The Impact of Heterogeneity and Geometry on the Proof Complexity of Random Satisfiability

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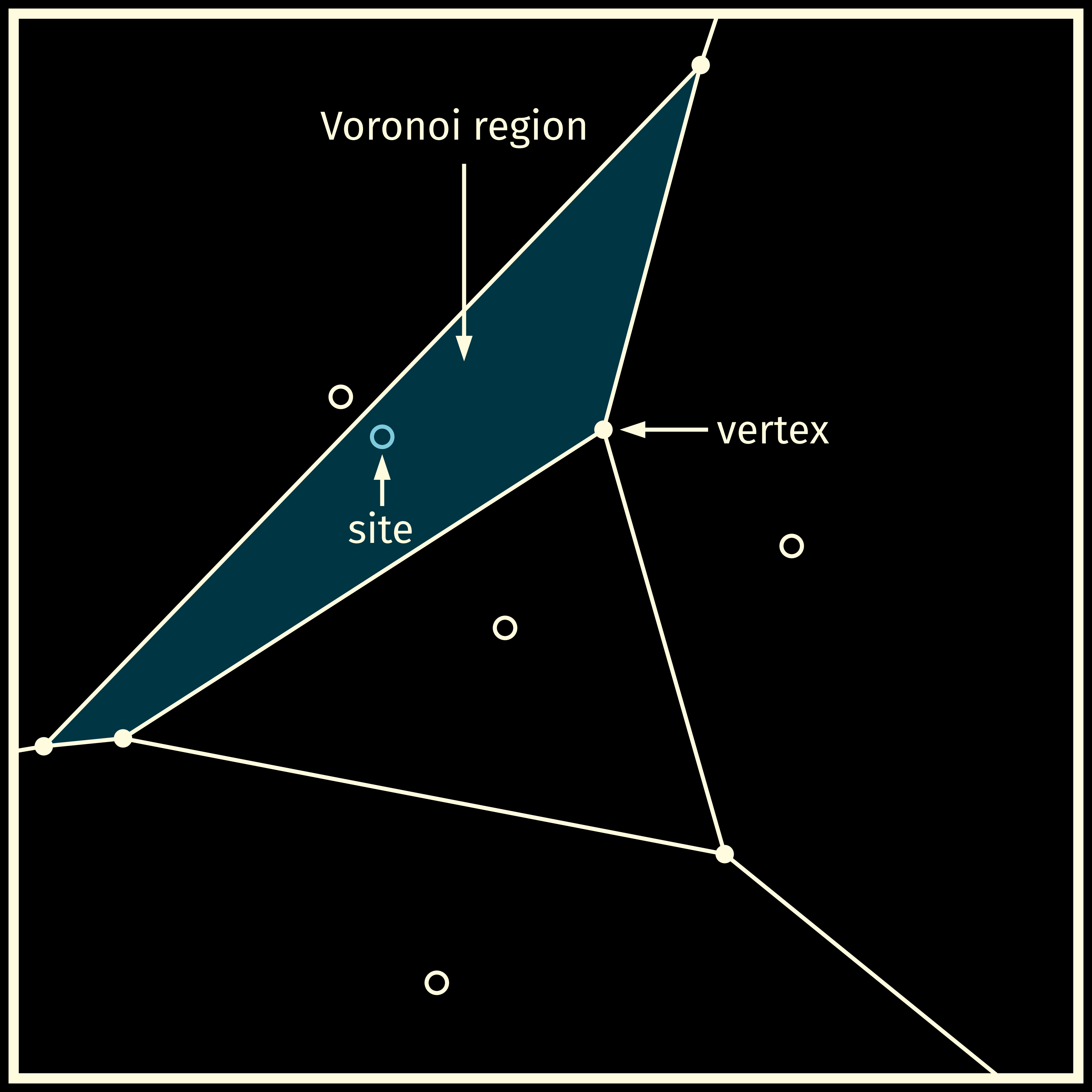
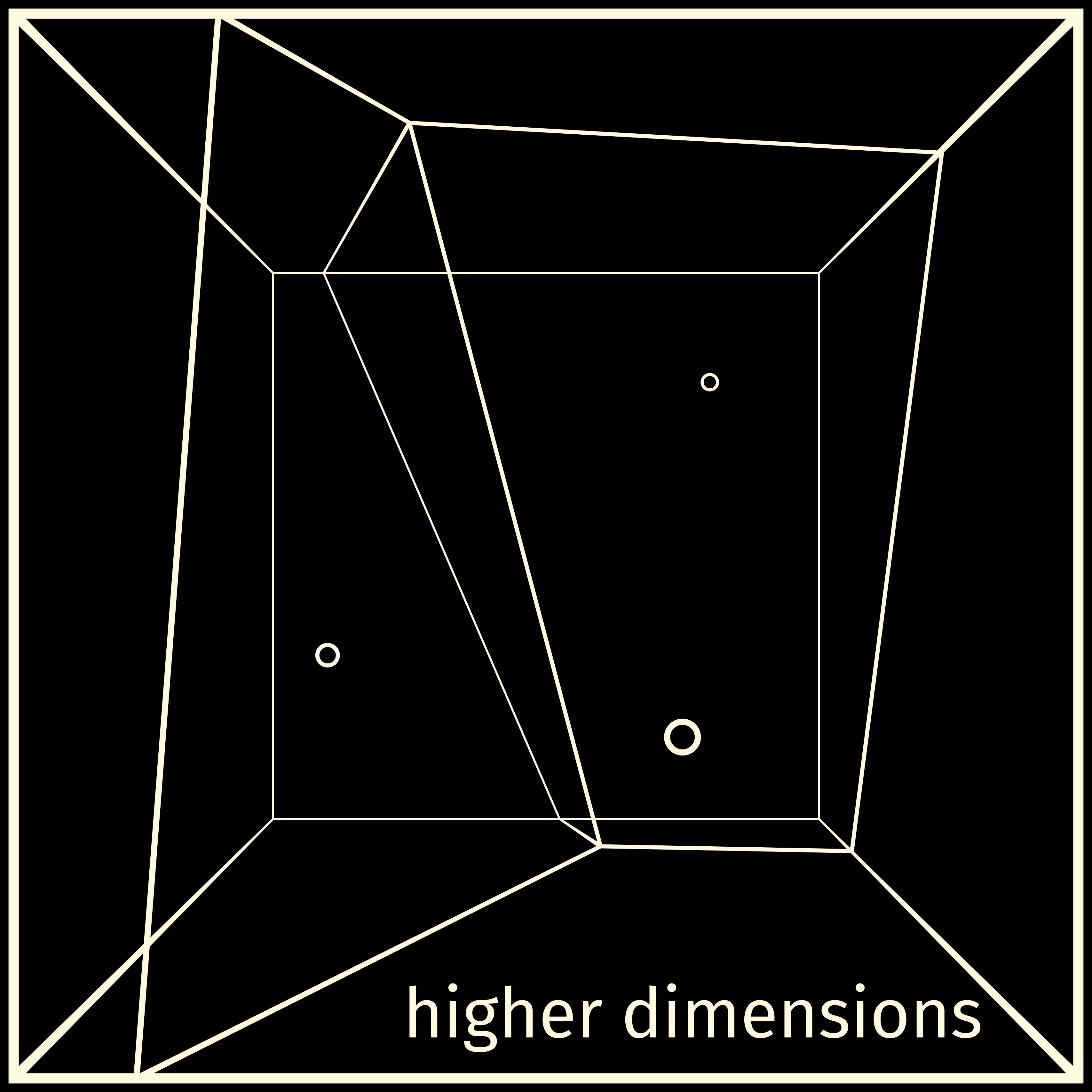
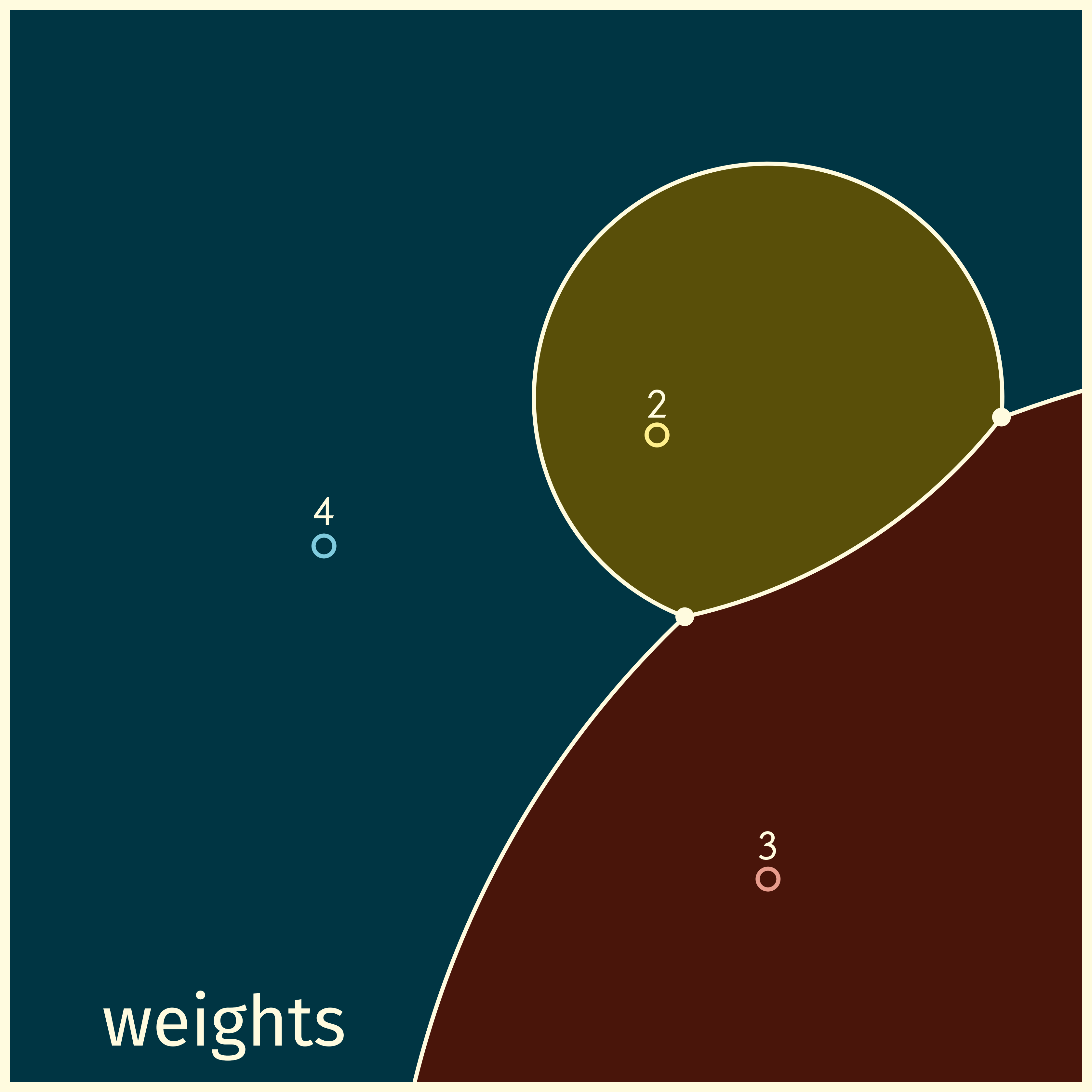
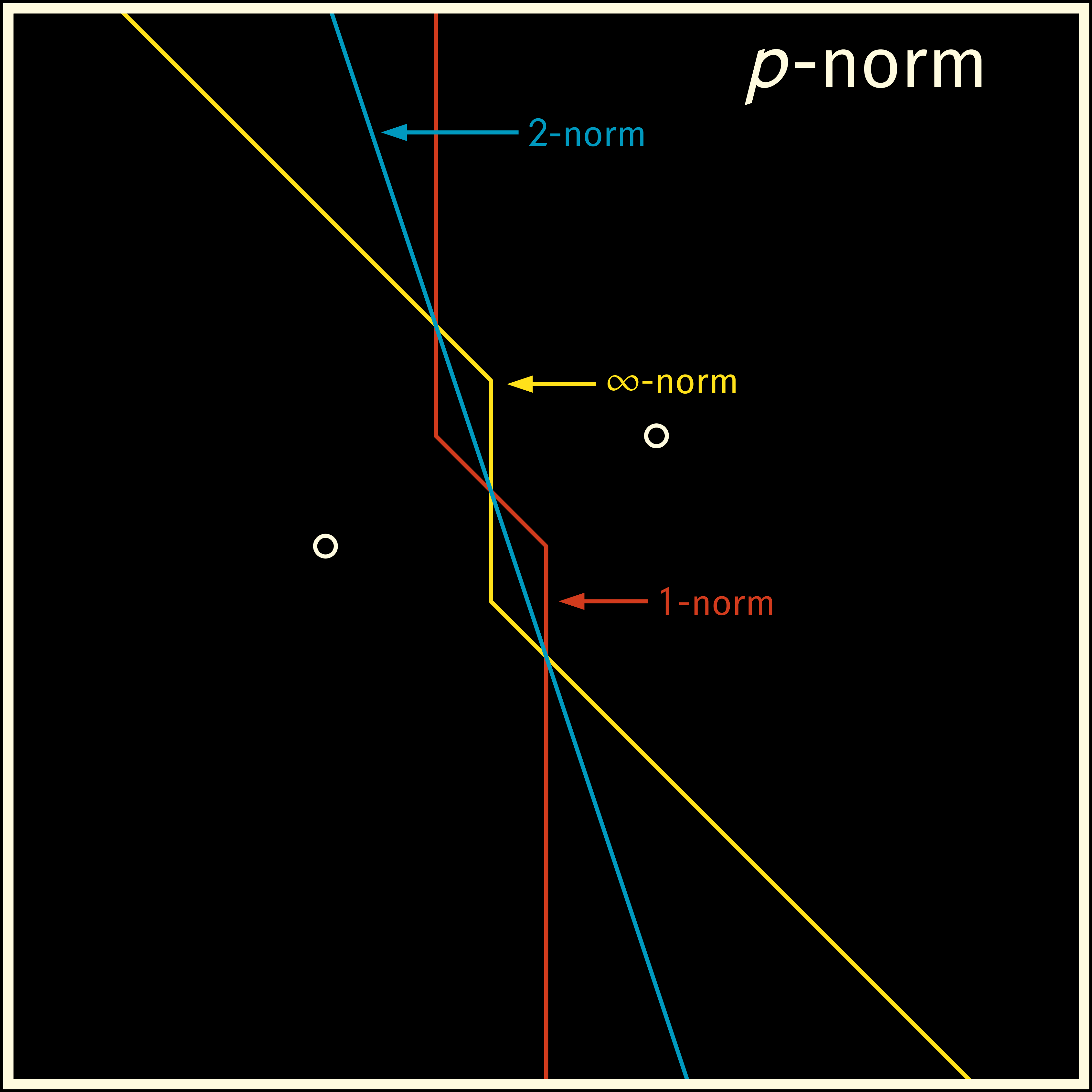
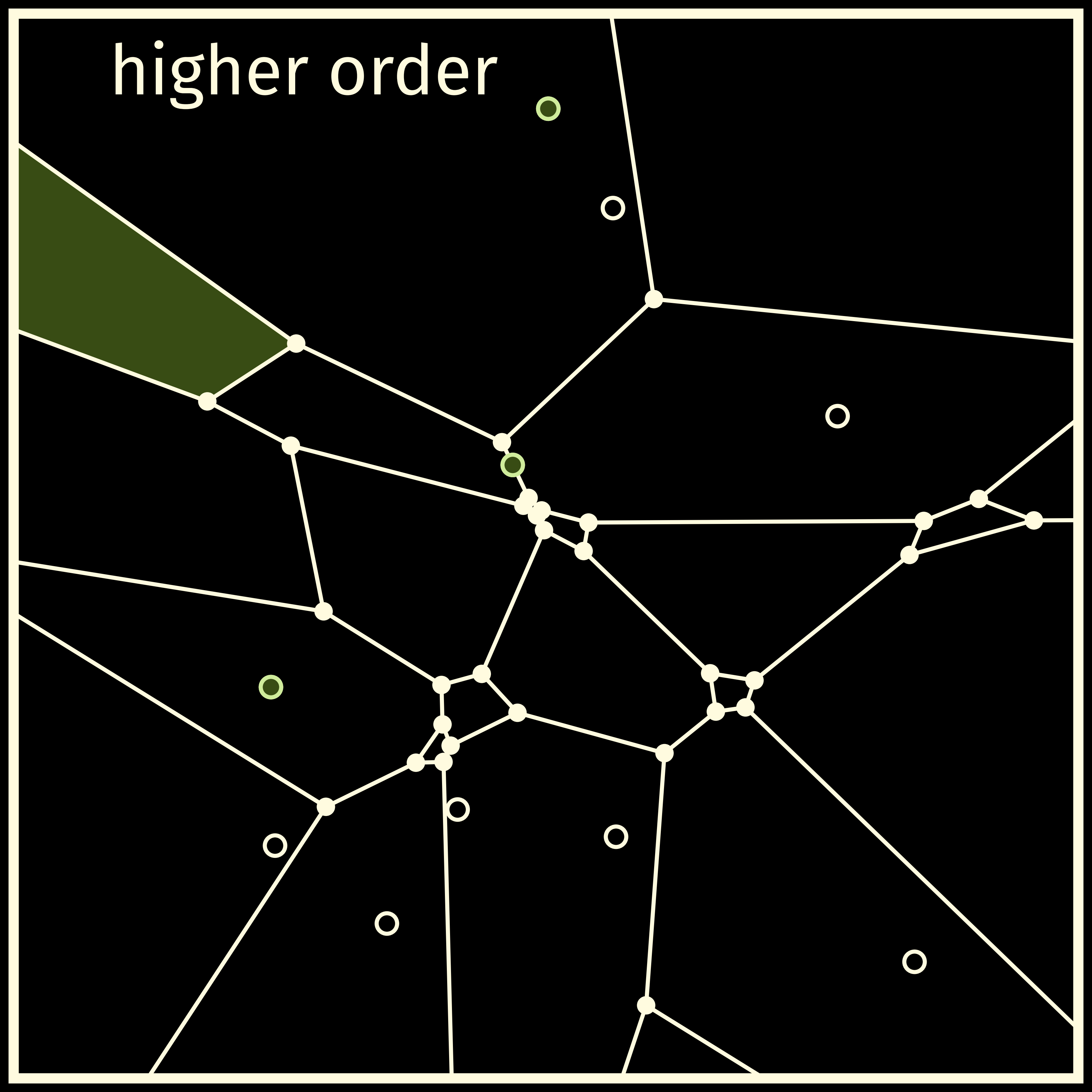
…, and the Complexity of Voronoi Diagrams

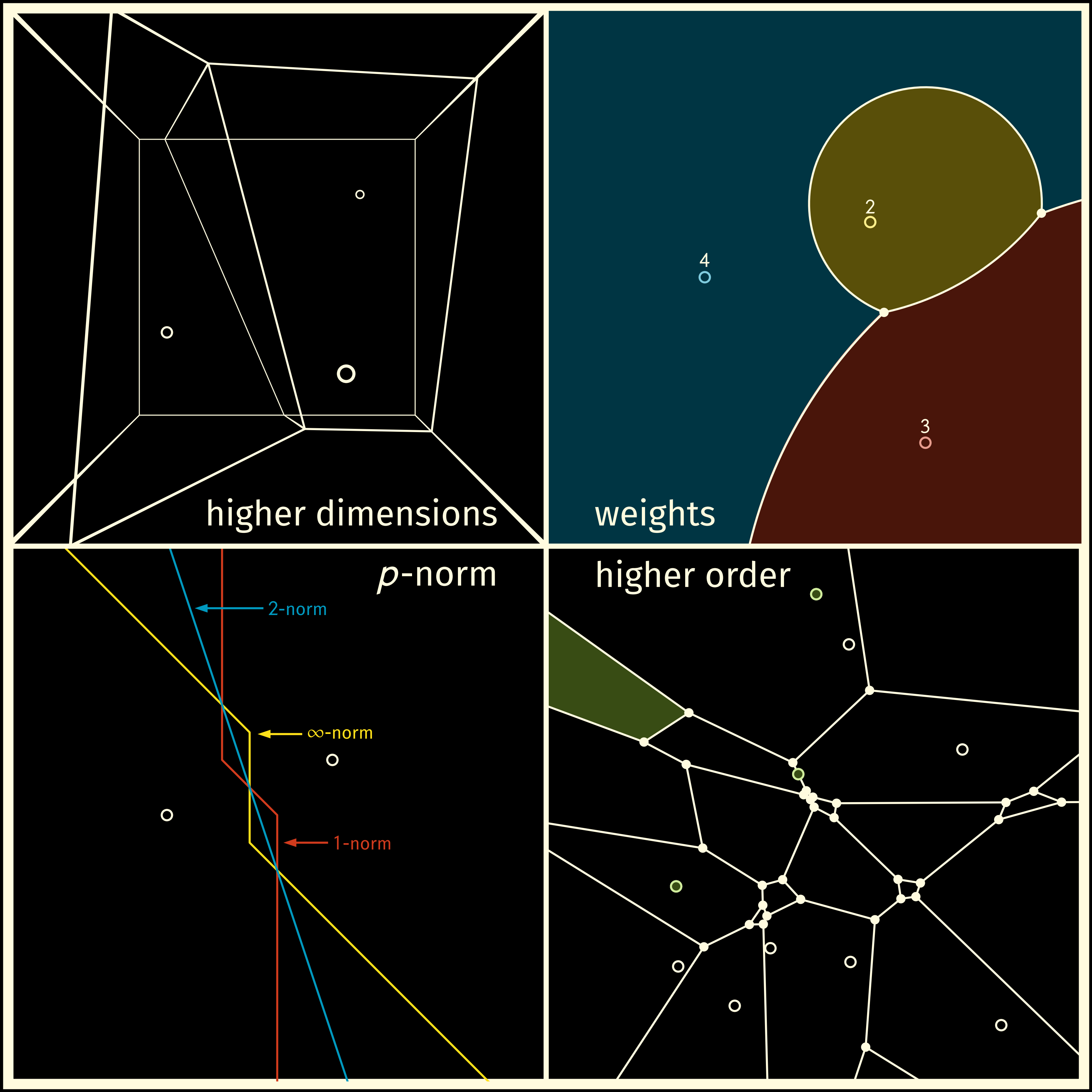

Voronoi region
vertex
site


higher dimensions

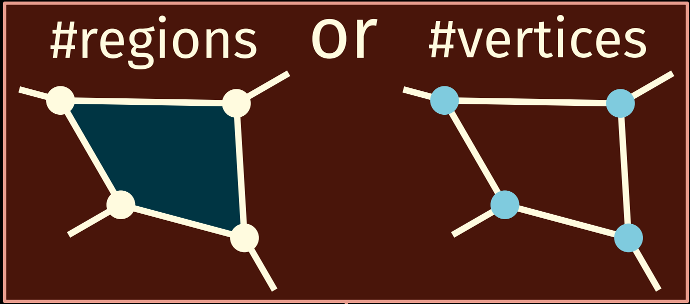

2
4
3
weights


p-norm
2-norm
∞-norm
1-norm


higher order



higher dimensions
weights
2
4
3
p-norm
2-norm
∞-norm
1-norm
higher order



$O(\text{total weight})$
weights ≥ 1

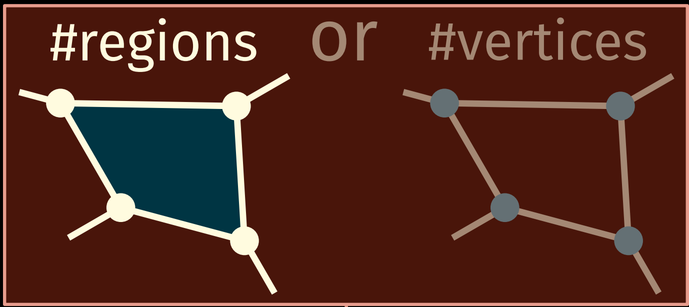
order $\in O(1)$

Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$

order	dim	norm	weights	complexity (region) (vertex)	
k	2	2	∘	$O\left(k(n-k)\right)$	[Lee 82]
k	2	$1, \infty$	∘	$O\left(\min\left\{k(n-k), (n-k)^2\right\}\right)$	[Liu, Papadopoulou, Lee 11]
k	2	abstract	∘	$\leq 2k(n-k)$	[Bohler, Cheilaris, Klein, Liu, Papadopoulou, Zavershynskyi 15]
k	d	2	∘	$O\left(n^{c(d)}\right)$	[Mulmuley 91]
1	3	2	∘	$\Theta\left(n^2\right)$	[Klee 80] [Seidel 87]
1	d	p	∘	$O\left(n^{c(d)}\right)$	[Lê 96]
1	d	∞	∘	$\Theta\left(n^{\lceil d/2 \rceil}\right)$	[Boissonnat, Sharir, Tagansky, Yvinec 98]
1	d	1	∘	$\Theta\left(n^2\right)$	[Boissonnat, Sharir, Tagansky, Yvinec 98]
1	d	2	∘	expected $O(n)$	[Bienkowski, Damerow, Meyer auf der Heide, Sohler 05]
1	2	2	✓	$\Omega\left(n^2\right)$	[Aurenhammer, Edelsbrunner 84]
1	2	2	✓ random	expected $O(n \text{ polylog } n)$	[Har-Peled, Raichel 15]



O(total weight)
weights ≥ 1

order $k \in O(1)$

Theorem

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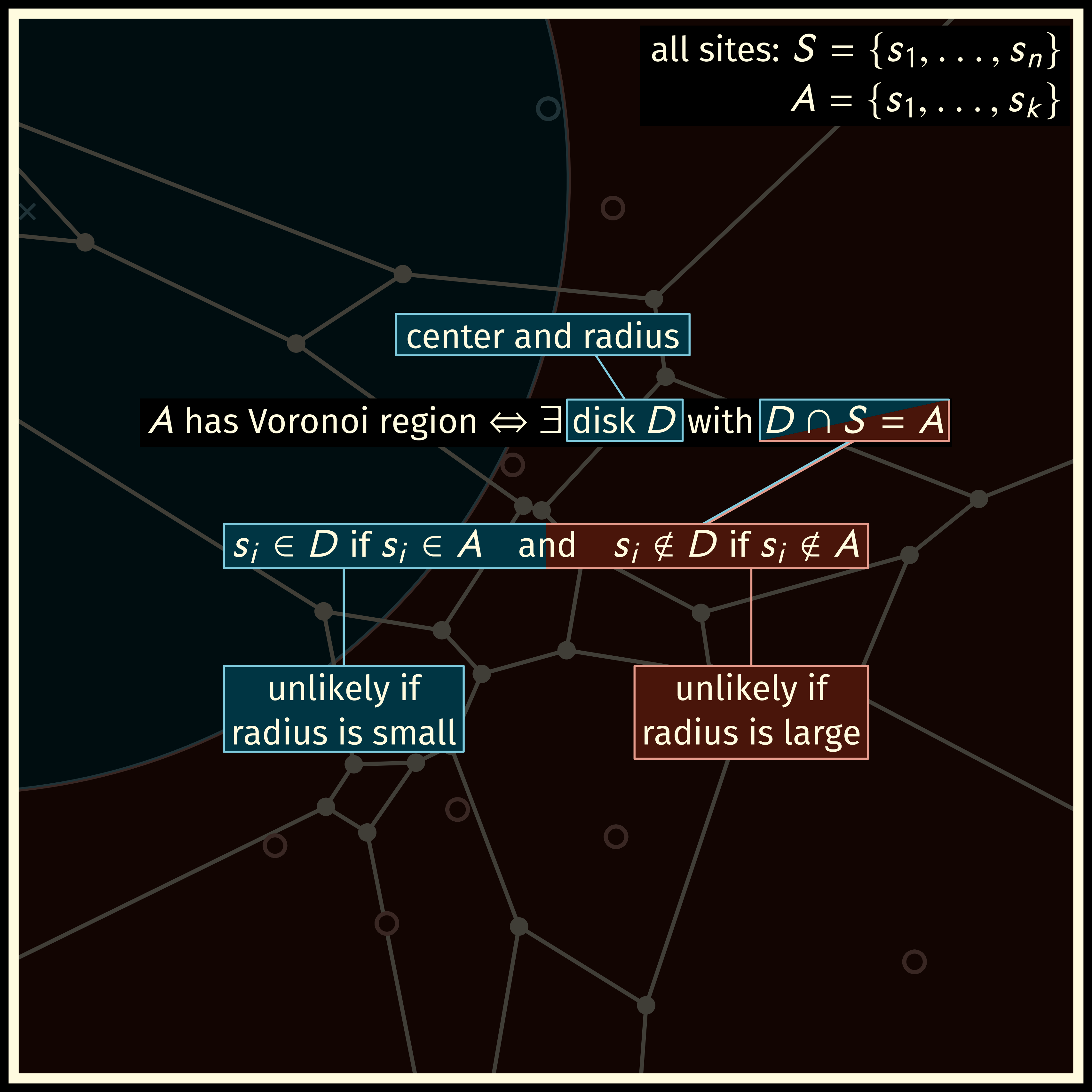
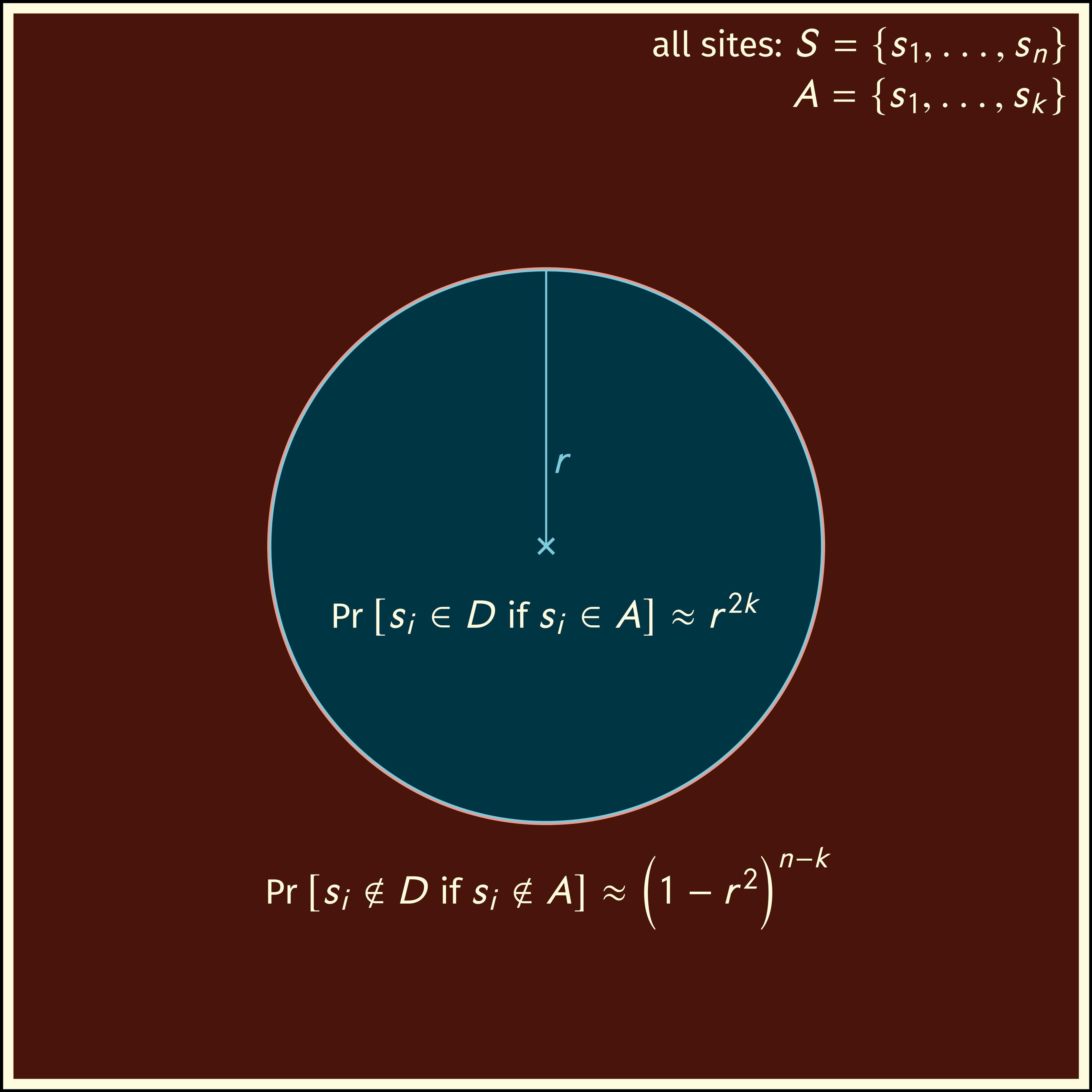
$$\mathbb{E}\left[\#\text{regions}\right] = \sum_{\substack{A \subseteq S \\ |A| = k}} \Pr\left[A \text{ has Voronoi region}\right]$$



all sites: $S = \{s_1, \ldots, s_n\}$
$A = \{s_1, \ldots, s_k\}$
center and radius
A has Voronoi region $\Leftrightarrow \exists$ disk D with $D \cap S = A$
$s_i \in D$ if $s_i \in A$ and $s_i \notin D$ if $s_i \notin A$
unlikely if radius is small
unlikely if radius is large


all sites: $S = \{s_1, \ldots, s_n\}$
$A = \{s_1, \ldots, s_k\}$
r
$\Pr[s_i \in D \text{ if } s_i \in A] \approx r^{2k}$
$\Pr[s_i \notin D \text{ if } s_i \notin A] \approx \left(1 - r^2\right)^{n-k}$

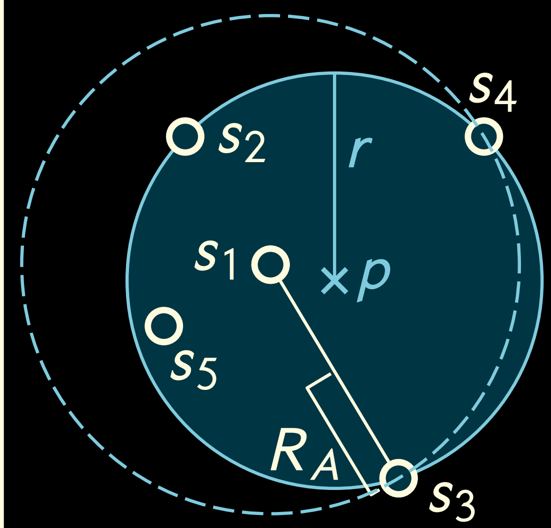
disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr\left[A \text{ has Voronoi region}\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A\right]$$

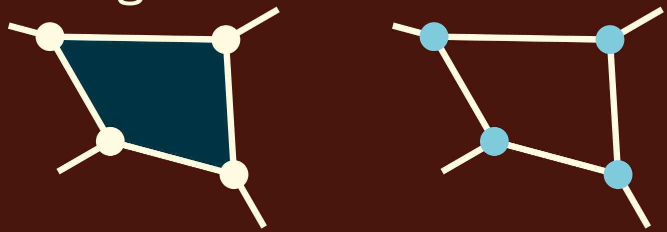
smallest r s.t. $\exists p\colon\ s_i \in D$ if $s_i \in A \Rightarrow r \geq R_A$



$$\Pr\left[A \text{ has Voronoi region}\right]$$
$$= \int_r \Pr\left[A \text{ has region} \mid R_A = r\right] f_{R_A}(r)\,\mathrm{d}r$$

$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$= \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$\leq \Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \wedge s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
$$\leq 4 \cdot \Pr\left[s_i \notin \text{ grid cell if } s_i \notin A\right] + \text{ something small}$$

#regions or #vertices



O(total weight)
weights ≥ 1

order $\in O(1)$

Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$

ℓ

$\Omega(\ell)$

Theorem

many vertices $\Longrightarrow$ many regions in higher order

dim d, order $k \longrightarrow$ order $(k + d)$

Theorem

many vertices (ℓ) $\Longrightarrow$ many regions ($\Omega(\ell)$) in higher order

dim d, order k $\longrightarrow$ order $(k + d)$