# Theoretical Study of Data Reduction in Real-World Hitting Set Instances

Master's Thesis of

Marcus Wunderlich

At the Department of Informatics
Institute of Theoretical Informatics (ITI)

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| Reviewer: | T.T.-Prof. Dr. Thomas Bläsius |
| Second reviewer: | PD Dr. Torsten Ueckerdt |
| Advisors: | T.T.-Prof. Dr. Thomas Bläsius |
| | Marcus Wilhelm |

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Karlsruher Institut für Technologie
Fakultät für Informatik
Postfach 6980
76128 Karlsruhe

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(Marcus Wunderlich)

# Abstract

Finding a minimum HittingSet is a fundamental optimization problem on hypergraphs. Despite being $\mathcal{NP}$-complete and even $W[2]$-complete (if parametrized by solution size), many real-world instances can be solved quickly due to two simple reduction rules by Weihe [Wei98]. Experiments show that *heterogeneity* and *locality* are both common properties of real-world instances as well as crucial factors for the effectiveness of these reduction rules.

In this thesis, we analyze a random model similar to 1D-GIRGs, that generates hypergraphs with adjustable degrees of heterogeneity and locality, from a theoretical perspective. For the model's *threshold variant*, where the locality is set to its maximum, we show that the reduction rules reduce the generated hypergraphs a.a.s. to a trivial kernel under weak conditions that depend on the degree of heterogeneity. We, additionally, provide a fix to solve these instances that do not satisfy this condition. For a general level of locality, we find that at least a significant constant percentage of vertices is expected to be eliminated by the reduction rules, depending on the exact degree of both heterogeneity and locality.

# Zusammenfassung

Das Finden von minimalen Hitting-Sets ist ein fundamentales Problem auf Hypergraphen. Obwohl das Problem $\mathcal{NP}$- und sogar $W[2]$-komplett (durch die Lösungsgröße parametrisiert) ist, können viele reale Instanzen schnell durch zwei einfache Reduktionsregeln nach Weihe [Wei98] gelöst werden. Experimentelle Studien zeigen, dass *Heterogenität* und *Lokalität* sowohl verbreitete Eigenschaften von realen Hypergraphen sind, als auch entscheidende Faktoren für die Effektivität der Reduktionsregeln.

In dieser Arbeit analysieren wir ein Zufallsmodell aus einer theoretischen Perspektive, das ähnlich zu 1D-GIRGs ist, und Hypergraphen mit variablen Graden von Heterogenität und Lokalität generieren kann. Für Hypergraphen, die mit maximaler Lokalität in diesem Modell erzeugt wurden, zeigen wir, dass die Reduktionsregeln die erzeugten Hypergraphen fast immer (a.a.s.) auf einen trivialen Kern reduzieren. Das geschieht unter schwachen Bedingungen, die vom Grad der Heterogenität abhängen. Für die Fälle, die diese Bedingung nicht erfüllen, stellen wir eine zusätzliche Verzweigungsregel auf, mit der diese Instanzen trotzdem vollständig gelöst werden können. Für ein allgemeines Maß an Lokalität zeigen wir, dass, abhängig vom genauen Grad der Heterogenität und Lokalität, im Erwartungswert ein konstanter aber nicht zu vernachlässigender Prozentsatz von Knoten durch die Reduktionsregeln eliminiert wird.

# Contents

# 1 Introduction

When solving $\mathcal{NP}$-hard problems in practice, data reduction is a key part for minimizing the run time of algorithms, especially on large instances. The idea of data reduction is to apply so-called *reduction rules* that shrink the given instance while retaining the ability to find an optimal solution in it [AK+22]. The better the reduction rules work on the passed input, the smaller the resulting instance; the so-called *kernel*. If the kernel is small enough, it can then be solved with an exact solver in reasonable time. In some cases, the reduction rules do not only shrink the instance but also split it into multiple disjoint sub-instances, decreasing the run time even further. In this thesis, we study a set of reduction rules for the HittingSet problem due to Weihe [Wei98] that have this behavior on many real-world instances.

In general, the *minimum hitting set problem*, or short HittingSet, is defined as follows: Given a hypergraph $H = (V, E)$, find a minimum set of vertices $S \subseteq V$ that intersects every hyperedge of $H$. The two reduction rules proposed by Weihe [Wei98] are based on the concept of dominance among vertices and hyperedges respectively. The *vertex domination rule* says that a vertex $a$, whose incident hyperedges are all incident to another vertex $b$, is *dominated* by $b$ and can thus be eliminated. On the other hand, the *hyperedge domination rule* allows to eliminate every hyperedge $a \in E$ that is a superset of another hyperedge $b \in E$.

Experiments show, that these reduction rules perform surprisingly well on real-world networks. One example of such networks are railroad networks or public transportation systems in general [Wei98][BFFS19]. These networks can be modeled as a hypergraph, where the stations are the vertices, and the connections, running through the stations, are the hyperedges. Bläsius et al. [BFFS19] studied the structure of several real railroad networks, and identified the *heterogeneity* of the vertices and the *locality* of the network as two key properties that cause the efficiency of the reduction rules on them. Heterogeneity is a measure of the diversity of vertex degrees. Roughly speaking, a heterogeneous hypergraph has many vertices with a small degree and only a few vertices with high degree, whereas homogeneous vertices all have about the same degree. Public transportation networks of metropolitan areas are good examples of heterogeneity, as they commonly have only a few highly frequented stations in the central city (e.g. transfer stations or stations near sights and event locations) and many stations in the suburbs using only a single connection headed to the center. Besides that, heterogeneity or power-laws (also called Zipf's Law or heavy-tail distribution) also occur in a whole host of other real-life phenomena, e.g., the distribution of city populations, computer file size, sales of books and music, as well as the number of citations among papers [New04]. Pinto et al. [PMM12] published a survey explaining many more phenomena following a power-law. This wide range of occurrences of power-law distributions underlines the importance of understanding networks with heterogeneity.

The second network property that benefits the efficiency of the HittingSet reduction rules, namely locality of the network, also has a good explanation for appearing in public transportation networks. In general, the locality of a network can be measured by the *bipartite cluster coefficient* [RA04], which is the probability that two vertices $A$ and $B$ share a hyperedge, if they have a common adjacent vertex $C$. In the context of railroad networks, this property presumably has its origin in the underlying geometry of the network [BFFS19]; which means that stations that are close to each other are also more likely to be connected. Combining these two properties yields networks in which the reduction rules cause large stations (vertices) to dominate the many smaller ones in their neighborhood. This may be one reason for the effectiveness of this set of reduction rules on instances with heterogeneity and locality. Because of this efficiency, the two reduction rules are a crucial part of many algorithmic approaches for different variants of HittingSet (e.g. [AK10][NR03]) and other similar covering problems (e.g. [GST14][DB11]). However, despite the many practical evidences for the efficiency of the reduction rules, there are only a few theoretical results on that; especially regarding networks with heterogeneity and locality.

These two properties were not only found in hypergraph-like networks but also in networks that can be modeled as (normal) graphs like social networks and internet infrastructure [BKL19]. It is not surprising, that studying the structure of such networks is of great interest. To support this study both practically and analytically, different random graph models were introduced; the most prominent of whom are the *hyperbolic random graphs* [Kri+10] and the more general *geometric inhomogeneous random graphs* (GIRGs) [BKL19]. These random graph models are able to generate synthetic complex graphs with varying levels of heterogeneity and locality for practical experiments, but are, on the other hand, simple enough to be analyzed mathematically. Therefore, they are suitable for both experimental and theoretical studies on complex graph-like networks.

For hypergraphs, there is a GIRG-like model due to Bläsius et al. [BFFS19], which we call *Hyper-GIRGs*. Similar to GIRGs, Hyper-GIRGs use a power-law distribution to control the level of heterogeneity among the vertex degrees, and a parameter called *temperature* $T \in (0, 1)$ to control the impact of the underlying geometry for locality. Further, Hyper-GIRGs are defined to have homogeneous hyperedge sizes. This is due to their origin in the context of public transportation systems, where the hyperedges have roughly the same size in practice [BFFS19]. In a special case, the *threshold variant* of the model, the temperature can be set to 0, which maximizes the locality. For every other temperature, it is called the *binomial variant*.

## 1.1 Contribution and Outline

In this thesis, we take Hyper-GIRGs as synthetic real-world HittingSet instances and analyze the efficiency of the reduction rules due to Weihe [Wei98] on them from a theoretical perspective. As Hyper-GIRGs mainly control heterogeneity and locality of the generated hypergraphs, we focus on the impact of those properties in particular. The practical perspective of this approach is covered by the experiments from Bläsius et. al [BFFS19], which are summarized in Figure 1.1.

These experiments show that, although both properties have positive a impact on the effectiveness of the reduction rules, heterogeneity alone is not enough to fully explain their efficiency. However, with a certain level of locality, an increasing heterogeneity of the vertices can decrease the kernel size. On the other hand, a high locality on its own is sufficient to achieve very small kernel sizes independent of the degree of heterogeneity. Our goal is to find proofs for the above suppositions on Hyper-GIRGs using the 1d-Torus as underlying geometry.

After showing basic properties of this model in Chapter 3, we first focus on hypergraphs generated by the threshold variant in Chapter 4. We show that the reduction rules reduce these hypergraphs to a trivial kernel with a probability tending to 1 for increasing hypergraph sizes. Here, the degree of heterogeneity impacts the speed of convergence of this probability.

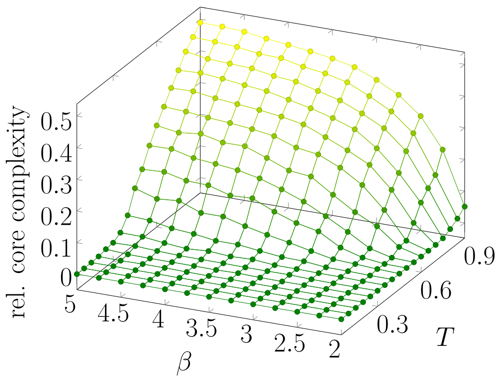


**Figure 1.1:** Figure 2 from [BFFS19]. It shows the reduction result for different values of heterogeneity and locality. The lower the $\beta$ or the $T$, the higher the heterogeneity or the locality, respectively (precise definition and explanation follow in Section 2.3). The relative core complexity is the fraction of the original vertices that remain after the application of the reduction rules.

However, for small and rather homogeneous hypergraphs there is a chance that the reduction rules do not reduce the hypergraph entirely. In this case, we provide a simple fix to solve these instances anyway. In Chapter 5, we consider the general binomial variant with $T \in (0; 1)$ and show a lower bound to the expected fraction of vertices that is dominated in these hypergraphs. We find that in hypergraphs with high locality and a bit heterogeneity, a significant percentage of the vertices is dominated.

## 1.2 Related Work

The minimum hitting set problem is one of Karp's original 21 $\mathcal{NP}$-complete problems [Kar72]. It is even $W[2]$-complete if parameterized by the solution size [DF12]. The variant of HittingSet where the hyperedges have a maximum size of $d$ is often referred to as $d$-HittingSet. Apart from that, HittingSet is connected to many other $\mathcal{NP}$-hard problems, e.g., it is the dual problem of SetCover and equivalent to *red-blue dominating set*.

In practice, a common way to solve HittingSet instances are ILP-solvers. However, Bläsius et al. recently developed a branch-and-bound solver that outperforms modern ILP-solvers by at least an order of magnitude on many instances [BFSW]. They use the above-mentioned reduction rules due to Weihe [Wei98], which were, besides that, also adapted for many variants of HittingSet and other covering problems. For example, Niedermeier and Rossmanith [NR03] as well as Faisal and Abu-Khzam [AK10] adapted these reduction rules for efficient algorithms for 3- and $d$-HittingSet, respectively. Further, there is a kernelization algorithm for planar red-blue dominating set due to Garnero et al. [GST14] that is based on Weihe's reduction rules. For a general overview of data reduction methods and open problems, Abu-Khzam et a. [AK+22] published a survey about recent advances in practical data reduction, which also contains a section about HittingSet.

The second main research area we touch in this thesis (besides data reduction) are GIRGs. They were proposed by Bringmann et al. [BKL19] as a more general and also easier-to-analyze form of the hyperbolic random graphs (HRGs) [Kri+10]. GIRGs (as well as HRGs) serve to generate graphs with real-world properties and adjustable levels of heterogeneity and locality.

Bringman et al. proved that GIRGs have the *small-world property*, i.e., they have a logarithmic diameter with high probability [BKL16], which is a well-known property of many practical graphs. Since GIRGs are especially used to understand the behavior of algorithms on complex networks, it is important to generate even large GIRGs in reasonable time. The currently best generator for GIRGs and HRGs is due to Bläsius et al. [Blä+22], which runs in expected linear time.

Many problems and algorithms have already been studied on GIRGs (and HRGs), e.g., MaximumFlow [BFW21], MinimumSpanningTree [BKW22], and VertexCover [BFFK23][BFK23] to name just a few examples. Furthermore, Chauhan et al. [CFR20] studied the behavior of greedy heuristics for DominatingSet, VertexCover, and IndependentSet on general scale-free networks. On the other hand, the Hyper-GIRG model is much less investigated. Besides the original paper [BFFS19] we are only aware of a study on the impact of heterogeneity and locality on the run time (proof complexity) among random SAT-instances [Blä+23]. Interestingly, both papers came to similar conclusions: heterogeneity is not sufficient to decrease the efficiency of the reduction rules or the proof complexity of SAT-instances, but supports the high positive impact of locality for both metrics.

# 2 Preliminaries

In this chapter, we provide all definitions, notations, and generally known statements, we need throughout this thesis. This includes basics of probability and graph theory, as well as a formal definition of the reduction rules due to Weihe [Wei98] and the Hyper-GIRG model due to Bläsius et al. [BFFS19]. Further, we state important inequalities and explain properties of the exponential function that we need in several places of this thesis. At last, we state the proof of a new inequality, which would otherwise break the flow of reading at its actual point of use.

## 2.1 Probability Theory

For probability theory, we mostly follow the notations by Mitzenmacher et al.[MU17]. In this subsection, we state the definition of probability and explain the difference between discrete and continuous random variables and distributions. We, further, give specific information and properties of the distributions that are either used in the Hyper-GIRG model itself or are necessary for its analysis. This includes the uniform distribution, the power-law distribution, and the (discrete) binomial distribution.

### 2.1.1 Basic Definitions and Notations on Probability

**Sets.** As a basis, we use the following notations regarding sets: Let $A$ be an arbitrary set. Then, the *power set* $2^A$ is the set of all subsets of $A$. The set of all subsets of $A$ with exactly $k \in \{0, \dots, |A|\}$ elements is denoted as $\binom{A}{k}$. If two sets $A$ and $B$ have no common element, they are called *disjoint*. Their union is then denoted as $A + B$. Based on that, a *partition* of a set $A$ is a set of pairwise mutually disjoint subsets $A_1, A_2, \dots \in 2^A \setminus \emptyset$ such that their union $\sum_i A_i$ is exactly the whole set $A$.

**Events and Probabilities.** A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of a *sample space* $\Omega$, an *event space* $\mathcal{F}$, and a *probability function* $\mathbb{P}$. The sample space is the set of all possible outcomes of the random process modeled by the probability space. These possible outcomes are called *atomic events*. The event space $\mathcal{F}$ is a set of subsets of the sample space. Its elements are called *events*. In this thesis, $\mathcal{F}$ will always be the whole power set $2^\Omega$. The probability function $\mathbb{P} \colon \mathcal{F} \to \mathbb{R}$ maps each event $E \in \mathcal{F}$ to its *probability* $\mathbb{P}[E]$. Per definition, $\mathbb{P}$ satisfies the following three conditions:

(1) $\mathbb{P}[\Omega] = 1$

(2) $\mathbb{P}[E] \in [0; 1]$ for every event $E \in \mathcal{F}$

(3) for any finite or countably infinite sequence $E_1, E_2, E_3, \dots$ of pairwise mutually disjoint events, $\mathbb{P}$ satisfies the *countable additivity property*, i.e.,

$$\mathbb{P}\left[\bigcup_{i \geq 1} E_i\right] = \sum_{i \geq 1} \mathbb{P}[E_i] . \tag{2.1}$$

Note that the right side of Equation (2.1) is always an upper bound for the left side even if the events $E_i$ are not mutually disjoint. This inequality is known as the *union bound*.

**Conditional Probability.** Let $A, B \in \mathcal{F}$ be two events with $\mathbb{P}[B] > 0$. The *conditional probability of A given B* is the probability that $A$ occurs given that $B$ occurs. It can be expressed as

$$\frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]},$$

which we denote as $\mathbb{P}[A \mid B]$. Two events $A, B \in \mathcal{F}$ are called *independent* if and only if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$. For two independent events $A$ and $B$ it holds that $\mathbb{P}[A \mid B] = \mathbb{P}[A]$ and $\mathbb{P}[B \mid A] = \mathbb{P}[B]$. Given these definitions, the *law of total probability* states: If $E_1, \dots, E_n$ is a partition of $\Omega$, then it holds for every event $A \in \mathcal{F}$ that

$$\mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A \mid E_i]\,\mathbb{P}[E_i].$$

**Abbreviations for Probabilities.** For a series of events $E = (E_n)_{n \in \mathbb{N}}$, we introduce the following common abbreviations. If the probability $\mathbb{P}[E_n]$ for $n \to \infty$ tends to 1, we say that $E$ occurs *asymptotically almost surely* (a.a.s.). If the probability $\mathbb{P}[E_n]$ additionally is in $1 - \mathcal{O}\left(\frac{1}{n}\right)$, the event is said to occur *with high probability* (w.h.p.). Finally, $E$ occurs *with overwhelming probability* (w.o.p.) if for every $c > 0$ it holds with a probability of at least $1 - \mathcal{O}\left(n^{-c}\right)$. We extend these definitions by an intermediate stage between "w.h.p" and "w.o.p". If an event $E$ occurs with a probability in $1 - \mathcal{O}\left(\frac{1}{n^c}\right)$ for some $c > 1$ we say it occurs with more than high probability.

### 2.1.2 Random Variables

In the following, we introduce the basics of *random variables*. Random variables are functions $X\colon \Omega \to \mathbb{R}$ that map the atomic event to the real numbers. One distinguishes between *discrete* and *continuous* random variables.

#### 2.1.2.1 Discrete Random Variables

Discrete random variables have a finite or infinite countable sample space $\Omega$. For this subsection, let $X\colon \Omega \to \mathbb{R}$ be a discrete random variable. The probability that $X$ takes a specific value $a \in \mathbb{R}$ is denoted as

$$\mathbb{P}[X = a] \coloneqq \mathbb{P}[\{\omega \in \Omega \mid X(\omega) = a\}] \coloneqq \sum_{\omega \in \Omega \mid X(\omega) = a} \mathbb{P}[\omega]. \tag{2.2}$$

The function that maps each value $x \in \mathbb{R}$ to the probability that $X$ takes this value, is called the *probability mass function* (in the discrete case). The *expected value* of $X$ is defined as

$$\mathbb{E}[X] \coloneqq \sum_{i} i\mathbb{P}[X = i] \tag{2.3}$$

where the summation is over the whole range of values of $X$. The function $\mathbb{E}$ is linear, i.e., for every finite set of random variables $X_1, \dots, X_n$, it holds that

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]. \tag{2.4}$$

This property is formally known as the *Linearity of Expectations*.

#### 2.1.2.2 Continuous Random Variables

In contrast to discrete random variables, continuous random variables can take values in a continuous range. As one cannot sum over an uncountable set of elements, the definition of a probability mass function is not applicable here. Instead, continuous random variables are defined by the *cumulative distribution function* (CDF). A random variable $X$ is said to be *continuous* if there is a continuous function $F\colon \mathbb{R} \to [0;1]$, i.e., the cumulative distribution function, such that

$$\forall x \in \mathbb{R}\colon F(x) = \mathbb{P}[X \leq x] \tag{2.5}$$

with $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. If there is a function $f(x)$ such that

$$\forall a \in \mathbb{R}\colon F(a) = \int_{-\infty}^{a} f(t)\,dt \tag{2.6}$$

then $f$ is called the *probability density function (PDF)* of $F$. The PDF is the continuous equivalent of the discrete probability mass function. Although the probability that a continuous random variable $X$ takes a single value $x \in \mathbb{R}$ is 0, $f(x)$ can informally be interpreted as: how likely is it for $X$ to take a value close to $x$. Formally, the PDF $f$ is used to calculate the probability that $X$ takes a value between two values $a, b \in \mathbb{R}$ with $a < b$. This probability is equivalent to the area under $f$ between $a$ and $b$, i.e.,

$$\mathbb{P}[a \leq X < b] = \int_{a}^{b} f(x)\,dx.$$

Note that this value does not change if "$\leq$" is exchanged by "$<$" since the probability for the edge cases is 0. If a random variable $X$ has a PDF $f_X$, then the *law of total probability* (in the continuous case) states that the probability for an event $A$ to occur is

$$\mathbb{P}[A] = \int_{-\infty}^{\infty} \mathbb{P}[A \mid X = x]\, f_X(x)\ \mathrm{d}x.$$

The *expected value* of a continuous random variable can be calculated analogously to the expected value of a discrete random variable:

$$\mathbb{E}[X] \coloneqq \int_{-\infty}^{\infty} x f(x)\,dx.$$

The linearity of expectations (Equation (2.4)) holds for continuous expectations too. Similar to the law of total probability there is also the law of total expection. The *law of total expectation* states for every two random variables $X, Y$ that

$$\mathbb{E}[X] = \mathbb{E}_Y[\mathbb{E}_X[X \mid Y]]. \tag{2.7}$$

If $Y$ has a PDF $f_Y$, then this is equal to

$$= \int_{-\infty}^{\infty} \mathbb{E}[X \mid Y = y]\, f_Y(y)\ \mathrm{d}y. \tag{2.8}$$

As we will often use probabilities like $\mathbb{P}[X \geq a] = 1 - F(a)$ in the course of this thesis, we also define the *complementary cumulative distribution function* (CCDF) of a continuous random variable as

$$\overline{F}(a) \coloneqq \mathbb{P}[X \geq a] = 1 - F(a) \quad (a \in \mathbb{R}).$$

In the following, we give the basic definitions and lemmata about the most important probability distributions that we use in this thesis.

### 2.1.3 Uniform Distribution

The *uniform distribution* is a continuous distribution. If the PDF $f$ of a random variable $X$ is constant over an interval $[a;b]$, we say that $X$ is *uniformly distributed* over $[a;b]$. For that, we write $X \sim U(a,b)$. The formal definition of the PDF is

$$f(x) \coloneqq \begin{cases} \frac{1}{b-a}, & x \in [a;b] \\ 0, & \text{otherwise} \end{cases}. \tag{2.9}$$

Then, the CDF of such an $X$ is

$$F(X) \coloneqq \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & x \in [a;b] \\ 1, & x > b \end{cases}. \tag{2.10}$$

**Lemma 2.1** (Lemma 8.3. in [MU17]): *If $n$ points $(x_i)_{i \in [n]}$ are uniformly randomly placed on the interval $[0;1]$, then the expected distance between an arbitrary pair of consecutive points on the torus $\mathbb{T}^1$ is $\frac{1}{n}$.*

### 2.1.4 Power Law Distribution

The *power law distribution $PL(\beta, x_{\text{min}})$* is a continuous distribution. It takes two parameters: the *power law exponent $\beta$* and *minimum $x_{\text{min}}$*. Its PDF is defined as

$$f(x) \coloneqq C_\beta x^{-\beta} \quad (x \in [x_{\text{min}}, \infty]), \tag{2.11}$$

where $C_\beta \coloneqq \frac{\beta-1}{x_{\text{min}}^{1-\beta}}$ is a normalization constant to fulfill the condition $\int_{x_{\text{min}}}^{\infty} f(x)\,dx = 1$. The CDF of this probability distribution is

$$F(u) = \int_{x_{\text{min}}}^{u} f(x)\,dx = 1 - \left(\frac{u}{x_{\text{min}}}\right)^{1-\beta}.$$

The complementary cumulative distribution function (CCDF) is thus

$$\overline{F}(u) \coloneqq 1 - F(u) = \left(\frac{u}{x_{\text{min}}}\right)^{1-\beta}. \tag{2.12}$$

In the following, we will state some important properties of a power law distribution $PL(\beta, x_{\text{min}})$. First, we look at the expected value.

**Lemma 2.2** ([New04]): *The expected value for a random variable $X \sim PL(\beta, x_{min})$ is*

$$\mathbb{E}[X] = \begin{cases} \infty, & \beta \leq 2 \\ \frac{1-\beta}{2-\beta} x_{min}, & \beta > 2 \end{cases}.$$

In this thesis, we only consider $\beta > 2$, which holds for almost all instances arising from practical phenomena. Most of them admit a power-law exponent between 2 and 3. Next, we state properties of a set of $n$ sample values $x_1, \ldots, x_n$ that were drawn independently from $PL(\beta, x_{\text{min}})$. The following Lemma 2.3 describes how the sum $X \coloneqq \sum_{i=1}^{n} x_i$ of all samples grows with increasing $n$.

**Lemma 2.3** (Lemma 4.2. of [BKL16]): *If $x_1, \ldots, x_n \sim PL(\beta, x_{min})$ are independently drawn and $X \coloneqq \sum_{i=1}^{n} x_i$, then $X \in \Theta(n)$.*

Note that the authors of [BKL16] did not mention with which probability this statement holds. However, this and similar statements are often assumed to be true in the context of GIRGs. The next Lemma 2.4 describes how the maximum of $n$ identically and independently drawn power-law samples behave.

**Lemma 2.4:** *If $x_1, \ldots, x_n \sim PL(\beta, x_{min})$ independently drawn, then the maximum sample is expected $n^{\frac{1}{\beta-1}}$ and w.h.p. smaller than $n^{\frac{2}{\beta-1}} x_{min}$.*

*Proof.* Let $x_{\max}$ denote the maximum sample value. Then $\mathbb{E}[x_{\max}] \sim n^{\frac{1}{\beta-1}}$ is proven in [New04]. The second property of $x_{\max}$, that is stated in the lemma, can be shown with the CDF $F_\beta$ of the power-law distribution. The probability that a single sample is smaller than $n^{\frac{2}{\beta-1}} x_{\min}$ is

$$\mathbb{P}\left[x_1 \leq n^{\frac{2}{\beta-1}}\right] = F_\beta\left(n^{\frac{2}{\beta-1}}\right) = 1 - \left(\frac{n^{\frac{2}{\beta-1}} x_{\min}}{x_{\min}}\right)^{1-\beta} = 1 - \frac{1}{n^2}.$$

This event thus occurs with more than high probability. Now, using a union bound, the probability that at least one sample is larger than $n^{\frac{2}{\beta-1}} x_{\min}$ is

$$\mathbb{P}\left[x_{\max} \geq n^{\frac{2}{\beta-1}}\right] = \mathbb{P}\left[\bigcup_{i \in n} [x_i \geq n^{\frac{2}{\beta-1}} x_{\min}]\right] \leq n\mathbb{P}\left[x_1 \geq n^{\frac{2}{\beta-1}}\right] = n\frac{1}{n^2} = \frac{1}{n}.$$

■

### 2.1.5 Binomial Distribution

The binomial distribution can be defined via Bernoulli trials. A *Bernoulli trial* is an experiment with exactly the two outcomes, namely *success* and *failure* (e.g. a coin flip). It can be defined as a random variable $X$ that takes either 1 (success) with a certain probability $p \in [0; 1]$ or 0 (failure) with probability $1 - p$. Such random variables are also called *indicator variables* and are often denoted as $\mathbf{1}_A$ where $A$ is the success event. A sequence of multiple independent and identically distributed Bernoulli trials is called a *Bernoulli process*. The distribution of the number of successes in a Bernoulli process is the *binomial distribution* $\mathrm{Bin}(n, p)$, where $n$ is the number of Bernoulli trials and $p$ is their common success probability. The PMF of the binomial distribution $\mathrm{Bin}(n, p)$ is

$$f(k) \coloneqq \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & k \in \{0, \ldots, n\} \\ 0, & \text{otherwise} \end{cases}$$

where $\binom{n}{k}$ is the binomial coefficient of $n$ over $k$. With this, but also with the linearity of expectation, it can be shown that the expected number of successful Bernoulli trials in this process is exactly $p \cdot n$. In contrast to the PMF, the CDF of Binomial Distributions does not have a closed form. To be able to bound the values for the CDF anyway, we will use the following concentration bound stated in Theorem 2.5.

**Theorem 2.5** (Chernoff Bounds for Binomial Distribution)**:** *Let $X_1, \ldots, X_n$ be independent random variables taking values in $[0;1]$ and $X \coloneqq \sum_{i=1}^{n} X_i$ their sum. Then*

*(1)* $\mathbb{P}[X > (1+\varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right)$ *holds for all* $\varepsilon \in (0;1)$.

*(2)* $\mathbb{P}[X > (1-\varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\varepsilon^2}{2}\mathbb{E}[X]\right)$ *holds for all* $\varepsilon \in (0;1)$.

*(3)* $\mathbb{P}[X > t] \leq 2^{-t}$ *holds for all* $t > 2e\mathbb{E}[X]$.

## 2.2 Graph Theory

In this section, we give basic notations and definitions on graphs and hypergraphs as we use them in this thesis. For that, we mostly follow the notations by Diestel [Die12]. Further, we define the HittingSet problem and state the reduction rules due to Weihe [Wei98] in the context of hypergraphs.

**Graphs.** An *(undirected) graph* $G$ is a pair $(V, E)$ of a finite set $V$ and a set $E \subseteq \binom{V}{2}$ of two-element sets of $V$. We call the elements of $V$ *vertices* and the elements in $E$ *edges.* Note that, according to the definition, neither multiple edges nor loops are allowed. Two vertices $v_1, v_2 \in V$ are called *adjacent* if there is an edge $\{v_1, v_2\} = e \in E$ in the graph. In this case, $v_1$ and $v_2$ are *incident* to the edge $e$, respectively. The *neighborhood* $N_G(v)$ of a vertex $v \in V$ is the set of all vertices that are adjacent to $v$ in $G$. The number of those vertices is called the *degree of* $v$ and is denoted as $\deg(v)$. A graph whose vertices can be partitioned into two sets $A + B = V$ such that every edge is incident to exactly one vertex in $A$ and one vertex in $B$ is called *bipartite.*

**Hypergraphs.** A *hypergraph* $H = (V, E)$ is a pair of a finite set of *vertices* $V$ and a multiset of *hyperedges* $e \in 2^V \setminus \{\emptyset\}$. Typically, we denote the number of vertices and hyperedges with $n \coloneqq |V|$ and $m \coloneqq |E|$, respectively. Note that, in contrast to graphs, we allow multiple (hyper-)edges to have the same set of vertices. A hypergraph $H = (V, E)$ is *Sperner* if there is no hyperedge containing another hyperedge.

The *incidence graph* of a hypergraph $H$ is the bipartite graph $G = (V + E, \mathcal{I})$ where $(v, e) \in V \times E$ is an edge in $G$ if and only if the hyperedge $e$ contains the vertex $v$ in $H$. We denote $E(v)$ for the set of hyperedges that contain a vertex $v \in V$. We define the *dual* of a (primal) hypergraph $H = (V, E)$ as the hypergraph $H^* = (V^*, E^*)$ with the primal hyperedges as dual vertices $V^* \coloneqq E$, and the sets $E(v)$ of all primal vertices $v \in V$ as the dual hyperedges.

**The HittingSet Problem.** The HittingSet problem is defined as follows: given a hypergraph $H = (V, E)$, find a minimum subset $S \subseteq V$ of vertices, such that every hyperedge contains at least one vertex from $S$. In this context, we call a hyperedge $e \in E$ *covered* if there exists a vertex $v \in S$ with $v \in e$. In this case, the vertex $v$ *covers* the hyperedges $e$.

The two reduction rules due to Weihe [Wei98] allow us to safely remove dominated vertices and hyperedges from the hypergraph, respectively. A vertex $v_1 \in V$ is said to *dominate* another vertex $v_2 \in V$ if $E(v_2) \subseteq E(v_1)$. A dominated vertex can safely be removed from the hypergraph since every hyperedge containing $v_2$ is already covered by $v_1$. Therefore, it is always better to select $v_1$ instead of $v_2$. We call this rule the *vertex domination rule.*

On the other hand, a hyperedge $e_1 \in E$ *dominates* another hyperedge $e_2 \in E$ if $e_1 \subseteq e_2$. Since every hyperedge needs to be covered, $e_2$ is always covered if $e_1$ is covered. Therefore, removing $e_2$ is safe. We call this rule the *hyperedge domination rule*.

If one of the rules is applied on one vertex or hyperedge, we call this an *application* of the rule on this specific vertex or hyperedge. If we remove all dominated vertices or hyperedges at once, we call this one *round* of the respective rule.

Besides the reduction rules due to Weihe [Wei98], we add an additional technical reduction rule that simplifies some explanations in Chapter 4. This reduction rule allows us to *select* a vertex, i.e., take it into the resulting hitting set, if there is a hyperedge $e$ that contains only this vertex. This vertex has to be part of the solution, as there is no other way to cover the hyperedge $e$. When a vertex is selected, all its incident hyperedges $E(v)$ and the vertex itself may be removed from the hypergraph. We adopt the name from Shi and Cai [SC10] and call it the *Unit Hyperedge Rule*. This rule is safe in the sense that the reduced hypergraph has a minimum hitting set of size $k-1$ if and only if the original hypergraph has a minimum hitting set of size $k$.

## 2.3 Random (Hyper-)Graph Models

In the following, we give the formal definition of the Hyper-GIRG model due to Bläsius et al. [BFFS19]. As its basis, we describe the GIRG model due to Bringmann, Keusch, and Lengler [BKL19] first. Finally, we provide technical notations that are formally necessary to properly define colloquial terms on a torus, which is the basis for both models.

### 2.3.1 GIRGs

The abbreviation GIRG stands for *Geometric inhomogeneous random graph*. It describes a random graph model that generates graphs $G = (V, E)$ with varying degrees of heterogeneity and locality. To achieve locality, it uses an underlying geometry (also called ground space), which usually is the $d$-dimensional *torus* $\mathbb{T}^d$ together with the *maximums metric*.

$$\operatorname{dist}(x, y) \coloneqq \operatorname{dist}_\infty(x, y) \coloneqq \max_{i=1,\dots,d} \min\{|x_i - y_i|, 1 - |x_i - y_i|\}.$$

A $d$-dimensional torus $\mathbb{T}^d$ is a $d$-dimensional cube $[0;1]^d$ where opposite sides are identified. In this thesis, we assume the 1-dimensional torus $\mathbb{T}^1$ to be the underlying geometry. Here, the minimal distance between two points $x, y$ on the $\mathbb{T}^1$ has the more simple form $\min\{|x_i - y_i|, 1 - |x_i - y_i|\}$. When a graph $G = (V, E)$ is generated, every vertex $v$ is assigned a position $x_v$ inside the torus. Those positions (or every coordinate) are independently drawn from a uniform distribution $U[0;1]$.

To achieve heterogeneity, every vertex $v$ is additionally assigned a *weight* $w_v$ that is independently drawn from a power-law distribution $\mathrm{PL}(\beta, w_{\min})$. This weight controls the degree of a vertex $v$ by scaling the probability that close vertices are adjacent. The probability for an edge to exist between two vertices $v, u \in V$ is called the *edge probability* $p_e(v, u)$ and is defined as

$$p_e(v, u) \coloneqq \min\left\{1, a \cdot \left(\frac{1}{\operatorname{dist}(x_v, x_u)^d} \cdot \frac{w_v w_u}{W}\right)^{\frac{1}{T}}\right\}$$

where $W$ is the sum of all weights and $a$ is a scaling parameter of the model. The parameter $T \in (0;1)$ is called the *temperature*. It controls the locality or the impact of the geometry. It describes how important the distance between the two vertices is to be connected. In a special case, the so-called *threshold variant*, the temperature can be set to $T = 0$, which maximizes the locality. Here, the edge probability degenerates to the step-function

$$p_e(v,u) = \begin{cases} 1, & \operatorname{dist}(x_v, x_u) \leq \left(\frac{w_v w_u}{W}\right)^{\frac{1}{d}} \\ 0, & \text{otherwise} \end{cases}.$$

For every other temperature $T \in (0;1)$ it is called the *binomial variant*. In conclusion, there are 4 parameters for GIRGs: the number of vertices $n$, the power-law exponent $\beta$, the minimal vertex weight $w_{\min}$, and the scaling parameter $a$. Note that many papers assume $\beta > 2$, which we also do in this thesis.

### 2.3.2 Hyper-GIRGs

The idea of the Hyper-GIRG model is to generate a bipartite graph and interpret it as the incidence graph $G = (V + E, \mathcal{I})$ of a hypergraph $H = (V, E)$. Thus, similar to the vertices, also every hyperedge $e \in E$ is assigned a position $x_e$ on the torus together with a weight $w_e$. All positions, of vertices as well as hyperedges, are independently drawn from a uniform distribution $U[0;1]$. A key difference to the GIRG model is the way the hyperedge weights are defined. Although it would be possible to apply a power-law distribution again, they are defined to be a constant $w_e$. Due to this decision, Hyper-GIRGs (in this form) generate hypergraphs with homogeneous hyperedge sizes.

The incidences in the Hyper-GIRG model are generated similarly to the edges in the GIRG model. For every incidence a coin decides whether this incidence exists or not. The probability by which an incidence $(v,e) \in \mathcal{I}$ exists is the *incidence probability*

$$p_I(v,e) \coloneqq \min \left\{ 1, \left( a \cdot \frac{1}{\operatorname{dist}(x_v, x_e)^d} \cdot \frac{w_v w_e}{W} \right)^{\frac{1}{T}} \right\} \tag{2.13}$$

with the same definitions for $a$, $d$, $T$, and $W$ as in the GIRG model. In contrast to GIRGs, the scaling parameter $a$ is inside the brackets in the Hyper-GIRG model. This is a design decision that ensures that $a$ has an impact on the incidence probability of the threshold variant, which is

$$p_I(v,e) = \begin{cases} 1, & \operatorname{dist}(x_v, x_e) \leq \left(a w_e \frac{w_v}{W}\right)^{\frac{1}{d}} \\ 0, & \text{otherwise} \end{cases}. \tag{2.14}$$

In both bases, we omit one variable in $p_I$ if it can be clearly inferred from the context.

In the following, we fix some notations and definitions about Hyper-GIRGs. We denote the model as $\mathcal{H}(n, m, \beta, w_{\min}, w_e, T, a)$ with the number of vertices $n \in \mathbb{N}$, the number of hyperedges $m \in \mathbb{N}$, the power-law exponent $\beta > 2$, the minimum vertex weight $w_{\min}$, the constant hyperedge weight $w_e \geq w_{\min}$, the temperature $T \in (0, 1)$, and a scaling coefficient $a > 0$ as parameters. As mentioned above, we set $d = 1$ throughout this thesis. Further, we define $W_c \coloneqq \frac{W}{n}$ to be the average vertex weight, and $\delta_{EV} \coloneqq \frac{m}{n}$ the *hyperedge-vertex-ratio* of a generated hypergraph. Unless otherwise stated, $H = (V, E)$ is a hypergraph and $G = (V + E, \mathcal{I})$ its incidence graph. If $H$ or $G$ is generated by the model, we write $H \sim \mathcal{H}(n, m, \beta, w_{\min}, w_e, T, a)$ or $G \sim \mathcal{H}(n, m, \beta, w_{\min}, w_e, T, a)$, respectively. In the context of Hyper-GIRGs, vertices and hyperedges of $H$ are interpreted as points $(x, w)$ with their randomly drawn position $x$ and weight $w$ as coordinates.

### 2.3.3 Technicalities on the Torus

The 1-dimensional torus is the geometric basis for the Hyper-GIRGs we study in this thesis. It can be described as the interval $[0; 1]$ with identified ends. On $[0; 1]$, it is easy to say that a subinterval $I$ has a *left end*, i.e., the smallest number in $I$, or *right end*, i.e., the largest number in $I$. However, these definitions are much harder on the torus.

In this thesis, we often try to give discrete values on the torus, e.g., the positions of hyperedges, an order. Formally, this is done via the concept of *cyclic orders* (e.g. [Blä15]). Informally, a cyclic order of a set $S = \{a_1, \ldots, a_n\}$ is a linear order $a_1 \prec a_2 \prec \cdots \prec a_n$ with the additional relation $a_n \prec a_1$. For a formal definition, let $S = \{a_1, \ldots, a_n\}$ be a set with a linear order $a_1 \prec a_2 \prec \cdots \prec a_n$. We denote this linear order as $\langle a_1, \ldots, a_n\rangle$. A *circular shift* is an operation that transforms a linear order $\langle a_1, \ldots, a_n\rangle$ into a linear order $\langle a_{n-i}, \ldots, a_n, a_1, \ldots, a_{n-i-1}\rangle$ for some $i \in \{0, \ldots, n-1\}$. Two linear orders are *cyclically equivalent* if one can be transformed into the other by a circular shift. The equivalence classes of this equivalence relation are called cyclic orders. We denote the cyclic order as a representative $\langle a_1, \ldots, a_n\rangle$ with a small circle in the superscript, i.e., $\prec^\circ \coloneqq \langle a_1, \ldots, a_n\rangle^\circ$. An *interval* in a cyclic order is a set of elements that are consecutive in some linear order of the cyclic order, e.g., $\{e, a, b\}$ is an interval in the cyclic order $\langle a, b, c, d, e\rangle^\circ$.

With these definitions, we are finally able to define the ends of an interval. For every interval $I$ that is neither empty nor contains all elements of $S$, the *left end* is the element in $I$ that does not have a predecessor in $I$ with respect to $\prec^\circ$. Similarly, the *right end* of $I$ is the element $I$ without a successor in $I$ with respect to $\prec^\circ$. In the example interval $\{e, a, b\}$ of $\langle a, b, c, d, e\rangle^\circ$, the left end is $e$ and the right end is $b$.

At last, we adapt these notations for the continuous torus itself. Instead of describing the torus as the interval $[0; 1]$ with identified borders, we write $[0; 1]^\circ$ to denote the cyclic characteristic. Sometimes, we describe the $\mathbb{T}^1$ over the interval $[-0.5, 0.5]$ instead of $[0; 1]$ to have a center point with more symmetry. In this case, we write $\mathbb{T}^1 = [-0.5; 0.5]^\circ$.

## 2.4 Properties of the Exponential Function

The exponential function is the function $f(x) = exp(x) = e^x$. Two well-known lower bounds to this function are stated in the following form of *Bernoulli's inequality*.

**Lemma 2.6** (Bernoulli's inequalities)**:** *For all real numbers $x \geq -1$ and $r \geq 1$ the following inequalities hold:*

$$1 + rx \leq (1 + x)^r \leq e^{rx}.$$

The first bound, i.e., $(1 + x)^r \leq e^{rx}$ is highly connected to the Euler limit, while the second bound, i.e., $1 + rx \leq e^{rx}$, is a linearization of the convex exponential function at $x = 0$. We will discuss these topics in the following two subsections.

### 2.4.1 The Euler Limit

The *Euler Limit* states that for every complex number $z \in \mathbb{C}$ the limit $\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$ is exactly $e^z$. The following Lemma 2.7 states the limits for a more general kind of series.

**Lemma 2.7:** *Let $b, c > 0$ be positive constants, and $(a_n)_{n\in\mathbb{N}}$ a number sequence with elements $a_n := \left(1 - \frac{c}{n^b}\right)^n$. Then $(a_n)_{n\in\mathbb{N}}$ has the following limits depending on $b$ and $c$:*

$$\lim_{n\to\infty} a_n = \begin{cases} 0, & b \in (0;1) \\ \frac{1}{e^c}, & b = 1 \\ 1, & b > 1 \end{cases} .$$

*Proof.* We prove each case separately. For $b = 1$, the elements of $(a_n)_{n\in\mathbb{N}}$ simplify to $\left(1 - \frac{c}{n}\right)^n$. This is exactly the form, that is used to define the complex powers of Euler's number, i.e., $e^z := \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$ for $z \in \mathbb{C}$. Using $z := -c$ yields the result.

To prove the two remaining cases, we use the two inequalities stated in Lemma 2.6. For $b > 1$, we show that $a_n \leq 1$ for every $n \in \mathbb{N}$ and use Bernoulli's inequality to find a lower bound for $a_n$ that tends to 1. The first part is easy to show as $\frac{c}{n^b}$ is positive and can thus be omitted in the definition of $a_n$ to get $1^n = 1$ as an upper bound. For the lower bound, we use Bernoulli's inequality $(1 + x)^r \geq 1 + rx$ with $x_n := -\frac{c}{n^b}$ and $r_n := n$. The condition $r_n \geq 1$ is satisfied as $n \in \mathbb{N}$. The condition $x_n \geq -1$ holds for every $n \geq \sqrt[b]{c}$, because

$$x_n = -\frac{c}{n^b} \geq -\frac{c}{\left(\sqrt[b]{c}\right)^b} = -1.$$

Therefore, we get the lower bound:

$$a_n = \left(1 - \frac{c}{n^b}\right)^n = (1 + x_n)^{r_n} \geq 1 + r_n x_n = 1 - \frac{c}{n^{b-1}} = 1 - cn^{1-b}.$$

As $b > 1$, the function $n^{1-b}$ has a negative exponent and thus tends to 0. As a result, $a_n$ tends to 1 for $n \to \infty$.

For the case $b \in (0;1)$, we find an upper bound, that tends to 0, and show that $a_n \geq 0$ for all sufficiently large $n$. Again, we use the substitutions $x_n := -\frac{c}{n^b}$ and $r_n := n$. In the case above, we already showed $x_n \geq -1$ for any $n \geq \sqrt[b]{c}$. As a consequence, $(1 + x_n)^{r_n} \geq 0$ for the same $n \geq \sqrt[b]{c}$. Therefore, the limit of $a_n$ can not be negative. For the following upper bound, we apply the other Bernoulli inequality $(1 + x_n)^{r_n} \leq e^{r_n x_n}$, for which we already showed the necessary conditions $r_n \geq 1$ and $x_n \geq -1$ in the case above:

$$a_n = \left(1 - \frac{c}{n^b}\right)^n = (1 + x_n)^{r_n} \leq e^{r_n x_n} = e^{-cn^{1-b}}.$$

Because $b < 1$, the function $n^{1-b}$ has a positive exponent and thus tends to infinity. As a result, the $a_n$ tend to 0. ∎

With Bernoulli's inequality, one can prove that the series $a_n$ from Lemma 2.7 with $b = 1$ approaches $e^{-c}$ from below. Therefore, every element of this series has $e^{-c}$ as an upper bound. However, in Section 5.2 we will need a lower bound of such terms containing the exponential function. As the proof of the following Lemma 2.8 would break the flow of reading there, we move it to the preliminaries.

**Lemma 2.8:** *Let $a, b > 0$ two constants. Then the following inequality holds for all $x > \frac{a}{0.68}$*

$$\left(1 - \frac{a}{x}\right)^{bx} \geq \exp(-ab) - \frac{0.542}{bx}.$$

*Proof.* We first find proof that $\exp(-ab)\frac{a^2 b}{x}$ is an upper bound to $\exp(-ab) - \left(1 - \frac{a}{x}\right)^{bx}$ and then show the inequality $\exp(-ab)\frac{a^2 b}{x} \leq \frac{0.542}{bx}$ for all $x > \frac{a}{0.68}$. For the first part, we find a lower bound for the term[1]

$$\left(1 - \frac{a}{x}\right)^{bx} = \exp\left(bx \cdot \ln\left(1 - \frac{a}{x}\right)\right).$$

We use the inequality $\ln(1 + y) \geq y - y^2$ that holds for all $y \geq -0.68$. Since $x > \frac{a}{0.68}$, we can substitute $y = -\frac{a}{x}$ and get the lower bound

$$\begin{aligned} &\geq \exp\left(bx \cdot \left(-\frac{a}{x} - \left(\frac{a}{x}\right)^2\right)\right) \\ &= \exp(-ab) \cdot \exp\left(-\frac{a^2 b}{x}\right). \end{aligned}$$

Therefore, we find the following upper bound for the difference

$$\begin{aligned} \exp(-ab) - \left(1 - \frac{a}{x}\right)^{bx} &\leq \exp(-ab)\left(1 - \exp\left(-\frac{a^2 b}{x}\right)\right) \\ &\leq \exp(-ab)\frac{a^2 b}{x} \end{aligned}$$

where we used Lemma 2.6 for the second bound. At last, we prove the inequality

$$\exp(-ab)\frac{a^2 b}{x} \leq \frac{0.542}{bx}.$$

For that, we reformulate the left side to

$$\exp(-ab)\frac{a^2 b}{x} = \exp(-ab)\frac{a^2 b^2}{bx}$$

and look at the function $e^{-y}y^2$. As this function has a global maximum just below 0.542, we get the upper bound

$$\leq \frac{0.542}{bx}$$

for $\exp(-ab) - \left(1 - \frac{a}{x}\right)^{bx}$. ∎

### 2.4.2 Convex Functions and Linearization

In this section, we prove the second bound, i.e., $1 + rx \leq e^{rx}$, from Lemma 2.6, and thus provide basic tools and notation used in this thesis along with it. An important property for this bound to be true is the convexity of the exponential function for all $r \in \mathbb{R}$.

[1] The inequality $\ln(1 + y) \geq y - y^2$ is from "Useful Inequalities": *https://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf*

For the following definitions and lemmata on convex functions, we refer to Stein [Ste18]. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *convex* if and only if its second derivative is non-negative for all $x \in \mathbb{R}$. This is called the $\mathcal{C}^2$*-characterization* of convex functions. The exponential function is a convex function since

$$\frac{\partial^2}{\partial x^2}\Big(\exp(rx)\Big) = r^2 \exp(rx) > 0.$$

All convex functions have the nice property, that all its tangents are lower bounds to the whole function. The formal statement is given in the following Lemma 2.9.

**Lemma 2.9** ($\mathcal{C}^1$-Characterization of Convex Functions (Lemma 2.1.40 in [Ste18])): *A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is convex if and only if*

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

*holds for all $x, x_0 \in \mathbb{R}$.*

The left side of the inequality in Lemma 2.9 is the tangent to the function $f$ at $x_0$. Using the tangent as a bound to the actual function is called *Linearization.* For the exponential function $e^{rx}$, the tangent at $x_0 \in \mathbb{R}$ is

$$t_{x_0}(x) := e^{rx_0} + re^{rx_0}(x - x_0), \tag{2.15}$$

and thus $t_0(x) = 1 + rx$. We will use the general form of a tangent later in Lemma 5.2.

# 3 Fundamentals of Hyper-GIRGs

In this chapter, we find basic properties of Hyper-GIRGs, e.g., the distributions for vertex degrees and hyperedge sizes. To do so, we first introduce basic notations on the geometric structure of vertices in Section 3.1, and describe, based on that, the visualizations we use for Hyper-GIRGs. A crucial part for many proofs in this thesis is to determine the probability by which a single vertex *covers* a specific interval on the torus, i.e., is incident to all hyperedges in that interval. In Section 3.2 we find a lower bound for that probability, that is tight in the threshold variant. From that, we infer how likely vertices are dominated based on how their incident hyperedges are located on the torus. The last Section 3.3 deals with the neighborhood of a vertex in Hyper-GIRGs from a geometrical as well as graph-theoretical perspective.

## 3.1 Notations and Visualizations

For the following definitions, let $v \coloneqq (x_v, w_v)$ be a vertex with a fixed weight $w_v \geq w_{\min}$. Without loss of generality, we assume that $v$ is positioned at $x_v = 0$ on $\mathbb{T}^1$ and that the torus is built upon the interval $[-0.5; 0.5]$. Therefore, $v$ is exactly in the center. In this section, we define basic terms around the geometric neighborhood of $v$, i.e., how its graph-theoretic neighbors are located on the torus $\mathbb{T}^1$.

Considering the incidence probability for the vertex $v$ (see Equation (2.13))

$$p_I(e) \coloneqq \min\left\{1, \left(a \cdot \frac{1}{|x_e|} \cdot \frac{w_v w_e}{W}\right)^{\frac{1}{T}}\right\},$$

there is an interval of hyperedge positions $x_e \in \mathbb{T}^1$ that is independent of $T$, such that $v$ is incident to every hyperedge in it. This interval has a radius of $aw_e\frac{w_v}{W}$ around the vertex position. In the threshold variant, where the temperature is at its minimum $T = 0$, this interval contains all incident hyperedges of $v$. We call this interval the *cold area* $\mathrm{NA}_{\mathrm{cold}}(v)$ of $v$ and all incidences to hyperedges inside it *cold* or *solid* incidences. For higher temperatures, there might be hyperedges outside the cold area. We call the incidences to these hyperedges *warm* or *fluid*. To stay true to the analogy, we call the radius of the cold area the *melting radius*

$$m_r(v) \coloneqq aw_e\frac{w_v}{W} \tag{3.1}$$

of $v$. In sketches of Hyper-GIRGs where certain incidences are important, we color the solid/cold incidences blue, and the fluid/warm incidences red as shown in Figure 3.2.

However, we mainly use the cold areas of vertices to visualize Hyper-GIRGs. Throughout this thesis, we use two different visualizations, each of which takes the weight $w_v$ and the position $x_v$ of a vertex $v$ as 2-dimensional coordinates. The difference between the two is the coordinate system they use; one uses the polar coordinate system and the other one the Euclidean. A sketch of both visualizations is shown in Figure 3.1.

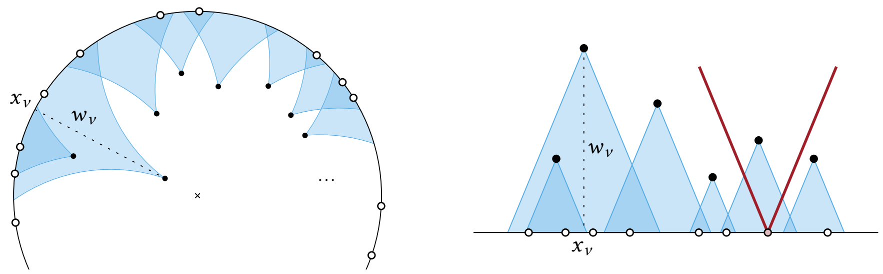


**Figure 3.1:** Visualizations of a generated hypergraph on $\mathbb{T}^1$ based on the polar coordinate system (left) and the Euclidean coordinate system (right). The black points are vertices and the points filled with white are the hyperedges. Each blue cone starting at a vertex $v$ marks the area in which another vertex lies if and only if it is dominated by $v$ in the threshold variant. The intervals on the circle (left) and the line (right) that are covered by the blue cones are the cold areas of the respective vertices. The red cone in the right sketch marks the area in which all vertices lie that contain the red hyperedge in their cold area.

If it is important, that the torus is circular, we use a polar coordinate system or the circular visualization (on the left side of Figure 3.1). This is based on a circle with a circumference of 1, where all positions $x \in [0; 1]$ of vertices and hyperedges can be placed on accordingly. To find a good radius to visualize a vertex with some weight $w_v$, we use a similar approach as in the default visualization of hyperbolic random graphs. Hyperbolic random graphs are a well-known subclass of GIRGs [BKL19], in which high-degree vertices are often displayed more towards the center of the visualization. Therefore, we use the weight of a vertex as the distance from the circle to its center. Vertices with a low weight are placed near the base circle, but not on the circle itself (since $w_{\min} > 0$). On the other hand, vertices with a high weight are closer to the center. Note that we ignore the fact that there might be weights tending to infinity. The reason for this is that we will use this visualization more for sketches rather than for precise mathematical calculations. To distinguish vertices and hyperedges in this visualization, we neglect the (constant) weight of hyperedges and only use their positions to place them directly onto the base circle. A sketch of this visualization can be seen on the left side of Figure 3.1. The intervals, at which the blue cones intersect the circle, mark the cold areas of the individual vertices.

The second visualization is the unrolled version of the first one. Instead of letting the position define the angle of the vertex in the polar coordinate system, we use the position as it is and draw it onto the interval $[0; 1]$, which we call the *baseline*. Everything else stays the same. The weight of each vertex defines its distance to the baseline, and the hyperedges are drawn directly onto the line. An example can be seen on the right side of Figure 3.1. The advantage of this visualization is that it can be better used for precise mathematical arguments, which we will use in the subsequent Section 3.2.

Before that, we apply these visualizations to illustrate the few remaining definitions. In the threshold variant, the area in which all incidences of $v$ lie is bounded by its cold area or the melting radius. However, this is not true for the binomial variant. In order to have a measure for the range of a vertex anyways, we define the *neighborhood area* $\mathrm{NA}(v)$ of $v$ to be the smallest interval of the torus that includes the positions of all hyperedges incident to $v$. Further, we define the positions of the hyperedges marking the left and the right end of the neighborhood area the be the *left border* $b_L(v)$ and *right border* $b_R(v)$ of $v$, respectively. The length of the neighborhood area of $v$ is its *diameter* $d(v) \coloneqq b_R(v) - b_L(v)$. If the vertex

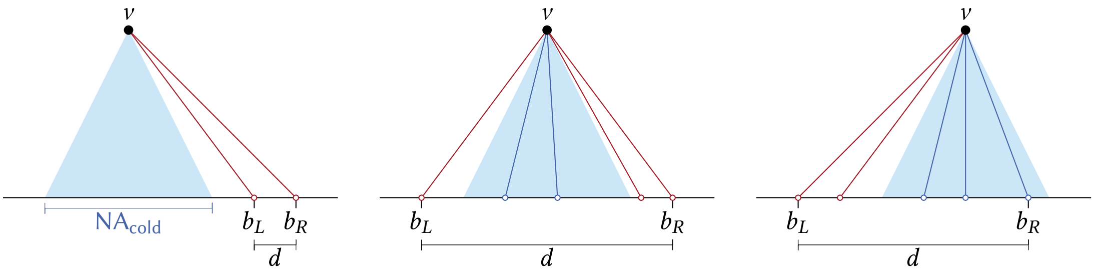


**Figure 3.2:** Three different cases of how the cold area $\mathrm{NA}_{\mathrm{cold}}(v)$ (light blue) of vertex $v$ can lie relative to its neighborhood area. The neighborhood area is the interval between the left border $b_L$ and right border $b_R$ of $v$. The distance between the two borders is the diameter $d$. Its bounds may be cold incidences (depicted as blue lines) but also warm incidences (depicted as red lines), depending on the case. The left sketch shows a neighborhood area that is disjoint to the cold area of $v$. The middle sketch shows a neighborhood area that includes the cold area entirely. The right sketch shows a neighborhood area that only intersects the cold area.

can be clearly inferred from the context, we omit the argument "$(v)$" in the above definitions. Figure 3.2 illustrates these definitions for different situations. It additionally shows that, at least in the binomial variant, the neighborhood area of a vertex might be completely independent from its cold area.

## 3.2 Covering Probability

In this thesis, we are often interested in the probability that a vertex $v$ *covers* a certain interval $I \subseteq \mathbb{T}^1$ with its cold area, i.e., $I \subseteq \mathrm{NA}_{\mathrm{cold}}(v)$. This probability is calculated in Lemma 3.1.

**Lemma 3.1:** *Let $I \subseteq \mathbb{T}^1$ be an interval on the torus, and $u = (X_u, W_u)$ a vertex with a weight $W_u \sim \mathrm{PL}(\beta, w_{min})$ and a position $X_u \sim U[0;1]$. Then the probability that $u$ covers $I$ entirely with its cold area is*

$$\begin{aligned}\mathbb{P}\left[I \subseteq \mathrm{NA}_{cold}(u)\right] &= 2\frac{aw_e}{W} w_{min}^{\beta-1} \frac{1}{\beta-2}\left(\left(\frac{W}{aw_e}\frac{|I|}{2}\right)^{2-\beta} - \left(\frac{W}{2aw_e} + \frac{W}{aw_e}\frac{|I|}{2}\right)^{2-\beta}\right) \\ &= \left(2\frac{aw_e}{W}\right)^{\beta-1} w_{min}^{\beta-1}\frac{1}{\beta-2}\left((|I|)^{-(\beta-2)} - (1+|I|)^{-(\beta-2)}\right)\end{aligned}$$

*for every $|I| \geq 2m_r(w_{min})$, and for every $|I| \leq 2m_r(w_{min})$*

$$= 2\left(m_r(w_{min}) - \frac{|I|}{2} + \frac{aw_e}{W} w_{min}^{\beta-1}\frac{1}{\beta-2}\left(w_{min}^{2-\beta} - \left(\frac{W}{2aw_e} + \frac{W}{aw_e}\frac{|I|}{2}\right)^{2-\beta}\right)\right).$$

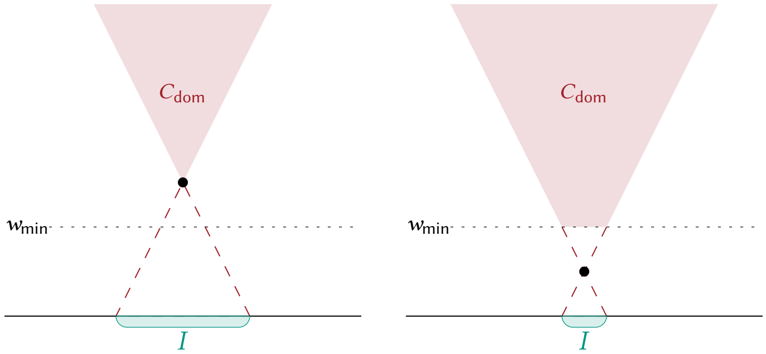


**Figure 3.3:** Two intervals $I$ on the torus. The green area indicates exactly the area in which a vertex lies that covers the interval $I$ entirely. If the interval is large enough, the area is a cone (left). Otherwise, the cone is truncated because of the minimum weight of the vertices (right). The black dot is the peak of the respective cone.

*Proof.* Every vertex $u = (x_u, w_u)$, that entirely covers $I$ with its cold area, satisfies the condition $I \subseteq [-m_r(u); m_r(u)]$. We assume w.l.o.g. that the position 0 is exactly in the middle of the interval $I$. Therefore, the left and right end of $I$ are exactly at $\pm\frac{|I|}{2}$ and thus

$$\begin{aligned} & I \subseteq [-m_r(u); m_r(u)] \\ \Leftrightarrow & \begin{cases} x_u - m_r(u) \leq -\frac{|I|}{2} \\ \frac{|I|}{2} \leq x_u + m_r(u) \end{cases} \\ \Leftrightarrow & \begin{cases} x_u + \frac{|I|}{2} \leq aw_e\frac{w_u}{W} \\ -x_u + \frac{|I|}{2} \leq aw_e\frac{w_u}{W} \end{cases} \\ \Leftrightarrow & \begin{cases} \frac{W}{aw_e}x_u + \frac{W}{aw_e}\frac{|I|}{2} \leq w_u \\ -\frac{W}{aw_e}x_u + \frac{W}{aw_e}\frac{|I|}{2} \leq w_u \end{cases} . \end{aligned}$$

As shown in Figure 3.3, those inequalities form a cone $C_{\text{dom}}$ with slopes $\pm\frac{W}{aw_e}$. Since the weight of the vertices is always above $w_{\min}$, the cone might be truncated for too small intervals. The peak of the cone is the crossing point of the linear equations

$$\begin{cases} (I): \frac{W}{aw_e}x_u + \frac{W}{aw_e}\frac{|I|}{2} = w_u \\ (II): -\frac{W}{aw_e}x_u + \frac{W}{aw_e}\frac{|I|}{2} = w_u \end{cases}$$

implied by the inequalities above. By adding and subtracting both equalities, we get the position- and the weight-coordinate of the peak

$$\begin{aligned} \Leftrightarrow & \begin{cases} (III) = (II) + (I): -\frac{W}{aw_e}x_u + \frac{W}{aw_e}\frac{|I|}{2} + \left(\frac{W}{aw_e}x_u + \frac{W}{aw_e}\frac{|I|}{2}\right) = w_u + w_u \\ (IV) = (II) - (I): -\frac{W}{aw_e}x_u + \frac{W}{aw_e}\frac{|I|}{2} - \left(\frac{W}{aw_e}x_u + \frac{W}{aw_e}\frac{|I|}{2}\right) = w_u - w_u \end{cases} \\ \Leftrightarrow & \begin{cases} (III): \frac{W}{aw_e}|I| = 2w_u \\ (IV): -2\frac{W}{aw_e}x_u = 0 \end{cases} . \end{aligned}$$

Therefore, the peak of the cone is at $x_u = 0$ and $w_u = \frac{W}{aw_e}\frac{|I|}{2}$. As a result, the cone is symmetric to $x_u = 0$ and its right border is described by the linear function

$$\ell(x) := \frac{W}{aw_e}x_u + \frac{W}{aw_e}\frac{|I|}{2}.$$

In the following, let $X_u$ and $W_u$ be random variables for the position and the weight of $u$ respectively. Then, the probability that $u$ covers the entire interval $I$ (with its cold area) is exactly the probability that $u$ is inside the cone $C_{\mathrm{dom}}$, i.e.,

$$\begin{aligned} \mathbb{P}\left[u \text{ covers } I\right] &= \mathbb{P}\left[u \in C_{\mathrm{dom}}\right] \\ &= \int_{-0.5}^{0.5} \mathbb{P}\left[u \text{ covers } I \mid X_u = x_u\right] f_{X_u}(x_u)\ \mathrm{d}x_u. \end{aligned}$$

Since $x_u$ is drawn from $U[-0.5; 0.5]$, its PDF $f_{X_u}(x_u)$ is always 1. Therefore, the above probability is equal to

$$= \int_{-0.5}^{0.5} \mathbb{P}\left[u \in C_{\mathrm{dom}} \mid X_u = x_u\right]\ \mathrm{d}x_u.$$

As $C_{\mathrm{dom}}$ is symetrical to $x = 0$, this is equal to

$$\begin{aligned} &= 2\int_{0}^{0.5} \mathbb{P}\left[u \in C_{\mathrm{dom}} \mid X_u = x_u\right]\ \mathrm{d}x_u \\ &= 2\int_{0}^{0.5} \mathbb{P}\left[W_v \geq \ell(X_u) \mid X_u = x_u\right]\ \mathrm{d}x_u. \end{aligned}$$

With $\overline{F}_{W_u}$ being the CCDF of the power law distribution $W_u$ is drawn from,

$$\mathbb{P}\left[W_u \geq l(X_u) \mid X_u = x_u\right] = \overline{F}_{W_u}(\ell(x_u))$$

holds for all $\ell(x_u) \geq w_{\min}$. For all other values, the CCDF is exactly 1. For intervals with $|I| \geq 2m_r(w_{\min})$ the CCDF is never (or only at the peak) 1. In this case, the integral resolves to

$$\begin{aligned} &= 2\int_{0}^{0.5} \overline{F}_{W_u}(\ell(x_u))\ \mathrm{d}x_u \\ &= 2\int_{0}^{0.5} w_{\min}^{\beta-1}\left(\frac{W}{aw_e}x_u + \frac{W}{aw_e}\frac{|I|}{2}\right)^{1-\beta}\ \mathrm{d}x_u \\ &= 2\frac{aw_e}{W}w_{\min}^{\beta-1}\frac{1}{\beta-2}\left(\left(\frac{W}{aw_e}\frac{|I|}{2}\right)^{2-\beta} - \left(\frac{W}{2aw_e} + \frac{W}{aw_e}\frac{|I|}{2}\right)^{2-\beta}\right) \\ &= \left(2\frac{aw_e}{W}\right)^{\beta-1} w_{\min}^{\beta-1}\frac{1}{\beta-2}\left((|I|)^{-(\beta-2)} - (1+|I|)^{-(\beta-2)}\right). \end{aligned} \tag{3.2}$$

For the other intervals, i.e., intervals $I$ with $|I| \leq 2m_r(w_{\min})$, we split the integral at the position $x$ where the cone is truncated, which is

$$\ell(x) = w_{\min} \iff x = m_r(w_{\min}) - \frac{|I|}{2}.$$

With this, we get that

$$\begin{aligned} \mathbb{P}\left[I \subseteq \mathrm{NA}_{\mathrm{cold}}(u)\right] &= \mathbb{P}\left[u \in C_{\mathrm{dom}}\right] \\ &= 2\left(\int_{0}^{m_r(w_{\min})-\frac{|I|}{2}} 1\ \mathrm{d}x_u + \int_{m_r(w_{\min})-\frac{|I|}{2}}^{0.5} \overline{F}_{W_u}(\ell(x_u))\ \mathrm{d}x_u\right) \\ &= 2\left(m_r(w_{\min}) - \frac{|I|}{2} + \frac{aw_e}{W}w_{\min}^{\beta-1}\frac{1}{\beta-2}\left(w_{\min}^{2-\beta} - \left(\frac{W}{2aw_e} + \frac{W}{aw_e}\frac{|I|}{2}\right)^{2-\beta}\right)\right). \end{aligned} \tag{3.3}$$

Note that Equation (3.2) and Equation (3.3) have the same value for $|I| = 2m_r(w_{\min})$, namely

$$2\frac{aw_e}{W}w_{\min}^{\beta-1}\frac{1}{\beta-2}\left(w_{\min}^{2-\beta}-\left(w_{\min}+\frac{W}{2aw_e}\right)^{2-\beta}\right).$$

∎

The interval $I$ from Lemma 3.1 can be interpreted as many things, for example as the neighborhood area of another vertex but also as the position of a hyperedge. If $v$ contains the neighborhood area of another vertex $u$, then $v$ dominates $u$ according to the vertex domination rule. Note that this is a sufficient but not necessary condition, as $v$ may also dominate $u$ using fluid incidences. However, we do not focus on such cases in this thesis. In Corollary 3.2, we derive the first basic statements of the Hyper-GIRG model when the interval $I$ from Lemma 3.1 is interpreted as the neighborhood area of a vertex.

**Corollary 3.2** (From Lemma 3.1)**:** *The following statements hold for every Hyper-GIRG*

*(1) each vertex $v$ with $\deg(v) > 0$ is dominated by the cold area of another vertex $u$ with a probability of*

$$\mathbb{P}\left[v \textit{ dom. cold}\right] = 1-\Big(1-\mathbb{P}\left[\mathrm{NA}(v)\subseteq \mathrm{NA}_{cold}(u)\right]\Big)^{n-1} < 1.$$

*(2) w.h.p. a vertex with diameter $d \in \Omega\left(n^{\frac{1}{\beta-2}-1}\right)$ is not dominated by the cold area of any another vertex.*

*Proof of (1):* Let $v$ be a vertex with $\deg(v) > 0$. To prove statement (1), we first derive the formula in the lemma and then show that it is constant in $n$. The probability that $v$ is dominated by the cold area of another vertex is

$$\begin{aligned}\mathbb{P}\left[v \text{ dom. cold}\right] &= \mathbb{P}\left[\bigcup_{u\in V\setminus\{v\}}\left[\mathrm{NA}(v)\subseteq \mathrm{NA}_{\mathrm{cold}}(u)\right]\right]\\ &= 1-\mathbb{P}\left[\bigcap_{u\in V\setminus\{v\}}\neg\left[\mathrm{NA}(v)\subseteq \mathrm{NA}_{\mathrm{cold}}(u)\right]\right]\\ &= 1-\Big(1-\mathbb{P}\left[\mathrm{NA}(v)\subseteq \mathrm{NA}_{\mathrm{cold}}(u)\right]\Big)^{n-1}\end{aligned}$$

To show that this expression is smaller than 1, we show that it is decreasing for increasing $n$. This is sufficient as it is a probability and thus always smaller or equal to 1. According to Lemma 2.7, this term is decreasing if $\mathbb{P}\left[\mathrm{NA}(v)\subseteq \mathrm{NA}_{\mathrm{cold}}(u)\right] \in \mathcal{O}\left(\frac{1}{n}\right)$ holds for every $v$. This probability gets larger the smaller the diameter of $v$. The smallest possible diameter for a vertex $v$ with $\deg(v) > 0$ is $d = 0$ (if there is exactly one incident hyperedge). Since $d = 0 \leq 2m_r(w_{\min})$, we use the second equation of Lemma 3.1 to determine $\mathbb{P}[\mathrm{NA}(v) \subseteq \mathrm{NA}_{\mathrm{cold}}(u)]$. Asymptotically, this is

$$\begin{aligned}
&\mathbb{P}[\mathrm{NA}(v) \subseteq \mathrm{NA}_{\mathrm{cold}}(u)] \\
&= 2\left(m_r(w_{\min}) - \frac{|\mathrm{NA}(v)|}{2} + \frac{aw_e}{W} w_{\min}^{\beta-1} \frac{1}{\beta-2}\left(w_{\min}^{2-\beta} - \left(\frac{W}{2aw_e} + \frac{W}{aw_e}\frac{|\mathrm{NA}(v)|}{2}\right)^{2-\beta}\right)\right). \\
&\overset{\Theta}{=} \frac{1}{W} - |\mathrm{NA}(v)| + \frac{1}{W}\left(w_{\min}^{2-\beta} - (W + W|I_d|)^{2-\beta}\right) \\
&\leq \frac{1}{W} + \frac{1}{W} w_{\min}^{2-\beta}.
\end{aligned}$$

Since $W \in \Theta(n)$, this expression is in $\mathcal{O}\left(\frac{1}{n}\right)$. As a result, $\mathbb{P}[v \text{ dom. cold}]$ can be bounded from above by a decreasing function that starts at 1 and is thus smaller than 1 for $n > 1$. ∎

*Proof of (2):* Let $v$ be a vertex with diameter $d \in \Omega\left(\frac{n^{\frac{1}{\beta-2}}}{n}\right)$ and thus $d \geq m_r(w_{\min})$. Then, the probability that a vertex $u$ dominates $v$ with its cold area is asymptotically

$$\begin{aligned}
\mathbb{P}[\mathrm{NA}(v) \subseteq \mathrm{NA}_{\mathrm{cold}}(u)] &\overset{\Theta}{=} \left(\frac{1}{n}\right)^{\beta-1}\left(d^{-(\beta-2)} - (1+d)^{-(\beta-2)}\right) \\
&\overset{\Theta}{=} \left(\frac{1}{n}\right)^{\beta-1} d^{-(\beta-2)} \\
&\in \mathcal{O}\left(\left(\frac{1}{n}\right)^{\beta-1}\left(\frac{n^{\frac{1}{\beta-2}}}{n}\right)^{-(\beta-2)}\right) \\
&\in \mathcal{O}\left(\frac{1}{n^2}\right).
\end{aligned}$$

Now, the probability that there is no such vertex $u$ is

$$\mathbb{P}\left[\neg \bigcup_{u \in V\setminus\{v\}} [\mathrm{NA}(v) \subseteq \mathrm{NA}_{\mathrm{cold}}(u)]\right].$$

With the same reformulations as in the proof of (1), this is equal to

$$= \left(1 - \mathbb{P}[\mathrm{NA}(v) \subseteq \mathrm{NA}_{\mathrm{cold}}(u)]\right)^{n-1}$$

which has a lower bound of

$$\begin{aligned}
&= 1 - (n-1)\mathbb{P}[\mathrm{NA}(v) \subseteq \mathrm{NA}_{\mathrm{cold}}(u)] \\
&\in 1 - \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}$$

∎

Part (1) of Corollary 3.2 states in particular that not even vertices with constant weight are dominated with high probability.

The interval $I$ from Lemma 3.1 can also be interpreted as the position of a hyperedge. In the following Corollary 3.3, we derive two basic properties of the hyperedge in the threshold variant from that interpretation.

**Corollary 3.3** (From Lemma 3.1): *The following statements hold for every Hyper-GIRG generated with the threshold variant*

*(1) the expected hyperedge size is constant.*

*(2) w.h.p. all hyperedges have a size in $\mathcal{O}\left(\ln(n)^2\right)$.*

*Proof of (1):* Let $e \in E$ be a hyperedge. As already shown in the proof of part (1) of Corollary 3.2, the probability for an interval of size 0 to be covered by the cold area of a vertex $u$ with random position and weight is in $\Theta\left(\frac{1}{n}\right)$. Therefore, each vertex $u$ is incident to $e$ with probability $\Theta\left(\frac{1}{n}\right)$. As a result, the size $S_e$ of $e$ follows a binomial distribution $\mathrm{Bin}\left(n, \Theta\left(\frac{1}{n}\right)\right)$. Part (1) is a direct consequence of that. ∎

*Proof of (2) (adapted from Lemma 4.5 (i) in [BKL16]):* Again, let $e \in E$ be a hyperedge and $S_e$ a random variable for its size. From part (1) we know that $S_e$ follows a binomial distribution $\mathrm{Bin}\left(n, \Theta\left(\frac{1}{n}\right)\right)$. We apply part (3) of the Chernoff bound for binomial distributions (see Theorem 2.5) for which, we need a constant $t \in \mathbb{R}$ with $t \geq \ln(n)^2$ and $t > 2e\mathbb{E}\left[S_e\right]$. Since $\mathbb{E}\left[S_e\right] \in \Theta(1)$, there is a constant $c > 1$ such that $t \coloneqq c\ln(n)^2$ fulfills both conditions. With that,

$$\mathbb{P}\left[S_e \geq t\right] \leq 2^{-t} = 2^{-c\ln(n)^2} = n^{-c\ln(2)\ln(n)} \in n^{-\Theta(\ln(n))}.$$

Now, a union bound over all hyperedges yields

$$\mathbb{P}\left[\bigcap_{e \in E}\left[S_e \leq t\right]\right] = 1 - \mathbb{P}\left[\bigcup_{e \in E}\left[S_e \geq t\right]\right] \geq 1 - n \cdot n^{-\Theta(\ln(n))} = 1 - n^{-\Theta(\ln(n))}.$$

That term $n^{-\Theta(\ln(n))}$ decreases asymptotically faster than $\frac{1}{n^1}$ for every constant $c$. ∎

Based on the result from the following Section 3.3, we find a similar bound to the expected hyperedge sizes in the binomial variant.

## 3.3 The Neighborhood of a Vertex

In this section, we analyze the neighborhood of a vertex $v \in V$ from a graph-theoretical and a geometrical perspective. In the graph-theoretical perspective (see Section 3.3.1), we study the degree of $v$ and determine the expected average degree of Hyper-GIRGs in the binomial variant. In Section 3.3.2, we focus on the diameter of $v$, which can be seen as the geometrical size of the vertex. For that, we derive the distribution functions for the left and the right end of $v$ and thus find an upper bound for the diameter depending on the temperature.

### 3.3.1 The Degree of a Vertex

The core problem of determining the degree of a vertex $v$ in Hyper-GIRGs is to find the probability by which $v$ is incident to a hyperedge. More generally, we are often interested in the probability by which a hyperedge with a uniformly random position lies left or right of a certain threshold and is incident to $v$. To simplify the following considerations, we assume without loss of generality that $v$ has a fixed weight $w_v \geq w_{\min}$ and is centered on the torus $\mathbb{T}^1 \coloneqq [-0.5; 0.5]^\circ$, i.e., $x_v = 0$. With that, we denote the set of incident hyperedges of $v$ that are left of a threshold $t \in [-0.5; 0.5]$ as

$$N_{\leq}^{v}(t) \coloneqq \{e = (x_e, w_e) \in E \mid x_e \leq t \wedge v \in e\}.$$

The sets $N^v_{\geq}(t)$, $N^v_{<}(t)$, and $N^v_{>}(t)$ are defined analogously to $N^v_{\leq}(t)$. For vertices with another position as $x_v = 0$, the definitions have to be shifted by $x_v$. Note that $N^v_{\leq}(0.5)$ and $N^v_{\geq}(-0.5)$ are always the same as the whole set of incident hyperedges of $v$ since their intervals cover the entire torus. Additionally, the set of cold incidences of a vertex can be expressed by $N^v_{\geq}(-m_r(v)) \cap N^v_{\leq}(m_r(v))$. With the following Lemma 3.4, we can derive the probability for a random hyperedge to be in one of the above sets.

**Lemma 3.4:** *Let $v = (x_v = 0; w_v)$ be a fixed vertex and $e = (x_e, w_e)$ a hyperedge with random position $x_e \in [-0.5; 0.5]$. Then the following holds for every $s, t \in [-0.5; 0.5]$ with $s < t$:*

$$\mathbb{P}\left[v \in e \wedge x_e \in [s;t]\right] = \int_s^t p_I(x) \, \mathrm{d}x.$$

*Proof.* To prove that, we interpret the hyperedge $e = (x_e, p) \in [-0.5; 0.5] \times [0; 1]$ as a pair of the position of the hyperedge on the torus $x_e \in [-0.5; 0.5]$, and a uniformly drawn probability $p \in [0; 1]$. This probability is used to define whether $v \in e$ in the following way: $v \in e$ if and only if $p \leq p_I(x_e)$. Using this, we can reformulate the left side of the equation to

$$\mathbb{P}\left[v \in e \wedge x_e \in [s;t]\right] = \mathbb{P}\left[p \leq p_I(x_e) \wedge x_e \in [s;t]\right].$$

Since $x_e$ is uniformly drawn on an interval of length 1, the law of total probability yields

$$= \int_s^t \mathbb{P}\left[p \leq p_I(x_e) \wedge x_e = x\right] \, \mathrm{d}x.$$

As $p$ is uniformly drawn in $[0; 1]$ and the values of $p_I$ never leave the interval $[0; 1]$ (since it is a probability), this is exactly

$$= \int_s^t p_I(x) \, \mathrm{d}x.$$

∎

As a direct consequence of Lemma 3.4, it holds that

$$\mathbb{P}\left[e \in N^v_{\leq}(t)\right] = \int_{-0.5}^t p_I(x) \, \mathrm{d}x = \mathbb{P}\left[e \in N^v_{\geq}(-t)\right] \tag{3.4}$$

since $p_I(x)$ is symmetrical to $x = 0$. Note that the same equalities hold for the strict sets $N^v_{<}(t)$ and $N^v_{>}(t)$. To get a better understanding of the values of these probabilities, we calculate the solution for the integral $\int_{-0.5}^t p_I(x) \, \mathrm{d}x$ in the following. For that, we first apply the substitution $u(x) \coloneqq \frac{x}{m_r(v)}$ with $u'(t) = \frac{1}{m_r(v)}$ where $m_r(v)$ is the melting radius of $v$ (defined in Equation (3.1)). This yields

$$\int_{-0.5}^t p_I(x) \, \mathrm{d}x = m_r(v) \cdot \int_{-\frac{1}{2m_r(v)}}^{\frac{t}{m_r(v)}} \min\left\{1, |u|^{-\frac{1}{T}}\right\} \, \mathrm{d}u.$$

Now, we define the functions

$$L(t) \coloneqq \int_{-\frac{1}{2m_r(v)}}^{\frac{t}{m_r(v)}} (-u)^{-\frac{1}{T}} \, \mathrm{d}u, \quad M(t) \coloneqq \int_{-1}^{\frac{t}{m_r(v)}} 1 \, \mathrm{d}u, \quad R(t) \coloneqq \int_{1}^{\frac{t}{m_r(v)}} u^{-\frac{1}{T}} \, \mathrm{d}u$$

which can be combined such that

$$\int_{-\frac{1}{2m_r(v)}}^{\frac{t}{m_r(v)}} \min\left\{1, |u|^{-\frac{1}{T}}\right\} \mathrm{d}u = \begin{cases} L(t), & t \in [-0.5; -m_r(v)] \\ L(-m_r) + M(t), & t \in [-m_r(v); m_r(v)] \\ L(-m_r) + M(m_r) + R(t), & t \in [m_r(v); 0.5] \end{cases}.$$

The function $M$ is a constant integral with the solution

$$M(t) = \int_{-1}^{\frac{t}{m_r(v)}} 1 \; \mathrm{d}u = \frac{t}{m_r(v)} + 1.$$

To solve the remaining integrals

$$L(t) = \int_{-\frac{1}{2m_r(v)}}^{\frac{t}{m_r(v)}} (-u)^{-\frac{1}{T}} \; \mathrm{d}u = \int_{-\frac{t}{m_r(v)}}^{\frac{1}{2m_r(v)}} u^{-\frac{1}{T}} \; \mathrm{d}u$$

and $R(t)$, we solve the general integral $\int_a^b u^{-\frac{1}{T}} \; \mathrm{d}u$ for all $1 \leq a \leq b \leq \frac{1}{2m(v)}$. The solution to this integral is

$$\int_a^b u^{-\frac{1}{T}} \; \mathrm{d}u = \frac{1}{\frac{1}{T} - 1} \left[u^{1-\frac{1}{T}}\right]_b^a = \tau \left(a^{-\frac{1}{\tau}} - b^{-\frac{1}{\tau}}\right)$$

where $\tau := \frac{T}{1-T}$. Putting all pieces together yields:

$$\begin{aligned} &\int_{-0.5}^{t} p_I(x) \; \mathrm{d}x \\ &= m_r(v) \cdot \begin{cases} L(t), & t \in [-0.5; -m_r(v)] \\ L(-m_r) + M(t), & t \in [-m_r(v); m_r(v)] \\ L(-m_r) + M(m_r) + R(t), & t \in [m_r(v); 0.5] \end{cases} \\ &= m_r(v) \cdot \begin{cases} \tau\left(\left(\frac{-t}{m_r(v)}\right)^{-\frac{1}{\tau}} - \left(\frac{1}{2m_r(v)}\right)^{-\frac{1}{\tau}}\right), & t \in [-0.5; -m_r(v)] \\ \tau\left(1 - \left(\frac{1}{2m_r(v)}\right)^{-\frac{1}{\tau}}\right) + \left(\frac{t}{m_r(v)} + 1\right), & t \in [-m_r(v); m_r(v)] \\ \tau\left(1 - \left(\frac{1}{2m_r(v)}\right)^{-\frac{1}{\tau}}\right) + 2 + \tau\left(1 - \left(\frac{t}{m_r(v)}\right)^{-\frac{1}{\tau}}\right), & t \in [m_r(v); 0.5] \end{cases}. \end{aligned}$$

which can be simplified to

$$= m_r(v) \cdot \begin{cases} \tau\left(-\frac{t}{m_r(v)}\right)^{-\frac{1}{\tau}} - \tau\left(2m_r(v)\right)^{\frac{1}{\tau}}, & t \in [-0.5; -m_r(v)] \\ \frac{t}{m_r(v)} + 1 + \tau\left(1 - (2m_r(v))^{\frac{1}{\tau}}\right), & t \in [-m_r(v); m_r(v)] \\ -\tau\left(\frac{t}{m_r(v)}\right)^{-\frac{1}{\tau}} + \tau\left(2 - (2m_r(v))^{\frac{1}{\tau}}\right) + 2, & t \in [m_r(v); 0.5] \end{cases}. \tag{3.5}$$

As already stated above, the set $N^v_{\leq}(0.5)$ is exactly the set of all hyperedges incident to $v$. With the help of Lemma 3.4 and the third case of Equation (3.5), we can now state the exact probability by which a hyperedge $e$ with uniformly drawn position $x_e$ is incident to the vertex $v = (x_v = 0, w_v)$. This is

$$\begin{aligned}\mathbb{P}\left[(v,e) \in \mathcal{I}\right] &= \mathbb{P}\left[e \in N^v_{\leq}(0.5)\right] \\ &= m_r(v) \cdot \left(-\tau\left(\frac{1}{2m_r(v)}\right)^{-\frac{1}{\tau}} + \tau\left(2 - (2m_r(v))^{\frac{1}{\tau}}\right) + 2\right) \\ &= m_r(v) \cdot \left(2(\tau+1) - 2\tau(2m_r(v))^{\frac{1}{\tau}}\right). \qquad (3.6)\end{aligned}$$

As a first application of Lemma 3.4, we use Equation (3.6) to calculate the expected degree of a vertex and the expected average degree of Hyper-GIRGs in the following Lemma 3.5.

**Lemma 3.5:** *Let $H = (V, E)$ be a Hyper-GIRG with average vertex weight $W_c$. Then*

*(1) the expected degree of a vertex $v = (x_v, w_v)$ is*

$$2\delta_{EV} a w_e \frac{w_v}{W_c} \frac{1}{1-T}\left(1 - T(2m_r(v))^{\frac{1-T}{T}}\right).$$

*(2) w.h.p. all vertices $v = (x_v, w_v)$ have a degree in $\mathcal{O}\left(w_v + \ln(n)^2\right)$.*

*(3) the expected average degree tends to the upper bound*

$$2d_{EV} a w_e \frac{1}{W_c} \frac{1}{1-T} \frac{\beta - 1}{\beta - 2} w_{min}.$$

*Proof of (1):* We show that the degree of $v$ follows the binomial distribution $\mathrm{Bin}(m, p)$ with $p = \mathbb{P}\left[(v,e) \in \mathcal{I}\right]$ from Equation (3.6). Let $\mathbf{1}_e$ be the indicator variable that is 1 if and only if the hyperedge $e \in E$ is incident to $v$. Then the degree $D_v$ of $v$ is the sum of those indicator variables, i.e.,

$$D_v = \sum_{e \in E} \mathbf{1}_e$$

and thus follows a binomial distribution with probability

$$\mathbb{P}\left[\mathbf{1}_e = 1\right] = \mathbb{P}\left[(v,e) \in \mathcal{I}\right].$$

Using Equation (3.6) as well as $W_c \coloneqq \frac{W}{n}$ and $\delta_{EV} = \frac{m}{n}$ from Section 2.3, we get

$$\begin{aligned}\mathbb{E}\left[D_v\right] &= m \cdot \mathbb{P}\left[(v,e) \in \mathcal{I}\right] \\ &= m \cdot a w_e \frac{w_v}{W} \cdot \left(2(\tau+1) - 2\tau(2m_r(v))^{\frac{1}{\tau}}\right) \\ &= 2\delta_{EV} a w_e \frac{w_v}{W_c} \cdot \left((\tau+1) + \tau(2m_r(v))^{\frac{1}{\tau}}\right).\end{aligned}$$

With $\tau + 1 = \frac{1}{1-T}$ this is equal to

$$= 2\delta_{EV} a w_e \frac{w_v}{W_c} \frac{1}{1-T}\left(1 - T(2m_r(v))^{\frac{1-T}{T}}\right).$$

∎

*Proof of (2) (adapted from Lemma 4.5 (i) in [BKL16]):* We first prove that the probability, that a single vertex $v = (x_v, w_v)$ has its degree in $\mathcal{O}\left(w_v + \ln(n)^2\right)$, is in $n^{-\Theta(\ln(n))}$, and later use a union bound to show that for all vertices.

For the first part, we make a case distinction over the weight $v$. If $w_v \in \mathcal{O}\left(\ln(n)^2\right)$, we apply the Chernoff bound (3) (see Theorem 2.5). For that, we need a constant $t \in \mathbb{R}$ such that $t \geq \ln(n)^2$ and $t > 2e\mathbb{E}\left[D_v\right]$ where $D_v$ is the degree of $v$. Since $w_v \in \mathcal{O}\left(\ln(n)^2\right)$ and thus $\mathbb{E}\left[D_v\right] \in \mathcal{O}\left(\ln(n)^2\right)$ (see part (1)), there is a constant $c > 1$ such that $t \coloneqq c\ln(n)^2$ fulfills both conditions. Therefore, it holds that

$$\mathbb{P}\left[D_v \geq t\right] \leq 2^{-t} = 2^{-c\ln(n)^2} = n^{-c\ln(2)\ln(n)} \in n^{-\Theta(\ln(n))}.$$

If, otherwise, $w_v \in \Omega\left(\ln(n)^2\right)$, we apply the Chernoff bound (1) (see Theorem 2.5). For any constant $\varepsilon \in (0,1)$ it holds that

$$\mathbb{P}\left[X > (1+\varepsilon)\mathbb{E}\left[D_v\right]\right] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}\left[D_v\right]\right) \in e^{-\Theta(\mathbb{E}[D_v])} = e^{-\Theta\left(\ln(n)^2\right)} = n^{-\Theta(\ln(n))}.$$

Now, we find the probability that $D_v \in \mathcal{O}\left(w_v + \ln(n)^2\right)$ holds for all vertices by using a union bound in the following way

$$\begin{aligned}
&\mathbb{P}\left[\bigcap_{v\in V}\left\{D_v \in \mathcal{O}\left(w_v + \ln(n)^2\right)\right\}\right] \\
&= 1 - \mathbb{P}\left[\bigcup_{v\in V}\left\{D_v \in \mathcal{O}\left(w_v + \ln(n)^2\right)\right\}\right] \\
&\geq 1 - n\cdot\mathbb{P}\left[D_v \in \mathcal{O}\left(w_v + \ln(n)^2\right)\right] \\
&\in 1 - n\cdot n^{-\Theta(\ln(n))} \\
&= 1 - n^{-\Theta(\ln(n))}.
\end{aligned}$$

Therefore, w.h.p. all vertices have a degree of $D_v \in \mathcal{O}\left(w_v + \ln(n)^2\right)$. ■

*Proof of (3):* Again, let $D_v$ be a random variable for the degree of a vertex $v$. Using the linearity of expectation and the fact that all vertices are independently drawn from identical distributions, we get that the expected average degree is equal to the expected degree of a vertex, i.e.,

$$\mathbb{E}\left[\frac{1}{n}\sum_{v\in V} D_v\right] = \frac{1}{n}\sum_{v\in V}\mathbb{E}\left[D_v\right] = \mathbb{E}\left[D_v\right].$$

In contrast to part (1) the weight of the vertex is unknown. Therefore, let $W_v \sim \mathrm{PL}(\beta, w_{\min})$ be the random variable for the weight of a fixed vertex $v \in V$. Then, the law of total expectation (Equation (2.8)) yields

$$\mathbb{E}\left[D_v\right] = \int_{w_{\min}}^{\infty} \mathbb{E}\left[D_v \mid W_v = w_v\right] f(w_v)\ \mathrm{d}w_v$$

where $f$ is the PDF of the power-law distribution of the vertex weights. For a known weight, we already determined the expected degree of a vertex in part (1). Putting these two things together yields:

$$= \int_{w_{\min}}^{\infty} 2\delta_{EV} a w_e \frac{w_v}{W_c}\frac{1}{1-T}\left(1 - T(2m_r(v))^{\frac{1-T}{T}}\right)\cdot\frac{\beta-1}{w_{\min}^{1-\beta}} w_v^{-\beta}\ \mathrm{d}w_v.$$

With $n \longrightarrow \infty$ the term $(2m_r(v))^{\frac{1-T}{T}}$ tends to 0. Therefore, we get

$$
\begin{aligned}
&\leq \int_{w_{\min}}^{\infty} 2\delta_{EV} a w_e \frac{w_v}{W_c} \frac{1}{1-T} \cdot \frac{\beta - 1}{w_{\min}^{1-\beta}} w_v^{-\beta} \, \mathrm{d}w_v \\
&= 2\delta_{EV} a w_e \frac{1}{W_c} \frac{1}{1-T} \frac{\beta - 1}{w_{\min}^{1-\beta}} \cdot \int_{w_{\min}}^{\infty} w_v^{1-\beta} \, \mathrm{d}w_v \\
&= 2\delta_{EV} a w_e \frac{1}{W_c} \frac{1}{1-T} \frac{\beta - 1}{w_{\min}^{1-\beta}} \cdot \left[ \frac{1}{2-\beta} w_v^{2-\beta} \right]_{w_{\min}}^{\infty} \\
&= 2\delta_{EV} a w_e \frac{1}{W_c} \frac{1}{1-T} \frac{\beta - 1}{\beta - 2} w_{\min}.
\end{aligned}
$$

■

A consequence of part (3) of Lemma 3.5 is that the expected hyperedge size is constant in the binomial variant. In this part, we did not state the solution to the actual integral, as it diverges for every $T > 1 - \frac{1}{\beta - 1}$. However, for small temperatures, the term $(2m_r(v))^{\frac{1-T}{T}}$ tends to 0 very fast as $\frac{1-T}{T}$ is large in this case. Therefore, this bound is close to the actual solution of the integral in the cases we focus on the most in this thesis. Additionally, the formula in part (3) has a nice property: If we set $W_c$ to be the expected average vertex weight, namely $\mathbb{E}[W_c] = \frac{\beta-1}{\beta-2} w_{\min}$, this values is canceled out. The resulting bound $\frac{2d_{EV} a w_e}{1-T}$ is completely independent of the chosen power-law exponent. As a result, the heterogeneity does not have a huge impact on the expected average degree of Hyper-GIRGs with low temperatures.

### 3.3.2 The Diameter of a Vertex

In this section, we find an upper bound to the diameter of a vertex $v \in V$ depending on the temperature and the weight $w_v \sim \mathrm{PL}(\beta, w_{\min})$ of $v$. As $D \coloneqq B_R - B_R$, deriving the distribution functions for the left border $B_L$ and right border $B_R$ is a crucial first step. This is done in the following Lemma 3.6.

**Lemma 3.6:** *Let $m \in \Theta(n)$ be the number of hyperedges in the hypergraph, and $v = (x_v = 0, w_v)$ a vertex. Then, for every $b_L, b_R \in [-0.5; 0.5]$, the CDFs of the left border $B_L$ and the right border $B_R$ of $v$ are*

$$
\begin{aligned}
F_{B_L}(b_L) &\coloneqq \mathbb{P}[B_L \leq b_L] = 1 - \left(1 - \mathbb{P}\left[e \in N_{\leq}^{v}(b_L)\right]\right)^m \\
F_{B_R}(b_R) &\coloneqq \mathbb{P}[B_R \leq b_R] = \left(1 - \mathbb{P}\left[e \in N_{\geq}^{v}(b_R)\right]\right)^m,
\end{aligned}
$$

*and their PDFs are*

$$
\begin{aligned}
f_{B_L}(b_L) &\coloneqq m \cdot p_I(b_L) \cdot \left(1 - \mathbb{P}\left[e \in N_{\leq}^{v}(b_L)\right]\right)^{m-1} \\
f_{B_R}(b_R) &\coloneqq m \cdot p_I(-b_R) \cdot \left(1 - \mathbb{P}\left[e \in N_{\geq}^{v}(b_R)\right]\right)^{m-1}.
\end{aligned}
$$

*Proof.* We first focus on $F_{B_L}(b_L)$ and later on $F_{B_R}(b_R)$. For a fixed $b_L \in [-0.5; 0.5]$, the inequality $B_L \leq b_L$ holds if and only if there is at least one hyperedge $e$ that is incident to $v$ and lies left to $b_L$. Therefore

$$\begin{aligned} F_{B_L}(b_L) &= \mathbb{P}\left[B_L \leq b_L\right] \\ &= \mathbb{P}\left[\bigcup_{e\in E}[e \in N^v_{\leq}(b_L)]\right] \\ &= 1 - \mathbb{P}\left[\bigcap_{e\in E}[e \notin N^v_{\leq}(b_L)]\right]. \end{aligned}$$

As all components of all hyperedges are drawn independently, this is equal to the product of their individual probabilities, i.e.,

$$= 1 - \left(1 - \mathbb{P}\left[e \in N^v_{\leq}(b_L)\right]\right)^m.$$

We get the PDF $f_{B_L}(b_L)$ of $B_L$ by taking the derivative of its CDF $F_{B_L}(b_L)$, which is

$$\begin{aligned} f_{B_L}(b_L) &= \frac{\partial}{\partial b_L}\left(1 - \left(1 - \mathbb{P}\left[e \in N^v_{\leq}(b_L)\right]\right)^m\right) \\ &= -m\left(1 - \mathbb{P}\left[e \in N^v_{\leq}(b_L)\right]\right)^{m-1} \cdot \frac{\partial}{\partial b_L}\left(1 - \mathbb{P}\left[e \in N^v_{\leq}(b_L)\right]\right). \end{aligned}$$

As $\mathbb{P}\left[e \in N^v_{\leq}(b_L)\right] = \int_{-0.5}^{b_L} p_I(x)\ \mathrm{d}x$ (see Equation (3.4)), the latter term is equal to $-p_I(b_L)$, and thus

$$= m \cdot p_I(b_L) \cdot \left(1 - \mathbb{P}\left[e \in N^v_{\leq}(b_L)\right]\right)^{m-1}.$$

The distribution functions of the right border $B_R$ that are stated in the lemma follow directly from the symmetry of the left and right border, i.e., $\mathbb{P}\left[B_R \leq t\right] = \mathbb{P}\left[B_L \geq -t\right]$ for every $t \in [-0.5; 0.5]$, and the fact $\mathbb{P}\left[e \in N^v_{\geq}(t)\right] = \mathbb{P}\left[e \in N^v_{\leq}(-t)\right]$ from Equation (3.4). ∎

Lemma 3.6 proves in particular, that the random variables $B_L$ and $-B_R$ follow the same distribution, since $f_{B_L}(x) = f_{B_R}(-x)$ holds for all $x \in [-0.5; 0.5]$. Based on these distribution functions, it is possible to derive a closed distribution function for the diameter itself. The PDF of the diameter is the convolution of the PDFs of the left and the right borders, which is not nice to handle mathematically. However, the distribution functions derived in Lemma 3.6 are sufficient to find the following upper bounds to the vertex diameter in Lemma 3.7.

**Lemma 3.7:** *Let $w_v$ denote the weight of a vertex $v$. Then,*

*(1) w.h.p. the diameter of a single vertex $v \in V$ with weight $w_v \geq w_{min}$ is in $\mathcal{O}\left(w_v^{\frac{1}{1-T}} n^{\frac{T}{1-T}-1}\right)$.*

*(2) w.h.p. all vertices in a generated hypergraph have a diameter in $\mathcal{O}\left(w_v^{\frac{1}{1-T}} n^{2\frac{T}{1-T}-1}\right)$.*

*Proof.* Let $v$ be a single vertex with $x_v = 0$ and weight $w_v \geq w_{\min}$. The probability that $v$ has a diameter of at most $d$ can be bounded from below by the probability that both ends have an absolute distance of at most $\frac{d}{2}$. Therefore, with $B_L$, $B_R$, and $D$ being random variables representing the left end, right end, and the diameter of $v$, it holds that

$$\mathbb{P}\left[D \leq d\right] \geq \mathbb{P}\left[B_L \geq -\frac{d}{2} \wedge B_R \leq \frac{d}{2}\right].$$

This is equal to the probability that $N_{\leq}^{v}\left(\frac{d}{2}\right)$ and $N_{\geq}^{v}\left(\frac{d}{2}\right)$ are both empty. Therefore, it holds that

$$\begin{aligned}
&= \mathbb{P}\left[N_{\leq}^{v}\left(-\frac{d}{2}\right) \cup N_{\geq}^{v}\left(\frac{d}{2}\right) = \emptyset\right] \\
&= \mathbb{P}\left[\bigcap_{e\in E} \neg\left[e \in N_{\leq}^{v}\left(\frac{d}{2}\right) \cup N_{\geq}^{v}\left(\frac{d}{2}\right)\right]\right].
\end{aligned}$$

Since both have the same size, this is equal to

$$= \left(1 - 2\mathbb{P}\left[e \in N_{\geq}^{v}\left(\frac{d}{2}\right)\right]\right)^{m}.$$

Therefore, it is sufficient to consider only one end of $v$. In the following, we show that $\mathbb{P}\left[B_R \leq c \cdot w_v^{\frac{1}{1-T}} n^{P\frac{T}{1-T}-1}\right] \in 1 - \mathcal{O}\left(\frac{1}{n^P}\right)$ holds for every constant $c > 0$ and $P \in \{1, 2\}$. With that, we later prove both statements above. We focus on the right end with its CDF $F_{B_R}$ is the CDF from Lemma 3.6. Then with $\frac{1}{1-T} = \frac{T}{1-T} + 1$ we get

$$\begin{aligned}
\mathbb{P}\left[B_R \leq c w_v^{\frac{1}{1-T}} n^{P\frac{T}{1-T}-1}\right] &= \mathbb{P}\left[B_R \leq c w_v^{\frac{T}{1-T}+1} n^{P\frac{T}{1-T}-1}\right] \\
&= F_{B_R}\left(c w_v^{\frac{T}{1-T}+1} n^{P\frac{T}{1-T}-1}\right) \\
&= \left(1 - \mathbb{P}\left[e \in N_{\geq}^{v}\left(c w_v^{\frac{T}{1-T}+1} n^{P\frac{T}{1-T}-1}\right)\right]\right)^{m} \\
&\geq 1 - m\mathbb{P}\left[e \in N_{\geq}^{v}\left(c w_v^{\frac{T}{1-T}+1} n^{P\frac{T}{1-T}-1}\right)\right].
\end{aligned}$$

Since $w_v^{\frac{T}{1-T}+1} n^{P\frac{T}{1-T}-1} \geq \frac{w_v}{n}$, we assume that the diameter is larger than the melting radius. Therefore, we can use the first case in Equation (3.5) to get the probability with the help of Equation (3.4). Therefore, with $d \coloneqq w_v^{\frac{T}{1-T}+1} n^{P\frac{T}{1-T}-1}$ this is equal to

$$\begin{aligned}
&= 1 - m \cdot m_r(v)\left(\frac{T}{1-T}\left(-\frac{d}{m_r(v)}\right)^{-\frac{1-T}{T}} - \underbrace{\frac{T}{1-T}\left(2m_r(v)\right)^{\frac{1-T}{T}}}_{\geq 0}\right) \\
&\overset{\Theta}{\geq} 1 - m \cdot m_r(v)\left(\frac{d}{m_r(v)}\right)^{-\frac{1-T}{T}}.
\end{aligned}$$

With $W \in \Theta(n)$ (see Lemma 2.3), we get that $m_r(v) \in \Theta\left(\frac{w_v}{n}\right)$. Additionally, with $m \in \Theta(n)$ it holds that $m \cdot m_r(v) \in \Theta(n)$. Therefore, the latter term is equal to

$$m \cdot m_r(v)\left(\frac{w_v^{\frac{T}{1-T}+1} n^{P\frac{T}{1-T}-1}}{m_r(v)}\right)^{-\frac{1-T}{T}} \overset{\Theta}{=} w_v\left(w_v^{\frac{T}{1-T}} n^{P\frac{T}{1-T}}\right)^{-\frac{1-T}{T}} = w_v\left(w_v^{-1} n^{-P}\right) = \frac{1}{n^P}.$$

As a result, we get

$$\mathbb{P}\left[B_R \leq c w_v^{\frac{T}{1-T}+1} n^{\frac{T}{1-T}-1}\right] \in 1 - \mathcal{O}\left(\frac{1}{n}\right) \quad \text{and} \quad \mathbb{P}\left[B_R \leq c w_v^{\frac{T}{1-T}+1} n^{2\frac{T}{1-T}-1}\right] \in 1 - \mathcal{O}\left(\frac{1}{n^2}\right).$$

The first relation proves the first statement. Additionally, using a union bound over all vertices, the second statement follows from the second relation. ∎

The first part of the Lemma 3.7 is only meaningful for $T \leq 0.5$. For every other $T > 0.5$, the term $n^{\frac{T}{1-T}-1}$ increases with $n$ and is thus too large to represent a diameter on a torus with constant length. The same applies to vertices with a weight in $\omega\left(n^{1-2T}\right)$. In both cases, $T = 0.5$ seems to be a temperature where asymptotics change. For the second bound if Lemma 3.7 this point of change is even lower at $T = \frac{1}{3}$. This bound has only impact for vertices with a weight in $\mathcal{O}\left(n^{1-3T}\right)$. If one considers that the maximum vertex weight is w.h.p. in $\mathcal{O}\left(n^{\frac{2}{\beta-1}}\right)$, the condition $T < \frac{1}{3}\frac{\beta-3}{\beta-1}$ is necessary for the second bound to be impactful for all vertices. Therefore, it is not applicable for $\beta < 3$.

# 4 The Threshold Variant

In this chapter, we study Hyper-GIRGs generated with the threshold variant and the behavior of the reduction rules on it. In the threshold variant, the temperature is $T = 0$ and the incidence probability is the step-function

$$p_I(v, e) = \begin{cases} 1, & \operatorname{dist}(x_v, x_e) \leq a\frac{w_v w_e}{W} \\ 0, & \text{otherwise} \end{cases}$$

for some vertex $v = (x_v, w_v)$ and a hyperedge $e = (x_e, w_e)$ (see Equation (2.14)). A consequence of this step-function is that all hyperedges inside the neighborhood area of $v$ are actually incident to $v$. Note that this is not necessarily the case in the binomial variant, as the neighborhood area of $v$ is just the interval between the most distant incident hyperedges, and $v$ may have an (unlikely but possible) fluid incidence to a distant hyperedge, possibly skipping many other hyperedges. This property of neighborhoods in the threshold area is sufficient to classify the generated hypergraphs as so-called *dual circular-arc hypergraphs*; a hypergraph class we introduce in Section 4.1. We find that the reduction rules always reduce these hypergraphs to a trivial kernel independent of their degree distribution if they have at least one gap (roughly speaking, there is no cycle going around the torus). In Section 4.2 we deal with the probability of gaps in Hyper-GIRGs depending on the heterogeneity. Our main result is that there is a.a.s. at least one gap in every Hyper-GIRG generated with the threshold variant. As a result, the reduction rules reduce such hypergraphs a.a.s. entirely. However, for small Hyper-GIRGs generated with low heterogeneity, it may be possible to not have a gap. In this case, we describe a branching rule that creates a gap in such hypergraphs deterministically (see Section 4.3).

## 4.1 (Dual) Circular-Arc Hypergraphs

The definitions of *circular-arc hypergraphs / CA-hypergraphs* and their duals are largely based on cyclic orders (defined in Section 2.3.3). A hypergraph $H = (V, E)$ is a CA-hypergraph if there is a cyclic order $\prec_V^\circ \coloneqq \langle v_1, \dots, v_n \rangle^\circ$ on the vertices, such that the vertices of every hyperedge form an interval in $\prec_V^\circ$. The hyperedge whose interval covers all vertices of $H$ is called the *universal hyperedge*. For all other hyperedges $e \in E \setminus \{V\}$, the *left end* and the *right end* of $e$ are the ends of its vertex interval with respect to $\prec_V^\circ$.

The duals of CA-hypergraphs can be defined in an analog way. A hypergraph $H = (V, E)$ is a *dual circular-arc hypergraphs / CA$^*$-hypergraph* if its dual $H^* = (V^*, E^*)$ is a CA-hypergraph. Thus, the roles of vertices and hyperedges swap. The hyperedges of a CA$^*$-hypergraph have a cyclic order $\prec_E^\circ \coloneqq \langle e_1, \dots, e_m \rangle^\circ$ in which all $E(v)$ (for $v \in V$) are intervals in $\prec_E^\circ$. In this context, we call $E(v)$ the *interval* of the vertex $v \in V$. Figure 4.1 shows an example of a CA-hypergraph together with its dual CA$^*$-hypergraph.

In the following Lemma 4.1 we show that all Hyper-GIRGs generated with the threshold variant are CA$^*$-hypergraphs.

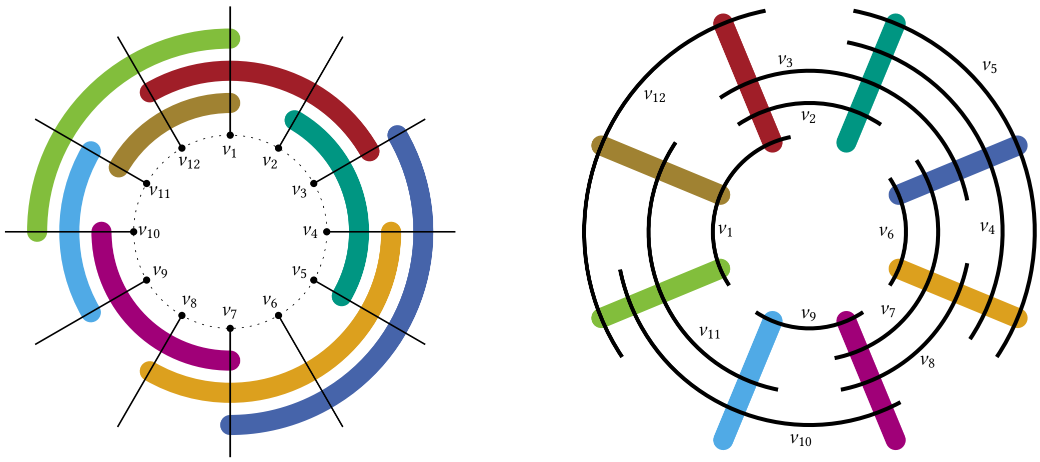


**Figure 4.1:** A CA-hypergraph (left) and its dual CA*-hypergraph (right). The hyperedges of the graphs are shown as circular arcs. The vertices are depicted as lines pointing towards the middle of the circle. A vertex is part of a hyperedge, if and only if its line intersects the circular arc of the hyperedge

**Lemma 4.1:** *Let $H = (V, E)$ be a Hyper-GIRG with $T = 0$. Then $H$ is a CA*-hypergraph.*

*Proof.* As mentioned above, a vertex with weight $w_v$ is part of an edge if and only if their distance on $\mathbb{T}^1$ is smaller than $a\frac{w_v w_e}{W}$. For each vertex $v \in V$, this area is convex. Therefore, all hyperedges $E(v)$ of $v$ lie inside an interval on the $\mathbb{T}^1$. As a result, if we define $<_E^\circ$ as the cyclic order of the "smaller than"-relation on the hyperedge positions $y_1, \ldots, y_m \in [0; 1]$ of $H$, all $E(v)$ are intervals on $<_E^\circ$. ∎

Note that we do not use any assumptions on the degree distribution in Lemma 4.1. Therefore, this result is independent of the power-law exponent of the generated hypergraph.

Due to Lemma 4.1, we assume for the remainder of this section that $H = (V, E)$ is a general CA*-hypergraph with cyclic edge order $<_E^\circ := \langle e_1, \ldots, e_m \rangle^\circ$. In our context of Weihe's reduction rules, the concept of gaps is important for CA*-hypergraph. Formally, a *gap* in $H$ is a pair of consecutive hyperedges $(e_g, e_{g+1})$ such that there is no vertex $v \in V$ that is part of both hyperedges, i.e., $e_g \cap e_{g+1} = \emptyset$. In the following Lemma 4.2 we show that the reduction rules are sufficient to reduce every CA*-hypergraph with a gap to a trivial kernel. For that result, we count the additional unit hyperedge rule to the set of reduction rules.

**Lemma 4.2:** *The reduction rules solve the* HittingSet *problem on CA*-hypergraphs with at least one gap entirely.*

*Proof.* [1]Let $H = (V, E)$ be an arbitrary CA*-hypergraph with cyclic edge order $\langle e_1, \ldots, e_m \rangle^\circ$ and at least one gap. Due to symmetry, we can assume that one gap is between the hyperedges $e_m$ and $e_1$. In case $H$ has more than one connected component, we only consider one component, as the others are reduced equivalently.

We prove the lemma by arguing that there is a sequence of reduction rules, that eliminates the vertices next to the gap. For the gap $(e_m, e_1)$, we only consider $e_1$, as $e_m$ can be eliminated equivalently. Let $\{v_1, \ldots, v_k\}$ be the vertices of $e_1$, i.e., $e = \{v_1, \ldots, v_k\}$. Since $H$ is a CA*-hypergraph, each vertex $v \in e$ is part of all hyperedges $e_1, e_2, \ldots$ up to an individual index

[1] A similar idea of the proof is briefly described in Proposition 1 of [BFFS19].

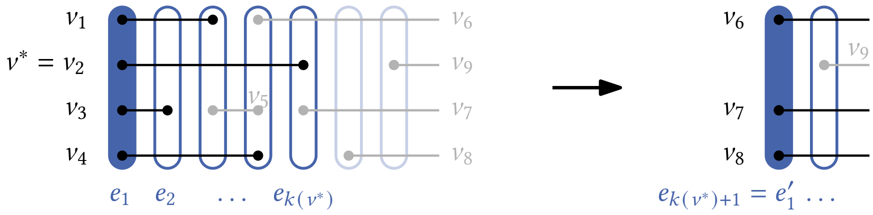


**Figure 4.2:** Sketch of the elimination of a hyperedge $e_1$ right to a gap. The hyperedges are arranged in a line instead of a circle. One can imagine the gap to be from the right end of the sketch to the left side of $e_1$ (or $e'$).

$k(v) \coloneqq \max\{j \mid v \in e_j\}$. Let $v^* \coloneqq \arg\max_{v \in e_1} k(v)$ be the vertex with the largest $k(\cdot)$, i.e., the vertex that is contained in the most hyperedges starting from $e_1$. This vertex dominates all other vertices in $e_1$ because $k(v) \leq k(v^*)$ holds for every $v \in e$ per definition of $v^*$. Therefore, after one round of the vertex domination rule, the hyperedge $e_1$ only consists of the vertex $v^*$. Consequently, the unit hyperedge rule is applicable. Therefore, $v^*$ has to be selected in a hitting set and can safely be deleted together with all its incident hyperedges.

Then, we can consider the sub-hypergraph with the remaining vertices and the hyperedges $e_{k(v^*)+1}, \dots, e_m$. This is again a CA*-hypergraph with a (bigger) gap and can thus be treated similarly. One step of this process is shown in Figure 4.2.

As a result, we get a sequence of reduction rule applications, that eliminate the hyperedges next to a gap step by step. Repeating this process eliminates all vertices and hyperedges of each connected component of $H$. ■

The algorithm described in Lemma 4.2 only works if there is at least one gap in the hypergraph either before or during the application of the reduction rules. For general CA*-hypergraphs, this might not be the case. Therefore, we provide two fixes to this problem in the following two sections. In Section 4.2 we motivate why the reduction rules often create gaps in Hyper-GIRGs with $T = 0$. Additionally, even if the reduction rules are not enough, we describe a branching rule in Section 4.3 that deterministically creates one in general CA*-hypergraphs.

## 4.2 Gap Probability

In this section, we describe how the reduction rules create gaps around vertices and determine how likely that happens. In particular, we prove that in every Hyper-GIRG $H$ there is a.a.s. at least one gap after one round of each reduction rule. For that, let $H_1$ be the Hyper-GIRG $H$ after one round of the vertex domination rule, and $H_2$ the hypergraph $H_1$ after an additional round of the hyperedge domination rule. Our goal for this section is to find the probability by which a gap exists in $H_2$ around sufficiently heavy vertices in $H$.

So far, we defined gaps to only exist between two hyperedges. To start this section, we explain what we mean by saying “a gap exists around a vertex”. For that, let $v = (x_v, w_v)$ be a (sufficiently large) vertex in $H_1$, i.e., $v$ is not dominated by any other vertex in $H$. As there cannot be any vertex domination in $H_1$, all other vertices (or more specifically their cold areas) contain at most one end of $v$. Therefore, we can refer to vertices $v' = (x'_v, w'_v)$ whose cold areas overlap with the one of $v$ as vertices that *reach into* $v$. Depending on whether they contain the left or right end of $v$, they reach into $v$ from the left or the right. Let $v_L$ and $v_R$ be the vertices that reach the furthest into $v$ from the left and the right, respectively. Based on this notation, we define the interval $I(v)$ for every vertex $v$ in $H_1$ to the interval from the

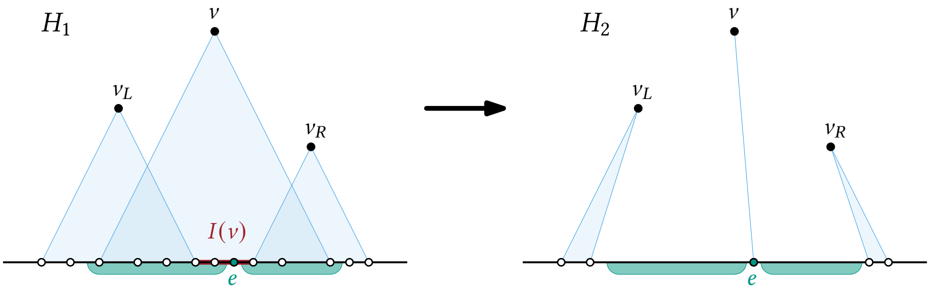


**Figure 4.3:** The process shows how gaps are formed after one round of the vertex domination rule (left) and after an additional round of the hyperedge domination rule (right). The vertex $v$ is sufficiently heavy to have an interval $I(v)$ (in red) that is only covered by $v$. The two remaining vertices in both sketches are the vertices that reach the furthest into the cold area of $v$ from both sides respectively. As there is one hyperedge in $I(v)$, all other hyperedges in the cold area of $v$ are dominated (inside the green intervals). Therefore they are removed by the hyperedge domination rule (right).

right end of $v_L$ to the left end of $v_R$. In case this interval exists, i.e., $R(v_L) < L(v_R)$, it is per definition only covered by $v$ in $H_1$. Any hyperedge $e \in E$ included in $I(v)$ only contains $v$ and thus dominates all other hyperedges containing $v$. As a result, $(v, e)$ is an isolated edge in the bipartite representation of $H_2$. As $v$ is the only vertex that can cover $e$, it has to be selected in every hitting set (unit hyperedge rule). Selecting $v$ allows to remove $e$ from $H_2$, too. This leads to a gap between the closest hyperedges next to the former position of $e$ that remain in $H_2$ after removing $e$. We call this gap to exist *around $v$ (or $e$)*. The whole process can be seen in Figure 4.3.

For the majority of this section, we argue with an $\varepsilon \in (0, 1)$ that we specify further at the end. This $\varepsilon$ defines the size of the interval $I \coloneqq \left[-\frac{1}{m^{1-\varepsilon}}; \frac{1}{m^{1-\varepsilon}}\right]$ centered around $x_v = 0$, which is sufficiently large enough to contain a.a.s. at least one hyperedge in $H$ (see Lemma 4.3). We will show in Lemma 4.4 that for any vertex $v = (x_v, w_v) \in V$ with $w_v \in \Omega\left(n^{\varepsilon}\right)$ there is a.a.s. no other vertex in $H_1$ reaching into that interval $I$. With those two lemmata, we later show that there is a.a.s. at least one gap in $H$ (Theorem 4.5). We start by showing in Lemma 4.3 that the interval $I$ is large enough to contain at least one hyperedge.

**Lemma 4.3:** *Let $\varepsilon \in (0, 1)$ and $c > 0$ be constants, and $I$ an interval with size $|I| = cm^{-(1-\varepsilon)}$. Then there is at least one hyperedge in $I$ a.a.s. with probability $1 - e^{-cn^{\varepsilon}}$.*

*Proof.* The probability that at least one out of the $m$ hyperedges is placed into $I$ can be expressed as:

$$\mathbb{P}\left[\exists e \in E : x_e \in I\right] = 1 - \mathbb{P}\left[\bigcap_{e \in E} \neg[x_e \in I]\right].$$

As all hyperedges are drawn independently, this is equal to

$$= 1 - (1 - \mathbb{P}\left[e \in I\right])^m$$

Since all hyperedges are uniformly sampled in $[0; 1]$, the probability for one hyperedge to lie in an interval is exactly the size of this interval, i.e., $\mathbb{P}\left[e \in I\right] = m^{-(1-\varepsilon)}$. Therefore, this is equal to

$$= 1 - \left(1 - \frac{c}{m^{1-\varepsilon}}\right)^m.$$

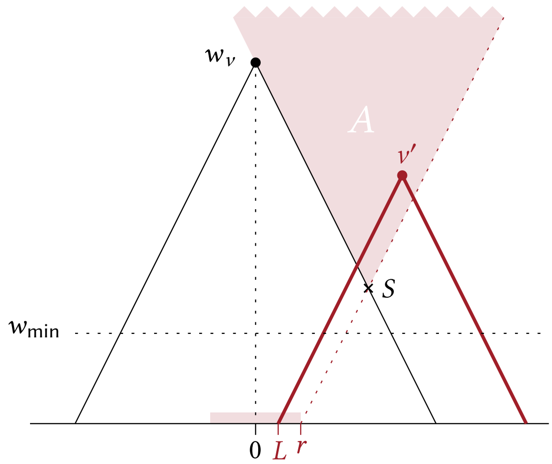


**Figure 4.4:** A sketch of Hyper-GIRG generated with $T = 0$. It shows a vertex $v$ and a random vertex $v'$ that covers a part of the interval $I$. This interval $I$ has a radius of $r$ around the position of vertex $v$, which is set to be 0. The area marked as "A" indicates all places in which such a vertex $v'$ can lie.

Using Lemma 2.7 with $b \coloneqq 1 - \varepsilon \in (0; 1)$, we find that the term $\left(1 - \frac{c}{m^{1-\varepsilon}}\right)^m$ tends to 0 for $m \longrightarrow \infty$ with

$$\geq 1 - e^{-cn^{\varepsilon}}.$$

As a result, $\mathbb{P}\left[\exists e \in E : e \in I\right]$ tends to 1 for $m \longrightarrow \infty$, and also for $n \longrightarrow \infty$ as we assumed $m \in \Theta(n)$. ■

For the rest of this section, let $v = (x_v = 0, w_v) \in V$ be a vertex in $H$ with a weight of $w_v \in \Omega(n^{\varepsilon})$. In the following Lemma 4.4, we show that $v$ is sufficiently heavy such that the interval $I$ from the last lemma is a.a.s. only covered by $v$ in $H_1$.

**Lemma 4.4:** *Let $H = (V, E) \sim \mathcal{H}(n, m, \beta, w_{min}, w_e, T = 0, a)$ and $v = (x_v, w_v) \in V$ a vertex in $H$ with a weight $w_v \in \Omega(n^{\varepsilon})$. Further, let $V_1$ be the set of vertices that remain after the first round of the vertex domination rule. Then, $v$ is a.a.s. the only vertex in $V_1$ that covers the interval $I$ with radius $r \coloneqq m^{-(1-\varepsilon)}$ around $x_v$. The exact probability is in $1 - \mathcal{O}\left(n^{-\varepsilon(\beta-2)}\right)$.*

*Proof.* Without loss of generality, we assume the vertex $v$ to be at position $x_v = 0$. We show, that there is a.a.s. no vertex $v' = (x'_v, w'_v)$ that (1) is not dominated by $v_{\max}$ and (2) has its left end $L$ to the left of the right interval bound $r$. With the same probability, there is no similar vertex on the other side of $v$. If both events occur a.a.s. then their conjunction occurs a.a.s. too.

The plan for this proof is to show that a single random vertex does not fulfill both properties (1) and (2) with more than high probability. Then, we find the result by using a union bound over all vertices. The area $A$ in Figure 4.4 visualizes all possible places, in which a vertex $v'$ could lie that fulfills both properties (1) and (2). The left border of $A$ ensures that $v'$ is not dominated by $v$. The right border ensures that the left end $L$ of $v'$ reaches into the interval $I$.

First, we find an upper bound to the probability $\mathbb{P}[v' \in A]$ with the help of Lemma 3.1. For that, we use the fact that $A$ is exactly the area in which all vertices lie that cover the interval between the right end of $I$, i.e., $r = m^{-(1-\varepsilon)}$, and the right end of $v$, i.e., the melting radius $m_r(v)$. Therefore, the interval $J$, we plug into Lemma 3.1, has a size of $m_r(v) - m^{-(1-\varepsilon)}$. The probability by which a vertex $v'$ with random position and weight covers $J$ is thus

$$\begin{aligned}
&\mathbb{P}[v' \in A] = \mathbb{P}[v' \text{ covers } J] \\
&= \left(2\frac{aw_e}{W}\right)^{\beta-1} w_{\min}^{\beta-1} \frac{1}{\beta-2} \left((|J|)^{-(\beta-2)} - (1+|J|)^{-(\beta-2)}\right) \\
&\leq \left(2\frac{aw_e}{W}\right)^{\beta-1} w_{\min}^{\beta-1} \frac{1}{\beta-2} |J|^{-(\beta-2)}.
\end{aligned}$$

With $W \in \Theta(n)$ and $|J| \in \Theta\left(n^{-(1-\varepsilon)}\right)$ this term is asymptotically

$$\begin{aligned}
&\in \Theta\left(n^{-(\beta-1)} n^{(1-\varepsilon)(\beta-2)}\right) \\
&= \Theta\left(n^{\varepsilon(2-\beta)-1}\right).
\end{aligned}$$

As we assume $\beta > 2$, the exponent $\varepsilon(2-\beta) - 1$ is strictly smaller than $-1$ for all $\varepsilon \in (0; 1)$. As a result, a random vertex $v'$ lies with more than high probability not in $A$. With a union bound over all vertices, we find that there is a.a.s. no vertex $v'$ that intersects the interval $J$ with radius $r := m^{-(1-\varepsilon)}$ around $x_v$.

$$\begin{aligned}
&\mathbb{P}\left[\bigcap_{v' \in V \setminus \{v\}} \neg[v' \in A]\right] \\
&= 1 - \mathbb{P}\left[\bigcup_{v' \in V \setminus \{v\}} [v' \in A]\right] \\
&\geq 1 - n\mathbb{P}[v' \in A] \\
&= 1 - \mathcal{O}\left(n^{-\varepsilon(\beta-2)}\right)
\end{aligned}$$

■

With Lemmas 4.3 and 4.4 we are able to prove that the reduction rules create a.a.s. at least one gap during their application in $H$. Together with Lemma 4.2 this results in the following Theorem 4.5.

**Theorem 4.5:** *The reduction rules reduce every Hyper-GIRG, that was generated with the threshold variant, to a trivial kernel a.a.s.*

*Proof.* Let $H = (V, E)$ be a Hyper-GIRG generated with the threshold variant, $H_1 = (V_1, E_1)$ the hypergraph $H$ after one round of the vertex domination rule, and $H_2$ the hypergraph $H_1$ after an additional round of the hyperedge domination rule. Lemma 4.2 shows that the reduction rules reduce $H$ to a trivial kernel if there is at least one gap before or during the execution of the reduction rules. In this proof, we show that there is a.a.s. a gap around the heaviest vertex of $H$ in $H_2$. For that, let $v_{\max} = (0, w_{\max})$ be the vertex with the maximum weight among all vertices in $H$. To bound the gap probability for $v_{\max}$ (and thus also for $H$), we use Lemmas 4.3 and 4.4 in the following way

$$\mathbb{P}\left[H \text{ has gap}\right] \geq \mathbb{P}\left[v_{\max} \text{ has gap}\right] \geq \mathbb{P}\left[\left[\exists e\colon x_e \in I_{\max}\right] \cap \bigcap_{v' \in V \setminus v_{\max}} \left[v' \notin A_{\max}\right]\right]$$

where $I_{\max}$ and $A_{\max}$ are defined as in the proof of Lemma 4.4 with respect to $v_{\max}$. Since hyperedges and vertices are generated independently, this is equal to

$$= \mathbb{P}\left[\exists e\colon x_e \in I_{\max}\right] \cdot \mathbb{P}\left[\bigcap_{v' \in V \setminus v_{\max}} \left[v' \notin A_{\max}\right]\right].$$

The two probabilities are given in Lemmas 4.3 and 4.4 respectively for a vertex with a weight in $\Omega\left(n^{\varepsilon}\right)$. In this proof, we set $\varepsilon = \frac{1}{\beta}$ and condition both events to the event that there is indeed a vertex with a weight in $\Omega\left(n^{\frac{1}{\beta}}\right)$. This yields the lower bound

$$\begin{aligned} &\geq \mathbb{P}\left[\exists e\colon x_e \in I \mid w_{\max} \geq n^{\frac{1}{\beta}}\right] \mathbb{P}\left[w_{\max} \geq n^{\frac{1}{\beta}}\right] \\ &\quad \cdot \mathbb{P}\left[\bigcap_{v' \in V \setminus v_{\max}} \left[v' \notin A_{\max}\right] \,\middle|\, w_{\max} \geq n^{\frac{1}{\beta}}\right] \mathbb{P}\left[w_{\max} \geq n^{\frac{1}{\beta}}\right]. \end{aligned} \tag{4.1}$$

In the following, we show that each of the factors tends to 1 for $n \to \infty$. According to Lemmas 4.3 and 4.4, the two conditional probabilities tend to 1, as

$$\mathbb{P}\left[\exists e\colon x_e \in I_{\max} \mid w_{\max} \geq n^{\frac{1}{\beta}}\right] \geq 1 - e^{-cn^{\frac{1}{\beta}}} \qquad (\textit{Lemma } 4.3)$$

$$\mathbb{P}\left[\bigcap_{v' \in V \setminus v_{\max}} \left[v' \notin A_{\max}\right] \,\middle|\, w_{\max} \geq n^{\frac{1}{\beta}}\right] \in 1 - \mathcal{O}\left(n^{-\frac{\beta-2}{\beta}}\right). \qquad (\textit{Lemma } 4.4)$$

Finally, it must be shown, that also $\mathbb{P}\left[w_{\max} \geq n^{\frac{1}{\beta}}\right]$ tends to 1 for $n \to \infty$. The maximum weight is above a threshold if and only if not all weights are below the threshold. Therefore, it holds that

$$\mathbb{P}\left[w_{\max} \geq n^{\frac{1}{\beta}}\right] = 1 - \mathbb{P}\left[\bigcap_{v \in V} \left[w_v < n^{\frac{1}{\beta}}\right]\right].$$

The probability $\mathbb{P}\left[w_v < n^{\frac{1}{\beta}}\right]$, that a single vertex with a randomly drawn weight is below a threshold, can be expressed with the CDF $F_\beta$ of the power-law distribution. As, additionally, all weights are drawn independently, we get

$$\begin{aligned} &= 1 - F(n^{\frac{1}{\beta}})^n. \\ &= 1 - \left(1 - \Theta\left(n^{\frac{1}{\beta}}\right)^{1-\beta}\right)^n \\ &= 1 - \left(1 - \Theta\left(\frac{1}{n^{\frac{\beta-1}{\beta}}}\right)\right)^n. \end{aligned}$$

According to Lemma 2.7, the latter term tends to 0, as $\frac{\beta-1}{\beta} < 1$ for all $\beta > 2$. As a result, all factors in Equation (4.1) tend to 1. Therefore, the probability that the reduction rules create a gap in $H$ tends to 1 for $n \to \infty$. ■

Note that the probability calculated in the proof of Theorem 4.5 decreases with increasing $\beta$. Therefore, it can happen that the reduction rules do not create any gap in $H$, especially for small $n$ and high $\beta$. In this case, we show a way in Section 4.3 to create gaps deterministically while retaining the ability to find an optimal solution in it.

## 4.3 Creating Gaps via Branching-Rule

In this section, we again consider the more general CA$^*$-hypergraph. However, the results that we find for these hypergraphs still hold for Hyper-GIRGs generated with the threshold variant due to Lemma 4.1. As usual in this setting, let $H = (V, E)$ be a CA$^*$-hypergraph with cyclic edge order $\prec_E^\circ \coloneqq \langle e_1, \dots, e_m \rangle^\circ$. The goal of this section is to describe a branching rule that creates a gap in sperner CA-hypergraphs. To be able to apply this rule to the given CA$^*$-hypergraph $H$, we first describe how the reduction rules turn $H$ into a sperner CA-hypergraph. We do this in two steps. In the first step, we show that one round of the vertex domination rule turns $H$ into a (primal) CA-hypergraph $H_1 = (V_1, E_1)$. The second step is to show that an additional round of the hyperedge domination rule makes the CA-hypergraph $H_1$ sperner. The following Lemma 4.6 shows step one, i.e., the transformation of a CA$^*$-hypergraph into a CA-hypergraph via the vertex domination rule.

**Lemma 4.6:** *After one round of the vertex domination rule, every CA$^*$-hypergraph is also a (primal) CA-hypergraph.*

*Proof.* Let $H = (V, E)$ be a CA$^*$-hypergraph after one round of the vertex domination rule. For the proof, we need to find a cyclic order of the vertices $\prec_V^\circ = \langle v_1, \dots, v_n \rangle^\circ$ such that all hyperedges contain a consecutive set of vertices in that order. Since $H$ is a CA$^*$-hypergraph there is already a cyclic order $\prec_E^\circ \coloneqq \langle e_1, \dots, e_m \rangle^\circ$ on the hyperedges. Based on that, we order the vertices $v \in V$ according to their left end $L(v)$, that is the hyperedge $e \in E(v)$ that does not have a direct predecessor in $E(v)$. Formally, $u \prec_V v$ holds, if the interval of $u$ contains the left end of $v$.

It remains to show that every hyperedge contains a set of consecutive vertices in this cyclic order $\prec_V^\circ$. For that, assume that there is an $e \in E$ with vertices $u, v \in e$ such that the interval from $u$ to $v$ as well as the interval from $v$ to $u$ in $\prec_V^\circ$ contains a vertex that is not in $e$. Let $w_u \in V$ be that vertex in the interval from $u$ to $v$ and $w_v \in V$ be that vertex in the interval from $v$ to $u$. This situation is shown in the left sketch of Figure 4.5. Since $H$ is a CA$^*$-hypergraph, $E(u)$ and $E(v)$ are intervals on $\prec_E^\circ$. We know that $u$ and $v$ share at least one edge, namely $e$. Therefore, their intervals overlap. Without loss of generality, let $L(u)$ be the left end of the interval $E(u) \cup E(v)$ and $R(v)$ the right end. This situation is shown on the right side of Figure 4.5. As $u, v \in e$, $e$ is in the interval $E(u) \cap E(v)$, i.e., the interval from $L(v)$ to $R(u)$. We now know, that in the joint interval $E(u) \cup E(v)$ there is the following ordering:

$$L(u) \prec_E^\circ L(w_u) \prec_E^\circ L(v) \prec_E^\circ e \prec_E^\circ R(u) \prec_E^\circ R(v). \tag{4.2}$$

Additionally, we know that $w_u \notin e$. Therefore $E(w_u)$ ends before $e$, i.e., $R(w_u) \prec_E^\circ e$. As a consequence of Equation (4.2), $R(w_u) \prec R(u)$ holds. This is a contradiction because $u$ dominates $w_u$, which is impossible after a round of the vertex domination rule.

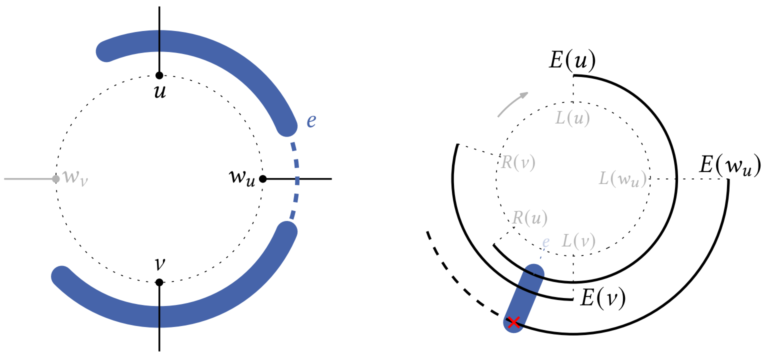


**Figure 4.5:** Proof sketch for Lemma 4.6. A (primal) CA-hypergraph with the vertices $u, w_u, v, w_v$ and one hyperedge that skips the vertex $w_u$ (left), and the same situation in the dual graph (right).

As a result, all hyperedges contain a set of consecutive vertices in $<_V^\circ$. Therefore, $H$ is a CA-hypergraph. ∎

As a result of Lemma 4.6, we know that $H_1$ is a (primal) CA-hypergraph. For the rest of this section let $<_V^\circ = \langle v_1, \dots, v_n \rangle^\circ$ be its cyclic vertex order, which is described in the proof. An additional round of the hyperedge domination rule makes $H_1$ sperner, since all hyperedges, that contain other hyperedges entirely, are dominated by those and thus eliminated in this step. Let $H_2 = (V_2, E_2)$ be the resulting sperner CA-hypergraph resulting from this step. Note that, $H_2$ is still a CA$^*$-hypergraph since the round of vertex domination only removed some vertices, which does not make the cyclic hyperedge order of the original $H$ invalid.

However, for the description of the following branching rule, we only need that $H_2$ is a sperner CA-hypergraph. The idea of this branching rule is to branch over all necessary hitting sets $\mathcal{S}_e$ of an arbitrary hyperedge $e = \{v_1, \dots, v_k\} \in E_2$. For the following notations, we shift the indices in the cyclic hyperedges order $<_V^\circ$ such that the left end of $e$ is $v_1$ and the right end is $v_k$. With that, we define $S_0 \coloneqq \{L(e), R(e)\}$ and $S_i \coloneqq \{v_i\}$ (for $i \in [k]$) to be elements of $\mathcal{S}_e$, i.e., all necessary hitting set candidates for $e$. For each $i \in \{0, \dots, k\}$ let $C_i$ be the child-hypergraph from the candidate $S_i$, that is the hypergraph $H_2$ without the vertices of $e$ and all hyperedges covered by the vertices in $S_i$. We prove in Lemma 4.7 that the branching rule is safe and the set $\mathcal{S}_e$ is complete. Further, we show in Lemma 4.8 that every child instance $C_i$ with $i \in \{0, \dots, k\}$ has a gap that can be used by the algorithm described in Lemma 4.2. Due to the last statement, the above branching rule earns its name *cut-branching-rule*. An example of the application of the cut-branching-rule is illustrated in Figure 4.6.

**Lemma 4.7:** *For each $e \in E$, $H_2$ has a minimum hitting set of size $h \in \mathbb{N}$ if and only if one of the graphs $C_i$ (for $i \in \{0, \dots, k\}$) has a minimum hitting set of size $h - |S_i|$.*

*Proof.* First, we prove the following claim: For every sperner CA-hypergraph $H = (V, E)$ and every hyperedge $e \in E$, there is a minimum hitting set $S$ of $H$, such that $S$ either contains exclusively one vertex in $e$ or both ends $L(e)$ and $R(e)$ of $e$. In the following, we describe a transformation of an arbitrary minimum hitting set $S_*$ of $H$ into a minimum hitting set $S'$ of $H$ with that property. For that, we make a case distinction over the size of $S_e^* \coloneqq S_* \cap e$. Since $e$ is covered by every hitting set, $S_e^*$ is not empty. If $|S_e^*| = 1$, $S_*$ contains exclusively one vertex in $e$ and thus already satisfies the condition for $S'$. Otherwise, if $|S_e^*| \geq 2$, then $S_* \setminus S_e^* \cup \{L(e), R(e)\}$ is also a minimum hitting set. Note that one can conclude from this,

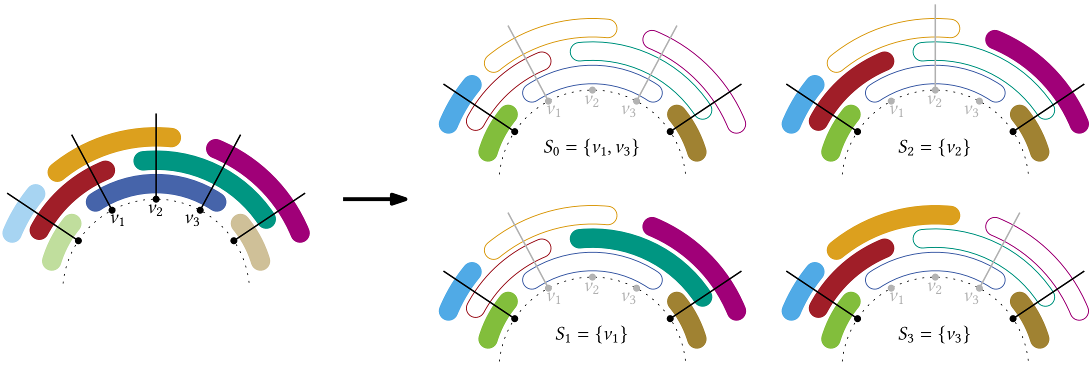


**Figure 4.6:** An example for the cut-branching-rule. The sub-hypergraph, which is shown on the left side, is the input hypergraph. As $H_2$ is a CA-hypergraph, there is a cyclic vertex order $<^{\circ}_{V}$ in which the vertices are depicted. The hyperedges of $H_2$ and its children (shown on the right) are displayed as arcs and represent their intervals on $<^{\circ}_{V}$. Hyperedges that are not important in $H_2$ are slightly lighter. The blue hyperedge is chosen for the cut-branching-rule, which can be seen from the indices that start with 1 at the left end of the blue hyperedge. In the child-hypergraphs, the hyperedges that were removed are outlined.

that every hyperedge and every hitting set of $H$ share at most two vertices. To prove that, we assume that $S_* \setminus S^*_e \cup \{L(e), R(e)\}$ is not a hitting set (since $|S^*_e| \geq |\{L(e), R(e)\}|$ this is equivalent to the assumption that $S_* \setminus S^*_e \cup \{L(e), R(e)\}$ is not a minimum hitting set). Then, $S^*_e$ has to cover some hyperedge $e' \in E$ that is not covered by $\{L(e), R(e)\}$. This is only possible, if $e' \subseteq e$, which is a contradiction to the assumption that $H$ is sperner. Therefore, $S_* \setminus S^*_e \cup \{L(e), R(e)\}$ is a hitting set of $H$, which proves the claim.

With this, we can prove the lemma. For the "$\Rightarrow$"-direction, let $S$ be a minimum hitting set of $H_2$ and $h \coloneqq |S|$. Because of the claim, we can assume that $S \cap e$ is one of the alternatives $\{L(e), R(e)\}, \{v_1\}, \dots, \{v_k\}$. Let $l \in \{0, \dots, k\}$ be the index such that $S \cap e = S_l$. We show that $S \setminus S_l$ is a minimum hitting set for the child-hypergraph $C_l$. Firstly, $S \setminus S_l$ is a hitting set of $C_l$, because $C_l$ contains exactly those hyperedges of $H_2$ that are not covered by $S_l$, and $S$ covers all hyperedges of $H_2$. Secondly, $S \setminus S_l$ is minimum, because every smaller hitting set of $C_l$ would induce a smaller hitting set than $S$ for $H_2$.

For the "$\Leftarrow$"-direction, let $C_*$ be one of the child-instances of $H_2$ that induces a minimum hitting set for $H_2$, i.e., the union of $S_*$, a minimum hitting set of $C_*$, and $S^*_e$, the respective hitting set for $e$, has minimum size among all child hypergraphs of $H_2$. This union $S_{\min} \coloneqq S_* \cup S^*_e$ is a hitting set for $H_2$ because all hyperedges that are not covered by $S^*_e$ are covered by $S_*$ by definition of $S_*$. We show that this union is also a minimum hitting set for $H_2$. For that, we assume that there is a smaller hitting set $S'_{\min}$ for $H_2$ with $S'_e \coloneqq e \cap S'_{\min}$ and $S' \coloneqq S_{\min} \setminus S'_e$. Because of the claim, we can assume that $S'_e$ is one of the alternatives $\{L(e), R(e)\}, \{v_1\}, \dots, \{v_k\}$. Thus, a hitting set with equal size as $S'_{\min}$ would have been found in the respective child hypergraph of $S'_e$. As a result, the union $S_{\min} \coloneqq S_* \cup S^*_e$ would not be minimal among the child hypergraphs of $H_2$, which is a contradiction to the choice of $C_*$. ∎

As a result of Lemma 4.7, we know that the set $\mathcal{S}_e$ of hitting set candidates for the hyperedge $e$ is complete. Additionally, we know that applying the branching rule described above retains the ability to find a minimum hitting set for the original hypergraph $H_2$ (and thus also for $H$).

To solve the resulting child instances $C_i$ (with $i \in \{0, \dots, k\}$), one can use the algorithm described in Lemma 4.2. For this to run correctly, we show that each of the child-hypergraphs has at least one gap. In fact, we show a slightly stronger statement in the following Lemma 4.8.

**Lemma 4.8:** *Let $H = (V, E)$ be a CA*-hypergraph after one round of the vertex domination rule, and $e \in E$ an arbitrary hyperedge in $H$. Then, removing all vertices in $e$ and all hyperedges covered by an arbitrary subset $s \in 2^e \setminus \emptyset$ results in a CA*-hypergraph with a gap around $e$.*

*Proof.* Let $e \in E$ and $s \in 2^e \setminus \emptyset$ be fixed, and let $H'$ be the hypergraph resulting from the removal of the vertices in $e$ and all hyperedges covered by the vertices in $s$. Further, let $\prec_E^\circ$ be the cyclic hyperedge order of $H$. As $H'$ is just a sub-hypergraph of $H$, $\prec_E^\circ$ is also valid for $H'$. Now assume that there is still a vertex $v \in V$ in $H'$, that is connected to the left and right neighbor $e_l, e_r \in E$ of $e$ with respect to $\prec_E^\circ$. Since $H$ is a CA$^*$-hypergraph, $E(v)$ is an interval on $\prec_E^\circ$. As $E(v)$ contains both direct neighbors of $e$ by definition of $v$, $e$ has to be in $E(v)$ too. This is a contradiction to the assumption that all vertices of $e$ were removed in $H'$. ∎

The work of this chapter can be concluded in an algorithm that solves HittingSet on CA$^*$-hypergraph in polynomial time using only the reduction rules and one application of the described cut-branching-rule.[2] This algorithm has no exponential runtime as the set of necessary hitting set candidates $\mathcal{S}_e$ for cut-branching-rule is only linear in the size of the chosen hyperedge. For Hyper-GIRGs from the threshold variant, a subclass of CA$^*$-hypergraph (see Lemma 4.1), this algorithm is even faster in expectation, since they are likely to either create gaps during the application of the reduction rules (see Section 4.2) or have small hyperedges afterward. In fact, Hyper-GIRGs have a constant expected hyperedge size (see Lemma 3.5), which is much better than the worst-case linear size in general CA$^*$-hypergraphs.

[2] Note that there is already a polynomial algorithm due to Elbassioni and Rauf [ER10] for EdgeCover on CA-hypergraphs, which is the dualized equivalent to HittingSet on CA$^*$-hypergraphs.

# 5 The First Round of Vertex Domination

In the first round of the vertex domination rule, there is a number of vertices $V_{\mathrm{dom}}$ that is dominated, i.e., eliminated by this rule. In part (1) of Corollary 3.2 we found, that not even vertices with constant weight or vertices with only one incidence are dominated with high probability by another vertex. In fact, they are dominated by only constant probability. As there are linearly many vertices with constant weight in Hyper-GIRGs independent of the degree of heterogeneity, there are thus also expected linearly many vertices that are dominated, i.e., $\mathbb{E}\left[|V_{\mathrm{dom}}|\right] \in \Theta\left(n\right)$. However, the constant in the Big-O notation highly depends on the parameters of the Hyper-GIRG, especially on the power-law exponent $\beta$ (heterogeneity) and the temperature $T$ (locality). This constant can be seen as the fraction of vertices that are dominated which we denote as $F_{\mathrm{dom}}$. The goal of this section is to find a lower bound to $\mathbb{E}\left[F_{\mathrm{dom}}\right]$, i.e., the expected fraction of vertices that are dominated in Hyper-GIRGs and are thus eliminated in the first round of the vertex domination rule. For that, we first focus on the threshold variant in Section 5.1 and then extend the result to the binomial variant in Section 5.2.

Both calculations are based on the same bound that we derive in the following. For that, let $\mathbf{1}_v$ be the indicator variable for the event that a vertex $v \in V$ is dominated in the generated hypergraph. Then, it holds that $F_{\mathrm{dom}} = \frac{1}{n}\sum_{v\in V}\mathbf{1}_v$. Therefore, the expected value of $F_{\mathrm{dom}}$ is exactly the probability for $v$ to be dominated if both the position and the weight of $v$ are randomly drawn from the respective distributions:

$$\mathbb{E}\left[\frac{\sum_{v\in V}\mathbf{1}_v}{n}\right] = \frac{1}{n}\sum_{v\in V}\mathbb{E}\left[\mathbf{1}_v\right] = \mathbb{P}\left[\mathbf{1}_v = 1\right].$$

As a result, we only need to focus on one vertex $v = (x_v = 0, W_v)$ with a random variable $W_v \sim \mathrm{PL}(\beta, w_{\min})$ as its weight. Using the PDF $f_{W_v}$ of $W_v$ (see Equation (2.11)), we can integrate over all possible weights of $v$ using the law of total probability, such that

$$\begin{aligned}&\mathbb{P}\left[\mathbf{1}_v = 1\right]\\ =& \int_{w_{\min}}^{\infty}\mathbb{P}\left[\mathbf{1}_v = 1 \mid W_v = w_v\right] f_{W_v}(w_v)\ \mathrm{d}w_v.\end{aligned} \tag{5.1}$$

Further, let $D$ be a random variable representing the diameter of $v$, and $f_D$ its (unknown) PDF. Then, we can again integrate over all possible diameters of $v$ to get

$$= \int_{w_{\min}}^{\infty}\int_{0}^{1}\mathbb{P}\left[\mathbf{1}_v = 1 \mid W_v = w_v, D = d\right] f_D(d \mid W_v = w_v) f_{W_v}(w_v)\ \mathrm{d}d\ \mathrm{d}w_v. \tag{5.2}$$

Based on Equation (5.2) we find lower bounds to $\mathbb{E}\left[F_{\mathrm{dom}}\right]$ for the threshold and the binomial variant in the following two sections. In both cases, we use simplifications to avoid the unknown PDF of the diameter $f_D$ and its second integral.

## 5.1 In the Threshold Variant

In this section, we use the fact that the diameter of a vertex cannot exceed $2m_r(v)$ in the threshold variant due to the step-function-like incidence probability (from Equation (2.14)). Since the probability for a vertex to be dominated decreases with increasing diameter, we therefore find the lower bound

$$\mathbb{E}\left[F_{\text{dom}}\right] \geq \int_{w_{\min}}^{\infty} \mathbb{P}\left[v \text{ is dom.} \mid W_v = w_v, D = 2m_r(v)\right] f_{W_v}(w_v)\ \mathrm{d}w_v. \tag{5.3}$$

Based on that bound, we derive the following Lemma 5.1.

**Lemma 5.1:** *In every Hyper-GIRG in the threshold variant, the expected fraction of vertices that are eliminated by the first round of the vertex domination rule is at least*

$$\mathbb{E}\left[F_{dom}\right] \geq F_\beta\left(w_{min} + \frac{W}{2aw_e}\right) - \frac{W_c(\beta-2)}{2aw_e}\left(1 - \exp\left(-\frac{2aw_e}{W_c(\beta-2)}\right)\right)$$

*where $F_\beta$ is the CDF of the power-law distribution that the vertex weights follow.*

*Proof.* We use Equation (5.3) to prove this bound. For that, let $v = (x_v, W_v)$ be a vertex with power-law distributed weight $W_v \sim \mathrm{PL}(\beta, w_{\min})$ and diameter $D = 2m_r(v)$. First, we find a lower bound to the probability $\mathbb{P}\left[v \text{ is dom.}\right]$ using part (1) of Corollary 3.2, which yields

$$\begin{aligned}\mathbb{P}\left[v \text{ dom. cold}\right] &= 1 - \Big(1 - \mathbb{P}\left[\mathrm{NA}(v) \subseteq \mathrm{NA}_{\text{cold}}(u)\right]\Big)^n \\ &\geq 1 - \exp\Big(-n\mathbb{P}\left[\mathrm{NA}(v) \subseteq \mathrm{NA}_{\text{cold}}(u)\right]\Big).\end{aligned}$$

where $u$ is a vertex with a randomly drawn position and weight. As we make the assumption $\mathrm{NA}(v) \subseteq \mathrm{NA}_{\text{cold}}(v)$, we can use the first equation from Lemma 3.1 using $|I| = 2m_r(v)$ and writing $c_\beta \coloneqq 2\frac{aw_e}{W_c}w_{\min}^{\beta-1}\frac{1}{\beta-2}$ to get

$$\begin{aligned}\mathbb{P}\left[\mathrm{NA}(v) \subseteq \mathrm{NA}_{\text{cold}}(u)\right] &\geq \mathbb{P}\left[\mathrm{NA}_{\text{cold}}(v) \subseteq \mathrm{NA}_{\text{cold}}(u)\right] \\ &= 2\frac{aw_e}{W}w_{\min}^{\beta-1}\frac{1}{\beta-2}\left(\left(\frac{W}{aw_e}\frac{|I|}{2}\right)^{2-\beta} - \left(\frac{W}{2aw_e} + \frac{W}{aw_e}\frac{|I|}{2}\right)^{2-\beta}\right) \\ &= \frac{c_\beta}{n}\left(w_v^{2-\beta} - \left(w_v + \frac{W}{2aw_e}\right)^{2-\beta}\right) \\ &\geq \frac{c_\beta}{n}\left(w_v^{2-\beta} - \left(w_{\min} + \frac{W}{2aw_e}\right)^{2-\beta}\right).\end{aligned}$$

This lower bound of the probability $\mathbb{P}\left[\mathrm{NA}(v) \subseteq \mathrm{NA}_{\text{cold}}(u)\right]$ is negative for $w_v \geq w_{\min} + \frac{W}{2aw_e}$. Since every probability is non-negative, we will use the following lower bound instead

$$\geq \frac{c_\beta}{n} \cdot \max\left\{w_v^{2-\beta} - \left(w_{\min} + \frac{W}{2aw_e}\right)^{2-\beta}, 0\right\}.$$

Therefore, we get for a vertex with fixed weight $w_v$ and diameter $2m_r(v)$ that

$$\mathbb{P}\left[v \text{ dom. cold}\right] \geq \frac{c_\beta}{n} \cdot \max\left\{w_v^{2-\beta} - \left(w_{\min} + \frac{W}{2aw_e}\right)^{2-\beta}, 0\right\}. \tag{5.4}$$

Putting all things together yields

$$\begin{aligned}
\mathbb{E}\left[F_{\mathrm{dom}}\right] &= \int_{w_{\min}}^{\infty}\int_0^1 \mathbb{P}\left[v \text{ is dom.} \mid W_v = w_v, D = d\right] f_D(d) f_{W_v}(w_v)\ \mathrm{d}d\ \mathrm{d}w_v \\
&\geq \int_{w_{\min}}^{\infty} \mathbb{P}\left[v \text{ dom. cold} \mid W_v = w_v, D = 2m_r(v)\right] f_{W_v}(w_v)\ \mathrm{d}w_v \\
&\geq \int_{w_{\min}}^{\infty} \left(1 - \exp\Big( - n\mathbb{P}\left[\mathrm{NA}_{\mathrm{cold}}(v) \subseteq \mathrm{NA}_{\mathrm{cold}}(u)\right]\Big)\right) \frac{\beta - 1}{w_{\min}^{1-\beta}} w_v^{-\beta}\ \mathrm{d}w_v \\
&= 1 - \int_{w_{\min}}^{\infty} \exp\Big( - n\mathbb{P}\left[\mathrm{NA}_{\mathrm{cold}}(v) \subseteq \mathrm{NA}_{\mathrm{cold}}(u)\right]\Big) \frac{\beta - 1}{w_{\min}^{1-\beta}} w_v^{-\beta}\ \mathrm{d}w_v \\
&\geq 1 - \int_{w_{\min}}^{\infty} \exp\left(-c_\beta \cdot \max\left\{ w_v^{2-\beta} - \left(w_{\min} + \frac{W}{2aw_e}\right)^{2-\beta}, 0\right\}\right) \frac{\beta - 1}{w_{\min}^{1-\beta}} w_v^{-\beta}\ \mathrm{d}w_v.
\end{aligned}$$

For the integrand, we find an upper bound that is easy to integrate by decreasing the exponents from $2 - \beta$ to $1 - \beta$. Thus, the new integral is

$$\begin{aligned}
&\geq 1 - \int_{w_{\min}}^{\infty} \exp\left(-c_\beta \cdot \max\left\{ w_v^{1-\beta} - \left(w_{\min} + \frac{W}{2aw_e}\right)^{1-\beta}, 0\right\}\right) \frac{\beta - 1}{w_{\min}^{1-\beta}} w_v^{-\beta}\ \mathrm{d}w_v \\
&= 1 - \left[\frac{\beta - 1}{w_{\min}^{1-\beta}} \exp\left(c_\beta \left(w_{\min} + \frac{W}{2aw_e}\right)^{1-\beta}\right)\right. \\
&\qquad \left. \cdot \int_{w_{\min}}^{w_{\min} + \frac{W}{2aw_e}} \exp\Big( - c_\beta w_v^{1-\beta}\Big) w_v^{-\beta}\ \mathrm{d}w_v \ + \ \overline{F_\beta}\left(w_{\min} + \frac{W}{2aw_e}\right)\right],
\end{aligned}$$

where $F_\beta$ is the CDF of $\mathrm{PL}(\beta, w_{\min})$. The solution of the remaining integral is

$$\int_{w_{\min}}^{w_{\min} + \frac{W}{2aw_e}} \exp\Big( - c_\beta w_v^{1-\beta}\Big) w_v^{-\beta}\ \mathrm{d}w_v = \frac{1}{c_\beta(\beta - 1)} \left[e^{-x}\right]_{c_\beta w_{\min}^{1-\beta}}^{c_\beta\left(w_{\min} + \frac{W}{2aw_e}\right)^{1-\beta}}. \tag{5.5}$$

As a result, we get the following lower bound for $\mathbb{E}\left[F_{\mathrm{dom}}\right]$

$$\begin{aligned}
&= F_\beta\left(w_{\min} + \frac{W}{2aw_e}\right) - \frac{1}{c_\beta w_{\min}^{1-\beta}}\left(1 - \exp\left(-c_\beta\left(w_{\min}^{1-\beta} - \left(w_{\min} + \frac{W}{2aw_e}\right)^{1-\beta}\right)\right)\right) \\
&\geq F_\beta\left(w_{\min} + \frac{W}{2aw_e}\right) - \frac{1}{c_\beta w_{\min}^{1-\beta}}\left(1 - \exp\left(-c_\beta w_{\min}^{1-\beta}\right)\right).
\end{aligned}$$

∎

The lower bound for $\mathbb{E}\left[F_{\mathrm{dom}}\right]$ from Lemma 5.1 is equal to

$$1 - \frac{W_c(\beta - 2)}{2aw_e}\left(1 - \exp\left(-\frac{2aw_e}{W_c(\beta - 2)}\right)\right)$$

with an additive error in $\mathcal{O}\left(n^{1-\beta}\right)$. This error comes from the fact that the probability for a vertex to have a weight linear in the combined weight of $n$ vertices is very small.

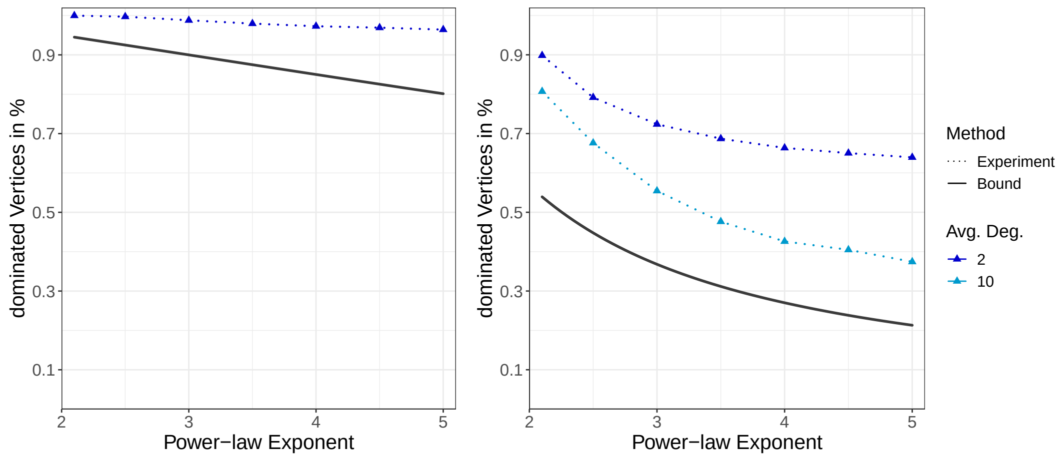


**Figure 5.1:** The diagrams compare the $F_{\text{dom}}$ from different experiments (in dotted lines) to the lower bound of the expected value from Lemma 5.1 for different power-law exponents. The bound uses the expected value for $W_c$. The left diagram has an $a$ of about 10, while Hyper-GIRGs for the right diagram have an $a$ of about 1.

The diagrams in Figure 5.1 show how tight this bound for $\mathbb{E}[F_{\text{dom}}]$ is in different settings and for different power-law exponents. The left diagram shows the setting presented in the main experiment of [BFFS19] about public transportation systems. The hyperedge-vertex-ration is set to $\delta_{EV} = 0.1$, and the scaling parameter is set to achieve a desired average degree of $\overline{\delta} = 2$. The right diagram uses $\overline{\delta} = 2$ with $\delta_{EV} = 1$ and $\overline{\delta} = 10$ with $\delta_{EV} = 0.2$. Both settings result in similar $a \approx 1$ and thus in a similar bound. Further, we used $w_{\min} = w_e = 1$ and $|E| = 10^4$ for all generated Hyper-GIRGs. For every real value in the diagrams we generated ten Hyper-GIRGs, and depicted the mean of the resulting $F_{\text{dom}}$. Especially for larger $\beta$ the single results do not deviate much from the respected average. As the bound from Lemma 5.1 needs a value for the average vertex weight $W_c$, we chose the expected value $\frac{\beta-1}{\beta-2} w_{\min}$.

The left diagram shows that in Hyper-GIRGs, similar to many public transportation systems, the bound predicts that a significant fraction of vertices are dominated, which coincides with the experiments. For $\beta = 2.1$ at least 94% are expected to be eliminated in the first round of the vertex domination round, and at least 80% even for the less heterogeneous Hyper-GIRGs. At $\beta = 5$, the bound is off by about 15%. However, when we consider the fraction of vertices that remain after the round of vertex domination, i.e., $1 - F_{\text{dom}}$, the bound is about five times worse than the experiments imply.

For the right diagram, we generated Hyper-GIRGs with different values for $\overline{\delta}$ and $\delta_{EV}$, which nevertheless have similar scaling parameters $a$. Therefore, the sizes of the cold areas are equal for vertices with equal weights (up to the difference in the generated $W_c$). The difference is the number of hyperedges that are in those cold areas. Note that, increasing solely the average degree would also increase the number of hyperedge in the cold areas. However, the hyperedge-vertex-ratio has to be changed along with it, to ensure similar value for $a$ and thus similar sizes for the cold areas. This is due to the fact that $a \approx \frac{\overline{\delta}}{\delta_{EV}} \frac{1}{2w_e}$ for $T = 0$ and $W_c \approx \mathbb{E}[W_c]$ (see Lemma 3.5 part (3)). The diagram shows that the bound is tighter the more hyperedges are in the cold area of equally weighted vertices. This is because the left and the right end tend to be closer to the borders of the cold area, which we assumed to be the exact left and right end in the proof of the bound from Lemma 5.1.

## 5.2 In the Binomial Variant

Now, we generalize the idea from Lemma 5.1 to find a lower bound to $\mathbb{E}\left[F_{\text{dom}}\right]$ for $T > 0$. To do so, we look at the vertices $v$ that have their neighborhood area inside their cold area,

$$\mathrm{NA}(v) \subseteq \mathrm{NA}_{\text{cold}}(v),$$

despite the temperature in the model. We call such vertices *nice*. In the following Lemma 5.2 we first find that, at least for low temperatures, the fraction of nice vertices is not negligible. Additionally, we use a lower bound of the fraction of nice vertices to find a lower bound for $\mathbb{E}\left[F_{\text{dom}}\right]$ with $T > 0$.

**Lemma 5.2:** *In every Hyper-GIRG in the binomial variant, the expected fraction of vertices that are eliminated by the first round of the vertex domination rule is at least*

$$\begin{aligned}\mathbb{E}\left[F_{dom}\right] \geq\ & \exp\left(-c_T w_{min}\right)\left(\frac{c_T}{\beta-2}\cdot\left[x^{2-\beta}\right]_{w_{min}}^{\frac{1}{c_T}+w_{min}} - \frac{(1+c_T w_{min})}{\beta-1}\left[x^{1-\beta}\right]_{w_{min}}^{\frac{1}{c_T}+w_{min}}\right)\\ & - \exp\left(c_\beta\left(w_{min}+\frac{W}{2aw_e}\right)^{2-\beta} - c_T w_{min}\right)\cdot\frac{1}{c_\beta(\beta-1)}\left[e^{-x}\right]_{c_\beta w_{min}^{1-\beta}}^{c_\beta\left(w_{min}+\frac{W}{2aw_e}\right)^{1-\beta}}\\ & - \frac{0.542}{m}\end{aligned}$$

*where* $c_\beta \coloneqq 2\frac{aw_e}{W_c}w_{min}^{\beta-1}\frac{1}{\beta-2}$ *and* $c_T \coloneqq 2\delta_{EV}a\frac{w_e}{W_c}\frac{T}{1-T}$.

*Proof.* We use the same approach as in Lemma 5.1. There, we had two assumptions that always hold in the threshold variant. The first one is that the size of the cold area is an upper bound to the diameter of a vertex. The second one is that a vertex $v$ is dominated by another vertex $u$ if and only if the cold area of $u$ covers the neighborhood area of $v$. We describe the latter scenario as a *cold domination* of $u$ on $v$. However, in the binomial variant, a vertex $u$ may dominate a vertex $v$ using a warm/liquid incidence. In the following, we rule those situations out and thus get the lower bound for Equation (5.1)

$$\mathbb{E}\left[F_{\text{dom}}\right] \geq \int_{w_{\min}}^{\infty} \mathbb{P}\left[v \text{ dom. cold} \mid W_v = w_v\right] f_{W_v}(w_v)\ \mathrm{d}w_v.$$

Now, we get a lower bound by only considering the nice vertices. Thus, the law of total probability yields

$$\geq \int_{w_{\min}}^{\infty} \mathbb{P}\left[v \text{ dom. cold} \mid v \text{ nice}, W_v = w_v\right]\mathbb{P}\left[v \text{ nice} \mid W_v = w_v\right] f_{W_v}(w_v)\ \mathrm{d}w_v.$$

For the first term $\mathbb{P}\left[v \text{ dom. cold} \mid v \text{ nice}, W_v = w_v\right]$, we can use the lower bound from Equation (5.4) derived in Lemma 5.1. This is possible since we (again) only consider nice vertices. Therefore, it holds that

$$\mathbb{P}\left[v \text{ dom. cold} \mid v \text{ nice}, W_v = w_v\right] \geq 1 - \exp\left(-c_\beta \max\left\{w_v^{2-\beta} - \left(w_{\min}+\frac{W}{2aw_e}\right)^{2-\beta}, 0\right\}\right)$$

where $c_\beta := 2\frac{aw_e}{W_c} w_{\min}^{\beta-1} \frac{1}{\beta-2}$. For the second term $\mathbb{P}[v \text{ nice} \mid W_v = w_v]$, we are interested in the probability that a vertex $v$ with fixed weight $w_v \geq w_{\min}$ is nice. This is equal to the probability that there are no incident hyperedges outside the cold area of $v$, i.e., $N^v_\leq(-m_r(v)) \cup N^v_\geq(m_r(v)) = \emptyset$. Since a hyperedge $e$ with uniformly random position is in $N^v_\leq(-m_r(v))$ and $N^v_\geq(m_r(v))$ with the same probability, it holds that:

$$\begin{aligned}
&\mathbb{P}[v \text{ nice} \mid W_v = w_v] \\
&= \left(1 - 2\mathbb{P}\left[e \in N^v_\leq(-m_r(v))\right]\right)^m .
\end{aligned}$$

The probability $\mathbb{P}\left[e \in N^v_\leq(-m_r(v))\right]$ is known from Lemma 3.4. Therefore, we get the following formula and its lower bound

$$\begin{aligned}
&= \left(1 - 2m_r(v)\frac{T}{1-T}\left(1 - (2m_r(v))^{\frac{1-T}{T}}\right)\right)^m \\
&\geq \left(1 - 2m_r(v)\frac{T}{1-T}\right)^m .
\end{aligned}$$

To get a lower bound for that, we use Lemma 2.8 with $a := 2m_r(v)\frac{T}{1-T}$ and $b := \delta_{EV}$:

$$\geq \exp(-ab) - \frac{0.542}{bn}.$$

In the following, we use $c_T := 2\delta_{EV} a \frac{w_e}{W_c}\frac{T}{1-T}$, such that the formula can be written as

$$= \exp(-c_T w_v) - \frac{0.542}{m}.$$

Plugging these two bounds in, yields

$$\begin{aligned}
&\mathbb{E}[F_{\text{dom}}] \\
&\geq \int_{w_{\min}}^{\infty} \left(1 - \exp\left(-c_\beta \max\left\{w_v^{2-\beta} - \left(w_{\min} + \frac{W}{2aw_e}\right)^{2-\beta}, 0\right\}\right)\right) \cdot \left(e^{-c_T w_v} - \frac{0.542}{m}\right) f_\beta(w_v)\,\mathrm{d}w_v \qquad (5.6) \\
&\geq \int_{w_{\min}}^{w_{\min}+\frac{W}{2aw_e}} \left(1 - \exp\left(-c_\beta\left(w_v^{2-\beta} - \left(w_{\min} + \frac{W}{2aw_e}\right)^{2-\beta}\right)\right)\right) \cdot \left(e^{-c_T w_v} - \frac{0.542}{m}\right) f_\beta(w_v)\,\mathrm{d}w_v.
\end{aligned}$$

For the rest of this proof, we find a solvable lower bound to this integral. To do so, we multiply out the integrand and get the following four integrals

$$\begin{aligned}
\geq\; & \underbrace{\int_{w_{\min}}^{w_{\min}+\frac{W}{2aw_e}} e^{-c_T w_v} f_\beta(w_v)\,\mathrm{d}w_v}_{(I)} - \underbrace{\int_{w_{\min}}^{w_{\min}+\frac{W}{2aw_e}} \frac{0.542}{m} f_\beta(w_v)\,\mathrm{d}w_v}_{II} \\
& \underbrace{- \int_{w_{\min}}^{w_{\min}+\frac{W}{2aw_e}} \exp\left(-c_\beta\left(w_v^{2-\beta} - \left(w_{\min} + \frac{W}{2aw_e}\right)^{2-\beta}\right)\right) e^{-c_T w_v} f_\beta(w_v)\,\mathrm{d}w_v}_{(III)} \\
& \underbrace{+ \int_{w_{\min}}^{w_{\min}+\frac{W}{2aw_e}} \exp\left(-c_\beta\left(w_v^{2-\beta} - \left(w_{\min} + \frac{W}{2aw_e}\right)^{2-\beta}\right)\right) \frac{0.542}{m} f_\beta(w_v)\,\mathrm{d}w_v}_{(IV)} .
\end{aligned}$$

To get a lower bound in total, we find a lower bound for the first integrals (I) and (IV), and an upper bound for the integrals (II) and (III).

For (I) we find a lower bound, by using the tangent to the convex function $f_1(x) \coloneqq \exp(-c_T x)$ instead of the function itself. The function $f_1$ is convex as

$$\frac{\partial^2}{\partial x^2}\Big(\exp(-c_T x)\Big) = c_T^2 \exp(-c_T x) > 0$$

(see $\mathcal{C}^2$-Characterization of convex functions). According to the $\mathcal{C}^1$-Characterization of convex functions, every tangent of this function is a lower bound for all $x \in \mathbb{R}$. The tangent to $f_1$ at $x = w_{\min}$ is

$$\begin{aligned} t_{w_{\min}}(x) \coloneqq\ & \exp(-c_T w_{\min}) - c_T \exp(-c_T w_{\min})(x - w_{\min}) \\ =\ & -c_T \exp(-c_T w_{\min})\, x + \exp(-c_T w_{\min})(1 + c_T w_{\min}). \end{aligned}$$

Since exponential functions are strictly positive, $\max(t_{w_{\min}}(w_v), 0)$ is a lower bound for $f_1$, and thus

$$\int_{w_{\min}}^{\infty} \exp(-c_T w_v)\, w_v^{-\beta}\, \mathrm{d}w_v \geq \int_{w_{\min}}^{\infty} \max(t_{w_{\min}}(w_v), 0) w_v^{-\beta}\, \mathrm{d}w_v.$$

The tangent $t_{w_{\min}}$ is only positive for $x \leq \frac{1}{c_T} + w_{\min}$. Therefore, the integral is equal to

$$= \int_{w_{\min}}^{\frac{1}{c_T}+w_{\min}} t_{w_{\min}}(w_v) w_v^{-\beta}\, \mathrm{d}w_v.$$

The solution of this integral is

$$\exp(-c_T w_{\min}) \left( \frac{c_T}{\beta - 2} \cdot \left[x^{2-\beta}\right]_{w_{\min}}^{\frac{1}{c_T}+w_{\min}} - \frac{(1 + c_T w_{\min})}{\beta - 1} \left[x^{1-\beta}\right]_{w_{\min}}^{\frac{1}{c_T}+w_{\min}} \right).$$

To get the lower bound for the whole integral (I), one has to multiply this bound by $\frac{\beta-1}{w_{\min}^{1-\beta}}$.

For integral (II), the solution is $\frac{0.542}{m} F_\beta\left(w_{\min} + \frac{W}{2aw_e}\right)$. Since weights linear in $n$ are highly unlikely, $F_\beta\left(w_{\min} + \frac{W}{2aw_e}\right)$ is close to 1. Therefore, we use the upper bound $\frac{0.542}{m}$ for the integral (II) instead.

For integral (III), i.e.,

$$\exp\left(\underbrace{c_\beta \left(w_{\min} + \frac{W}{2aw_e}\right)^{2-\beta}}_{\coloneqq A}\right) \int_{w_{\min}}^{w_{\min}+\frac{W}{2aw_e}} \exp\left(-c_\beta w_v^{2-\beta}\right) e^{-c_T w_v} f_\beta(w_v)\, \mathrm{d}w_v$$

we get an upper bound by decreasing the absolute value of the exponent in the following way

$$\begin{aligned} &\leq \exp(A) \cdot \int_{w_{\min}}^{w_{\min}+\frac{W}{2aw_e}} \exp\left(-c_\beta w_v^{2-\beta}\right) e^{-c_T w_{\min}} f_\beta(w_v)\, \mathrm{d}w_v \\ &\leq \exp(A - c_T w_{\min}) \cdot \int_{w_{\min}}^{w_{\min}+\frac{W}{2aw_e}} \exp\left(-c_\beta w_v^{1-\beta}\right) f_\beta(w_v)\, \mathrm{d}w_v. \end{aligned}$$

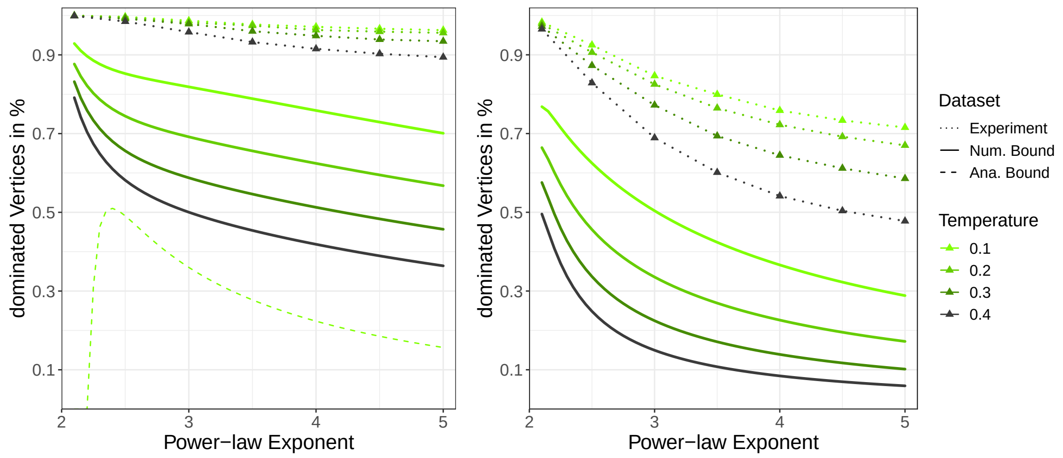


**Figure 5.2:** The diagrams compare the lower bound from Equation (5.6) (in solid lines) with the actual percentages of vertices being eliminated in the first round of the vertex domination rule (in dotted lines). The triangles represent the mean percentage over ten generated hypergraphs. The dashed line in the left diagram shows the analytical bound from Lemma 5.2 for $T = 0.1$. The hypergraphs in the left diagram were generated with $\delta_{EV} = 1$, and the ones in the right with $\delta_{EV} = 0.1$.

This is the same integral as in Equation (5.5). Therefore, integral (III) resolves to

$$= \exp\left(c_\beta\left(w_{\min} + \frac{W}{2aw_e}\right)^{2-\beta} - c_T w_{\min}\right) \cdot \frac{1}{c_\beta(\beta-1)}\left[e^{-x}\right]_{c_\beta w_{\min}^{1-\beta}}^{c_\beta\left(w_{\min}+\frac{W}{2aw_e}\right)^{1-\beta}}.$$

Integral (IV) has a factor of $\frac{0.542}{m}$, which is small enough to bound it to 0. ∎

Similar to Section 5.1, we plot the bound from Lemma 5.2 for some example settings and compare it to real values of $F_{\text{dom}}$ from experiments, to get an understanding of how good this bound is. Note that, we do not use the closed formula from Lemma 5.2 but a numerical solution to the corresponding integral in Equation (5.6) from its proof. The reason for this is, that the closed formula has a large difference to the actual solution of the integral for small power-law exponents. This difference comes largely from the inaccurate estimate we found for integral (III). For comparison, the analytical bound from Lemma 5.2 is shown for an example value of $T = 0.1$ as a dashed line in the left diagram of Figure 5.2.

For each setting, the bound and the values from the experiment are plotted for a range of $\beta \in [2.1; 5]$ and four different temperatures $T = \{0.1, 0.2, 0.3, 0.4\}$. The left diagram shows the setting presented in the main experiment of [BFFS19] about public transportation systems, i.e., $\delta_{EV} = 0.1$ and $\overline{\delta} = 2$. The right diagram uses $\delta_{EV} = 1$ and $\overline{\delta} = 5$. All remaining parameters are equal for both diagrams, which are $w_{\min} = w_e = 1$ and $|V| = 10^4$. Again, we chose $W_c$ to be the expected value $\frac{\beta-1}{\beta-2} w_{\min}$ to calculate the bound values.

The left diagram in Figure 5.2 shows that in the context of public transportation systems, the bound can not entirely explain the high percentages of dominated vertices, which are above 90% for all shown $\beta$ and $T$. Additionally, the absolute difference between the bound and the experimental values increases with the temperature. For $T = 0.1$, the highest difference is at about 25%. For $T = 0.4$, this difference increases to over 50%. However, this bound allows to expect that at least half of all vertices are eliminated in the first round of the vertex domination rule for all $\beta \in (2, 3)$ and $T \leq 0.4$.

This does not hold for the setting in the right diagram. Here, the absolute difference is significantly larger; even for low temperatures like $0.1$ and $0.2$, where the maximal difference is over 40% for high power-law-exponents. Further, the absolute difference for $T = 0.3, 0.4$ is even over 60% for $\beta = 3$. However, it is not surprising that the difference increases with increasing $T$, since we only consider nice vertices, which become less and less likely with increasing temperature.

On the other hand, both diagrams indicate that the bound correctly predicts a decreasing $F_{\text{dom}}$ for an increasing level of heterogeneity and a decreasing level of locality. Unfortunately, this bound is not very tight for many settings. Even for low temperatures, there is a significant difference, that even increases when the temperature rises.

# 6 Conclusion & Outlook

In this thesis, we studied the efficiency of Weihe's HittingSet reduction rules on Hyper-GIRGs from a theoretical perspective. The Hyper-GIRG model served as a representative for real-world networks with adjustable degrees of heterogeneity and locality.

First, we introduced basic terminology and notations around the neighborhood of a vertex and stated basic properties of Hyper-GIRGs. In particular, we analyzed the impact of the vertex weight on its size. We derived simple asymptotic statements for both the degree of a vertex and its diameter, which is the space on the torus that its neighborhood takes up.

In the second part, we focused on the threshold variant of Hyper-GIRGs, in which the incident hyperedges of each vertex are consecutive on the torus. Based on that property, we proved them to be a subset of the more general dual circular arc hypergraphs (CA$^*$-hypergraphs). We showed that the reduction rules reduce all hypergraphs of this class to a trivial kernel if they have at least one gap either before or during the reduction. Additionally, we showed that the reduction rules create a.a.s. at least one gap during their application. Here, the exact probability decreases with increasing $\beta$. For the unlikely case, that there is still no gap after the application of the reduction rules, we describe a safe branching rule that branches over linearly many local hitting sets of a hyperedge and creates gaps in each child instance. These two results can be combined into a polynomial algorithm for HittingSet on CA$^*$-hypergraphs, and can thus also be applied to Hyper-GIRGs generated with the threshold variant. However, for Hyper-GIRGs from the threshold variant, the reduction rules are often enough, as they often create gaps during their application on such Hyper-GIRGs with heterogeneity.

In the third part of this thesis, we focused on the first round of the vertex domination rule and analyzed how many vertices remain after its application. In particular, we were interested in the expected value of the fraction of vertices that can be eliminated by this rule. For this expected value, we found lower bounds for both the threshold variant as well as the noisier binomial variant of the Hyper-GIRG model. These bounds show that a significant fraction of vertices is expected to be eliminated in heterogeneous Hyper-GIRGs with low temperature.

## Future Work

However, these bounds from Chapter 5 can be improved further in future work on this topic. One approach could be to solve or at least approximate the double integral from Equation (5.2), which describes the expected fraction of dominated vertices exactly. This includes, in particular, finding the PDF of the diameter of a vertex and variants that can be handled mathematically. Besides that, the diameter PDF can also be used to find an asymptotic lower bound for the diameter of the vertex depending on the temperature. Additionally, the matching upper bounds from Lemma 3.7 could thereby also be improved.

In general, the question of how the reduction rules perform on Hyper-GIRGs with high temperatures has not been fully resolved from the theoretic perspective; especially as Lemma 5.2 only yields a lower bound of $F_{\text{dom}}$ close to 0. Experiments suggest that the fraction of dominated vertices indeed tends to 0 for $T \to 1$. Therefore, an upper bound for this value would be interesting in these cases. However, this would not prove that the reduction rules do not have a huge impact on Hyper-GIRGs with high temperature, as it only approximates the behavior of the first round of the vertex domination rule. Showing that an exhaustive application of both reduction rules does not strongly alter high-temperature Hyper-GIRGs would be valuable for the theoretical perspective of this topic.

For small temperatures, on the other hand, the bounds from Chapter 5 contribute to the explanation of the effectiveness of the reduction rules. This approach can be developed further, by finding a similar statement for the number of hyperedges that are eliminated in the first round of the hyperedge domination rule. The resulting hypergraphs could admit interesting structural properties that perhaps allow (depending on the temperature) some steps of the algorithm for the threshold Hyper-GIRGs from Lemma 4.2, and therefore further reductions. Especially for temperatures tending to 0, one approach could be to bound the number of nice vertices and analyze how non-nice vertices break the flow of this algorithm. Preliminary experiments on Hyper-GIRGS with low temperature imply that (at least for some sets of parameters) the few fluid incidences do have a huge impact on the reduction.

The algorithm from Lemma 4.2 is strong evidence for the effectiveness of the reduction rules on Hyper-GIRGs generated with the threshold variant. However, one could improve the statement on the probability of gaps in these hypergraphs, which are necessary for the algorithm. One approach for that is to additionally take the positions of the hyperedges into account. This is not done in Section 4.2 since we assumed the cold areas of the vertices to be their neighborhood area, which can greatly increase the diameter in some cases.

Another interesting challenge for future work is to extend the findings from this thesis to Hyper-GIRGs in higher dimensions. As far as we know, there are neither experimental nor theoretical studies on the effectiveness of the reduction rules higher dimensional Hyper-GIRGs. Figure 6.1 shows an example of a Hyper-GIRG on the $\mathbb{T}^2$ generated with the threshold variant. This example shows that the reduction rules do not deterministically reduce these hypergraphs entirely even for $T = 0$. This is probably due to the fact, that small cycles (of size 3 or 4) are likely to be generated in this 2d-setting. However, if threshold Hyper-GIRGs should be reduced entirely, one could think about changing the way Hyper-GIRGs are generated in higher dimensions. One approach could be to take the structure of rail networks as a guide, in which mainly one-dimensional objects are distributed and connected across the two-dimensional plane. But also other changes to the model like heterogeneous hyperedge sizes and other geometries would be interesting.

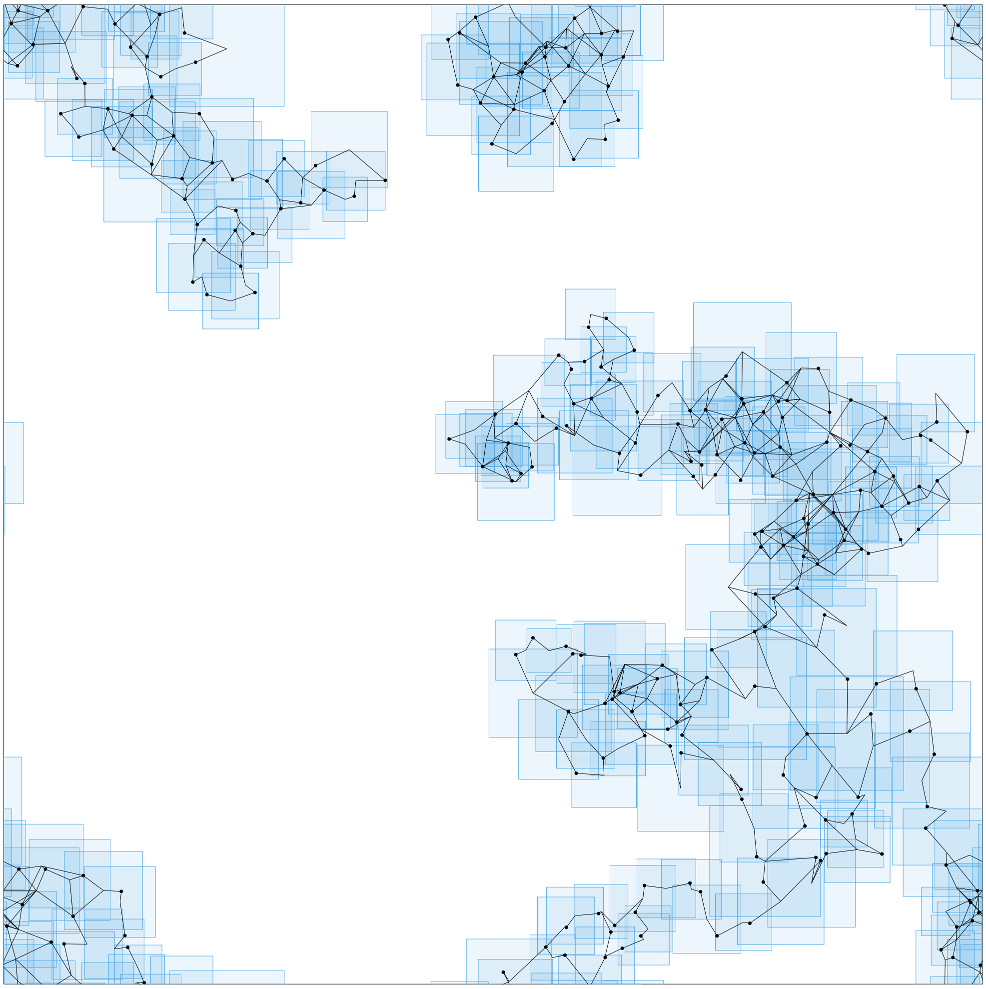

**Figure 6.1:** Kernel of a Hyper-GIRG on the $\mathbb{T}^2$ generated with the threshold variant (including the unit hyperedge rule). The blue squares mark the area around a vertex position in which all hyperedges are incident to the vertex with probability 1. The black dots are the positions of all remaining hyperedges. All incidences are depicted as black lines between the position of the respective hyperedge and vertex. The parameters for this Hyper-GIRG are $n = m = 3000$, $\overline{\delta} = 5$, and $\beta = 3.5$.

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