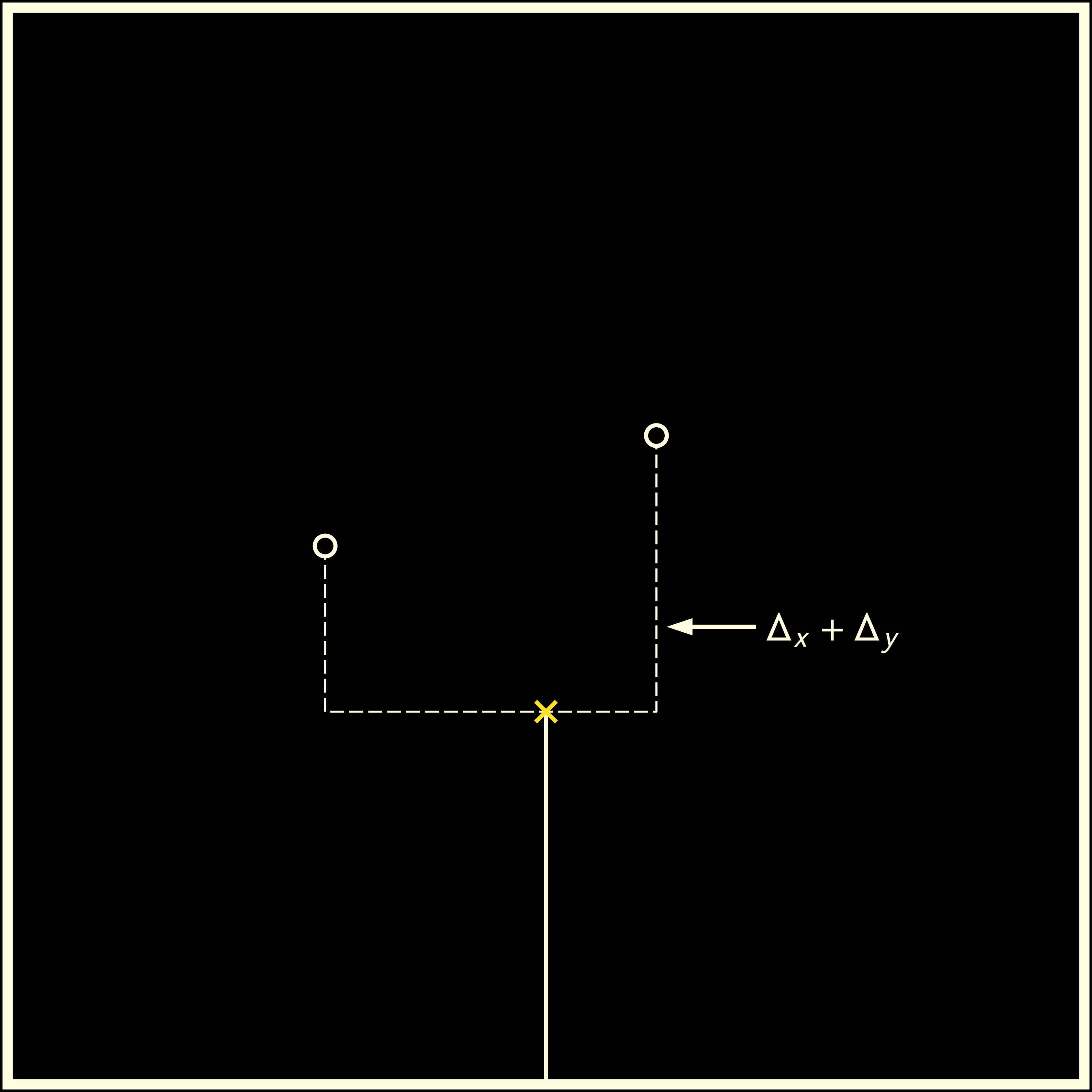

$\Delta_x + \Delta_y$

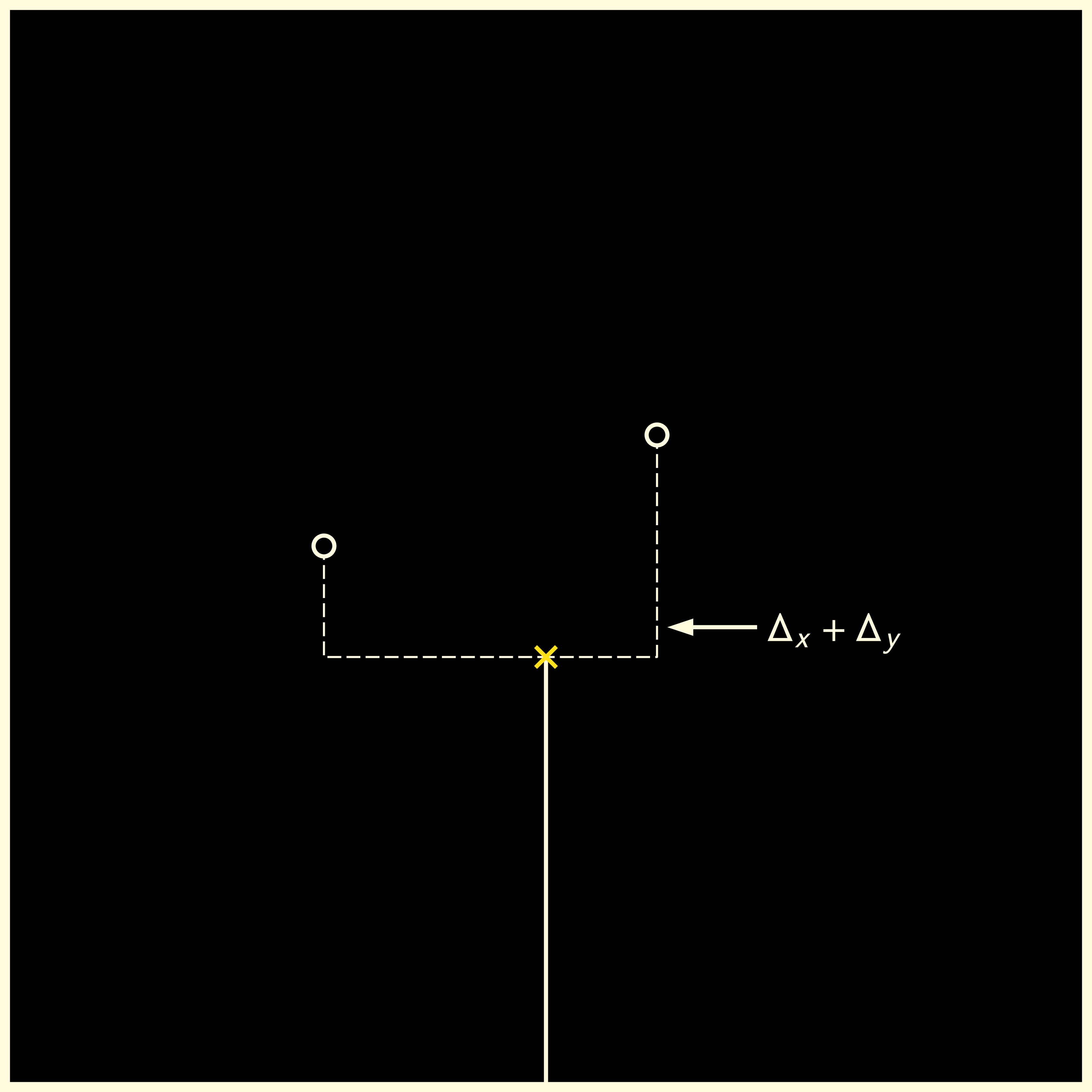

$\Delta_x + \Delta_y$

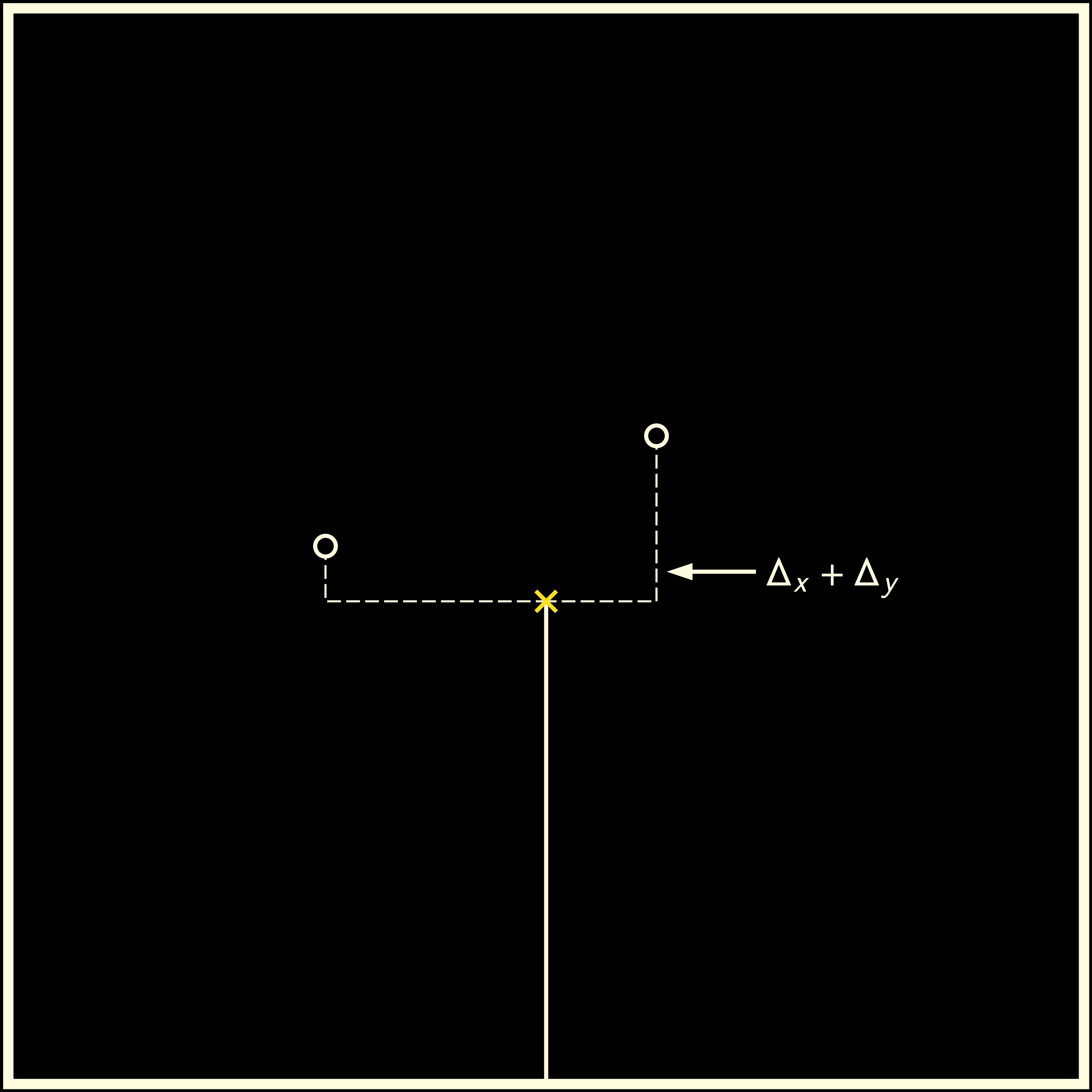


$\Delta_x + \Delta_y$

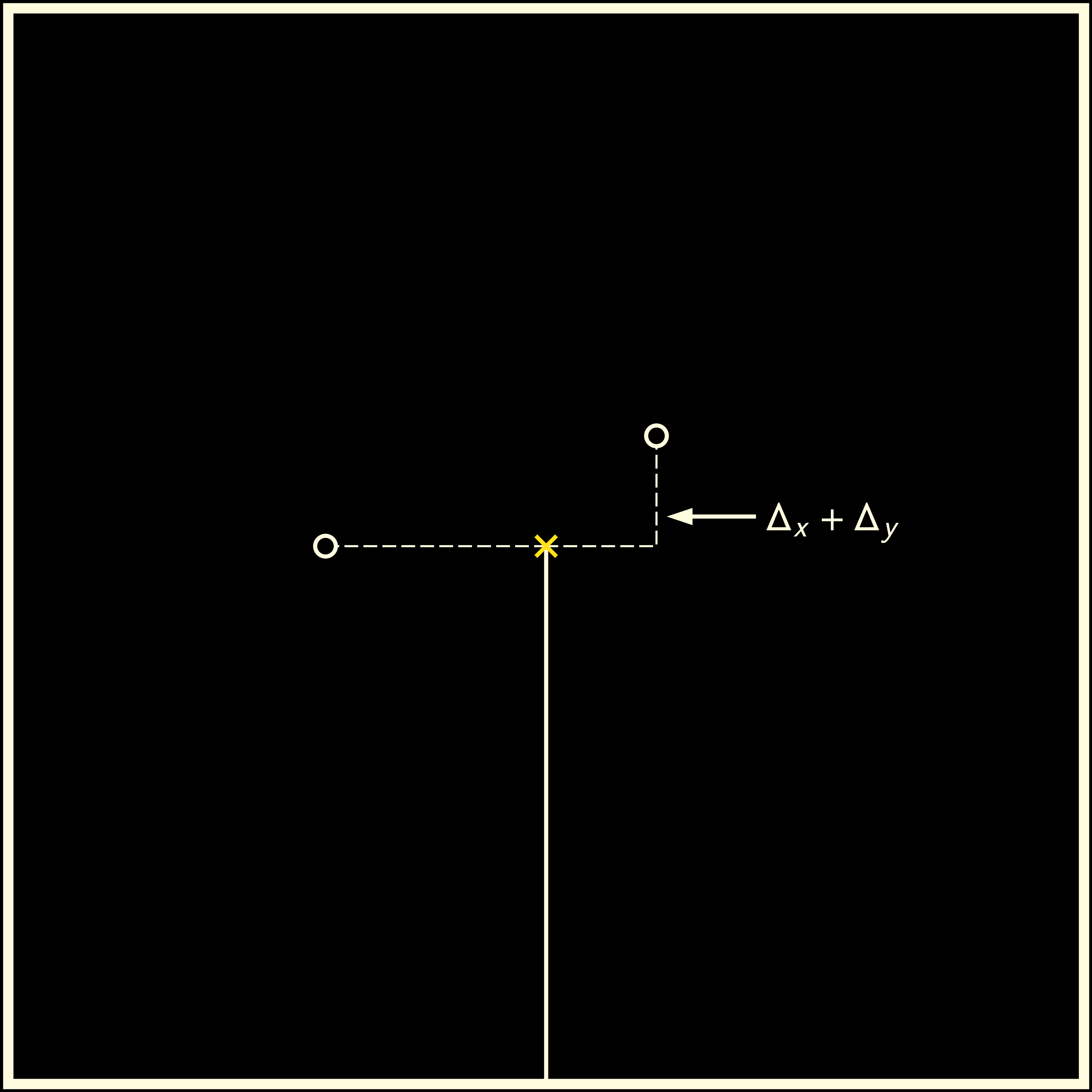

$\Delta_x + \Delta_y$

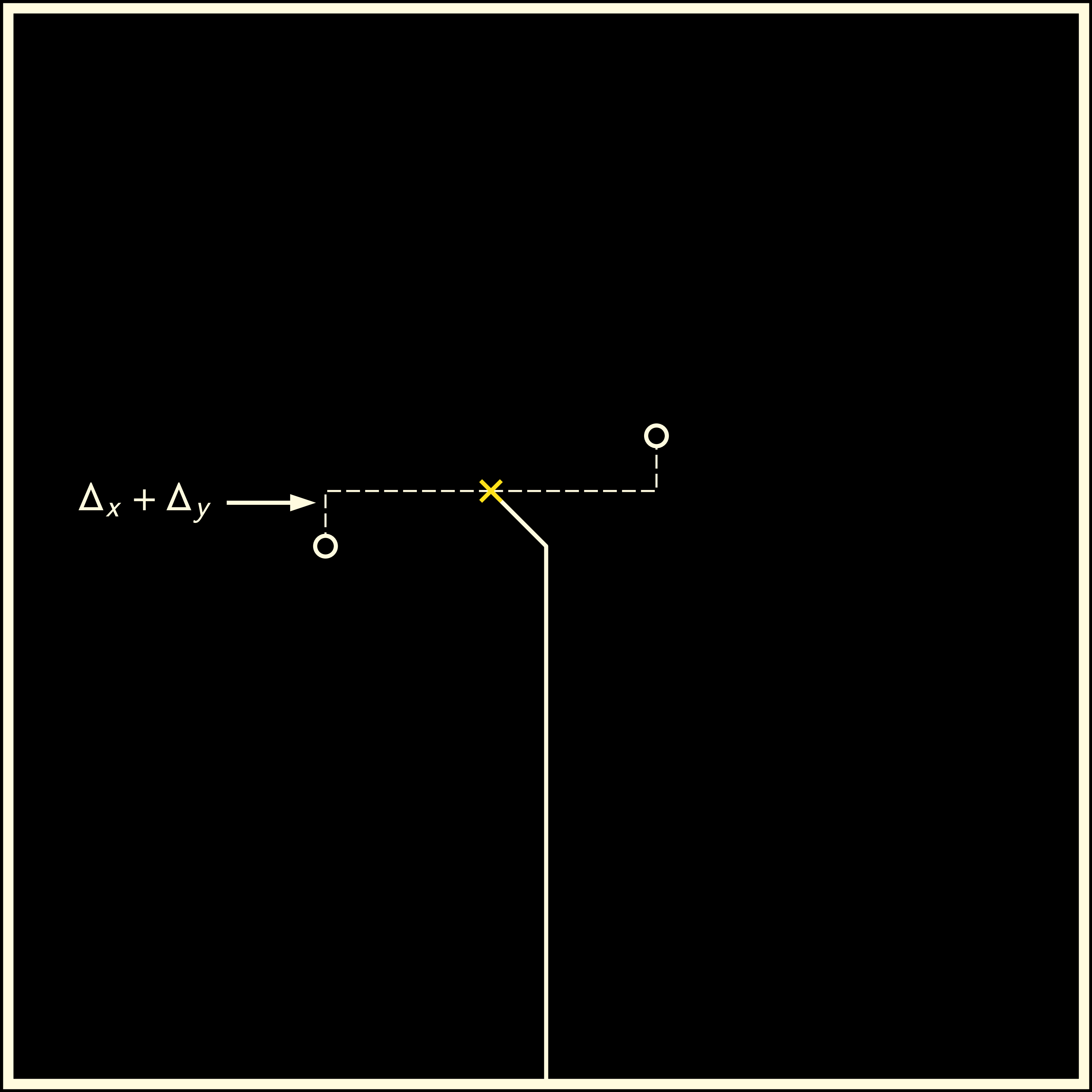

Δx + Δy

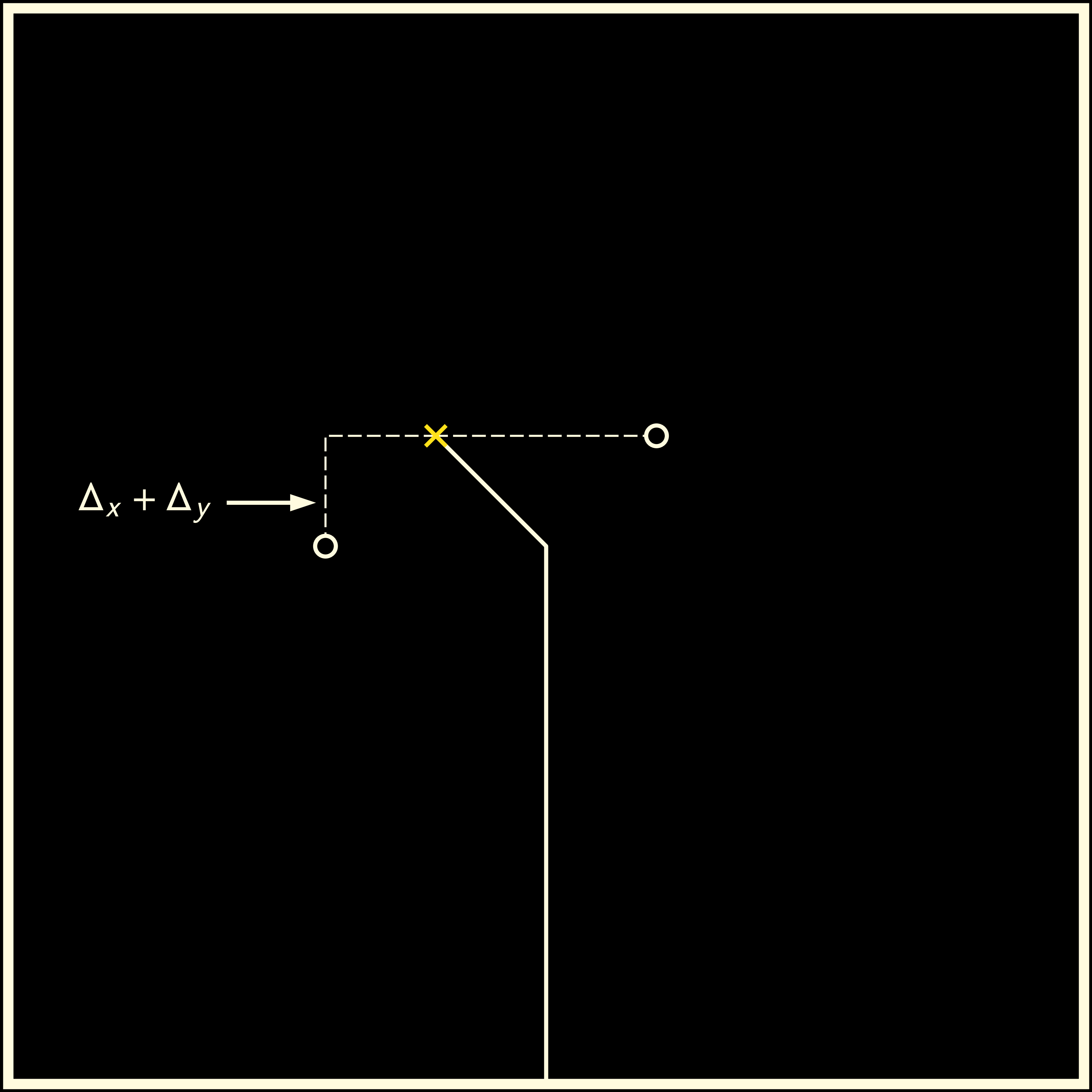

$\Delta_x + \Delta_y$



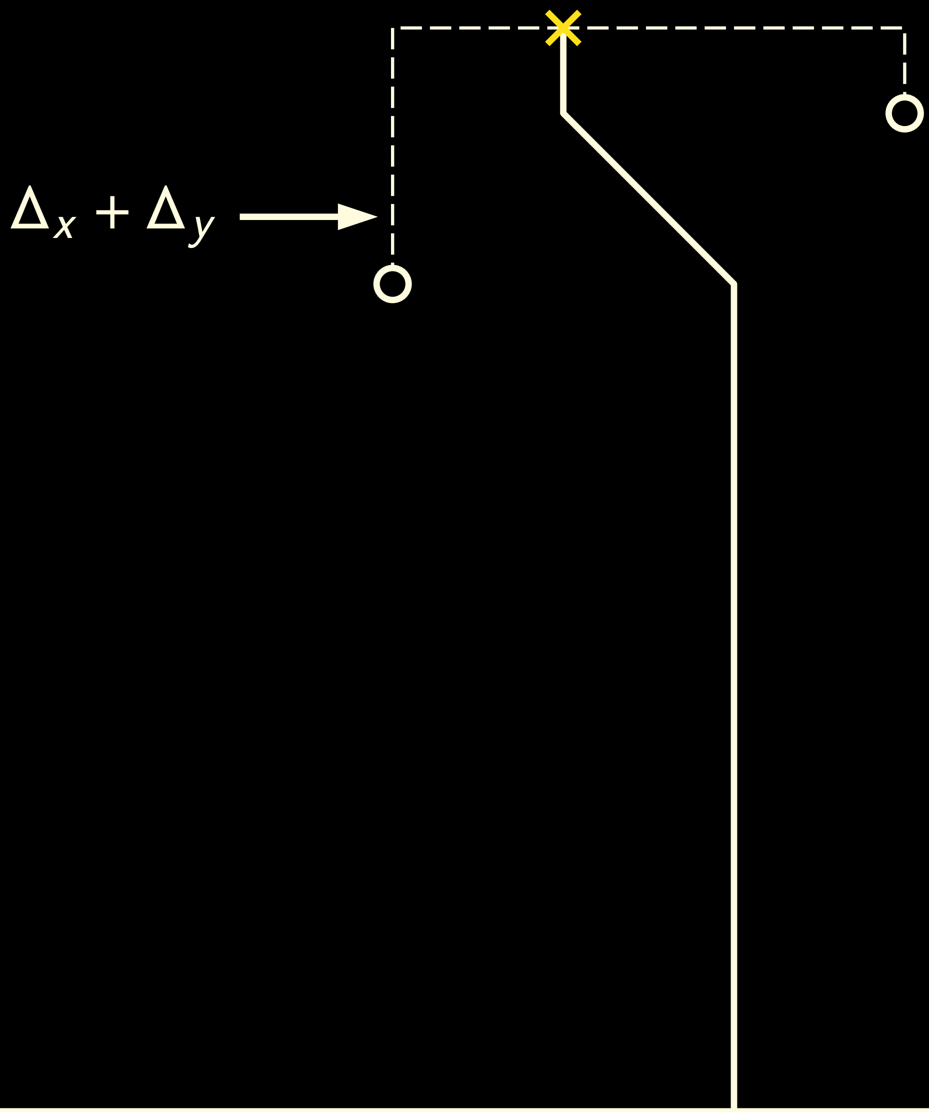
$\Delta_x + \Delta_y$

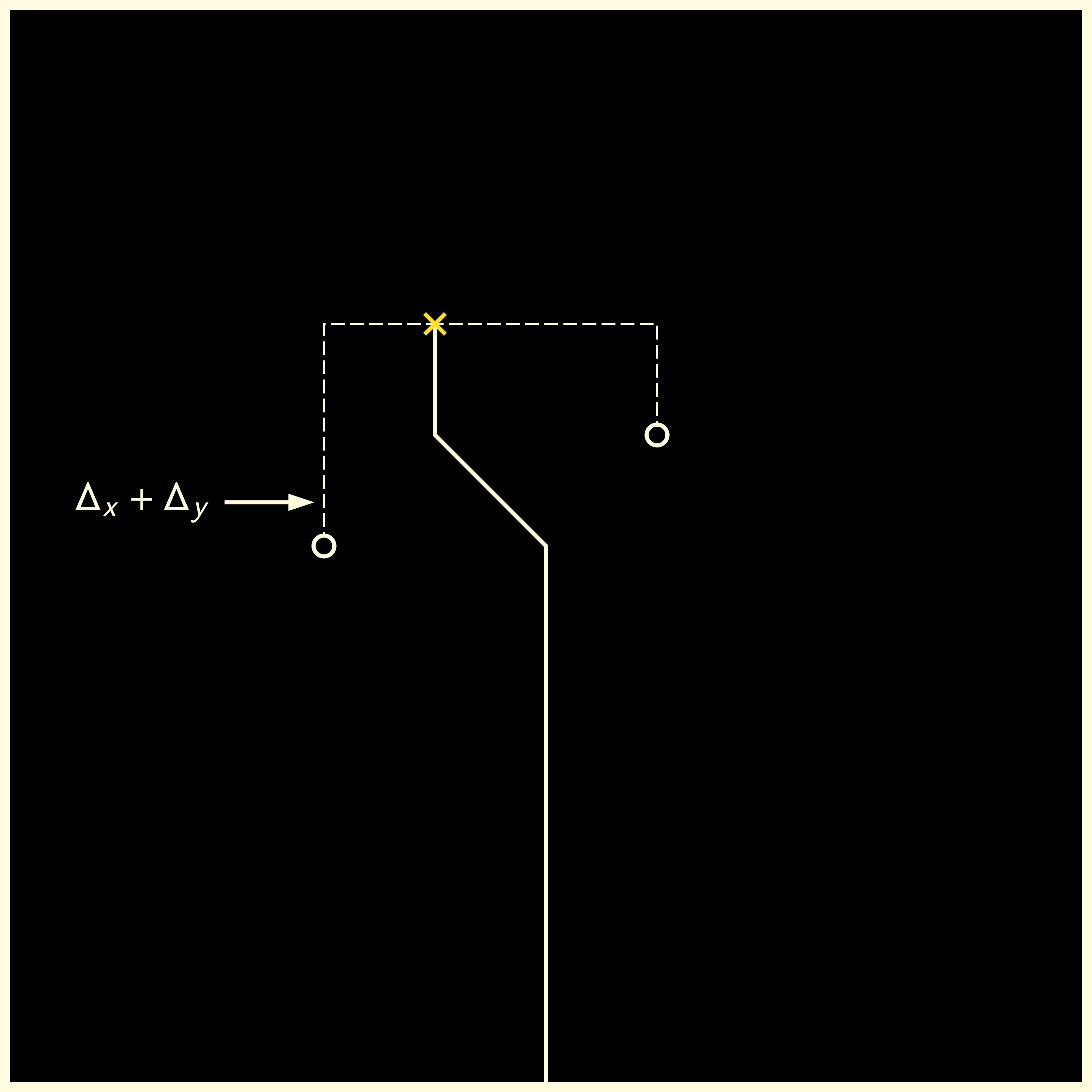

$\Delta_x + \Delta_y$

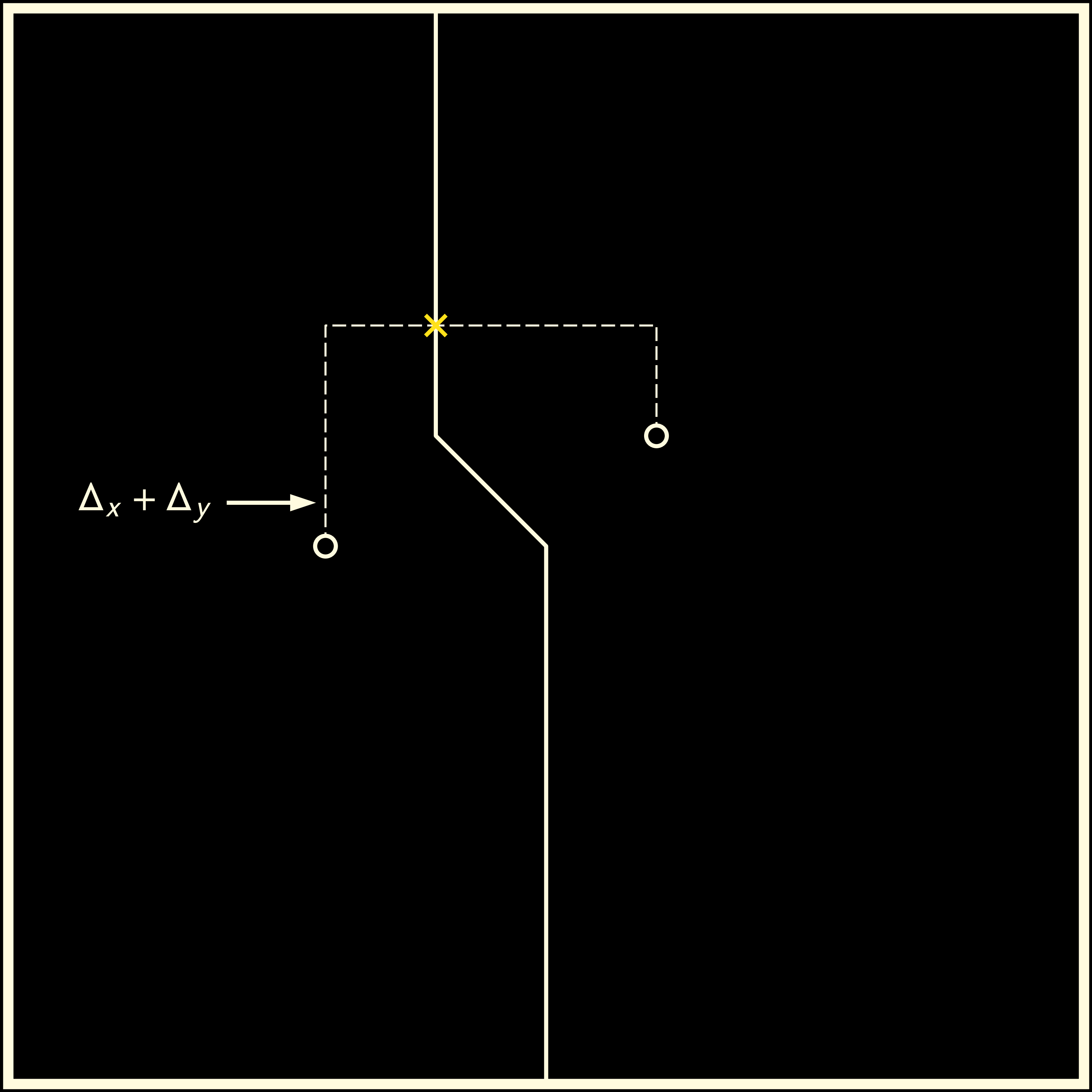

$\Delta_x + \Delta_y$

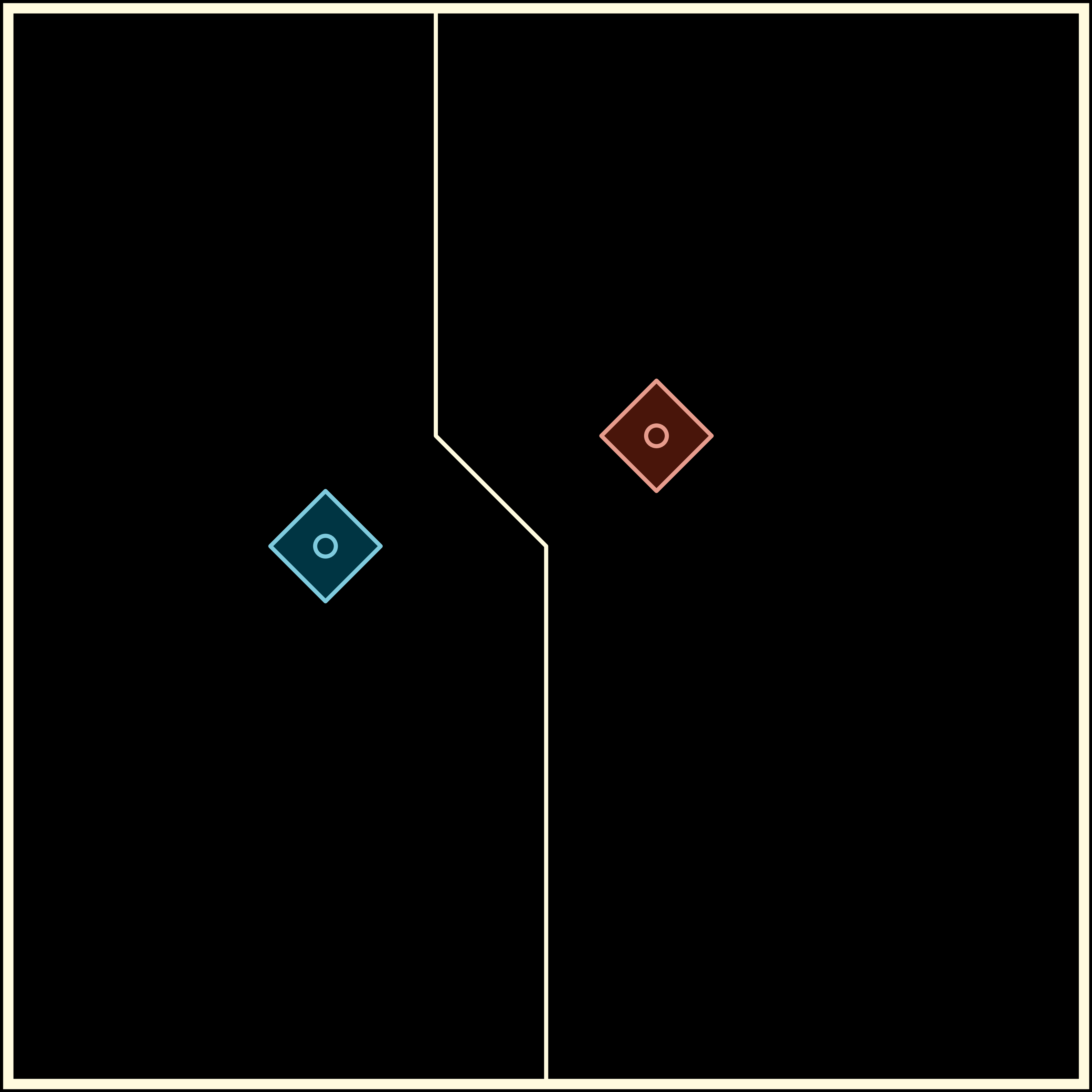


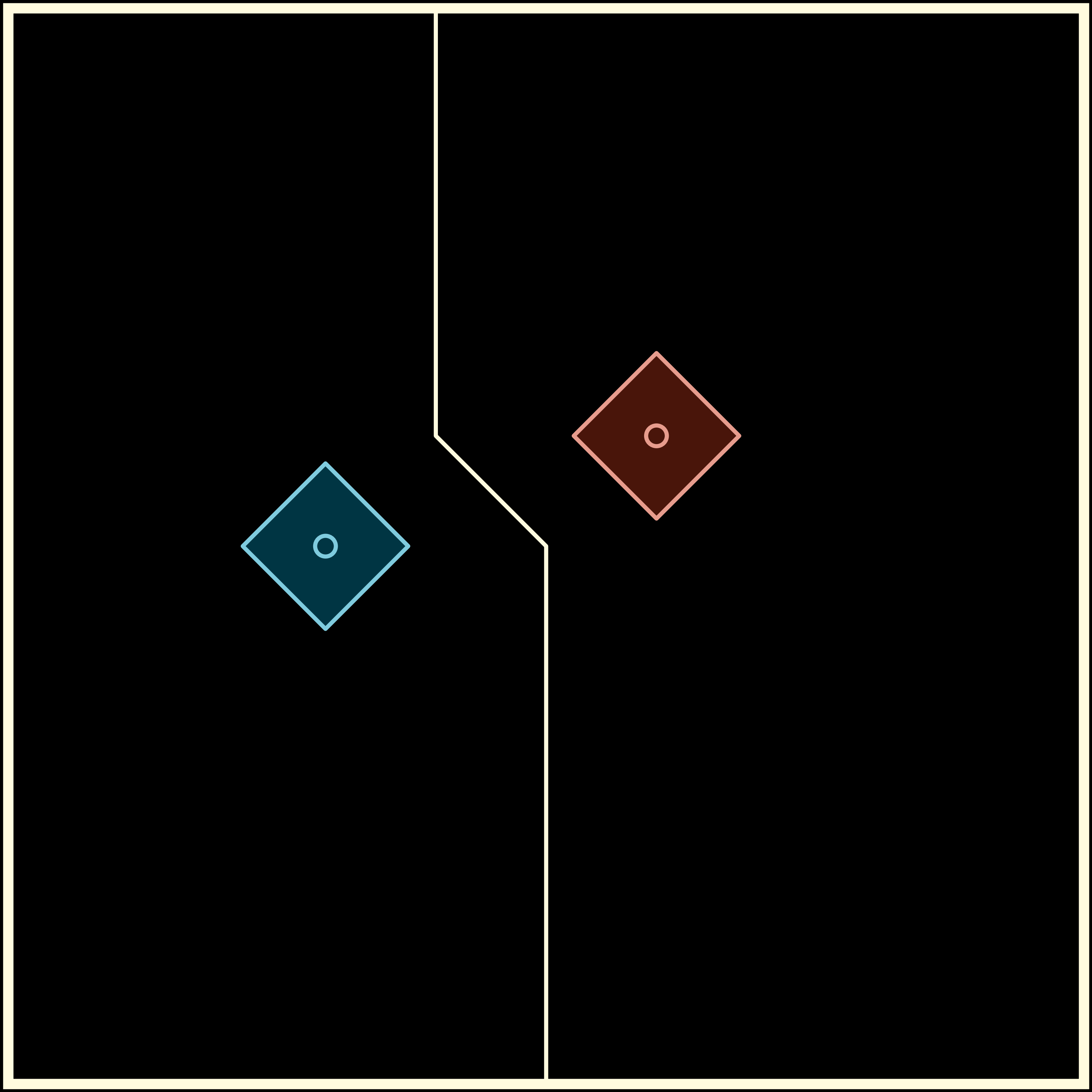
$\Delta_x + \Delta_y$

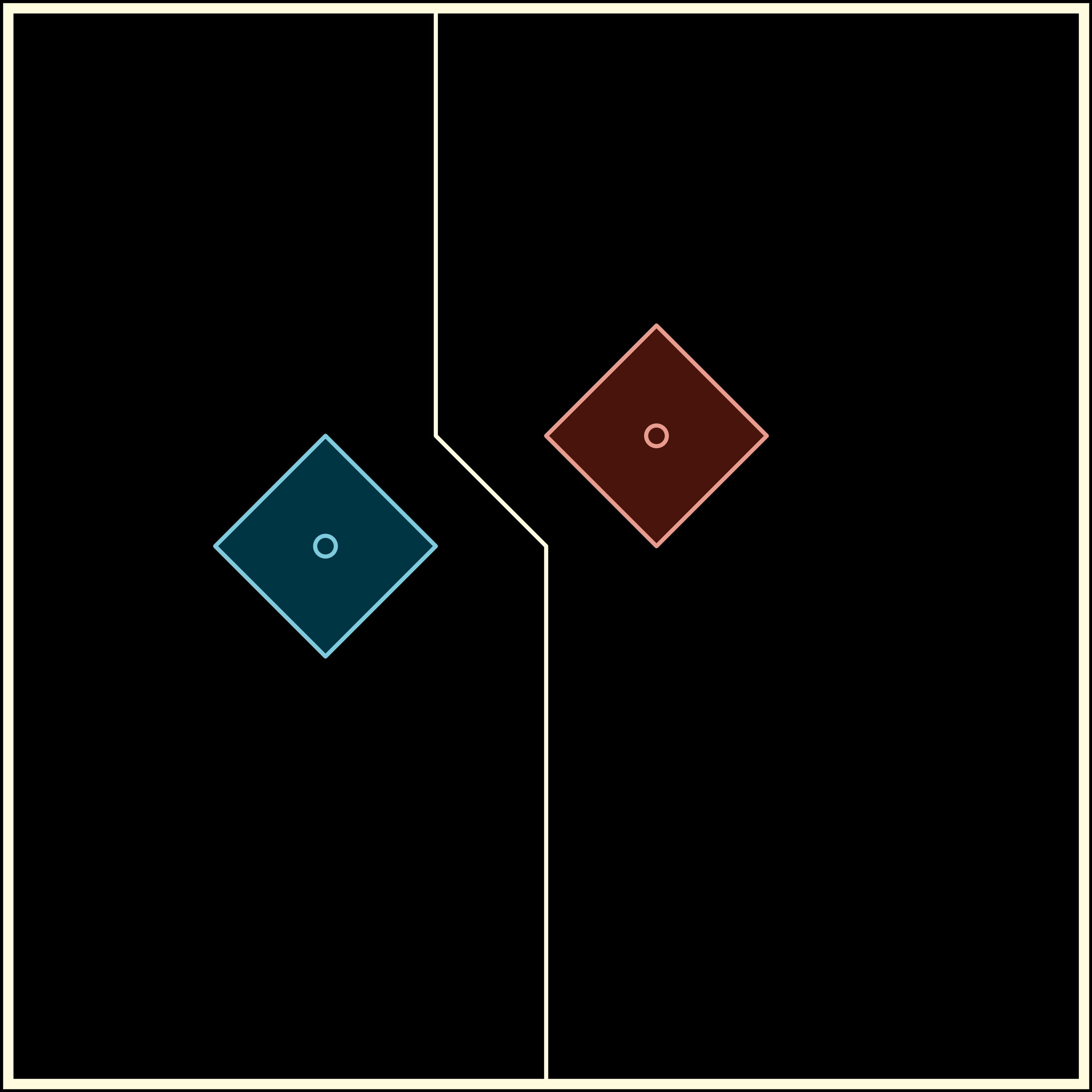

$\Delta_x + \Delta_y$

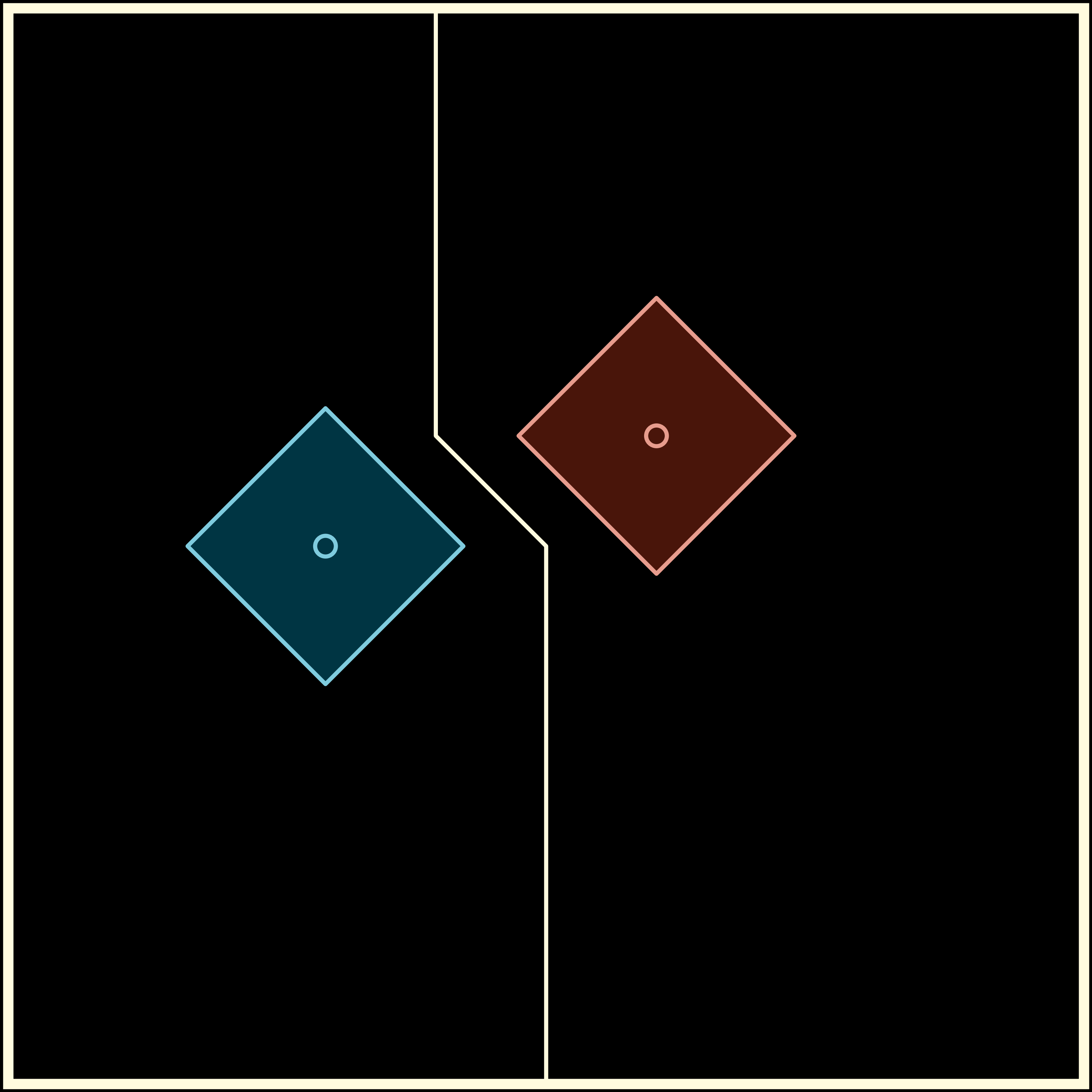

$\Delta_x + \Delta_y$

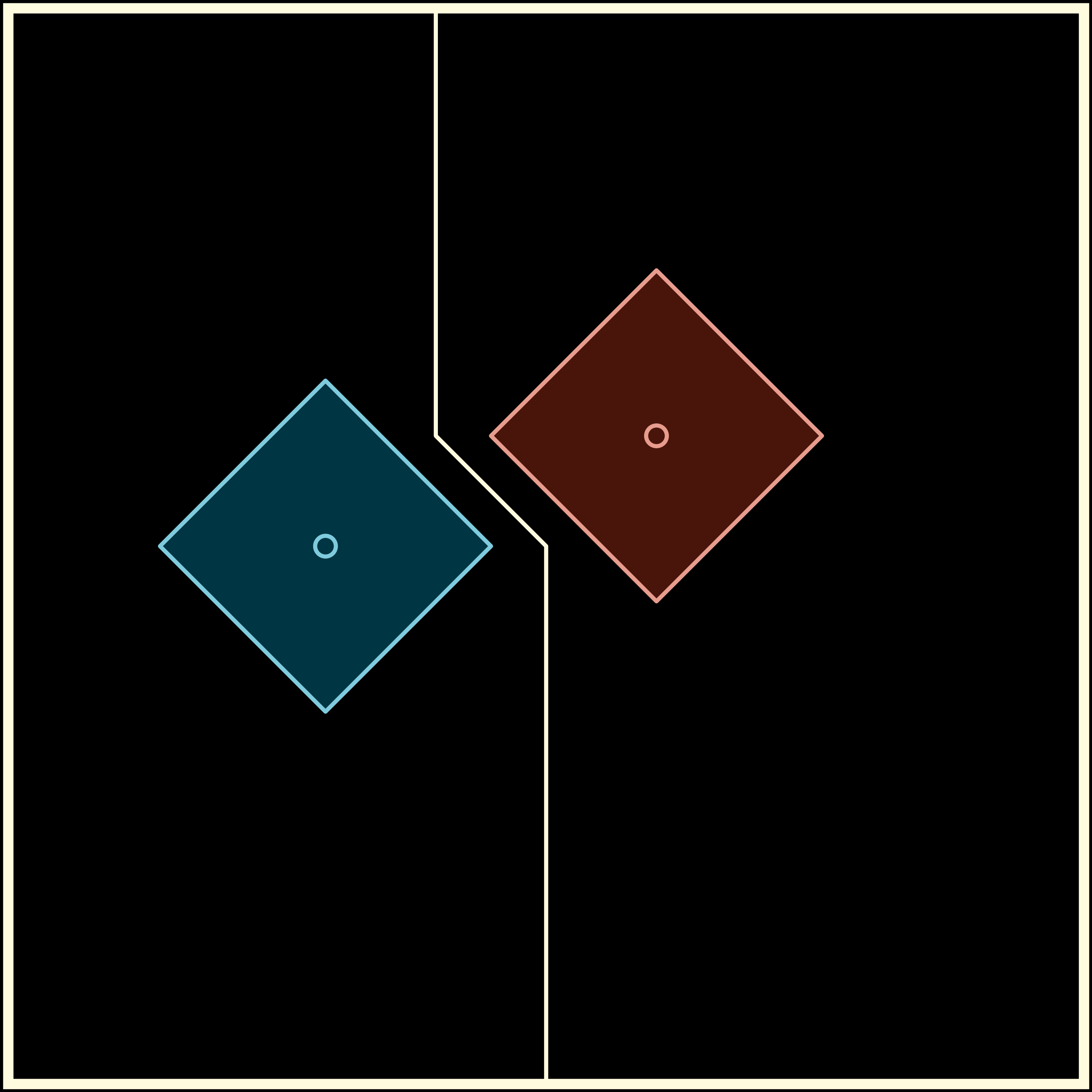

$\Delta_x + \Delta_y$

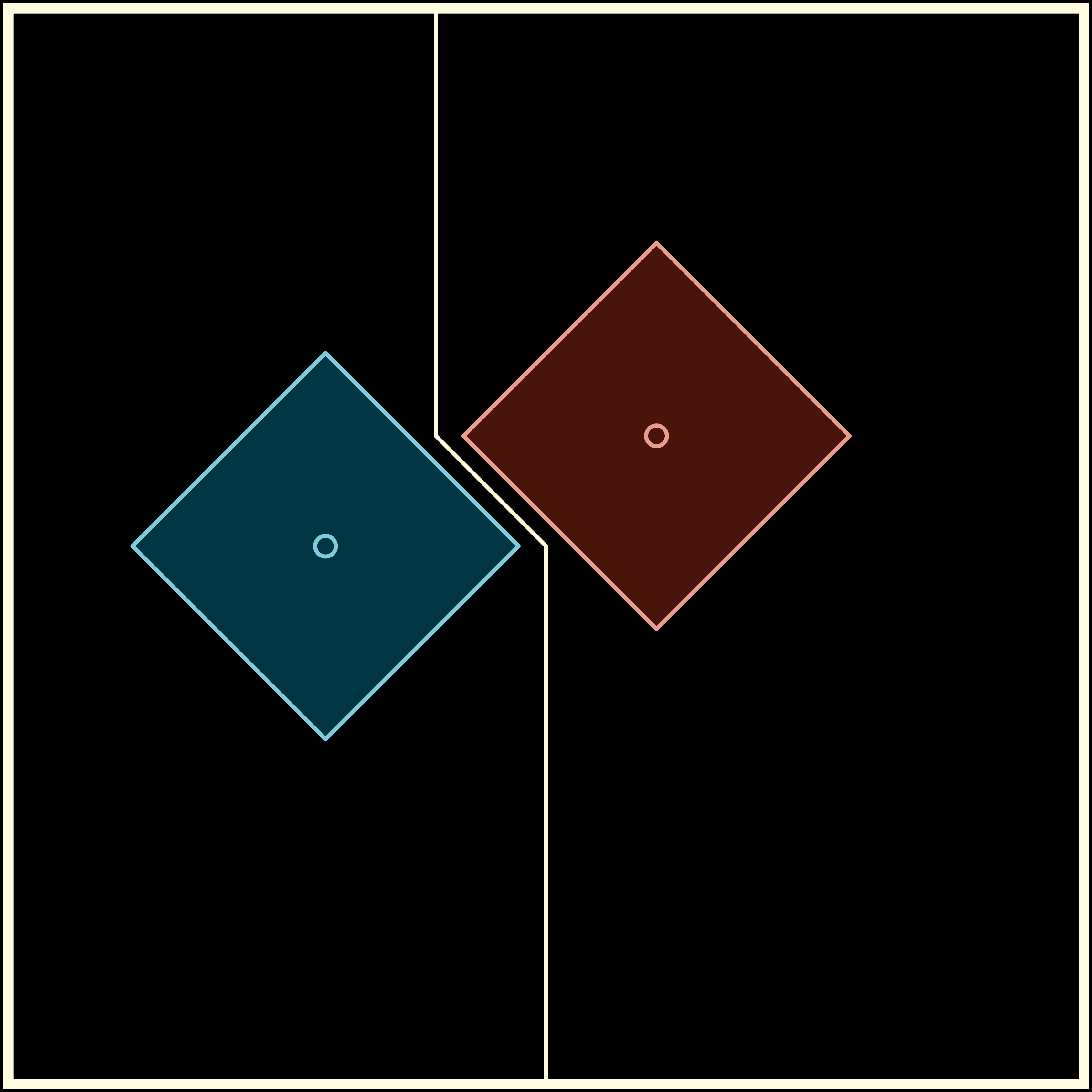


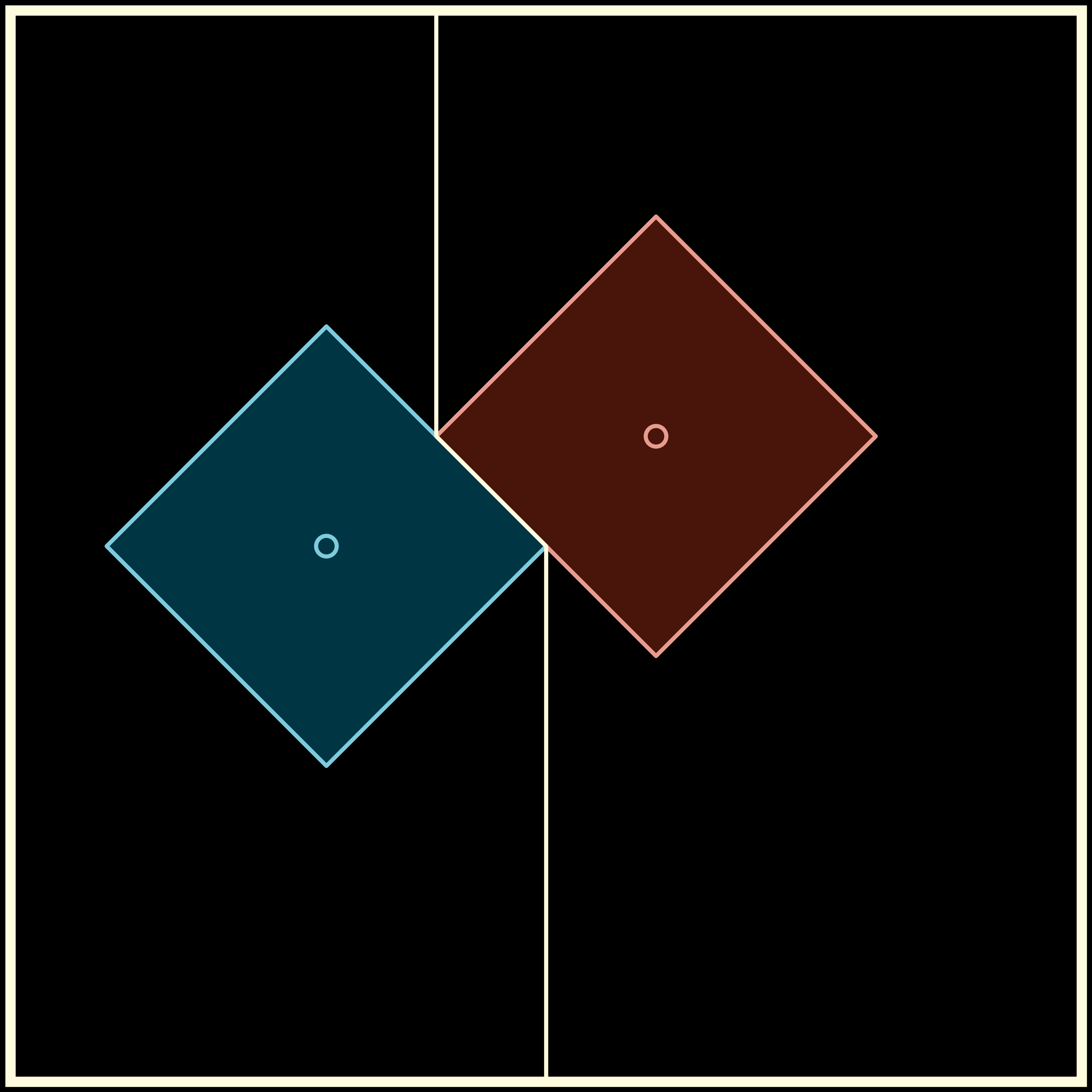


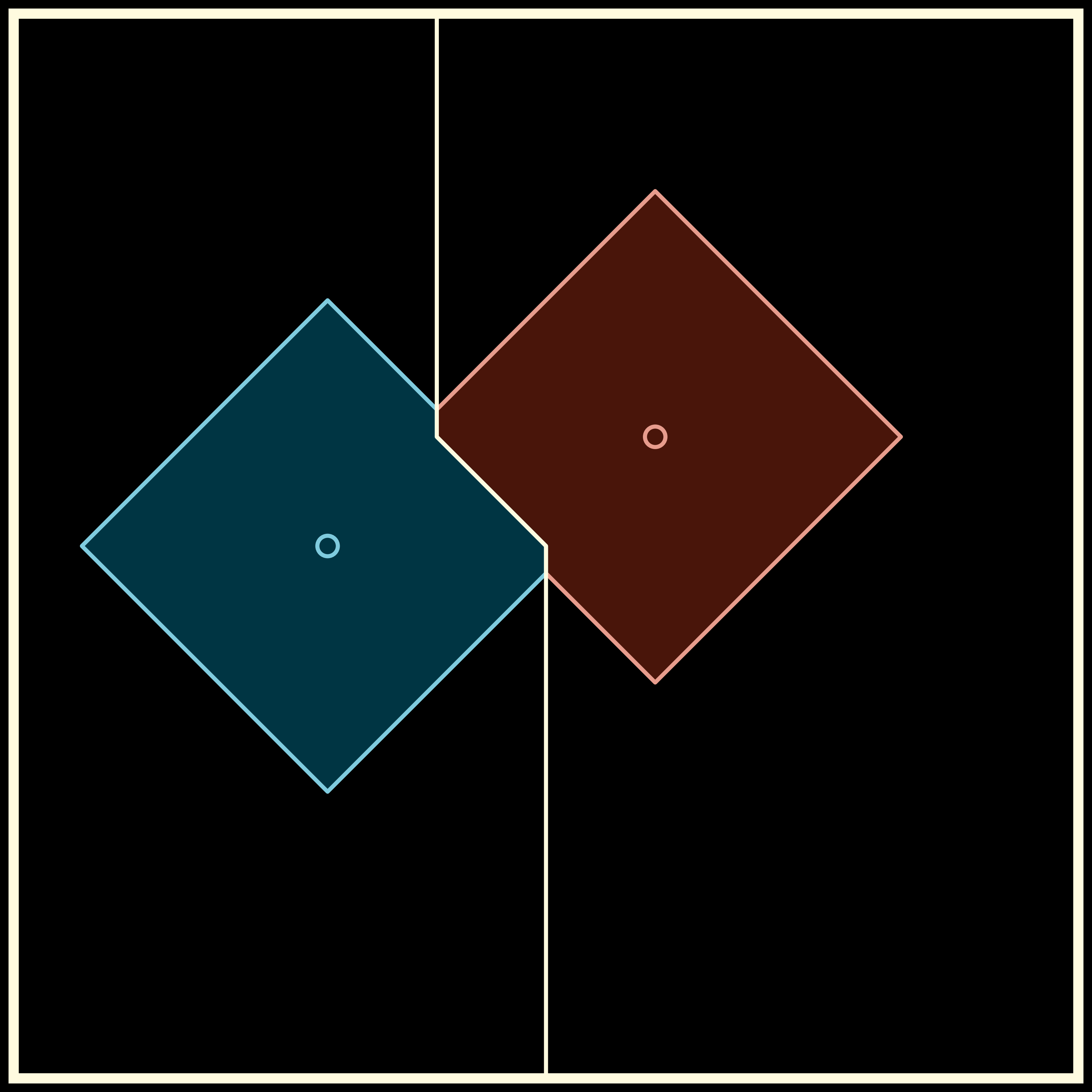


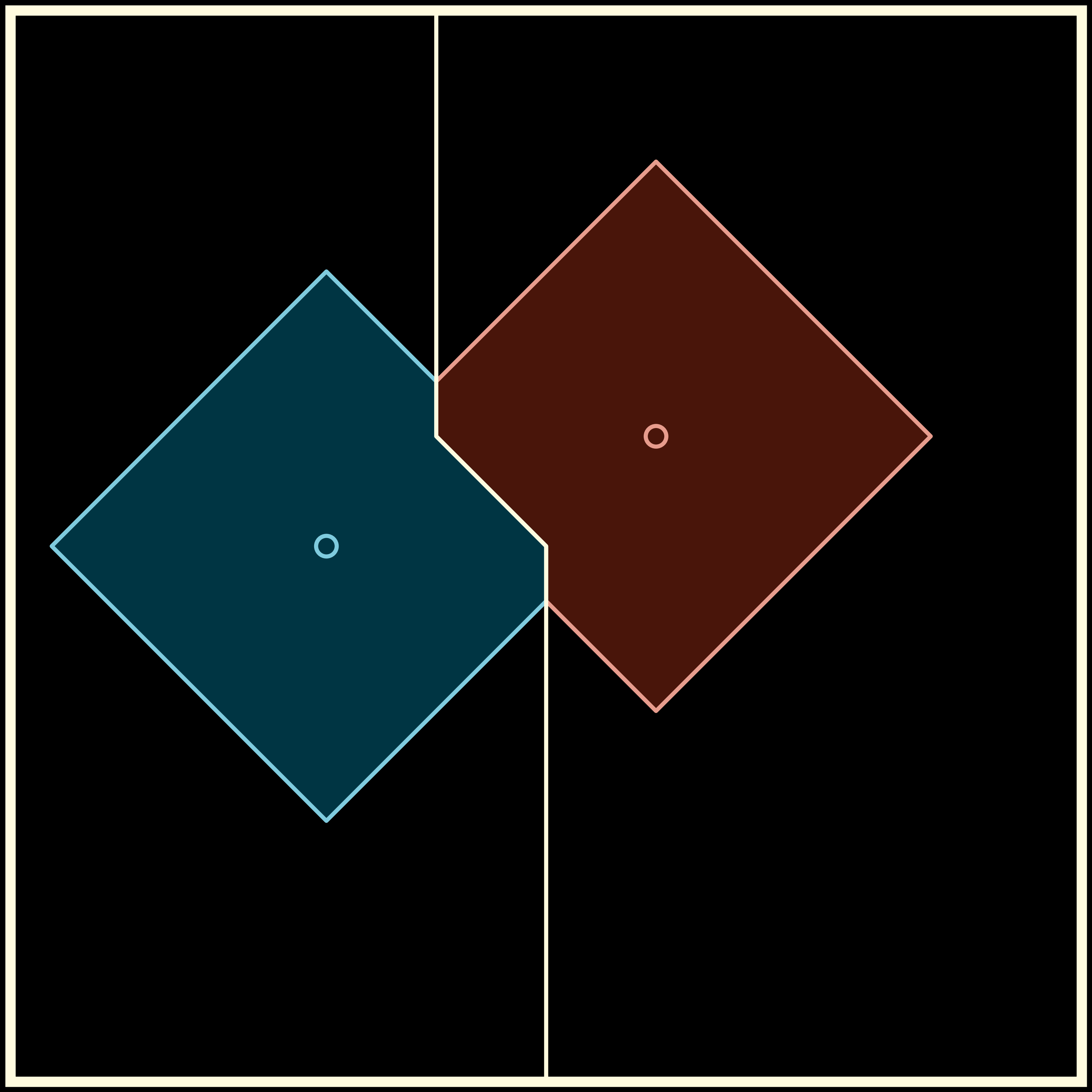


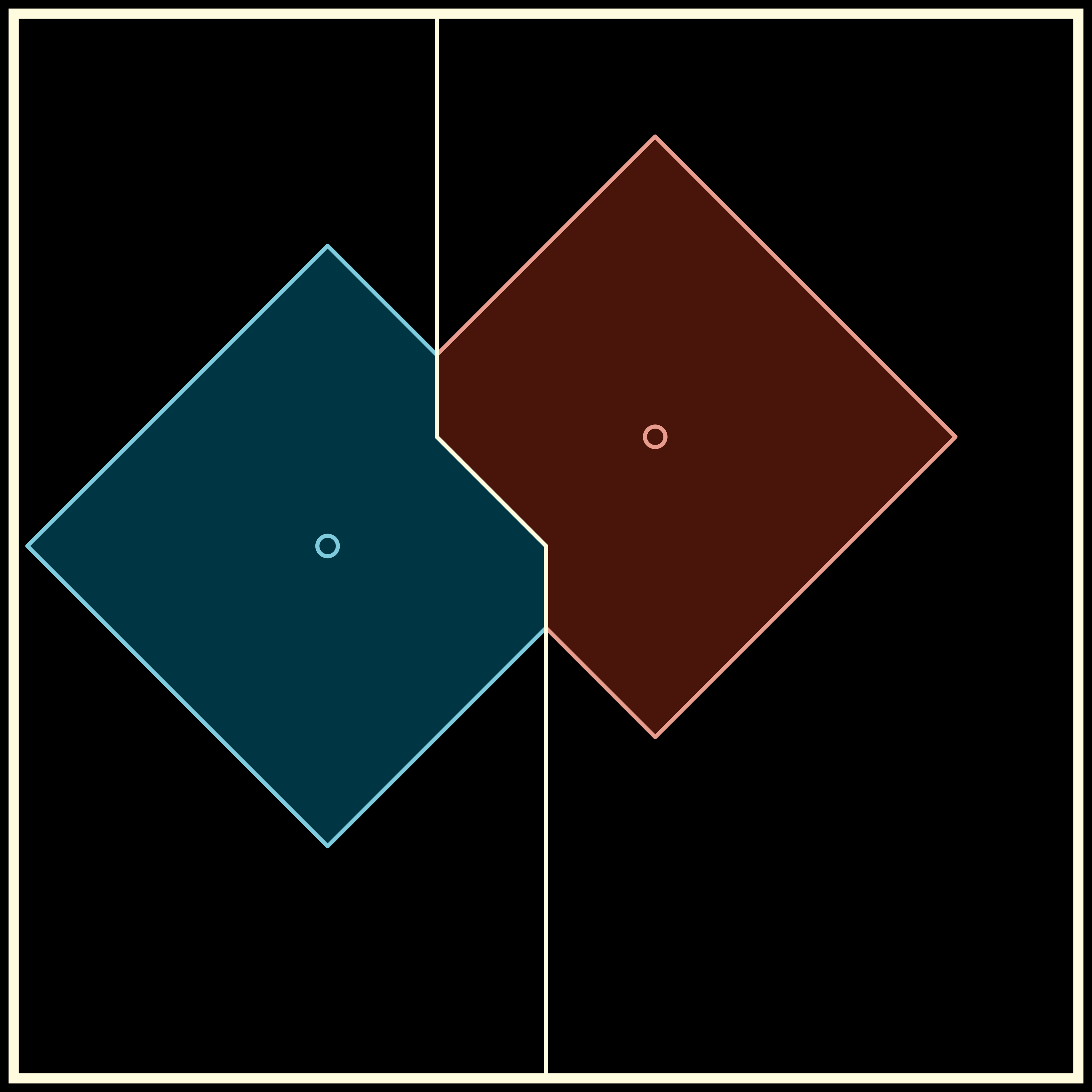


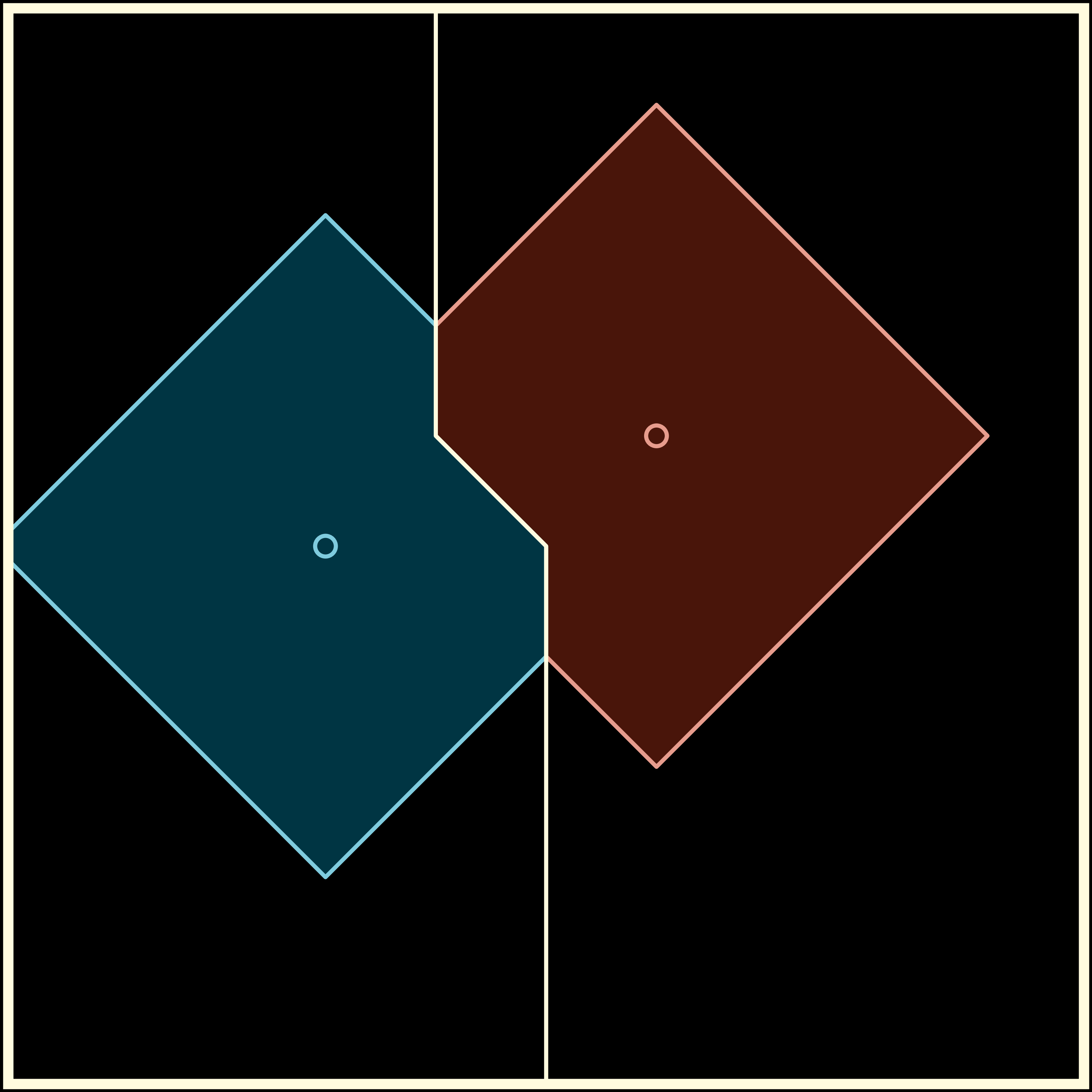


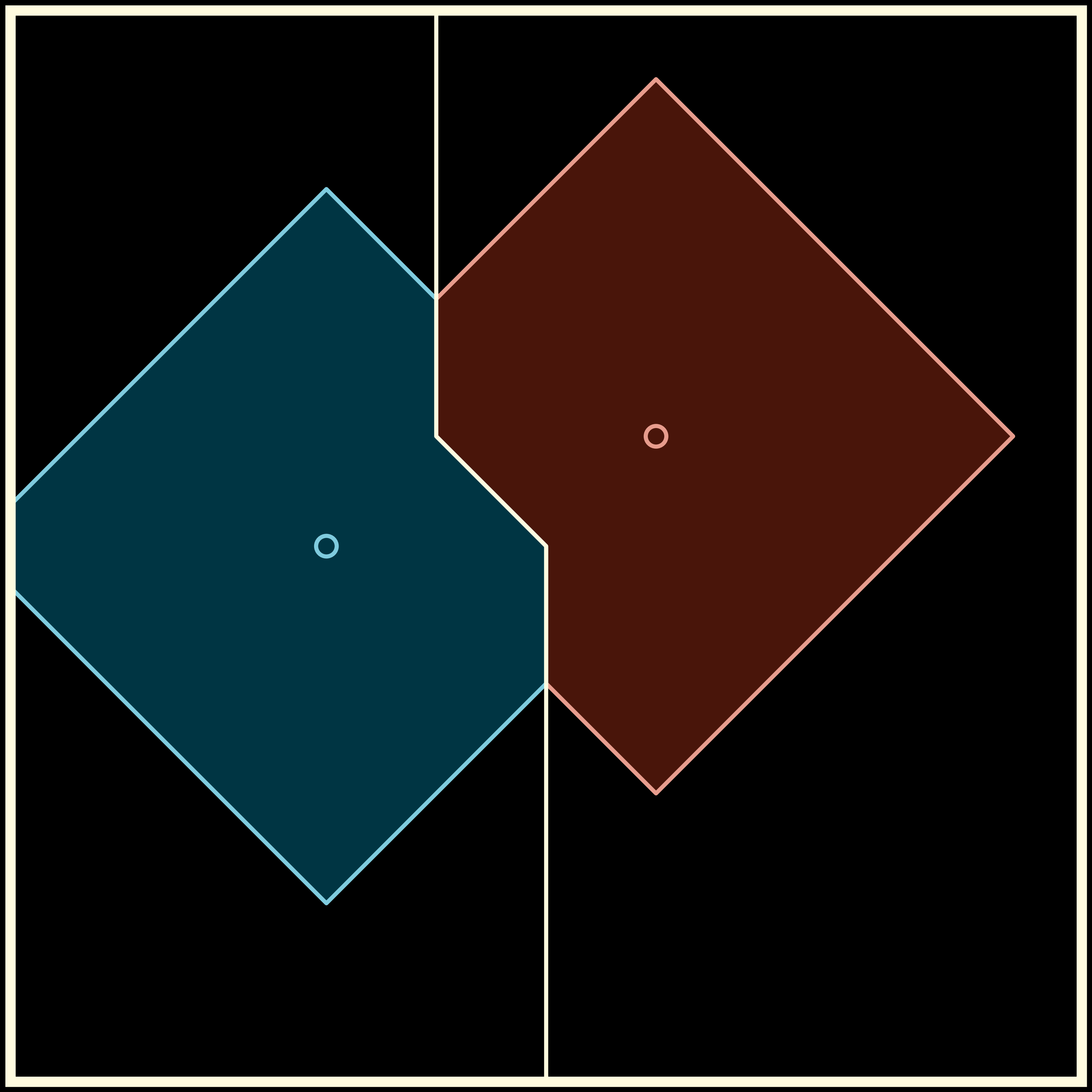


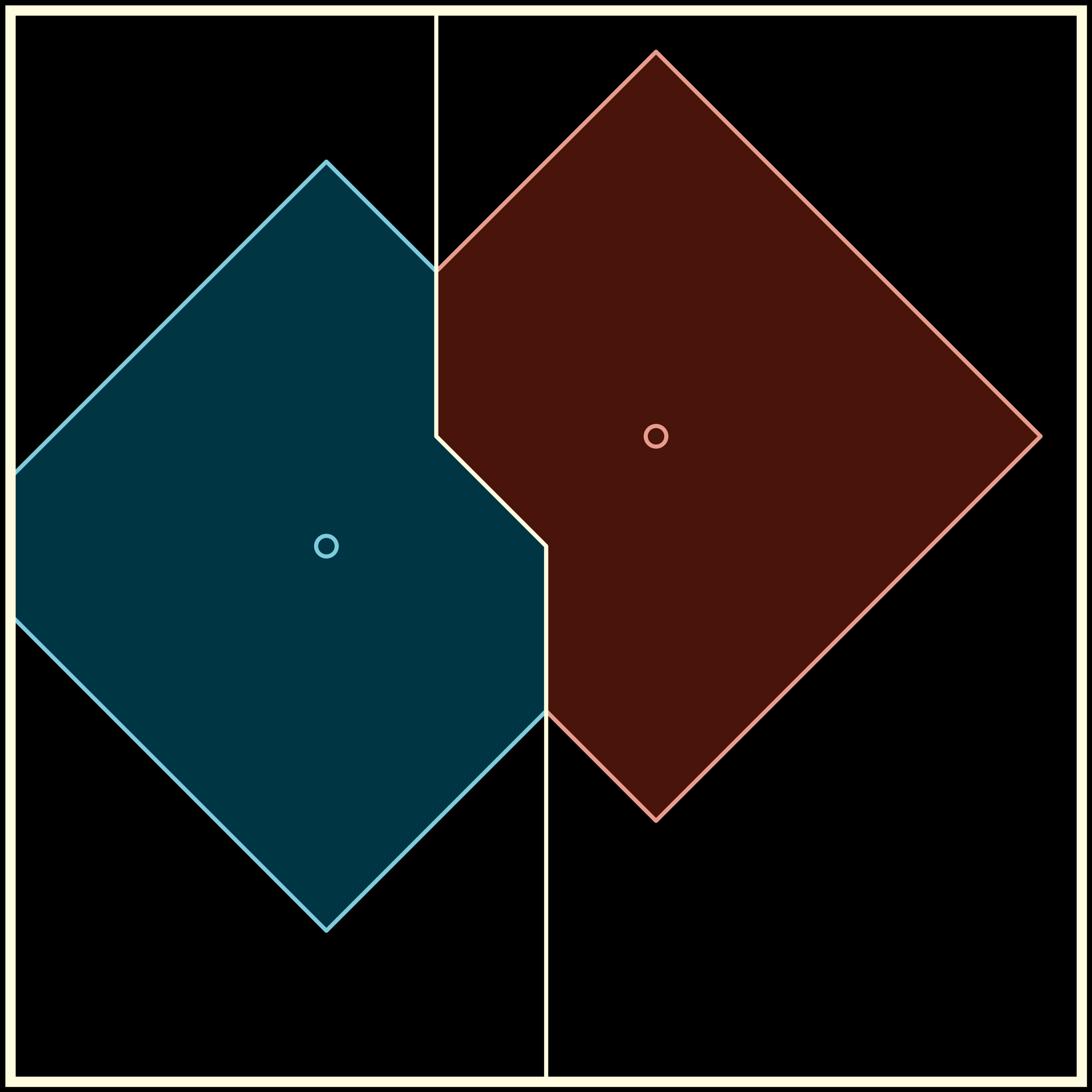


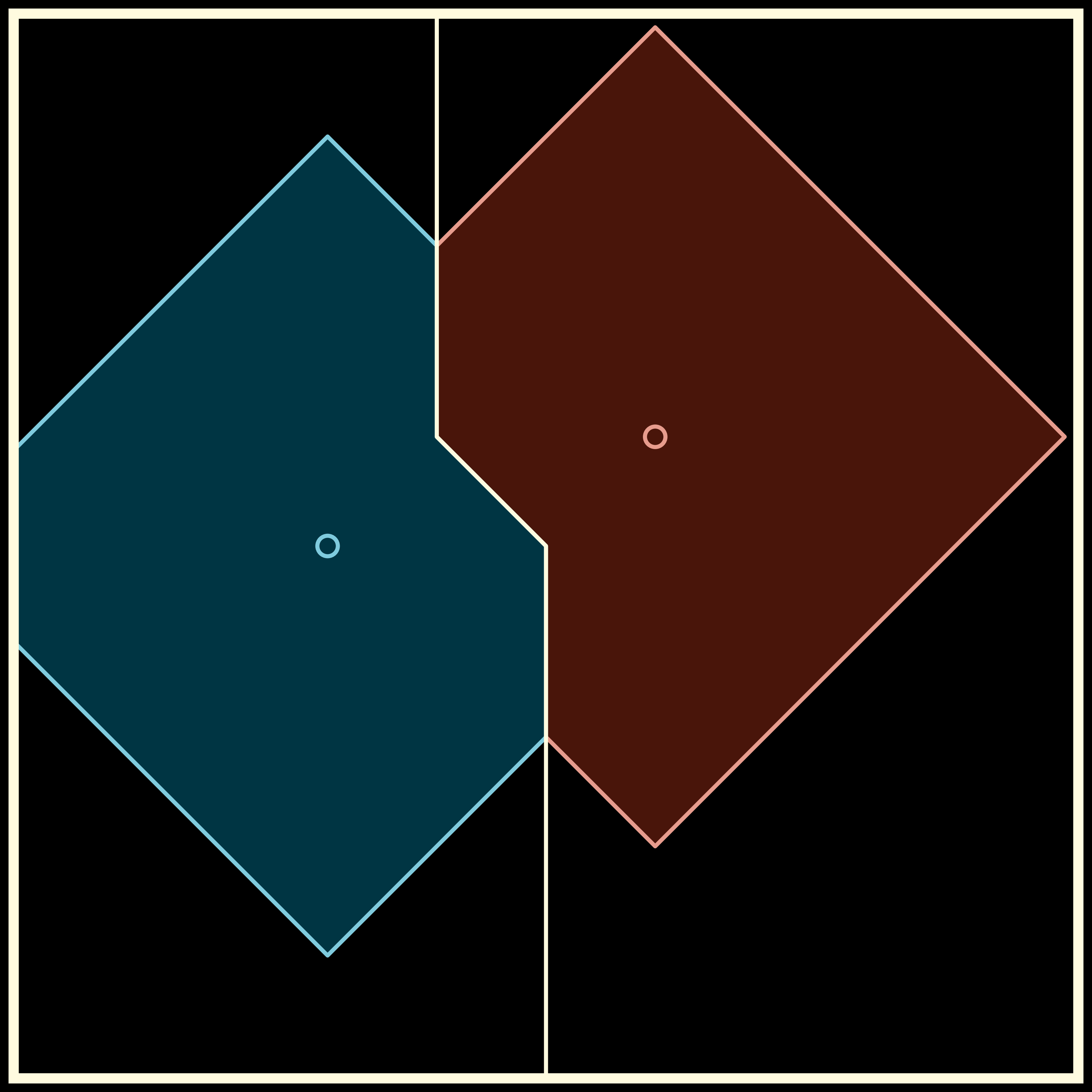


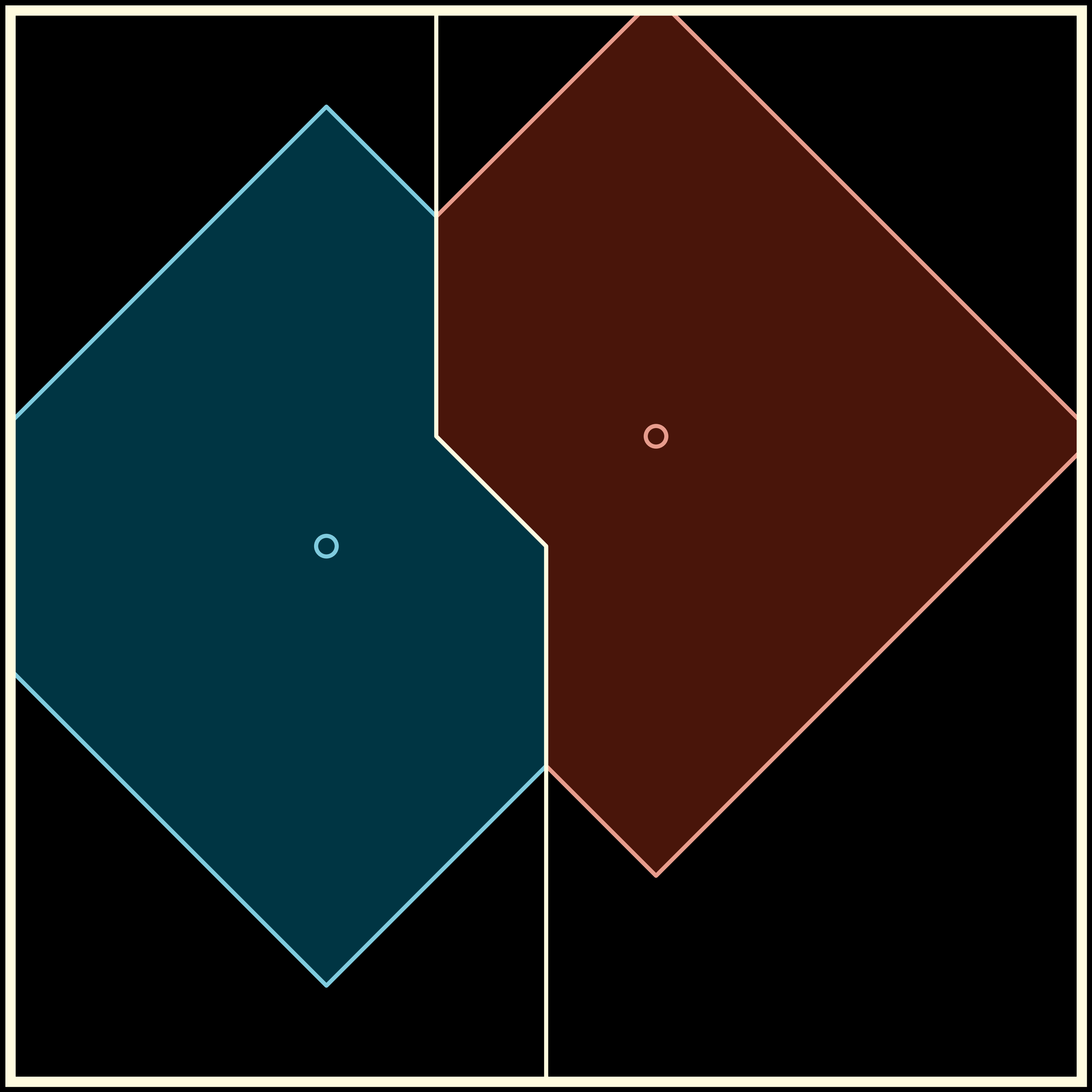


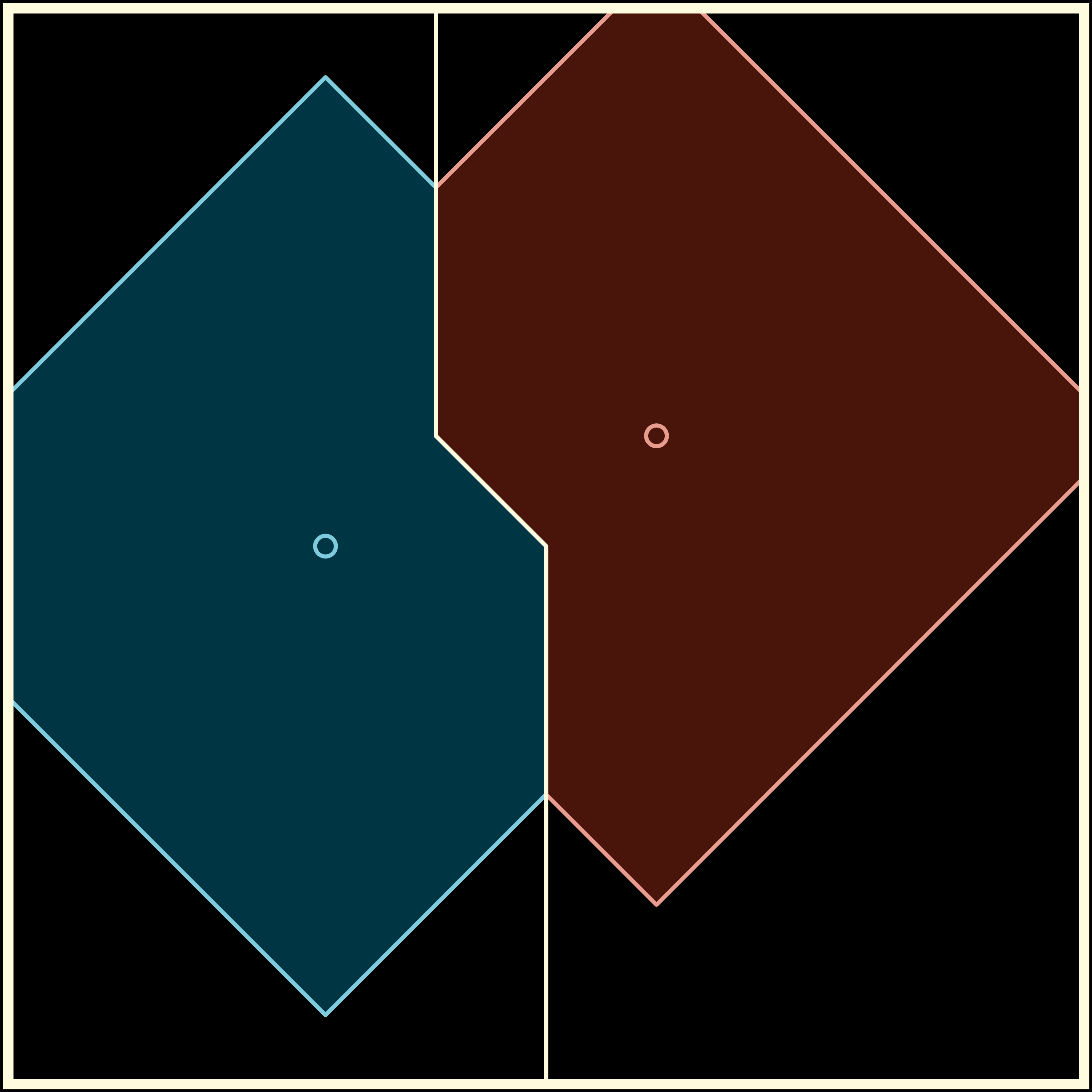


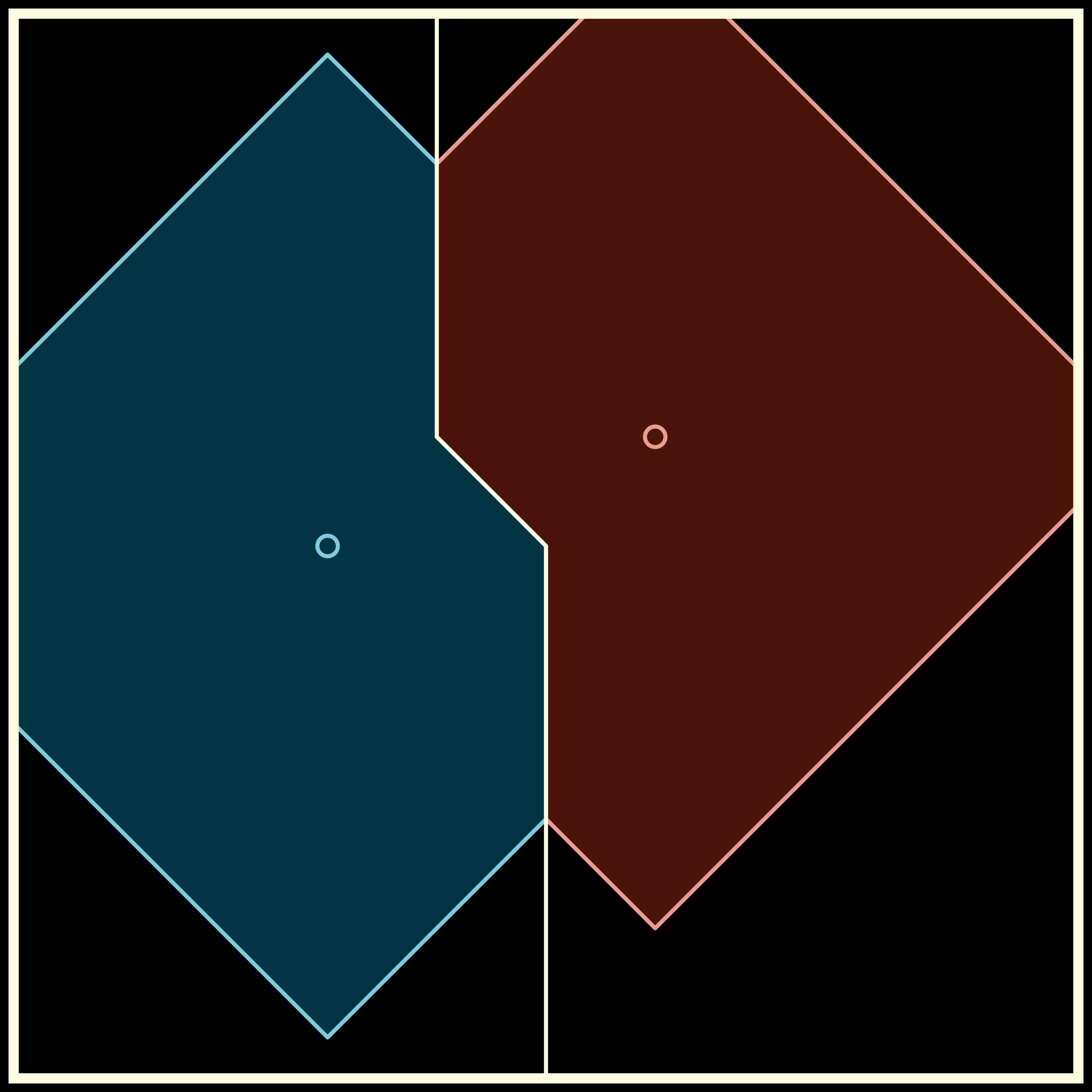


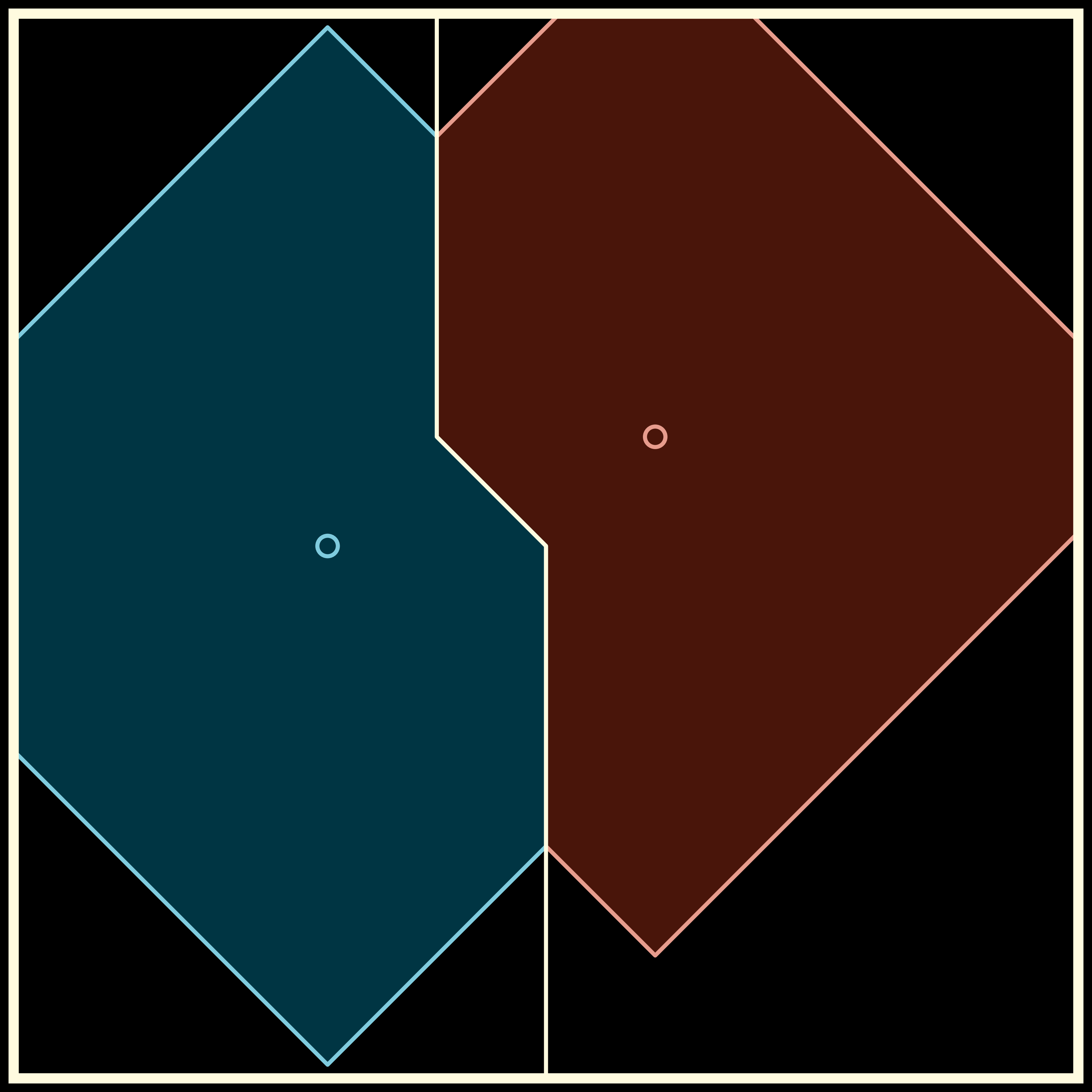


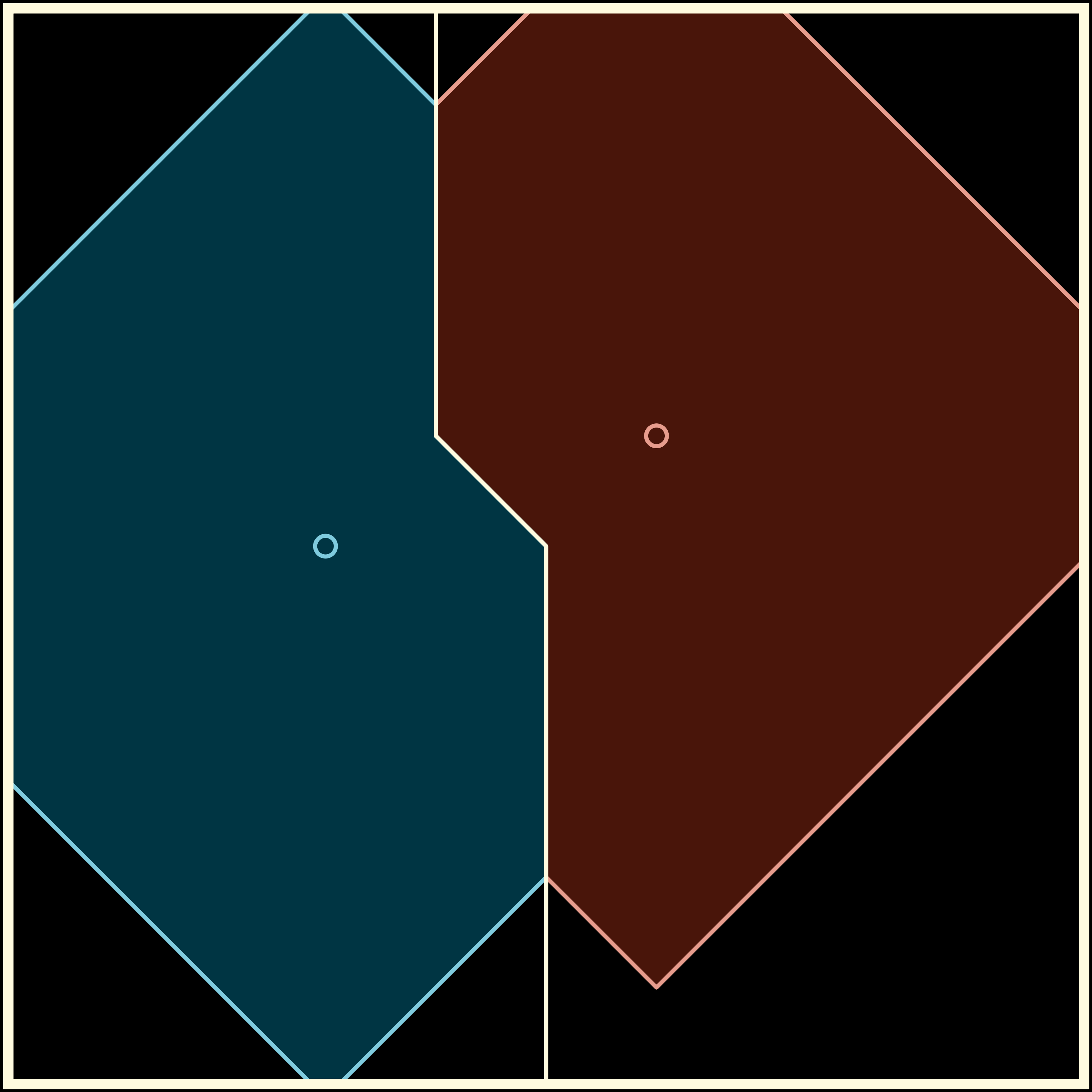


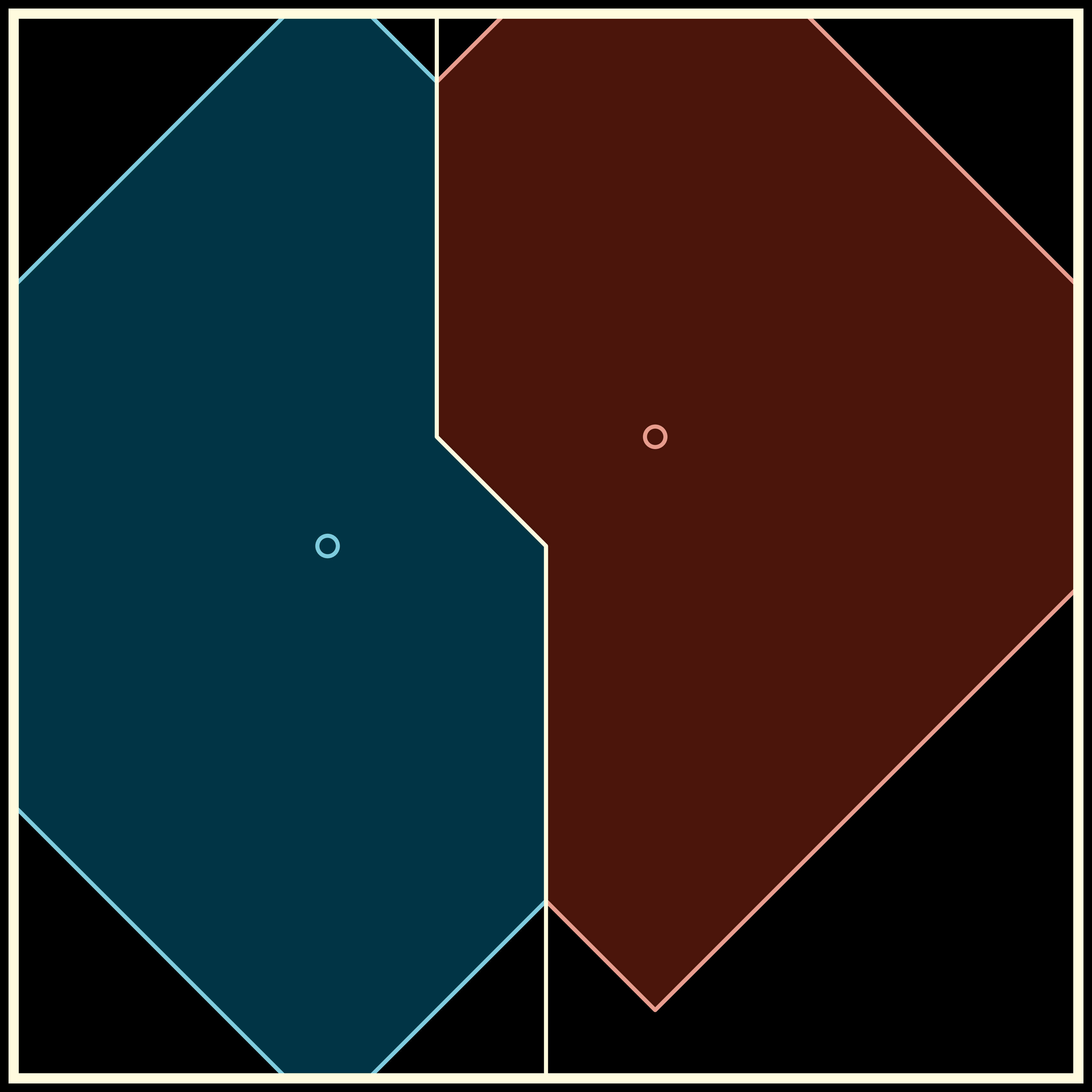


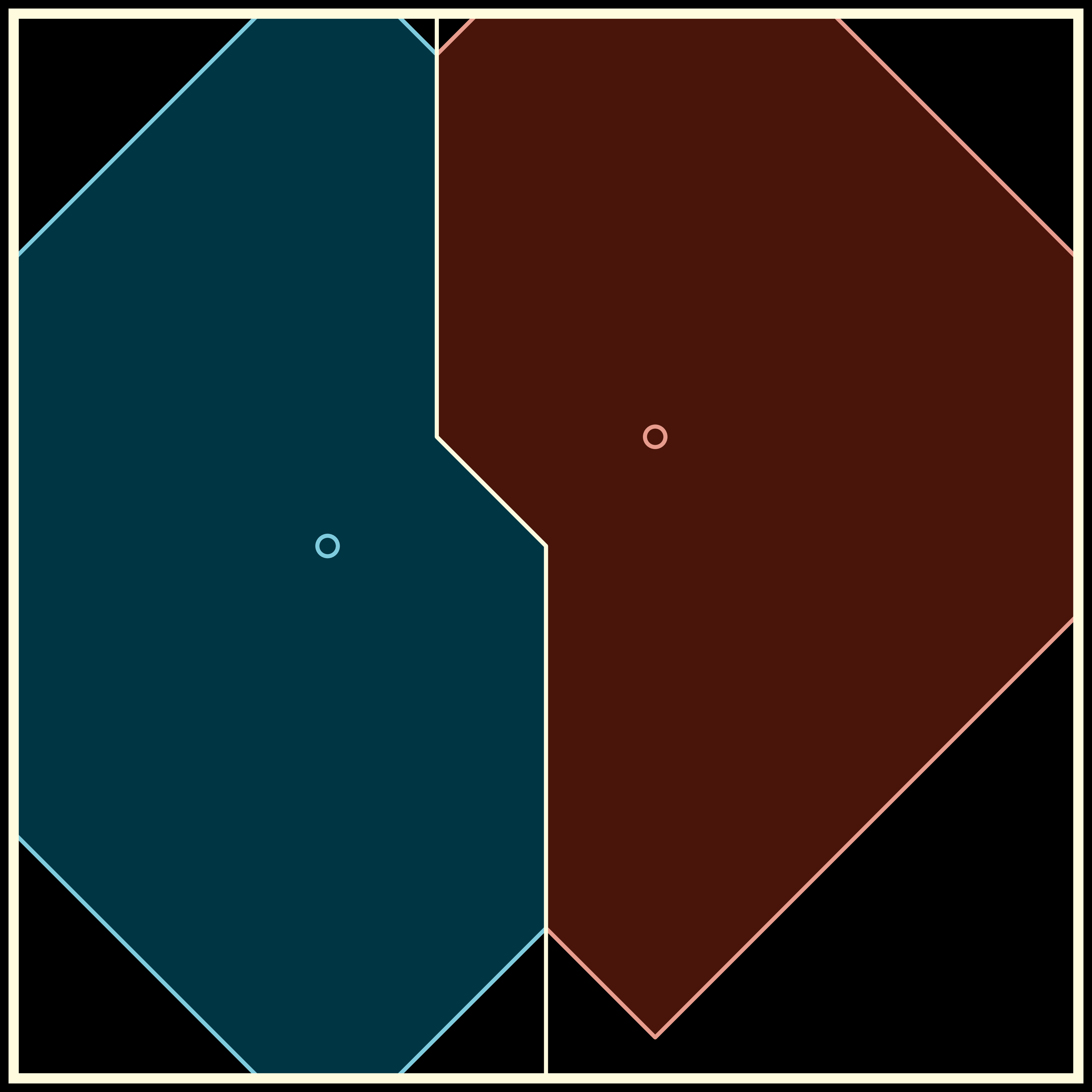


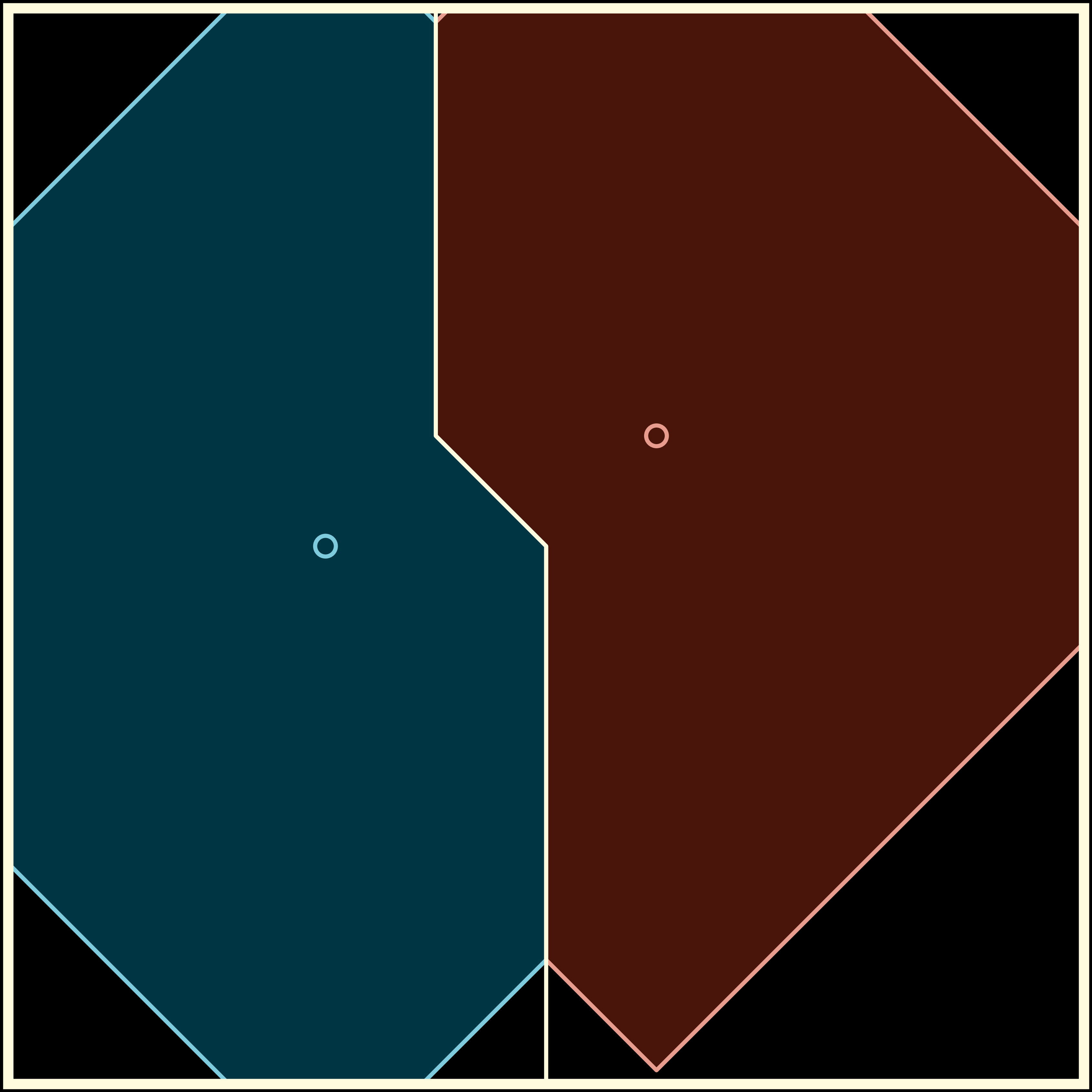


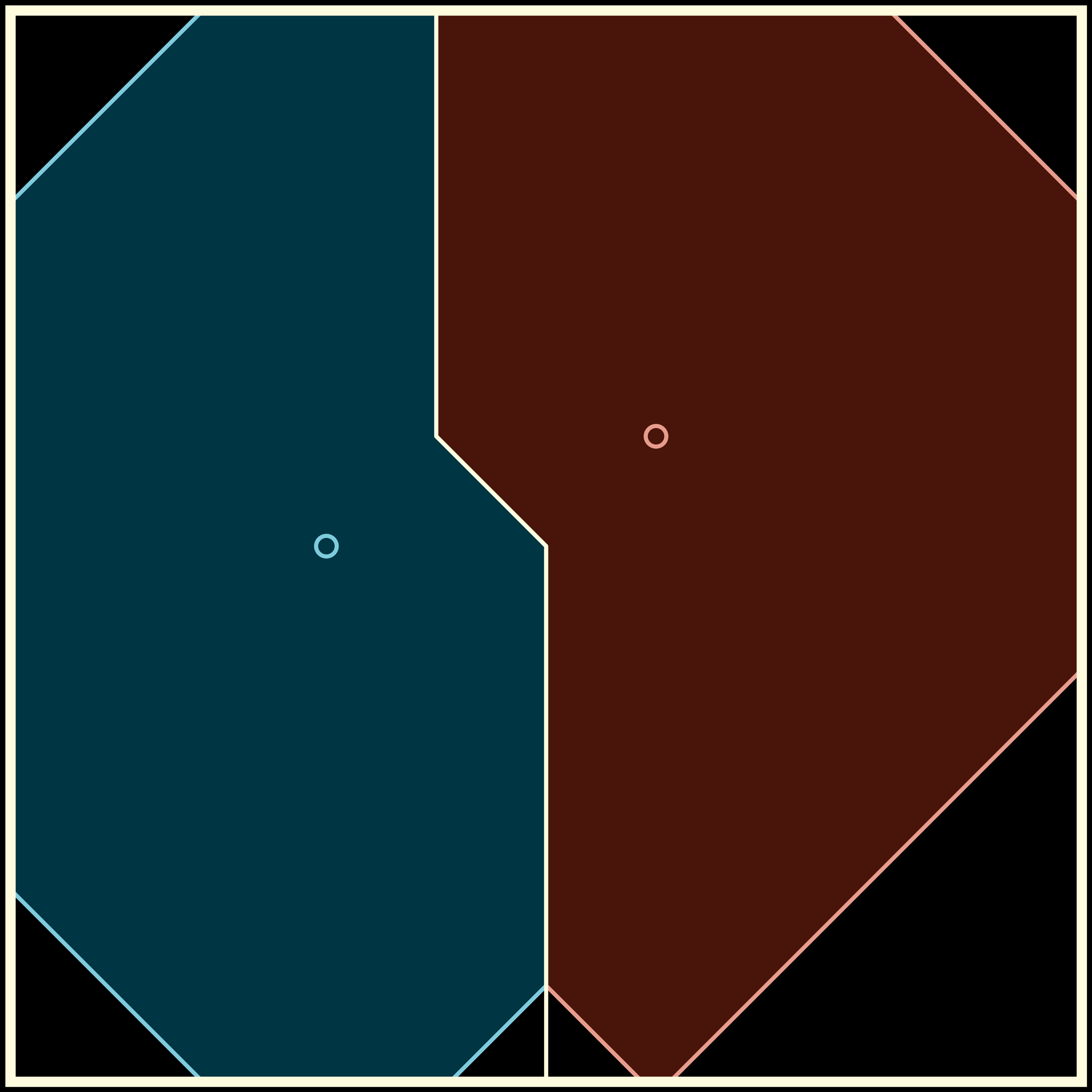


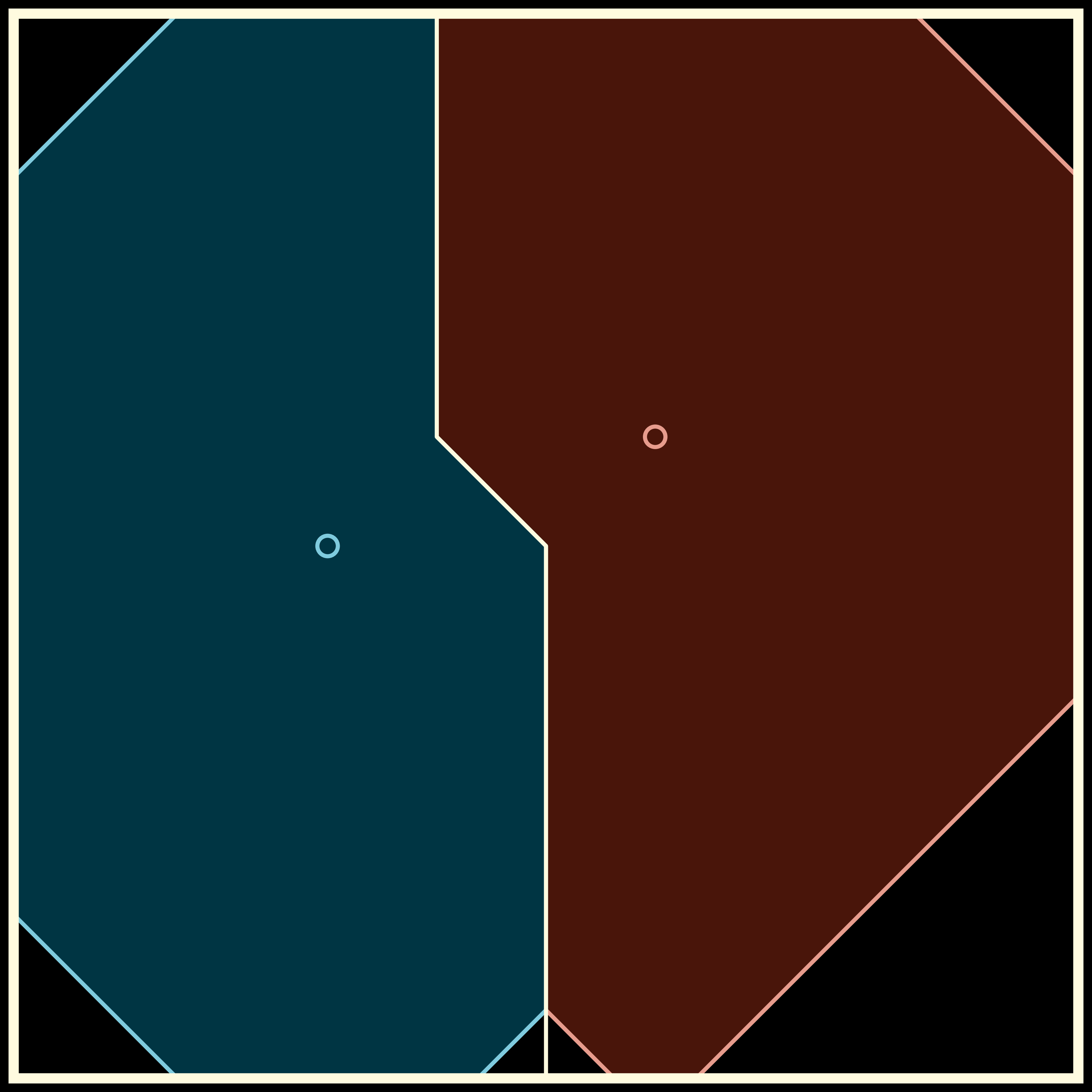


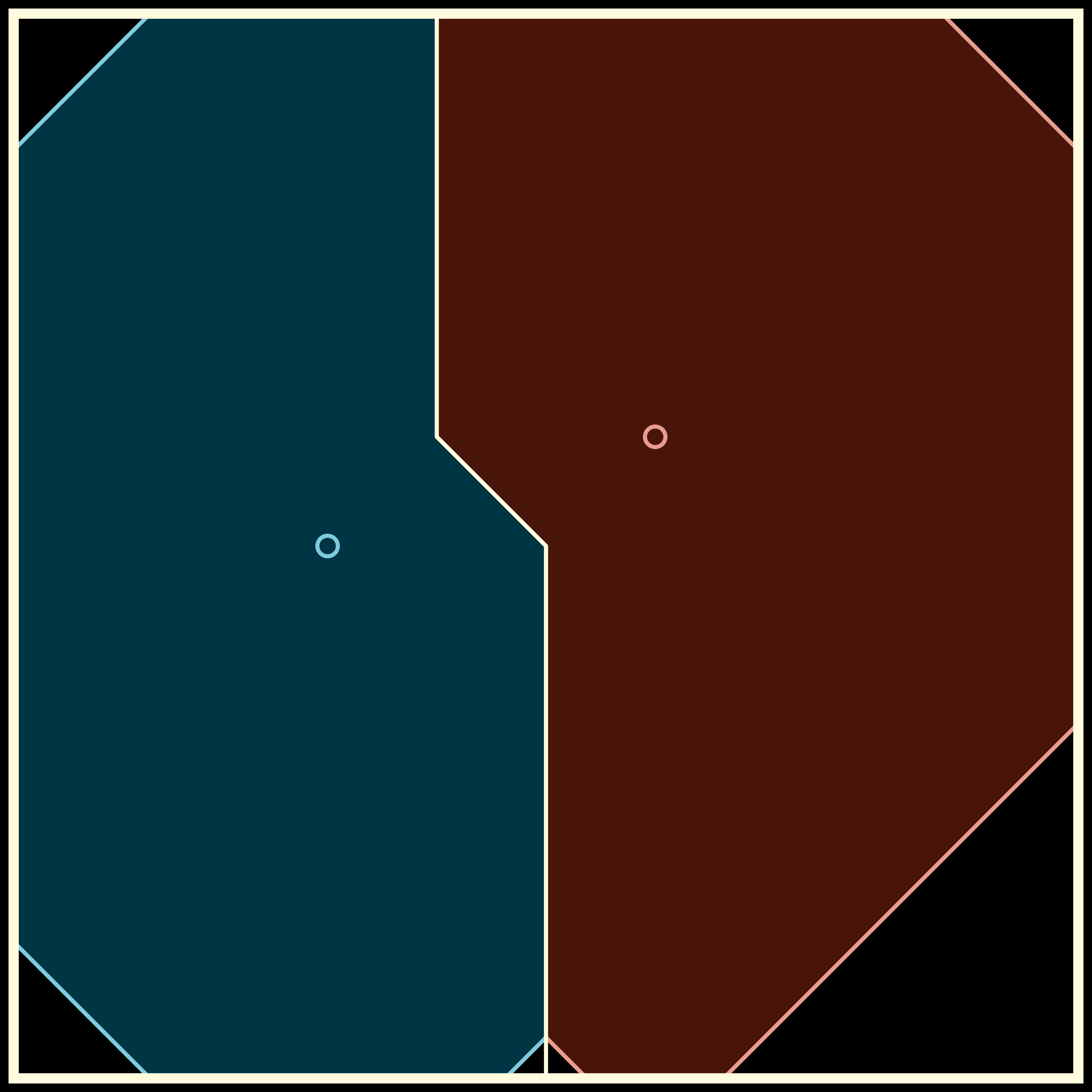


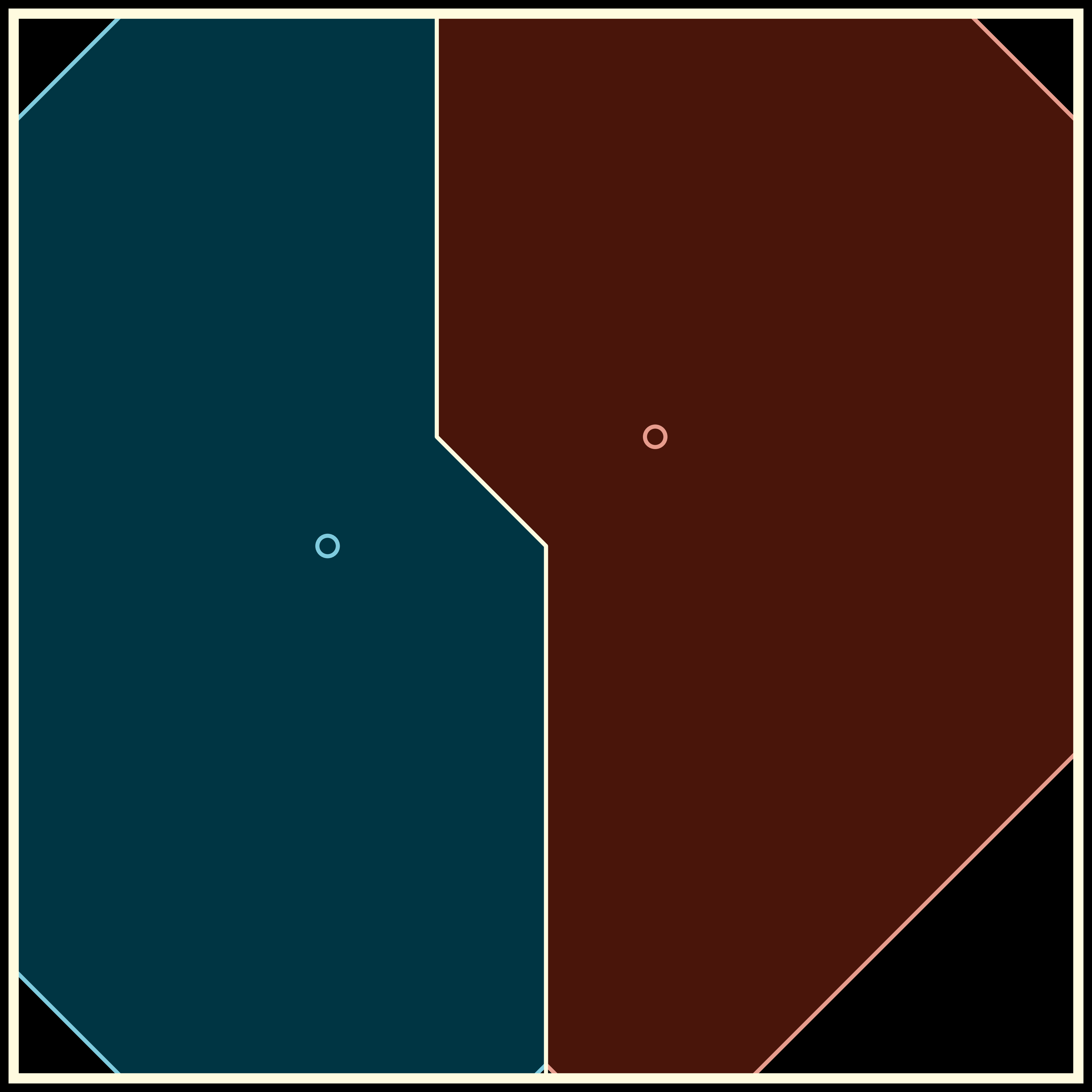


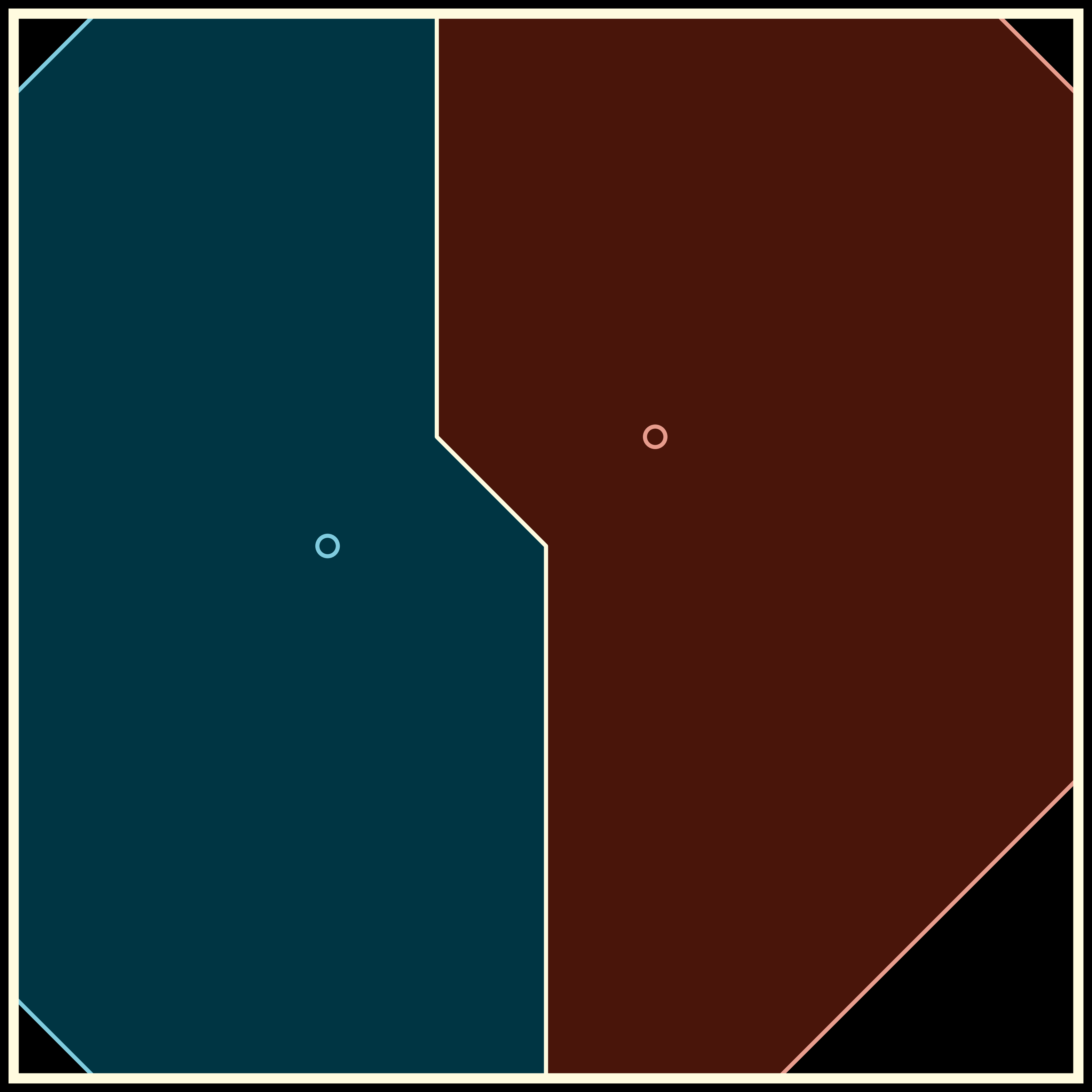


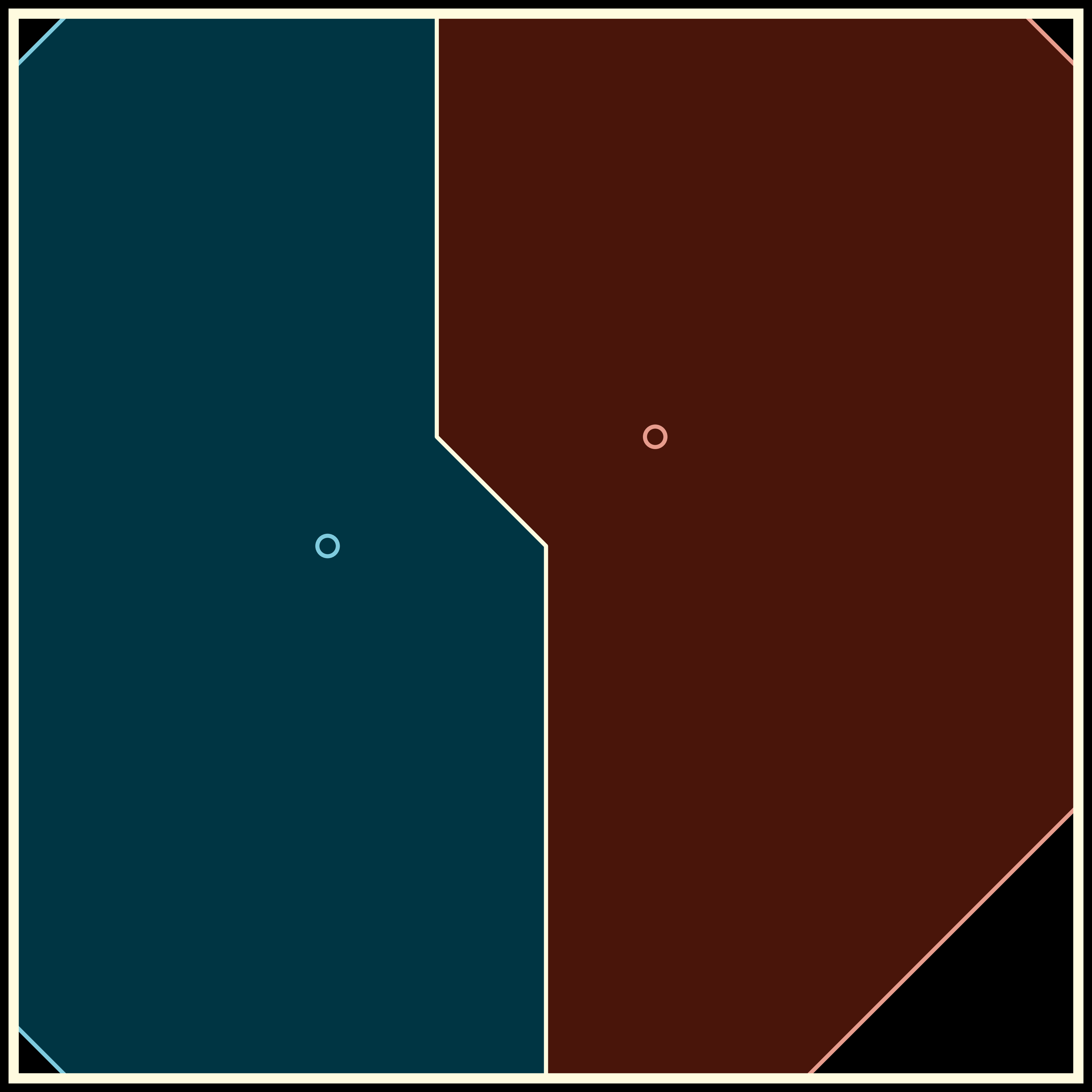


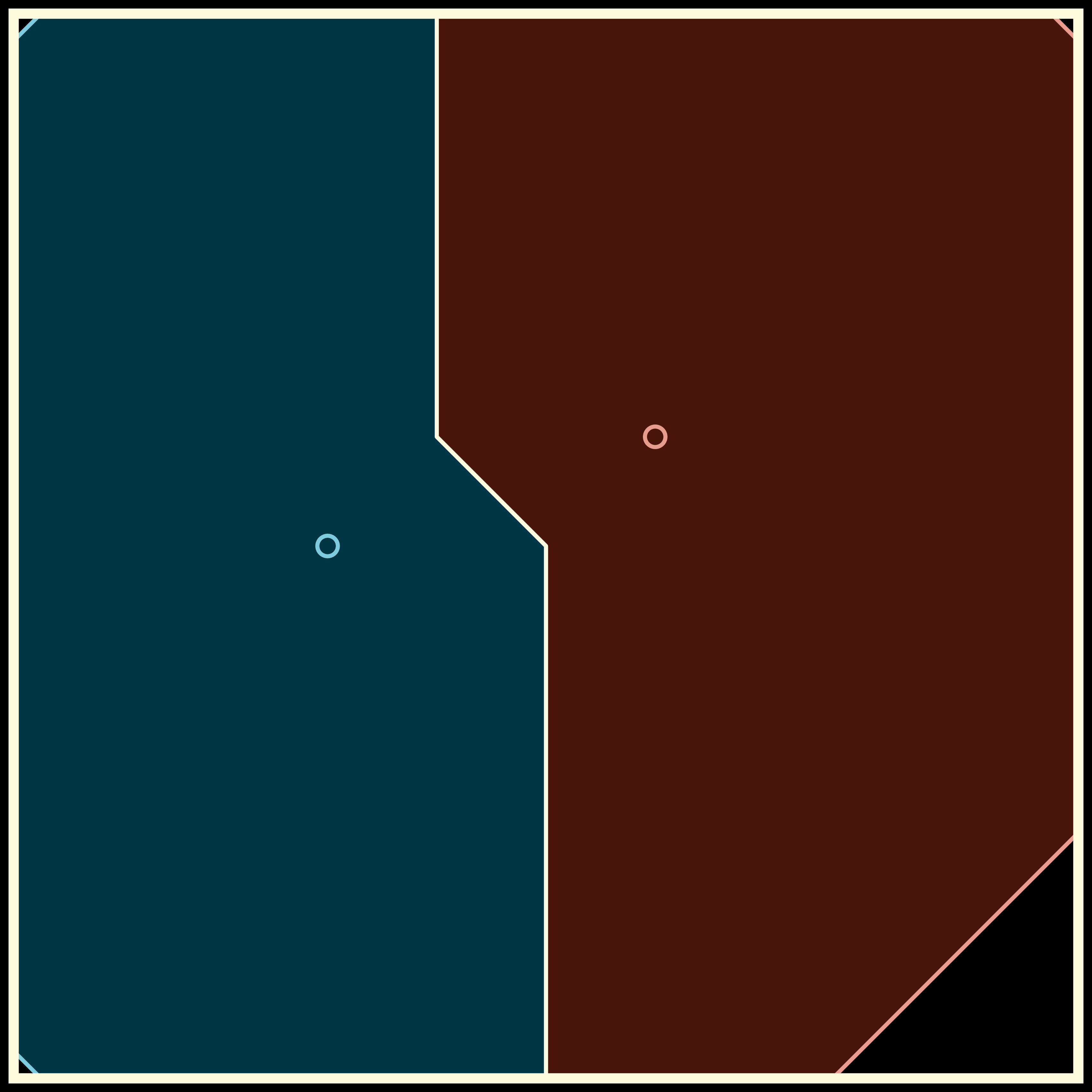


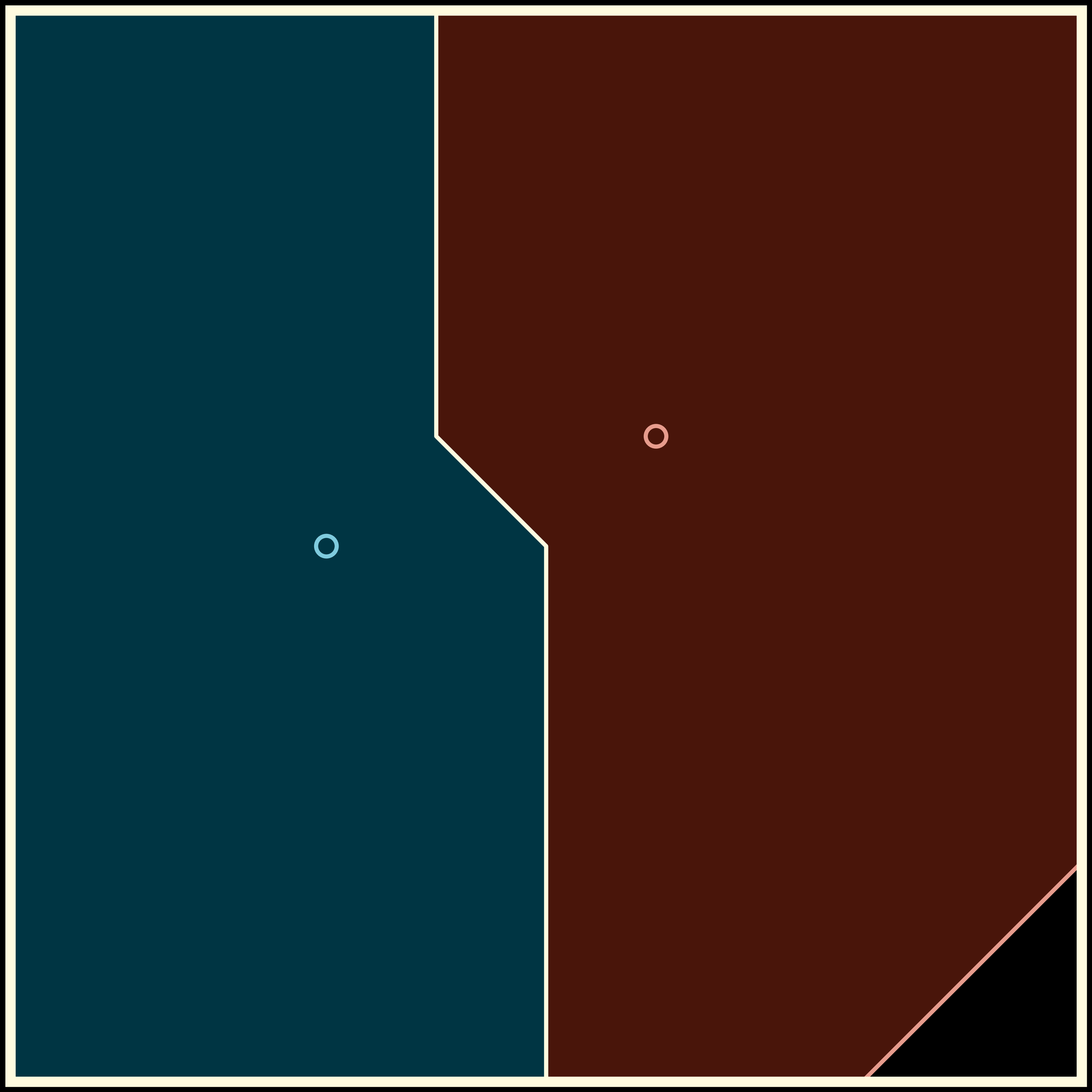


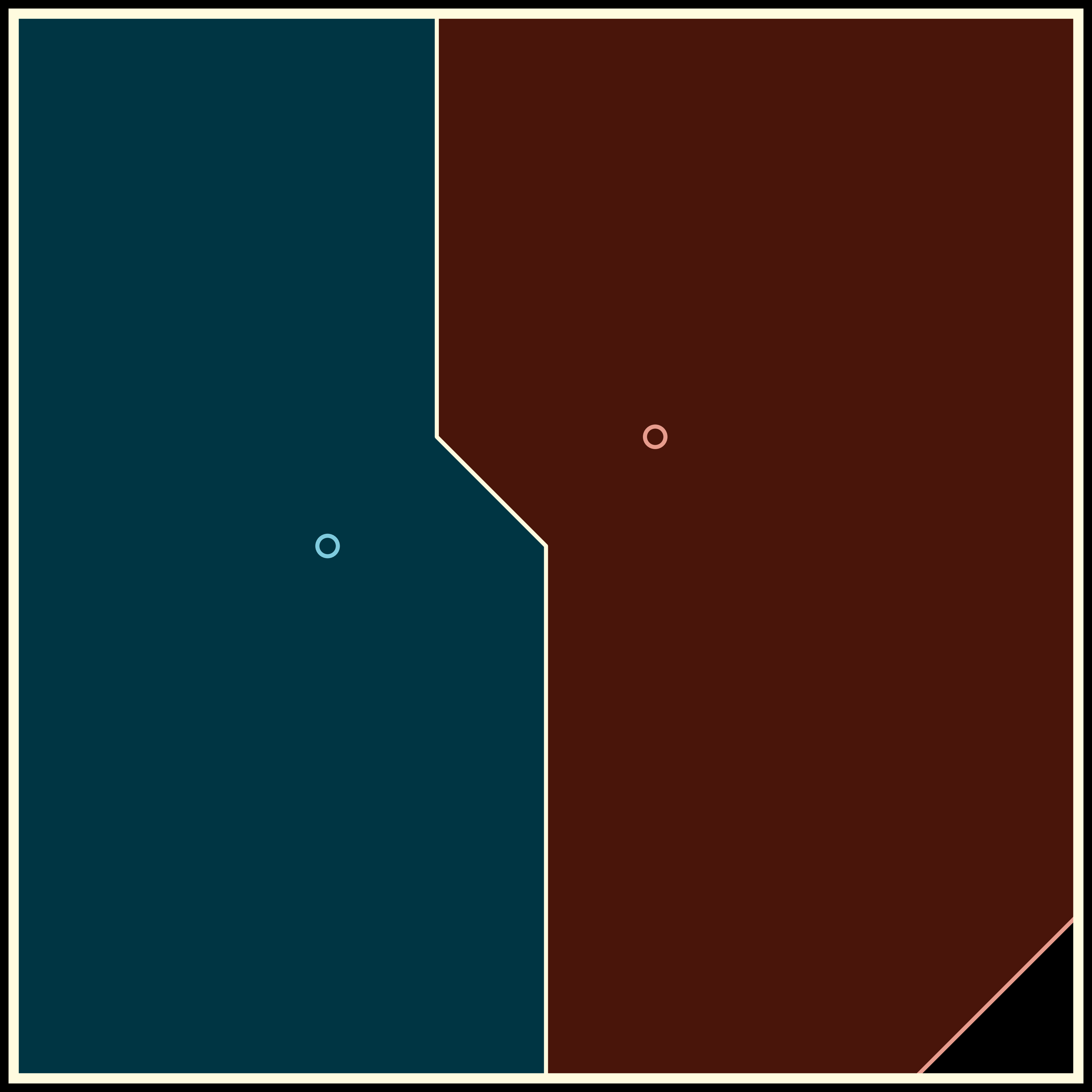


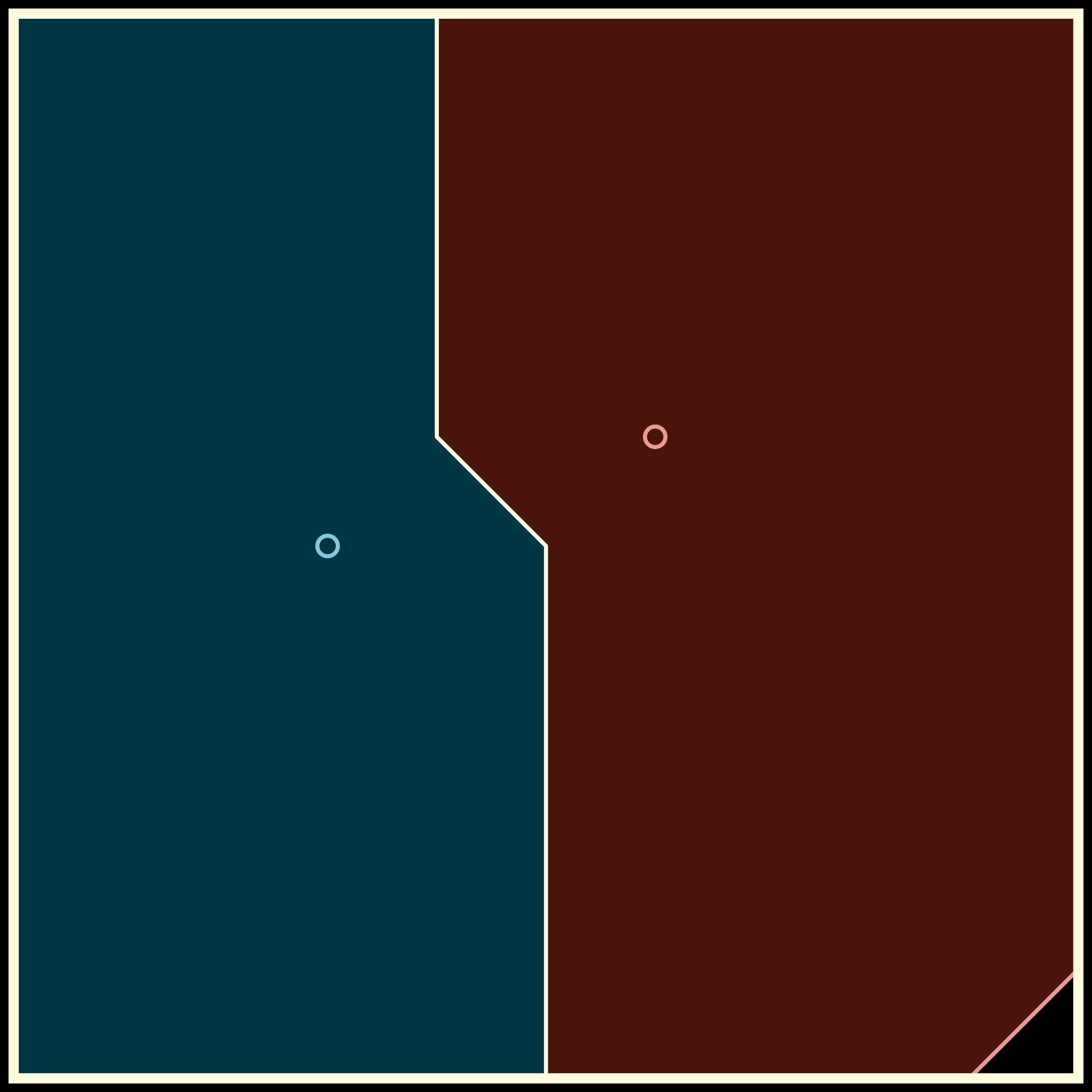


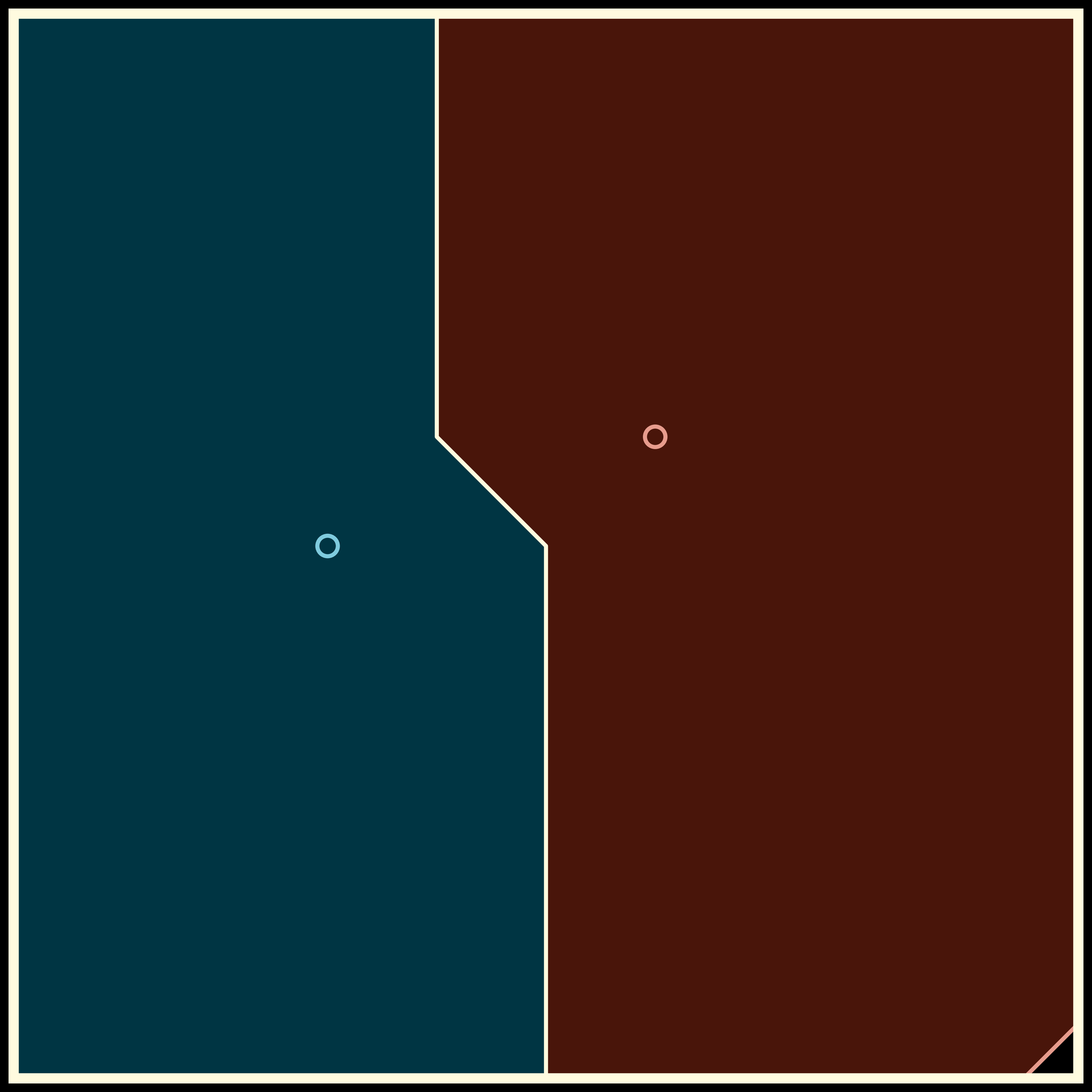


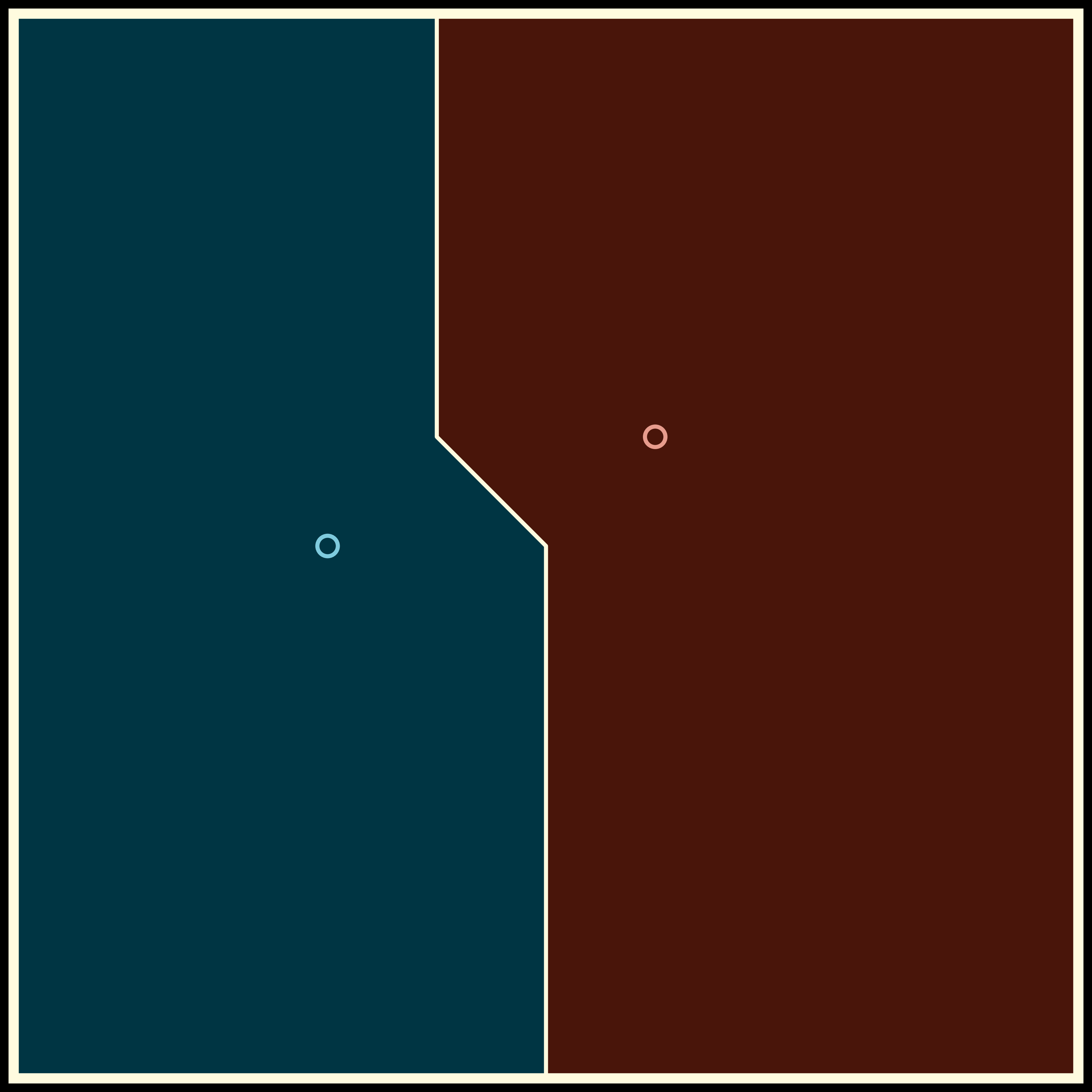


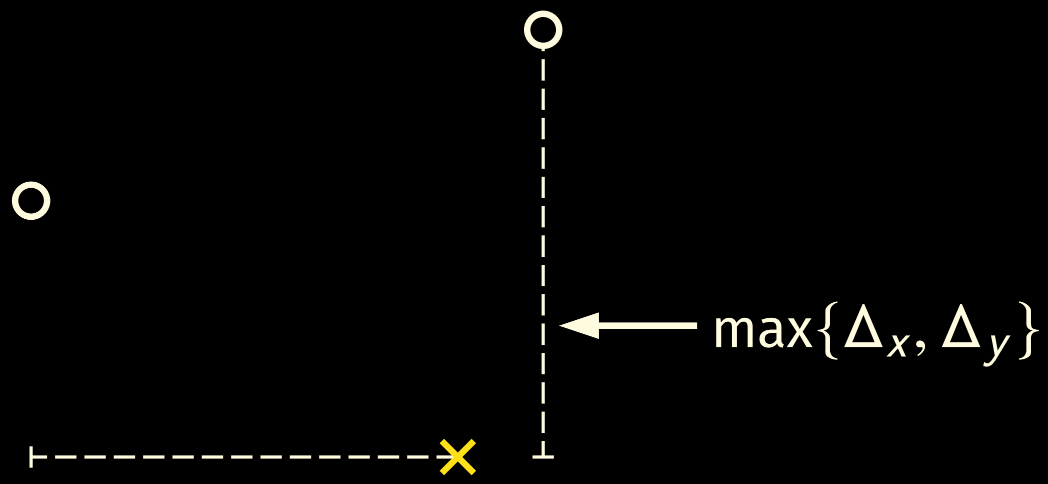


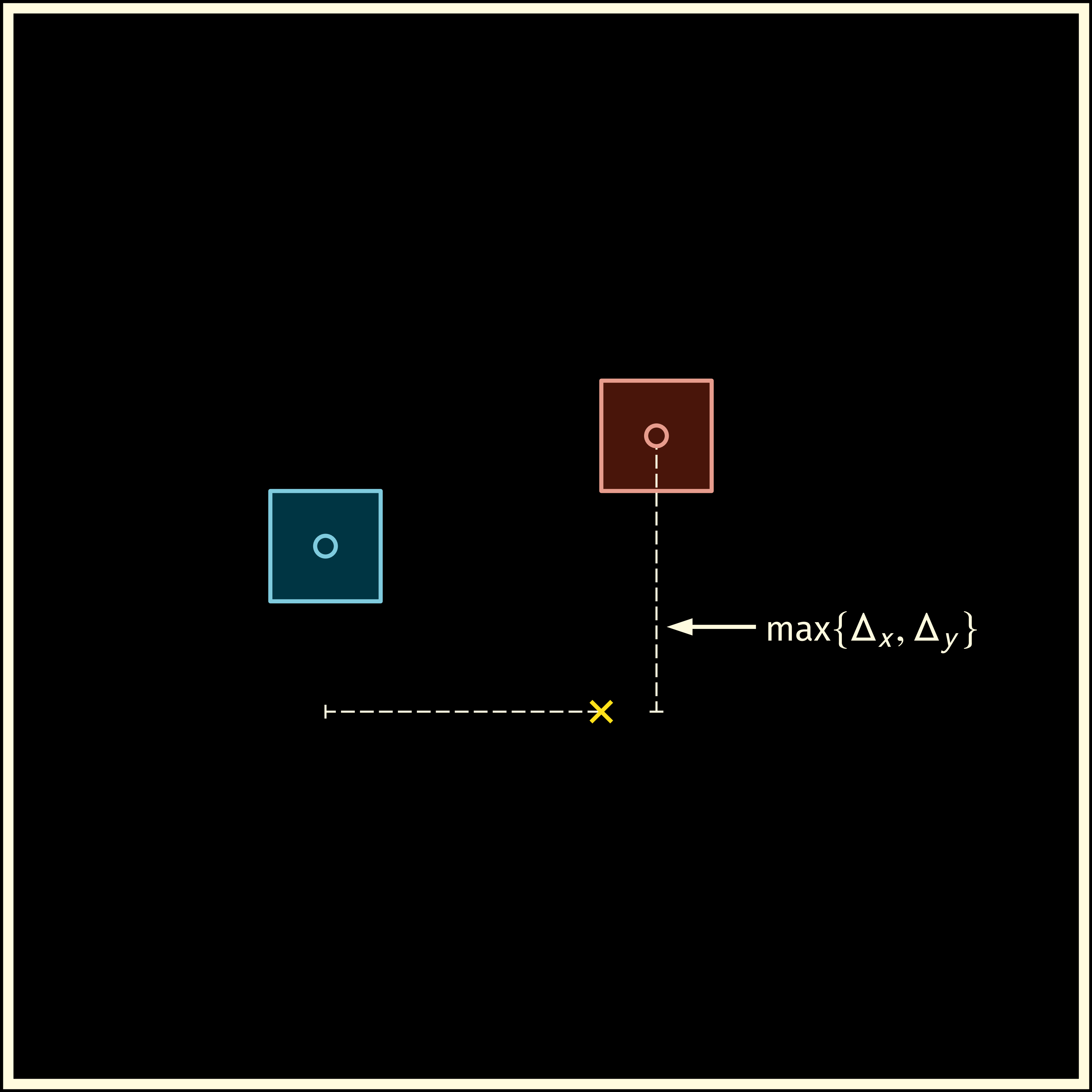


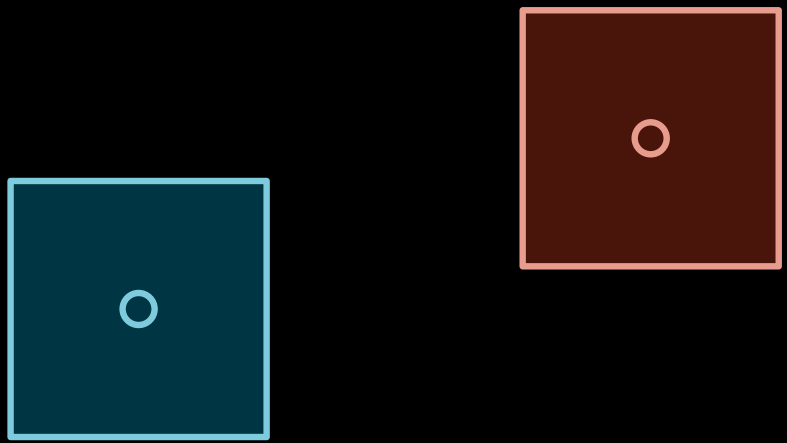


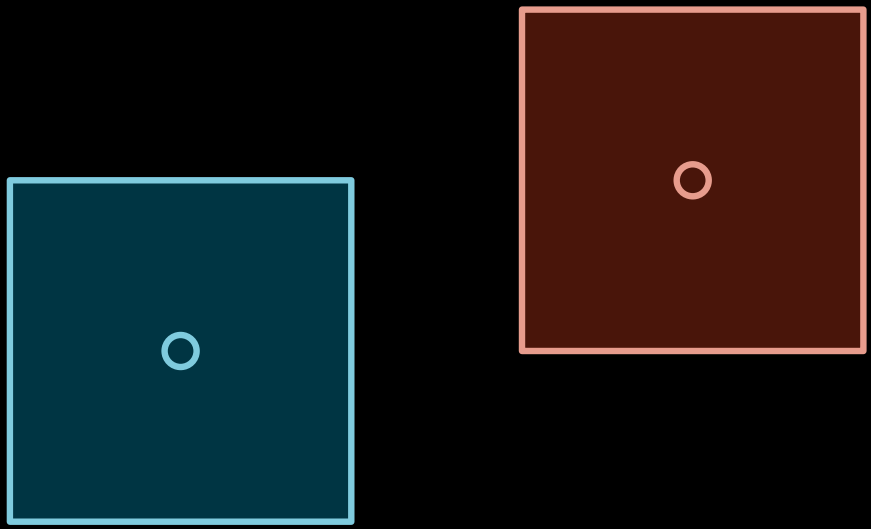


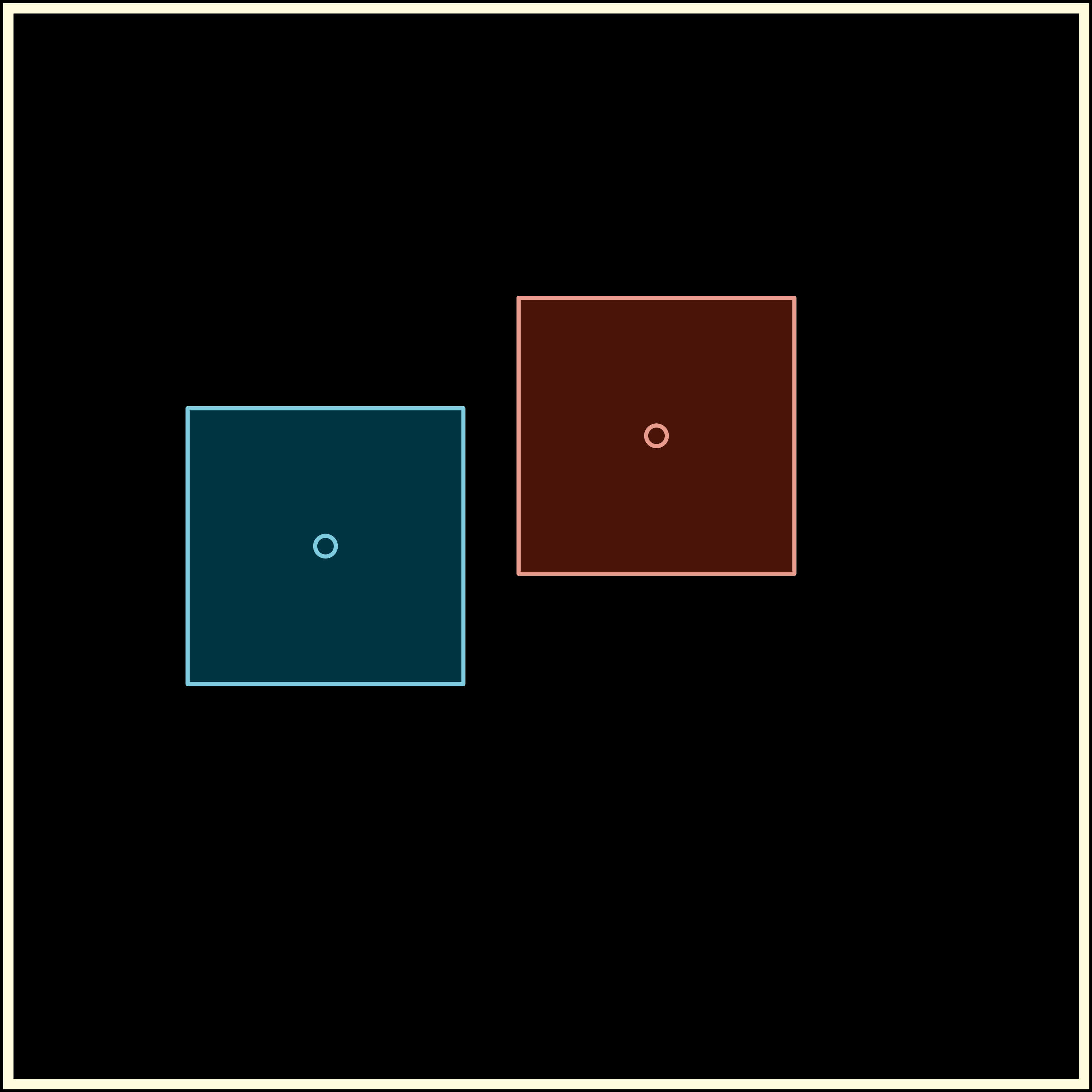


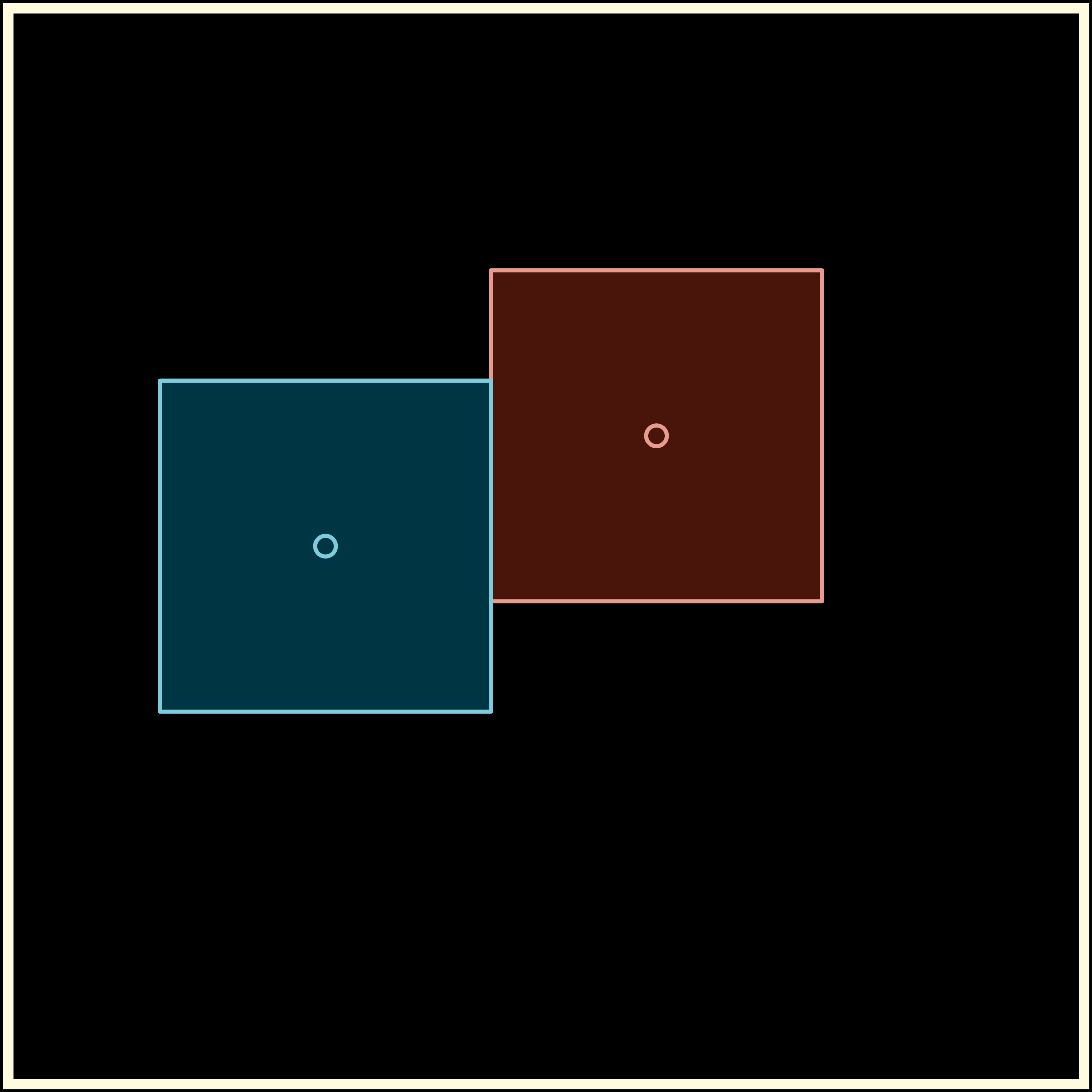

max{Δ_x, Δ_y}

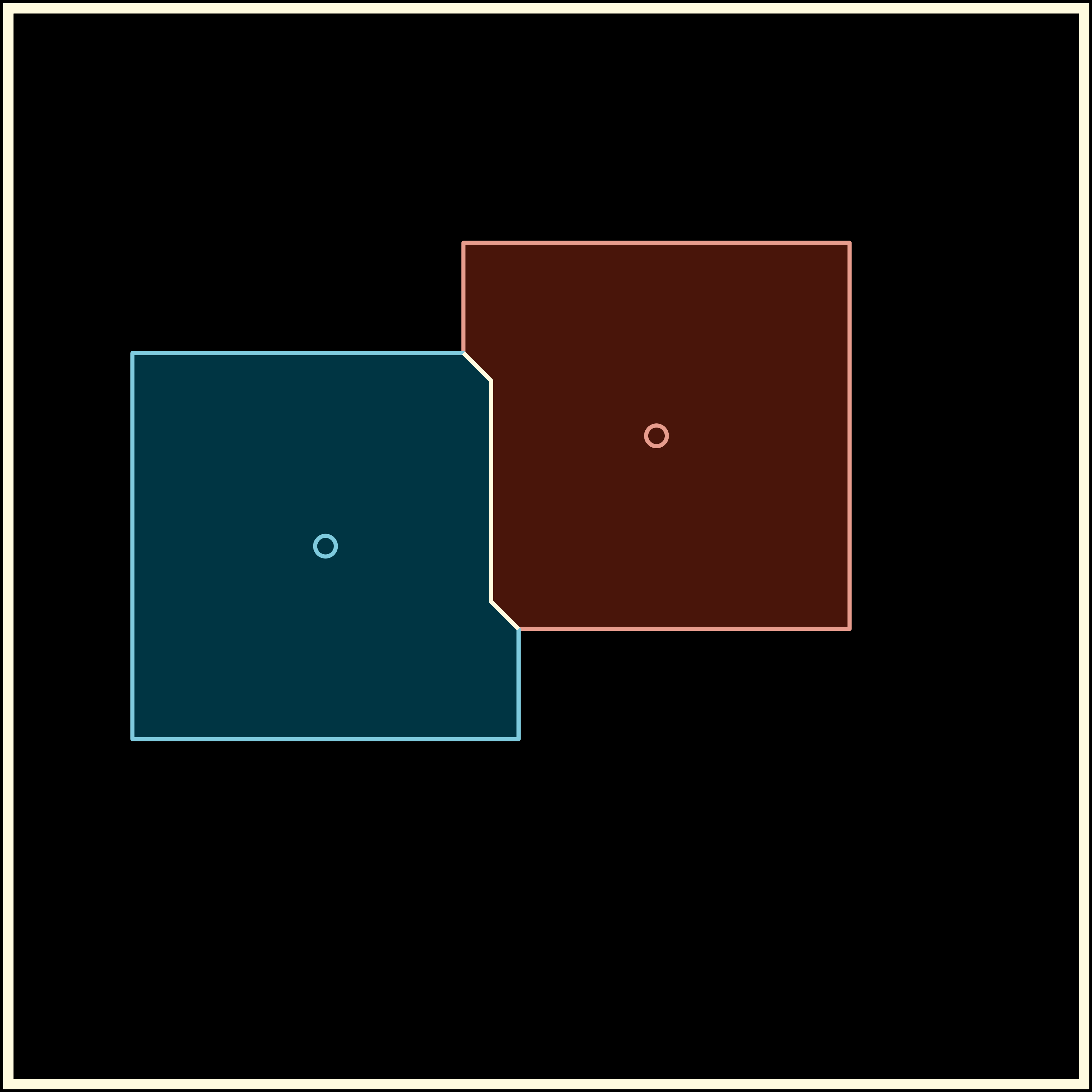

max{Δ_x, Δ_y}

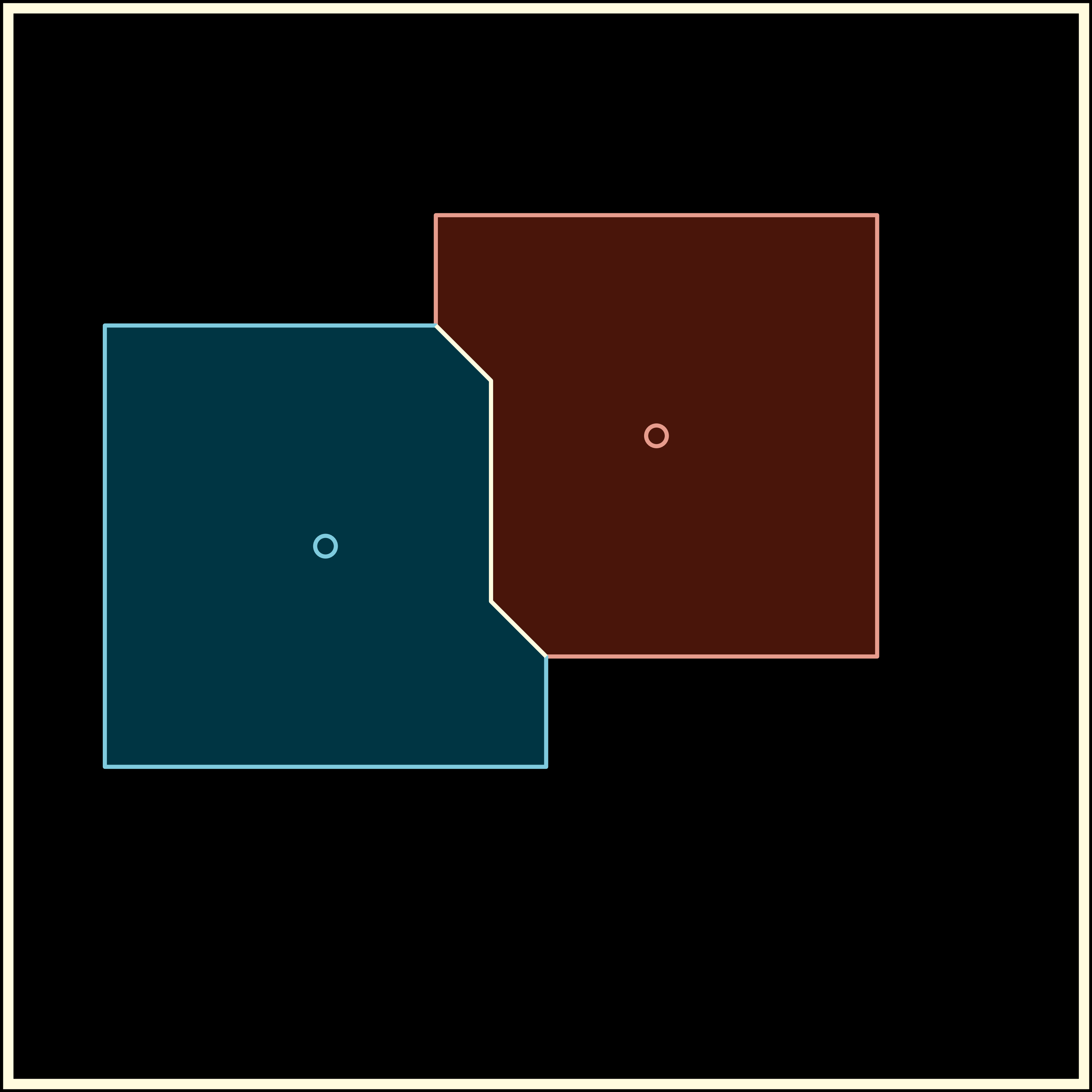


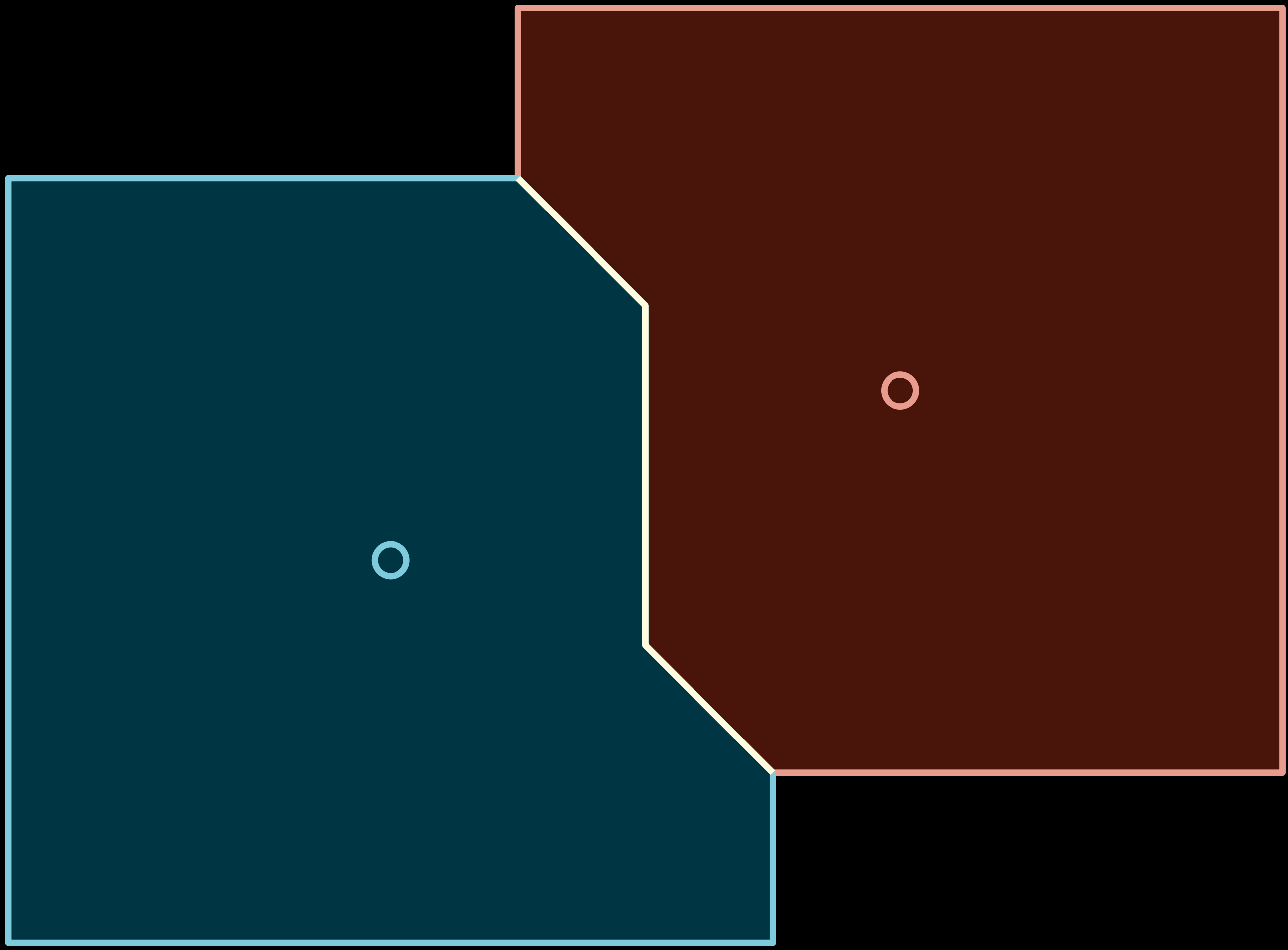


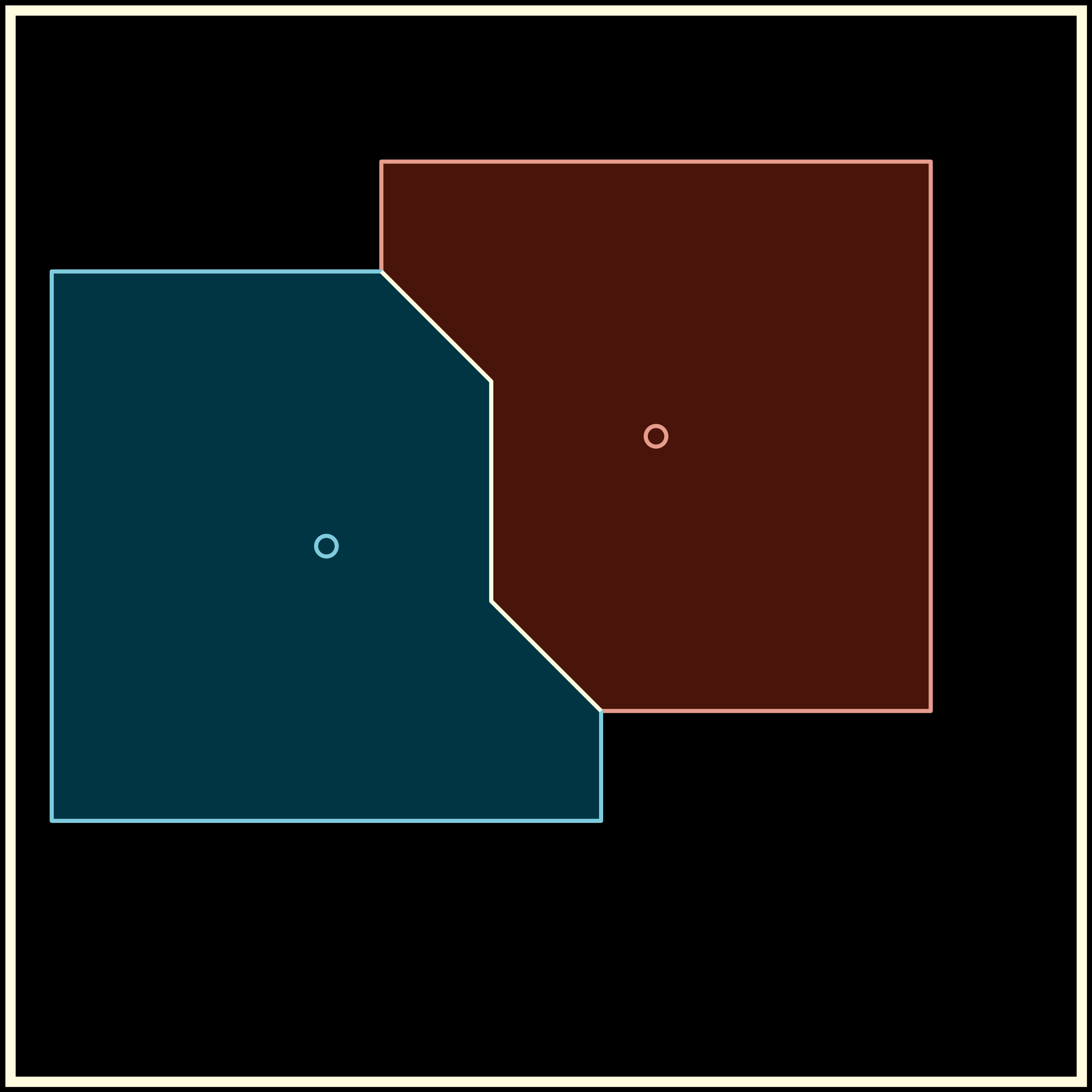


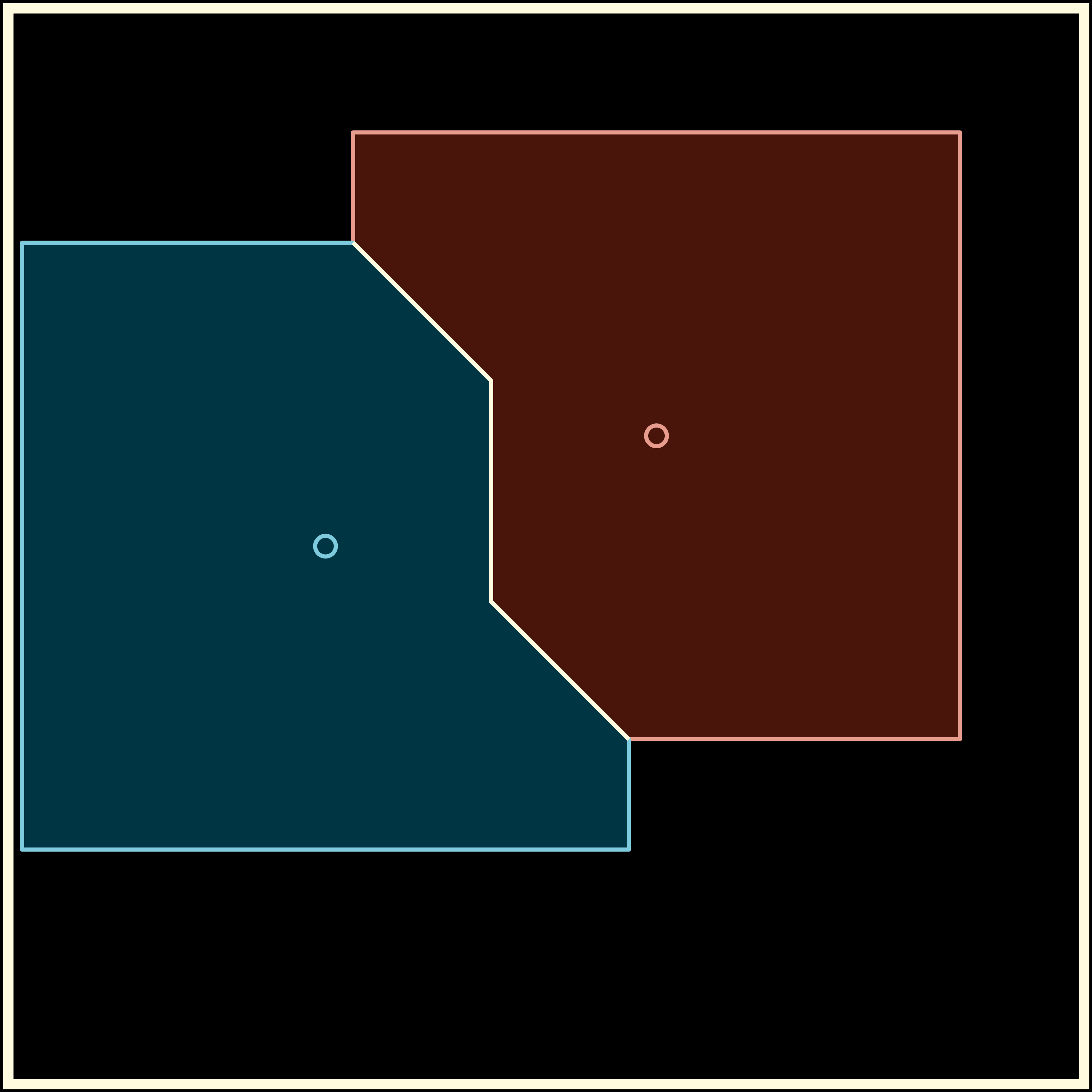


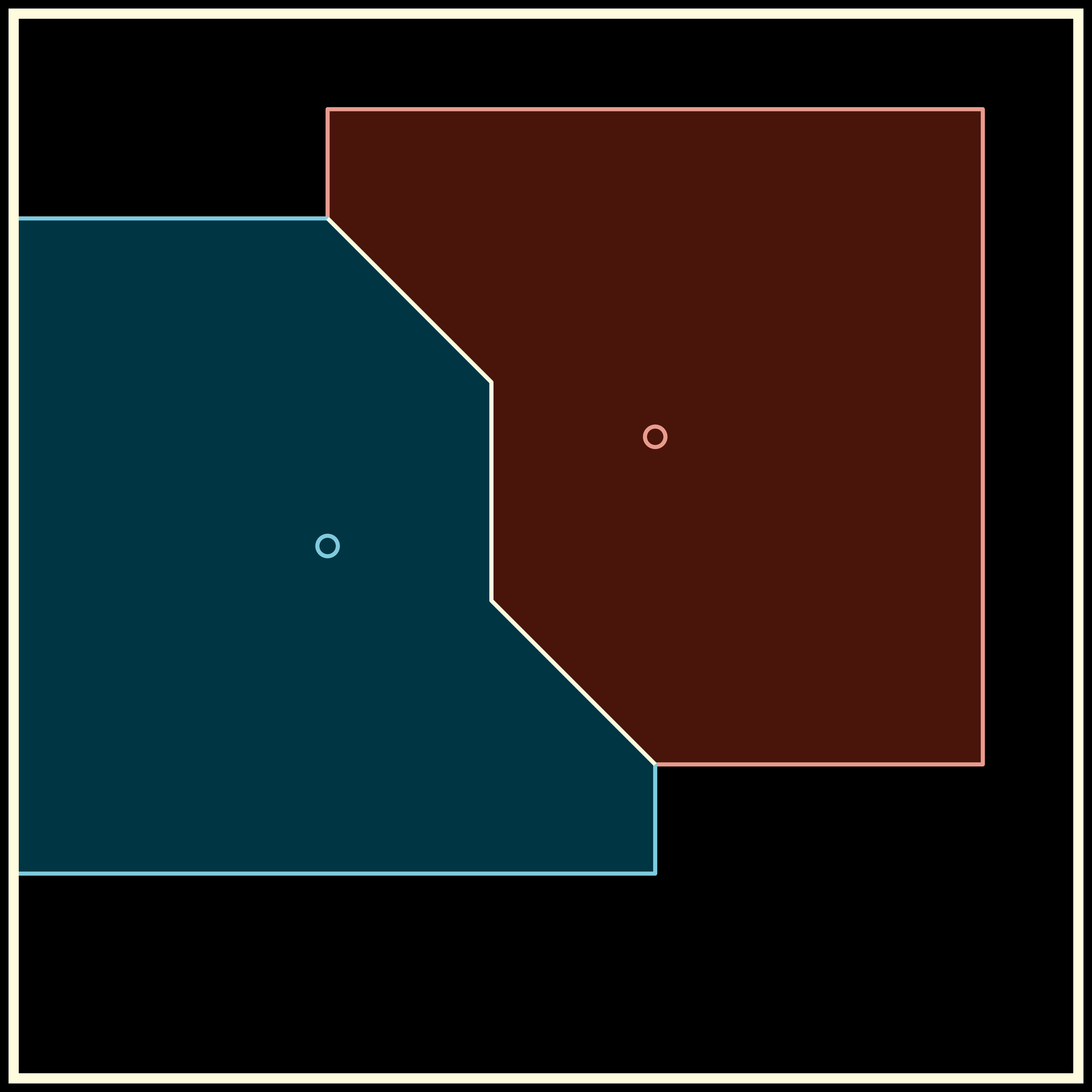


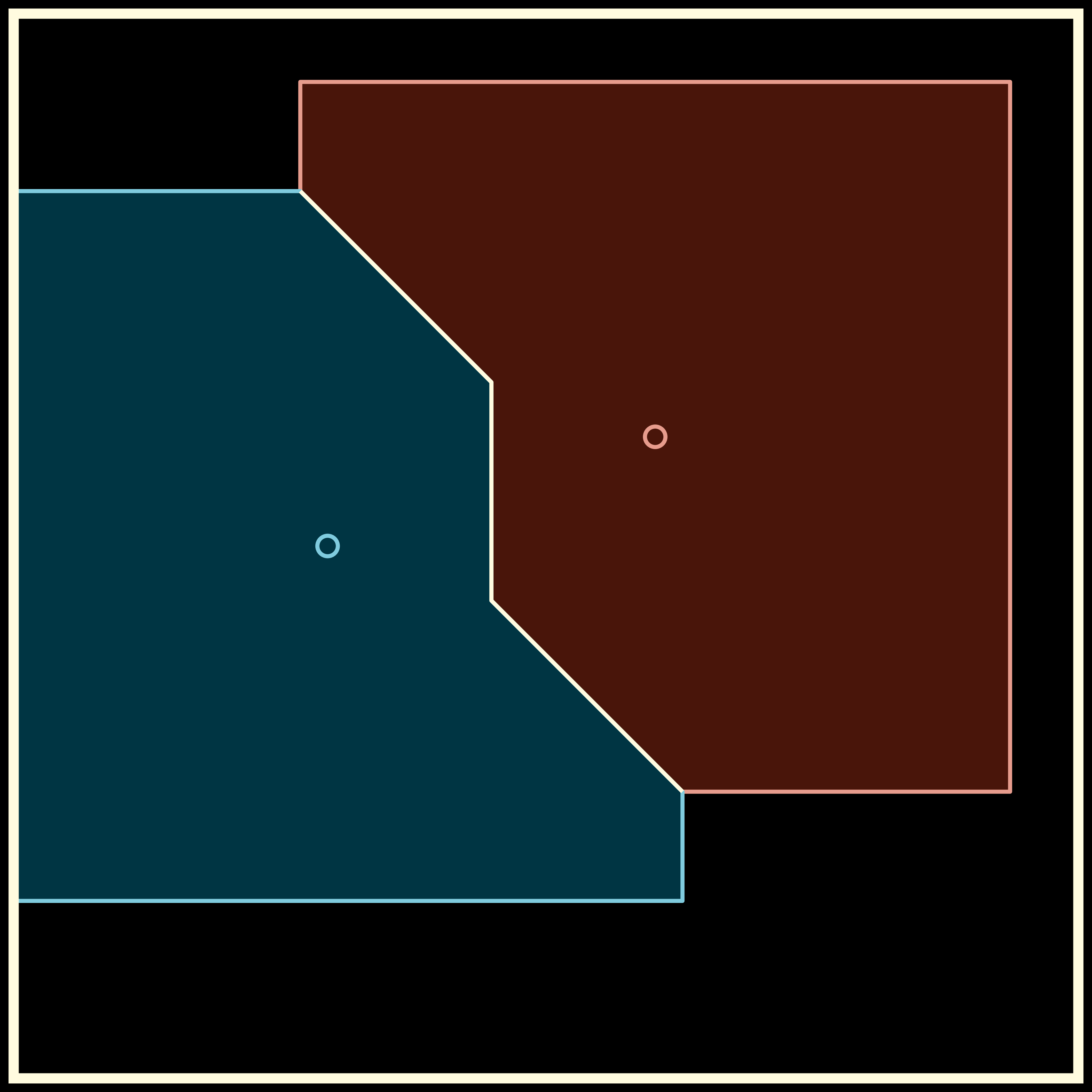


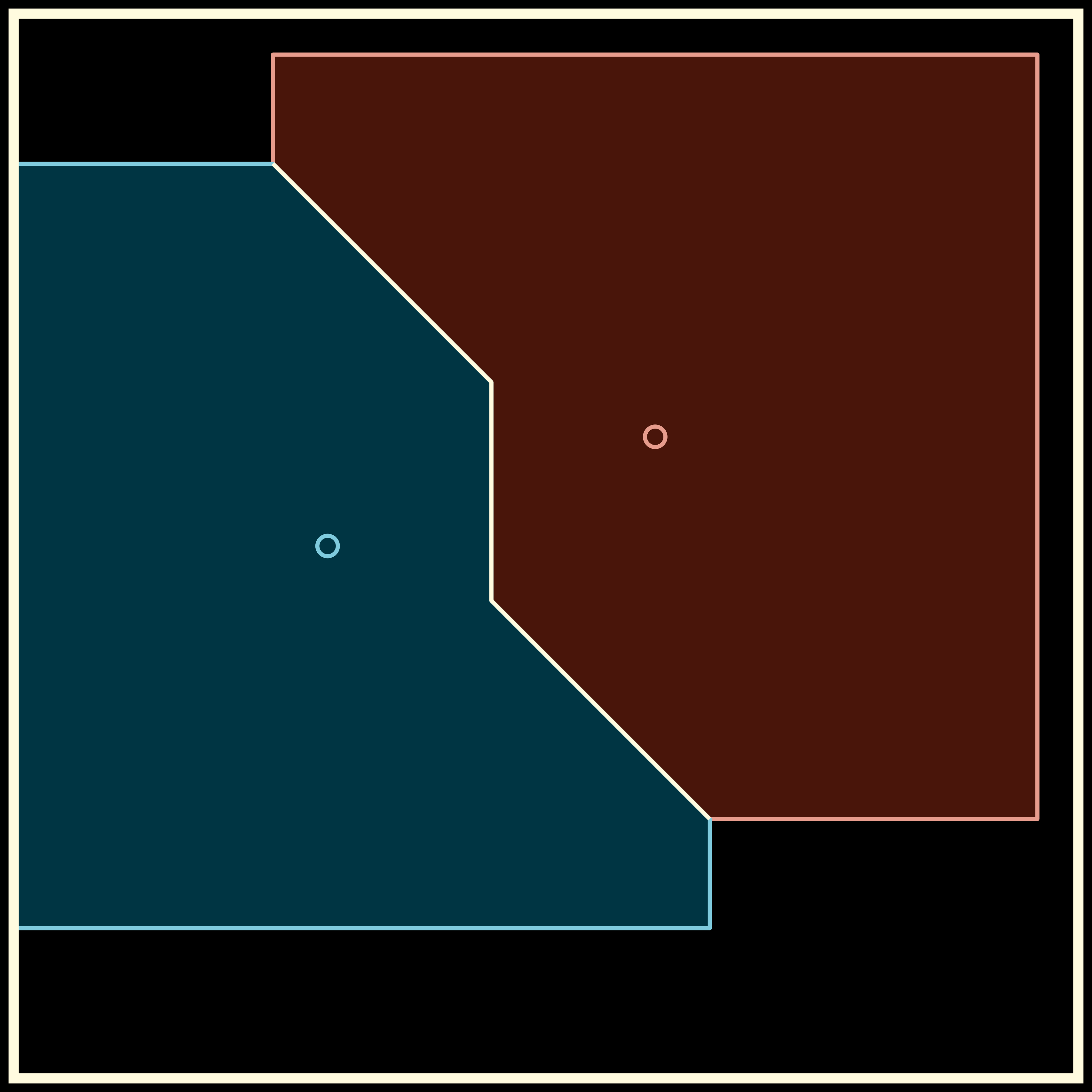


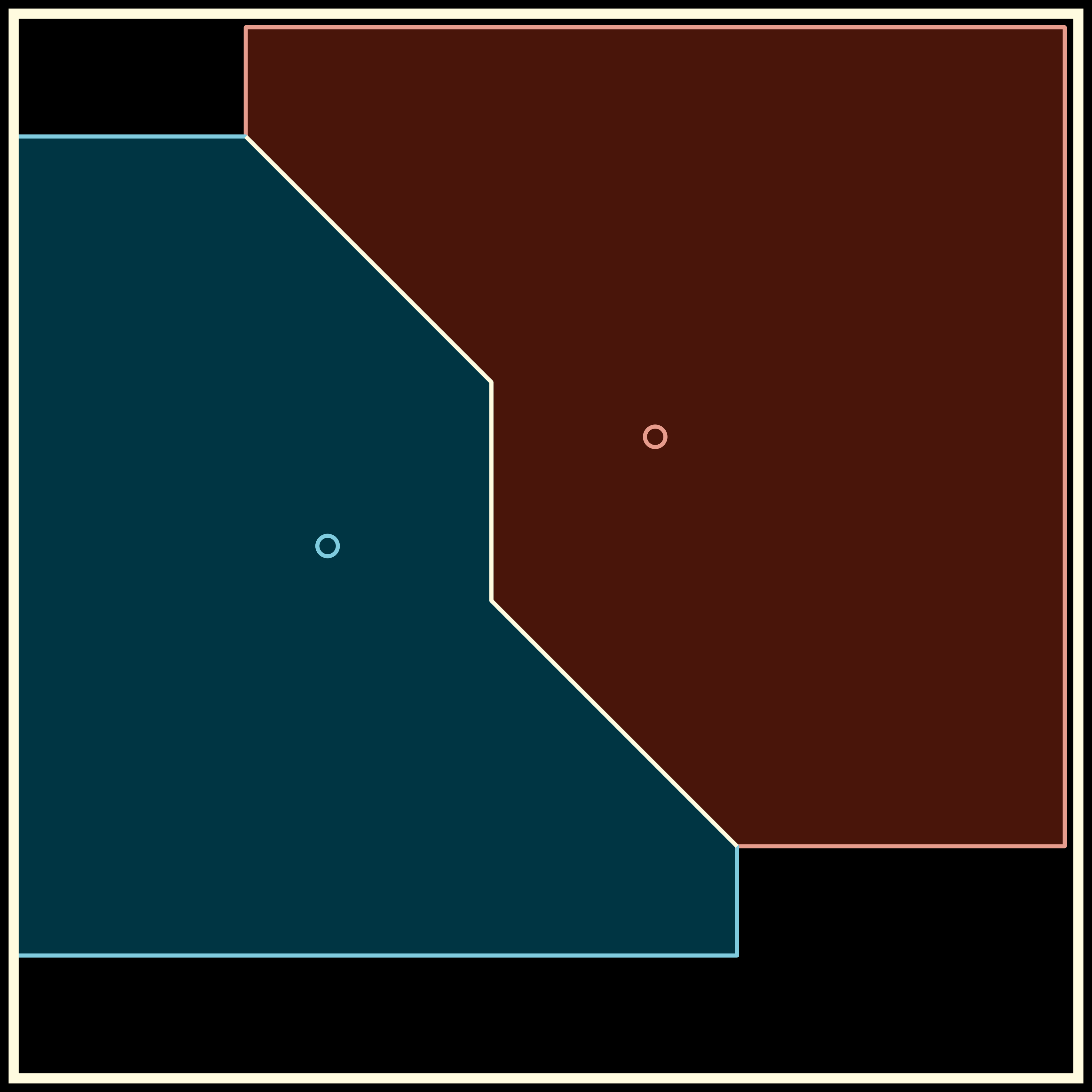


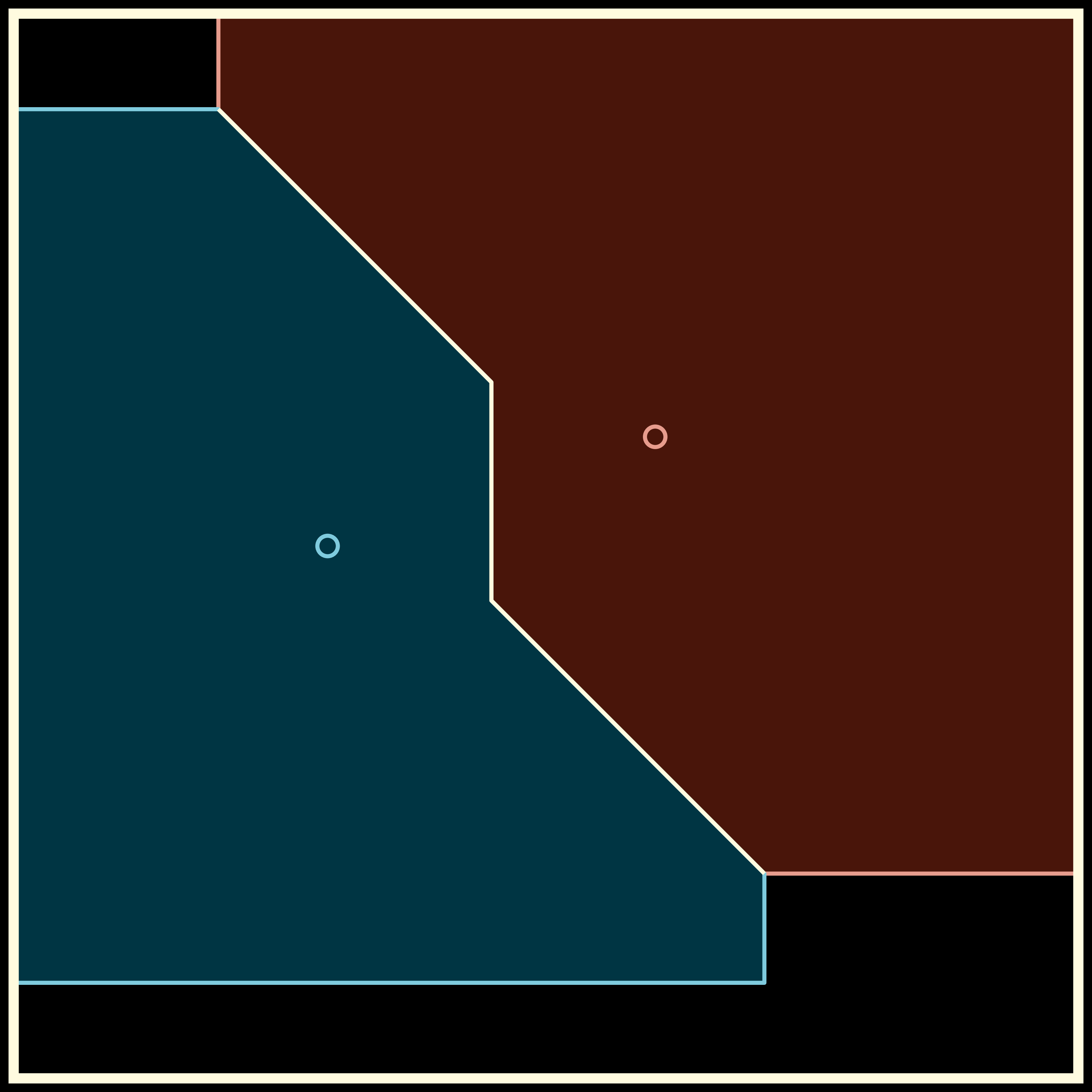


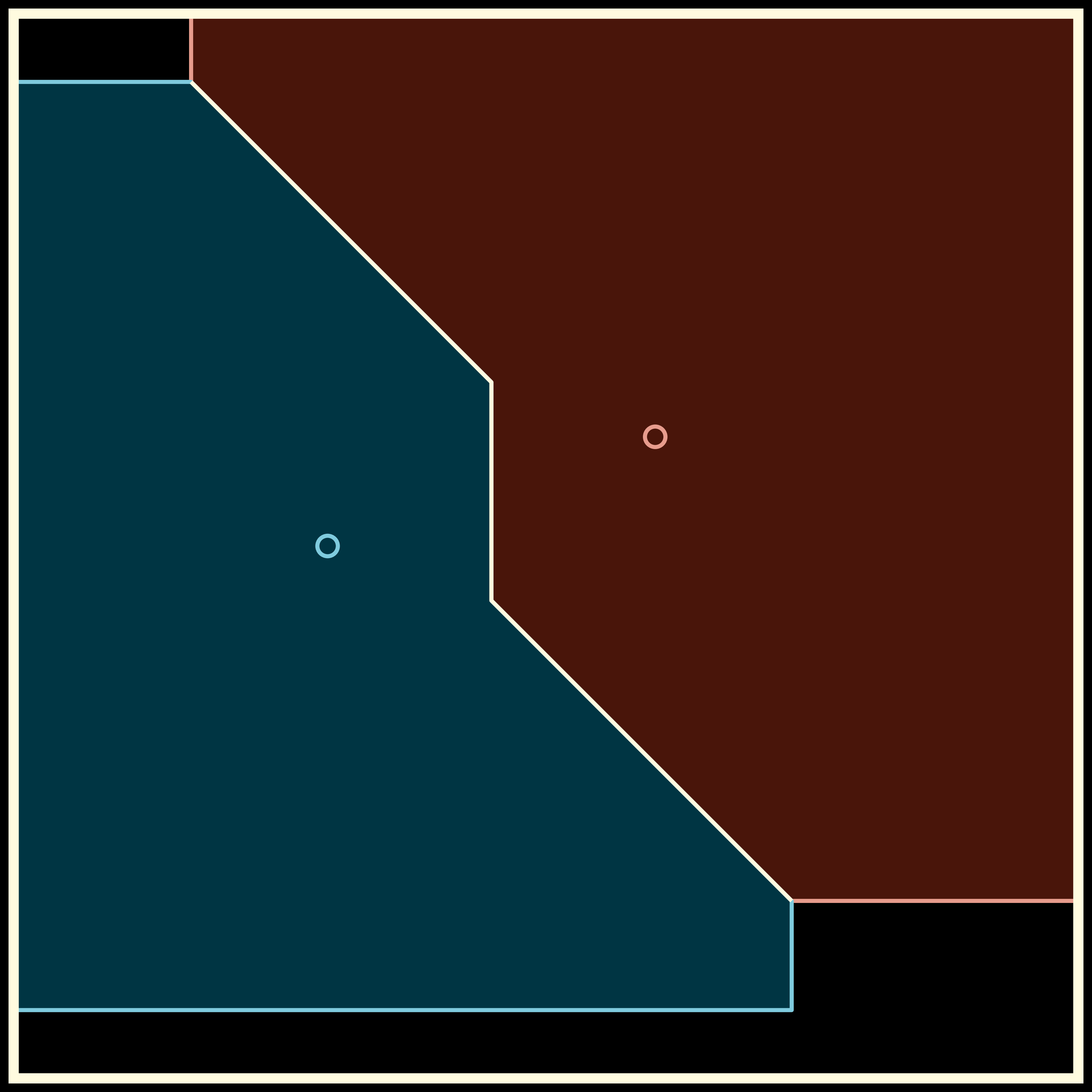


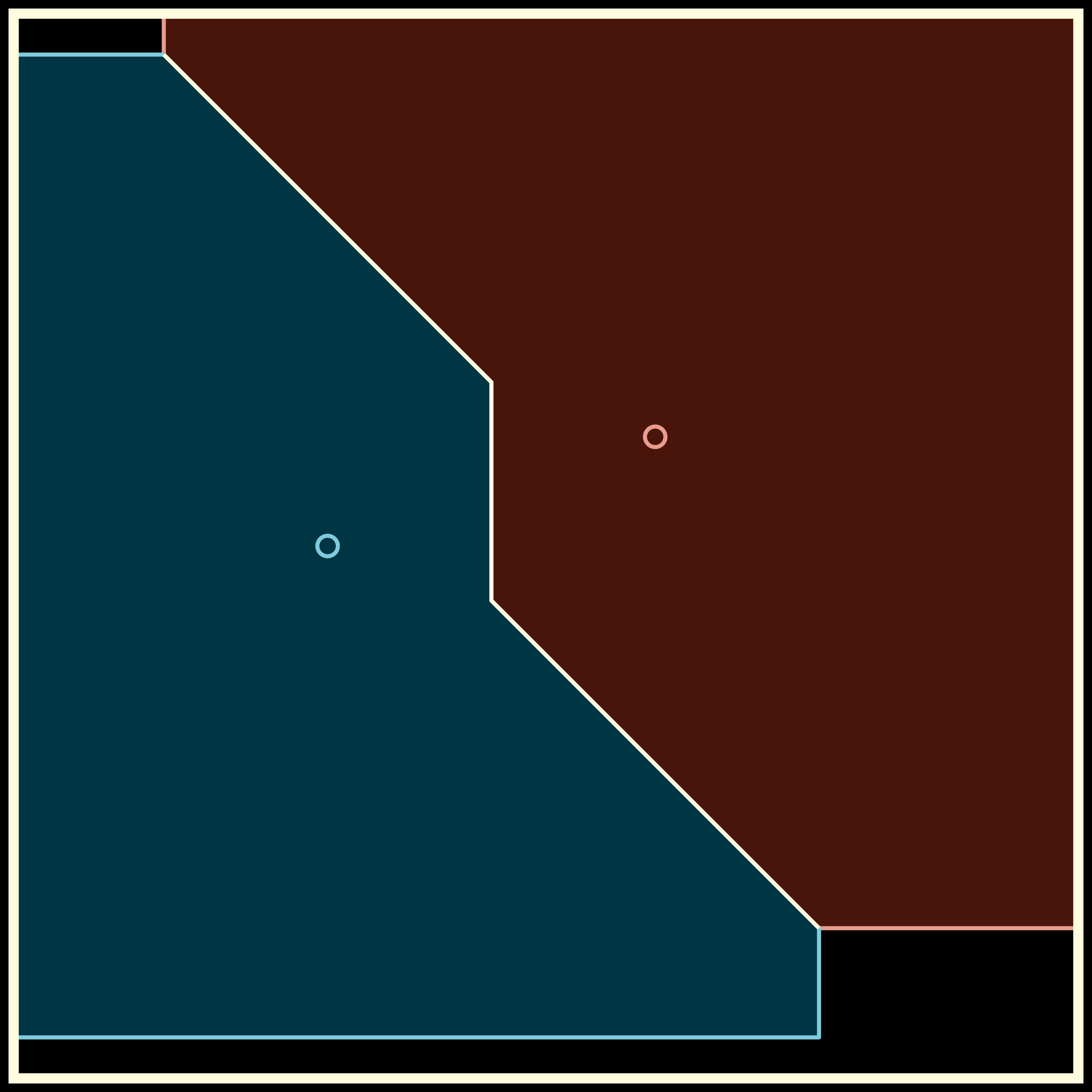


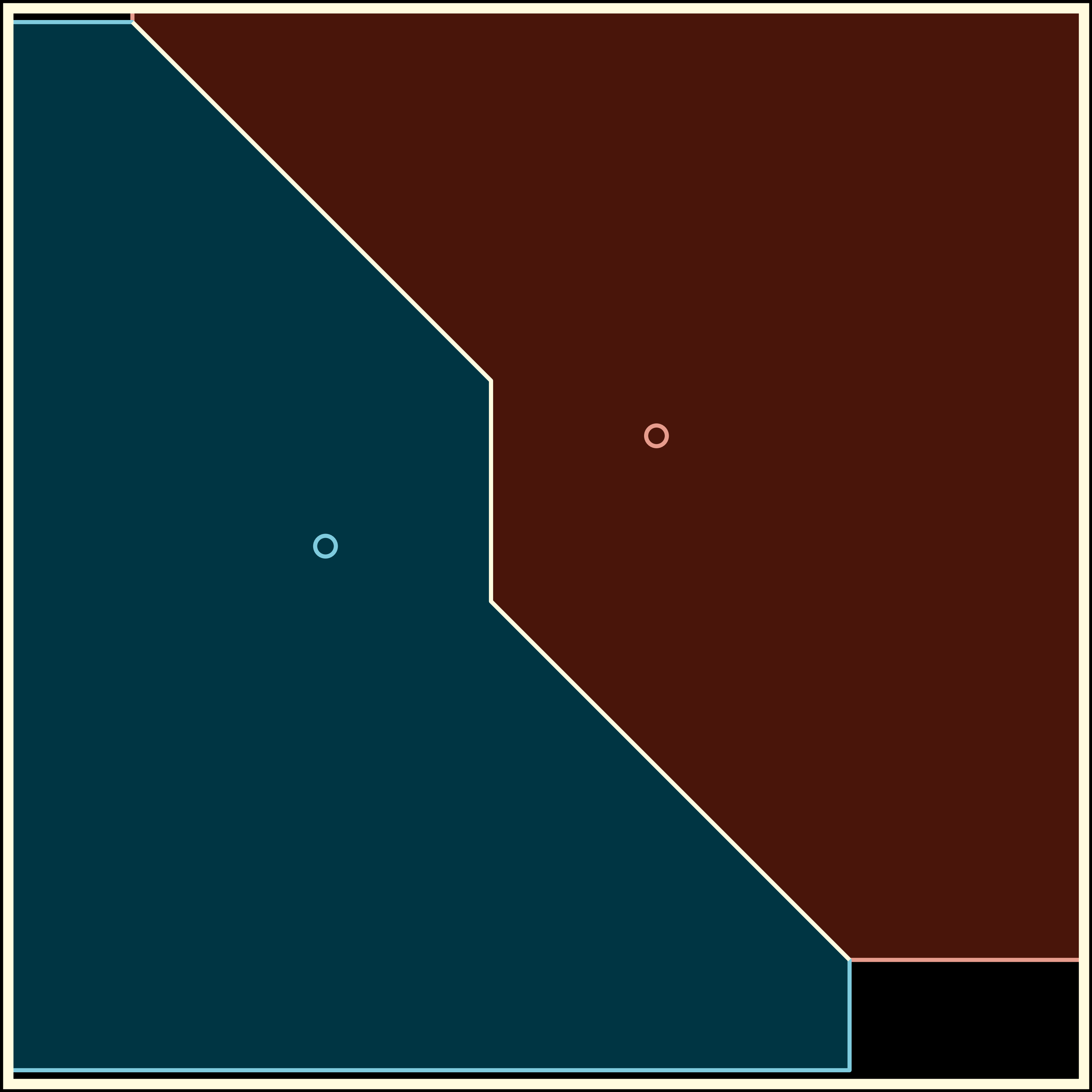


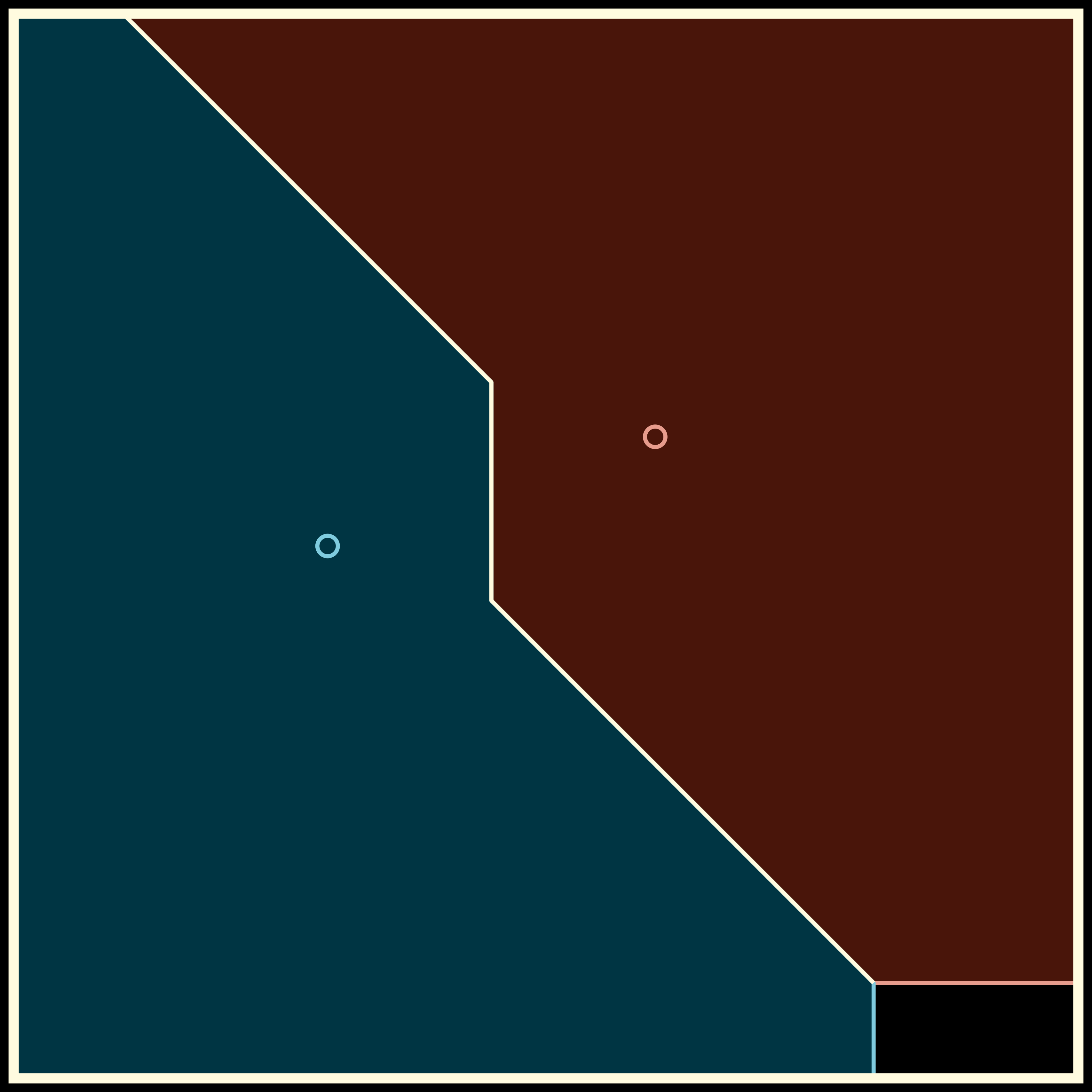


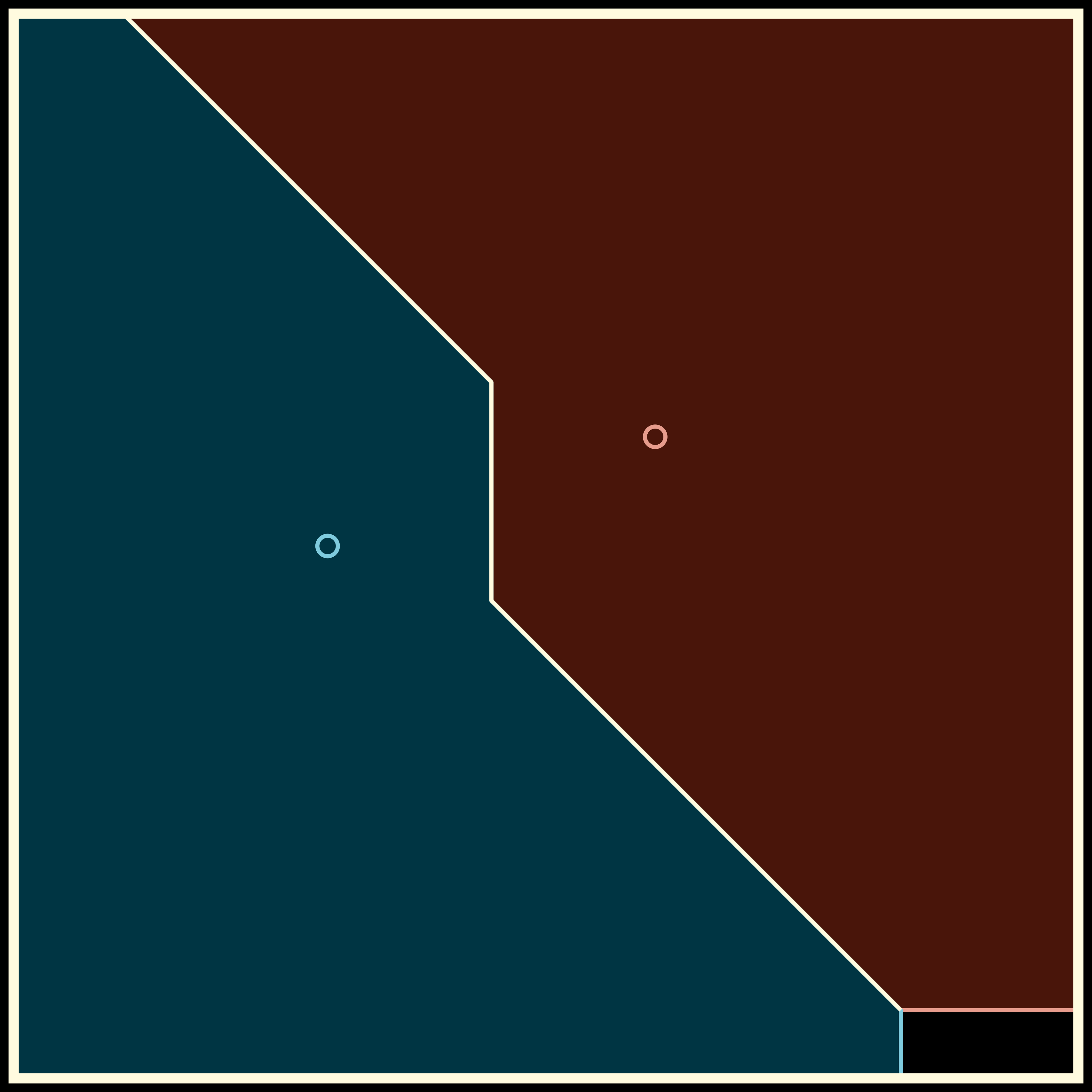


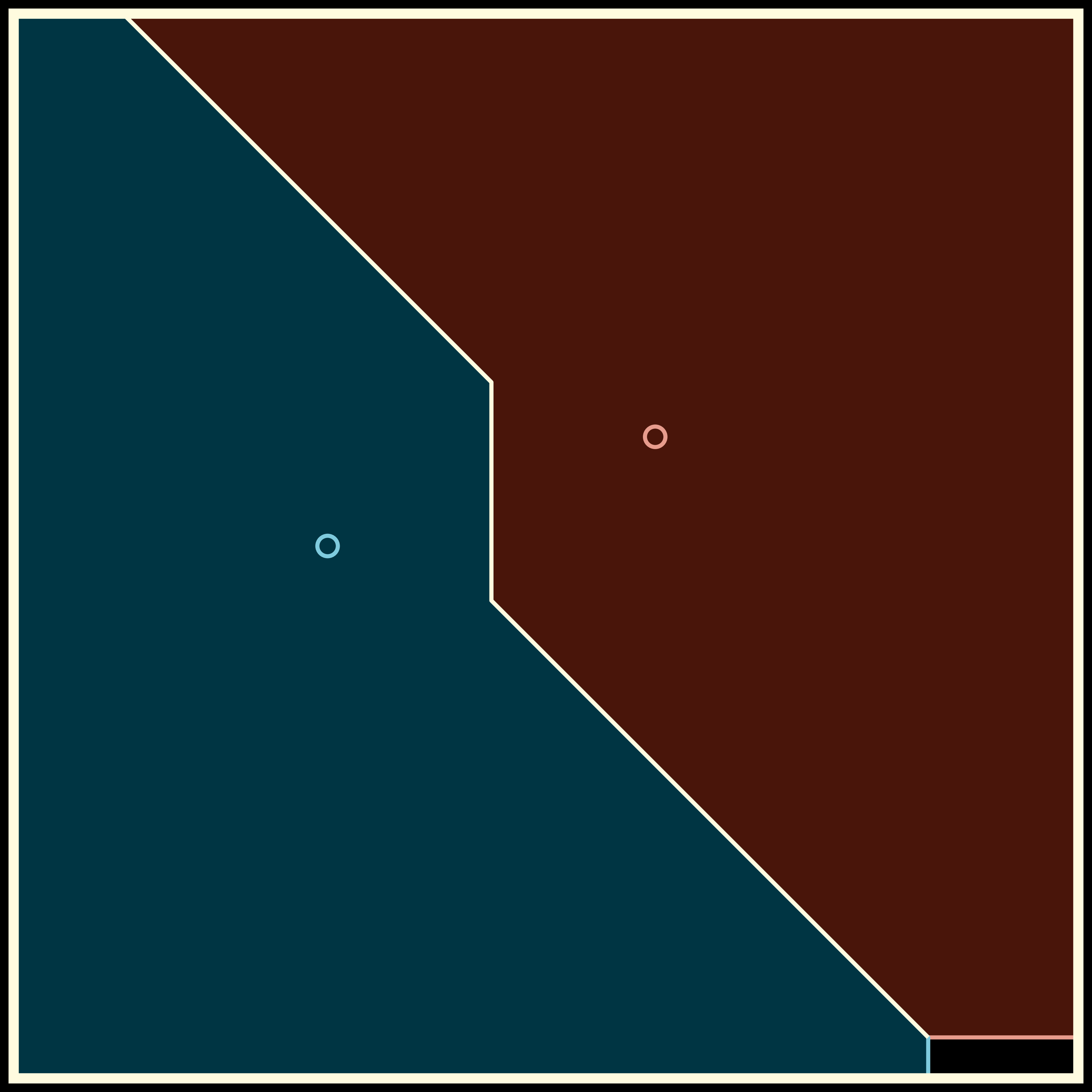


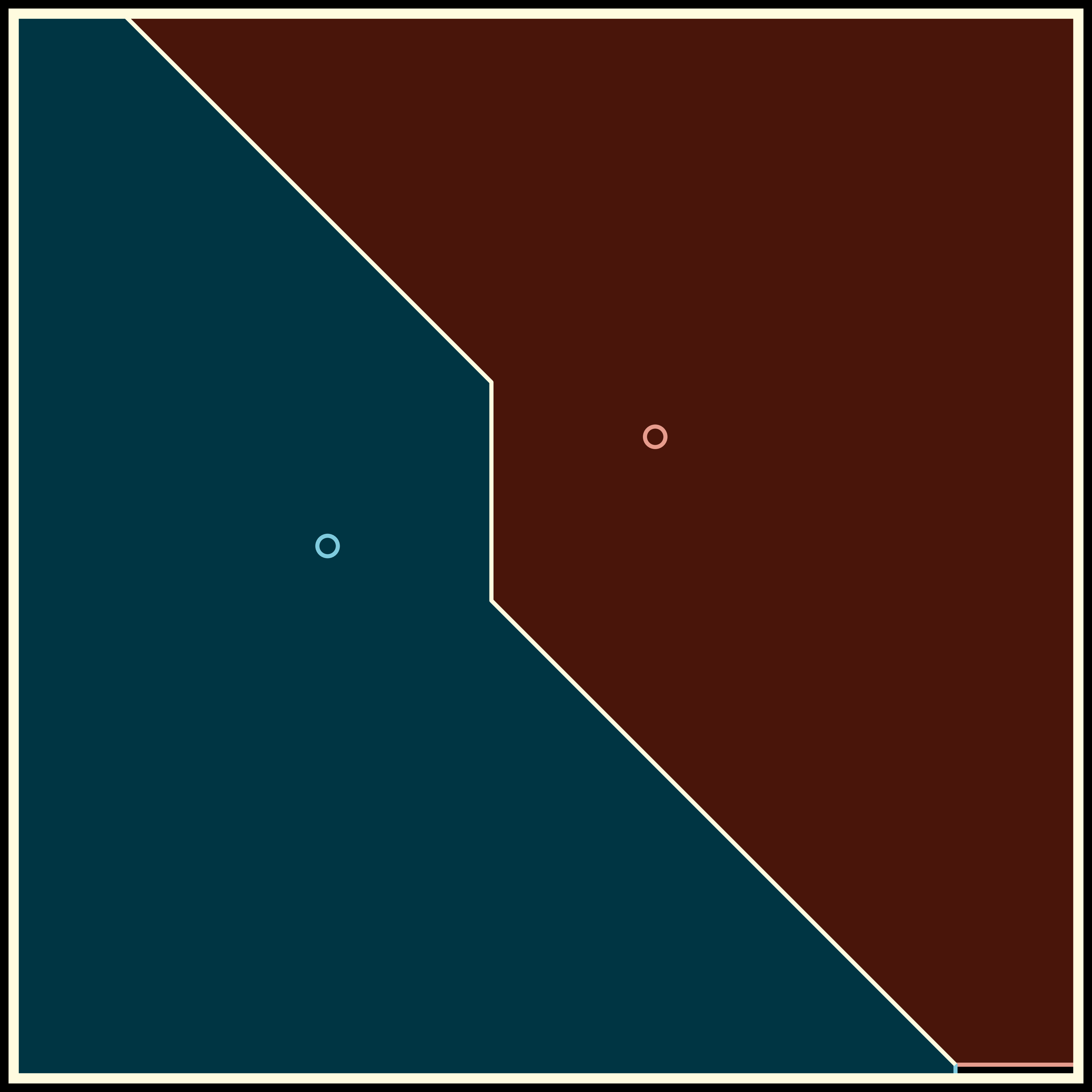


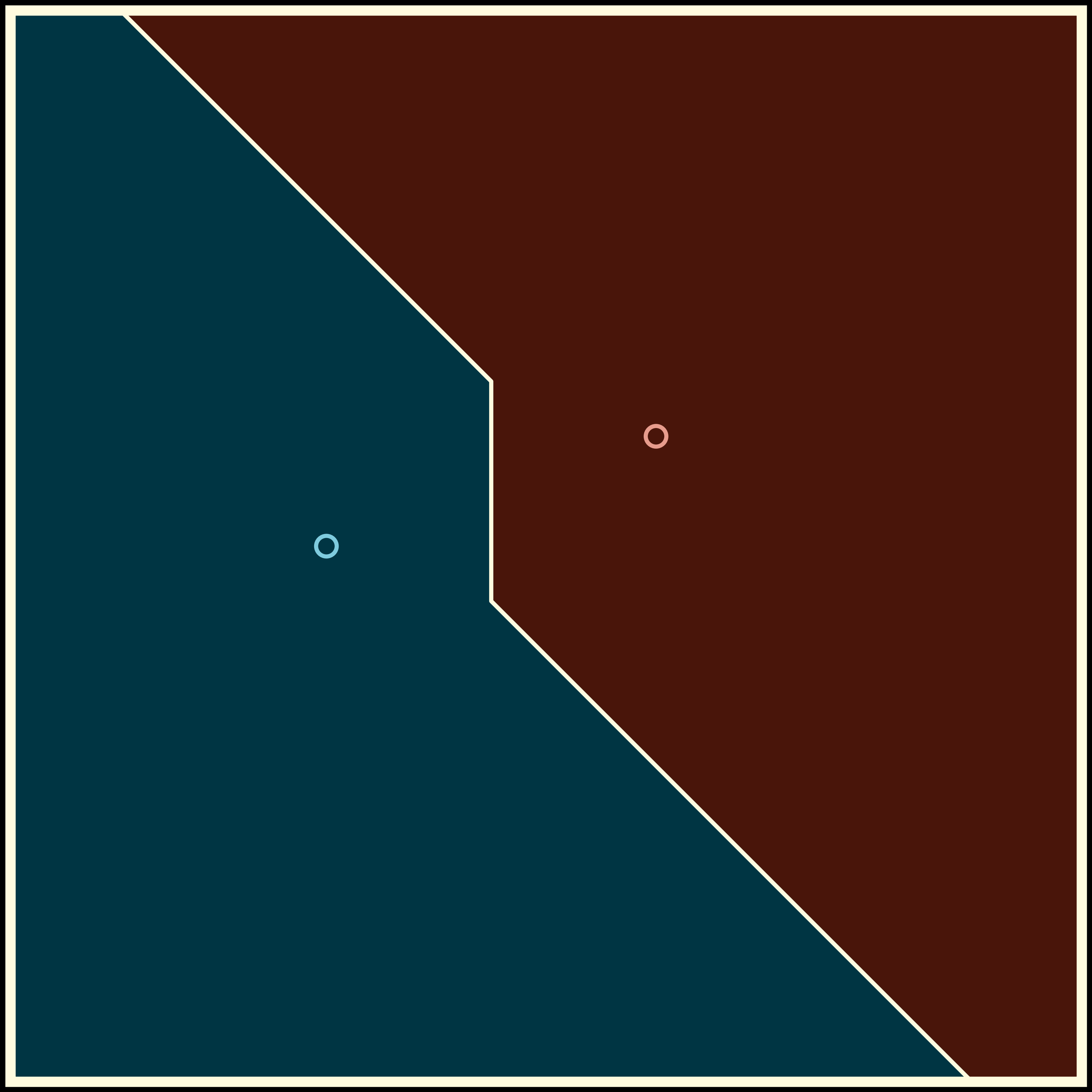


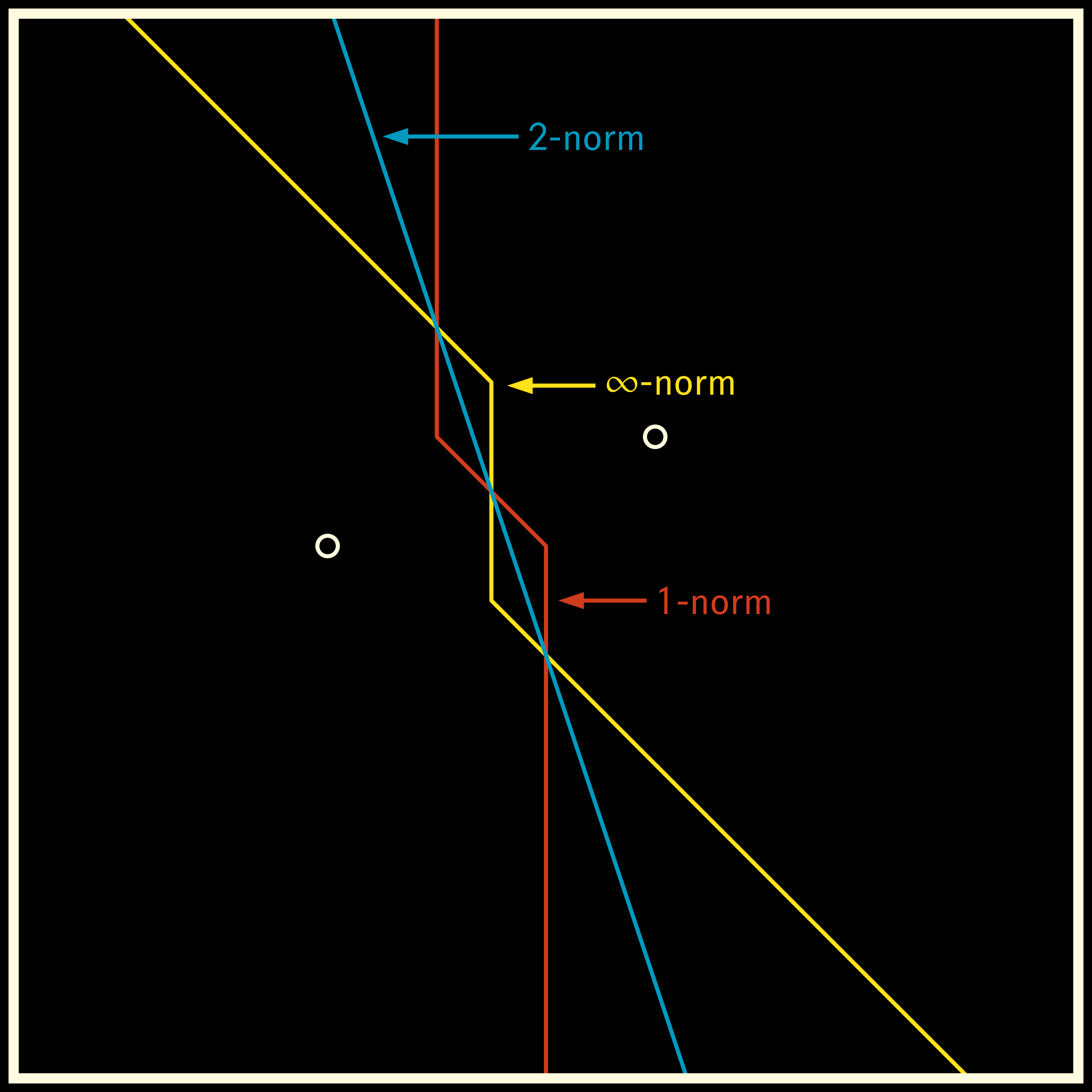


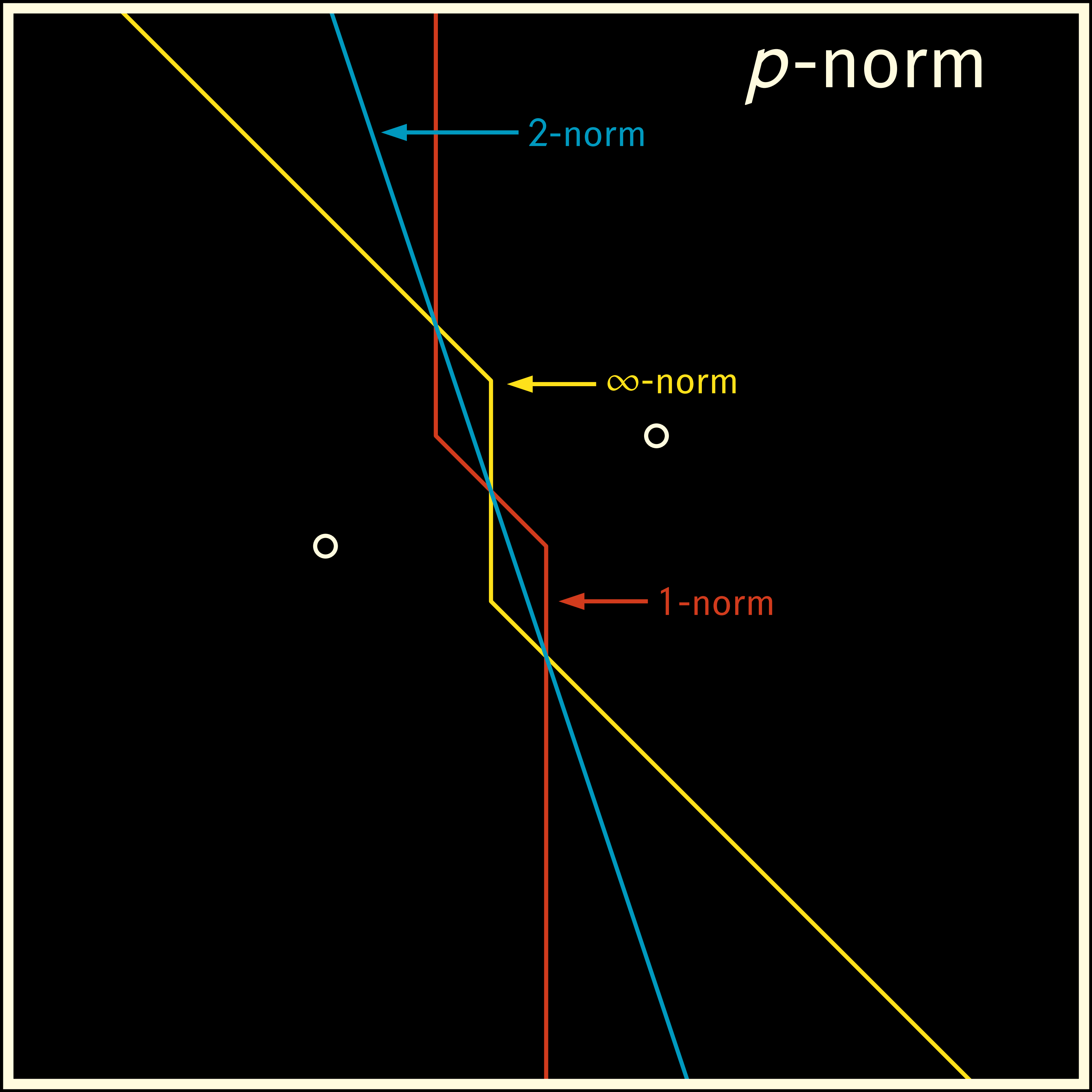




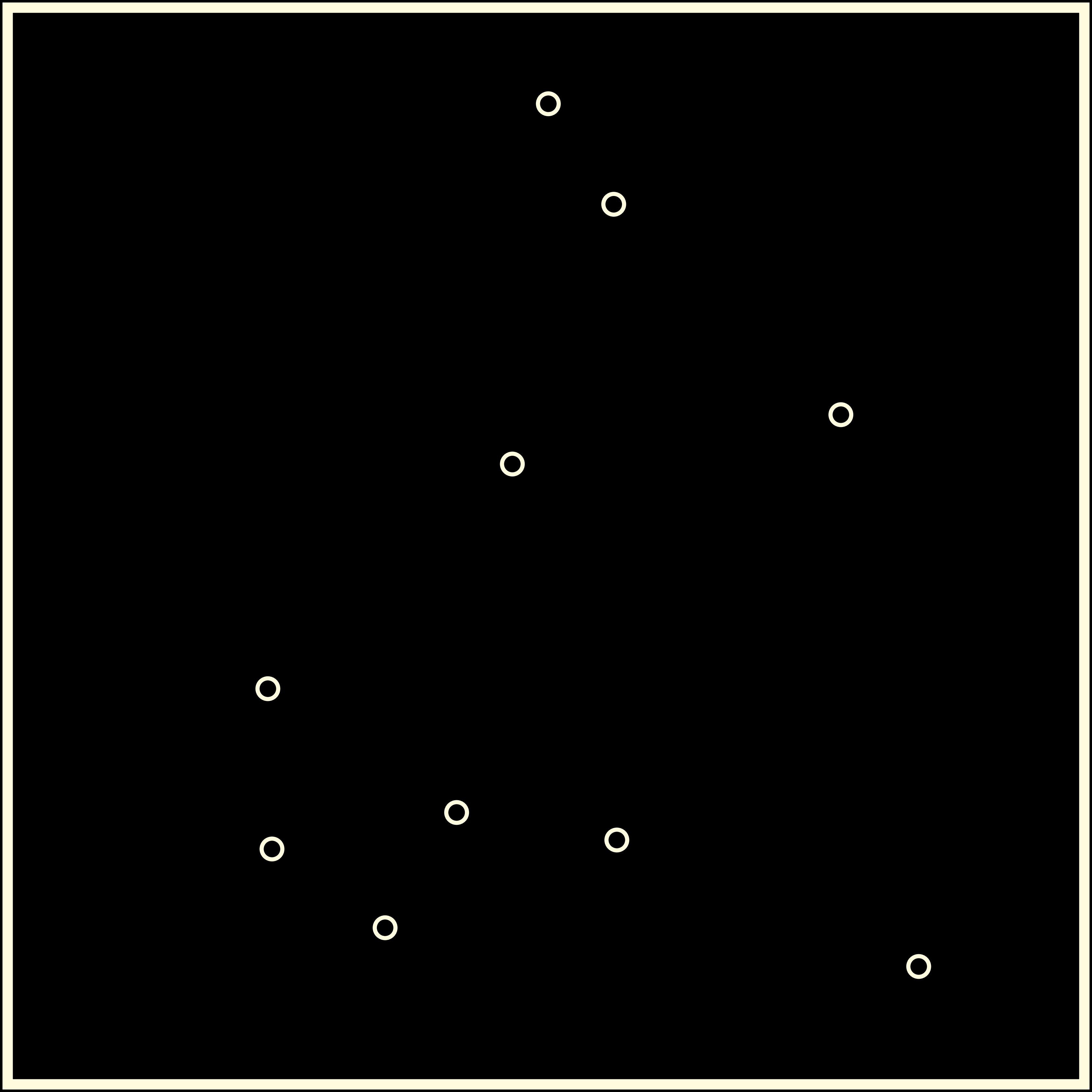


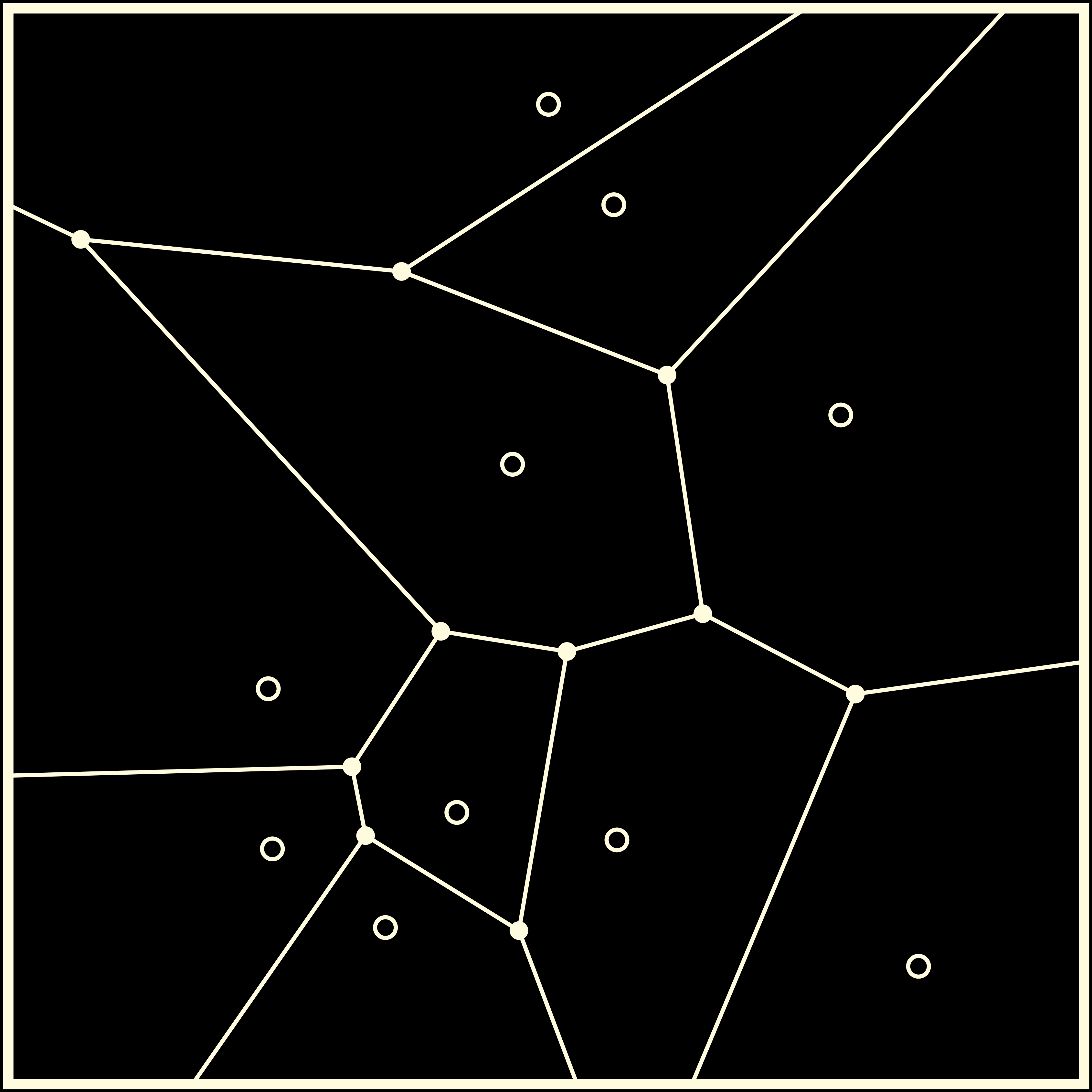


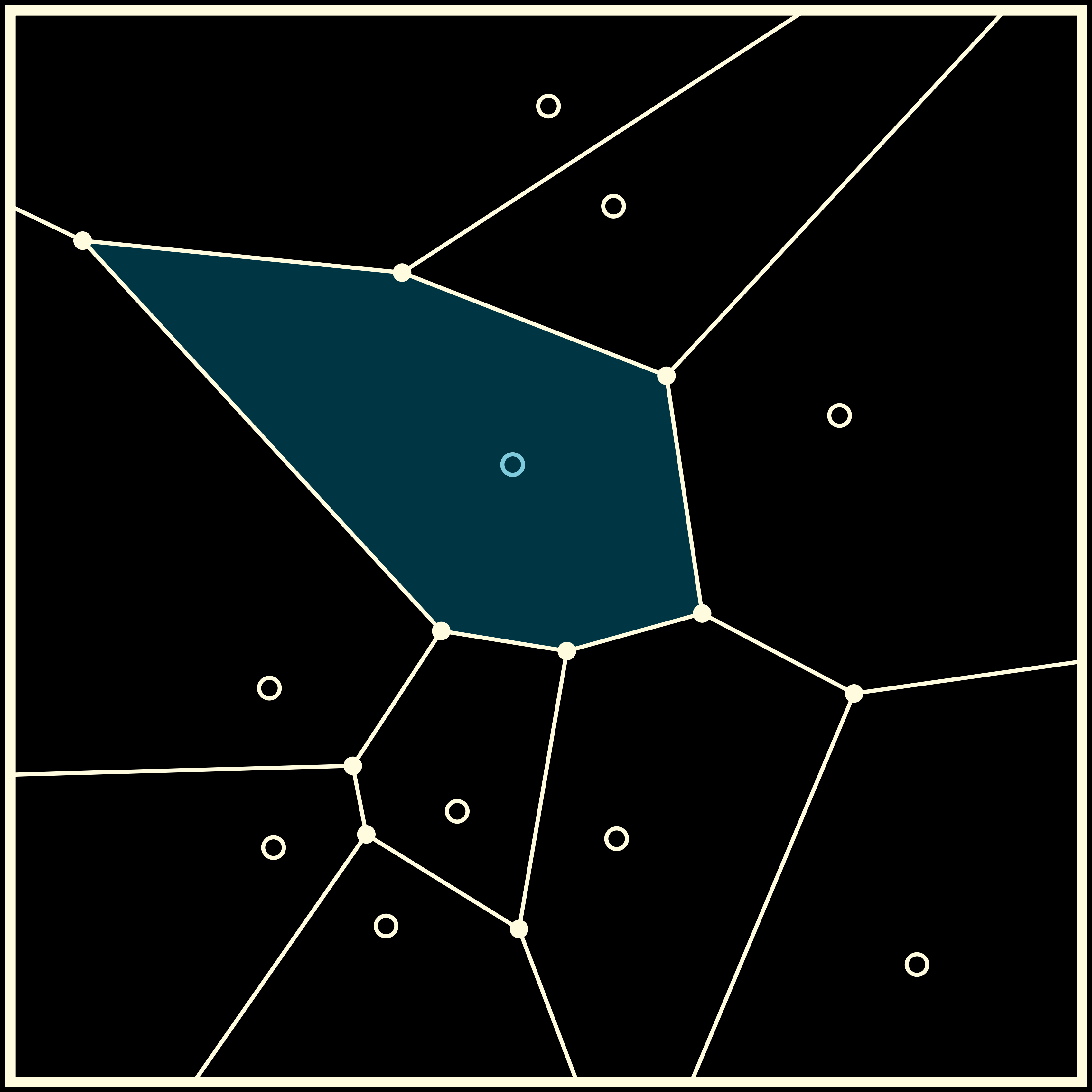

2-norm
∞-norm
1-norm

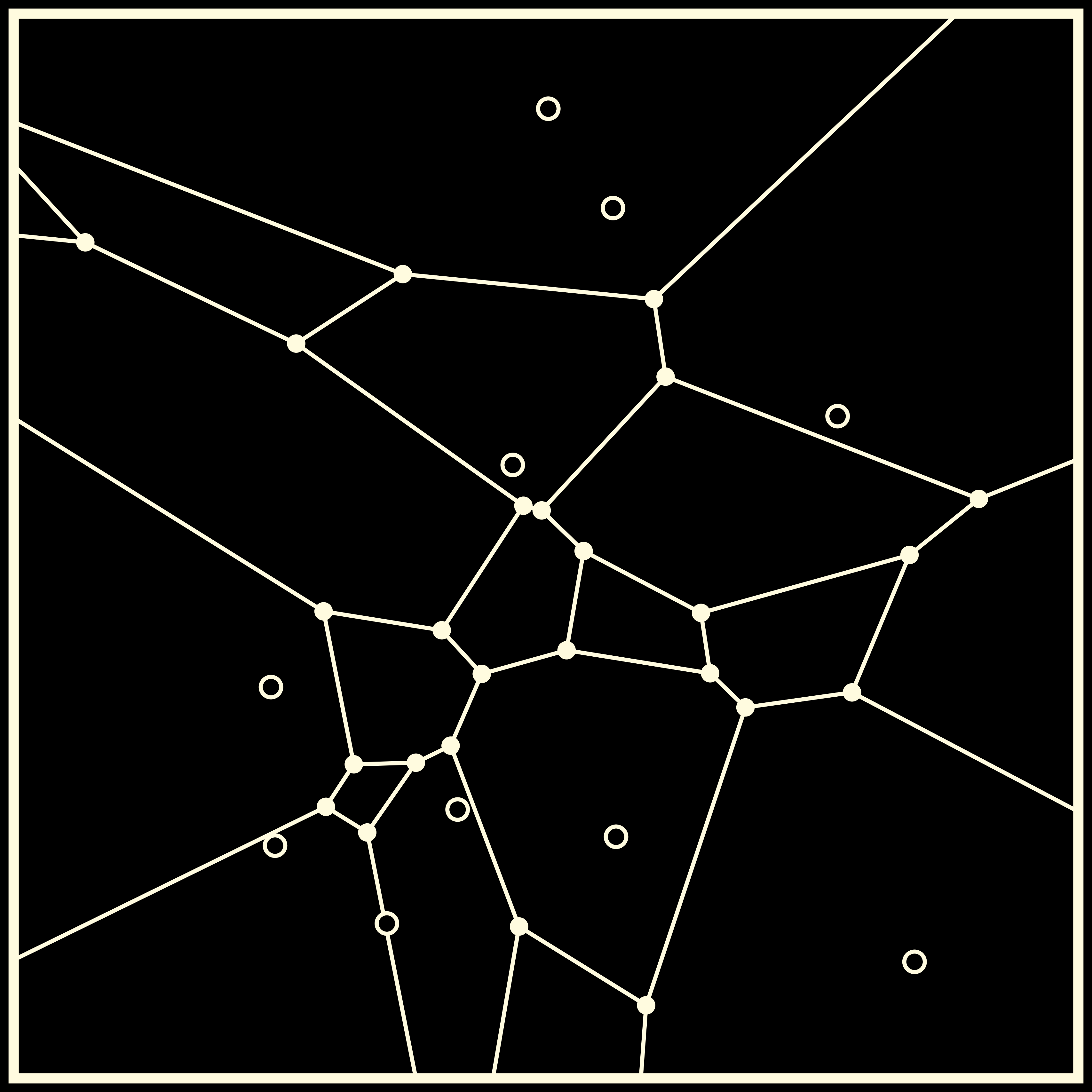


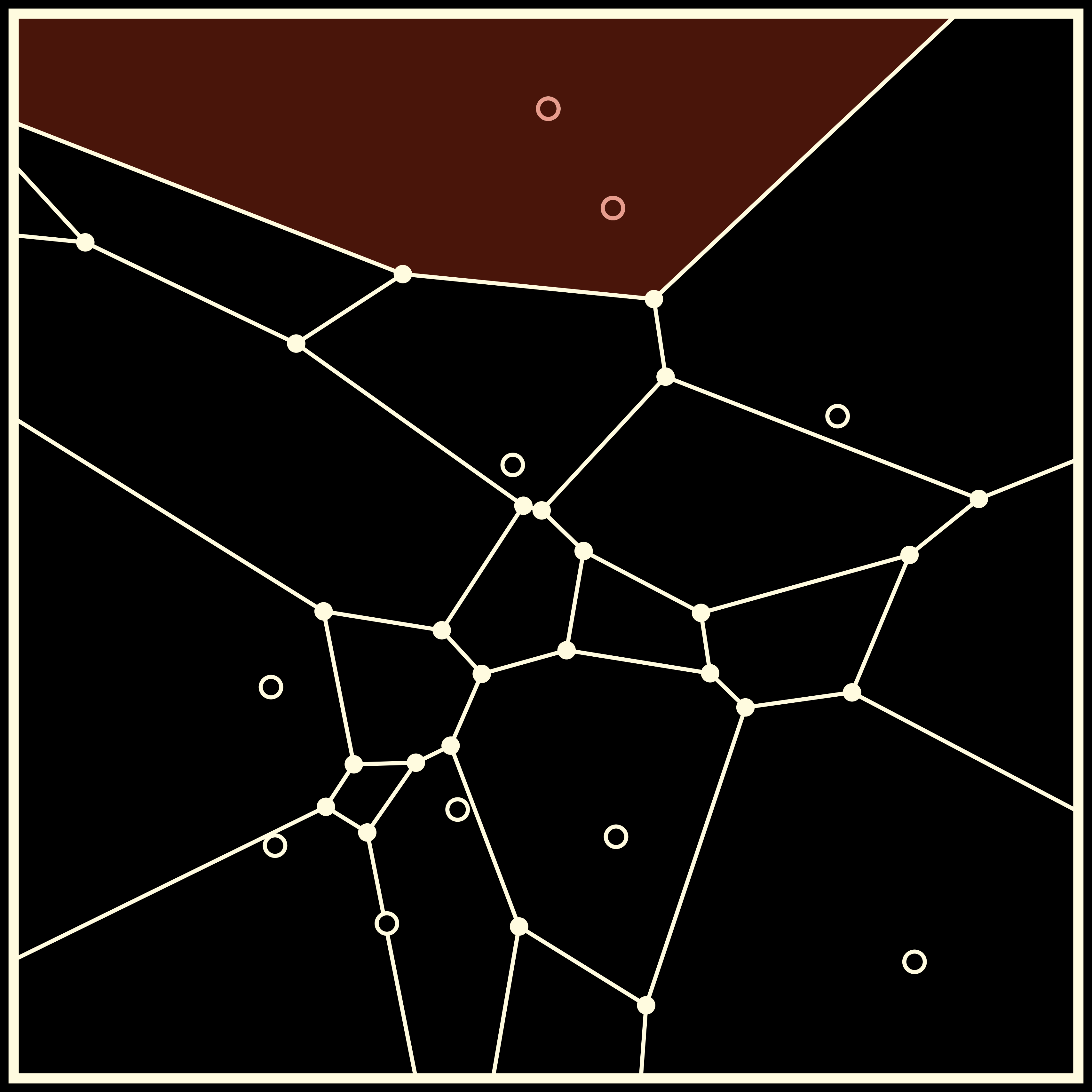
p-norm
2-norm
∞-norm
1-norm

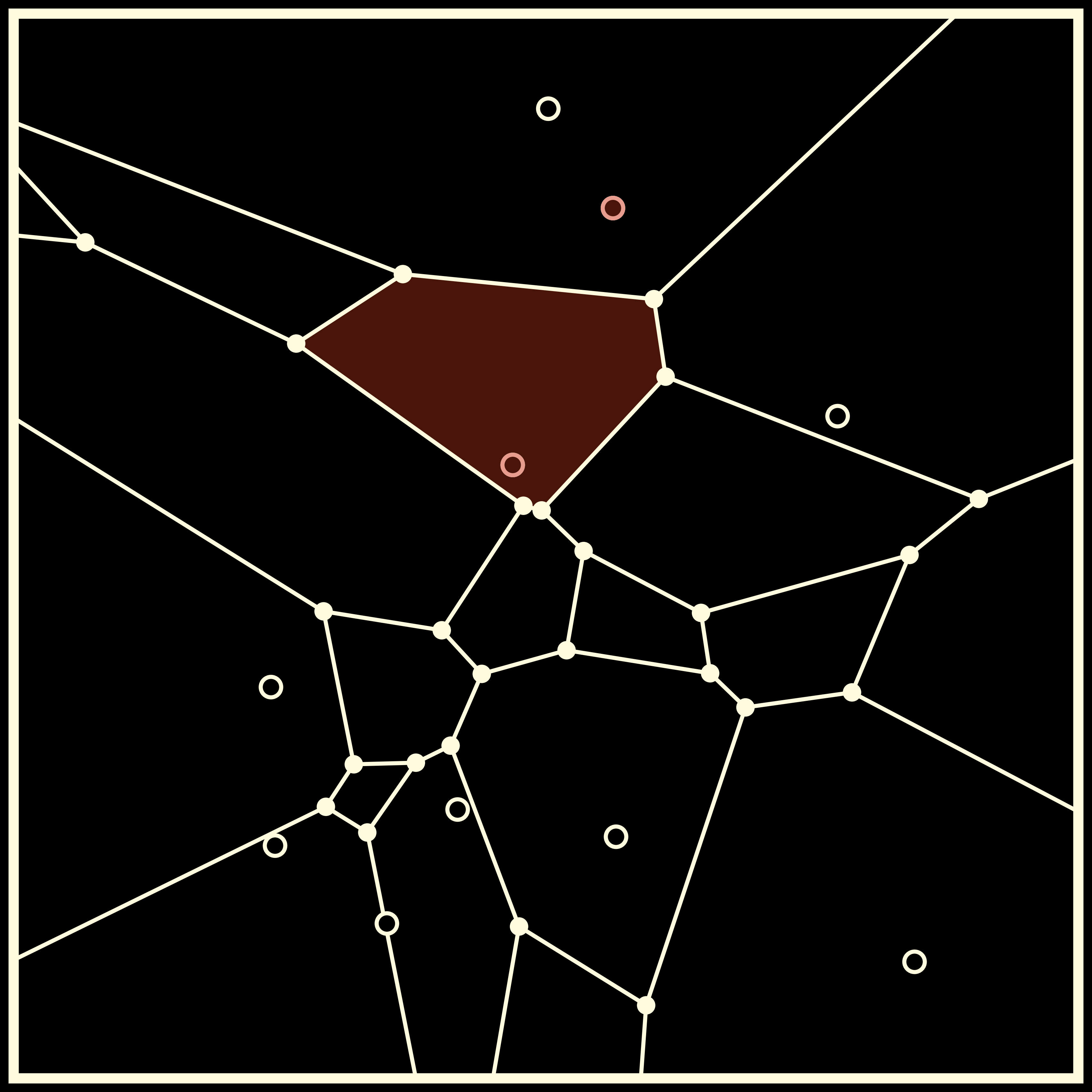


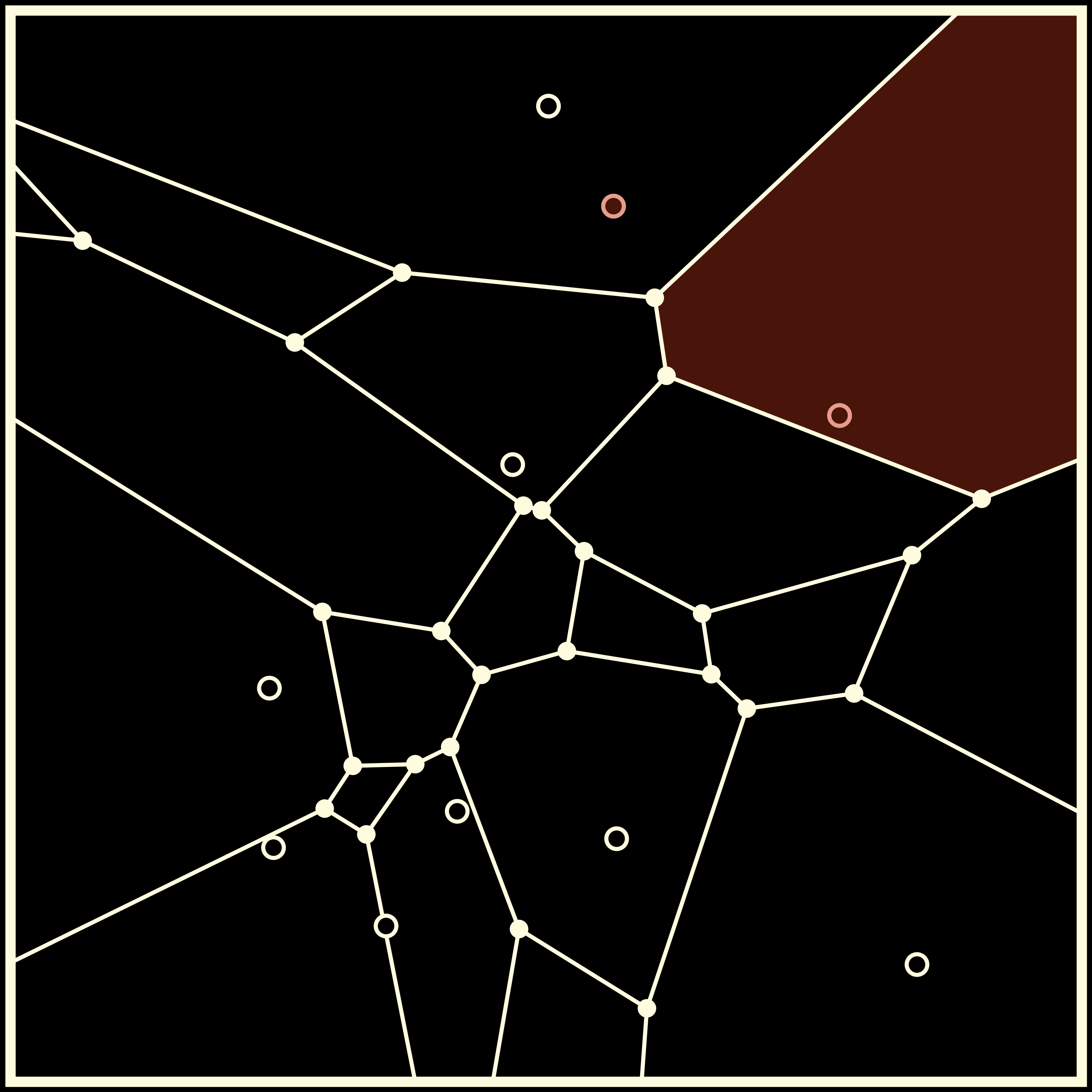


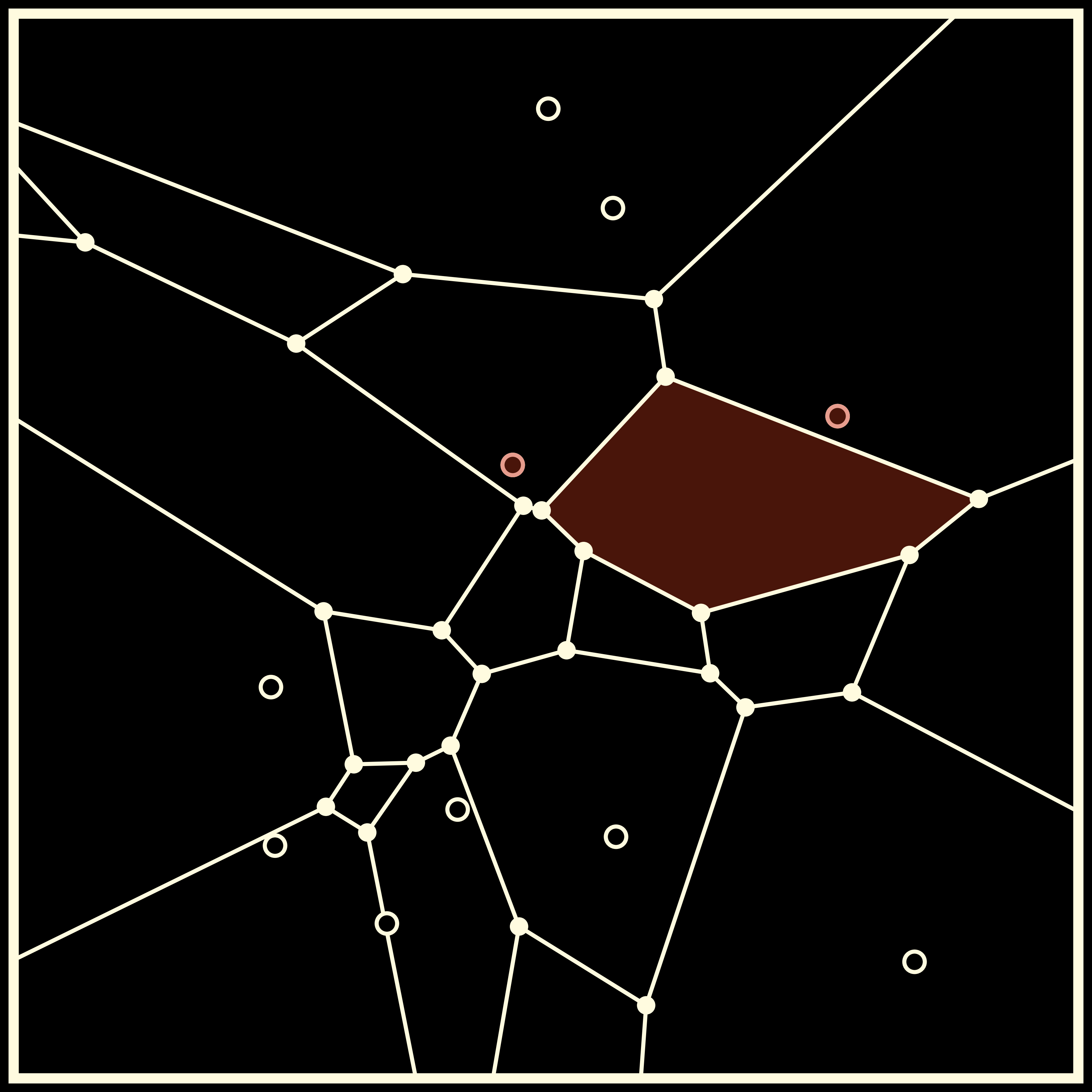


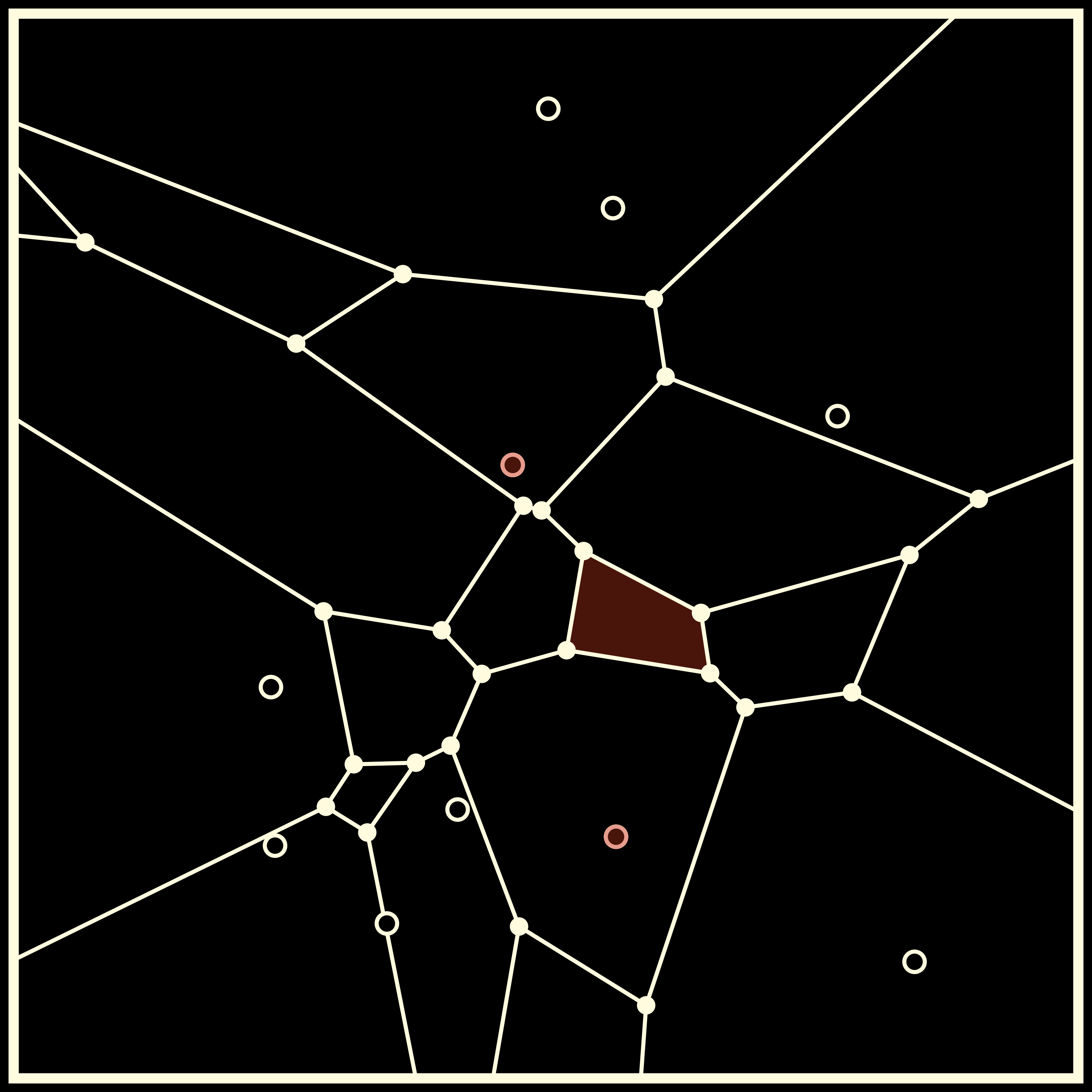


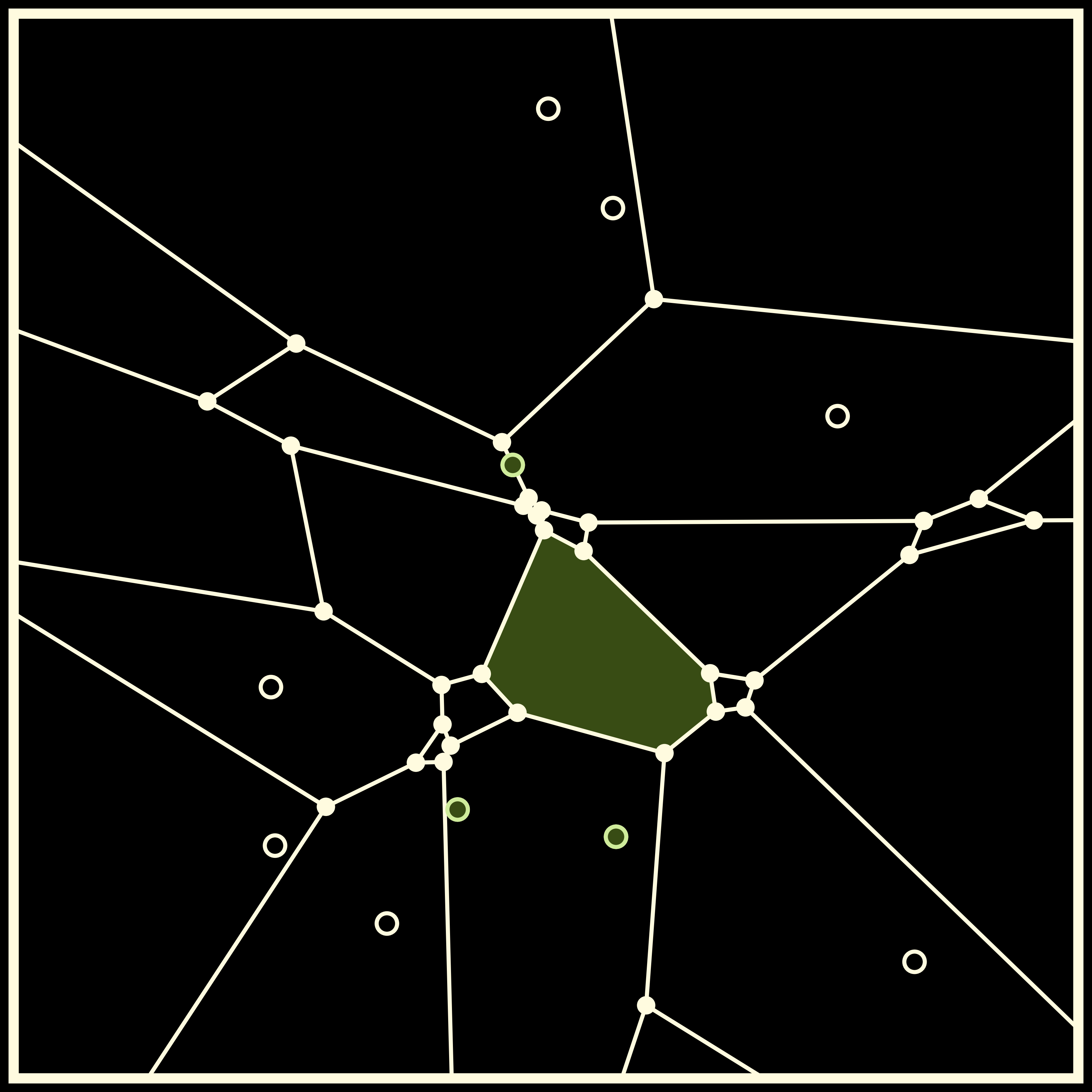


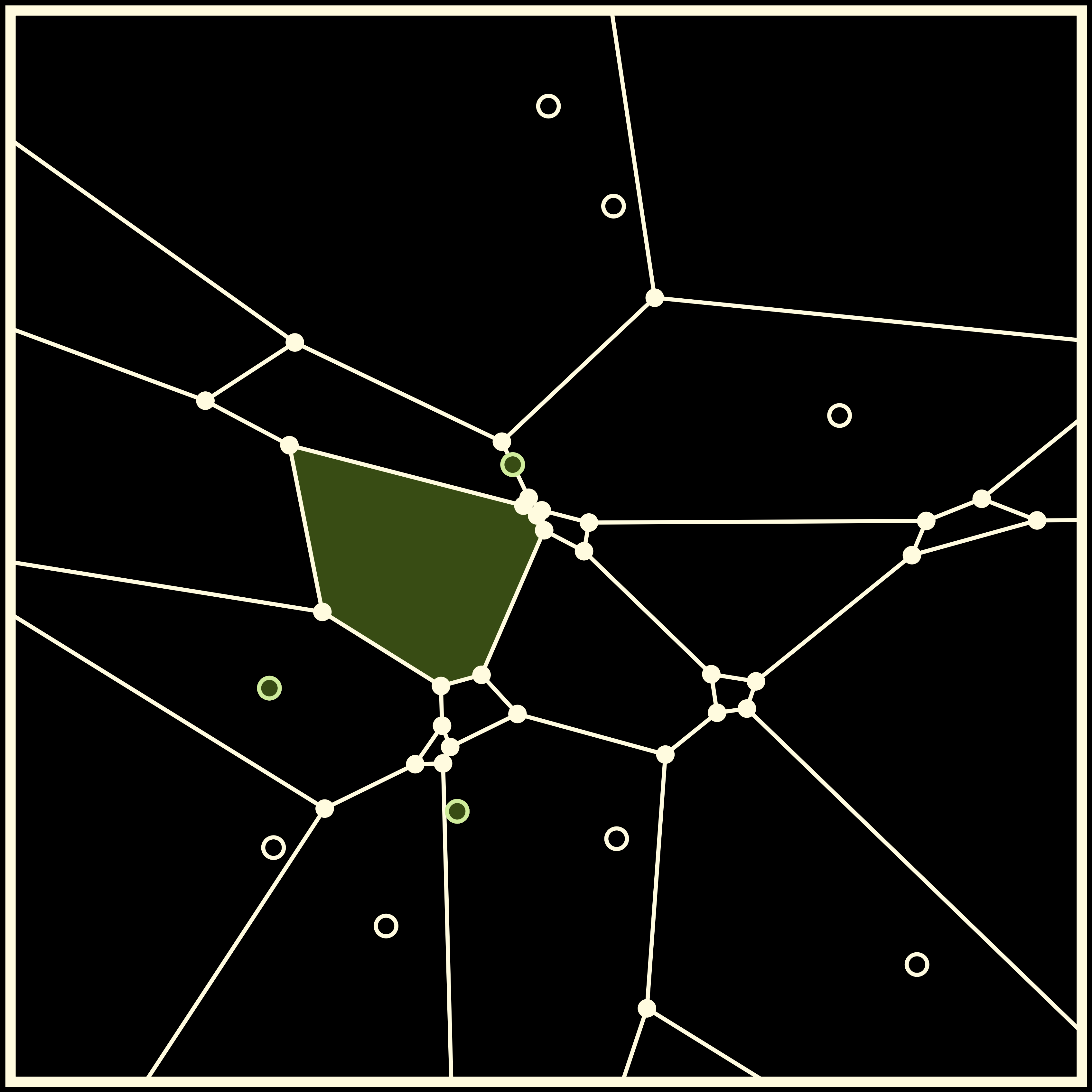


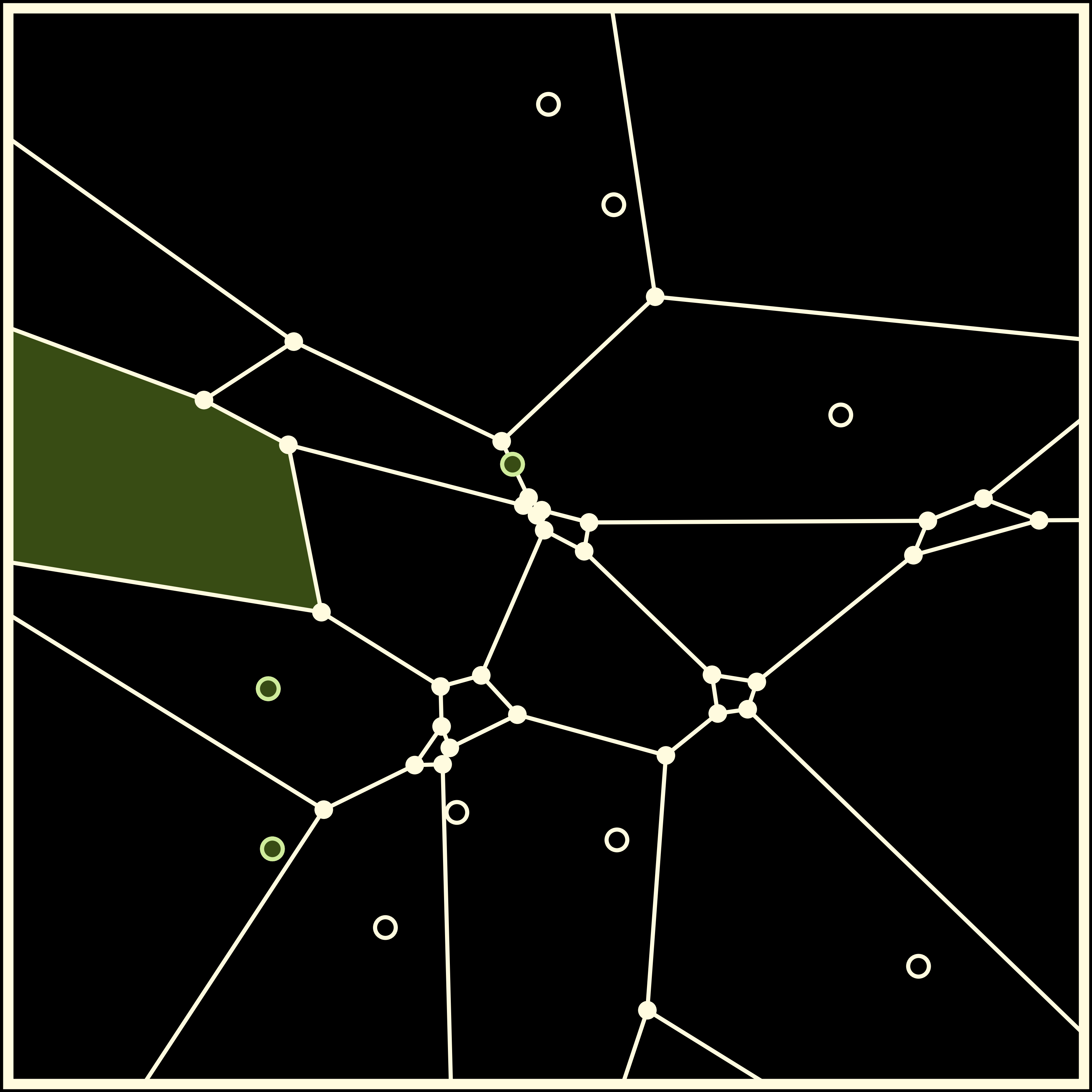


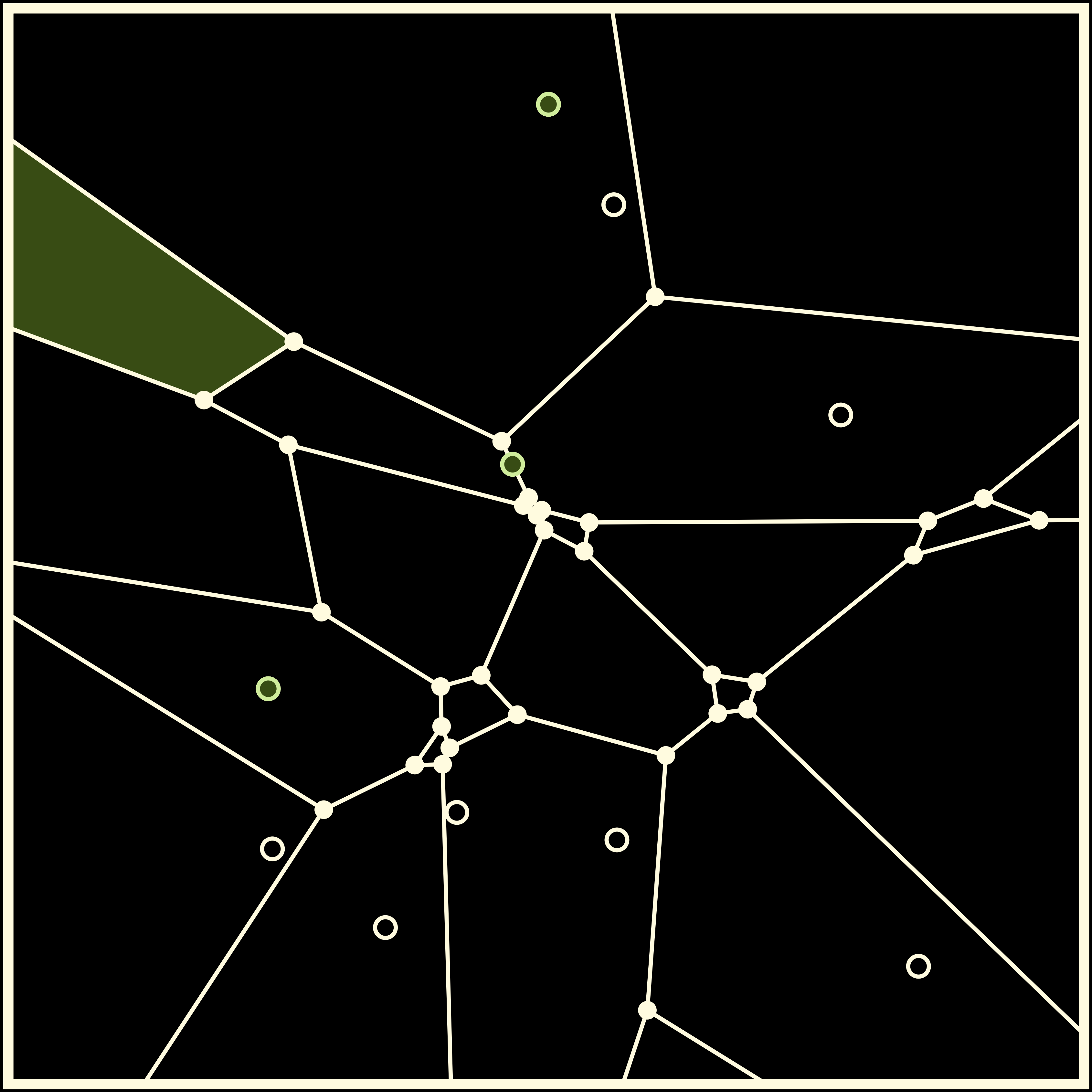


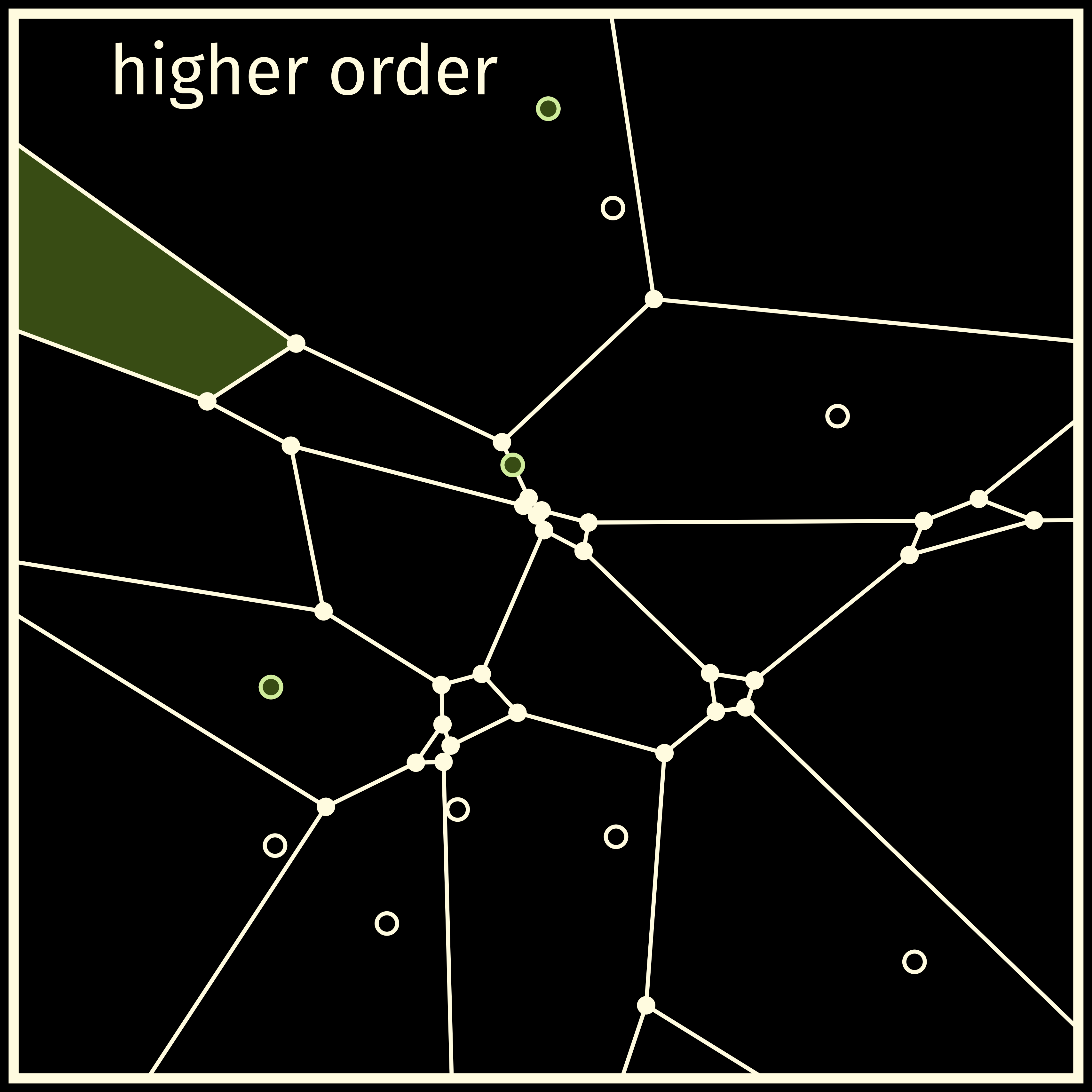


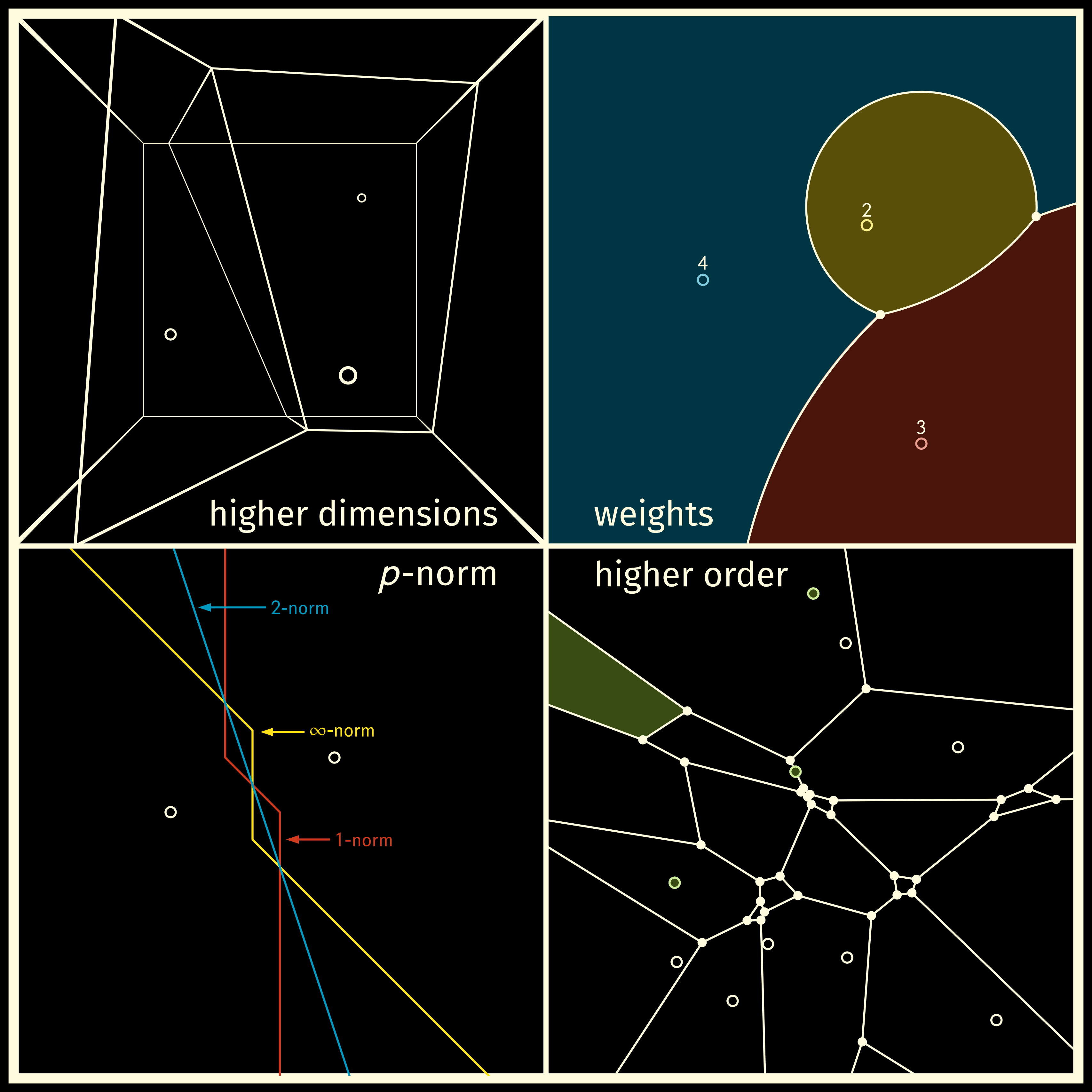




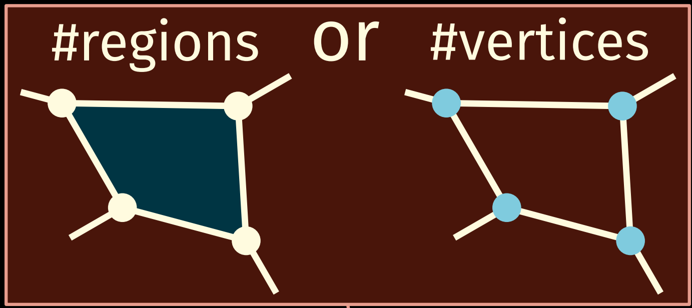





higher order

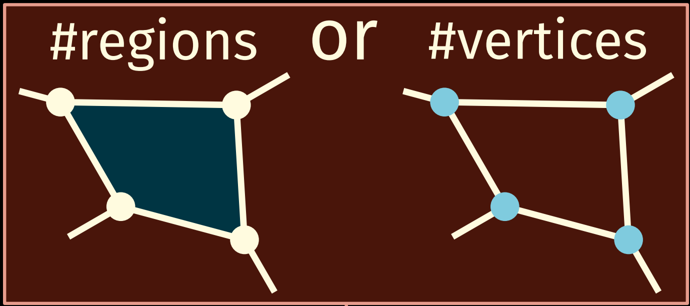

higher dimensions
weights
4
2
3
p-norm
2-norm
∞-norm
1-norm
higher order

Theorem
random sites $\Longrightarrow$ low complexity Voronoi diagram



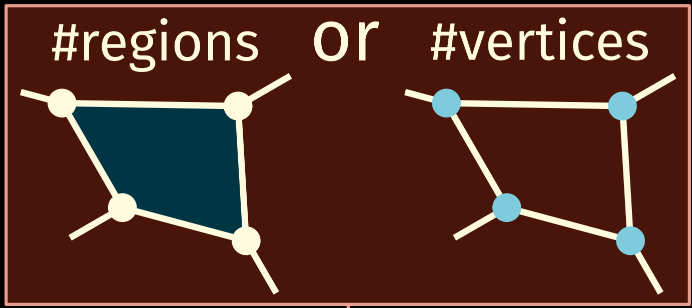
Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram



$O(\text{total weight})$
weights ≥ 1

Theorem
random sites $\Rightarrow$ low complexity Voronoi diagram

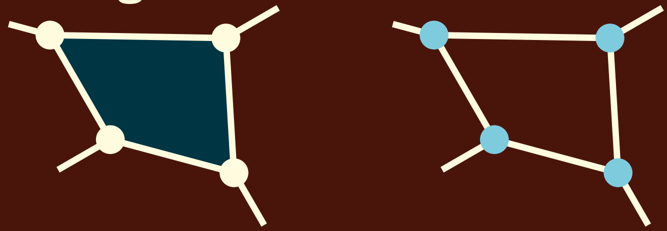


O(total weight)
weights ≥ 1

order $\in O(1)$

Theorem
random sites $\Longrightarrow$ low complexity Voronoi diagram

#regions or #vertices



O(total weight)
weights ≥ 1

order $\in O(1)$

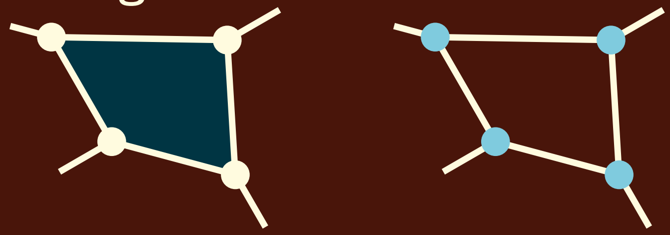
Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$

order	dim	norm	weights	complexity (region) (vertex)	
k	2	2	$\circ$	$O\left(k(n-k)\right)$	[Lee 82]
k	2	$1, \infty$	$\circ$	$O\left(\min\left\{k(n-k), (n-k)^2\right\}\right)$	[Liu, Papadopoulou, Lee 11]
k	2	abstract	$\circ$	$\leq 2k(n-k)$	[Bohler, Cheilaris, Klein, Liu, Papadopoulou, Zavershynskyi 15]
k	d	2	$\circ$	$O\left(n^{c(d)}\right)$	[Mulmuley 91]
1	3	2	$\circ$	$\Theta\left(n^2\right)$	[Klee 80] [Seidel 87]
1	d	p	$\circ$	$O\left(n^{c(d)}\right)$	[Lê 96]
1	d	∞	$\circ$	$\Theta\left(n^{\lceil d/2 \rceil}\right)$	[Boissonnat, Sharir, Tagansky, Yvinec 98]
1	d	1	$\circ$	$\Theta\left(n^2\right)$	[Boissonnat, Sharir, Tagansky, Yvinec 98]
1	d	2	$\circ$	expected $O(n)$	[Bienkowski, Damerow, Meyer auf der Heide, Sohler 05]
1	2	2	✓	$\Omega\left(n^2\right)$	[Aurenhammer, Edelsbrunner 84]
1	2	2	✓ random	expected $O(n \text{ polylog } n)$	[Har-Peled, Raichel 15]

#regions or #vertices



O(total weight)
weights ≥ 1

order $k \in O(1)$

Theorem

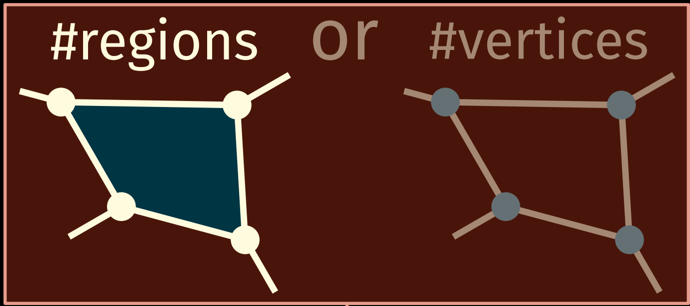
random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$

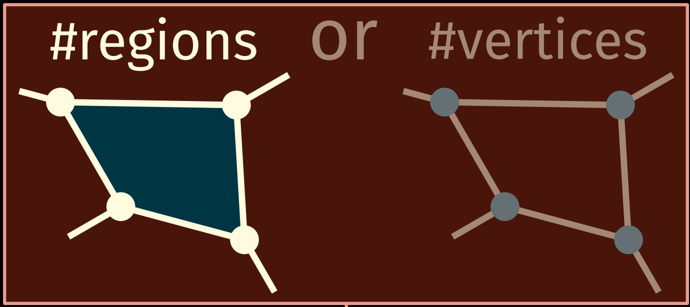
Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

- random sites: hypercube with p-norm; dimension $\in O(1)$
- low: O(total weight); weights ≥ 1
- complexity:



- Voronoi diagram: order $k \in O(1)$



O(total weight)
weights ≥ 1

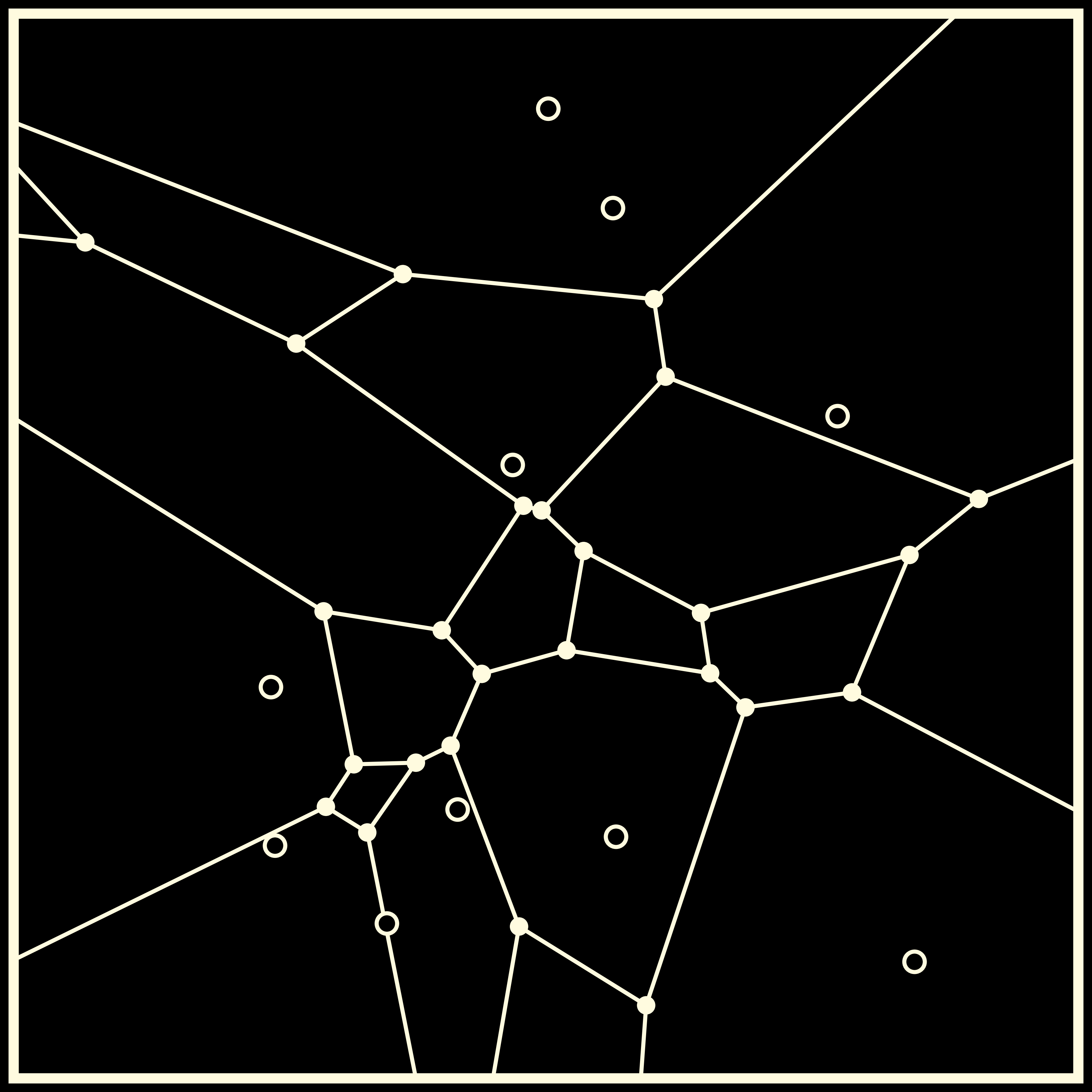
order $k \in O(1)$

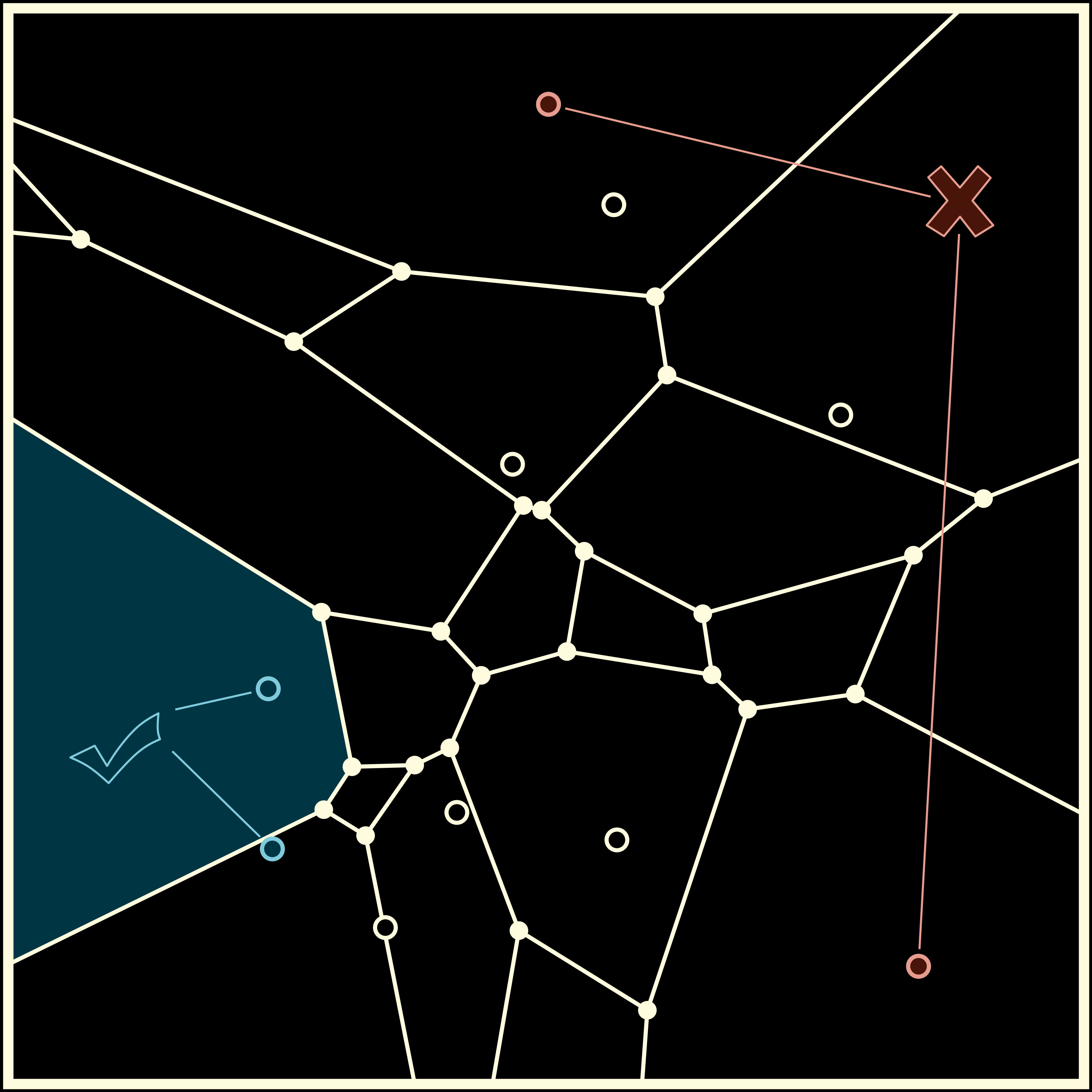
Theorem

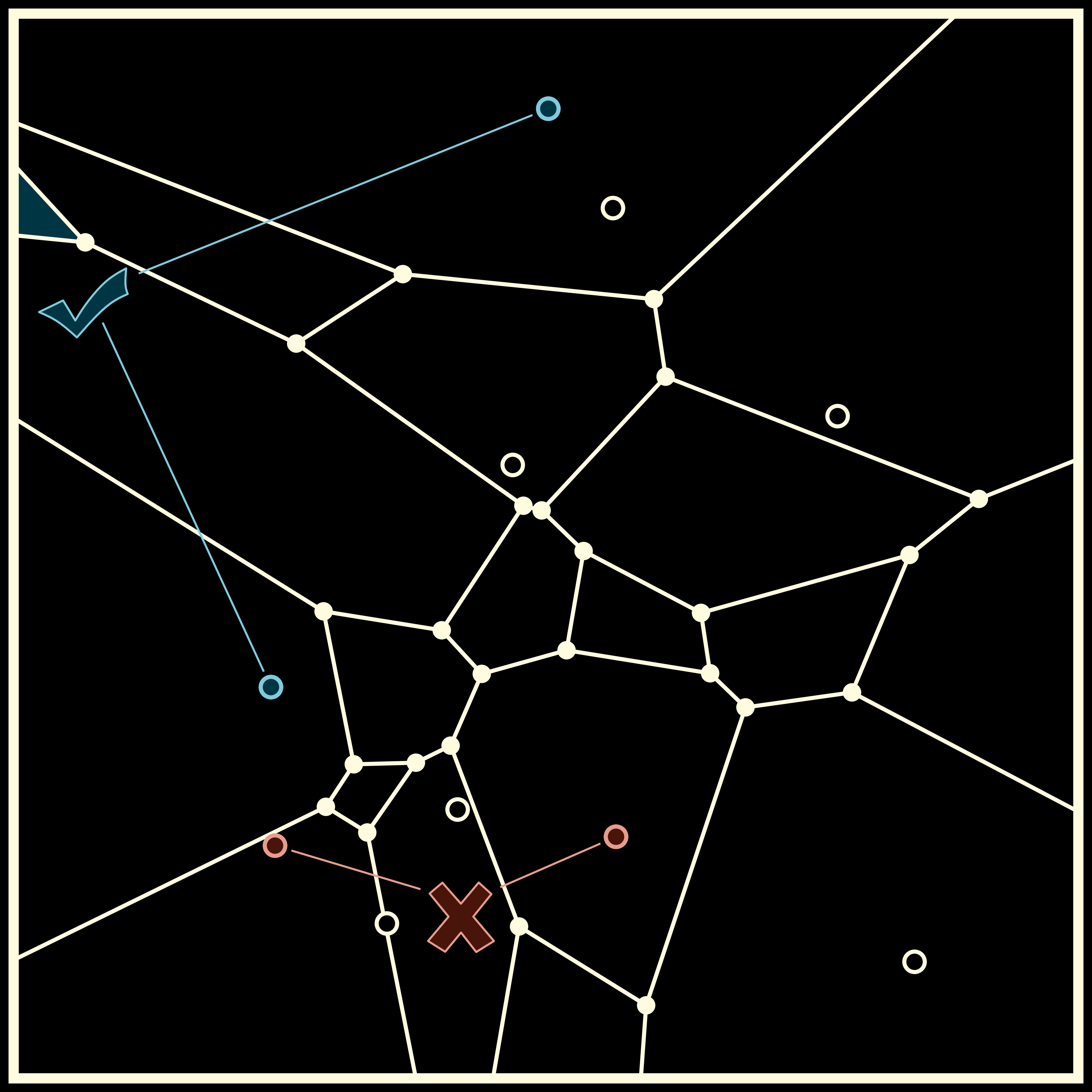
random sites $\Longrightarrow$ low complexity Voronoi diagram

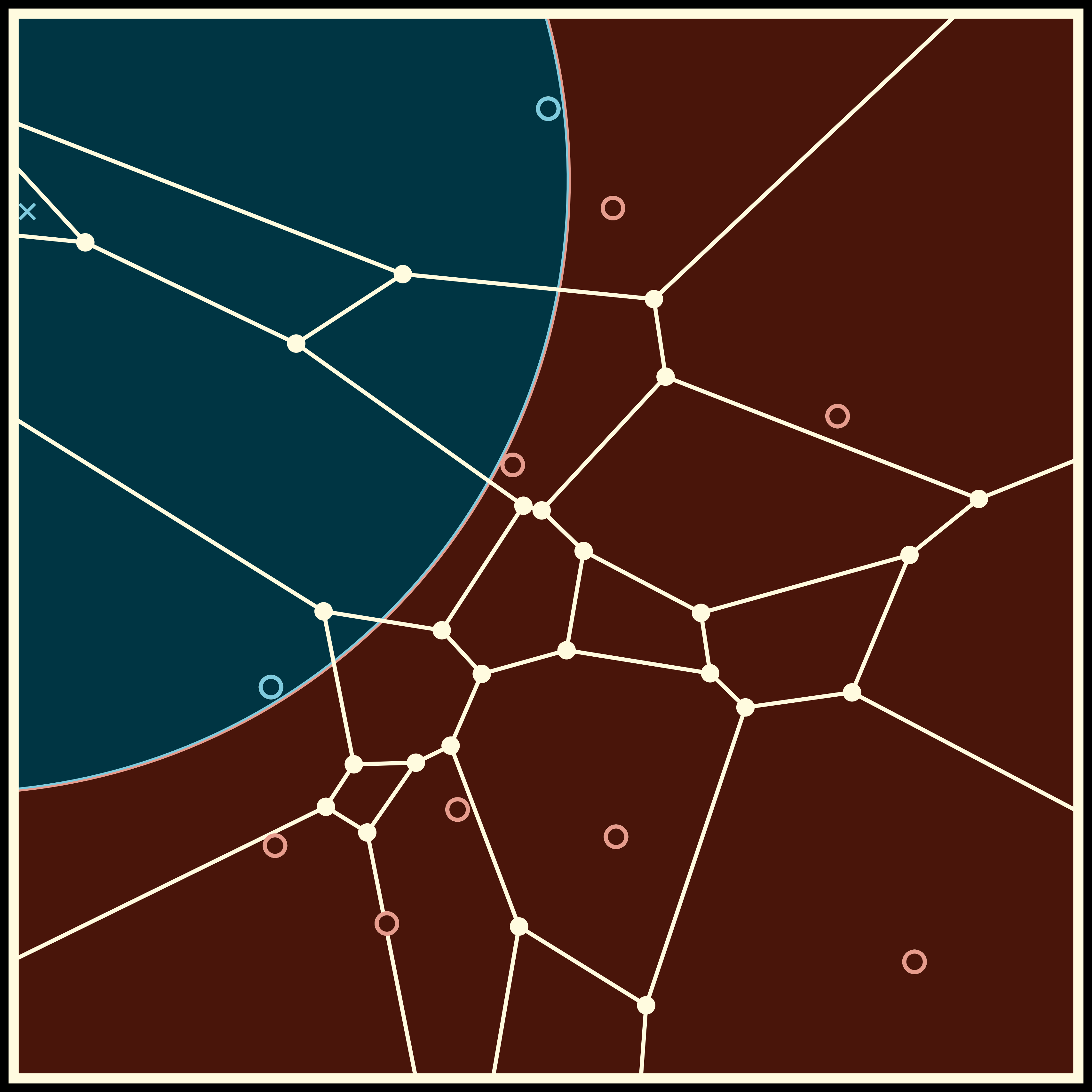
hypercube with p-norm
dimension $\in O(1)$

$$\mathbb{E}\left[\#\text{regions}\right] = \sum_{\substack{A \subseteq S \\ |A| = k}} \Pr\left[A \text{ has Voronoi region}\right]$$



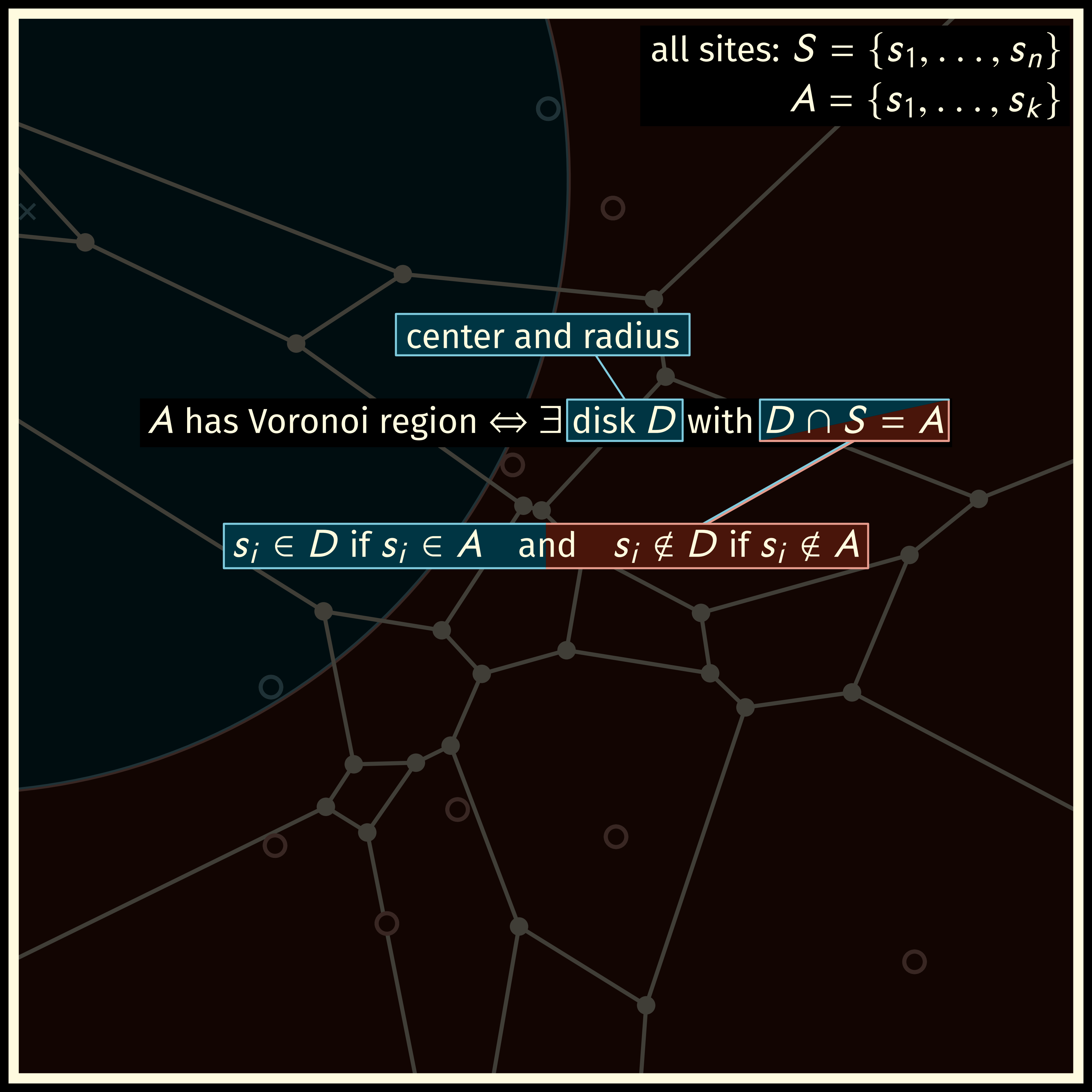




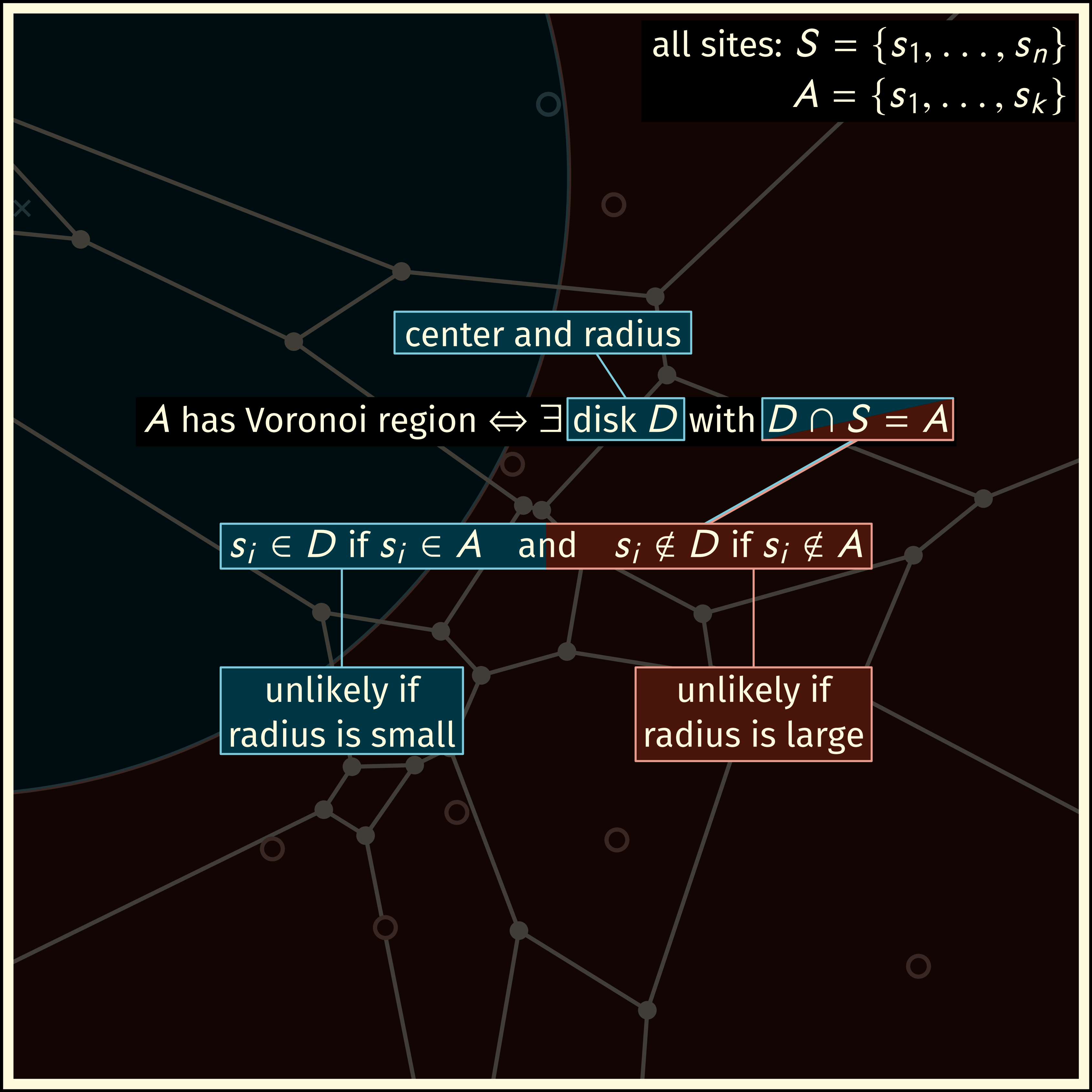


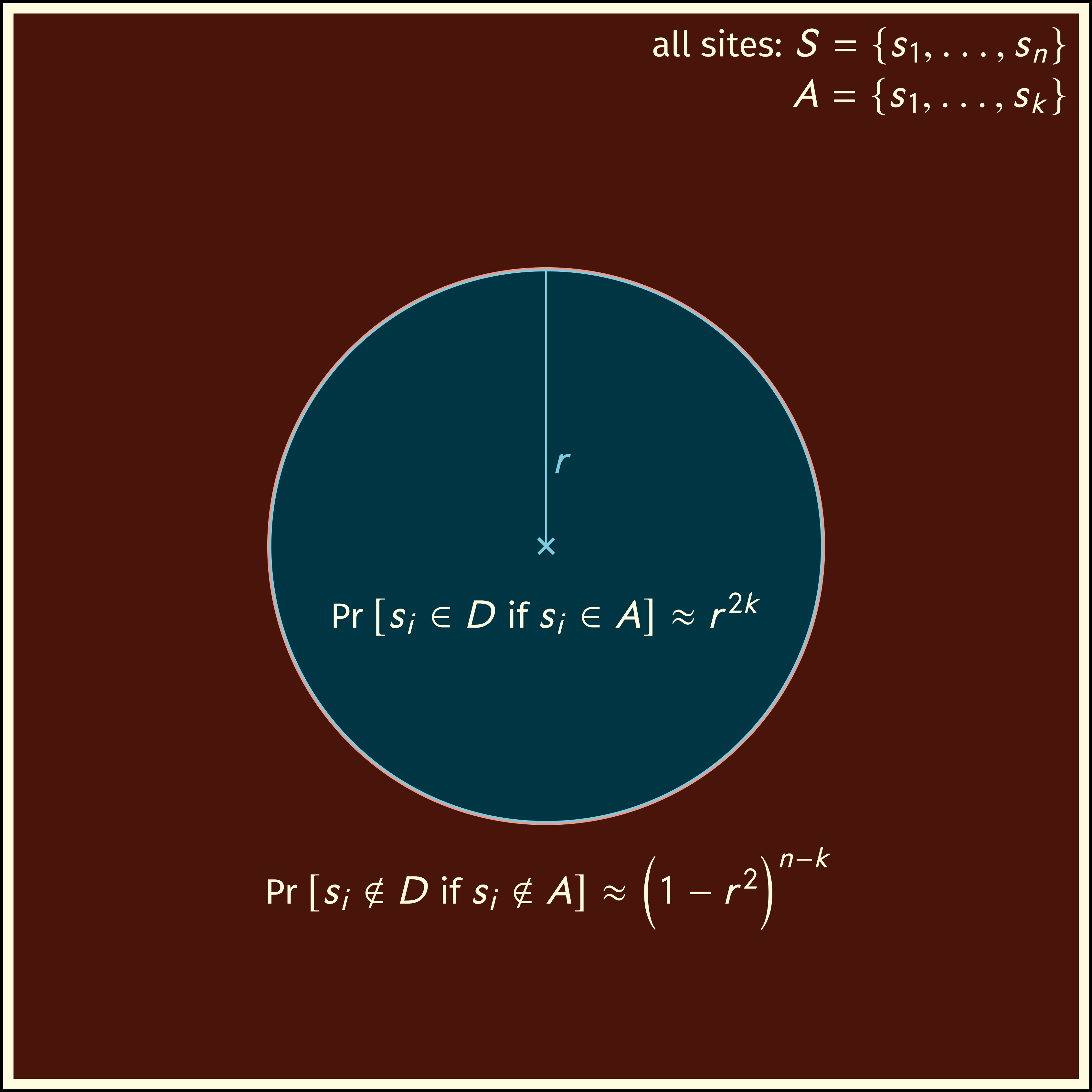

all sites: $S = \{s_1, \ldots, s_n\}$
$A = \{s_1, \ldots, s_k\}$
A has Voronoi region $\Leftrightarrow$ $\exists$ disk D with $D \cap S = A$

all sites: $S = \{s_1, \ldots, s_n\}$
$A = \{s_1, \ldots, s_k\}$
center and radius
A has Voronoi region $\Leftrightarrow \exists$ disk D with $D \cap S = A$



all sites: $S = \{s_1, \ldots, s_n\}$
$A = \{s_1, \ldots, s_k\}$
center and radius
A has Voronoi region $\Leftrightarrow \exists$ disk D with $D \cap S = A$
$s_i \in D$ if $s_i \in A$ and $s_i \notin D$ if $s_i \notin A$


all sites: $S = \{s_1, \ldots, s_n\}$
$A = \{s_1, \ldots, s_k\}$
center and radius
A has Voronoi region $\Leftrightarrow \exists$ disk D with $D \cap S = A$
$s_i \in D$ if $s_i \in A$ and $s_i \notin D$ if $s_i \notin A$
unlikely if radius is small
unlikely if radius is large


all sites: $S = \{s_1, \ldots, s_n\}$
$A = \{s_1, \ldots, s_k\}$
r
$\Pr\left[s_i \in D \text{ if } s_i \in A\right] \approx r^{2k}$
$\Pr\left[s_i \notin D \text{ if } s_i \notin A\right] \approx \left(1 - r^2\right)^{n-k}$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, ..., s_n\}$

$A = \{s_1, ..., s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$

$$= \Pr\left[\exists r\ \exists p:\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$$

smallest r s.t. $\exists p$: $s_i \in D$ if $s_i \in A$

disk D: center p and radius r

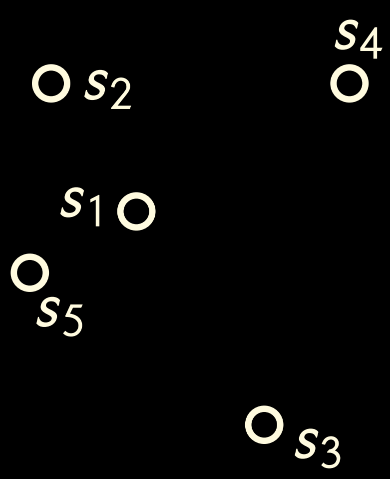
all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$\Pr[A \text{ has Voronoi region}]$

$= \Pr\left[\exists r\ \exists p:\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$

smallest r s.t. $\exists p:\ s_i \in D$ if $s_i \in A$



disk D: center p and radius r

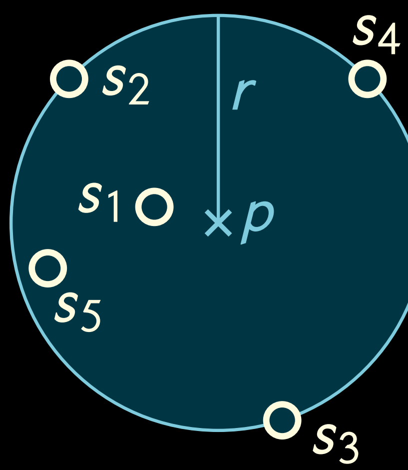
all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$\Pr[A \text{ has Voronoi region}]$

$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$

smallest r s.t. $\exists p\colon\ s_i \in D \text{ if } s_i \in A$



disk D: center p and radius r

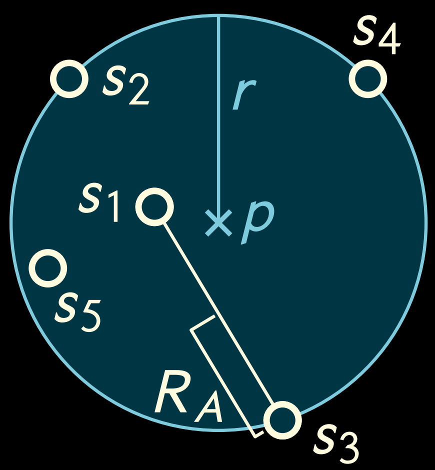
all sites: $S = \{s_1, \dots, s_n\}$

$A = \{s_1, \dots, s_k\}$

$\Pr[A \text{ has Voronoi region}]$

$= \Pr[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A]$

smallest r s.t. $\exists p\colon\ s_i \in D$ if $s_i \in A$



disk D: center p and radius r

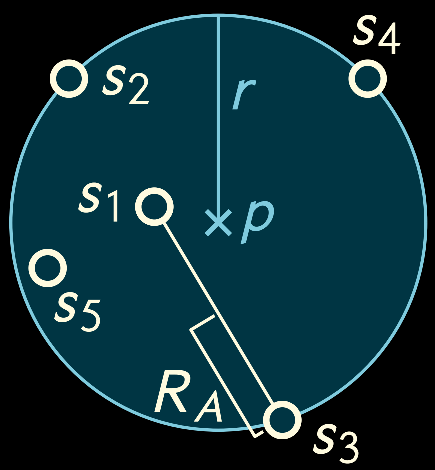
all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$\Pr[A \text{ has Voronoi region}]$

$= \Pr[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A]$

smallest r s.t. $\exists p\colon\ s_i \in D \text{ if } s_i \in A \Rightarrow r \geq R_A$



disk D: center p and radius r

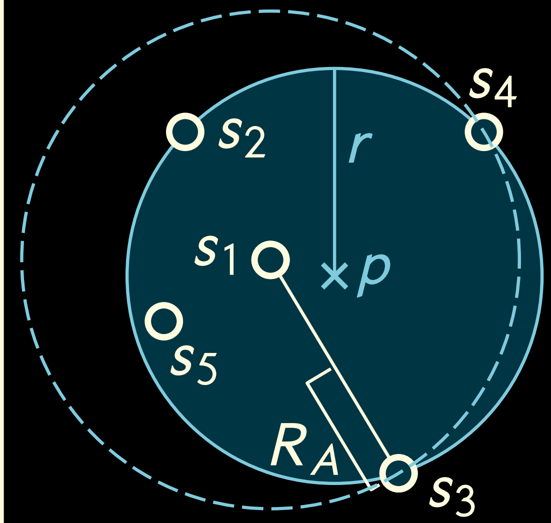
all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$

$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$$

smallest r s.t. $\exists p\colon\ s_i \in D$ if $s_i \in A \Rightarrow r \geq R_A$



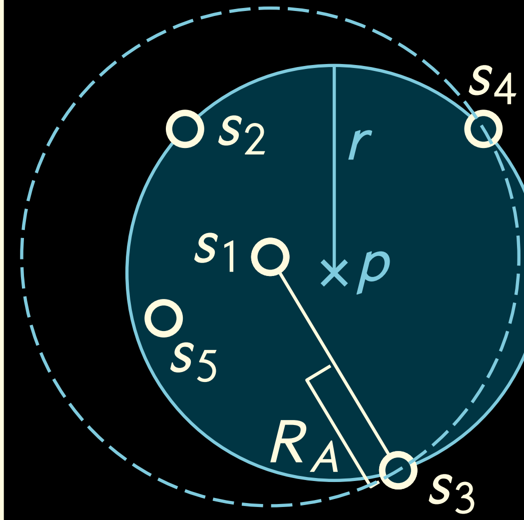
disk D: center p and radius r

all sites: $S = \{s_1, \dots, s_n\}$

$A = \{s_1, \dots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$$

smallest r s.t. $\exists p\colon\ s_i \in D$ if $s_i \in A \Rightarrow r \geq R_A$



$$\Pr[A \text{ has Voronoi region}]$$
$$= \int_r \Pr\left[A \text{ has region} \mid R_A = r\right] f_{R_A}(r)\, \mathrm{d}r$$

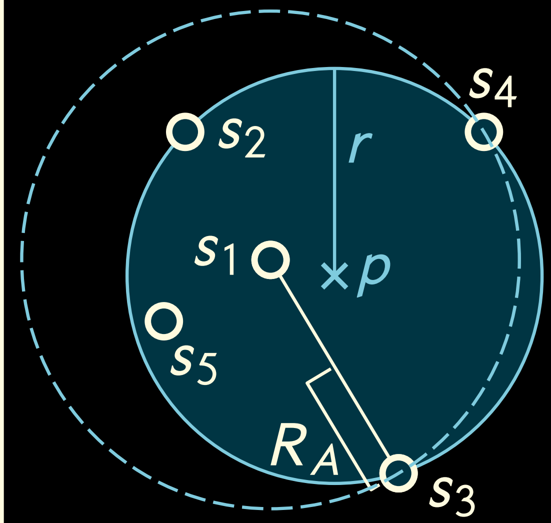
disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$$

smallest r s.t. $\exists p\colon\ s_i \in D \text{ if } s_i \in A \Rightarrow r \geq R_A$



$$\Pr[A \text{ has Voronoi region}]$$
$$= \int_r \Pr\left[A \text{ has region} \mid R_A = r\right] f_{R_A}(r)\,\mathrm{d}r$$

$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

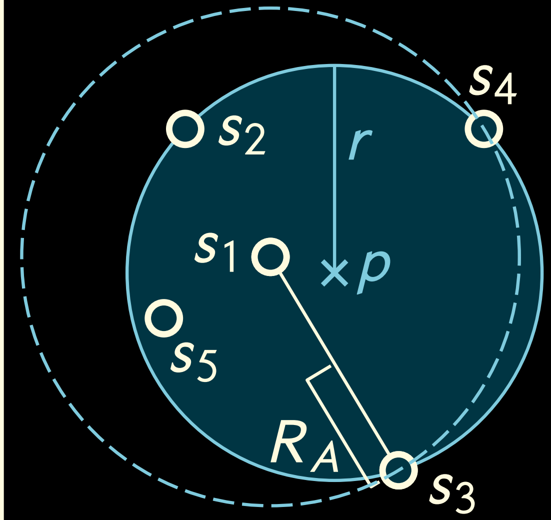
disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$$

smallest r s.t. $\exists p\colon\ s_i \in D$ if $s_i \in A \Rightarrow r \geq R_A$



$$\Pr[A \text{ has Voronoi region}]$$
$$= \int_r \Pr\left[A \text{ has region} \mid R_A = r\right] f_{R_A}(r)\, \mathrm{d}r$$

$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

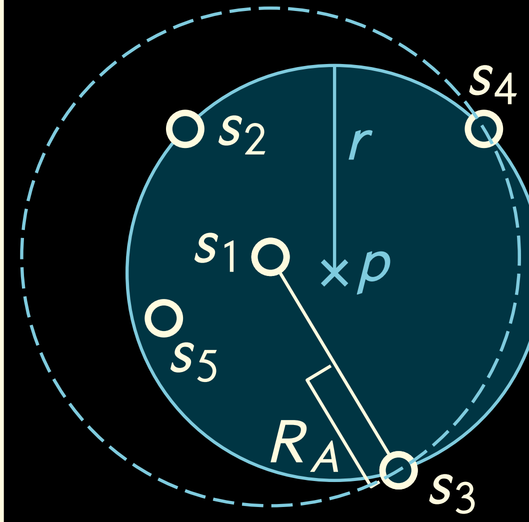
disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A\right]$$

smallest r s.t. $\exists p\colon\ s_i \in D$ if $s_i \in A \Rightarrow r \geq R_A$



$$\Pr[A \text{ has Voronoi region}]$$
$$= \int_r \Pr\left[A \text{ has region} \mid R_A = r\right] f_{R_A}(r)\,\mathrm{d}r$$

$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

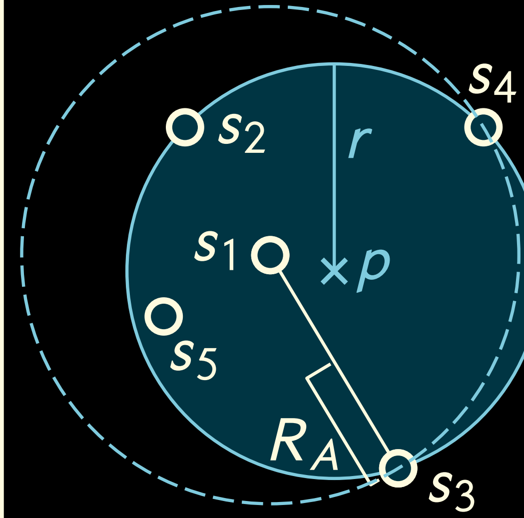
disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$$

smallest r s.t. $\exists p\colon\ s_i \in D \text{ if } s_i \in A \Rightarrow r \geq R_A$



$$\Pr[A \text{ has Voronoi region}]$$
$$= \int_r \Pr\left[A \text{ has region} \mid R_A = r\right] f_{R_A}(r)\,\mathrm{d}r$$

$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A\ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

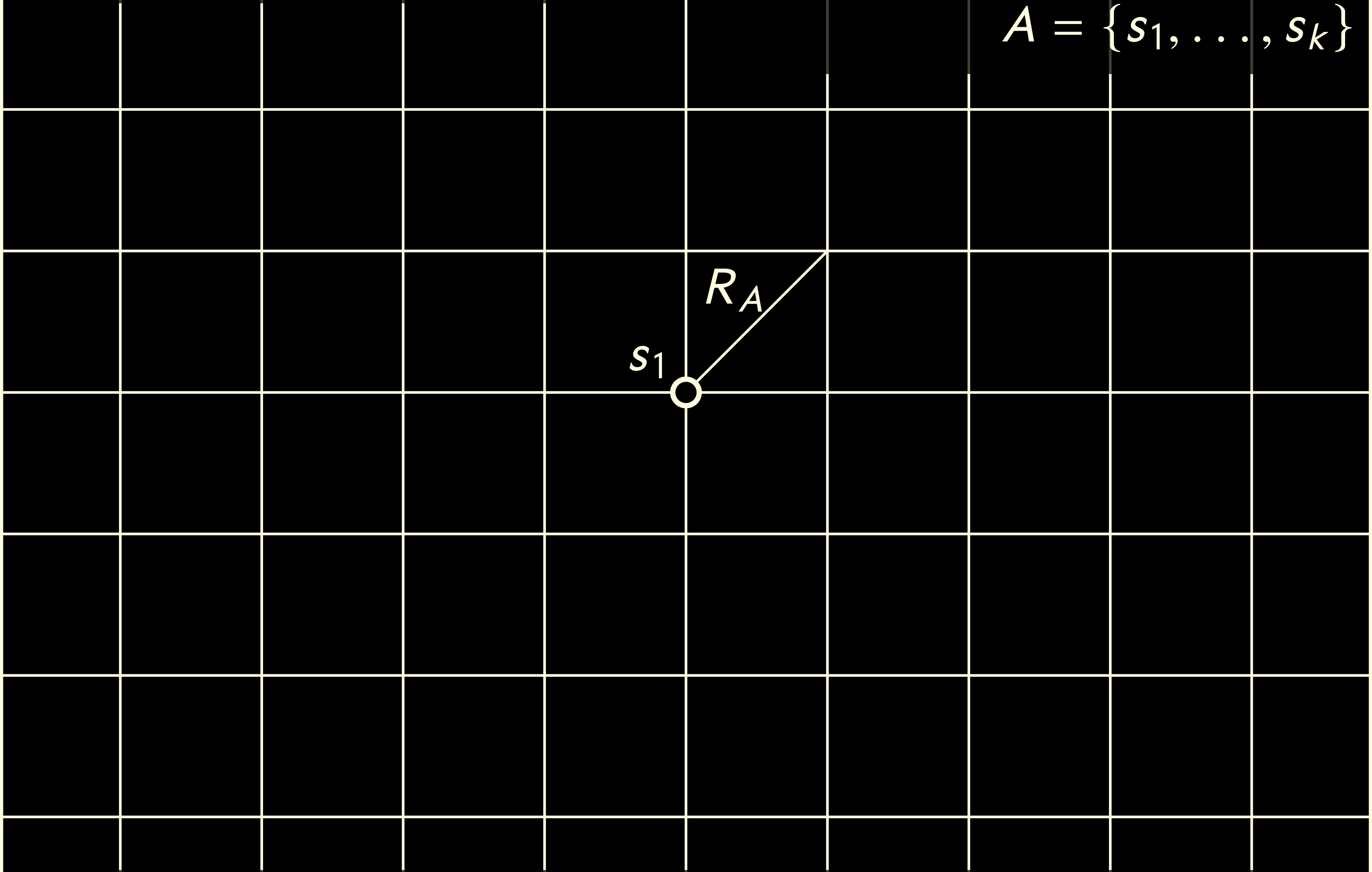
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A\ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D\ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

$$= \Pr\left[\exists r\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

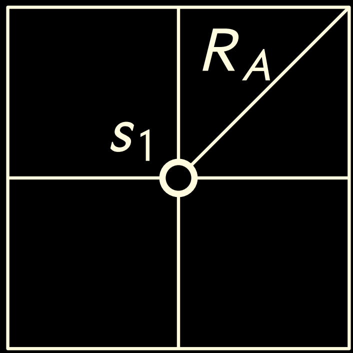
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

$$\leq \Pr\left[\exists \boxed{r \geq R_A}\ \exists p\colon\ s_1 \in D \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid \boxed{R_A}\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

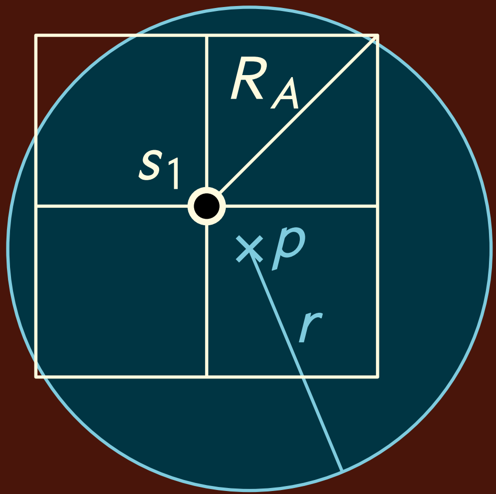
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \dots, s_n\}$
$A = \{s_1, \dots, s_k\}$

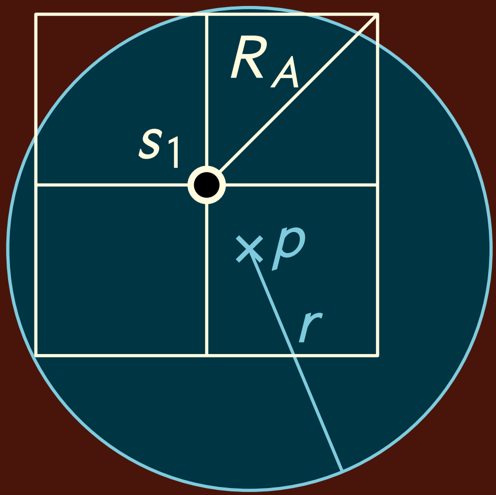


$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \dots, s_n\}$

$A = \{s_1, \dots, s_k\}$

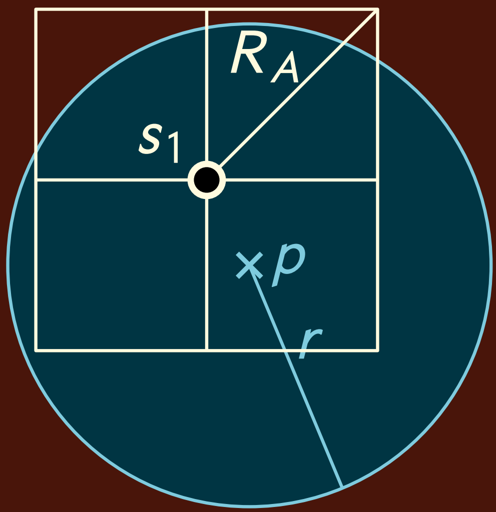


$$
\begin{aligned}
&\Pr\left[A \text{ has Voronoi region} \mid R_A\right] \\
= {}&\Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
= {}&\Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
\leq {}&\Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]
\end{aligned}
$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

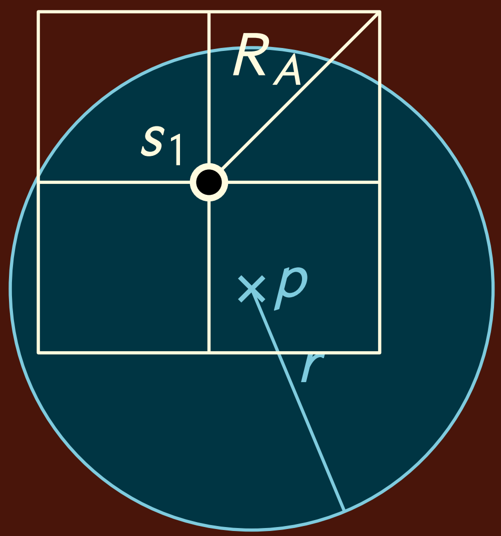
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

$$= \Pr\left[\exists r\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

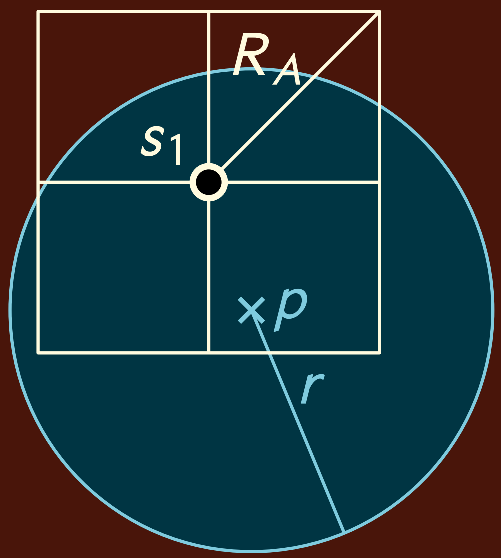
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

$$\leq \Pr\left[\exists \boxed{r \geq R_A}\ \exists p\colon\ s_1 \in D \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid \boxed{R_A}\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

$$= \Pr\left[\exists r\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

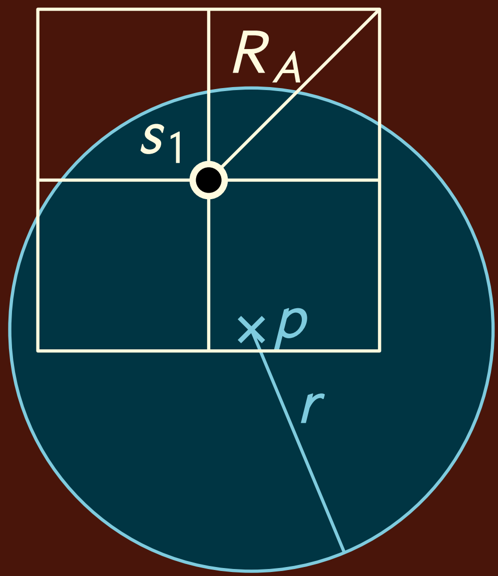
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

$$\leq \Pr\left[\exists \boxed{r \geq R_A}\ \exists p\colon\ s_1 \in D \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid \boxed{R_A}\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

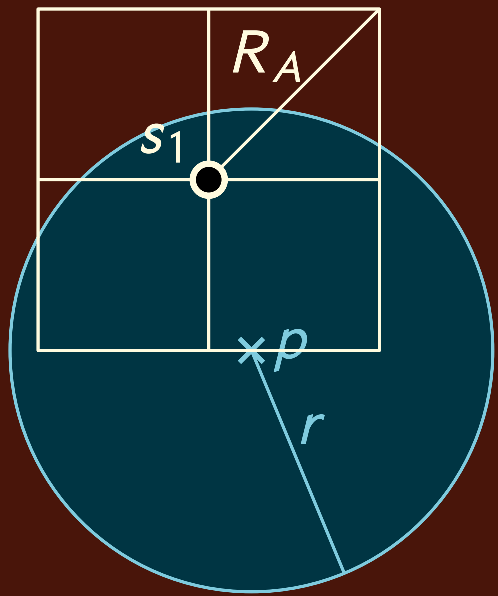


$$
\begin{aligned}
&\Pr\left[A \text{ has Voronoi region} \mid R_A\right] \\
= &\Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
= &\Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
\leq &\Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]
\end{aligned}
$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

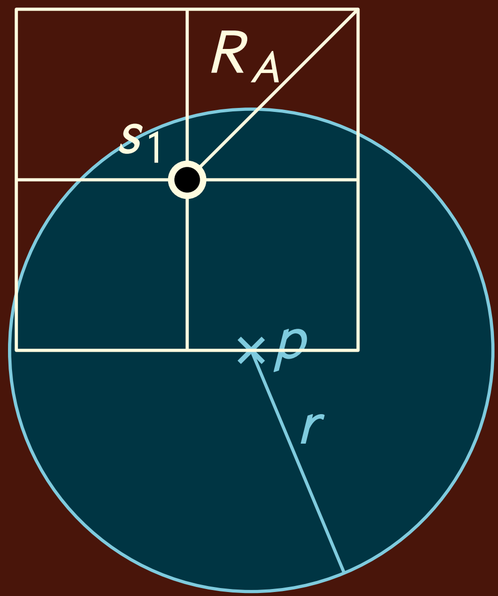


$$
\begin{aligned}
&\Pr\left[A \text{ has Voronoi region} \mid R_A\right] \\
&= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
&= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
&\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]
\end{aligned}
$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

$$= \Pr\left[\exists r\, \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

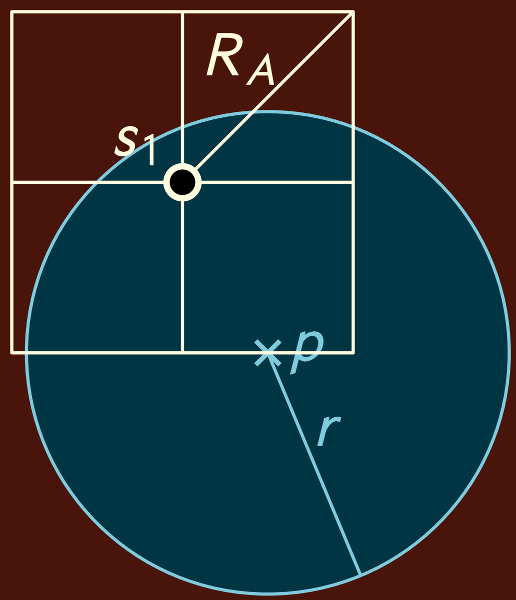
$$= \Pr\left[\exists r \geq R_A\, \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

$$\leq \Pr\left[\exists r \geq R_A\, \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

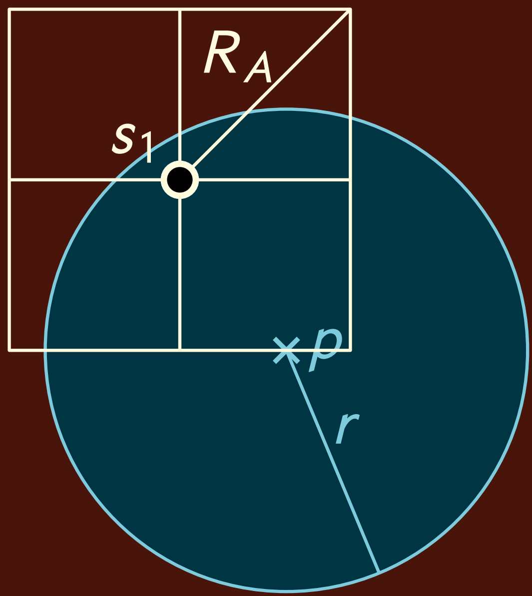


$$
\begin{aligned}
&\Pr\left[A \text{ has Voronoi region} \mid R_A\right] \\
&= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
&= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
&\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]
\end{aligned}
$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

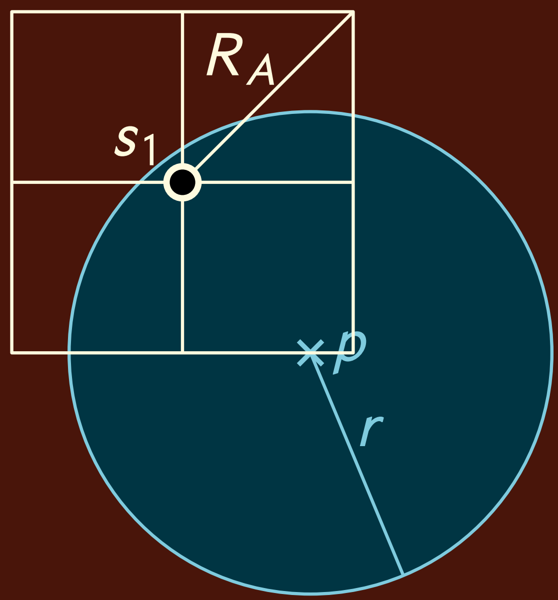
$A = \{s_1, \ldots, s_k\}$



$$
\begin{aligned}
&\Pr\left[A \text{ has Voronoi region} \mid R_A\right] \\
= &\Pr\left[\exists r\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right] \\
= &\Pr\left[\exists r \geq R_A\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right] \\
\leq &\Pr\left[\exists \boxed{r \geq R_A}\ \exists p\colon\ s_1 \in D \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid \boxed{R_A}\right]
\end{aligned}
$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$
$A = \{s_1, \ldots, s_k\}$

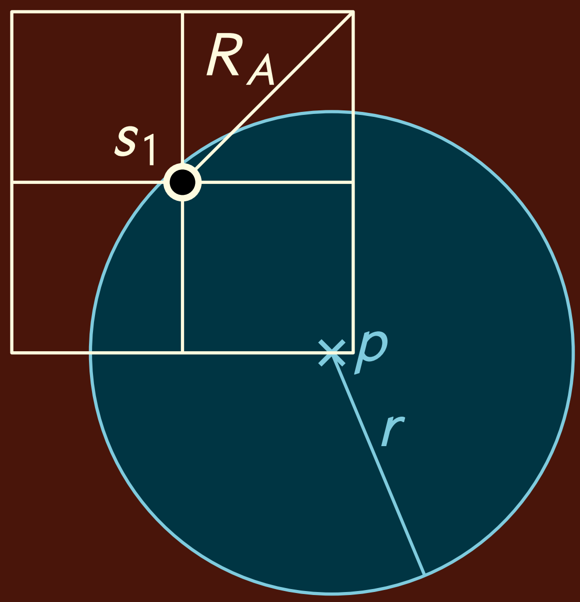


$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$
$$\leq \Pr\left[\exists \boxed{r \geq R_A}\ \exists p\colon\ s_1 \in D \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid \boxed{R_A}\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

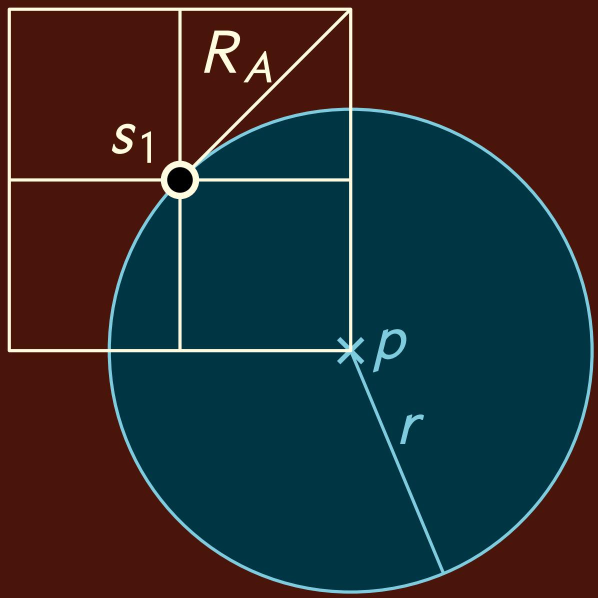


$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

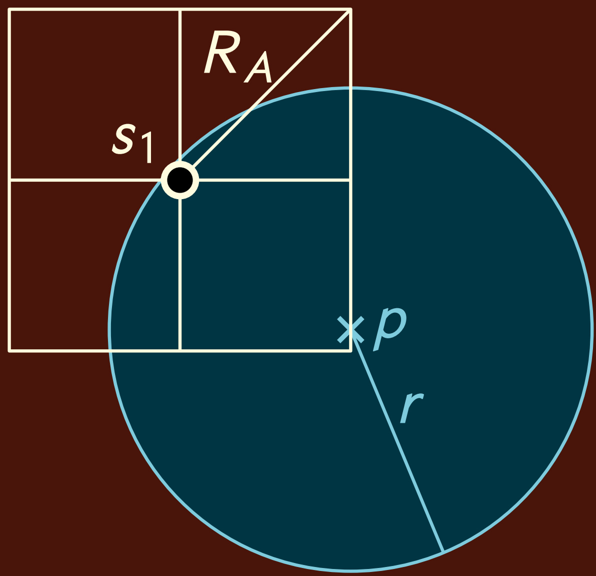


$$
\begin{aligned}
&\Pr\left[A \text{ has Voronoi region} \mid R_A\right] \\
&= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
&= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
&\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]
\end{aligned}
$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

$$= \Pr\left[\exists r\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

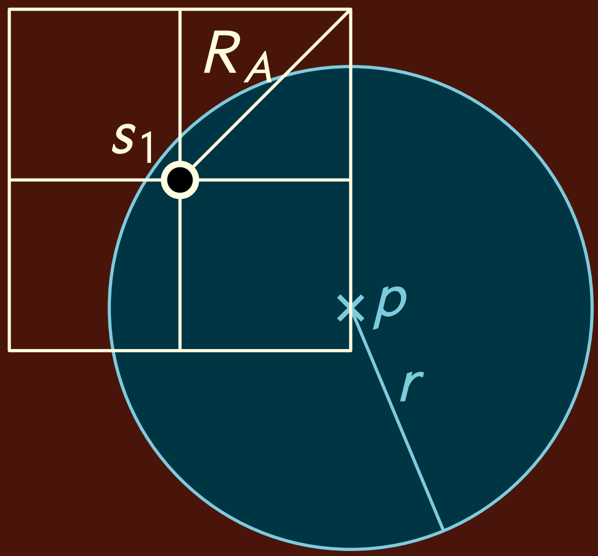
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

$$\leq \Pr\left[\exists \boxed{r \geq R_A}\ \exists p\colon\ s_1 \in D \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid \boxed{R_A}\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

$$= \Pr\left[\exists r\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

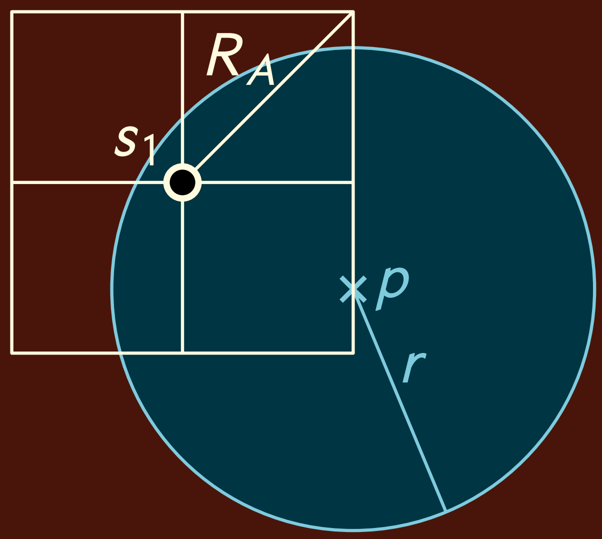
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

$$\leq \Pr\left[\exists \boxed{r \geq R_A}\ \exists p\colon\ s_1 \in D \qquad \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid \boxed{R_A}\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \dots, s_n\}$

$A = \{s_1, \dots, s_k\}$

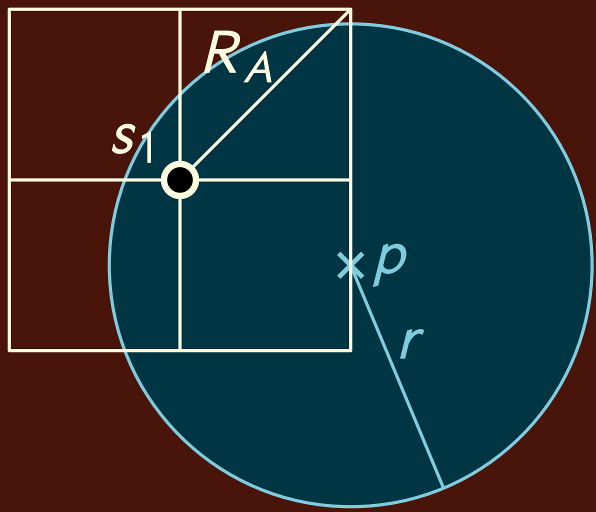


$$
\begin{aligned}
&\Pr\left[A \text{ has Voronoi region} \mid R_A\right] \\
&= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
&= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
&\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]
\end{aligned}
$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

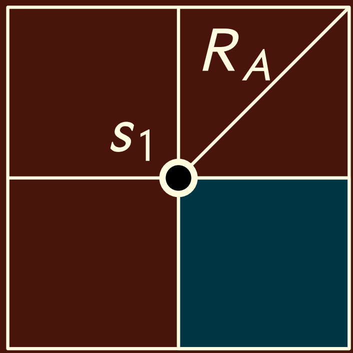
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

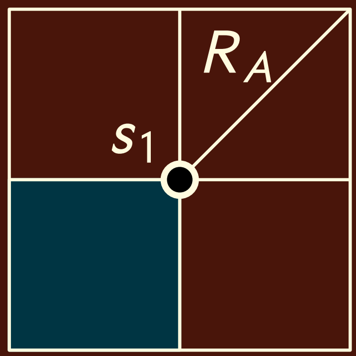
$$= \Pr\left[\exists r\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

$$\leq \Pr\left[\exists \boxed{r \geq R_A}\ \exists p\colon\ s_1 \in D \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid \boxed{R_A}\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$
$A = \{s_1, \ldots, s_k\}$

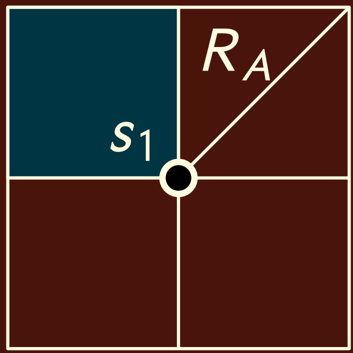


$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A\ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A\ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D\ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

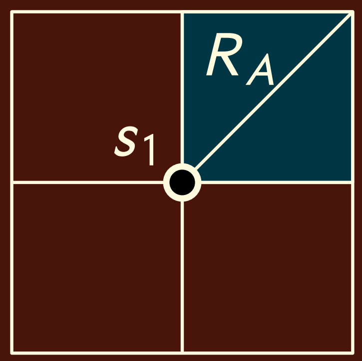


$$
\begin{aligned}
& \Pr\left[A \text{ has Voronoi region} \mid R_A\right] \\
= & \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
= & \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
\leq & \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]
\end{aligned}
$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

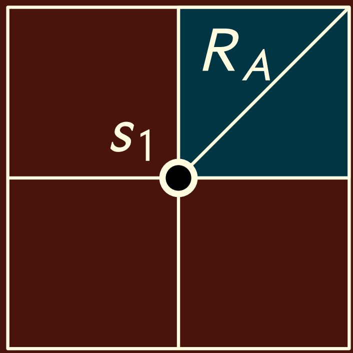


$$
\begin{aligned}
& \Pr\left[A \text{ has Voronoi region} \mid R_A\right] \\
= & \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
= & \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
\leq & \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right] \\
\leq & 4 \cdot \Pr\left[s_i \notin \text{ grid cell if } s_i \notin A\right] +
\end{aligned}
$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$



$$
\begin{aligned}
&\Pr\left[A \text{ has Voronoi region} \mid R_A\right] \\
&= \Pr\left[\exists r\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right] \\
&= \Pr\left[\exists r \geq R_A\ \exists p\colon\ \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right] \\
&\leq \Pr\left[\exists \boxed{r \geq R_A}\ \exists p\colon\ s_1 \in D \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid \boxed{R_A}\right] \\
&\leq 4 \cdot \Pr\left[s_i \notin \text{ grid cell if } s_i \notin A\right] + \text{ something small}
\end{aligned}
$$

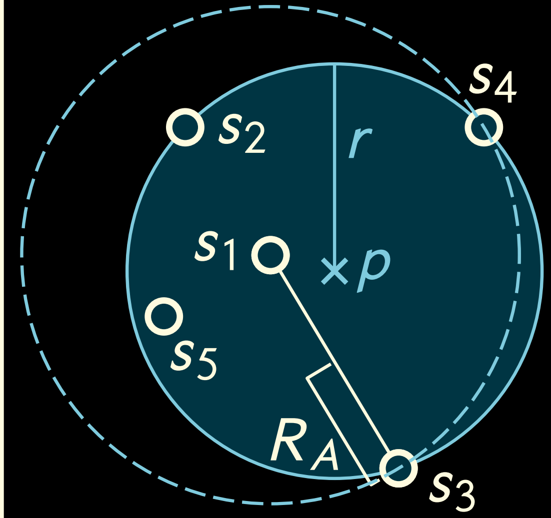
disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$$

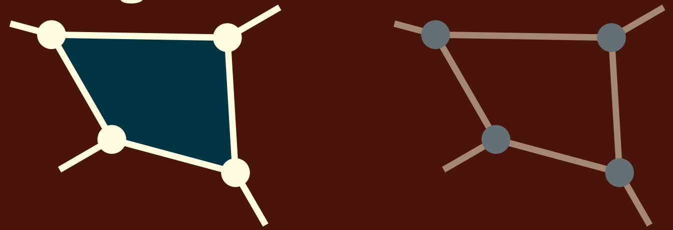
smallest r s.t. $\exists p\colon\ s_i \in D$ if $s_i \in A \Rightarrow r \geq R_A$



$$\Pr[A \text{ has Voronoi region}]$$
$$= \int_r \Pr\left[A \text{ has region} \mid R_A = r\right] f_{R_A}(r)\, \mathrm{d}r$$

$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$\leq 4 \cdot \Pr[s_i \notin \text{ grid cell if } s_i \notin A] + \text{ something small}$$

#regions or #vertices



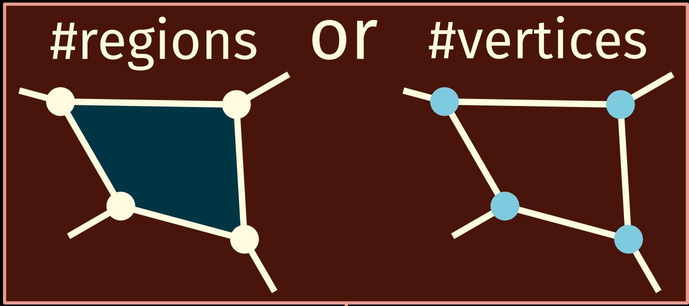
O(total weight)
weights ≥ 1

order $\in O(1)$

Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$



O(total weight)
weights ≥ 1

order $\in O(1)$

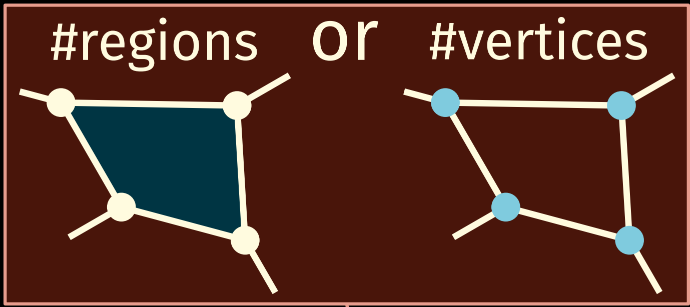
Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$

Theorem

many vertices $\Longrightarrow$ many regions in higher order



O(total weight)
weights ≥ 1

order $\in O(1)$

Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$

ℓ

$\Omega(\ell)$

Theorem

many vertices $\Longrightarrow$ many regions in higher order

dim d, order $k \longrightarrow$ order $(k+d)$

Theorem
many vertices $\Longrightarrow$ many regions in higher order

Theorem

many vertices (ℓ) $\implies$ many regions ($\Omega(\ell)$) in higher order

Theorem

ℓ — $\Omega(\ell)$

many vertices $\Longrightarrow$ many regions in higher order

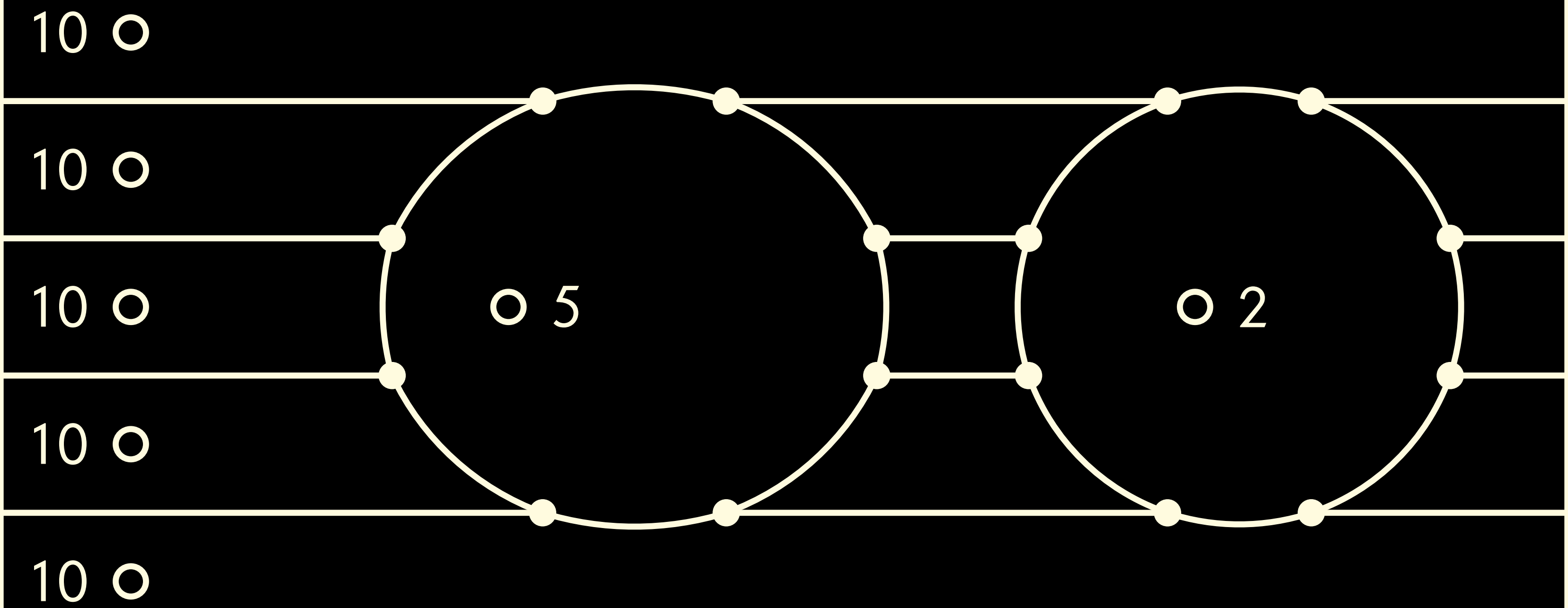
dim d, order $k \longrightarrow$ order $(k+d)$

Theorem

many vertices $\Longrightarrow$ many regions in higher order

ℓ — $\Omega(\ell)$

dim d, order k $\longrightarrow$ order $(k+d)$



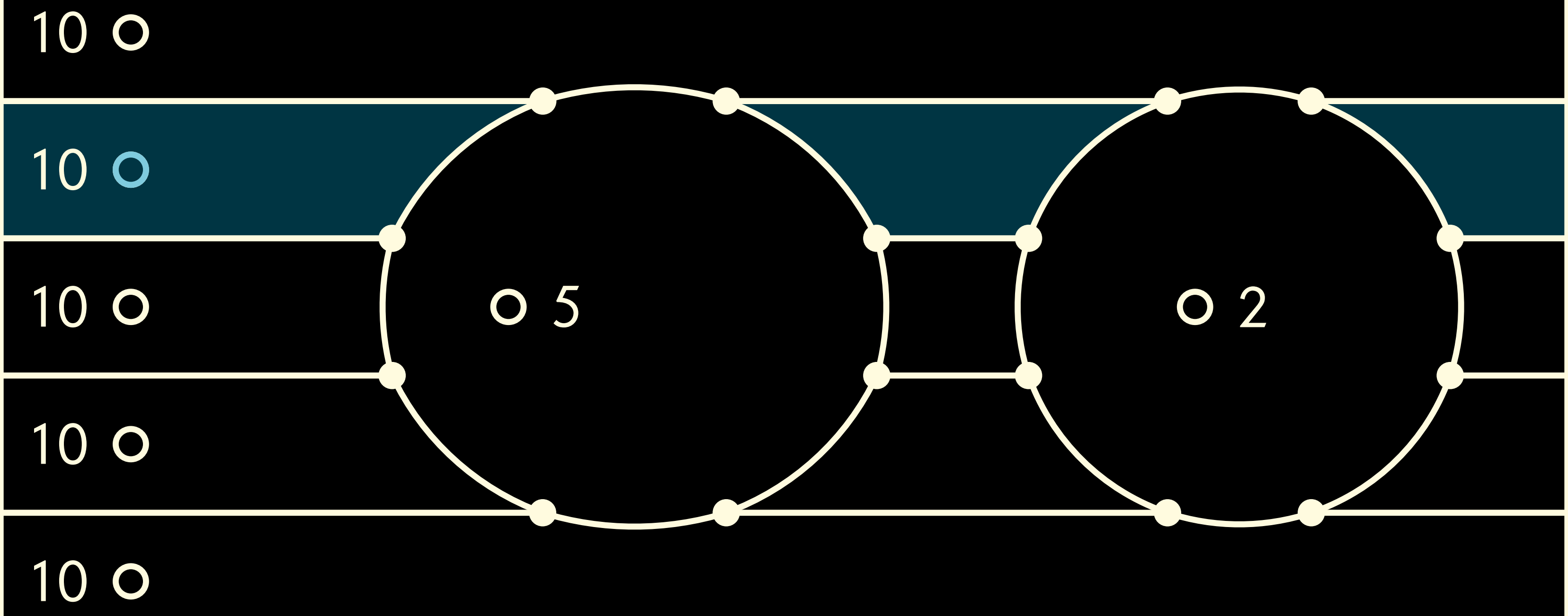
Theorem

many vertices $\Longrightarrow$ many regions in higher order

ℓ

$\Omega(\ell)$

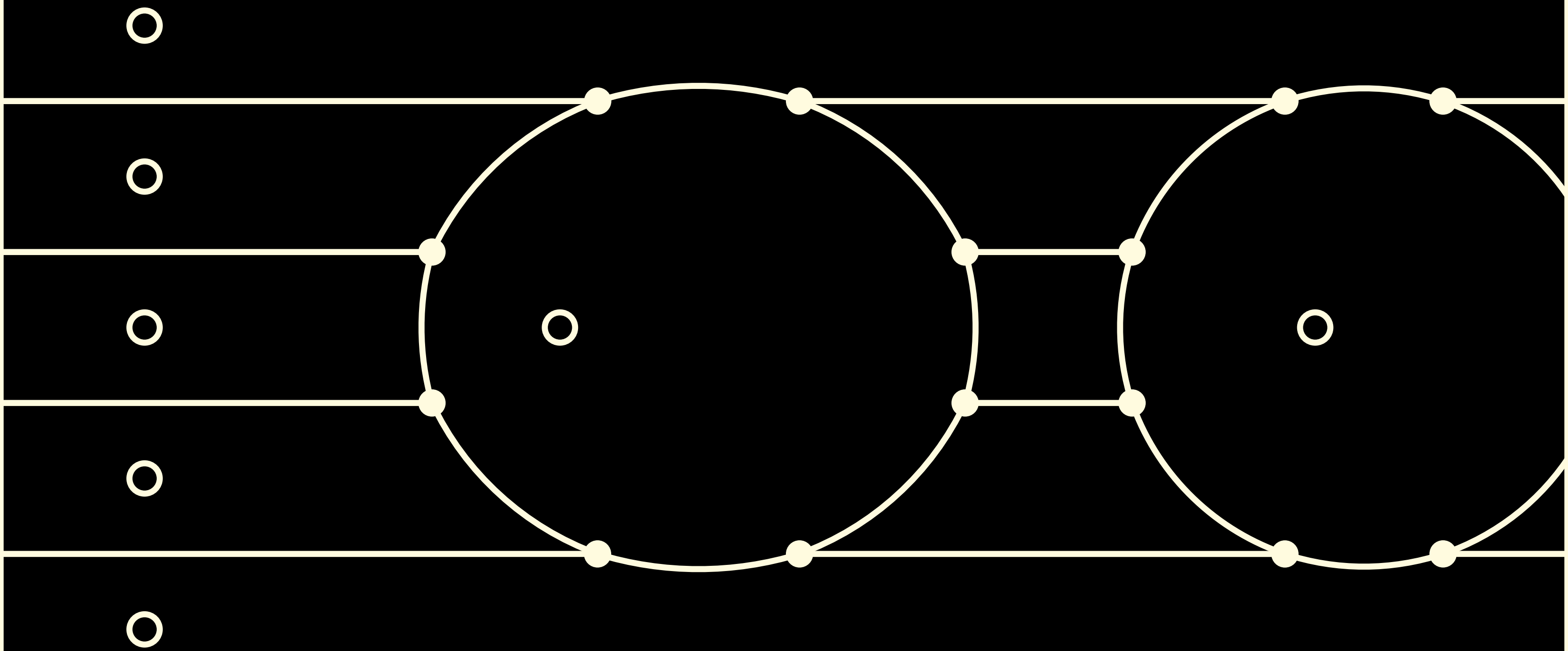
dim d, order $k \longrightarrow$ order $(k + d)$



Theorem

many vertices (ℓ) $\Longrightarrow$ many regions ($\Omega(\ell)$) in higher order

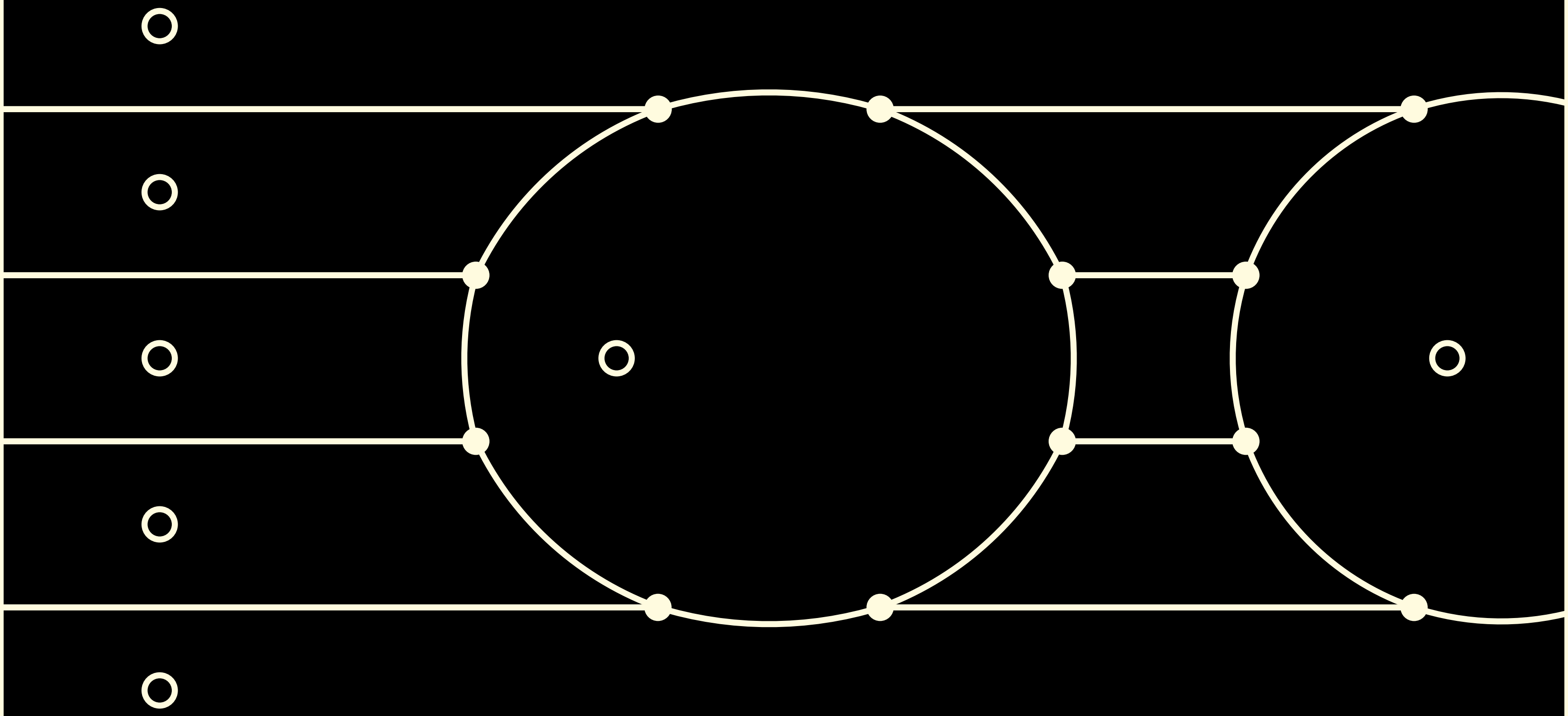
dim d, order k $\longrightarrow$ order $(k + d)$



Theorem

many vertices [ℓ] $\Longrightarrow$ many regions [$\Omega(\ell)$] in higher order

dim d, order $k \longrightarrow$ order $(k+d)$

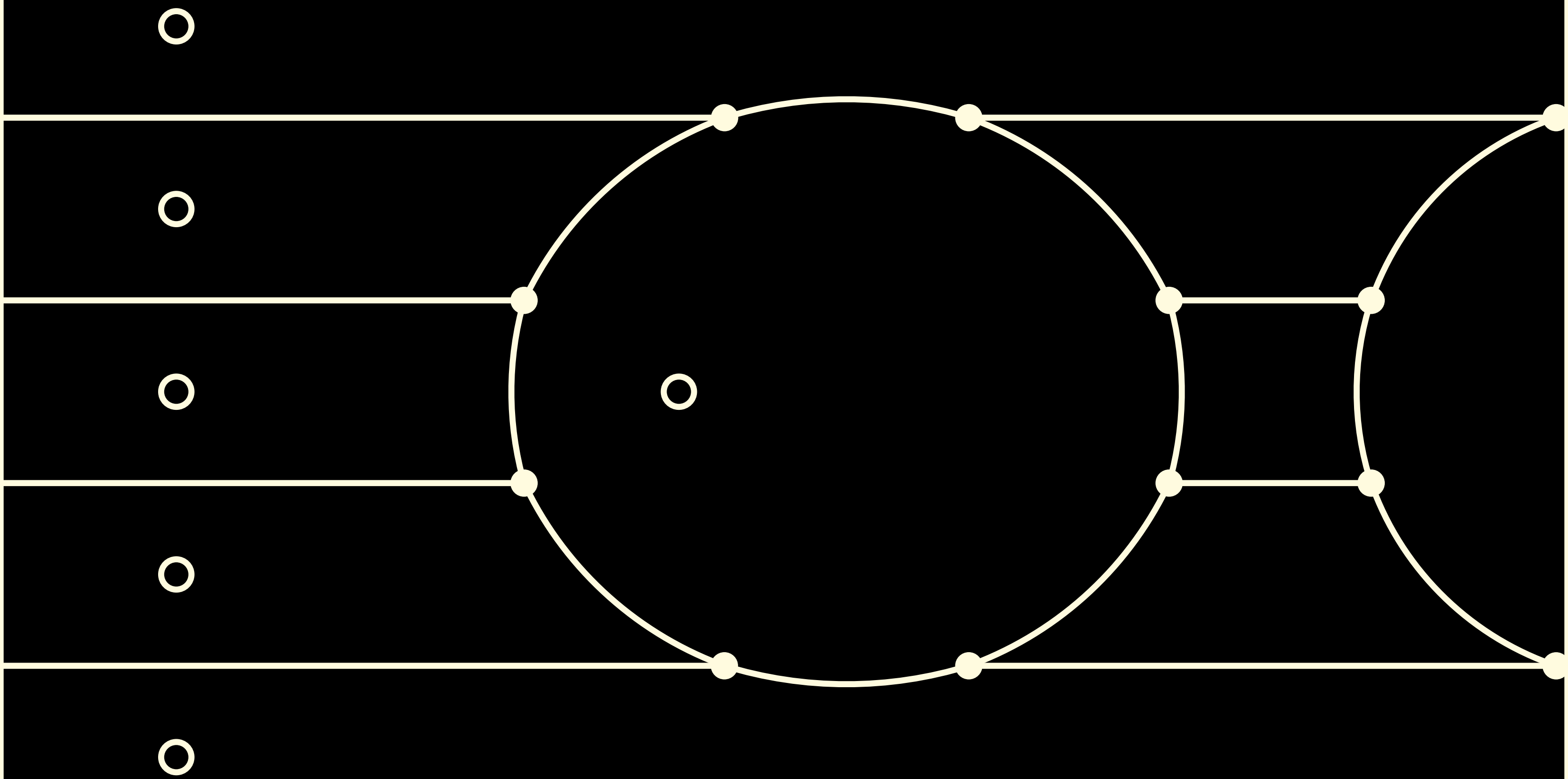


Theorem

many vertices $\Rightarrow$ many regions in higher order

ℓ (many vertices) — $\Omega(\ell)$ (many regions)

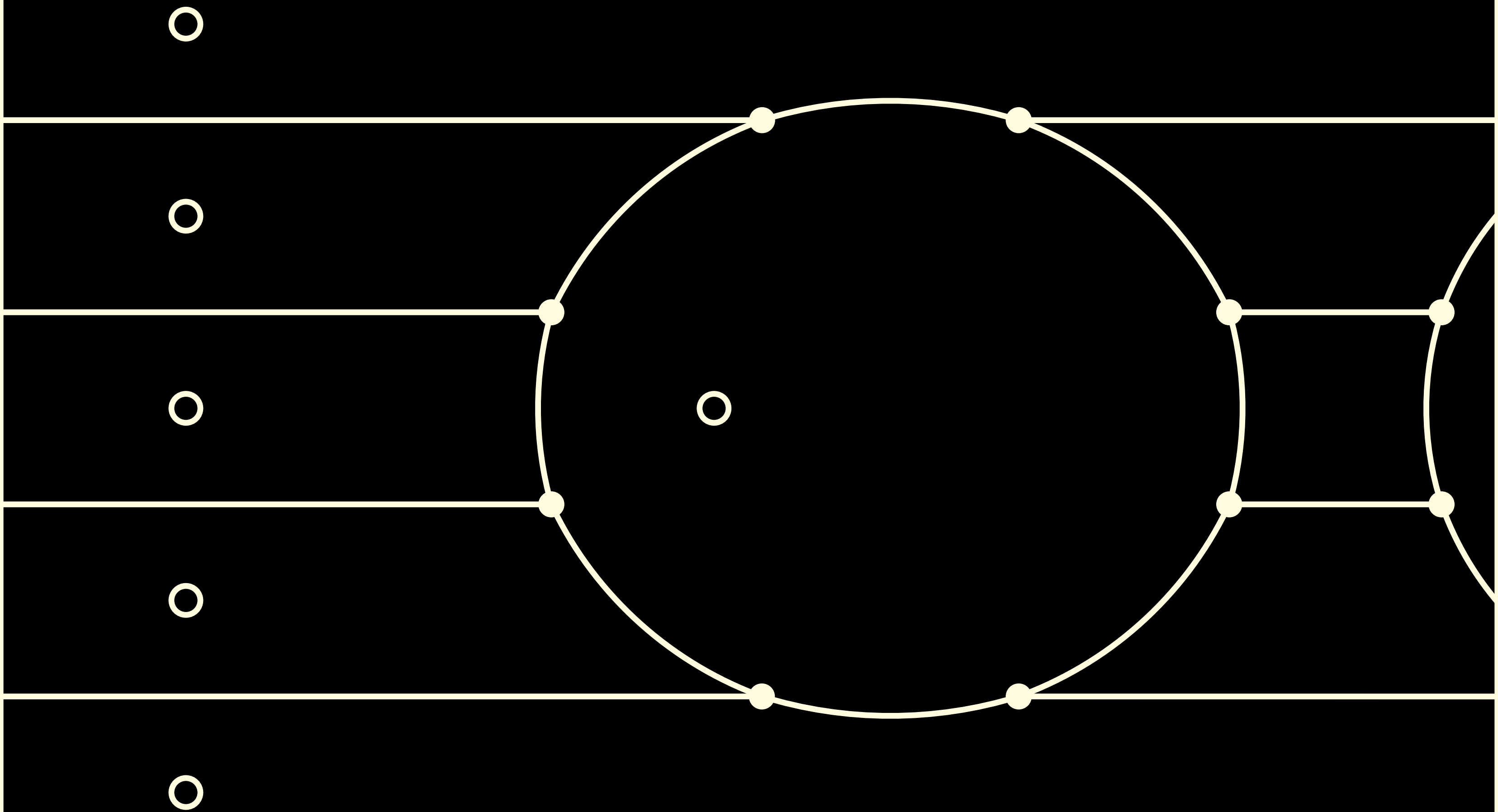
dim d, order $k \longrightarrow$ order $(k+d)$



Theorem

many vertices (ℓ) $\Longrightarrow$ many regions ($\Omega(\ell)$) in higher order

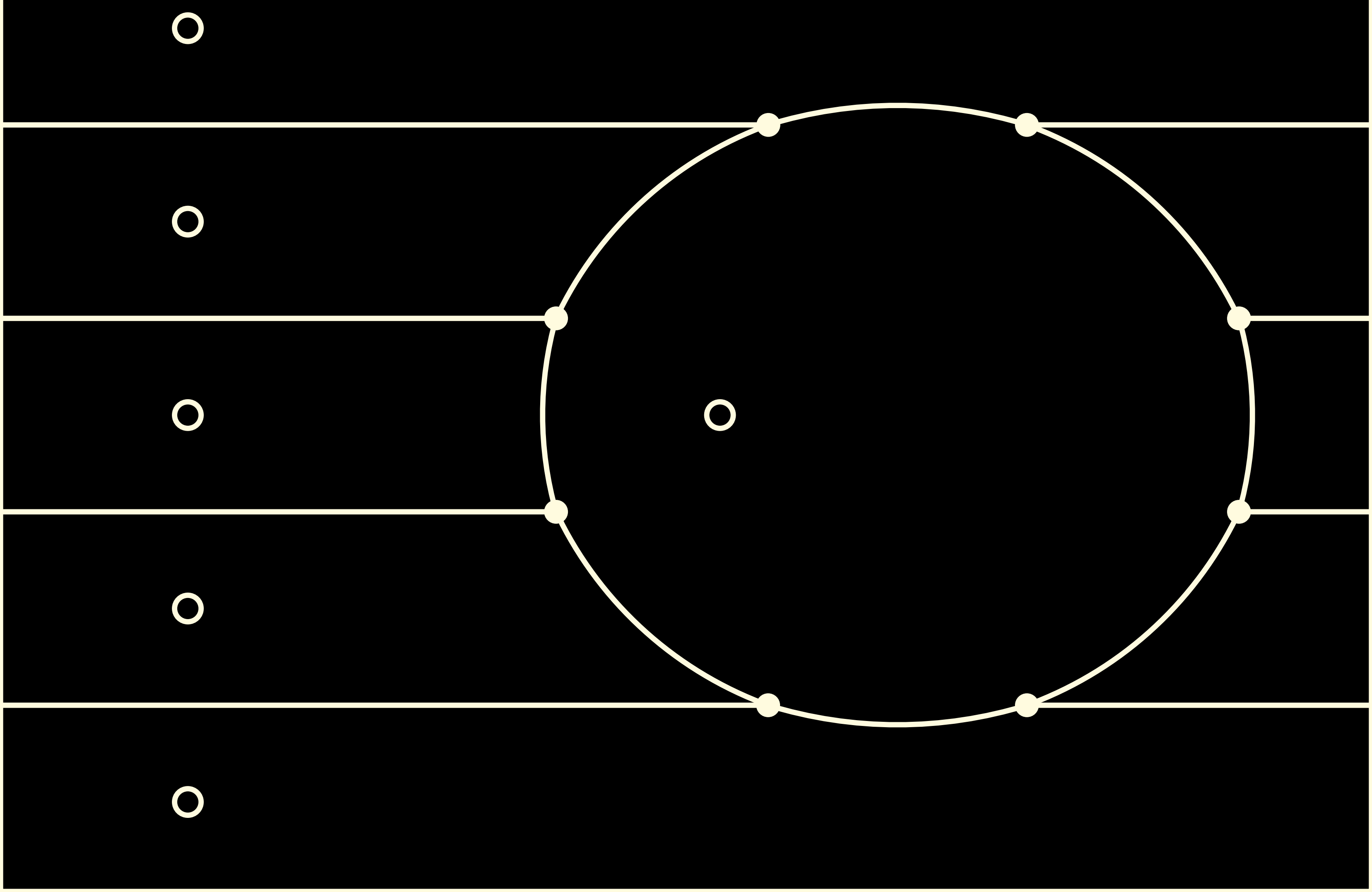
dim d, order $k \longrightarrow$ order $(k + d)$



Theorem

many vertices (ℓ) $\Longrightarrow$ many regions ($\Omega(\ell)$) in higher order

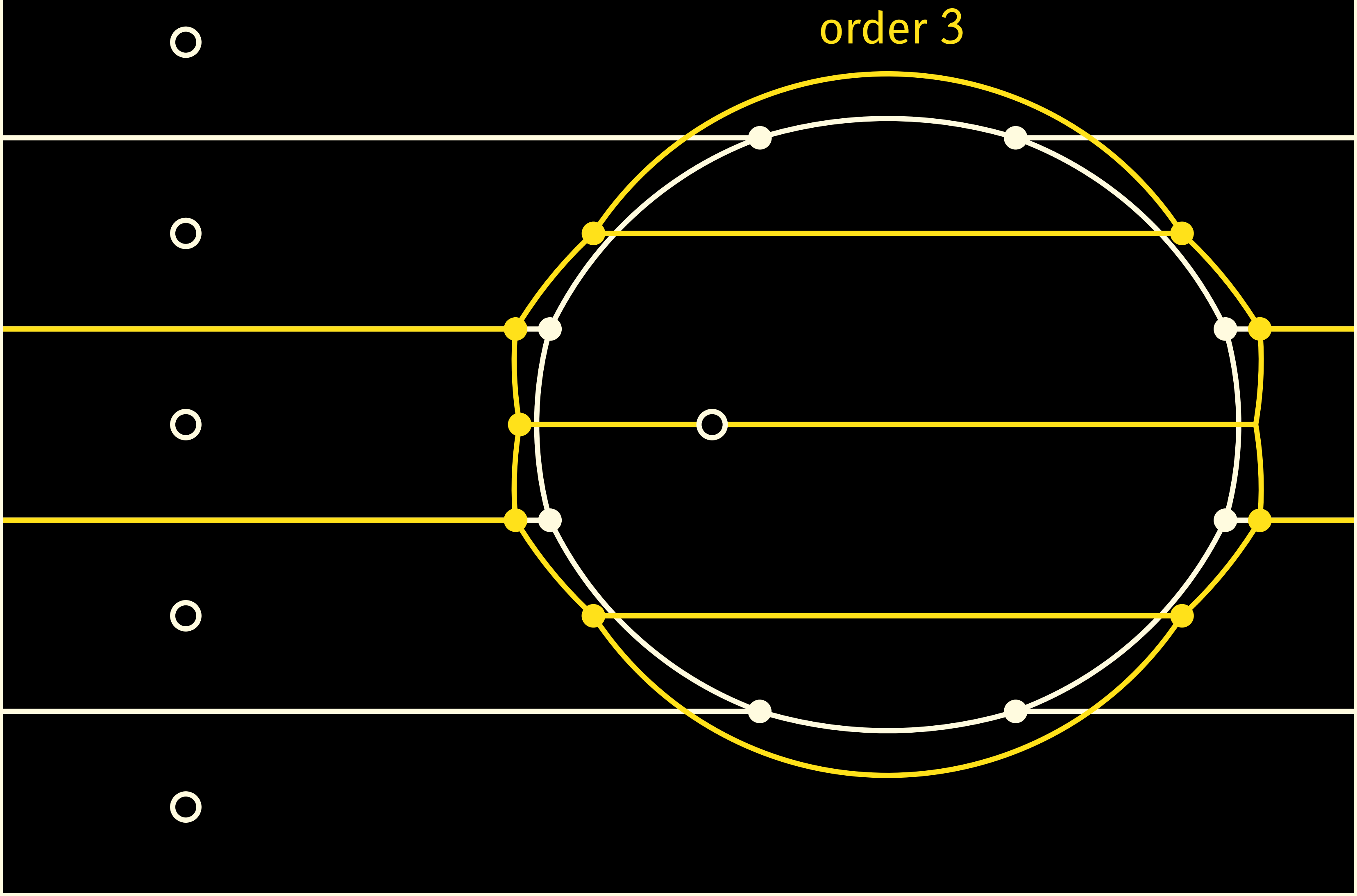
higher order: dim d, order $k \longrightarrow$ order $(k+d)$



Theorem

many vertices (ℓ) $\Longrightarrow$ many regions ($\Omega(\ell)$) in higher order

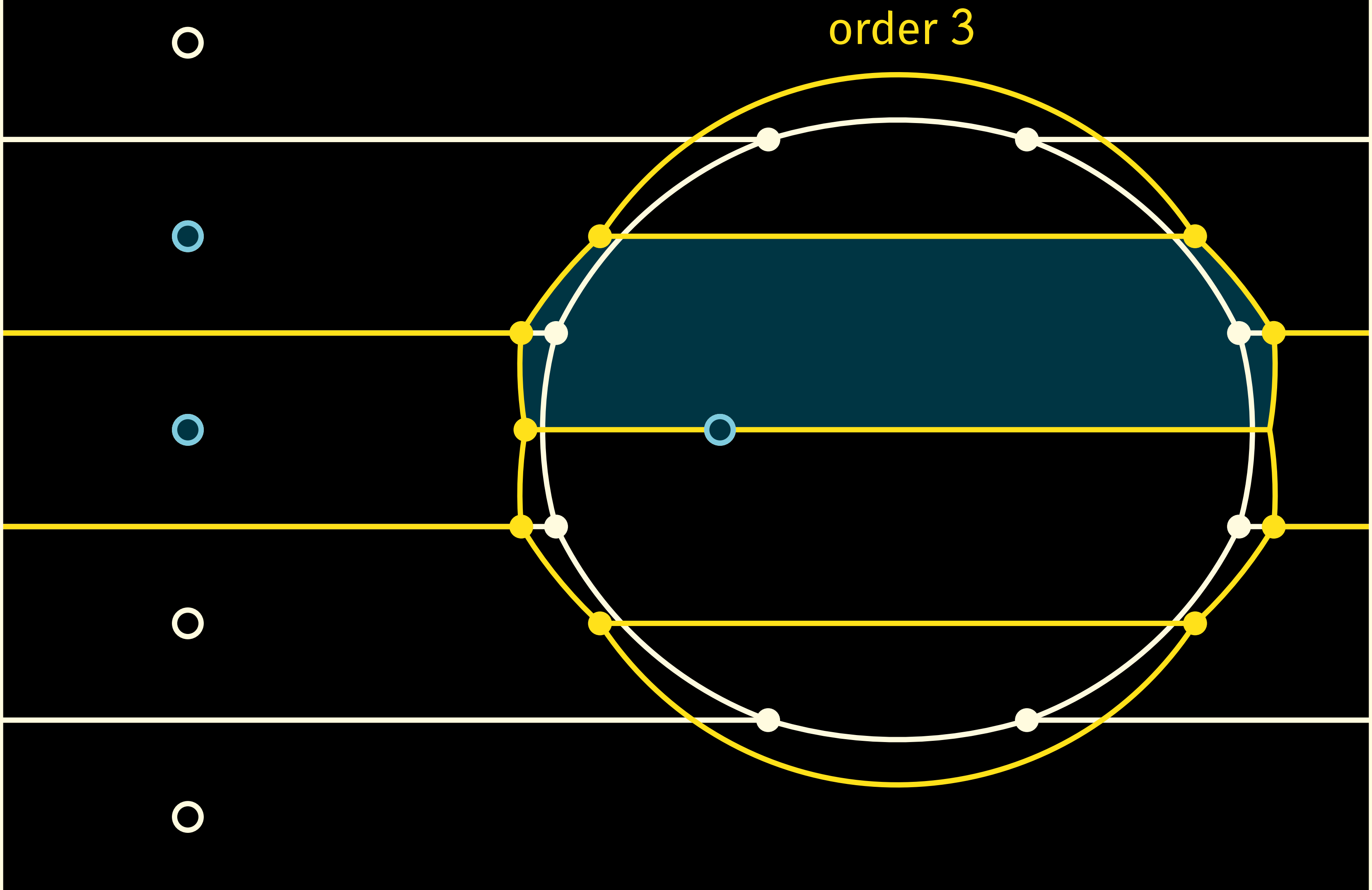
dim d, order k $\longrightarrow$ order $(k+d)$



Theorem

many vertices (ℓ) $\Longrightarrow$ many regions ($\Omega(\ell)$) in higher order

dim d, order k $\longrightarrow$ order $(k + d)$



Theorem

many vertices (ℓ) $\Longrightarrow$ many regions ($\Omega(\ell)$) in higher order

dim d, order k $\longrightarrow$ order $(k+d)$